

Numerical Solution of Stochastic Differential Equations with Jumps in Finance

A Thesis Submitted for the Degree of
Doctor of Philosophy

by

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July 31, 2007

Certificate

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirement for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

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Acknowledgements

I would like to thank my supervisor, Professor Eckhard Platen, for his guidance throughout my PhD studies. I am very grateful to him for sharing his deep knowledge in stochastic numerical methods and finance, for teaching me the principles of research and academic status, and for giving me the opportunity of several academic visits abroad. I would also like to thank my co-supervisor, Associate Professor Erik Schlögl, for his additional supervision.

I have also benefited from discussions with Giovanni Barone-Adesi, Lorenzo Casavecthia, Carl Chiarella, Ernst Eberlein, Paul Embrechts, Damir Filipovic, David Heath, Christina Nikitopoulos-Sklivosios, Massimo Piccardi, Andrea Roncoroni and Wolfgang Runggaldier. A special thank goes to Hardy Hulley, who has given me many useful advices on several technical aspects of this thesis.

I would like to thank Sandro Salsa and Emilio Barucci for their hospitality during several visits at Politecnico di Milano.

The financial assistance received with the scholarships of the Quantitative Finance Research Centre and of Bocconi University is greatly appreciated. Additionally, I acknowledge the financial support that I received from the School of Finance and Economics, from the Financial Integrity Research Network, from the Faculty of Business and from the Vice-Chancellor to attend various conferences.

Finally, I would like to thank my family for their continual support and visits from another continent, and Ellie for being always supportive and for her love.

Contents

Basic Notation	ix
Abstract	xi
1 Introduction	1
1.1 Brief Survey of Results	1
1.2 Motivation	3
1.3 Literature Review	7
1.3.1 Strong Approximations	7
1.3.2 Weak Approximations	9
2 Stochastic Differential Equations with Jumps	11
2.1 Introduction	11
2.2 Existence and Uniqueness of Strong Solutions	15
3 Stochastic Expansions with Jumps	17
3.1 Introduction	17
3.2 Multiple Stochastic Integrals	18
3.2.1 Multi-Indices	18
3.2.2 Multiple Integrals	19
3.3 Coefficient Functions	24

3.4	Hierarchical and Remainder Sets	27
3.5	Wagner-Platen Expansions	28
3.6	Moments of Multiple Stochastic Integrals	30
3.7	Weak Truncated Expansions	51
4	Regular Strong Taylor Approximations	55
4.1	Introduction	55
4.2	Euler Scheme	58
4.3	Order 1.0 Taylor Scheme	60
4.4	Commutativity Conditions	69
4.5	Convergence Results	74
4.6	Lemma on Multiple Itô Integrals	77
4.7	Proof of Theorem 4.5.1	87
5	Regular Strong Itô Approximations	95
5.1	Introduction	95
5.2	Derivative-Free Order 1.0 Scheme	96
5.3	Drift-Implicit Schemes	103
5.3.1	Drift-Implicit Euler Scheme	104
5.3.2	Drift-Implicit Order 1.0 Scheme	105
5.4	Predictor-Corrector Schemes	109
5.4.1	Predictor-Corrector Euler Scheme	109
5.4.2	Predictor-Corrector Order 1.0 Scheme	110
5.5	Convergence Results	114
5.5.1	Derivative-Free Schemes	117
5.5.2	Drift-Implicit Schemes	120

5.5.3	Predictor-Corrector Schemes	125
6	Jump-Adapted Strong Approximations	131
6.1	Introduction	131
6.2	Taylor Schemes	134
6.2.1	Euler Scheme	134
6.2.2	Order 1.0 Taylor Scheme	136
6.2.3	Order 1.5 Taylor Scheme	137
6.3	Derivative-Free Schemes	140
6.3.1	Derivative-Free Order 1.0 Scheme	140
6.3.2	Derivative-Free Order 1.5 Scheme	141
6.4	Drift-Implicit Schemes	142
6.4.1	Drift-Implicit Euler Scheme	142
6.4.2	Drift-Implicit Order 1.0 Scheme	143
6.4.3	Drift-Implicit Order 1.5 Scheme	143
6.5	Predictor-Corrector Schemes	144
6.5.1	Predictor-Corrector Euler Scheme	144
6.5.2	Predictor-Corrector Order 1.0 Scheme	145
6.6	Exact Schemes	147
6.7	Convergence Results	148
7	Numerical Results on Strong Schemes	155
7.1	Introduction	155
7.2	The Case of Low Intensities	156
7.3	The Case of High Intensities	158
8	Strong Schemes for Pure Jump Processes	163

8.1	Introduction	163
8.2	Pure Jump Model	164
8.3	Jump-Adapted Schemes	165
8.4	Euler Scheme	166
8.5	Wagner-Platen Expansion	167
8.6	Order 1.0 Strong Taylor Scheme	171
8.7	Order 1.5 and 2.0 Strong Taylor Schemes	172
8.8	Convergence Results	174
9	Regular Weak Taylor Approximations	181
9.1	Introduction	181
9.2	Euler Scheme	182
9.3	Order 2.0 Taylor Scheme	182
9.4	Commutativity Conditions	189
9.5	Convergence Results	191
10	Jump-Adapted Weak Approximations	199
10.1	Introduction	199
10.2	Taylor Schemes	200
10.2.1	Euler Scheme	200
10.2.2	Order 2.0 Taylor Scheme	201
10.2.3	Order 3.0 Taylor Scheme	204
10.3	Derivative-Free Schemes	206
10.4	Predictor-Corrector Schemes	208
10.4.1	Order 1.0 Predictor-Corrector Scheme	209
10.4.2	Order 2.0 Predictor-Corrector Scheme	210

<i>CONTENTS</i>	vii
10.5 Exact Schemes	212
10.6 Convergence of Jump-Adapted Weak Taylor Approximations	213
10.7 Convergence of Jump-Adapted Weak Approximations	223
10.7.1 Simplified and Predictor-Corrector Schemes	227
11 Numerical Results on Weak Schemes	231
11.1 Introduction	231
11.2 The Case of a Smooth Payoff	232
11.3 The Case of a Non-Smooth Payoff	235
12 Efficiency of Implementation	243
12.1 Introduction	243
12.2 Simplified Weak Schemes	244
12.3 Multi-Point Random Variables and Random Bit Generators	247
12.4 Software Implementation	249
12.4.1 Random Bit Generators in C++	249
12.4.2 Experimental Results	251
12.5 Hardware Accelerators	264
12.5.1 System Architecture	264
12.5.2 FPGA Implementation	267
12.5.3 Experimental Results	269
13 Conclusions and Further Directions of Research	277
13.1 Conclusions	277
13.2 Further Directions of Research	277
A Appendix: Inequalities	281

A.1	Finite Inequalities	281
A.2	Integral Inequalities	281
A.3	Martingale Inequalities	283

Basic Notation

x^\top	transpose of a vector or matrix x ;
$x = (x^1, \dots, x^d)^\top$	column vector $x \in \mathbb{R}^d$ with i th component x^i ;
$ x $	absolute value of x or Euclidean norm;
$A = [a^{i,j}]_{i,j=1}^{k,d}$	$(k \times d)$ -matrix A with ij th component $a^{i,j}$;
$\mathbb{N} = \{1, 2, \dots\}$	set of natural numbers;
$\mathbb{R} = (-\infty, \infty)$	set of real numbers;
$\mathbb{R}^+ = [0, \infty)$	set of nonnegative real numbers;
\mathbb{R}^d	d -dimensional Euclidean space;
(a, b)	open interval $a < x < b$ in \mathbb{R} ;
$[a, b]$	closed interval $a \leq x \leq b$ in \mathbb{R} ;
Ω	sample space;
\emptyset	empty set;
Δ	time step size of a time discretization;
$n! = 1 \cdot 2 \cdot \dots \cdot n$	factorial of n ;
$\binom{i}{l} = \frac{i!}{l!(i-l)!}$	combinatorial coefficient;
$[a]$	largest integer not exceeding $a \in \mathbb{R}$;
$(\text{mod } c)$	modulo c ;
$(a)^+ = \max(a, 0)$	maximum of a and 0;

$\ln(a)$	natural logarithm of a ;
i.i.d.	independent identically distributed;
a.s.	almost surely;
$f : Q_1 \rightarrow Q_2$	function f from Q_1 into Q_2 ;
f'	first derivative of $f : \mathbb{R} \rightarrow \mathbb{R}$;
f''	second derivative of $f : \mathbb{R} \rightarrow \mathbb{R}$;
$\frac{\partial u}{\partial x^i}$	i th partial derivative of $u : \mathbb{R}^d \rightarrow \mathbb{R}$;
$\partial_{x^i}^k u$ or $\left(\frac{\partial}{\partial x^i}\right)^k u$	k th order partial derivative of u with respect to x^i ;
$C^k(\mathbb{R}^d, \mathbb{R})$	set of k times continuously differentiable functions;
$C_P^k(\mathbb{R}^d, \mathbb{R})$	set of k times continuously differentiable functions which, together with their partial derivatives of order up to k , have polynomial growth;
$\mathbf{1}_A$	indicator function for event A to be true;
$\mathcal{N}(\cdot)$	Gaussian distribution function;
\mathcal{A}	collection of events, sigma-algebra;
$\underline{\mathcal{A}}$	filtration;
$E(X)$	expectation of X ;
$E(X \mathcal{A})$	conditional expectation of X under \mathcal{A} ;
$P(A)$	probability of A ;
$\mathcal{B}(U)$	smallest sigma-algebra on U ;
SDE	stochastic differential equation;

Letters such as $K, \tilde{K}, C, \tilde{C}, \dots$ represent finite positive real constants that can vary from line to line. All these constants are assumed to be independent of the time step size Δ . The remaining notation is either standard or will be introduced when used.

Abstract

This thesis concerns the design and analysis of new discrete time approximations for stochastic differential equations (SDEs) driven by Wiener processes and Poisson random measures. In financial modelling, SDEs with jumps are often used to describe the dynamics of state variables such as credit ratings, stock indices, interest rates, exchange rates and electricity prices. The jump component can capture event-driven uncertainties, such as corporate defaults, operational failures or central bank announcements. The thesis proposes new, efficient, and numerically stable strong and weak approximations. Strong approximations provide efficient tools for problems such as filtering, scenario analysis and hedge simulation, while weak approximations are useful for handling problems such as derivative pricing, the evaluation of moments, and the computation of risk measures and expected utilities. The discrete time approximations proposed are divided into regular and jump-adapted schemes. Regular schemes employ time discretizations that do not include the jump times of the Poisson measure. Jump-adapted time discretizations, on the other hand, include these jump times.

The first part of the thesis introduces stochastic expansions for jump diffusions and proves new, powerful lemmas providing moment estimates of multiple stochastic integrals. The second part presents strong approximations with a new strong convergence theorem for higher order general approximations. Innovative strong derivative-free and predictor-corrector schemes are derived. Furthermore, the strong convergence of higher order schemes for pure jump SDEs is established under conditions weaker than those required for jump diffusions. The final part of the thesis presents a weak convergence theorem for jump-adapted higher order general approximations. These approximations include new derivative-free, predictor-corrector, and simplified schemes. Finally, highly efficient implementations of simplified weak schemes based on random bit generators and hardware accelerators are developed and tested.