Portfolio Analysis and Equilibrium Asset Pricing with Heterogeneous Beliefs

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Thesis submitted for Doctor of Philosophy at the University of Technology, Sydney ·

October 2010 ·

Certificate

I certify that this thesis has not previously been submitted for a degree nor has it been submitted as part of requirement for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

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Date . . . .

Acknowledgments

The completion of this thesis would not be possible without the encouragement, guidance and support from my principal supervisor Associate Professor Xue-Zhong (Tony) He. The discussions we had throughout my PhD candidature have proven to be most helpful. His positive attitude towards research and work will continue to be a motivation for my future professional career. I would also like to thank my alternate supervisor Professor Carl Chiarella for his continuous support and the staff and fellow PhD students in the School of Fiance and Economics for providing such a wonderful environment. I appreciate the financial support from the school, Financial Integrity Research Network (FIRN), the Australian Research Council (ARC) and both of my supervisors in contributing to cost of attending numerous national and international conferences and workshops both in Australia and overseas which provided constant and timely feedback for my research. Last but not least, I am grateful to the Australia Commonwealth Government for the APA scholarship, which is of great financial assistance.

Contents

Abstract

The representative agent paradigm with homogeneous expectations has been the dominant framework for the development of theories in portfolio analysis, equilibrium asset pricing and derivative pricing. Homogeneous expectations is the major assumption underlining the most widely used financial models including the *Capital Asset Pricing Model* (CAPM), Lucas's general equilibrium model and the Black-Scholes option-pricing formula. These models are popular because of their clear economic intuition and simplicity. However, this paradigm fails to take into account the heterogeneity, bounded rationality and speculative behaviour of different agents in the economy, which results in models with predictions that lack empirical support. There exist several empirical inconsistencies; (i) The CAPM predicts all investors hold an efficient portfolio in equilibrium and every asset's expected return is linearly related to the market portfolio by the asset's beta. However, it has been found that investors underdiversify in some situations; furthermore, other factors including value, size, momentum and dispersion of analyst forecast also predict future returns, which contradicts CAPM. (ii) Lucas' model predicts that the average equity premium should be proportional to relative risk aversion and covariance between equity return and aggregate consumption; however, the observed equity premium implies an implausibly high relative risk aversion. This is termed the *equity premium puzzle.* (iii) The Black-Scholes model of option pricing predicts that the implied volatility of option prices is independent of time to maturity and strike prices, but the implied volatility in real markets is observed to be skewed. This feature of option prices is called the volatility skew. Although the postulate of unbounded rationality has dominated economic modelling for several decades, empirical evidence, unconvincing justification of the assumption of unbounded rationality and investor psychology have led to the incorporation of heterogeneity in beliefs and bounded rationality into financial modelling. Heterogeneity can have profound consequences for the interpretation of empirical evidence and the formulation of economic policy. Heckman (2001), the 2001 Nobel Laureate in economics, said, "*The most important discovery was the evidence on pervasiveness of heterogeneity and*

diversity in economic life. When a full analysis was made of heterogeneity in response, a variety of candidate averages emerged to describe the "average" person, and the long-standing edifice of the representative consumer was shown to lack empirical support." The aim of this thesis is to use a framework of heterogeneous agents to examine the impact of heterogeneous beliefs on portfolio analysis and asset pricing and explore the potential to explain the observed phenomenon mentioned above. The agents have heterogeneous beliefs regarding future outcomes in the market and the belief of the "average" agent is characterised by the *consensus belief*. The thesis consists of three main components:

- The impact of heterogeneous beliefs on the cross section of asset returns, the geometric tangency relation between the portfolio frontier and the capital market line, and the portfolio efficiency of investors' subjectively optimal portfolios. This is the focus of Chapters 2 and 3. In Chapter 2, investors are assumed to have heterogeneous beliefs about asset payoffs while in Chapter 3 is based on the assumption of heterogeneous beliefs about the rates of return. We find that the tangency relation in the standard portfolio analysis does not hold in general and that adding a riskless asset in zero net supply can increase the marginal utility of the market in some situations. Subjectively optimal portfolios are mean-variance (MV) inefficient in general, depending on the various aspects of heterogeneity amongst investors.

- The relationship between heterogeneity and market risk premium and risk-free rate. This is the focus of Chapters 4 and 5. Chapter 4 considers a multi-asset economy in a static mean-variance framework and Chapter 5 considers a Lucas-type continuous-time general equilibrium model with one risky asset and one riskless asset. In a multi-asset economy, we find that various combinations of heterogeneity can increase the market risk premium and reduce the risk-free rate. The effect is significant in some cases and insignificant in others. In a pure exchange economy, we find that the impact of heterogeneity on the equity premium and interest rate can be magnified under a relative consumption framework.

- The pricing of options under heterogeneous beliefs. Chapter 6 develops a binomial lattice to model investors' subjective beliefs in a multi-period discrete time setting and provides an option-pricing formula under heterogeneous beliefs. The framework is able to replicate various patterns of the implied volatilities observed in the market and provides some economic intuitions and explanations.

The three components together contribute to the growing literature of asset pricing under heterogeneous beliefs by improving the understanding of the impact of heterogeneity in preferences and beliefs on portfolio analysis and equilibrium asset prices.

Chapter 1

Introduction

1.1 Literature Review and Motivation

The representative agent paradigm with homogeneous expectations has been the dominant framework for the development of theories in portfolio analysis, equilibrium asset pricing and derivative pricing. However, despite its simplification power the assumption of homogeneous expectations is unrealistic. Differences of opinion amongst market participants is a pervasive feature of financial markets (see Heckman (2001)) and its importance in determining equilibrium asset prices has been well recognised (see Basak (2005)). Heterogeneous beliefs and their impact on equilibrium asset prices have taken centre stage of the theoretical asset-pricing literature in recent years. There are three main driving forces for the development of the literature in heterogeneous beliefs.

The first one is motivated by the failure of the CAPM to explain the cross section of expected asset returns, in particular the value and size premium originated from Fama and French (1992). More recently, Carhart (1997) and Diether, Malloy and Scherbina (2002) found that momentum in stock return and dispersion in analysts' forecast have explanatory power for the cross section. Furthermore, there has been strong evidence that conditional betas are time-varying; see for example, Kothari, Shanken and Sloan (1995), Campbell and Vuolteenaho (2004) and Adrian and Franzoni (2005). According to Jagannathan and Wang (1996), a conditional CAPM that takes into account conditional expectations provides a convenient way to incorporate time-varying beta and displays superiority in explaining the cross section of returns and anomalies. There is much literature on the estimation of time-varying beta models, mostly in the class of the GARCH model introduced by Engle (1982) and Bollerslev (1986). There is another growing body of literature on heterogeneous agent models (HAMs), which consider the financial market as a nonlinear expectation-feedback system, introduced by Brock and Hommes (1997, 1998) and Lux (1995, 2009). Agents are assumed to follow certain myopic rules of thumb and may switch between strategies based on some performance measures. This class of models is able to generate realistic features of asset returns, including volatility clustering, fat tails and decaying in volatility autocorrelations. Chiarella, Dieci and He (2010*a*, 2010*b*) introduced a multi-asset CAPM framework for analysing the impact of heterogeneous beliefs on asset prices by the construction of a consensus belief and found that heterogeneity becomes part of an asset's systematic risk. Chiarella et. al (2010*c*) used this framework in a dynamic setting and demonstrated the stochastic behavior of time-varying betas and showed that there is an inconsistency between ex-ante and ex-post estimates of asset betas when beliefs are heterogeneous. This suggests that the methods for estimating asset betas currently used in the literature can be inappropriate. Regarding the impact of biased beliefs on asset prices, some argue that investors biases should on average cancel each other out and thus have no impact on asset prices. For example, Levy, Levy and Benita (2006) assume that investors are mean-variance maximisers and differ only in their beliefs of the expected future asset returns. They show that the standard CAPM holds if there is an infinite number of investors and risky assets and investors are on average unbiased. Miller (1977) argues that stocks with a great divergence of opinions regarding their future payoff should have a higher equilibrium price and a lower expected return compared to an otherwise similar stock. His reason is that a short-sale constraint prevents investors with a pessimistic view of the future payoff from affecting the equilibrium asset price. Miller's claim has found both empirical support (Diether et. al (2002)) and theoretical support (Johnson (2004)); however, short-sale constraints do not seem to explain a significant part of the observed phenomenon. Anderson, Ghysels and Juegens (2005) attempt to estimate a disagreement asset-pricing model by using data on the dispersion of analyst forecast to test whether heterogeneous beliefs is a pricing factor. They show that heterogeneity matters for asset pricing, however, the effect is not significant. Anderson, Ghysels and Juegens (2010) found that uncertainty also matters, but by no means does it provide a complete explanation of the cross section of stock returns. Exactly how does investors' heterogeneity impacts on the cross section of expected returns of multiple assets in general remains an unsolved issue.

Secondly, the development of disagreement models is largely due to the equity risk premium and the risk-free rate puzzle discovered by Mehra and Prescott (1985) and Weil (1989). Essentially, the puzzle arises from the fact that the observed equity premium is too high and the risk-free rate is too low if we simplify the market as one representative agent endowed with the total output of the economy and with a reasonable risk-aversion coefficient. Since then, many models have assumed investors to be bounded rational and attempted to explain the puzzles. For example, Abel (2002) shows that pessimism and doubt at the aggregate level can increase the equity premium and reduce the risk-free rate. Jouini and Napp (2006) show that a positive correlation between risk tolerance and pessimism/doubt at the individual level can also help to solve the puzzles. David (2008) assumes that investors are the same except their beliefs about the growth rate of the endowment and dividend processes both follow a two-state Markov chain; calibration shows that half of the equity premium can be explained. The problem is that most of these models require implausible parameter values to explain the puzzles, or the explanatory power is not significant. There are also attempts to resolve the puzzles by means other than assuming heterogeneous beliefs, which include habit formation (Constantinides (1990), Sundaresan (1989) and Abel (1990,1999)), separating risk aversion and intertemporal substitution (Epstein and Zin (1989,1991) and Weil (1989)), irrational expectations, based on the theory of behavioural finance (Barberis and Thaler (2003) and Shefrin (2005)), liquidity premium (Bansal and Coleman (1996)), borrowing constraints (Constantinides, Donaldson and Mehra (2002)) and idiosyncratic risk in labour income (Constantinides and Duffie (1996) and Freeman (2002)). However, most if not all the models either have difficulty explaining both puzzles simultaneously or can only partially explain them. Some have dealt with market imperfections in a continuous-time framework. For example, Basak and Cuoco (1998) dealt with restricted market participation, Basak (2000) with non-fundamental risks and Detemple and Serrat (2003) with liquidity constraints. We refer to the survey papers by Campbell (2003) and Mehra and Prescott (2003) for more discussions along these lines.

The third driving force for the study of heterogeneous beliefs concerns the survival of irrational agents. Friedman (1953) argues that irrational investors will consistently lose money and be driven out of the market by rational investors and therefore they have no price impact. Based on a partial equilibrium model, DeLong, Shleifer, Summers and Waldmann (1991) suggest that traders with erroneous beliefs may hold portfolios with higher growth rates and therefore can eventually outgrow the rational traders and survive in the long run. In contrast, Sandroni (2000) and Blume and Easley (2006) use a general equilibrium approach with intermediate consumption and show that irrational traders do not survive in the long run if the market is complete. When investors only care about their terminal consumption and irrational investors have constant beliefs about the drift of the endowment process, Kogan, Ross, Wang and Westerfield (2006) demonstrate that survival and price impact are two independent concepts. They show that survival is not a necessary condition for the irrational trader to influence long-run prices. It is often found in this literature that, irrational agents may not survive, and instead become extinct after a long time (such as hundreds of years). Therefore, they can have an impact on the market before becoming extinct. Most of the models on agents' survival assume the market is frictionless and at least one agent in the market must be completely rational, which is a strong assumption. It is not clear which agent will survive if none of the agents are completely rational.

Another recent development in asset-pricing models under heterogeneous beliefs is motivated by option pricing. The famous Black and Scholes (BS) hedging argument implies that options are redundant securities. However, the enormous trading volume in the derivatives market suggests otherwise. Empirically, it is observed that the volatility inferred from option prices is neither constant with respect to the strike price nor time to maturity, which violates the key assumption underlying the BS model. Several pricing models have been proposed to overcome these problems. These include stochastic volatility models (Hull and White (1987), Wiggins (1987), Melino and Turnbull (1990), Heston (1993)), GARCH models (Duan (1995), Heston and Nandi (2000) and models with jumps in the underlying price process (Merton (1976), Bates (1991)). By modifying the stochastic process followed by the underlying asset price, some of these models can be calibrated to the currently observed volatility surface. However, these models do not provide any direct economic explanations for this phenomenon observed in the option market. Another strand of literature proposes to solve the problem by assuming incomplete information, model uncertainty and rational learning. In Guidolin and Timmerman (2003, 2007), one representative Bayesian learner is assumed, whereas others assume heterogeneous agents with different priors learn rationally from observed quantities (Buraschi and Jiltsov (2006), Li (2007) and Cao and Ou-Yang (2009)). Buraschi and Jiltsov (2006) assume that investors observe the dividend process and a signal that correlates with the growth rate. They use the model to explain open interest in the option market since options are non-redundant in an incomplete market. Li (2007) assumes investors have different time preferences as well as heterogeneous beliefs about the dividend process. Cao and Ou-Yang (2009) analyses the effects of differences of opinion on the dynamics of trading volume in stocks and options. They find that differences in the mean and precision of the terminal stock payoff have impacts on the trading of stocks and options. In general, models with uncertainty and learning provide a better explanation for the observed implied volatilities than the models mentioned in the beginning of this paragraph in terms of economic intuition and fitting the volatility surface. The current equilibrium option-pricing models typically assume that investors agree on the model except for the drift of the dividend process. However, in reality investors may have completely different models for the price dynamics of the underlying asset. Hence, one requires a simple framework which is general enough to take into account investors' diverse beliefs about the future evolution of the underlying asset price.

This thesis is largely motivated by the above literature, in particular, the study of the impact of heterogeneous beliefs on market equilibrium through defining and constructing a consensus belief, which aggregates individuals' subjective beliefs, through the work of Chiarella, Dieci and He (2010*a*, 2010*b*). Within a mean-variance framework, they developed a CAPM under heterogeneous beliefs and show that the consensus belief reflects the belief of the aggregate market regarding future asset returns and determines equilibrium asset prices. The consensus belief is determined endogenously by investors' heterogeneity in taste (risk aversion), beliefs, and endowments. The aim of this thesis is to examine the impact of heterogeneity on market equilibrium through its impact on the consensus belief. The concept of consensus belief allows us to understand the complex effect of heterogeneity on market equilibrium while keeping the models parsimonious and tractable. This thesis shows that different combinations of heterogeneity can have different effects on portfolio analysis, equilibrium asset prices and option prices, and provides explanations about these mentioned market anomalies and puzzles.

1.2 Structure of the Thesis

The thesis consists of three main components. The first part, consisting of Chapters 2 and 3, is devoted to portfolio analysis and cross-sectional analysis under heterogeneity. The second part, consisting of Chapters 4 and 5, focuses on the impact of different combinations of heterogeneity in beliefs and preferences at the micro level on endogenous quantities at the aggregate level, including the risk-free rate and the market risk premium. Chapter 4 considers a static multi-asset economy whereas Chapter 5 considers a pure-exchange Arrow-Debreu economy in continuous time. The option pricing under heterogeneous beliefs is studied in Chapter 6 under a binomial lattice framework. Chapter 7 summarises the main results of the thesis and related future research is discussed. All proofs are collected in Appendix A (unless specified otherwise).

1.2.1 Mean-Variance (MV) Analysis with Heterogeneous Agents

Markowitz's mean-variance (MV) analysis plays an important role in portfolio analysis and development of asset pricing. It suggests that all agents should invest in MV efficient portfolios that lie on the efficient portfolio frontier. When beliefs regarding the joint probability distribution of future asset returns are common, the (MV) efficiency of agents' optimal portfolios are identical. Furthermore, under modern portfolio theory, there exists a tangency portfolio between portfolio frontiers with and without a riskless asset; this tangency portfolio is the market portfolio in equilibrium. However, these properties may no longer hold when beliefs are heterogeneous, in particular about the expected values and variance/covariances of future asset returns. Chapters 2 and 3 of this thesis are devoted to address the following questions; (i) How is the MV efficiency of agents' subjectively optimal portfolios affected by differences in their beliefs? (ii) What is the impact of heterogeneity on the geometric tangency relation and what are the conditions under which the tangency relation still holds? (iii) If investors are on average unbiased with respect to a benchmark belief, when is the benchmark belief also the belief of the heterogeneous market? (iv) Does the market benefit from diversity in beliefs? (v) What is the relationship between divergence in opinions and expected asset returns? To shed light on these questions, we adopt and generalise the MV framework of Chiarella et al. (2010*a*, 2010*b*), in which a consensus belief is introduced to transform the original heterogeneous economy into an equivalent homogeneous economy under which the effect of heterogeneity on market equilibrium is examined. We also generalise their results by relaxing the assumption of a riskless asset in the economy and derive a zero-beta CAPM under heterogeneous beliefs.

1.2.2 Differences in Opinion and Asset Pricing

In the standard representative agent approach, all investors are assumed to know the true probability distribution of either future asset returns or the aggregate endowment. When agents are allowed to have heterogeneous beliefs, the belief of the market can be represented by the consensus belief, which if held by all the agents generates the same equilibrium asset prices. Heterogeneity has an impact on equilibrium asset prices when the consensus belief does not conform to the objective belief given by the true probability distribution. Chapters 4 and 5 analyse this impact within a static mean-variance framework and a continuous-time equilibrium model, respectively. In a simplified economy with two agents and two risky assets, we are able to characterise the consensus belief explicitly given the mean preserving spreads on agents' biased beliefs and risk tolerance and analyse the impact of heterogeneity on market equilibrium in terms of the risk-free rate, market risk premium, market portfolio, market volatility and the Sharpe ratio. Analytical results are obtained when we have homogeneous beliefs about the variance/covariances of asset returns. Results are also extended to a continuum of agents and a consumption market. In Chapter 5, heterogeneous beliefs are introduced in a continuous-time relative consumption framework where agents' objective is to maximise expected utility of relative consumption. Under this framework, agents not only agree to disagree, but also incorporate their knowledge about the other agent's belief to form their optimal portfolios. We show that even "small" heterogeneity in belief can have a significant impact on asset prices under this framework.

1.2.3 Heterogeneity and Option Prices

The celebrated Black-Scholes (BS) option-pricing formula is based on a hedging argument such that the option price is purely determined by the prices of the underlying asset and a risk-free bond. The BS option price is also the equilibrium price in an economy where agents have a *constant relative risk aversion* (CRRA) utility function and homogeneous beliefs regarding the probability distributions of the terminal stock price and consumption. When beliefs of the underlying asset price dynamics are heterogeneous, the risk-free rate is no longer constant, but rather a wealth-weighted average of the risk-free rate under each agent's subjective belief (see Detemple and Murthy (1994)), which violates the underlying the assumption of the BS model. Furthermore, when agents disagree on the actual outcome of asset return in each state, each agent perceives different state prices, which means that even though the market is complete and an option written on the risky asset can be replicated, the price of the option is different under the subjective belief of each investor[1]. The question is, under which belief should one price the options written on the risky asset? Chapter 6 is largely motivated by this question. It uses the concept of a consensus belief within a binomial lattice framework. A *fair* belief is defined under which every agent's wealth share is a martingale. This fair belief is then used for pricing options and contingency claims in general.

[1] We are not looking at the pricing of options in an incomplete market under which there is an interaction between the option and equity market.

Chapter 2

Portfolio Analysis under Heterogeneous Beliefs about Payoffs

2.1 Introduction

The Capital Asset Pricing Model (CAPM) developed by Sharpe (1964), Lintner (1965) and Mossin (1966) is perhaps the most influential equilibrium model in modern finance. It is based on the assumptions that investors have homogeneous beliefs about the means and variances/covariances of risky assets and there is unrestricted borrowing and lending of a risk-free asset. To relax these unrealistic assumptions, Lintner (1969) extends the CAPM by incorporating heterogeneous beliefs among investors. To provide a theoretical explanation of the early empirical tests of the CAPM, Black (1972) removes the risk-free asset and develops the well-known zero-beta CAPM. As a matter of fact, there is no absolute risk-free asset in financial markets and therefore it is important to examine markets without a risk-free asset. Equilibrium models have been developed in the literature to examine the impact of heterogeneity amongst investors on market equilibrium[1]. Assuming that investors are bounded rational, heterogeneity may be caused by differences in information or differences in opinion[2].

In the mean-variance (MV) literature, the impact of heterogeneous beliefs is mostly

[1]Some have considered the problem in discrete time (for example, see Lintner (1969), Rubinstein (1976), Fan (2003), Sun and Yang (2003), Chiarella et al (2010*b*) and Sharpe (2007)) and others in continuous time (for example, see Williams (1977), Detemple and Murthy (1994) and Zapatero (1998)), and more recently Jouni and Napp (2006, 2007), Hara (2009) and Brown and Rogers(2009). Some models are in the MV framework (see, Lintner (1969), Williams (1977) and Sun and Yang (2003)), others are in the Arrow-Debreu contingent claims economy (see, for example Rubinstein (1976) and Abel (2002)).

[2]In the first case, investors may update their beliefs as new information become available, Bayesian updating rule is often used (see, for example, Williams (1977) and Zapatero (1998)). In the second case, investors agree to disagree and may revise their portfolio strategies as their views of the market change over time (see, for example, Lintner (1969), Rubinstein (1975) and Brown and Rogers(2009)). For a discussion on the difference of the two cases, we refer the reader to a survey paper by Kurz(2009).

studied in the context of a portfolio of one risky asset and one risk-free asset. Lintner (1969) is the first to consider the CAPM with heterogeneous beliefs and without a risk-free asset and shows that heterogeneity does not change the structure of capital asset prices in any significant way, and removing risk-free asset is a mere extension of the case with a risk-free asset. Surprisingly, this significant contribution from Lintner has not been paid much attention until recent years[3]. The main obstacle in dealing with heterogeneity is the complexity and heavy notation involved when the number of assets and the dimension of the heterogeneity increase, it makes analysis of the impact of heterogeneity on the market equilibrium prices complicated and hard to follow (see Lintner (1969)). Recently, Sun and Yang (2003) provided conditions for the existence of the market equilibrium and showed that the zero-beta CAPM still holds under heterogeneous beliefs within the MV framework. However, they do not provide the market equilibrium price and examine the impact of heterogeneity on the market equilibrium price, including MV efficiency of the optimal portfolios of heterogeneous investors. When investors have heterogeneous beliefs about the means and variances/covariances of asset returns, in general it is expected that the subjectively optimal portfolios are no longer MV efficient. If we treat managed funds as subjectively optimal portfolios, the MV inefficiency would imply the under-performance of the managed funds. This chapter is devoted to presenting an explicit equilibrium price formula, and examining the impacts of the heterogeneous beliefs on the MV efficiency of the optimal portfolios and on the market equilibrium in general.

Recently, Chiarella, Dieci and He (2010*b*) use the concept of *consensus belief* to show that, when there is a riskless asset, the market consensus belief can be constructed explicitly as a weighted average of the heterogeneous beliefs. They show that the market equilibrium price vector is a weighted average of the equilibrium price vector perceived by each investor. They also establish a CAPM-like relation under heterogeneous beliefs. The concept of a consensus belief was first introduced by Lintner (1969) in a mean-variance framework and then Rubinstein (1974, 1975), who provided an aggregation theorem for constructing a consensus belief in a state-preference framework and later used it for the study of market efficiency. In this chapter, we first extend the analysis of Chiarella et al. (2010*b*) to a case where there is no riskless asset and then obtain a zero-beta CAPM-like relation under heterogeneous beliefs. It is well known that the geometric tangency relation of traditional portfolio theory plays a very important role in the establishment of the CAPM [4]. We demonstrate that this geometric relationship does not hold under heterogeneous beliefs.

[3]See, for example, Wenzelburger(2004), Böhm and Chiarella (2005), Böhm and Wenzelburger (2005), and Chiarella et al. (2005, 2007, 2009, 2010*b*).

[4]The market portfolio remains the same and MV efficient with or without the existence of a riskless security

This chapter is related to the work of Jouni and Napp (2006, 2007), who investigate the impact of beliefs heterogeneity on the consumption CAPM and the risk-free rate by constructing a consensus belief and consumer. They show how pessimism and doubt at the aggregate level result from pessimism and doubt at the individual level. The construction of the consensus belief in this chapter shares some similarity (in a much simpler and more explicit way within the MV framework) to that in Jouni and Napp; however, our focus is on the portfolio analysis and MV efficiency of the subjectively optimal portfolios, rather than on the risk premium. In other words, the focus of Jouni and Napp is on the impact of the aggregation of heterogeneous beliefs on the market, while we focus on the impact of the aggregation on the MV efficiency of individuals' optimal portfolio. Also, we compare the market MV frontiers with and without a riskless asset and focus on the impact of the heterogeneous beliefs on the geometric relation of the frontiers. Interestingly, a similar result on the MV efficiency of the optimal portfolio of heterogeneous beliefs is found in Easley and O'Hara (2004), where the heterogeneous beliefs are due to information asymmetry[5]. With a rational expectations equilibrium model, they show that the average market portfolio is MV efficient, but not necessarily for investors with different information. In our setup, investors are bounded rational in the sense that they choose their optimal portfolio based on their own beliefs and the market consensus belief is endogenously determined by all market participants. We show that the market portfolio is always MV efficient (by the construction of the consensus belief) and the subjectively optimal portfolios of investors are inefficient in general.

This chapter is structured as follows. In Section 2.2, we introduce and construct the market consensus belief linking the heterogeneous market with an equivalent homogeneous market, and present an explicit market equilibrium price formula. Consequently, a zero-beta CAPM under the heterogeneous beliefs is derived. In Section 2.3, we examine the impacts of different aspects of heterogenous beliefs on the market equilibrium. Through some numerical examples, Section 2.4 examines the implications of heterogeneity on the MV efficiency of the optimal portfolios of heterogeneous investors and the geometric tangency relation of the portfolios with and without a riskless asset. Section 2.5 extends numerical analysis to a market with many investors and examines the impact of heterogeneity on MV efficiency and on the market when the belief dispersions are characterised by a mean-preserved spread. Section 2.6 summarises and concludes the chapter.

[5]The heterogeneity can be due to either asymmetric information or different interpretation about the same information among investors in general. In the first case, certain structures on information and learning (such as Bayesian updating and learning) are imposed, while in the second case, the heterogeneous beliefs are associated with certain trading strategies used in financial markets (such as the momentum and contrarian strategies).

2.2 MV Equilibrium Asset Prices Under Heterogeneous Beliefs

When a financial market consists of investors with different views on the future movement of the market, it is important to understand how market equilibrium is obtained and the roles played by different investors. Within the standard MV framework, in this section, we first introduce heterogeneous beliefs among investors and a concept of market consensus belief to reflect the market belief when the market is in equilibrium. By constructing the consensus belief explicitly, we characterise the equilibrium asset prices. Consequently, we obtain a zero-beta CAPM-like relation under heterogeneous beliefs.

2.2.1 Heterogeneous Beliefs

Following Lintner (1969) and Black (1972), we extend the static MV model with homogeneous belief and consider a market in which there are many risky assets but there is no risk-free asset and investors have heterogeneous beliefs about the future payoffs of risky assets. This set up of the market is similar to that in Chiarella et al. (2010*b*) except that they assume there is a risk-free asset.

Consider a market with N risky assets, indexed by $j = 1, 2, \cdots, N$ and I investors indexed by $i = 1, 2, \cdots, I$. Let $\tilde{\mathbf{x}} = (\tilde{x}_1, \cdots, \tilde{x}_N)^T$ be the random payoff vector of the risky assets, which are jointly normally distributed. Assume that each investor has his/her own set of beliefs about the market in terms of means, variances and covariances of the payoffs of the assets, denoted by

$$y_{i,j} = E_i[\tilde{x}_j], \quad \sigma_{i,jk} = Cov_i(\tilde{x}_j, \tilde{x}_k) \qquad \text{for } 1 \leq i \leq I, \quad 1 \leq j, k \leq N. \tag{2.1}$$

For investor i, we define the mean vector and covariance matrix of the payoffs of N assets as follows, $\mathbf{y}_i = \mathbb{E}_i(\tilde{\mathbf{x}}) = (y_{i,1}, y_{i,2}, \cdots, y_{i,N})^T$ and $\Omega_i = (\sigma_{i,jk})_{N\times N}$. Denote $\mathcal{B}_i = (\mathbb{E}_i(\tilde{\mathbf{x}}), \Omega_i)$ the set of subjective beliefs of investor i. Let $\mathbf{z}_i = (z_{i,1}, z_{i,2}, \cdots, z_{i,N})^T$ be the portfolio in the risky assets (in quantity) and $W_{i,o}$ be the initial wealth of investor i. Then the end-of-period portfolio wealth of investor i is given by $\tilde{W}_i = \tilde{\mathbf{x}}^T\mathbf{z}_i$. Under the belief $\mathcal{B}_i$, the mean and variance of the portfolio wealth $\tilde{W}_i$ of investor i are given, respectively, by

$$\mathbb{E}_i(\tilde{W}_i) = \mathbf{y}_i^T\mathbf{z}_i, \qquad \sigma_i^2(\tilde{W}_i) = \mathbf{z}_i^T\Omega_i\mathbf{z}_i. \tag{2.2}$$

As in the standard MV framework, we assume that investor i has a constant absolute risk aversion (CARA) utility function $U_i(w) = -e^{-\theta_i w}$, where θ_i is the CARA coefficient, and the end-of-period wealth $\tilde{W}_i$ of investor i is normally distributed. Under these assumptions, maximising investor i's expected utility of wealth is equivalent to maximising his/her certainty equivalent end-of-period wealth

$$Q_i(\mathbf{z}_i) := \mathbb{E}_i(\tilde{W}_i) - \frac{\theta_i}{2}\sigma_i^2(\tilde{W}_i) = \mathbf{y}_i^T\mathbf{z}_i - \frac{\theta_i}{2}\mathbf{z}_i^T\Omega_i\mathbf{z}_i$$

subject to the wealth constraint

$$\mathbf{p}_0^T\mathbf{z}_i = W_{i,o}, \tag{2.3}$$

where $\mathbf{p}_0$ is the market price vector of the risky assets. Applying the first-order conditions, we obtain the following lemma on the optimal portfolio of the investor.

Lemma 2.1 *For the given market price vector $\mathbf{p}_0$ of risky assets, the optimal risky portfolio $\mathbf{z}_i^*$ of investor i is uniquely determined by*

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}[\mathbf{y}_i - \lambda_i^*\mathbf{p}_0], \tag{2.4}$$

where

$$\lambda_i^* = \frac{\mathbf{p}_0^T\Omega_i^{-1}\mathbf{y}_i - \theta_i W_{i,o}}{\mathbf{p}_0^T\Omega_i^{-1}\mathbf{p}_0}. \tag{2.5}$$

Lemma 2.1 implies that the optimal demand of investor i depends on his/her constant absolute risk aversion (CARA) coefficient (θ_i), the expected payoffs and covariance matrix of the risky asset payoffs, the Lagrange multiplier (λ_i^*), as well as the market price of the risky assets. Following Lintner (1969), λ_i^* is a shadow price, measuring the *marginal real (riskless) certainty-equivalent of investor i's end-of-period wealth.* In fact, applying the first-order condition, we obtain $\frac{\partial Q_i(\mathbf{z}_i^*)}{\partial \mathbf{z}_i} = \lambda_i^*\mathbf{p}_o$, which leads to

$$\lambda_i^* = \frac{1}{p_{oj}}\frac{\partial Q_i(\mathbf{z}_i^*)}{\partial z_{ij}} \qquad \text{for all} \quad j = 1, 2, \cdots, N. \tag{2.6}$$

More precisely, equation (2.6) indicates that λ_i^* actually measures *investor i's optimal marginal certainty equivalent end-of-period wealth per unit of asset j relative to its market price* and it is a constant across all risky assets. In general, the shadow price is not necessarily the same for all investors; however, it becomes the same when there exists a risk-free asset in the market. In fact, let the current price of the risk-free asset f be 1 and its payoff be $R_f = 1 + r_f$. Applying (2.6) to the risk-free asset leads to $\lambda_i^* = R_f$ for all investors, that is, the shadow price is equal to the payoff of the risk-free asset.

2.2.2 Consensus Belief and Equilibrium Asset Prices

We define the market equilibrium asset-price vector $\mathbf{p}_o$ of the risky assets as the price vector under which the individual's optimal demands (2.4) satisfy the market aggregation condition

$$\sum_{i=1}^{I} \mathbf{z}_i^* = \sum_{i=1}^{I} \bar{\mathbf{z}}_i := \mathbf{z}_m, \tag{2.7}$$

where $\bar{\mathbf{z}}_i$ is the endowment portfolio of investor i. Correspondingly, $\mathbf{z}_m$ is the market portfolio of the risky assets. It then follows from (2.7) and (2.4) that the market equilibrium price $\mathbf{p}_o$ is given, in terms of the heterogeneous beliefs of the investors, by

$$\mathbf{p}_0 = \left(\sum_{i=1}^{I} \theta_i^{-1} \lambda_i^* \Omega_i^{-1} \right)^{-1} \left[\left(\sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} \mathbf{y}_i \right) - \mathbf{z}_m \right]. \tag{2.8}$$

This expression defines the market equilibrium price $\mathbf{p}_o$ implicitly since λ_i^* depends on $\mathbf{p}_o$ as well. For the existence of the market equilibrium price, we refer to Sun and Yang (2003) and the references cited there. The concept of consensus belief has been used to characterise the market when investors are heterogeneous in different context, (such as Jouini and Napp (2006, 2007) and Chiarella et al. (2010*b*)). It is closely related to but significantly different from the concept of a representative investor in the standard finance literature. It is endogenously determined through the market aggregation and reflects a weighted average of heterogeneous beliefs. We now introduce the concept of a consensus belief in a market with heterogeneous beliefs.

Definition 2.2 *A belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$, defined by the expected payoff of the risky assets $\mathbb{E}_a(\tilde{\mathbf{x}})$ and the covariance matrix of the risky asset payoffs Ω_a, is called a market* **consensus belief** *if the market equilibrium price under the heterogeneous beliefs is also the market equilibrium price under the homogeneous belief $\mathcal{B}_a$.*

When a consensus belief exists, the market with heterogeneous beliefs can be treated as a market with a homogeneous consensus belief and then the standard Markowtiz portfolio analysis can be applied. Due to the complexity of heterogeneity, the existence and identification of such a consensus belief is an obstacle that makes the examination of the impact of the heterogeneity difficult. In the following, we construct the consensus belief explicitly, from which market equilibrium prices $\mathbf{p}_0$ can be determined explicitly in terms of the consensus belief. It is the explicit construction of the consensus belief that makes it easy to examine the role of heterogeneous beliefs in determining the market equilibrium price and to derive the zero-beta CAPM relation.

Proposition 2.3 *Let*

$$\theta_a := \left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\right)^{-1}, \qquad \lambda_a^* := \frac{1}{I}\theta_a\sum_{i=1}^{I}\theta_i^{-1}\lambda_i^*.$$

Then

(i) the consensus belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$ *is given by*

$$\Omega_a = \theta_a^{-1}\lambda_a^*\left(\frac{1}{I}\sum_{i=1}^{I}\lambda_i^*\theta_i^{-1}\Omega_i^{-1}\right)^{-1}, \tag{2.9}$$

$$\mathbf{y}_a := \mathbb{E}_a(\tilde{\mathbf{x}}) = \theta_a\Omega_a\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\mathbb{E}_i(\tilde{\mathbf{x}})\right); \tag{2.10}$$

(ii) the market equilibrium price $\mathbf{p}_o$ *is determined by*

$$\mathbf{p}_0 = \frac{1}{\lambda_a^*}\left[\mathbf{y}_a - \frac{1}{I}\theta_a\Omega_a\mathbf{z}_m\right]; \tag{2.11}$$

(iii) the equilibrium optimal portfolio of investor i *is given by*

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}\left[(\mathbf{y}_i - \frac{\lambda_i^*}{\lambda_a^*}\mathbf{y}_a) + \frac{\lambda_i^*}{I\lambda_a^*}\theta_a\Omega_a\mathbf{z}_m\right]. \tag{2.12}$$

Proposition 2.3 shows how the consensus belief can be constructed explicitly from heterogeneous beliefs. Under the consensus belief, the market equilibrium prices of the risky assets are determined as in the standard way with no risk-free asset. Intuitively, Proposition 2.3 indicates that the market consensus belief is a weighted average of heterogeneous beliefs. More precisely, the market risk tolerance $(1/\theta_a)$ is simply an average of the risk tolerance of the heterogeneous investors, according to Huang and Lizenberger (1988), $\theta_a/I = (\sum_i^I \theta_i^{-1})^{-1}$ is called the *aggregate absolute risk-aversion*. The weighted average behaviour can also be viewed in the following way. Let $\tau_i = 1/\theta_i$ be the risk tolerance of investor i and $\tau_a = \sum_{i=1}^I \tau_i$ be the market aggregate risk tolerance. Then

$$\lambda_a^* = \sum_{i=1}^{I}\frac{\tau_i}{\tau_a}\lambda_i^*, \qquad \Omega_a^{-1} = \sum_{i=1}^{I}\frac{\tau_i\lambda_i}{\tau_a\lambda_a}\Omega_i^{-1}, \qquad \mathbb{E}_a(\tilde{\mathbf{x}}) = \Omega_a\sum_{i=1}^{I}\frac{\tau_i}{\tau_a}\Omega_i^{-1}\mathbb{E}(\tilde{\mathbf{x}}).$$

Hence the precision matrix (Ω_a^{-1}) for the market reflects a weighted average of the precision matrices of all investors and the market's expected payoff is a weighted average of the expected payoffs of the investors. The market equilibrium prices are determined such that each investor can choose their optimal portfolio subjectively and the market is cleared. It follows from (2.4) in Lemma 2.1 that $\mathbf{p}_0 = \frac{1}{\lambda_i^*}(\mathbf{y}_i - \frac{1}{\tau_i}\Omega_i \mathbf{z}_i^*)$ for $i = 1, \cdots, I$. However, if the entire market acts as an aggregate investor, then for the market to clear, the prices must be determined by the consensus belief, as in (2.11). This suggests that the consensus belief $\mathcal{B}_a$ must correspond to the belief of the aggregate market such that the market portfolio is an optimal portfolio. The expressions in Proposition 2.3 provide explicit relationships between the heterogeneous belief and the market consensus belief under the market aggregation. Their usefulness will be revealed when we derive a zero-beta CAPM-like relation and examine the impacts of heterogeneity on the market equilibrium in the following subsections.

2.2.3 The Zero-Beta CAPM under Heterogeneous Beliefs

As a corollary of Proposition 2.3, we now show that a zero-beta CAPM-like relation holds under the constructed consensus belief with no risk-free asset.

Let the future payoff of the market portfolio $\mathbf{z}_m$ be given by $\tilde{W}_m = \tilde{\mathbf{x}}^T \mathbf{z}_m$ and its current market value be $W_{m,o} = \mathbf{z}_m^T \mathbf{p}_0 = \sum_{i=1}^I W_{i,o}$. Hence, under the consensus belief $\mathcal{B}_a$, $\mathbb{E}_a(\tilde{W}_m) = \mathbf{y}_a^T \mathbf{z}_m$ and $\sigma_a^2(\tilde{W}_m) = \mathbf{z}_m^T \Omega_a \mathbf{z}_m$. Define the return vector $\tilde{\mathbf{r}} = (\tilde{r}_1, \cdots, \tilde{r}_N)^T$ with $\tilde{r}_j = \tilde{x}_j / p_{j,o} - 1$ and $\tilde{r}_m = \tilde{W}_m / W_{m,o} - 1$. Under the market consensus belief $\mathcal{B}_a$, we set

$$\mathbb{E}_a(\tilde{r}_j) = \frac{\mathbb{E}_a(\tilde{x}_j)}{p_{j,o}} - 1, \qquad \mathbb{E}_a(\tilde{r}_m) = \frac{\mathbb{E}_a(\tilde{W}_m)}{W_{m,o}} - 1 \qquad \sigma_a^2(\tilde{r}_m) = \frac{\sigma_a^2(\tilde{W}_m)}{W_{m,o}^2}$$

and

$$Cov_a(\tilde{r}_j, \tilde{r}_m) = \frac{1}{p_{j,o} W_{m,o}} Cov_a(\tilde{x}_j, \tilde{W}_m), \quad Cov_a(\tilde{r}_j, \tilde{r}_k) = \frac{1}{p_{j,o} p_{k,o}} Cov_a(\tilde{x}_j, \tilde{x}_j).$$

Then we have the following result.

Corollary 2.4 *In market equilibrium, the relation between expected return and risk under the heterogeneous beliefs can be expressed as*

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - (\lambda_a^* - 1)\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1)], \tag{2.13}$$

where

$$\lambda_a^* = \frac{\mathbf{z}_m^T \mathbf{y}_a - \theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m,o}}, \tag{2.14}$$

$$\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1) = \frac{\theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m,0}} = \frac{1}{\tau_a} W_{m,o} \sigma_a^2(\tilde{r}_m) > 0 \tag{2.15}$$

and $\mathbf{1} = (1, 1, \cdots, 1)^T$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_N)^T$ *with*

$$\beta_j = \frac{Cov_a(\tilde{r}_m, \tilde{r}_j)}{\sigma_a^2(\tilde{r}_m)} = \frac{W_{mo}}{p_{oj}} \frac{Cov_a(\tilde{x}_j, \tilde{W}_m)}{\sigma_a^2(\tilde{W}_m)}, \qquad j = 1, \cdots, N.$$

The equilibrium relation (2.13) is the standard zero-beta CAPM except that the mean and variance/covariances are calculated based on the consensus belief $\mathcal{B}_a$. We refer it as the *Zero-beta Heterogeneous Capital Asset Pricing Model* (ZHCAPM). For each risky asset, relation (2.13) is equivalent to

$$\mathbb{E}_a[\tilde{r}_j] - (\lambda_a^* - 1) = \beta_j [\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1)], \qquad \text{for} \qquad j = 1, \cdots, N. \tag{2.16}$$

The zero-beta rate, $\lambda_a^* - 1$, corresponds to the expected return of the zero-beta portfolio of the market portfolio, where λ_a^* is the market shadow price. As in the standard case, the market risk premium, given by equation (2.15), is positively proportional to the aggregate relative risk-aversion $W_{m,o}/\tau_a$ and the variance of the market portfolio returns $\sigma_a^2(\tilde{r}_m)$. The market price of risk under the consensus belief is given by $\phi = (\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1))/\sigma_a(\tilde{r}_m) = W_{m,0}\sigma_a(\tilde{r}_m)/\tau_a$, which is proportional to the level of volatility of the market and the aggregate relative risk-aversion.

As discussed earlier, investor i's shadow price becomes R_f across all investors when there exists a risk-free asset in the market. That is, $\lambda_i^* = \lambda_a^* = R_f$. Substituting this into Proposition 2.3 and Corollary 2.4 leads to the main results in Chiarella et al. (2010*b*).

2.3 The Impact of Heterogeneity

In this section, we use Proposition 2.3 and Corollary 2.4 to examine the impact of the heterogeneous beliefs on the market consensus belief and equilibrium price. To simplify the analysis, we focus on some special cases.

2.3.1 The Shadow Prices and the Aggregation Property

We first examine the relationship between individual shadow prices and the market consensus shadow price. Following (2.3), let $\lambda_a^* = f(\lambda_1^*, \lambda_2^*, \cdots, \lambda_I^*; \theta_1, \theta_2, \cdots, \theta_I)$. Then it is easy to see that $\frac{\partial f}{\partial \lambda_i^*} = \frac{\theta_a \theta_i^{-1}}{I} > 0$, showing that the market consensus shadow price increases as the shadow price of investor i increases, and the rate of increase depends on

θ_i. It follows from $\frac{\partial^2 f}{\partial \lambda_i^* \partial \theta_i} = \frac{1}{I}\theta_i^{-3}\theta_a(\frac{1}{I}\theta_a - \theta_i)$ and $I\theta_i^{-1} > \theta_a^{-1}$ that $\frac{\partial^2 f}{\partial \lambda_i^* \partial \theta_i} < 0$. Therefore the market consensus shadow price is more sensitive to the change of the shadow price of the investor who is less risk-averse.

According to Huang and Litzenberger (1988), when investors have homogeneous beliefs, time-additive and state-independent utility functions with linear risk tolerance and a common cautiousness coefficient, the market equilibrium prices are independent of the distribution of the initial wealth among investors and, if this is the case, we say that the market satisfies the aggregation property. In a general two-period economy without specifying the type of utility function for any investors, Fan (2003) shows that the *Second Welfare Theorem* holds. The theorem states that investors with large capital endowments would have lower marginal utilities of capital endowments and a stronger influence on the market equilibrium. In our case, the utility is measured by $Q_i(\mathbf{z}_i)$. From (2.6), the marginal utility of investor i is represented by the shadow price (λ_i^*). It then follows from (2.5) that a large initial wealth or capital endowment leads to a lower marginal utility. Also, from the expression of the equilibrium price vector in (2.8), it can be seen that (λ_i^*) is inversely related to the price vector. This suggests that an investor with a lower shadow price or marginal utility has a stronger impact on the market equilibrium prices, and hence an investor with more capital is more influential in the market. This is consistent with the *Second Welfare Theorem.* In other words, generally the aggregation property does not hold in our case. However, if there is a risk-free asset in the market, then the shadow prices or marginal utilities are constant across all investors. Correspondingly, the market prices are independent of the initial wealth distribution. Summarising the above analysis, we have the following corollary.

Corollary 2.5 *With heterogeneous beliefs and no risk-free asset, the aggregation property does not hold. Furthermore, investors with lower shadow prices or marginal utilities have a stronger impact on the market equilibrium prices, and hence investors with larger capital are more influential in the market. However, if there is a risk-free asset, the aggregation property holds.*

2.3.2 The Impact of Heterogeneous CARA Coefficients

Proposition 2.3 indicates that the heterogeneous CARA coefficients or risk tolerance and beliefs have a complicated joint impact on the market equilibrium price. To disentangle the effects of preference heterogeneity from belief heterogeneity, we first consider a special case when investors are homogeneous in the expected payoffs and covariance matrix but heterogeneous in CARA, that is, $\Omega_i = \Omega_a := \Omega_o, \mathbf{y}_i = \mathbf{y}_a := \mathbf{y}_o$ for all i. Accordingly, the equilibrium price vector can be written as

$$\mathbf{p}_0 = \frac{1}{\lambda_a^*}\Big[\mathbf{y}_o - \frac{1}{I}\theta_a \Omega_o \mathbf{z}_m\Big], \qquad \lambda_a^* = \frac{\mathbf{z}_m^T \mathbf{y}_o - \theta_a \mathbf{z}_m^T \Omega_o \mathbf{z}_m / I}{W_{m,0}}. \tag{2.17}$$

Equation (2.17) implies that, when the risk-aversion coefficient is the only source of heterogeneity, the market equilibrium prices are independent of the initial wealth distribution amongst individuals and hence the aggregation property holds. For any risky asset j, (2.17) becomes

$$p_{0,j} = \frac{1}{\lambda_a^*}\Big[y_{o,j} - \frac{1}{I}\theta_a Cov(\tilde{x}_j, \tilde{W}_m)\Big].$$

This, together with the market shadow price in equation (2.17), leads to $\frac{\partial p_{0,j}}{\partial \theta_a} = \frac{\sigma^2(\tilde{r}_m)(1-\beta_j)}{I\lambda_a^*}$. In the presence of a risk-free asset with payoff R_f, this becomes $\frac{\partial p_{0,j}}{\partial \theta_a} = -\frac{\sigma^2(\tilde{r}_m)\beta_j}{IR_f}$. It should be noted that, in this case, the equilibrium prices and expected returns are inversely related since the expected payoff is given. Together with the fact that $\frac{\partial \theta_a}{\partial \theta_i} = (\theta_a^{-1}\theta_i^{-1})^2/I > 0$ and $\frac{\partial^2 \theta_a}{\partial \theta_i^2} = -2\frac{\partial \theta_a}{\partial \theta_i}(\frac{\partial \theta_a}{\partial \theta_i}\theta_a^{-1} + \theta_i^{-1}) < 0$, this analysis leads to the following corollary.

Corollary 2.6 *In a market with homogeneous beliefs and no risk-free assets,*

$$(\beta_j - 1)\frac{\partial p_{0,j}}{\partial \theta_i} < 0, \qquad (\beta_j - 1)\frac{\partial \mathbb{E}_o(\tilde{r}_j)}{\partial \theta_i} > 0$$

for $\beta_j \neq 1$ *and* $\frac{\partial p_{0,j}}{\partial \theta_i} = \frac{\partial \mathbb{E}_o(\tilde{r}_j)}{\partial \theta_i} = 0$ *for* $\beta_j = 1$. *If there exists a risk-free asset, then*

$$\beta_j \frac{\partial p_{0,j}}{\partial \theta_i} < 0, \qquad \beta_j \frac{\partial \mathbb{E}_o(\tilde{r}_j)}{\partial \theta_i} > 0$$

for $\beta_j \neq 0$ *and* $\frac{\partial p_{0,j}}{\partial \theta_i} = \frac{\partial \mathbb{E}_o(\tilde{r}_j)}{\partial \theta_i} = 0$ *for* $\beta_j = 0$. *The rate of change for both the equilibrium price and expected return is greater when the investor is less risk averse.*

Corollary 2.6 indicates that the impact of CARA on the market equilibrium depends on the beta of the asset. When there is no risk-free asset, if an asset is riskier than the market ($\beta_j > 1$), an increase in CARA for any investor increases the price and decreases the expected future return of the asset, and vice versa for a less risky asset. However, if there is a risk-free asset, the changes depend on the sign of the return correlation of the asset with the market. If the returns of the asset and market are positively correlated, an increase (decrease) in CARA of any investor leads to a lower (higher) market equilibrium price and a higher (lower) expected return for the asset. In addition, changing the CARA of less risk-averse investors has a more significant impact on market equilibrium price and expected return. The market is dominated by less risk-averse investors, because the market average risk-aversion coefficient θ_a is a harmonic mean of θ_is, which aggravates the impact of the small θ_is. This suggests that, when there is no risk-free asset in the market and when the risk-aversion coefficients of the investors become more divergent with a given average, the aggregate CARA would be reduced, resulting in a lower (higher) equilibrium price and a higher (lower) expected return for assets with betas are below (above) the market level. However, when there is a risk-free asset, the reduction of the market aggregate risk-aversion leads to lower (higher) equilibrium price and higher (lower) expected return for assets that are negatively (positively) correlated with the market.

2.3.3 The Impact of Heterogeneous Expected Payoffs

We now assume that investors agree on the variances and covariances of asset payoffs, say $\Omega_i = \Omega_o$, but disagree on the expected future payoffs of the assets. Consequently, $\Omega_a = \Omega_o$ and the equilibrium price for asset j becomes

$$p_{0,j} = \frac{1}{\lambda_a^*}\left[y_{a,j} - \frac{1}{\tau_a}Cov_o(\tilde{x}_j, \tilde{W}_m)\right], \tag{2.18}$$

where $\lambda_a^* = [\mathbf{z}_m^T \mathbf{y}_a - \mathbf{z}_m^T \Omega_o \mathbf{z}_m / \tau_a]/W_{m,0}$ and $y_{a,j} = \sum_{i=1}^{I} (\tau_i/\tau_a) y_{i,j}$. This, together with (2.18), leads to

$$\frac{\partial p_{0,j}}{\partial y_{a,j}} = \frac{1-\alpha_j}{\lambda_a^*}, \tag{2.19}$$

where $\alpha_j = p_{0,j} z_{m,j}/W_{m,0}$ is the market share of asset j in wealth. If there is a risk-free asset in the market with payoff R_f, then (2.19) simply becomes $\partial p_{0,j}/\partial y_{a,j} = 1/R_f$. Note that

$$\frac{\partial y_{a,j}}{\partial y_{i,j}} = \frac{1}{I}\frac{\theta_a}{\theta_i} y_{i,j} > 0, \qquad \frac{\partial^2 y_{a,j}}{\partial y_{i,j} \partial \theta_i} = \theta_a \theta_i^{-3} \frac{y_{i,j}}{I}\left(\frac{\theta_a}{I} - \theta_i\right) < 0. \tag{2.20}$$

Because $\alpha_j \in [0, 1]$, equations (2.19) and (2.20) indicate that investor i's subjective belief about the expected payoff of asset j is positively related to its equilibrium price. This is also true when there is a risk-free asset in the market. The positive correlation between the subjective beliefs about the expected payoff and the equilibrium price for asset j does not necessarily lead to a negative correlation between the subjective beliefs about the expected payoff and the market expected return for asset j. To see the exact relation, the fact that $\mathbb{E}_a(\tilde{r}_j) = y_{a,j}/p_{o,j} - 1$ leads to

$$\frac{\partial \mathbb{E}_a(\tilde{r}_j)}{\partial y_{a,j}} = \frac{p_{o,j} - (1-\alpha_j)y_{a,j}/\lambda_a^*}{p_{o,j}^2}. \tag{2.21}$$

This expression is negative if and only if $(1+\mathbb{E}_a(\tilde{r}_j))(1-\alpha_j) > \lambda_a^*$. When this condition holds, the expected return decreases when the expected payoff increases for asset j. When there is a risk-free asset, $\lambda_a^* = R_f$ and equation (2.21) becomes $\frac{\partial \mathbb{E}_a(\tilde{r}_j)}{\partial y_{a,j}} = \frac{p_{o,j} - y_{a,j}/R_f}{p_{o,j}^2}$, which is negative if and only if $\mathbb{E}_a(\tilde{r}_j) > r_f$. When this condition holds, the expected return decreases when the heterogeneous belief about the expected payoff increases for asset j. Summarising the above analysis, we obtain the following corollary.

Corollary 2.7 *In a market with homogeneous beliefs about covariance matrix and no risk-free assets, if*

$$(1+\mathbb{E}_a(\tilde{r}_j))(1-\alpha_j) > \lambda_a^* \tag{2.22}$$

for asset j, then the market expected payoff increases and the expected return decreases when the heterogeneous belief about the expected payoff of any investor increases for asset j. When there is a risk-free asset, the condition (2.22) *becomes* $\mathbb{E}_a(\tilde{r}_j) > r_f$.

The following discussion is devoted to Miller's hypothesis (Miller (1977)) that assets with high dispersion in beliefs have higher market price and lower expected future return than otherwise similar stocks. Empirical tests performed in Diether, Malloy and Scherbina (2002) support Miller's hypothesis. Intuitively, optimistic investors would increase the price of the asset and then reduce its expected future returns. We now provide an explanation of this hypothesis. Let us consider a market in which investors have homogeneous beliefs about the covariance matrix but heterogeneous beliefs about the expected payoffs of two risky assets j and j'. Let the expected payoffs be $\mathbf{y}_j = (y_{1,j}, y_{2,j}, \cdots, y_{I,j})^T$ and $\mathbf{y}_{j'} = (y_{1,j'}, y_{2,j'}, \cdots, y_{I,j'})^T$ for asset j and j', respectively. Assume $y_{i,j'} = y_{i,j} + \epsilon_{i,j}$, where $\{\epsilon_{1,j}, \epsilon_{2,j}, \cdots, \epsilon_{I,j}\}$ is a set of real numbers such that $\sum_{i=1}^n \epsilon_{i,j} = 0$ and $\frac{1}{I}\sum_{i=1}^I (y_{i,j'} - \bar{y})^2 \geq \frac{1}{I}\sum_{i=1}^I (y_{i,j} - \bar{y})^2$, where $\bar{y} = (1/I)\sum_{i=1}^I y_{i,j}$. This condition implies that investors have a greater divergence of opinions in the expected payoff for asset j' than asset j. According to Miller's hypothesis, asset j' would have a higher market price and a lower expected future return than asset j. To see if this is true, we consider the following simple example when $I = 2$.

Example 2.8 *Let $I = 2$. Given $\epsilon > 0$, consider two assets j and k with $y_{2,j} < y_{1,j}$, and $y_{1,k} = y_{1,j} + \epsilon$ and $y_{2,k} = y_{2,j} - \epsilon$. This specification indicates that the divergence of opinion about the asset's expected payoff is greater for asset k than for asset j. Then*

$y_{a,j} = \frac{\theta_1^{-1}}{\theta_1^{-1}+\theta_2^{-1}} y_{1,j} + \frac{\theta_1^{-1}}{\theta_1^{-1}+\theta_2^{-1}} y_{2,j}$ *and* $y_{a,k} = \frac{\theta_1^{-1}}{\theta_1^{-1}+\theta_2^{-1}}(y_{1,j}+\epsilon) + \frac{\theta_1^{-1}}{\theta_1^{-1}+\theta_2^{-1}}(y_{2,j}-\epsilon)$. *Hence* $y_{a,j} - y_{a,k} = \frac{\epsilon}{\theta_1^{-1}+\theta_2^{-1}}(\theta_2^{-1} - \theta_1^{-1})$. *Accordingly,* $y_{a,j} < y_{a,k}$ *if and only if* $\theta_1 < \theta_2$. *This implies that if an investor who is optimistic about the asset's expected payoff is less risk averse, then a divergence of opinion among two investors for the expected payoff for asset* k *leads to a high expected payoff for the asset in equilibrium. This suggests that divergence of opinion on the asset's expected payoffs generates a higher market expected payoff if belief about the assets' expected future payoffs is negatively correlated to risk-aversion for any investor* i. *It then follows from Corollary 2.7 that, when both assets* j *and* k *satisfy the condition* (2.22)*, the divergence of opinion on the asset's expected payoffs generates a lower expected future return for the asset.*

To summarise, if our model is to be consistent with Miller's hypothesis that divergence of opinion causes asset price to increase and expected return to decrease, we require the investor with an optimistic view of the asset's future payoff to be less risk-averse compared with the relatively pessimistic investor, and for the asset return to satisfy condition (2.22).

2.4 MV Efficiency and Geometric Relationship of MV Frontiers

In this section, we examine the MV efficiency of the optimal portfolios of investors and the geometric relationship of the MV frontiers with and without a riskless asset in market equilibrium. Following the standard Markowitz method, we can construct the MV portfolio frontier based on the consensus belief. Because the consensus belief reflects the market belief when it is in equilibrium, we call this frontier the market equilibrium MV frontier. A portfolio is MV efficient if it is located on the market equilibrium MV frontier. When investors are homogeneous in their beliefs, it is well known that the optimal portfolios of investors are always MV efficient and also the market portfolio is the unique tangency portfolio between the MV frontiers with and without a riskless asset. When investors' beliefs are heterogeneous, we would expect that the subjectively optimal portfolios of investors are MV inefficient. However, it is not clear if the geometric relationships of the MV frontiers with and without a riskless asset still hold. In this section, we first show that the subjectively optimal portfolios need not be MV efficient and then show that the geometric relationship breaks down in general under heterogeneous beliefs.

2.4.1 MV Efficiency under Heterogeneous Beliefs

In the market we set up in Section 2.2, investors are bounded rational . Based on investors' subjective beliefs, we can construct the MV frontiers (in the standard deviation and expected return space) by using the standard Markowitz method. Of course, the optimal portfolios of the investors will be located on the efficient MV frontiers under their subjective beliefs. Similarly, based on the consensus belief, the market equilibrium MV frontier can be constructed. Due to the market-clearing condition and frontier construction, the market portfolio is always located on the market equilibrium frontier, hence is always efficient. The question is whether the optimal portfolios of individual investors are MV efficient. This is a very important question both theoretically and empirically. If the answer to the question is yes, then the optimal portfolio of the bounded rational heterogeneous investors are MV efficient under market aggregation. Otherwise, the market fails to provide MV efficiency for the investors. If we refer to heterogeneous investors as fund managers and the market portfolio as the market index, the MV efficiency of the optimal portfolios will have important implications as to whether fund managers can out perform the market index based on the MV criteria.

To answer this question, we consider a consensus investor with the market consensus beliefs $\mathcal{B}_a$, risk-aversion coefficient θ_i and initial wealth $W_{i,o}$. Then the optimal portfolio of the investor given by equation (2.12) becomes

$$\mathbf{z}_i^* = \left(1 - \frac{\lambda_i^*}{\lambda_a^*}\right)\theta_i^{-1}\Omega_a^{-1}\mathbf{y}_a + \frac{1}{I}\theta_i^{-1}\theta_a\frac{\lambda_i^*}{\lambda_a^*}\mathbf{z}_m. \tag{2.23}$$

Equation (2.23) shows that any consensus investor will divide his/her investment into two portfolios, namely, $\Omega_a^{-1}\mathbf{y}_a$ and the market portfolio $\mathbf{z}_m$, which is consistent with the *Two Fund Separation Theorem* (see Huang and Lizenberger (1988) Chapter 4, page 83) and such portfolios must be MV efficient due to the construction, which means that the portfolios $\Omega_a^{-1}\mathbf{y}_a$ and $\mathbf{z}_m$ must be two MV frontier portfolios. It is easy to verify from (2.23) that the aggregate position of the portfolio $\Omega_a^{-1}\mathbf{y}_a$ of all investors is $\sum_i(1-\frac{\lambda_i^*}{\lambda_a^*})\theta_i^{-1}\Omega_a^{-1}\mathbf{y}_a = \mathbf{0}$ when the market-clearing condition (2.7) is satisfied.

However, when investor i's subjective belief ($\mathcal{B}_i$) differs from the market belief ($\mathcal{B}_a$), the optimal portfolio of investor i can be expressed as

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}(\mathbf{y}_i - \frac{\lambda_i^*}{\lambda_a^*}\mathbf{y}_a) + \frac{1}{I}\theta_i^{-1}\theta_a\frac{\lambda_i^*}{\lambda_a^*}\Omega_i^{-1}\Omega_a\mathbf{z}_m. \tag{2.24}$$

Then the composition of the portfolio depends also on disagreement defined by the belief deviations (from the consensus belief) $\mathbf{y}_i - \mathbf{y}_a$ and $\Omega_i^{-1}\Omega_a$ of the investor i from the market. This may suggest that individual optimal portfolio needs not be MV efficient in market equilibrium. Proposition 2.9 provides a sufficient condition for a portfolio to be MV efficient.

Proposition 2.9 *A portfolio* $\mathbf{z}$ *is MV efficient if*

$$\mathbf{z} = c_1 \Omega_a^{-1} \mathbf{y}_a + c_2 \mathbf{z}_m / I \tag{2.25}$$

where c_1 *and* c_2 *are constants, and* $\mathbf{y}_a^T(\mathbf{z}_i^* - \mathbf{z}_{MVP}) \geq 0$ *where*

$$\mathbf{z}_{MVP} = \frac{\mathbf{z}^T \mathbf{p}_0}{\mathbf{p}_0^T \Omega_a^{-1} \mathbf{p}_0} \frac{1}{\lambda_a^*} \big(\Omega_a^{-1} \mathbf{y}_a - \theta_a \ \mathbf{z}_m / I\big). \tag{2.26}$$

Analytically, it seems difficult to check whether the optimal portfolio of investor i lies on the market equilibrium MV frontier. However, through Example B.1 in Appendix B, we can show that the optimal portfolios of investors are not located on the market equilibrium MV frontier in general. In this example, we consider a market with two investors and three risky assets. Given individuals' risk-aversion coefficients, subjective beliefs and initial wealth, we first form the consensus belief and calculate the equilibrium price vector. Using the equilibrium price, we convert the consensus belief about asset payoffs to the consensus belief about asset returns and obtain the market expected returns and variances/covariances of asset returns. With the information provided in Table B.3 in Appendix B, we can construct the portfolio frontiers for each investor and for the market equilibrium frontier in the mean-standard deviation space, and locate the optimal portfolios for individual investors as well as the market portfolio. Figure 2.1 exhibits the resulting MV frontiers under the heterogeneous and the market equilibrium beliefs.

Fig. 2.1 shows two interesting and important features. Firstly, the market equilibrium MV frontier can be located between two individual's MV frontiers. In this example, it is closer to that of investor 2. Intuitively, this may be due to the fact that investor 2 is less risk averse and more optimistic about the market in the sense that he/she perceives higher expected payoffs and smaller standard deviations on the asset payoffs based on Table B.3 and hence dominates the market. Secondly, it is demonstrated that the optimal portfolios of the two investors are always located on their MV efficient frontiers based on their own beliefs and the market portfolio is located on the market MV efficient frontier under the consensus belief. However, in market equilibrium, the optimal portfolios of the two investors are strictly located inside of the market equilibrium MV frontier. This may be hard to view in Fig. 2.1. We provide a zoom-in version in Fig. 2.2 to verify this observation.

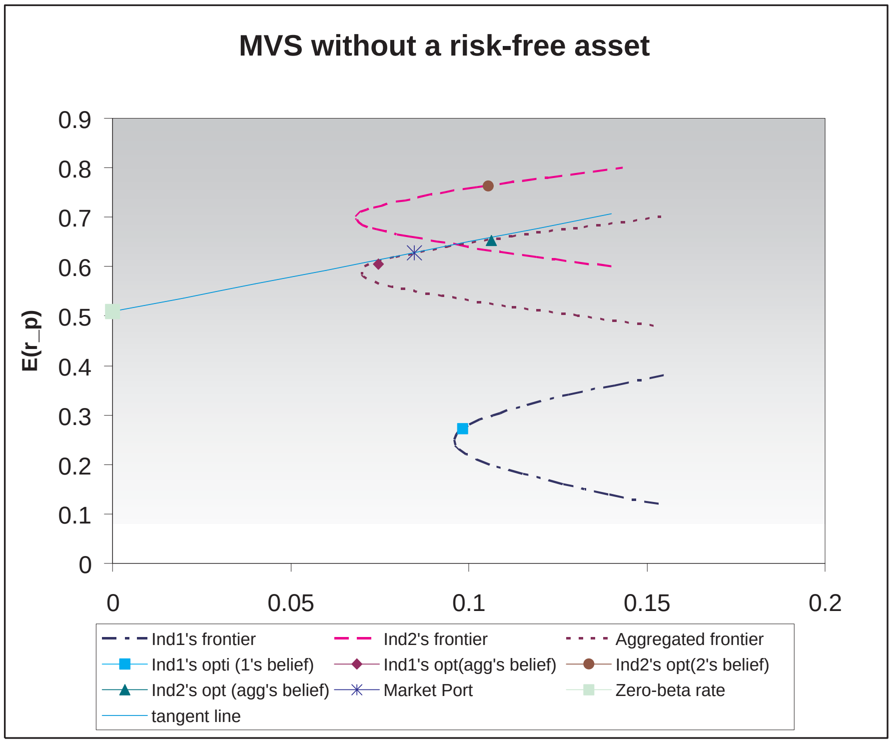


Figure 2.1: The mean variance frontiers under the heterogeneous beliefs and the market equilibrium consensus beliefs. The tangency line corresponding to the consensus belief has the market portfolio as the tangency portfolio and the expected return of the zero-beta portfolio of the market as the intercept with the expected return axis.

Fig. 2.2 clearly demonstrates that the optimal portfolios of the two investors are not located on the MV frontier, though they are very close to it, and hence are MV inefficient. Intuitively, because of the bounded rationality and the fact that the market consensus belief is jointly determined by all market participants, no investors have knowledge about the "correct" market belief. Therefore, both investors made "wrong guesses" about the market, investor 1 being pessimistic and investor 2 being optimistic. Their optimal portfolios suffer from those "wrong guesses" in terms of MV efficiency. Therefore, the individual optimal portfolios need not be MV efficient in market equilibrium in general.

2.4.2 The Geometric Relation of the Equilibrium MV Frontiers

To examine the *tangency relationship* of the traditional portfolio theory with heterogeneous beliefs, we consider the situation under which a riskless asset exists with future payoff R_f. Under the homogeneous belief, the classic portfolio theory tells us that the efficient portfolio frontier collapses to a straight line when a risk-free asset is added to the market. This straight line has one tangency point with the original frontier without

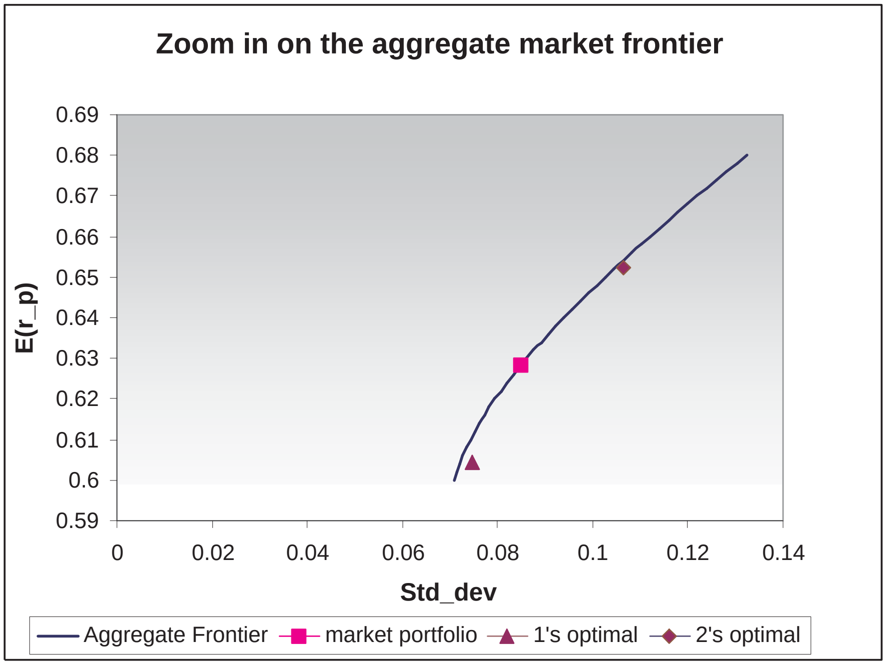


Figure 2.2: Close-up of the locations of individuals' optimal portfolios and the market portfolio relative to the market frontier when the market is in equilibrium.

a risk-free asset. This *tangency portfolio* is exactly the market portfolio when both the risk-free and equity markets are in equilibrium. This equilibrium tangency relationship may not hold under heterogeneous beliefs and we now examine this through the following example.

Example 2.10 *Consider the case with* $I = 2$ *investors with beliefs* $\mathcal{B}_i = (\Omega_i, \mathbf{y}_i)$ *for* $i = 1, 2$. *There are* $N = 3$ *risky assets and a risk-free asset with payoff* R_f. *Let the absolute risk-aversion coefficients* $(\theta_1, \theta_2) = (5, 1)$, *investors' initial wealth* $W_{1,o} = W_{2,o} =$ \$10, *market endowment of risky assets* $\mathbf{z}_m = (1, 1, 1)^T$, *and* $\mathbf{y}_o = (6.60, 9.35, 9.78)^T$, $\mathbf{1} = (1, 1, 1)^T$ *and* $\Omega_o = D_o C D_o$ *where*[6]

$$D_o = \begin{pmatrix} 0.7933 & 0 & 0 \\ 0 & 0.8770 & 0 \\ 0 & 0 & 1.4622 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0.2233 & 0.1950 \\ 0.2233 & 1 & 0.1163 \\ 0.1950 & 0.1163 & 1 \end{pmatrix},$$

in which D_o *corresponds to the standard deviation matrix and* C *is the correlation matrix. Assume that investors' beliefs are given by* $\mathbf{y}_i = (1 + \delta_i)\mathbf{y}_o$ *and* $\Omega_i = D_i C D_i$, *where* $D_i = (1 + \epsilon_i) D_o$ *for* $i = 1, 2$. *This implies that investors agree on the correlation of asset*

[6]We can think of $\mathbf{y}_o$ and Ω_o as the average belief of two investors about the expected asset payoffs and variance/covaraince of asset payoffs respectively.

payoffs, but disagree about the volatilities and expected payoffs. Investors may disagree on both the volatilities and correlations in general, this simplified assumption is aimed to disentangle their joint impact and to facilitate our numerical calculation. Next we aggregate individuals' beliefs according to Proposition 2.3, first without a risk-free asset, then with a risk-free asset. The risk-free payoff R_f is determined such that the risk-free asset is in net-zero supply in equilibrium. To examine the tangency relationship, we plot the MV frontiers and optimal portfolios under the market consensus belief with and without a risk-free asset for different values of δ_i and ϵ_i. Plots are shown in Figure 2.3.

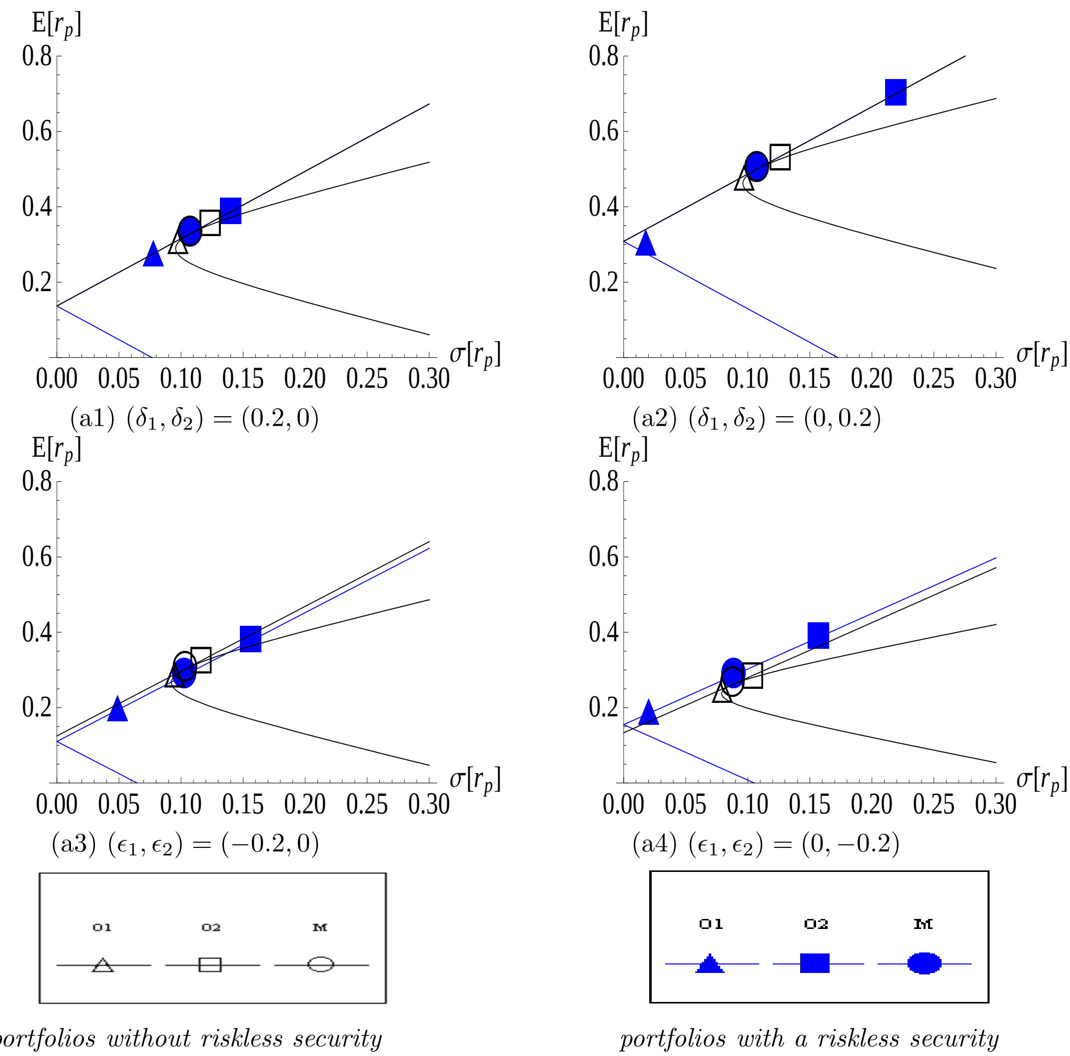


Figure 2.3: Comparison of the geometric relationships between market MV frontiers with and without a risk-free asset, when the risk-free asset is in net-zero supply. In (a1) and (a2), $\mathbf{y}_1 \neq \mathbf{y}_2, \Omega_1 = \Omega_2$; in (a3) and (a4), $\mathbf{y}_1 = \mathbf{y}_2, \Omega_1 \neq \Omega_2$.

In this example, when investors are homogeneous about the variances and covariances but heterogeneous about the expected payoffs of the risky asset, Fig. 2.3 (a1) and (a2) show that the tangency relation still holds. This is not surprising. Because of the homogeneous belief of the variance-covariance matrix $\Omega_i = \Omega_o$, the consensus variance-covariance matrix is given by $\Omega_a = \Omega_o$. From the construction of the consensus belief, the expected payoff $\mathbf{y}_a$ is a risk tolerance weighted average of the heterogeneous beliefs about the expected payoffs. Therefore, the consensus belief $\mathcal{B}_a$ remains the same when a risk-free asset is added to the market. Furthermore, since the risk-free asset is in net-zero supply, it follows from equation (2.11) in Proposition 2.3 that

$$W_{m,o} = \mathbf{z}_m^T \mathbf{p}_0 = \frac{1}{\lambda_a^*}\left[\mathbf{y}_a^T \mathbf{z}_m - \frac{1}{I}\theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m\right] = \frac{1}{R_f}\left[\mathbf{y}_a^T \mathbf{z}_m - \frac{1}{I}\theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m\right].$$

Consequently, the riskless payoff R_f must equal to the zero-beta payoff λ_a^*. This implies that both the market's optimal marginal certainty equivalent wealth (CEW) and the equilibrium prices do not change when a risk-free asset is added to the market. Therefore, the tangency relationship of the two market equilibrium frontiers with and without a risk-free asset holds with the market portfolio as the tangency portfolio. However, the efficiency of the optimal portfolios of the two investors depends on their expectations and risk-aversion coefficients. On the one hand, when the more risk averse investor is optimistic and the less risk-averse investor is pessimistic about the expected payoffs, Fig. 2.3 (a1) indicates that the optimal portfolios of both investors are located closer to the market portfolio and market MV frontiers. On the other hand, when the more risk-averse investor is pessimistic and the less risk-averse investor is optimistic about the expected payoffs, Fig. 2.3 (a2) indicates that the optimal portfolios of both investors are located far away from the market portfolio and the equilibrium market MV frontier. In particular, the optimal portfolio of the pessimistic investor may become even more inefficient when the risk-free asset is available. This means that adding a risk-free asset in this situation may help optimistic investor to achieve a higher expected return for his optimal portfolio by sacrificing the MV efficiency of the optimal portfolio of pessimistic investor.

When investors are heterogeneous in the variances of the asset payoffs but homogeneous in their expected payoffs, Fig. 2.3 (a3) and (a4) illustrate that the tangency relation breaks down. The risk-free payoff is no longer guaranteed to equal to the zero-beta payoff, which results in a change in the market's optimal CEW and also the equilibrium prices. In particular, when the relatively less risk-averse investor, investor 2 in this case, is more confident (as measured by the smaller variance), Fig. 2.3 (a4) indicates that the existence of a risk-free asset actually pushes up the MV frontier, leading to a higher expected return for the market portfolio. If one would believe that it is more likely that the less risk averse investor would be more confident in general, this implies that adding a risk-free asset would be may push the portfolio frontier line above the tangency line of the frontier without the risk-free asset, leading to a higher market expected return. This observation may help us to explain the risk premium puzzle[7] However, when the relatively more risk-averse investor, investor 1 in this case, is more confident, Fig. 2.3 (a3) implies that the existence of a risk-free asset may push down the MV frontier, lower the expected return of the market portfolio. This is an unexpected and surprising result. In the standard homogeneous case, the expected return of the market portfolio is independent of the existence of the risk-free asset which is in zero-net supply. The above analysis demonstrates that this needs not to be the case when investors are heterogeneous. Based on Fig. 2.3 (a4), we observe that a restriction to the access of the risk-free asset may lead to a lower market expected return, a phenomenon we have experienced in the global financial crisis, and we leave further development along this line to Chapter 3.

2.5 The Impact of Heterogeneity on the Market with Many Investors

In this section, we extend the numerical analysis in the previous section to a market consisting of many different investors whose beliefs are characterised by mean-preserved spreads that can be either univariate or multivariate. With large number of investors with mean-zero disagreement, we are interested in the MV efficiency of the investors. The heterogeneity is either in the expected payoffs or the variances of the payoffs. We introduce the following definition of a truncated normal distribution to measure the belief distribution of heterogeneous agents.

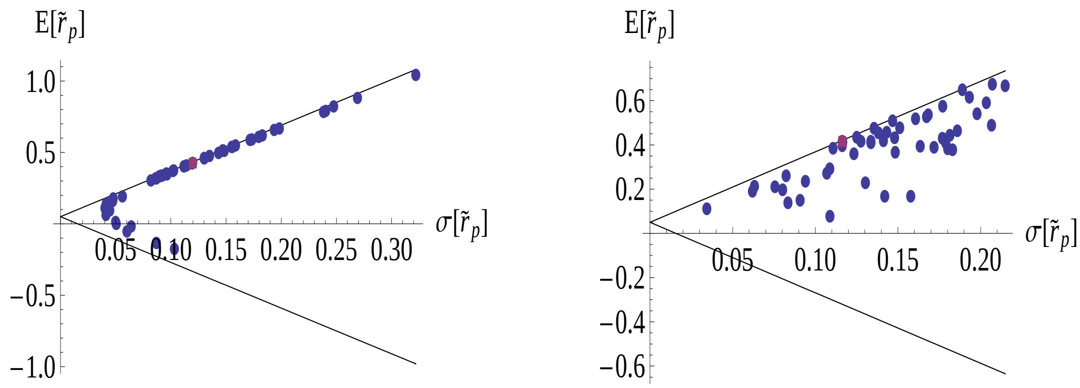
Definition 2.11 *Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim X|a < X < b$ has a truncated normal distribution, denote $\mathcal{TN}(\mu, \sigma^2; (a, b))$, with probability density function*

$$f(x; \mu, \sigma, a, b) = \frac{\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})},$$

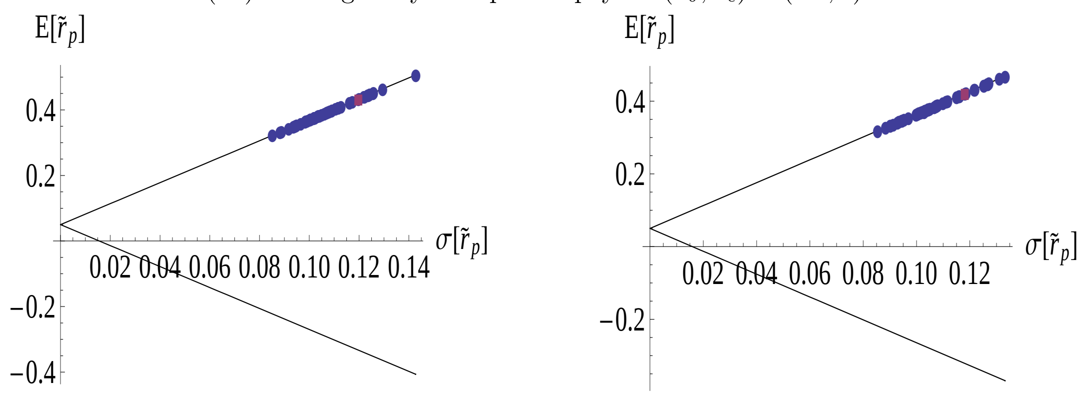
where $\phi(\cdot)$ is the probability density function of the standard normal distribution and $\Phi(\cdot)$ is its cumulative distribution function.

Example 2.12 *Let the number of investors $I = 50$, number of risky assets $N = 3$, and market portfolio of risky assets is given by $\mathbf{z}_m = (25, 25, 25)^T$ (so that the average number of each stock per investor stays at 0.5 as in the previous example). Assume that there is*

[7] A detailed analysis on the conditions under which the market generates a higher market risk premium and a lower risk-free rate in market equilibrium when there are two assets and two beliefs will be discussed in Chapter 4.



(b1) Heterogeneity in expected payoffs $(\sigma_\delta, \sigma_\epsilon) = (0.2, 0)$.



(b2) Heterogeneity in variances $(\sigma_\delta, \sigma_\epsilon) = (0, 0.03)$.

Figure 2.4: The optimal portfolios of all 50 investors and their relative position to the CML when investors' beliefs in terms of asset payoffs are homogeneous in variances and heterogeneous in expected payoffs in (b1) or homogeneous in expected payoffs and heterogeneous in variances in (b2). The left (right) panels correspond to univariate (multivariate) distribution in beliefs.

a risk-free asset with payoff $R_f = 1.05$. Investors' initial wealth $W_{0,i} = \$10$, the CARA coefficients $\theta_i \overset{iid}{\sim} \mathcal{TN}(\theta_o, \sigma_\theta^2; (0, \infty))$ with $\theta_o = 3$ and $\sigma_\theta = 0.3$ for $i = 1, 2, \cdots, I$. Consider two types of probability distributions for investors' beliefs;

(i). $\mathbf{y}_i = (1 + \delta_i)\mathbf{y}_o$ *and* $\Omega_i = D_i C D_i$, $D_i = (1 + \epsilon_i) D_o$ *for* $i = 1, \cdots, 50$, *where* C, $\mathbf{y}_o$ *and* D_o *are defined in Example 2.10 and* $\delta_i \overset{iid}{\sim} \mathcal{TN}(0, \sigma_\delta^2; (-1, \infty))$ *and* $\epsilon_i \overset{iid}{\sim} \mathcal{TN}(0, \sigma_\epsilon^2; (-1, \infty))$;

(ii). $\mathbf{y}_i = \boldsymbol{\delta}_i + \mathbf{y}_o$ *and* $\Omega_i = D_i C D_i$, $D_i = Diag[\boldsymbol{\epsilon}_i + (0.7933, 0.8770, 1.4622)^T]$ *for* $i = 1, \cdots, 50$, *where* $\delta_{i,j} \overset{iid}{\sim} \mathcal{TN}(0, \sigma_\delta; (-6.60, \infty))$ *and* $\epsilon_{i,j} \overset{iid}{\sim} \mathcal{TN}(\mathbf{0}, \sigma_\epsilon; (-0.7933, \infty))$ *for each risky asset* j.

We refer to (i) as the case with univariate belief dispersions since the beliefs are characterised by univariate random variables, and (ii) as the case with multivariate belief dispersions since beliefs are characterised by multivariate random variables.

The results for the two cases are plotted in Figure 2.4 in which the optimal portfolios of all 50 investors and their relative position to the *capital market line* CML are plotted. Fig. 2.4(b1) illustrates the case when investors are homogeneous in variances but heterogeneous in the expected payoffs, while Fig. 2.4(b2) illustrates the case in reverse. The left panels correspond to the case with univariate belief dispersions and the right panels correspond to the case with multivariate dispersions. Fig. 2.4 leads to the following interesting observations. (i) The optimal portfolios of all investors may become MV efficient when investors are heterogeneous in variances but homogeneous in the expected payoffs, illustrated by Fig. 2.4(b2). The same effect is observed for univariate spread (left panel) and multivariate spreads (right panel). This shows that heterogeneity in variances, characterised by mean preserved spreads, may play an insignificant role in the MV efficiency of the optimal portfolios of investors. (ii) The heterogeneity in expected payoff may have a significant impact on the MV efficiency of the optimal portfolios of the investors, as illustrated in Fig. 2.4(b1). The optimal portfolios may become less MV efficient, in particular, when the belief dispersions are multivariate normally distributed (the right panel). In this example, some optimal portfolios are far below the CML, and even have a lower expected return than the risk-free rate (the left panel). In addition, when the belief dispersions are univariate, the optimal portfolios seem to form a hyperbolic curve below the equilibrium market MV efficient frontier (the left panel). However, when the divergence of opinions is not the same for each asset, optimal portfolios may scattered under the MV frontier without any significant pattern. This example demonstrates that heterogeneity in expected payoff may have a more significant impact on the MV efficiency of optimal portfolios of investors than the heterogeneity in variances.

Based on the above example, which comprises many investors, we find that heterogeneity in expected payoffs may have a significant impact on the MV efficiency of the optimal portfolios of investors, but the impact may be insignificant from the heterogeneity in variances, and different mean preserved spreads in beliefs have a different impact. However, based on the analysis in the previous section, heterogeneity in expected payoffs (variances) may have an insignificant (significant) impact on the geometric relation of the frontiers with and without a riskless asset. Overall, we can see that, due to the heterogeneous beliefs, the market may fail to provide investors with a MV efficient portfolio. This feature is not what we would expect in a homogeneous market. It shows that heterogeneous investors can never beat the market when performance is measured by MV efficiency.

2.6 Conclusion

Within the MV framework, by assuming that investors are heterogeneous, this chapter examines the impact of the heterogeneity on the market equilibrium prices and the equilibrium MV frontier in a market with many risky assets and no riskless assets. Heterogeneity is measured by risk-aversion coefficients, expected payoffs, and covariance matrices of risky assets of heterogeneous investors. Investors are bounded rational in the sense that, based on their beliefs, they make optimal portfolio decisions. To characterise the market equilibrium prices of the risky assets, we introduce the concept of a consensus belief about the market and show how the consensus or market belief can be constructed from heterogeneous beliefs. Basically, under the market aggregation, the consensus belief is a weighted average of the heterogeneous beliefs. Explicit formulae for the market equilibrium prices of the risky assets are derived. As a by-product of the consensus belief and equilibrium price formula, we show that Black's standard zero-beta CAPM still holds with heterogeneous beliefs. The impact of heterogeneity on market equilibrium, portfolio frontiers and MV efficiency of the optimal portfolios of the investors is analysed. In particular, through some numerical examples, we show that, in market equilibrium, the biased beliefs (from the market belief) of an investor may make his/her optimal portfolio MV inefficient (although they may be very close to the MV efficient frontier). This indicates that bounded rational investors may never achieve their MV efficiency in market equilibrium. If we refer the heterogeneous investors as fund managers and the market portfolio as a market index, then our result offers an explanation on the empirical finding that, according to the MV criterion, managed funds under-perform the market indices on average. We also offer an explanation on Miller's proposition that "divergence of opinion corresponds to lower future asset returns" and the subsequent empirical findings on this. Furthermore, we show that the well-known *tangency relation* of frontiers with and without a risk-free asset under the homogeneous beliefs may break down under heterogeneous beliefs, in particular when investors are heterogeneous in variances. Adding a risk-free asset to a market with many risky assets can have a very complex effect on the market in general. In a homogeneous market, the expected return of the market portfolio is independent of the existence of the risk-free asset. However, in a heterogeneous market, adding a risk-free asset can have a different impact on the expected return for the market portfolio in equilibrium. This result can be used to explain the risk premium puzzle and global financial crisis. In addition, the heterogeneity in the expected payoffs has a significant impact on the MV efficiency of subjectively optimal portfolios but insignificantly for the geometric relation. However, it is the other way around for heterogeneity in variance.

Chapter 3

Portfolio Analysis under Heterogeneous Beliefs about Returns

3.1 Introduction

In the bounded rationality literature, investors can agree to disagree on either asset payoffs or asset rate of returns. The difference of the two types of disagreement can impact market differently. In this chapter, different from Chapter 2, we assume that investors' beliefs are formed about the probability distribution of rates of return on risky assets.We show that the impact of this different setup becomes significant when beliefs are heterogeneous. Essentially, when beliefs are formed about the distribution of future rates of return, investors' subjective optimal portfolios are independent of current prices, which are determined in equilibrium[1]. This favourable property allows us to construct a consensus belief and compute equilibrium prices explicitly with or without a riskless asset, which is crucial when we extend the model to a continuum of investors. Furthermore, the risk-free rate can also be calculated explicitly, as in Chiarella, Dieci and He (2010*a*).

Comparing with the results obtained in Chapter 2, we make three contributions in this chapter. First, we provide sufficient conditions under which heterogeneous beliefs do not have an impact on the market equilibrium. Fama and French (2007) argue that CAPM pricing still works if the misinformed investors who hold erroneous beliefs (beliefs that are different to the objective one) in aggregate hold the market portfolio. However, no further conditions are given. By assuming that investors have disagreement on expected future asset returns, Levy, Levy and Benita (2006) show that CAPM is intact, with an

[1]This is not the case in Chapter 2, investor's demand function depends on the current prices, optimal portfolio weights and beliefs about the distribution of rates of return also depend on the current asset prices.

infinite number of assets and investors[2]. When the risk-free rate is determined endogenously, we provide some sufficient conditions for the consensus belief to conform to the unbiased belief of the investors. Secondly, we provide some insights into the tangency relation between the efficient frontiers with and without a riskless asset or on existence of a tangency portfolio under the consensus belief. This issue has been investigated in Chapter 2, but only in the case of two investors, because the equilibrium prices need to be numerically determined without a riskless asset and it becomes a computationally difficult task with many investors in the market. We solve this problem in this chapter by assuming the beliefs are formed about the distribution of future rates of return on risky assets and we show that equilibrium prices can be computed in closed form, and the impact of heterogeneity in beliefs and risk tolerance can be examined in the case of a continuum of investors. Some have examined the effect of heterogeneity under short-sale constraints for individual assets, see Jarrow (1980) and Gallmeyer and Holifield (2007). We are interested in knowing when riskless borrowing/lending is still a non-binding constraint for the aggregate market as in the homogeneous case, or equivalently when the consensus belief is invariant to the assumption of a riskless security. Lastly, we analyse the performance of investors' subjectively optimal portfolios. It is already known that subjectively optimal portfolios of investors are not mean-variance efficient in general, see Wenzelburger (2004), Böhm and Chiarella (2005), Böhm and Wenzelburger (2005), Sharpe (2007) and Horst and Wenzelburger (2008). Chapter 2 also investigates this issue, but with only two investors when a riskless asset does not exist or many investors when a riskless asset exists because of the obstacle mentioned previously. In this chapter, we extend the analysis to a continuum of investors. To measure the inefficiency we focus on the average loss of expected returns of investors' portfolios due to their disagreement from an average belief.

The chapter is structured as follows. In Section 3.2, we introduce and construct the consensus belief of rate of asset returns linking the heterogeneous market with an equivalent homogeneous market, and present an explicit market equilibrium price formula. Consequently, a zero-beta CAPM under heterogeneous beliefs is derived. The results are then extended from a finite number of investors to a continuum of agents. In Section 3.3, we examine conditions under which the consensus beliefs of heterogeneous beliefs of investors conforms to the average belief of the investors. In Section 3.4, we introduce a measure to examine the impact of heterogeneous beliefs on the tangency portfolio. In Section 3.5, we measure the performances of investors' subjectively optimal portfolios relative to a portfolio on the efficient frontier. Section 3.6 concludes.

[2]They invest only in their perceived tangency portfolio and do not invest in the riskless asset, the risk-free rate is exogenously given.

3.2 Heterogeneous Beliefs about Returns

In this section, we introduce the heterogeneous beliefs of investors and then the consensus belief of the market to characterise the market equilibrium under the heterogeneous beliefs within the standard mean-variance framework. Following Lintner (1969) and Black (1972), we consider a static mean-variance model with many risky assets, but no riskless asset. Investors have heterogeneous beliefs regarding the distributions of future asset returns.

Consider a market with N risky assets, indexed by $j = 1, 2, \cdots, N$ and I investors indexed by $i = 1, 2, \cdots, I$. Let $\tilde{\mathbf{r}} = (\tilde{r}_1, \cdots, \tilde{r}_N)^T$ be the random rate of return vector of the risky assets that are jointly normally distributed. Assume that each investor has his/her own set of beliefs about the market in terms of means, variances and covariances of the future asset returns, denoted by

$$\mu_{i,j} = E_i[1 + \tilde{r}_j], \quad V_{i,jk} = Cov_i(\tilde{r}_j, \tilde{r}_k) \qquad \text{for } 1 \leq i \leq I, \quad 1 \leq j, k \leq N. \tag{3.1}$$

We let $\boldsymbol{\mu}_i$ be the expected asset return vector and V_i be the covariance matrix of asset returns for investor i. Denote $\mathcal{B}_i = (V_i, \boldsymbol{\mu}_i)$ as the subjective belief of investor i, and let $\boldsymbol{\pi}_i = (\pi_{i,1}, \pi_{i,2} \cdots, \pi_{i,N})^T$ be the portfolio of investor i where $\pi_{i,j}$ is the dollar amount invested in asset j, and $W_{i,0}$ be the initial wealth of investor i. Under investor i's subjective belief $\mathcal{B}_i$, the expected and the variance of his/her end-of-period wealth is given respectively by

$$\mathbb{E}_i(\tilde{W}_i) = \boldsymbol{\pi}_i^T \boldsymbol{\mu}_i, \qquad \sigma_i^2(\tilde{W}_i) = \boldsymbol{\pi}_i^T V_i \boldsymbol{\pi}_i. \tag{3.2}$$

3.2.1 Individual's Portfolio Selection

Investor i maximises the expected utility of his/her end-of-period wealth. Under the assumptions of constant absolute risk aversion (CARA) utility function $U_i(\tilde{W}) = -e^{-\tilde{W}/\tau_i}$, where $1/\tau_i$ is the CARA coefficient and hence τ_i measures investor i's risk tolerance. Following the assumption made above, the end-of-period wealth $\tilde{W}_i$ of investor i is normally distributed, investor i's portfolio selection problem becomes $\max_{\boldsymbol{\pi}_i} Q_i(\boldsymbol{\pi}_i)$ subject to the wealth constraint $\boldsymbol{\pi}_i^T \mathbf{1} = W_{i,0}$, where $Q_i(\boldsymbol{\pi}) = \boldsymbol{\mu}_i^T \boldsymbol{\pi}_i - \frac{1}{2\tau_i} \boldsymbol{\pi}_i^T V_i \boldsymbol{\pi}_i$ is the certainty equivalent end-of-period wealth of investor i. Solving this optimisation problem yields the following result.

Lemma 3.1 *The optimal portfolio of investor i (in dollar amount) is given by*

$$\boldsymbol{\pi}_i^* = \tau_i V_i^{-1}(\boldsymbol{\mu}_i - \lambda_i^* \mathbf{1}), \tag{3.3}$$

where

$$\lambda_i^* = \frac{\mathbf{1}^T V_i^{-1}\boldsymbol{\mu}_i - W_{i,0}/\tau_i}{\mathbf{1}^T V_i^{-1}\mathbf{1}}. \tag{3.4}$$

From Lemma 3.1, the Lagrange multiplier $\lambda_i^* = \partial Q_i(\boldsymbol{\pi}_i^*)/\partial \pi_{i,j}$ is the shadow price perceived by investor i, measuring the *optimal marginal certainty equivalent end-of-period wealth (CEW) per dollar investment in asset* j. It is the same across all risky assets, but different across different investors. Obviously, if there exists a risk-free asset (f) with return r_f, then it must be true that $\lambda_i^* = 1 + r_f = R_f$ for all investors. Equation (3.3) shows that when investor i's belief is formed about rates of return, his/her portfolio choice does not depend on the market prices of risky assets. However, this is not the case when beliefs were about future asset payoffs, see Chiarella et al. (2010*b*) and Chapter 2 of this thesis.

3.2.2 The Consensus Belief, Market Equilibrium, and Zero-Beta CAPM

We first introduce the concept of consensus belief for the market with the heterogeneous beliefs, which is the same as in Chiarella et al. (2010*a*).

Definition 3.2 *A belief* $\mathcal{B}_a(V_a, \boldsymbol{\mu}_a)$ *is called a market* **consensus belief** *if the market equilibrium price under heterogeneous beliefs* $\mathcal{B}_i = (V_i, \boldsymbol{\mu}_i)$, $i = 1, 2, \cdots, I$ *is also the market equilibrium price under the homogeneous belief* $\mathcal{B}_a$.

We now show that a *consensus belief* $\mathcal{B}_a(V_a, \boldsymbol{\mu}_a)$ can be constructed. The market aggregate condition is given by

$$\boldsymbol{\pi}_m = \sum_{i=1}^{I} \boldsymbol{\pi}_i^*$$

with the total market initial wealth $W_{m,o} = \boldsymbol{\pi}_m^T \mathbf{1}$. In this case the equilibrium price vector is determined by

$$\mathbf{p}_0 = Z^{-1} \sum_{i=1}^{I} \tau_i V_i^{-1}(\boldsymbol{\mu}_i - \lambda_i^* \mathbf{1}), \tag{3.5}$$

where Z is an $N \times N$ diagonal matrix with diagonal elements z_j representing the market supply in the number of shares in asset j. We denote $\bar{Z} = Z/I$ the average supply per investor.

Proposition 3.3 *Let* $\tau_a := \frac{1}{I}\sum_{i=1}^{I}\tau_i$ *and* $\lambda_a^* := \frac{1}{I}\tau_a^{-1}\sum_{i=1}^{I}\tau_i\lambda_i^*$. *Then*

(i) the consensus belief $\mathcal{B}_a$ *is given by*

$$V_a = \tau_a\lambda_a^*\bigg(\frac{1}{I}\sum_{i=1}^{I}\lambda_i^*\tau_i V_i^{-1}\bigg)^{-1}, \tag{3.6}$$

$$\boldsymbol{\mu}_a = \mathbb{E}_a(\mathbf{1}+\tilde{\mathbf{r}}) = \tau_a^{-1}V_a\bigg(\frac{1}{I}\sum_{i=1}^{I}\tau_i V_i^{-1}\boldsymbol{\mu}_i\bigg), \qquad \mathbb{E}_a(\tilde{\mathbf{r}}) = \boldsymbol{\mu}_a - \mathbf{1}; \tag{3.7}$$

(ii) the market equilibrium price $\mathbf{p}_o$ *is determined by*

$$\mathbf{p}_0 = \tau_a\bar{Z}^{-1}V_a^{-1}(\boldsymbol{\mu}_a - \lambda_a^*\mathbf{1}); \tag{3.8}$$

(iii) the Zero-beta CAPM relation

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - (\lambda_a^* - 1)\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - (\lambda_a - 1)], \tag{3.9}$$

holds, where

$$\tilde{r}_m = \frac{{\boldsymbol{\pi}_m}^T\tilde{\mathbf{r}}}{W_{m,0}}, \qquad \boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_N)^T, \quad \beta_j = \frac{Cov_a(\tilde{r}_m, \tilde{r}_j)}{\sigma_a^2(\tilde{r}_m)}, \quad j = 1, \cdots, N.$$

Proposition 3.3 shows that both the equilibrium price vector and consensus belief can be calculated explicitly given investors' subjective belief, risk tolerance and initial wealth. In contrast, in Chapter 2 equilibrium asset prices were calculated implicitly because the shadow prices depended on asset prices. If there exists a risk-free security (f) with return r_f, then the consensus belief will be as derived in Chiarella *et al* (2010a). For completeness, we include their results in the following corollary.

Corollary 3.4 *Assume there exists a riskless security (f) with return* r_f *in net zero supply. Let* $\tau_a = \sum_{i=1}^{I}\tau_i$. *Then*

(i) the consensus belief $\mathcal{B}_a$ *is given by*

$$V_a = \tau_a\bigg(\frac{1}{I}\sum_{i=1}^{I}\tau_i V_i^{-1}\bigg)^{-1}, \tag{3.10}$$

$$\boldsymbol{\mu}_a = \mathbb{E}_a(\mathbf{1}+\tilde{\mathbf{r}}) = \tau_a^{-1}V_a\bigg(\frac{1}{I}\sum_{i=1}^{I}\tau_i V_i^{-1}\boldsymbol{\mu}_i\bigg), \qquad \mathbb{E}_a(\tilde{\mathbf{r}}) = \boldsymbol{\mu}_a - \mathbf{1}; \tag{3.11}$$

(ii) the market equilibrium price $\mathbf{p}_0$ *is determined by*

$$\mathbf{p}_0 = \tau_a \bar{Z}^{-1} V_a^{-1} (\mathbb{E}_a(\tilde{\mathbf{r}}) - r_f \mathbf{1}); \tag{3.12}$$

(iii) the Zero-beta CAPM relation

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - r_f \mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - r_f] \tag{3.13}$$

holds, where the equilibrium risk-free rate is given by

$$r_f = \frac{\mathbf{1}^T V_a^{-1} \mathbb{E}_a(\tilde{\mathbf{r}}) - \bar{W}_0/\tau_a}{\mathbf{1}^T V_a^{-1} \mathbf{1}}, \tag{3.14}$$

where $\bar{W}_0 = W_{m,0}/I$ *is the average initial wealth.*

Corollary 3.4 and Proposition 3.3 show that the consensus belief $\mathcal{B}_a$ is not invariant to the existence of a riskless security, which is in net-zero supply. Hence the portfolio weights and the MV efficiency of the market portfolio as well as investors' optimal portfolios can differ depending on whether investors have access to riskless borrowing and lending when beliefs are heterogeneous.

3.2.3 Economy with a Continuum of Investors

As the number of investors increase in the above model, increasing dimensionality can make the model infeasible. In order to have a parsimonious model, but at the same time incorporate the realism of a large number of investors, we extend the finite-agent model to a model with a continuum of agents. We show that, when the heterogeneity in risk tolerance, initial wealth and beliefs of investors are described by probability distributions, we are also able to construct a consensus belief.

Consider a continuum of investors with unit mass indexed by $e \in [0, 1]$; the economy is defined by a measurable function $(\tau, W_0, V^{-1}, \boldsymbol{\mu}) : [0, 1] \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{N \times N} \times \mathbb{R}^N$ where $(\tau_e, W_{e,0}, V_e^{-1}, \boldsymbol{\mu}_e)$ is the risk tolerance, initial wealth, precision matrix and the belief of expected returns for investor e. The consensus belief $\mathcal{B}_a$ in this economy can be treated as the limits of equation (3.6) and (3.7) or (3.10) and (3.11) as the number of investors approaches infinity, and therefore can be expressed as integrals[3],

$$\tau_a = \int_0^1 \tau_e de, \tag{3.15}$$

$$V_a^{-1} = \frac{1}{\tau_a \lambda_a^*} \int_0^1 \tau_e \lambda_e^* \ V_e^{-1} \ de, \tag{3.16}$$

$$\boldsymbol{\mu}_a = \frac{V_a}{\tau_a} \int_0^1 \tau_e \ V_e^{-1} \boldsymbol{\mu}_e \ de, \tag{3.17}$$

where $\lambda_e^* = (\mathbf{1}^T V_e^{-1} \boldsymbol{\mu}_e - W_{0,e}/\tau_e)/(\mathbf{1}^T V_e^{-1} \mathbf{1})$ is the marginal utility per dollar for investor e and the average intial wealth $\bar{W}_0 = \int_0^1 W_{e,0} \ de$. When a risk-free asset exists, (3.16) becomes

$$V_a^{-1} = \frac{1}{\tau_a} \int_0^1 \tau_e \ V_e^{-1} \ de. \tag{3.18}$$

Furthermore, we assume that risk tolerance τ_e and belief $\mathcal{B}_e$ are i.i.d random variables for each investor e. With a continuum of investors, we can write the consensus belief $\mathcal{B}_a$ in Proposition 3.3 as

$$V_a = \tau_a \lambda_a^* \mathbb{E}[\tilde{\lambda}^* \tilde{\tau} \tilde{V}^{-1}]^{-1}, \qquad \boldsymbol{\mu}_a = \tau_a^{-1} V_a \mathbb{E}[\tilde{\tau} \tilde{V}^{-1} \tilde{\boldsymbol{\mu}}],$$

where

$$\tau_a = \mathbb{E}[\tilde{\tau}], \qquad \lambda_a^* = \tau_a^{-1} \mathbb{E}[\tilde{\tau} \tilde{\lambda}^*].$$

When a riskless asset exists, the consensus belief can be expressed as

$$V_a = \tau_a \mathbb{E}[\tilde{\tau} \tilde{V}^{-1}]^{-1}, \qquad \boldsymbol{\mu}_a = \tau_a^{-1} V_a \mathbb{E}[\tilde{\tau} \tilde{V}^{-1} \tilde{\boldsymbol{\mu}}],$$

which are independent of the initial wealth distribution. Note that the expectation operator is on the possible beliefs amongst investors.

3.3 Disagreement and the CAPM

In this section, we consider an average belief[4] defined by $\mathcal{B}(V_o, \boldsymbol{\mu}_o)$, that is $\int_e V_e de = V_o$ and $\int_e \boldsymbol{\mu}_e de = \boldsymbol{\mu}_o$. In the case V_e and $\boldsymbol{\mu}_e$ are *i.i.d* random variables for every agent e, we can write $V_o = \mathbb{E}(\tilde{V})$ and $\boldsymbol{\mu}_o = \mathbb{E}(\tilde{\boldsymbol{\mu}})$. By using the results developed in the previous section, we examine the conditions under which the consensus beliefs $\mathcal{B}_a$ under

[3] In the same spirit of Admati (1985), we define the integral of a random vector $\int_0^1 \tilde{\mathbf{X}}_e \ de \equiv \mathbf{0}$ if for every sequence $\{e_i\}$ of distinct indices from $[0,1]$, $1/I \sum_i^I \tilde{\mathbf{X}}_{e_i} \to \mathbf{0}$. Suppose that $\tilde{\mathbf{X}}_e = \tilde{\mathbf{Y}}_e - \mathbb{E}[\tilde{\mathbf{Y}}_e]$, since $\int_0^1 \tilde{\mathbf{X}}_e \ de = \mathbf{0}$, it is natural to define $\int_0^1 \tilde{\mathbf{Y}}_e \ de \equiv \int_0^1 \mathbb{E}[\tilde{\mathbf{Y}}_e] \ de$ which equals $\mathbb{E}[\tilde{\mathbf{Y}}]$ if $\tilde{\mathbf{Y}}_e$ are i.i.d.

[4] One can think of the average belief as the objective belief about the distribution of future asset returns.

heterogeneous beliefs conform to the average belief $\mathcal{B}_o$. We use normal ($\mathcal{N}$) and truncated normal ($\mathcal{TN}$)[5] distributions to characterise the risk tolerance and beliefs of investors.

Corollary 3.5 *Assume that an investor's beliefs about the covariance matrix are homogenous, that is $V_e = V_o$, and the beliefs about the expected future asset returns are given $\boldsymbol{\mu}_e = \boldsymbol{\mu}_o + \boldsymbol{\alpha}_e$ where $\boldsymbol{\alpha}_e$ is a random vector with each component $\tilde{\alpha}_{e,j} \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2_{\alpha_j})$, the correlation*[6] *is given by $\rho(\tilde{\alpha}_j, \tilde{\alpha}_k)$. Furthermore, investors' risk tolerance $\tau_e \overset{i.i.d}{\sim} \mathcal{TN}(\tau_o, \sigma^2_\tau; (0, 2\tau_o))$*[7]*, The correlation between risk tolerance and beliefs about expected return of asset j is given by $\rho(\tilde{\tau},\ \tilde{\alpha}_j)$, which is the same for all investors. If we treat $\mathcal{B}_o(V_o, \boldsymbol{\mu}_o)$ as an average belief, then the consensus belief is the average belief if investors' beliefs about expected asset returns are uncorrelated with their risk tolerance. This holds with or without the existence of a riskless security. That is to say,*

$$\rho(\tilde{\tau},\ \tilde{\alpha}_j) = 0 \ \textit{for all}\ j \qquad \Rightarrow \qquad \boldsymbol{\mu}_a \ = \ \boldsymbol{\mu}_o. \tag{3.19}$$

Hence, the equilibrium prices for risky assets are the same as their average values, and the (zero-beta) CAPM relation under the heterogeneous beliefs hold under the average belief $\mathcal{B}_o$.

When investors are homogeneous about the the covariance matrix but heterogeneous about the expected returns, Corollary 3.5 shows that, when investors' risk tolerance and their subjective beliefs about the expected future asset returns are uncorrelated, the consensus belief $\mathcal{B}_a$ conforms to the average belief $\mathcal{B}_o$. In this case, investors are on average unbiased relative to the average belief and the disagreement among the investors are cancelled out in market equilibrium, and therefore the disagreement among the investors do not have an impact on the market equilibrium. However, this result no longer holds when the risk tolerance and beliefs about expected returns among investors are correlated, measured by $\rho(\tilde{\tau},\ \tilde{\alpha}_j)$. In fact, the consensus belief becomes optimistic (pessimistic) relative to the average belief when $\rho(\tilde{\tau},\ \tilde{\alpha}_j) > 0$ ($\rho(\tilde{\tau},\ \tilde{\alpha}_j) < 0$)[8].

We can further extend the result in Corollary 3.5 by allowing "small" heterogeneity in the beliefs of the covariance matrix when assuming the existence of a riskless security.

Corollary 3.6 *Assume there is a riskless security in the market. Assume that an investor's belief about the covariance matrix is formed by $V_e = V_o + \delta_e\ X$, where X is a semi-positive definite matrix. $\delta_e \overset{i.i.d}{\sim} \mathcal{TN}(0, \sigma^2_\delta; (-\varepsilon, \varepsilon))$ and ε is small. Moreover, investors'*

[5] Truncated normal distribution is defined in Chapter 2, Definition 2.11.
[6] This means that investor e's belief about the expected return for asset j and k could be correlated.
[7] This is to ensure that investors' risk tolerance is τ_i is positive and $\mathbb{E}(\tau_i) = \tau_o$ for all i.
[8] When there is no riskless asset, even if agents are homogeneous about expected asset returns (so that $\boldsymbol{\mu}_e = \boldsymbol{\mu}_o$), the heterogeneity in variance/covariances will make $\boldsymbol{\mu}_a$ different from $\boldsymbol{\mu}_o$.

beliefs about the expected returns is given by $\boldsymbol{\mu}_e = \boldsymbol{\mu}_o + \alpha_e \times \mathbf{1}$, *where* $\alpha_e \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma_\alpha^2)$. *Assume that investors' risk tolerance* $\tau_e \overset{i.i.d}{\sim} \mathcal{TN}(\tau_o, \sigma_\tau^2; (0, 2\tau_o))$. *Then the consensus belief* $\mathcal{B}_a$ *conforms to the average belief* $\mathcal{B}_o$ *if* δ_e, α_e *and* τ_e *are independent.*

Corollary 3.6 shows that, when there is a riskless asset in the market, the consensus belief would conform approximately to the average belief if disagreement about the covariance matrix is "small" and investors' risk tolerance and beliefs are independent. We conclude this section by noting from Proposition 3.3 that the heterogeneous beliefs have a significant impact on the market equilibrium in general. In particular, when risk tolerance and heterogeneous beliefs are correlated, they become part of the beta coefficients, which measure the systematic risk of risky assets, Chapter 4 provides further analysis of this.

3.4 The Tangency Relation under Heterogeneous Beliefs

It follows from Proposition 3.3 that the consensus belief $\mathcal{B}_a$ is not invariant to the existence of a riskless security. This suggests that the portfolio weights for the market portfolio denoted by $\boldsymbol{\omega}_m$ will depend on whether a riskless security exists in the market, hence the tangency geometric relation in the standard portfolio theory may not hold under heterogeneous beliefs. In this section, we derive conditions under which the tangency relation does hold in a heterogeneous economy and measure the impact of heterogeneity on the market portfolio when the tangency relation does not hold. We first consider a finite-agent economy and then extend the discussion to a continuum-agent economy.

Corollary 3.7 *Assume a homogeneous belief about the variance/covariances of future asset returns, that is* $V_i = V_o$, *for all investor* $i = 1, 2, \cdots, I$ *or* $V_e = V_o$ *for any agent* $e \in [0, 1]$. *Then the consensus belief about expected future asset return is independent of the existence of a riskless security. Hence, the tangency relation of the portfolio frontiers with and without the riskless asset still holds; that is the equilibrium market prices for risky assets remain the same with or without a riskless security in the market.*

When investors are homogeneous about their beliefs of covariance matrix, the following equation

$$\boldsymbol{\mu}_{a,f} = \boldsymbol{\mu}_{a,z} = \frac{1}{I}\sum_{i=1}^{I}\frac{\tau_i}{\tau_a}\boldsymbol{\mu}_i,$$

shows that, independent of the existence of the riskless asset, the consensus belief about expected asset returns is simply a risk tolerance weighted average of subjective beliefs about expected returns. Furthermore, if we define $W_{i,0}/\tau_i$ as investor i's *relative risk* *aversion*, and $\bar{W}_0/\tau_a$ as the aggregate relative risk aversion, the following equation

$$\lambda_a - 1 = r_f = \frac{\mathbf{1}^T V_o^{-1}\mathbb{E}_a(\tilde{\mathbf{r}}) - \bar{W}_0/\tau_a}{\mathbf{1}^T V_o^{-1}\mathbf{1}}$$

shows that the optimal CEW for the aggregate market only depends on the aggregate relative risk aversion in this case, where $\mathbb{E}_a(\tilde{\mathbf{r}}) = \boldsymbol{\mu}_{a,f} - 1 = \boldsymbol{\mu}_{a,z} - 1$. However, when investors' beliefs about the covariance matrix are heterogeneous, the tangency relation no longer holds. In the rest of this section, through two numerical examples, one with finite agents and one with infinite agents, we examine the impact of different aspects of heterogeneity on the market portfolio and the tangency relation.

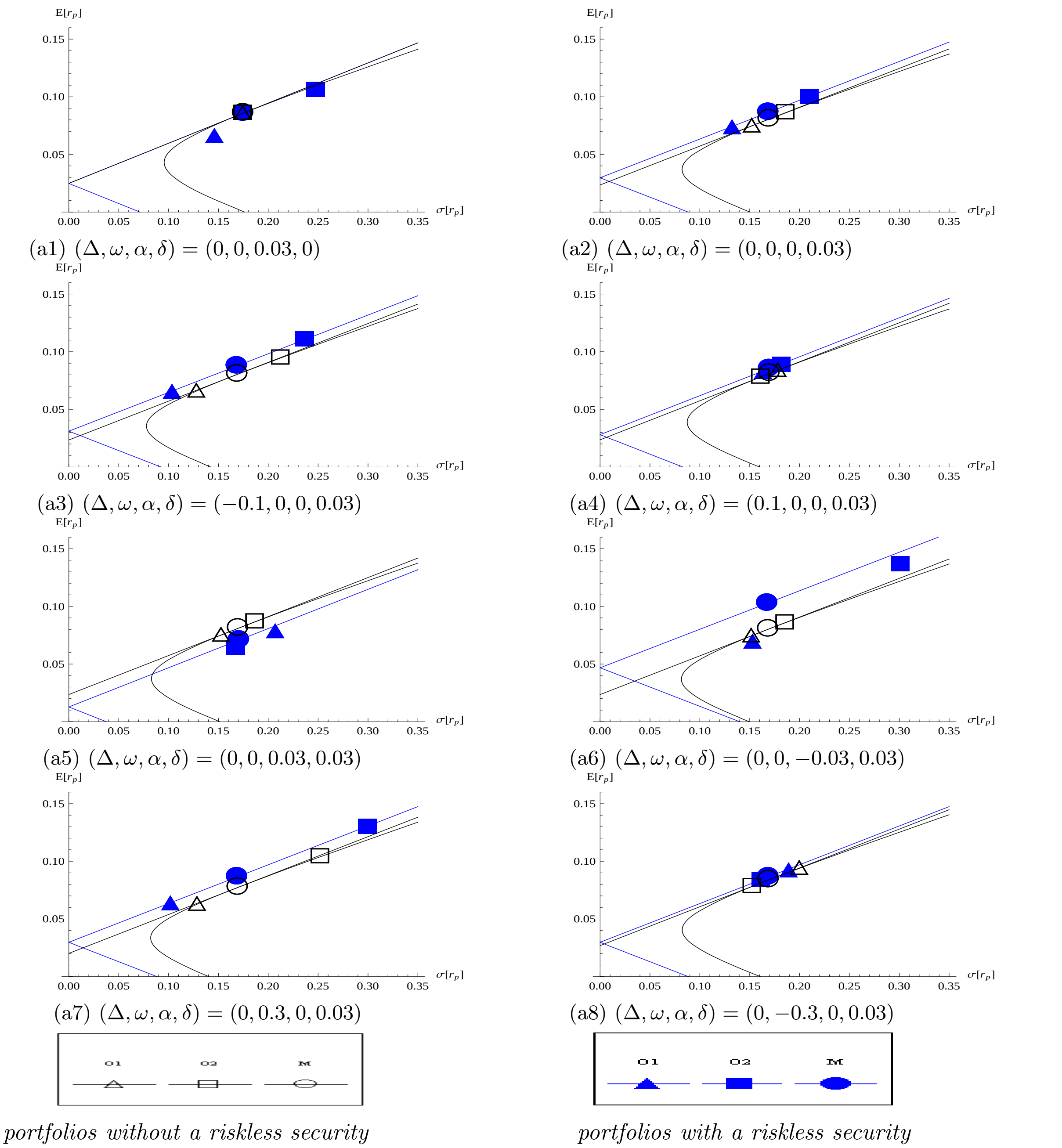
Example 3.8 *Consider the case of two agents with beliefs $\mathcal{B}_i = (V_i, \boldsymbol{\mu}_i)$ for $i = 1, 2$ and $N = 3$ risky assets in the market. For the two investors, let their risk tolerances $(\tau_1, \tau_2) = (\tau_o + \Delta, \tau_o - \Delta)$ and initial wealth $W_{1,o} = W_o + \omega$ and $W_{2,o} = W_o - \omega$, where $\tau_o = 0.5$ and $W_o = 1$. Assume that the market endowment of risky assets is given by $Z = diag(1,\ 1,\ 1)$. We take $\mathcal{B}_o(V_o, \boldsymbol{\mu}_o)$ as an average belief with $\boldsymbol{\mu}_o = (1.05,\ 1.08,\ 1.15)^T$ and $V_o = d_o C d_o$, where*

$$d_o = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix},$$

in which d_o corresponds to the standard deviation matrix and C is the correlation matrix.

We impose mean-preserving spreads of the average belief as the heterogeneous beliefs of the two investors by assuming that investors' beliefs are given by $(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = (\boldsymbol{\mu}_o + \alpha \times \mathbf{1}, \boldsymbol{\mu}_o - \alpha \times \mathbf{1})$ and $V_i = d_i C d_i$ for $i = 1, 2$, where $(d_1, d_2) = (\delta\ \ diag(1,1,1) + d_o, d_o - \delta\ \ diag(1,1,1))$. This implies that investors agrees on the correlation of future asset returns, but disagree about the expected and volatility of asset returns. Next we aggregate individuals' beliefs according to Proposition 3.3, first without a risk-free asset, then with a risk-free asset (f) using Corollary 3.4. It is found that the risk-free rate under the benchmark case is given by $r_f = 0.0249$.

To examine the tangency relationship, we plot the portfolio frontiers, the optimal portfolios and the market portfolio under the market consensus belief with and without a risk-free asset for different values of $(\Delta, \omega, \alpha, \delta)$ in Figure 3.1. We assume $\delta > 0$, that is, investor 2 is more confident about his/her belief of future asset returns than investor 1 in the sense that his/her belief about the standard deviations of asset returns are smaller than that of investor 1. Note that according to Corollary 3.7, when $\delta = 0$, the tangency relation holds. We also calculate the consensus expected returns of the three risky assets, the portfolio weights and expected return of the market portfolio, and the riskless/zero-beta return with/without the riskless asset in Table 3.1.



(a1) $(\Delta, \omega, \alpha, \delta) = (0, 0, 0.03, 0)$
(a2) $(\Delta, \omega, \alpha, \delta) = (0, 0, 0, 0.03)$
(a3) $(\Delta, \omega, \alpha, \delta) = (-0.1, 0, 0, 0.03)$
(a4) $(\Delta, \omega, \alpha, \delta) = (0.1, 0, 0, 0.03)$
(a5) $(\Delta, \omega, \alpha, \delta) = (0, 0, 0.03, 0.03)$
(a6) $(\Delta, \omega, \alpha, \delta) = (0, 0, -0.03, 0.03)$
(a7) $(\Delta, \omega, \alpha, \delta) = (0, 0.3, 0, 0.03)$
(a8) $(\Delta, \omega, \alpha, \delta) = (0, -0.3, 0, 0.03)$

portfolios without a riskless security *portfolios with a riskless security*

Figure 3.1: Comparing the optimal and market portfolios and geometric relationships of market equilibrium frontiers with and without a risk-free asset when the risk-free asset is in zero net supply.

Figure 3.1 (a1) confirms that in the case $\delta = 0$, that is, when beliefs about the covariance matrix are homogeneous, the tangency relation holds, that is, the portfolio weights and the MV efficiency of the market portfolio are unchanged when a riskless security is added to the market, as verified in Table 3.1 (b1) (with the riskless asset) and (c1) (without riskless asset). Also, the CEW of the aggregate market remains the same, indicating that the marginal utility per dollar investment for the market remains identical. However, the tangency relation breaks down when investors have heterogeneous beliefs about the covariance matrix, see Fig. 3.1 (a2)- Fig. 3.1 (a8). In the case of Fig. 3.1 (a2), the *capital market line* (CML) is disjoint from the portfolio frontier (PF). The market derives a larger marginal utility per dollar investment with a riskless security than without it, since the implied equilibrium risk-free return r_f in Table 3.1 (b2) is higher than the zero-beta rate $\lambda_a - 1$ in Table 3.1 (c2) under the consensus belief. The expected return of the market portfolio under the consensus belief increases while standard deviation remains at approximately the same level (as an observation) when a zero-net supply riskless security is added to the market. We refer to Table 3.1 (b2) and (c2) for further insights. Note that although the beliefs about expected returns are homogeneous, without a riskless security the consensus belief of expected returns does not conform to the average belief; it is actually pessimistic with respect to the average belief. Intuitively, this is because the consensus belief of the covariance matrix is weighted not only by the risk tolerance but also by investors' subjective marginal utility per dollar invested. In this case, when a riskless security exists the market invests more into assets 1 and 3 and improves the expected return of the market portfolio under the consensus belief.

The impact of heterogeneity is probably the strongest in Fig. 3.1 (a5) and (a6); the plots show two quite distinctive scenarios. First, when the more confident investor is also more optimistic about future asset returns, the CML is again above the PF, see Fig. 3.1 (a6). The gap is the greatest compared to all other cases, suggesting that this combination of heterogeneity is most helpful in improving the market's marginal utility per dollar investment and the MV efficiency, since the expected return of the market portfolio increases under the consensus belief while standard deviation remains approximately unchanged. Table 3.1 (b6) and (c6) show that the consensus belief of the expected asset return is above (below) the ones under the average belief $\mathcal{B}_o$ with (without) a riskless security. In this case, the market portfolio consists more of assets 1 and 3 when a riskless security exists. On the other hand, Fig. 3.1 (a5) shows that the CML can be below the PF in the case where the more confident investor is less optimistic about future asset return, leading to lower marginal utility per dollar investment (risk-free rate) and a lower expected return for the market portfolio under the consensus belief after a riskless security is added. Table 3.1 (b5) and (c5) show that with or without a riskless security, the consensus belief of expected returns is lower than the average values; however, it is more so with a riskless security present in the market. Furthermore, the aggregate market

	$(\Delta,\omega,\alpha,\delta)$	$\mathbb{E}_a(\tilde{\mathbf{r}})$	$\boldsymbol{\omega}_m$	$\mathbb{E}_a(\tilde{r}_m)$	r_f
	(0,0,0,0)	(0.05, 0.08, 0.15)	(0.4143, 0.3286, 0.2571)	0.0856	0.0249
(b1)	(0,0,0.03,0)	(0.05, 0.08, 0.15)	(0.4143, 0.3286, 0.2571)	0.0856	0.0249
(b2)	(0,0,0,0.03)	(0.05, 0.08, 0.15)	(0.4132, 0.3200, 0.2668)	0.0862	0.0297
(b3)	(-0.1,0,0,0.03)	(0.05, 0.08, 0.15)	(0.3921, 0.3336, 0.2743)	0.0874	0.0311
(b4)	(0.1,0,0,0.03)	(0.05, 0.08, 0.15)	(0.4332, 0.3078, 0.2590)	0.0851	0.0280
(b5)	(0,0,0.03,0.03)	(0.0328, 0.0661, 0.1315)	(0.3584, 0.3909, 0.2506)	0.0706	0.0127
(b6)	(0,0,-0.03,0.03)	(0.0672, 0.0939, 0.1685)	(0.4680, 0.2490, 0.2830)	0.1025	0.0467
(b7)	(0,0.3,0,0.03)	(0.05, 0.08, 0.15)	(0.4132, 0.3200, 0.2668)	0.0863	0.0297
(b8)	(0,-0.3,0,0.03)	(0.05, 0.08, 0.15)	(0.4132, 0.3200, 0.2668)	0.0863	0.0297
	$(\Delta,\omega,\alpha,\delta)$	$\mathbb{E}_a(\tilde{\mathbf{r}})$	$\boldsymbol{\omega}_m$	$\mathbb{E}_a(\tilde{r}_m)$	$\lambda_a - 1$
	(0,0,0,0)	(0.05, 0.08, 0.15)	(0.4143, 0.3286, 0.2571)	0.0856	0.0249
(c1)	(0,0,0.03,0)	(0.05, 0.08, 0.15)	(0.4143, 0.3286, 0.2571)	0.0856	0.0249
(c2)	(0,0,0,0.03)	(0.0436, 0.0748, 0.1430)	(0.3932, 0.3460, 0.2608)	0.0803	0.0235
(c3)	(-0.1,0,0,0.03)	(0.0422, 0.0735, 0.1414)	(0.3675, 0.3655, 0.2670)	0.0801	0.0234
(c4)	(0.1,0,0,0.03)	(0.0454, 0.0763, 0.1449)	(0.4188, 0.3265, 0.2547)	0.0808	0.0236
(c5)	(0,0,0.03,0.03)	(0.0438, 0.0752, 0.1436)	(0.3931, 0.3460, 0.2609)	0.0807	0.0235
(c6)	(0,0,-0.03,0.03)	(0.0434, 0.0744, 0.1424)	(0.3931, 0.3460, 0.2609)	0.0799	0.0235
(c7)	(0,0.3,0,0.03)	(0.0401, 0.0719, 0.1392)	(0.3822, 0.3602, 0.2576)	0.0771	0.0201
(c8)	(0,-0.3,0,0.03)	(0.0471, 0.0776, 0.1468)	(0.4041, 0.3318, 0.2641)	0.0836	0.0269

Table 3.1: Consensus belief of expected future asset returns $\mathbb{E}_a(\tilde{\mathbf{r}})$, portfolio weights for the market portfolio, $\boldsymbol{\omega}_m$ expected return for the market portfolio under the consensus belief $\mathbb{E}_a(\tilde{r}_m)$ and the risk-free rate r_f or the zero-beta rate $\lambda_a - 1$ for different parameters values assigned to $(\Delta,\omega,\alpha,\delta)$. The top half of the table corresponds to the case where a riskless security exists in the market with zero net supply and the bottom half of the table corresponds to the case where a riskless security does not exist in the market.

invests more wealth into asset 2 and less in assets 1 and 3 when a riskless security exists. An intuition might be that riskless borrowing/lending is a more binding constraint for the more confident investor than the less confident. Therefore, when the more confident investor is more (less) optimistic about future stock returns, the consensus belief will reflect the belief of the more confident investor more strongly if a riskless security exists, thus marginal utility per dollar and expected return under consensus belief will improve (worsen) when a riskless security is added to the market.

When we combine heterogeneity in investors' risk tolerance and beliefs about the variance and covariance matrix, Fig. 3.1 (a3) and (a4) show that the existence of a riskless security has a significant impact on market equilibrium when the more confident investor is more risk tolerant. Table 3.1 (b3)-(c3) and (b4)-(c4) show that when the confident investor is more risk tolerant, the market invests less into asset 1 and more into assets 2 and 3. The consensus belief of the expected asset returns conform to the average with a riskless security, whereas without a riskless security, the consensus belief of expected asset returns is below the average belief. Intuitively, since riskless borrowing/lending is more binding for the more confident investor, if he/she is also more risk tolerant and has greater demand for risky assets, then the consensus belief will reflect his/her belief more than the other investor. However, the effect of heterogeneous risk tolerance on the aggregate market becomes insignificant in this case because beliefs about the expected future asset returns are homogeneous. A similar intuition can be used to explain Fig. 3.1 (a7) and (a8) and results in Table 3.1 (b7)-(c7) and (b8)-(c8). In those cases, we combine heterogeneous beliefs about covariance matrix with heterogeneity in investors' initial wealth. Results show that the impact of heterogeneity is stronger when the more confident investor has less initial wealth (see Fig. 3.1 (a7)); the impact becomes insignificant when the more confident investor has relatively more initial wealth than the other investor. This maybe be due to the fact that investors' demands are independent of their initial wealth when a riskless security exists; however, without a riskless security the riskless borrowing/lending constraint is more binding for the more confident investor when he/she has less initial wealth. This is also reflected in the fact that an investor's subjective CEW or marginal utility per dollar is negatively related to his/her initial wealth, see equation (3.4), hence as an investor's initial wealth increases, his/her impact on the consensus belief reduces.

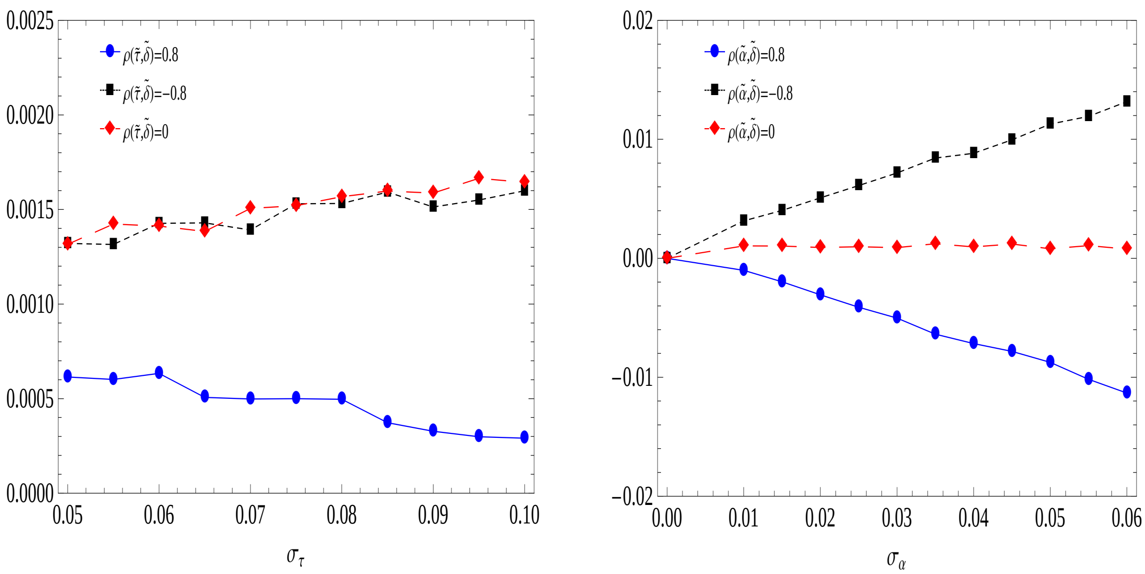
To extend the above analysis of a case with two investors to a case with a continuum of investors.

Example 3.9 *We simulate the economy with a continuum of investors with $I = 10,000$ investors and three risky assets. Assume that the average supply of shares per investor is given by $\bar{Z} = \frac{1}{2}diag(1,1,1)$ (in order to be consistent with the average share in Example 3.8). Investors' initial wealth $W_{0,i} = 1$ so that average initial wealth $\bar{W}_0 = 1$. Let the risk tolerance $\tau_i \overset{i.i.d}{\sim} \mathcal{TN}(\tau_o, \sigma_\tau^2, (0,1))$, where $\tau_o = 0.5$. Investors' beliefs about the covariance matrix $V_i = V_o + \delta_i\ X$ where X is a 3×3 matrix with all elements equal to 1, and $\delta_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma_\delta^2)$ and V_o is given in Example 3.8. Investors' beliefs about the expected future returns are given by $\boldsymbol{\mu}_i = \alpha_i \times \mathbf{1} + \boldsymbol{\mu}_o$, where $\boldsymbol{\mu}_o$ is as defined in Example 3.8 and $\alpha_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma_\alpha^2)$. Lastly, $\rho(\tilde{\tau}, \tilde{\delta})$ and $\rho(\tilde{\alpha}, \tilde{\delta})$ denote the correlation between investors' confidence and their risk tolerance and beliefs about expected returns respectively. We assume that risk tolerance and beliefs about expected returns are uncorrelated, that is $\rho(\tilde{\tau}, \tilde{\alpha}) = 0$. We fix the dispersion in the covariance matrix $\sigma_\delta = 0.2\%$ and set the dispersion in risk tolerance $\sigma_\tau \in [0.05, 0.1]$ and dispersion in the belief of expected future returns $\sigma_\alpha \in [0, 0.06]$.*

To facilitate our analysis of the tangency relation, we define the difference in the expected return of the market portfolio under the consensus belief between the case with a riskless security and the case without a riskless security as

$$\Delta(\tilde{r}_m) = \mathbb{E}_a^f(\tilde{r}_m) - \mathbb{E}_a^z(\tilde{r}_m),$$

where $\mathbb{E}_a^f(\tilde{r}_m)$ is the expected return of the market portfolio with a riskless security and $\mathbb{E}_a^z(\tilde{r}_m)$ is the expected return of the market portfolio without a riskless security. If the tangency relation holds, for example, when beliefs about the covariance matrix are homogeneous, $\Delta(\tilde{r}_m) = 0$. Otherwise, the tangency relation does not hold under heterogeneous beliefs. In particular, $\Delta(\tilde{r}_m) > 0$ suggests that the market benefits from the existence of a riskless security for a given level of dispersion in risk tolerance and beliefs of expected returns, since the market as an aggregate investor believes that his/her optimal portfolio (the market portfolio) is able to achieve a higher expected return when a riskless security exists. It is the reverse when $\Delta(\tilde{r}_m) < 0$. Therefore, we refer to $\Delta(\tilde{r}_m)$ as the *Benefit of Riskless Borrowing/Lending* (BRBL) to the aggregate market measured in expected returns. Fig. 3.2 illustrates the impact of heterogeneity on BRBL.



(d1) Impact of dispersion in risk tolerance (d2) Impact of dispersion in expected returns

Figure 3.2: Impact of dispersion in risk tolerance (d1) and beliefs in expected returns (d2) on the BRBL $\Delta(\tilde{r}_m)$ when correlation between risk tolerance and confidence $\rho(\tilde{\tau}, \tilde{\delta})$ is zero, positive or negative (in (d1)), and when correlation between optimism and confidence $\rho(\tilde{\alpha}, \tilde{\delta})$ is zero, positive or negative (in (d2)).

It is evident from Figure 3.2 that the dispersion in the beliefs of expected future asset returns measured by σ_α has a much stronger effect on the BRBL $\Delta(\tilde{r}_m)$ than dispersion in risk tolerance measured by σ_τ. It can be seen from Fig. 3.2(d2) that there is a higher BRBL with an increasing level of dispersion in the beliefs of expected future asset returns when confident investors are also more optimistic about expected future returns than less confident investors, that is $\rho(\tilde{\alpha}, \tilde{\delta}) < 0$. This relationship is inverted when more confident investors are less optimistic, that is $\rho(\tilde{\alpha}, \tilde{\delta}) > 0$. This observation is consistent with the results obtained in Example 3.8 in Figure 3.1 (a5) and (a6) where there were only two investors in the market. Intuitively, because riskless borrowing/lending constraint is more binding for the confident investor, the consensus belief will reflect more of the confident investor's belief about expected asset return when a riskless security exists. This intuition now carries over to the case with a continuum of investors, as a positive correlation between confidence and optimism can increase the BRBL whereas a negative correlation reduces BRBL. Note that the confidence level of investors has no significant impact on the BRBL when $\rho(\tilde{\alpha}, \tilde{\delta}) = 0$. Furthermore, the dispersions in expected returns have little or no effect on the BRBL when there is a zero correlation between investors' confidence and optimism. Figure 3.2 (d1) shows that dispersion in investors' risk tolerance has a slight positive effect on the BRBL when the correlation between investors' risk tolerance level and their level of confidence is zero or positive; increasing the level of correlation does not seem to magnify the effect. When the more confident investors are less risk tolerant, that is $\rho(\tilde{\tau}, \tilde{\delta}) > 0$, increasing dispersion in risk tolerance σ_τ actually causes the BRBL to reduce almost to zero. Again, the results are consistent with the two investors case, see Figure 3.1 (a3) and (a4), and the same reasoning applies.

3.5 MV Efficiency under Heterogeneous Beliefs

In this section, we examine the MV efficiency of the optimal portfolios of the investors under their heterogeneous beliefs. It is clear in general that the subjective optimal portfolios of investors can be MV inefficient. Figure 3.1 shows that the subjectively optimal portfolio of the two investors are quite close to the PF or CML in all of the cases except when investors disagree on the expected future asset return when riskless borrowing/lending is allowed, see Figures 3.1(a1), (a5), and (a6). This observation is quite fascinating, because this suggests that the market is able to provide both investors with nearly MV efficient portfolios, as in the situation when investors' beliefs about expected future returns are homogeneous or when a riskless borrowing/lending is not available. It would be interesting to see how this observation extends to the case with a continuum of investors whose risk tolerance and heterogeneous beliefs are assumed to be generated from certain mean-preserving distributions. We investigate this issue through the following numerical example in which two types of mean-preserving distributions are considered.

Example 3.10 *Consider the case in Example 3.9, where we assume that investors' beliefs about the covariance matrix are homogeneous, that is $V_i = V_o$ for all i, and consider two types of probability distributions for investors' beliefs of expected future asset returns;*

(i). $\boldsymbol{\mu}_i = \tilde{\alpha}_i \times \mathbf{1} + \boldsymbol{\mu}_o$, *where* $\boldsymbol{\mu}_o$ *is as defined in Example 3.8.* $\tilde{\alpha}_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma_\alpha^2)$.

(ii). $\boldsymbol{\mu}_i = \tilde{\boldsymbol{\alpha}}_i + \boldsymbol{\mu}_o$ *where each component* $\tilde{\alpha}_{i,j} \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma_\alpha^2)$*. We assume components* $\tilde{\alpha}_{i,j}$ *are uncorrelated and have the same mean and variance.*

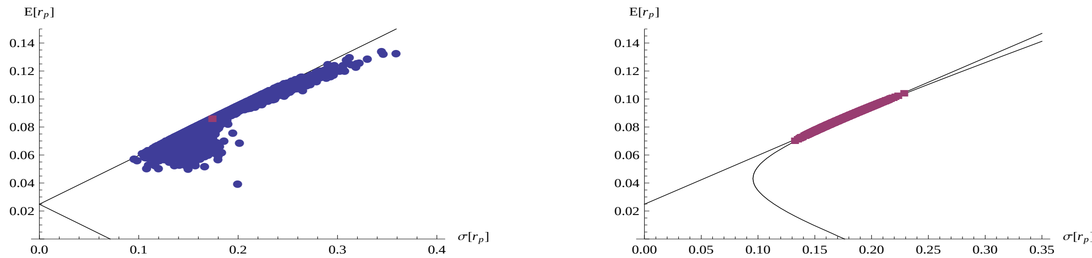
We simulate this economy with $10,000$ *investors.*

We first consider the case with $\rho(\tilde{\tau}, \tilde{\alpha}) = 0$, that is, the investor's risk tolerance is uncorrelated with his/her belief about the expected asset returns. Note that in this case, according to Corollary 3.5, the consensus belief $\mathcal{B}_a$ conforms to the average belief $\mathcal{B}_o$. Consequently, the tangency relation holds under the heterogeneous beliefs. In the following, we are interested in the MV efficiency of the optimal portfolio of the investors under their heterogeneous beliefs.

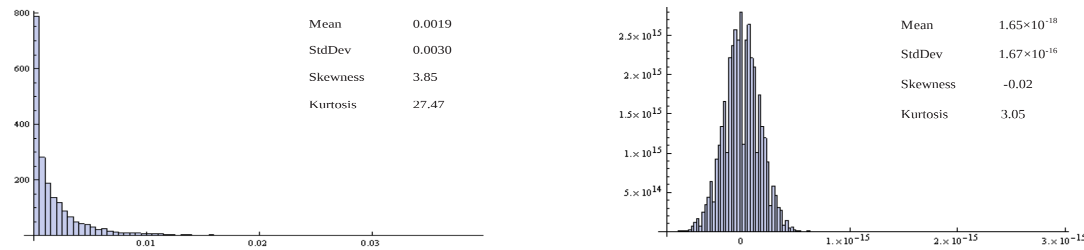
Case (i) corresponds to the case where investors' divergence of opinion is the same across each risky asset, because heterogeneity in beliefs of expected asset returns is characterised by an i.i.d univariate random variable $\tilde{\alpha}_i$. This means that if investor i is for example optimistic with respect to the average belief for one risky asset, that is, $\alpha_i > 0$, then he/she is also optimistic about the other two risky assets by the same amount. When the divergence of opinions is the same across each risky asset, Figure 3.3 (b1) plots the subjectively optimal portfolios with the *capital market line* (CML) and the *portfolio frontier* (PF), respectively. It is interesting to see that investors' subjectively optimal portfolios all lie very close to the efficient frontier without the riskless security in the market. This means that the market is able to provide most of the investors with almost MV efficient portfolios in equilibrium, which is a somewhat a surprising result. When a riskless security exists, subjectively optimal portfolios may no longer be MV efficient, though the majority of them lie close or exactly on the CML with a few significantly further away from from the CML. To quantify the MV efficiency of the subjectively optimal portfolios, we define the *Loss of Expected Portfolio Return*(LEPR) for investor i as the difference between his/her expected return under the consensus belief and the expected return of a portfolio on the CML with the same standard deviation. That is, given the expected return and standard deviation of investor i portfolio $(\sigma_a(\tilde{r}_p), \mathbb{E}_a(\tilde{r}_p))$, the loss of expected portfolio return is defined as $\mathbb{E}_a^*(\tilde{r}_p) - \mathbb{E}_a(\tilde{r}_p)$, where

$$\mathbb{E}_a^*(\tilde{r}_p) = \frac{A}{C} + \sqrt{\frac{D}{C}\sigma_a^2(\tilde{r}_p) + \left(\frac{A}{C}\right)^2 - \frac{B}{C}}$$

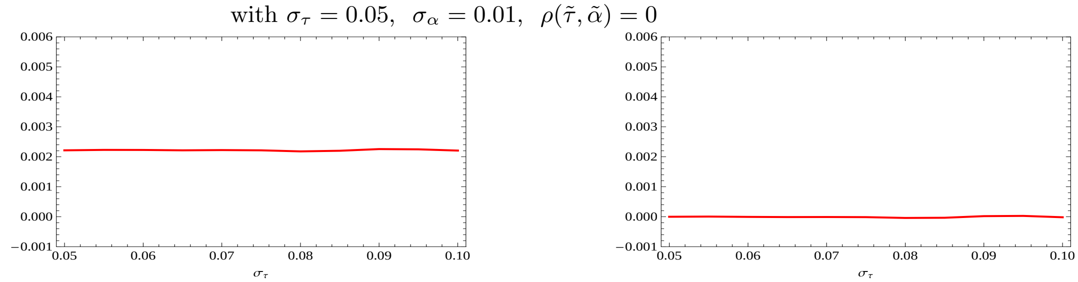
and $A = \mathbf{1}^T V_a^{-1} \mathbb{E}_a(\tilde{\mathbf{r}})$, $B = \mathbb{E}_a(\tilde{\mathbf{r}})^T V_a^{-1} \mathbb{E}_a(\tilde{\mathbf{r}})$, $C = \mathbf{1}^T V_a^{-1} \mathbf{1}$ and $D = BC - A^2$ in the case where a riskless security does not exist in the market (see Huang and Litzenberger



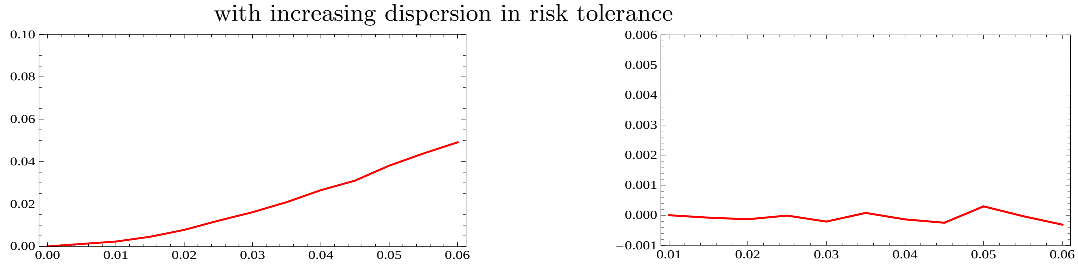
(b1) MV efficiency of the subjectively optimal portfolios
with $\sigma_\tau = 0.05, \quad \sigma_\alpha = 0.01, \quad \rho(\tilde{\tau}, \tilde{\alpha}) = 0$



(b2) Distribution of the loss of expected portfolio returns
with $\sigma_\tau = 0.05, \quad \sigma_\alpha = 0.01, \quad \rho(\tilde{\tau}, \tilde{\alpha}) = 0$



(b3) The average loss of expected portfolio returns
with increasing dispersion in risk tolerance



(b4) The average loss of expected portfolio returns with
increasing dispersion in beliefs of expected asset returns

Figure 3.3: MV efficiency of the subjectively optimal portfolios and the average loss of expected portfolio returns (LEPR) (i) with a riskless security (left panel) and (ii) without a riskless security (right panel). Divergences of opinion are the same across each risky assets.

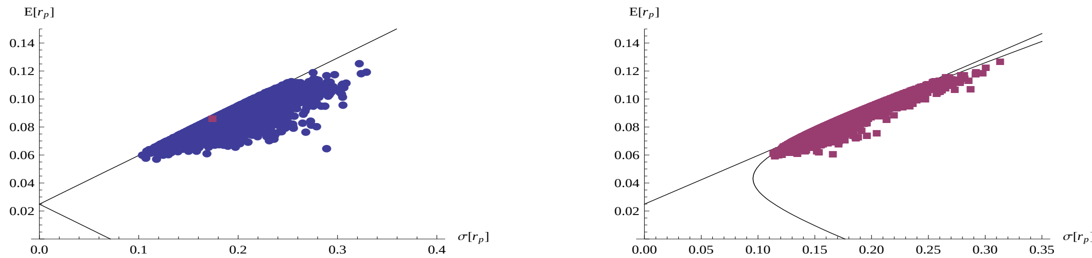
(1988)). When a riskless security exists, we have

$$\mathbb{E}_a^*(\tilde{r}_p) = r_f + \sqrt{H} \;\; \sigma_a(\tilde{r}_p),$$

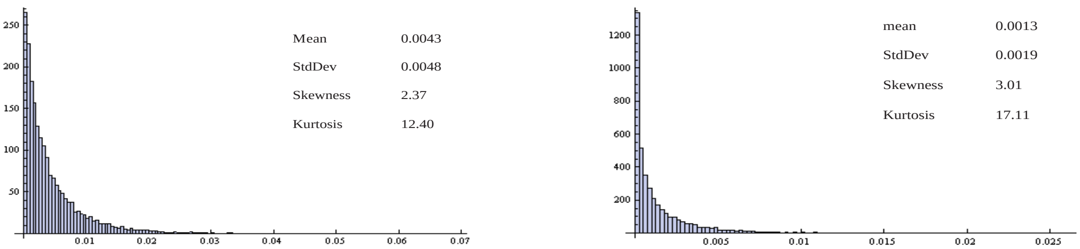
where $H = (\mathbb{E}_a(\tilde{\mathbf{r}}) - r_f)^T V_a^{-1} (\mathbb{E}_a(\tilde{\mathbf{r}}) - r_f)$. Theoretically, LEPR is non-negative. The loss of expected portfolio return indicates how much expected return investor i has lost due to his biased belief about the expected returns of risky assets. Fig. 3.3 (b2) (right-panel) shows that without a riskless security, the mean and standard deviation of the LEPR is very close to zero[9], which is consistent with the result given in Fig. 3.3 (b1) (right-panel). This means that, in this case, there are almost no loss of expected returns for any investor's optimal portfolios. Fig. 3.3 (b2) (left-panel) shows that when riskless borrowing and lending are allowed, the distribution of LEPR is heavily skewed to the right with a large kurtosis, suggesting that there are a few investors with very inefficient portfolios compared to other investors. Investors on average loses 0.19% with a standard deviation of 0.30%, given dispersion in belief of expected returns $\sigma_\alpha = 0.01$ and dispersion in risk tolerance $\sigma_\tau = 0.05$. Figs 3.3 (b3) and (b4) show the impact of dispersion in risk tolerance and belief about expected returns on the average LEPR, respectively. We observe that dispersion in risk tolerance has little or no effect on the average LEPR, while an increase in dispersion in beliefs about expected returns also increases the average LEPR by a significant amount. That is, when σ_α increases to 0.06, the portfolio of an average investor in the market is expected to lose 5% return against a portfolio on the CML with the same standard deviation. However, the dispersion in the belief of expected returns has an insignificant effect on the average LEPR in the case without a riskless security, and the average LEPR is virtually zero.

Case (ii) corresponds to the case where investors' divergence of opinions are different across each risky asset, because heterogeneity about beliefs of expected asset returns is characterised by i.i.d multivariate random variable $\tilde{\boldsymbol{\alpha}}_i$. Therefore a different realisation $\alpha_{i,j}$ is applied to each asset j, such that investor i could be optimistic about the future return with respect to the average belief for one risky asset, but pessimistic about the future return of another risky asset. When the divergence of opinions is different for each risky asset, Figure 3.4 (c1) plots the subjectively optimal portfolios with the CML and the PF, respectively. It shows that the optimal portfolios are scattered below the efficient frontiers; without access to riskless borrowing and lending there are more investors with their subjectively optimal portfolios close to the efficient frontier, but nowhere as close as in Fig. 3.3 (b1) (right-panel). Fig. 3.4 (c2) plot the distribution of the LEPR for the 10,000 subjectively optimal portfolios with a riskless security (left-panel) and without a riskless security (right-panel), respectively. Both distributions are skewed heavily to the right with a high kurtosis and have a similar feature to the plot in Fig. 3.3(b2). The

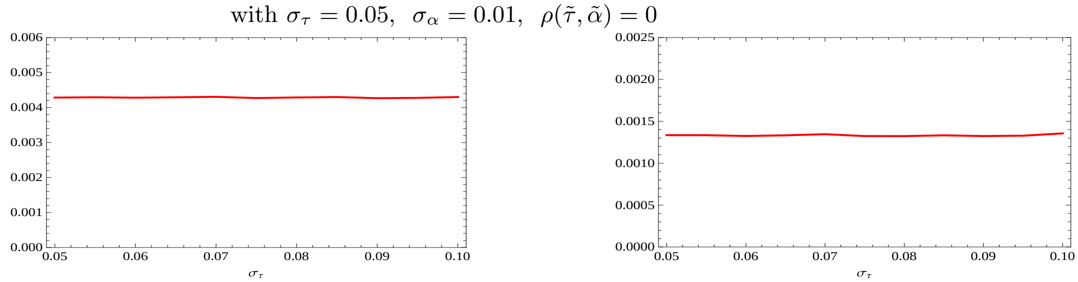
[9] Any negative LEPR that have resulted from the simulations in Fig. 3.3 (b4) (right-panel) are due to numerical errors from simulations, their magnitude is small enough to be ignored.



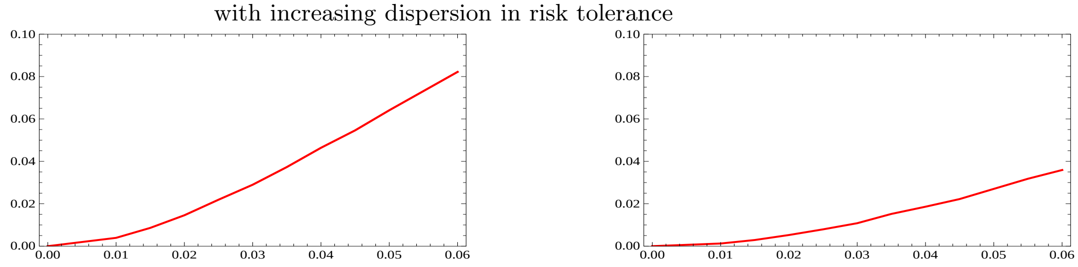
(c1) MV efficiency of the subjectively optimal portfolios
with $\sigma_\tau = 0.05, \quad \sigma_\alpha = 0.01, \quad \rho(\tilde{\tau}, \tilde{\alpha}) = 0$



(c2) Distribution of the loss of expected portfolio returns
with $\sigma_\tau = 0.05, \quad \sigma_\alpha = 0.01, \quad \rho(\tilde{\tau}, \tilde{\alpha}) = 0$



(c3) The average loss of expected portfolio returns
with increasing dispersion in risk tolerance



(c4) The average loss of expected portfolio returns with
increasing dispersion of beliefs about expected asset returns

Figure 3.4: MV efficiency of the subjectively optimal portfolios (i) with a riskless security (left panel) and (ii) without a riskless security (right panel). Divergences of opinion are different for each risky assets.

average loss of the expected returns is 0.45% with a riskless asset and 0.13% without a riskless asset. These numbers are significantly higher than those in Fig. 3.3 (b2), where divergence of opinions is the same across each risky asset, especially in the case without a riskless security. The average loss of expected returns remains constant with increasing dispersions in risk tolerances, see Fig. 3.4 (c3). However, Fig. 3.4 (c4) shows that the average loss of expected returns increases with higher dispersions of beliefs about expected future asset returns. When the dispersion of beliefs in expected returns $\sigma_\alpha = 6\%$, an average investor is expected to lose 8% return against an MV efficient portfolio with the existence of a riskless security in the market. When riskless borrowing/lending is not allowed, the average investor would be expected to lose 3.5%. Again, these numbers are significantly larger than those in Fig. 3.3 (b4). This suggests that investors' disagreement about expected future asset returns have a greater negative impact on the MV efficiencies of their subjectively optimal portfolio when divergence of opinions differs across each risky asset.

Throughout the above analysis, we have assumed that an investor's risk tolerance is uncorrelated with his/her beliefs about the expected asset returns. By redoing the analysis when the correlation $\rho(\tilde{\tau}, \tilde{\alpha}) > 0$ and $\rho(\tilde{\tau}, \tilde{\alpha}) < 0$, we found that the average LEPR is not significantly affected by the correlation between risk tolerance and beliefs about expected returns.

3.6 Conclusion

This chapter examines the impact of heterogeneity on the MV efficiency of subjectively optimal portfolios, the geometric tangency relation, and the market equilibrium in general under a mean-variance framework when the beliefs are formed with respect to the rate of asset returns. We construct a consensus belief in a similar way to that of Chapter 2, but in terms of future rate of returns. When the riskless asset exists, we recover the CAPM under heterogeneous beliefs obtained in Chiarella et al. (2010*a*). We show that when beliefs about the covariance matrix are homogeneous and when beliefs about expected returns and risk tolerance are characterised by uncorrelated mean-preserving spreads distributions, the consensus belief conforms to the average beliefs independently of the existence of the riskless asset. When there is a riskless asset, this result is also extended to allow "small" heterogeneity in the covariance matrix.

In general, the market may benefit from investing in the riskless asset. By considering a case with two investors and also a case with a continuum of investors in the market, we show that the *Benefit of Riskless Borrowing/Lending* (BRBL) to the market portfolio by allowing investment in the riskless asset increases significantly with increasing dispersion in the belief of expected future asset returns when more confident investors are also optimistic about future asset returns. However, when the correlation between confidence levels and optimism is negative, the BRBL can become negative and decrease further with increasing dispersion of beliefs about expected returns. It is found that the dispersion in risk tolerance does not seem to have a significant impact on the BRBL. The BRBL is zero, with homogeneous belief about the variance/covariances of asset returns, thus the tangency relation holds. Regarding the MV efficiency of portfolios, we demonstrate that investors' subjectively optimal portfolios may not be MV efficient, mainly caused by their disagreement about expected future asset returns. The average *loss of expected portfolio return* (LRP) increases with higher dispersion of belief about expected returns. The effect becomes stronger when the divergence of opinions differs across each risky asset.

Chapter 4

Differences in Opinion and Risk Premium

4.1 Introduction

To better explain market anomalies, puzzles and various market phenomena, economics and finance are witnessing an important paradigm shift, from a representative, rational agent approach towards a behavioural, agent-based approach in which economy and markets are populated with bounded rational agents who have heterogeneous beliefs (Conlisk (1996)). A large number of studies are motivated by the impact of various aspects of heterogeneity on the equity premium and the risk-free rate. Most of the theoretical models that relate heterogeneous beliefs to equity premium and the risk-free rate assume that investors know the payoff of risky assets in each state of the world, but disagree on the probability of each state occurring. This is of course a convenient way to formulate the problem because it limits the dimension of heterogeneity; however, in reality investors may need to make prediction about the entire joint probability distribution of asset returns, which complicates the problem immensely. To simplify the analysis, we consider in this chapter a static financial market with two risky assets, one risk-free asset, and two agents with different risk preferences and heterogeneous beliefs about the joint probability distribution of asset returns. We also extend the analysis to allow for a continuum of investors. We assume that agents agree on the expected return and risk (measured by the standard deviation) for one risky asset but disagree on that of the other risky asset. They may also disagree on the correlation coefficient of the returns of the two assets. The setup of the economy is similar to that of Chapter 3, the difference being that by considering the simple case with two risky assets and a riskless asset, we are able to characterise the consensus belief explicitly in terms of the mean-preserving spreads on the risk tolerance and beliefs of the investors. This allows us to analyse more explicitly the impact of heterogeneity on the market risk premium and the risk-free rate, which are the focus of this chapter. This is in contrast to Chapter 3, which focuses on portfolio analysis in a general economy with many risky assets and heterogeneous beliefs. Different to the standard rational expectation equilibrium, the market equilibrium under the consensus belief reflects the bounded rationality of the agents in the sense that the market equilibrium is achieved when agents make their optimal decision based on their subjective beliefs. We call such equilibrium a *boundedly rational equilibrium*, when the impact of investors' disagreement "cancel out" and the consensus belief conforms to the average belief. We show that the "cancel out" effect holds when the different aspects of heterogeneity, including risk tolerance, optimism/pessimism and confidence/doubt, are uncorrelated. However, they do not cancel out when different aspects of heterogeneity are correlated and have a significant effect on the endogenous variables such as the market risk premium (equity premium), risk-free rate, market volatility and the portfolio weights of the market portfolio. This chapter aims to improve our understanding of the impact of differences in opinion on the market equilibrium, in particular, the market risk premium and the risk-free rate (rather than trying to address the equity premium and risk-free rate puzzles in the standard setup of maximisation of intertemporal utility of consumption). It will become clear that the impact of heterogeneity on market equilibrium with two risky assets is significantly different to the case with one risky asset. For example, when the more risk-tolerant investor is less optimistic about the future return for one of the risky assets, the market indeed becomes pessimistic about the asset's future return, consistent with the findings in Jouni and Napp (2006, 2007). However, we show that this does not necessarily imply a higher market risk premium and lower risk-free rate, as one would expected in the case with only one risky asset. The additional dimension of heterogeneity induced by having two correlated risky assets plays an important role in the market risk premium and the risk-free rate when combined with disagreements in expected asset returns. We also extend the model to include consumption, and agents have heterogeneous beliefs about assets' future payoffs, different risk tolerance but the same patience for time-1 consumption. In this case, prices and returns of risky assets co-vary in equilibrium as beliefs change and the price effect gives us new conditions for increasing the market risk premium and reducing the risk-free rate in equilibrium when reasonable levels of consumption growth and volatility are assumed.

This chapter is structured as follows. In Section 4.2, we set up the economy and describe the aggregation problem when agents have heterogeneous preferences and beliefs.

We show how the different risk tolerances and heterogeneous beliefs can be aggregated through a consensus belief. In Section 4.3, as a benchmark of our analysis, we include the traditional CAPM under the homogeneous belief. In Section 4.4, by introducing different risk preferences and disagreement in beliefs among agents, we examine the joint impact of heterogeneity on the equity premium and risk-free rate in market equilibrium both analytically and numerically. In particular, we explore the conditions of the disagreement in beliefs to achieve a high equity premium and a low risk-free rate. The analysis with two agents is then extended to a continuum of agents. In Section 4.5, we extend the analysis to include a consumption market at time zero. The chapter concludes in Section 4.6.

4.2 Heterogeneous Beliefs and Boundedly Rational Equilibrium

In this section, we first set up the stylised economy with heterogeneous beliefs and then characterise the market equilibrium.

4.2.1 The Economy

We consider an identical two-day economy to that of Chapter 3, but with two risky assets, a riskless asset and two investors. The risky assets are indexed by $j = 1, 2$, the riskless asset by f and the agents are indexed by $i = 1, 2$. The rate of return for asset $j (j = 1, 2)$ is denoted by $\tilde{r}_j$ and the return of the riskless asset is denoted by r_f. The probability distribution of the returns of the risky assets are assumed to be jointly normal. Agents have heterogeneous beliefs about the expected returns and variances/covariance of asset returns. For agent $i\,(i = 1, 2)$, let τ_i be his risk tolerance, and

$$\boldsymbol{\mu}_i = \begin{pmatrix} \mu_{i,1} \\ \mu_{i,2} \end{pmatrix} \qquad \text{and} \qquad V_i = \begin{pmatrix} \sigma_{i,1}^2 & \rho_i \sigma_{i,1} \sigma_{i,2} \\ \rho_i \sigma_{i,1} \sigma_{i,2} & \sigma_{i,2}^2 \end{pmatrix}$$

be his beliefs about the means and covariance matrix, respectively, where

$$\mu_{i,j} = 1 + \mathbb{E}_i(\tilde{r}_j), \qquad \sigma_{i,j}^2 = Var_i(\tilde{r}_j), \qquad \rho_i = Correl_i(\tilde{r}_1, \tilde{r}_2)$$

for $i, j = 1, 2$. We use $\mathcal{B}_i := (\mu_{i,1}, \mu_{i,2}, \sigma_{i,1}, \sigma_{i,2}, \rho_i)$ to denote the belief of agent i.

4.2.2 Optimal Portfolio Problem

The terminal wealth[1] of agent i is given by

$$\tilde{W}_i = W_{i,0}(1 + r_f) + \boldsymbol{\pi}_i^T(\tilde{\mathbf{r}} - r_f\mathbf{1}),$$

where $W_{i,0}$ is the initial wealth of agent i, $\boldsymbol{\pi}_i = (\pi_{i,1}, \pi_{i,2})^T$ is the vector of the dollar amount invested in each risky asset and $\mathbf{1} = (1, 1)^T$. We assume that agent i maximises expected utility $\mathbb{E}_i[U_i(\tilde{W}_i)] = \mathbb{E}_i[-\tau_i \exp\{-\tilde{W}_i/\tau_i\}]$, given by

$$(1 + r_f)(W_{i,0} - \mathbf{1}^T\boldsymbol{\pi}_i) + \boldsymbol{\pi}_i^T\boldsymbol{\mu}_i - \frac{1}{2\tau_i}\boldsymbol{\pi}_i^T V_i\boldsymbol{\pi}_i,$$

where τ_i is the risk-tolerance. The optimal portfolio is given by

$$\boldsymbol{\pi}_i^* = \tau_i V_i^{-1}(\boldsymbol{\mu}_i - R_f\mathbf{1}), \tag{4.1}$$

where $R_f = 1 + r_f$.

4.2.3 Consensus Belief and Boundedly Rational Equilibrium

The consensus belief and market equilibrium can be characterised by using Corollary 3.4.

Without loss of generality, we assume $W_0 = W_{1,0} + W_{2,0} = 1$, then the market clearing condition is given by[2]

$$\boldsymbol{\pi}_1^* + \boldsymbol{\pi}_2^* = \boldsymbol{\pi}_m,$$

where $\boldsymbol{\pi}_m$ denotes the market portfolio weights of risky assets (proportion of market initial wealth invested in each risky asset)[3]. We also assume that the riskless asset is in net zero supply, which implies that $\mathbf{1}^T\boldsymbol{\pi}_m = 1$. Next we apply Proposition 3.3 to characterise market equilibrium. Let $\tau_a := \frac{\tau_1+\tau_2}{2}$, the consensus belief $\mathcal{B}_a$ is given by

$$V_a^{-1} = \frac{1}{2\tau_a}\big[\tau_1\ V_1^{-1} + \tau_2\ V_2^{-1}\big], \qquad \boldsymbol{\mu}_a = \mathbb{E}_a(\mathbf{1} + \tilde{\mathbf{r}}) = \frac{1}{2\tau_a}\big[\tau_1(V_aV_1^{-1})\boldsymbol{\mu}_1 + \tau_2(V_aV_2^{-1})\boldsymbol{\mu}_2\big]; \tag{4.2}$$

and the market portfolio

$$\boldsymbol{\pi}_m = \tau_a V_a^{-1}(\mathbb{E}_a(\tilde{\mathbf{r}}) - r_f\mathbf{1}). \tag{4.3}$$

[1]Note that within the two-period model, the terminal wealth and consumption of agents are the same.
[2]We assume that there is one unit of market initial wealth.
[3]The market clearing condition simply says that the market capitalisation of each risky asset must equal the total wealth invested in that asset. Of course $\boldsymbol{\pi}_m$ is endogenous in our model; if one wishes to place an exogenous quantity on the right-hand side of the market clearing condition, we can rewrite it as $P_0^{-1}(\boldsymbol{\pi}_1^* + \boldsymbol{\pi}_2^*) = \mathbf{z}_m$ where $\mathbf{z}_m$ is the total supply of risky asset in number of shares and P_0 is a diagonal matrix of equilibrium prices for risky assets. From this condition, one can compute the equilibrium asset prices.

Furthermore, the CAPM holds under the consensus belief with $\boldsymbol{\beta} = V_a\boldsymbol{\pi}_m/\boldsymbol{\pi}_m^T V_a \boldsymbol{\pi}_m$ and the risk-free rate is given by

$$r_f = \frac{\mathbf{1}^T V_a^{-1}\mathbb{E}_a(\tilde{\mathbf{r}}) - \frac{1}{\tau_a}}{\mathbf{1}^T V_a^{-1}\mathbf{1}}. \tag{4.4}$$

Since the market equilibrium is obtained based on the fact that both agents make their optimal portfolio decision under their subjective beliefs, rather than the objective belief, we call such equilibrium *Boundedly Rational Equilibrium (BRE)*.

The impact of the heterogeneity on the market equilibrium, CAPM relation, market risk premium, and risk-free rate can be complicated in general. By focusing on the case of two assets and two agents in the following discussion, we are able to examine explicitly the impact of heterogeneity on the market equilibrium. To compare this with the traditional CAPM, we first consider the homogeneous belief as the benchmark case in the next section.

4.3 A Benchmark Case under a Homogeneous Belief

To examine the impact of heterogeneity on the market equilibrium and compared with the market equilibrium under a homogeneous belief, in this section we consider a benchmark case under the standard rational expectation in which both agents may have different risk tolerance, but they have the same beliefs about returns[4], denoted by $\mathcal{B}_o = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$, that is $\mathcal{B}_i = \mathcal{B}_o$ for $i = 1, 2$. For this benchmark case, we have

$$V_a = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} := V_o, \qquad \boldsymbol{\mu}_a = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} := \boldsymbol{\mu}_o.$$

Consequently, the market portfolio is simply given by

$$\hat{\boldsymbol{\pi}}_m := \frac{1}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \begin{pmatrix} \tau_a(\mu_1 - \mu_2) + \sigma_2(\sigma_2 - \rho\sigma_1) \\ \tau_a(\mu_2 - \mu_1) + \sigma_1(\sigma_1 - \rho\sigma_2) \end{pmatrix}, \tag{4.5}$$

[4] The benchmark beliefs $\mathcal{B}_o$ can be treated as either an objective belief or a benchmark homogeneous belief.

the market risk-premium, risk-free return, market variance and asset betas are given, respectively, by

$$\begin{aligned}
\hat{\mathbb{E}}(\tilde{r}_m - r_f) &:= \frac{(\mu_1-\mu_2)^2\tau_a^2 + (1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_a(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)},\\
\hat{R}_f &:= \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\left(\frac{\mu_1}{\sigma_1^2}+\frac{\mu_2}{\sigma_2^2}-\frac{\rho(\mu_1+\mu_2)}{\sigma_1\sigma_2}-\frac{1}{\tau_a}(1-\rho^2)\right),\\
\hat{\sigma}^2(\tilde{r}_m) &:= \frac{(\mu_1-\mu_2)^2\tau_a^2 + (1-\rho^2)\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)},\\
\hat{\beta}_1 &:= \frac{\tau_a(\rho\sigma_1\sigma_2-\sigma_1^2)(\mu_2-\mu_1)+(1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_a^2(\mu_2-\mu_1)^2+(1-\rho^2)\sigma_1^2\sigma_2^2},\\
\hat{\beta}_2 &:= \frac{\tau_a(\sigma_2^2-\rho\sigma_1\sigma_2)(\mu_2-\mu_1)+(1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_a^2(\mu_2-\mu_1)^2+(1-\rho^2)\sigma_1^2\sigma_2^2}.
\end{aligned} \tag{4.6}$$

It is easy to see that $\tau_a = \hat{\sigma}^2(\tilde{r}_m)/\hat{\mathbb{E}}(\tilde{r}_m - r_f)$, hence the market risk-tolerance measures the market variance per unit of market risk premium. Equations (4.5) and (4.6) show that the market risk premium and the risk-free rate are quite complex expressions of the benchmark belief $\mathcal{B}_o$. Next we use a numerical example to illustrate the limitation of the benchmark case in generating a high risk premium and low risk-free rate.

Example 4.1 *Let the two risky assets in the economy have expected returns* $(\mu_1, \mu_2) = (1.06, 1.09)$*, standard deviations* $(\sigma_1, \sigma_2) = (0.08, 0.3)$*, and correlation coefficient* $\rho = 0.8$*. Both agents hold the benchmark belief, that is,* $\mathcal{B}_i = \mathcal{B}_o = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$*.*

By choosing a reasonable level of risk-tolerance, say $\tau_i = 0.5\,(i = 1, 2)$ (and hence $\tau_a = 0.5$), we have from $\tau_a = \hat{\sigma}^2(\tilde{r}_m)/\hat{\mathbb{E}}(\tilde{r}_m - r_f)$ that the market in equilibrium requires 2% expected excess return (above the risk-free rate) for 10% standard deviation. Consequently, we have from equations (4.5) and (4.6) that the market portfolio is given by $\boldsymbol{\pi}_m = (0.962,\ 0.038)^T$ and

$$\hat{r}_f = 4.62\%, \qquad \hat{\mathbb{E}}(\tilde{r}_m - r_f) = 1.49\%, \qquad \text{and} \qquad \hat{\sigma}_m = 8.63\%.$$

Note that the risk-free rate is rather high, the risk-premium and market volatility are rather low. Intuitively, because asset 2 has a much larger volatility relative to asset 1, but there is not enough compensation in terms of expected return, every investor knows about this (homogeneous beliefs), therefore the market portfolio has a small holding in asset 2, resulting in low market volatility and market risk premium. If both agents become more risk averse so that the risk tolerance τ_i reduces, say to $\tau_a = 0.1$, then

$$\hat{r}_f = 0\%, \qquad \hat{\mathbb{E}}(\tilde{r}_m - r_f) = 6\%, \qquad \text{and} \qquad \hat{\sigma}_m = 8\%.$$

The market portfolio becomes $\boldsymbol{\pi}_m = (1,\ 0)^T$ (asset 2 is redundant). In this case, reducing the risk tolerance can certainly increase the market risk premium and reduce the risk-free rate; however, this requires a higher risk aversion coefficient (which is 10 in this example). To examine the impact of heterogeneity on the market risk premium and risk-free rate, we introduce disagreement into investors' beliefs in the following discussion. If the disagreement "cancel out", then the consensus belief $\mathcal{B}_a$ would conform to the average belief defined by the homogeneous benchmark belief $\mathcal{B}_o$, and heterogeneity would not matter in determining the endogenous variables when the market is in equilibrium. However, we will show that this is generally not the case and heterogeneity in beliefs can have a significant impact on the market equilibrium. In particular, we show that certain correlations among the disagreement in beliefs can generate a high market risk premium and a low risk-free rate without having to decrease the risk tolerance level.

4.4 The Impact of Heterogeneity

To better understand the impact of heterogeneity, we assume that agents agree about the expected return and the variance of the first risky asset, asset 1, but disagree about the expected return, standard deviation of the second asset, asset 2, and the correlation coefficient of the returns of the risky assets. To see whether disagreement in investors' beliefs indeed "cancel out" and have no effect on market equilibrium, disagreements are characterised by mean-preserving spreads about the benchmark belief. We assume the beliefs about the expected return and the standard deviation of the first asset for agents are given by the benchmark beliefs: $(\sigma_{i,1}, \mu_{i,1}) = (\sigma_1, \mu_1)$ for $i = 1, 2$, while the risk tolerance and the beliefs of the agents in the expected return and standard deviation of the second asset and the return correlation of the assets are mean-preserving spreads of the benchmark belief $\mathcal{B}_o$ and risk tolerance τ_o. More precisely, we assume that the risk-tolerances of the agents are given, respectively, by

$$\tau_1 = \tau_o(1 - \Delta), \qquad \tau_2 = \tau_o(1 + \Delta), \qquad \Delta \in (-1, 1); \tag{4.7}$$

the beliefs about the standard deviation of asset 2 are given by

$$\sigma_{1,2} = \sigma_2(1 - \delta), \qquad \sigma_{2,2} = \sigma_2(1 + \delta), \qquad \delta \in (-1, 1); \tag{4.8}$$

the beliefs about the correlation between asset returns are given by

$$\rho_1 = \rho(1-\varepsilon), \qquad \rho_2 = \rho(1+\varepsilon), \qquad \varepsilon \in (-1,1); \tag{4.9}$$

and the beliefs about expected returns of asset 2 are given by

$$\mu_{1,2} = \mu_2(1-\alpha), \qquad \mu_{2,2} = \mu_2(1+\alpha), \qquad \alpha \in (-1,1). \tag{4.10}$$

The mean-preserving spreads in disagreements imply that, *on average*, risk-tolerance and belief in this heterogeneous economy is exactly the same as the benchmark homogeneous economy. However, the consensus belief is not necessarily the same as the benchmark belief. As a result, the market portfolio, market risk-premium, risk-free rate and the market volatility may also differ from the homogeneous benchmark economy. For this setup, the different aspects of heterogeneity are characterised by Δ, δ, ε and α. To examine the joint impact of risk tolerance, optimism/pessimism, and confidence/doubt on the market, we consider four different combinations of these parameters in the following.

4.4.1 Case 1: Risk Tolerance and Optimism/Pessimism

We first consider the case where the agents have different risk-tolerance and heterogeneous beliefs regarding the expected future return of asset 2, that is

$$\delta = 0, \qquad \varepsilon = 0, \qquad \Delta \in (-1,1), \qquad \alpha \in (-1,1). \tag{4.11}$$

This means that agent 1 is less (more) risk tolerant than agent 2 when $\Delta > (<)0$ and agent 1 is more pessimistic (optimistic) than agent 2 when $\alpha > (<)0$. In particular, when $\Delta\alpha > 0(< 0)$, the risk tolerance and optimism of the agents are positively (negatively) correlated, meaning that the more risk-tolerant agent is optimistic, while the less risk-tolerant agent is pessimistic about future return of asset 2. In this case, we obtain the following result.

Corollary 4.2 *Under* (4.11)*, the consensus belief is given by*

$$\tau_a = \tau_o, \qquad V_a = V_o, \qquad \boldsymbol{\mu}_a = (\mu_1, \;\; \mu_2(1+\alpha\Delta))^T. \tag{4.12}$$

Consequently,

(i) the change in market portfolio is given by

$$\boldsymbol{\pi}_m - \hat{\boldsymbol{\pi}}_m = \left(-\frac{\alpha\Delta \;\; \tau_o\mu_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \;\; \frac{\alpha\Delta \;\; \tau_o\mu_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \right)^T; \tag{4.13}$$

(ii) the change in risk-premium is given by

$$(\mathbb{E}(\tilde{r}_m) - r_f) - (\hat{\mathbb{E}}(\tilde{r}_m) - \hat{r}_f) = \alpha\Delta\mu_2 \frac{\sigma_1(\rho\sigma_2 - \sigma_1) + \tau_o(\mu_2 - \mu_1)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.14}$$

(iii) the change in risk-free rate is given by

$$\hat{r}_f - r_f = \alpha\Delta\sigma_1\mu_2 \frac{\rho\sigma_2 - \sigma_1}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.15}$$

(iv) the change in market volatility is given by

$$\sigma_m^2 - \hat{\sigma}_m^2 = \alpha\Delta\tau_o^2\mu_2 \frac{(\mu_2 - \mu_1) + (\mu_2(1+\alpha\Delta) - \mu_1)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.16}$$

(v) asset betas are given by

$$\begin{aligned} \beta_1 &= \frac{\tau_o(\rho\sigma_1\sigma_2 - \sigma_1^2)(\mu_2(1+\alpha\Delta) - \mu_1) + (1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_o^2(\mu_2(1+\alpha\Delta) - \mu_1)^2 + (1-\rho^2)\sigma_1^2\sigma_2^2}, \\ \beta_2 &= \frac{\tau_o(\sigma_2^2 - \rho\sigma_1\sigma_2)(\mu_2(1+\alpha\Delta) - \mu_1) + (1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_o^2(\mu_2(1+\alpha\Delta) - \mu_1)^2 + (1-\rho^2)\sigma_1^2\sigma_2^2}. \end{aligned} \tag{4.17}$$

Corollary 4.2 characterises explicitly the impact of the heterogeneity on the market. It is easy to see that if both agents have either the same risk preference (so that $\Delta = 0$) or the same benchmark belief about the expected return of both assets (so that $\alpha = 0$), then $\alpha\Delta = 0$ and the results for the heterogeneous beliefs are reduced to those for the benchmark homogeneous case. Consequently, heterogeneity among the agents is cancelled out. Otherwise, the impact of heterogeneity in the case of (4.11) depends on the sign of $\alpha\Delta$ and the return correlation ρ. Corollary 4.2 leads to the following three implications.

Firstly, when risk-tolerance and the optimism of agents about future returns are positively (negatively) correlated[5], that is $\alpha\Delta > (<)0$, it follows from (4.12) that the aggregate market is optimistic (pessimistic) about the expected return of the second asset. Consequently, the aggregate market, indicated by the market portfolio in (4.13), invests more (less) into asset 2 and less (more) into asset 1 and the market volatility measured by σ_m is high (low) following (4.16). This observation that the market becomes pessimistic when risk tolerance and pessimism are positively correlated is also found in Jouini and Napp (2006).

[5] In the sense that the more risk-tolerant agent is optimistic while the less risk-tolerant agent is pessimistic.

Secondly, compared with the benchmark belief case, we have from (4.14) that the market with disagreement among the agents increases the market risk premium when either

$$\alpha\Delta > 0 \qquad \text{and} \qquad \mu_2 - \mu_1 > \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o \tag{4.18}$$

or

$$\alpha\Delta < 0 \qquad \text{and} \qquad \mu_2 - \mu_1 < \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o. \tag{4.19}$$

Similarly, from (4.15), the risk-free rate under disagreement in beliefs is reduced when either

$$\alpha\Delta > 0 \qquad \text{and} \qquad \rho > \sigma_1/\sigma_2, \tag{4.20}$$

or

$$\alpha\Delta < 0 \qquad \text{and} \qquad \rho < \sigma_1/\sigma_2. \tag{4.21}$$

This observation implies that disagreement in beliefs can increase the market premium and reduce the risk-free rate either (i) when the risk tolerance and optimism of the agent are positively correlated, the returns of the two assets are highly positively correlated (so that $\rho > \sigma_1/\sigma_2$), and the disagreement in asset expected returns is large ($\mu_2 - \mu_1 > \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o$); or (ii) when the risk tolerance and pessimism of the agent are positively correlated, (that is, the risk tolerance and optimism are negatively correlated) the returns of the assets are less (even negatively) correlated (so that $\rho < \sigma_1/\sigma_2$), and the disagreement in asset expected returns is small ($\mu_2 - \mu_1 < \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o$). Within the framework of heterogeneous beliefs, Abel (2002) and Jouini and Napp (2006) argue that a positive correlation between risk tolerance and pessimism is sufficient to generate a high equity premium and a low risk-free rate. However, our analysis shows that correlation between risk tolerance and optimism/pessism may not be sufficient, depending on the disagreement dispersion and return correlation. In particular, we show in Section 4.3 that, in certain situations, disagreement about return correlation can generates a significantly high market equity premium and a low risk-free rate.

Thirdly, equation (4.17) implies that the betas under the consensus belief are different to the standard betas under the benchmark belief. It is clear that, when the disagreement in the expected return disappears, the betas become the standard betas under the benchmark belief. However, under disagreement in beliefs, the betas β_j in equation (4.17) depends on the product $\alpha\Delta$. The impact of α and Δ on $\boldsymbol{\beta}$, is complicated, however we derive the following conditions; the beta of asset 1 is reduced if

$$\rho\sigma_2 > \sigma_1 \;\text{ and }\; \left[(\mu_2 - \mu_1) + \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_o\sigma_1(\rho\sigma_2 - \sigma_1)}\right]^2 > \frac{(1-\rho^2)\sigma_1^2\sigma_2^2(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}{\tau_o^2(\rho\sigma_2 - \sigma_1)^2} \tag{4.22}$$

or

$$\rho\sigma_2 < \sigma_1 \text{ and } \left[(\mu_2 - \mu_1) + \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_o\sigma_1(\rho\sigma_2 - \sigma_1)}\right]^2 < \frac{(1-\rho^2)\sigma_1^2\sigma_2^2(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}{\tau_o^2(\rho\sigma_2 - \sigma_1)^2}. \quad (4.23)$$

Similarly, beta of asset 2 is reduced if

$$\rho\sigma_1 > \sigma_2 \text{ and } \left[(\mu_2 - \mu_1) + \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_o\sigma_2(\sigma_2 - \rho\sigma_1)}\right]^2 < \frac{(1-\rho^2)\sigma_1^2\sigma_2^2(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}{\tau_o^2(\sigma_2 - \rho\sigma_1)^2}. \quad (4.24)$$

or

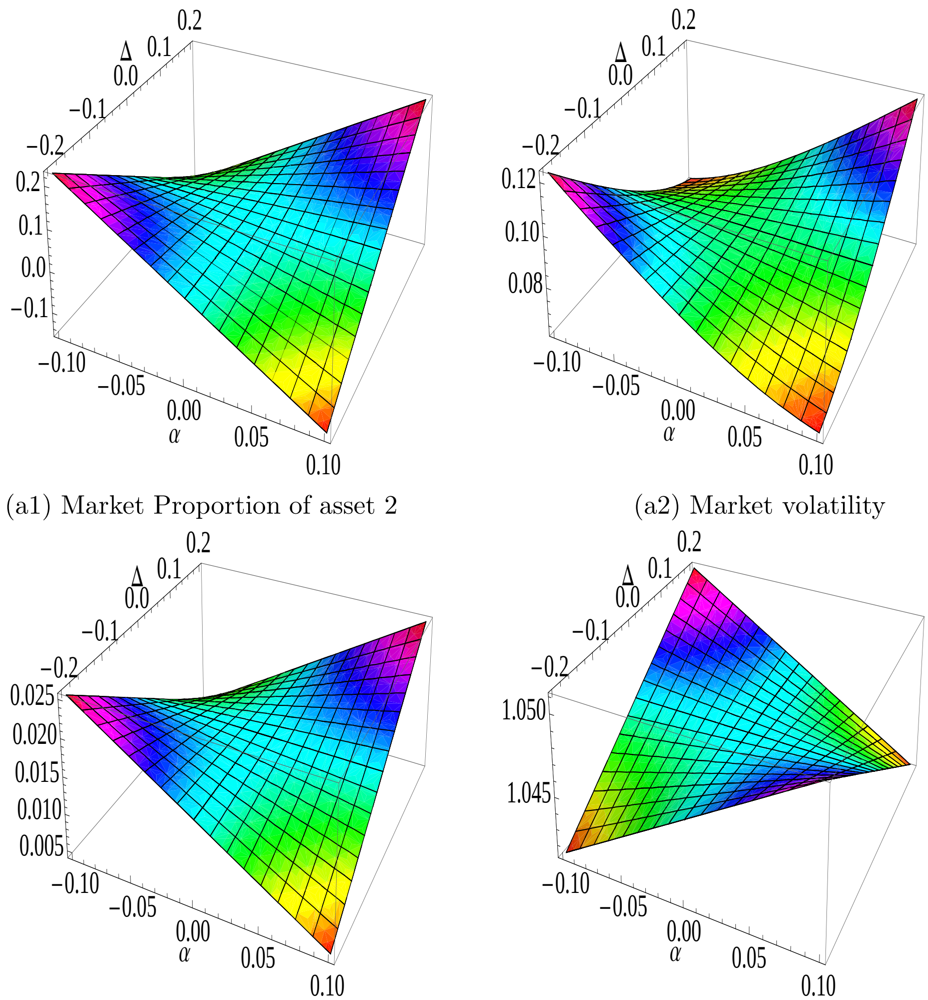
$$\rho\sigma_1 < \sigma_2 \text{ and } \left[(\mu_2 - \mu_1) + \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_o\sigma_2(\sigma_2 - \rho\sigma_1)}\right]^2 > \frac{(1-\rho^2)\sigma_1^2\sigma_2^2(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}{\tau_o^2(\sigma_2 - \rho\sigma_1)^2}. \quad (4.25)$$

To assess the impact, we now conduct a numerical analysis. Based on the numerical values provided in Example 4.1, we show graphically in Fig. 4.1 the impact of heterogeneity in terms of α and Δ on the change of market portfolio (in terms of the second risky asset in the market portfolio) in Fig. 4.1(a), the market volatility in Fig. 4.1(b), the expected market return in Fig. 4.1(c), and the risk-free rate in market equilibrium in Fig. 4.1(d). For the chosen values, we have $\rho > \sigma_1/\sigma_2$. The plots are symmetrical reflecting the fact that the effect of the heterogeneity depends on the product $\alpha\Delta$ rather than α and Δ individually. We see that, when the product $\alpha\Delta$ increases, the market portfolio consists more of asset 2, which leads to higher market return and volatility; at the same time the risk-free rate reduces and the risk-premium increases. In addition, the Sharpe ratio of the market portfolio increases, suggesting that heterogeneity of $\alpha\Delta$ improves the mean-variance efficiency of the aggregate market.

Cases	$(\Delta, \delta, \varepsilon, \alpha)$	$\pi_{m,2}$	$\sigma(\tilde{r}_m)$	$\mathbb{E}(\tilde{r}_m - r_f)$	r_f	$\frac{\mathbb{E}(\tilde{r}_m)-r_f}{\sigma_m}$
Benchmark	$(0,0,0,0)$	0.038	8.63%	1.49%	4.62%	0.1727
Case 1	$(0.2,0,0,0.1)$	0.2258	12.3%	2.54%	4.14%	0.2061
Case 2	$(0,-0.2,0,0.1)$	0.7511	24.15%	3.88%	4.37%	0.1606
Case 3	$(0,0,0.2,0.1)$	0.5415	19.31%	5.69%	1.94%	0.2947
Case 4	$(-0.2,0.2,0,0)$	0.1124	10.00%	1.77%	4.57%	0.1770

Table 4.1: Effects of heterogeneity on the market proportion of asset 2 ($\pi_{m,2}$), market volatility ($\sigma(\tilde{r}_m)$), market risk-premium ($\mathbb{E}(\tilde{r}_m - r_f)$), the risk-free rate ($r_f$), and the Sharpe ratio ($\mathbb{E}(\tilde{r}_m - r_f)/\sigma_m$ for the four cases, compared with the benchmark homogeneous case. Numerical values for the benchmark belief and risk tolerance are given in Example 4.1.

To quantify the impact on the market, based on the numerical values provided in Example 4.1, we choose $(\Delta, \delta, \varepsilon, \alpha) = (0.2, 0, 0, 0.1)$. Compared with the benchmark homogeneous belief, the results for Case 1 in Table 4.1 show that heterogeneity in the



(a1) Market Proportion of asset 2

(a2) Market volatility

(a3) Risk premium

(a4) Risk-free rate

Figure 4.1: Effect of heterogeneity on risk-tolerance Δ and in beliefs of expected return α on the market proportion of asset 2 (a1), market volatility (a2), market risk-premium (a3) and the risk-free rate (a4).

risk tolerance and the expected return helps to increase the market risk premium and reduce the risk-free rate when $\alpha\Delta > 0$. However, the overall effect is not significant for the chosen parameters. The risk premium increases moderately by 1% and the risk-free rate is merely reduced by less than half of a percent. This is mainly due to the market becoming over-optimistic with respect to asset 2's future return, which offsets the increase in aggregate volatility.

4.4.2 Case 2: Optimism/Pessimism and Confidence/Doubt

In the second case, we focus on the impact of optimism/pessimism (measured by α) and confidence/doubt (measured by δ) for asset 2 on the market in equilibrium by letting

$\Delta = 0,\ \varepsilon = 0$. Measured by beliefs about standard deviation, agent 1 becomes confident (doubtful) when $\delta > 0$. In this case, we obtain the following result.

Corollary 4.3 *For the second case when $\Delta = 0,\ \varepsilon = 0$ and $\delta,\ \alpha \in (-1, 1)$, the consensus belief $\mathcal{B}_a = (\mu_{a,1}, \mu_{a,2}, \sigma_{a,1}, \sigma_{a,2}, \rho_a)$ is given by $\tau_a = \tau_o$,*

$$\mu_{a,1} = \mu_1 - \alpha\delta\mu_2 \frac{\rho\sigma_1}{\sigma_2(1+\delta^2-\rho^2)}, \qquad \mu_{a,2} = \mu_2\big(1 - \frac{\alpha\delta(2-\rho^2)}{1-\rho^2+\delta^2}\big) \tag{4.26}$$

and

$$\sigma_{a,1}^2 = \sigma_1^2\big[1 - \frac{\delta^2\rho^2}{1+\delta^2-\rho^2}\big], \quad \sigma_{a,2}^2 = \sigma_2^2 \frac{(1-\delta^2)^2(1-\rho^2)}{1+\delta^2-\rho^2},$$
$$\rho_a = \rho\big[1 - \frac{\rho^2\delta^2}{1+\delta^2-\rho^2}\big]\frac{\sigma_1\sigma_2}{\sigma_{a,1}\sigma_{a,2}}. \tag{4.27}$$

Corollary 4.3 gives the explicit impact of disagreement about the expected return and standard deviation for the second asset among the agents. One special case is particularly interesting; when there is no disagreement about the standard deviation of the second asset (that is $\delta = 0$). In this case, we see from (4.26) and (4.27) that there is no difference between the heterogeneous case with disagreement about expected returns on the second asset and the benchmark belief case; so the effect from the disagreement about the expected return of the asset 2 is cancelled out and has no impact on the market. In general, based on (4.26) and (4.27), we see that the disagreement about the expected returns on asset 2 have an impact on the market expected return, but not the standard deviations and correlation. However, disagreement about the standard deviation on the return of asset 2 affect the expected returns, standard deviation, and correlation of assets when the asset returns are correlated. This effect vanishes when $\rho = 0$. Corollary 4.3 reflects a joint impact of the optimism/pessimism and confidence/doubt on the market. From equations (4.27), one can see that the aggregate market becomes over-confident when agents have disagreement regarding the variance of asset 2's return so that, for $0 < \delta < 1$, we have $\sigma_{a,1} < \sigma_1,\ \ \sigma_{a,2} < \sigma_2$ and $\rho_a\sigma_{a,1}\sigma_{a,2} < \rho\sigma_1\sigma_2$. From (4.26), when $\alpha\delta < 0$, that is when the optimistic (pessimistic) agent is confident (doubtful) about the future returns on asset 2, the market perceives a higher expected return for assets.

To examine the impact on the market, we let $\delta = -0.2$ and $\alpha = 0.1$. This means that the second (first) agent is optimistic (pessimistic) and confident (doubtful) about future returns of the second asset, so that $\alpha\delta < 0$. The numerical results for case 2 in Table 4.1 show a dramatic increase in the market's holding of asset 2. Therefore the market gains in risk premium but also becomes much more volatile. This is due to the fact that the increase in expected return is much higher for asset 2 than for asset 1 and the value of $(\rho\sigma_1/\sigma_2)$ is small relative to $(2-\rho^2)$, see (4.26). The risk-free rate reduces only slightly. Intuitively, although the market consists more of the riskier asset, the market also becomes over-confident and over-optimistic, which drives up the risk-free rate. This observation is consistent with the survey result in Giordani and Söderlind (2006) that doubt is not an adequate explanation of the high equity premium and the amount of pessimism provides only a small improvement. The Sharpe ratio drops compared to the benchmark case, suggesting that the gain in the risk premium cannot compensate for the higher volatility.

4.4.3 Case 3: Optimism/Pessimism and Biased Correlations

In the third case, we examine the joint impact of heterogeneity in the expected returns on asset 2 and the correlation coefficient by letting $\Delta = 0, \delta = 0$ and considering the effect of (ε, α). When $\epsilon > (<)0$, agent 1 believes that the return correlation is lower (higher), while agent 2 believes that the return correlation is higher (lower). In this case, we obtain the following result.

Corollary 4.4 *For the case that $\Delta = 0, \delta = 0$ and $\varepsilon, \alpha \in (-1, 1)$, the consensus belief $\mathcal{B}_a = (\mu_{a,1}, \mu_{a,2}, \sigma_{a,1}, \sigma_{a,2}, \rho_a)$ is given by $\tau_a = \tau_o$,*

$$\mu_{a,1} = \mu_1 - \alpha\varepsilon\frac{\rho\sigma_1}{(1-\rho^2)\sigma^2}\mu_2, \qquad \mu_{a,2} = \mu_2\big[1 + \alpha\varepsilon\frac{\rho^2}{1-\rho^2}\big], \tag{4.28}$$

$$\sigma_{a,1}^2 = \sigma_1^2\big[1 - \frac{\varepsilon^2\rho^2}{1-\rho^2}\big], \quad \sigma_{a,2}^2 = \sigma_2^2\big[1 - \frac{\varepsilon^2\rho^2}{1-\rho^2}\big], \quad \frac{\rho_a\sigma_{a,1}\sigma_{a,2}}{\rho\sigma_1\sigma_2} = 1 + \frac{\varepsilon^2\rho^2}{1-\rho^2}. \tag{4.29}$$

The proof of Corollary 4.4 is omitted since it is analogous to the proof of Corollary 4.3; it shows the impact of the optimism/pessimism and disagreement in correlation on the market. Disagreement about the expected returns of asset 2 affect the market expected returns of assets, but not the variances and covariance. However, disagreement about the return correlation affect both the first and second moments of the market returns of assets as well as the return correlation. It is easy to see that, for $0 < \varepsilon < 1$, we have $\sigma_{a,1} < \sigma_1, \sigma_{a,2} < \sigma_2$ and $\rho_a\sigma_{a,1}\sigma_{a,2} > \rho\sigma_1\sigma_2$. This indicates that in aggregate the market becomes more confident about future returns of assets but perceives a higher return covariance compared to the benchmark belief case. For $\alpha\varepsilon > 0$, when the optimistic agent also believes in a higher correlation between asset returns, we see from equation (4.28) that the market perceives a higher (lower) expected return for asset 2 (asset 1) when $\rho > 0$ and vice versa when $\rho < 0$. Intuitively, when $\rho > 0$ and $\alpha\epsilon > 0$, the market invests more into asset 2 because of the higher perceived expected return. As a result, the aggregate market expected return and volatility increase. However, in contrast to the previous cases, because (4.29) indicates that $\rho_a > \rho$, there is a markedly less diversification effect and consequently one should expect a significant reduction in the risk-free rate.

To examine the impact of heterogeneity on expected returns and correlation, we choose $\epsilon = 0.2$ and $\alpha = 0.1$ so that $\alpha\varepsilon > 0$. The results are given for Case 3 in Table 4.1, which shows the most desirable result, with a high market risk premium and low risk-free rate. The risk-free rate in this case is reduced significantly by nearly 3%, while the risk premium increased significantly by more than 4%. Most noticeably, the Sharpe ratio in this case becomes 0.2497, the highest amongst all cases including the homogeneous belief benchmark by far, implying that the aggregate market becomes the most mean-variance efficient when $\alpha\varepsilon > 0$.

4.4.4 Case 4: Risk-tolerance and Confidence/Doubt

In the fourth case, we examine the joint impact of heterogeneity in risk-tolerance (measured by Δ) and confidence/doubt (measured by δ) by letting $\alpha = 0$ and $\varepsilon = 0$. In this case, we obtain the following result.

Corollary 4.5 *For the case that $\alpha = 0, \varepsilon = 0$ and $\Delta, \delta \in (-1, 1)$, the consensus belief $\mathcal{B}_a = (\mu_{a,1}, \mu_{a,2}, \sigma_{a,1}, \sigma_{a,2}, \rho_a)$ is given by $\tau_a = \tau_o, \boldsymbol{\mu}_a = (\mu_1, \mu_2)^T$ and*

$$\sigma_{a,1}^2 = \sigma_1^2 \left[\frac{(1+\delta^2 - 2\Delta\delta)(1-\rho^2)}{(1+\delta^2-2\Delta\delta) - (1-\Delta\delta)^2\rho^2} \right],$$
$$\sigma_{a,2}^2 = \sigma_2^2 \left[\frac{(1-\delta^2)^2(1-\rho^2)}{(1+\delta^2-2\Delta\delta) - (1-\Delta\delta)^2\rho^2} \right],$$
$$\rho_a = \rho \left[\frac{(1-\delta^2)^2(1-\Delta\delta)(1-\rho^2))}{(1+\delta^2-2\Delta\delta) - (1-\Delta\delta)^2\rho^2} \right] \frac{\sigma_1\sigma_2}{\sigma_{a,1}\sigma_{a,2}}.$$

Proof of Corollary 4.5 is omitted since it is analogous to the proof of Corollary 4.3. When there is no disagreement in the standard deviation (so that $\delta = 0$), the consensus belief is reduced to the benchmark belief. Jouni and Napp (2006) argue that a positive correlation between risk tolerance and doubt can contribute to a high equity premium and a low risk-free rate. In our example, if we choose $\Delta = -0.2$ and $\delta = 0.2$ so that $\delta\Delta < 0$, that is, the more risk-tolerant agent is more confident about the future return on asset 2, we report the numerical results in Tab 4.1 for Case 4. We can see that the market risk premium increases and the risk-free rate reduces but the magnitude of the changes is not very significant. On the one hand, this result is inconsistent with the finding of Jouni and Napp (2006), suggesting that results from a single risky asset case do not necessarily carry over to the case with two risky assets, and typically the impact of heterogeneity depends on the correlation structure of the asset returns. On the other hand, consistent with the survey result in Giordani and Söderlind (2006), this illustrates that doubt may not be an adequate explanation for the equity premium puzzle.

4.4.5 Disagreements in the "Safe" Stock

In the previous cases, agents are assumed to have disagreement over the distribution of the returns on a "riskier" stock, in the sense that the stock has a higher expected return and higher risk. We now show that, when agents have heterogeneous beliefs about a "safe" stock or a less risky stock, which has a lower expected return and a lower risk, the impact on the market can be different. The is illustrated by considering the following numerical example[6].

Example 4.6 *Let the risky assets in the economy have expected returns* $(\mu_1, \mu_2) = (1.09, 1.06)$ *and standard deviations* $(\sigma_1, \sigma_2) = (0.3, 0.08)$ *and correlation coefficient* $\rho = 0.8$. *Agents have heterogeneous beliefs about the return on asset 2, and different risk tolerance. Their heterogeneity is characterised by parameters* Δ, δ, ε *and* α, *as described earlier in this section.*

Cases	$(\Delta, \delta, \varepsilon, \alpha)$	$\pi_{m,1}$	$\sigma(\tilde{r}_m)$	$\mathbb{E}(\tilde{r}_m - r_f)$	r_f	$\frac{\mathbb{E}(\tilde{r}_m)-r_f}{\sigma_m}$
Benchmark Case	$(0,0,0,0)$	0.038	8.63%	1.49%	4.62%	0.1727
Case 1	$(-0.2,0,0,0.02)$	0.0744	9.29%	2.12%	4.11%	0.228
Case 2	$(0,0.2,0,0.02)$	−0.088	6.78%	2.04%	3.70%	0.3010
Case 3	$(0,0,-0.2,0.02)$	0.4506	17.24%	4.61%	2.74%	0.2676
Case 4	$(-0.2,0.2,0,0)$	0.067	9.14%	1.34%	4.86%	0.1467

Table 4.2: Effects of heterogeneity on the market proportion of asset 2 ($\pi_{m,2}$), market volatility ($\sigma(\tilde{r}_m)$), market risk-premium ($\mathbb{E}(\tilde{r}_m - r_f)$), the risk-free rate ($r_f$) and the Sharpe ratio $(\mathbb{E}(\tilde{r}_m) - r_f)/\sigma_m$ for the four cases, compared with the benchmark homogeneous case.

We redo the numerical analysis in Table 4.1 for the four cases and present the results in Table 4.3. We assume lower disagreement in the expected return, that is $\alpha = 0.02$ since the second asset has a small standard deviation of 8%. We find that when asset 2 has a lower standard deviation and expected return than asset 1, heterogeneity has a positive impact on the market risk premium and a negative effect on the risk-free rate when $\alpha\Delta < 0$, $\alpha\delta > 0$ and $\alpha\varepsilon < 0$. In Case 1 when $\alpha\Delta < 0$, the more risk tolerant agent is less optimistic about the future return of asset 2, according to Corollary 4.2,

[6] Basically, we swap the two risky assets and still consider the disagreement about the second asset.

the aggregate market becomes less optimistic about the future of asset 2 and invest more wealth into asset 1, the market expected return and volatility increases and asset betas decrease as a result. Although the disagreement in expected return and the change in the market portfolio are small, the increase in the risk premium and the decrease in risk-free rate are still significant. This is because in contrast to Case 1 in Table 4.1, the market is pessimistic rather than optimistic about expected equity returns overall and more willing to invest in the risk-free security. This is consistent with the result obtained in Jouini and Napp (2006), in addition, we show that the contribution to the increase in the risk premium and the reduction in the risk-free rate is significant in this scenario. In Case 2 when $\alpha\delta > 0$, the more optimistic agent is less confident about the future returns of asset 2. The results are not satisfactory since the proportion of market initial market invested in asset 1 is negative[7]. Case 3 provides the most desirable result similar to that in Table 4.1 Case 3 except here we require $\alpha\varepsilon < 0$, that is the more optimistic agent to perceive a lower asset return correlation than the less optimistic agent. This is because the market perceives a lower (higher) expected return for asset 2 (asset 1) when $\alpha\varepsilon < 0$, thus investing more into asset 1 which has a higher expected return and volatility, hence the market expected return and volatility both increase. The market also perceives a higher correlation ($\rho_a > \rho$) since $\varepsilon > 0$, therefore the risk-free rate is significantly reduced.

4.4.6 Extension to a Continuum of Investors

Similar to Chapter 3, we extend the previous model of two agents to a model with a continuum of investors. In this case, we are able to characterise investors' heterogeneity in risk tolerance and beliefs through probability distributions and obtain similar results to those of the two agent economy.

Consider a continuum of investors indexed by $e \in [0, 1]$. The economy is defined by a measurable function $(\tau_e, \sigma_{1,e}, \sigma_{2,e}, \rho_e, \mu_{1,e}, \mu_{2,e}) : [0, 1] \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times [-1, 1] \times \mathbb{R} \times \mathbb{R}$, where $(\tau_e, \sigma_{e,1}, \sigma_{e,2}, \rho_e, \mu_{e,1}, \mu_{e,2})$ is the risk tolerance, belief about standard deviations, return correlation and belief about expected returns for investor e. The consensus belief $\mathcal{B}_a$ in this economy is the limits of equation (4.2) as the number of investors approaches infinity, thus we can rewrite the consensus belief as

$$\tau_a = \int_0^1 \tau_e \ de, \qquad V_a^{-1} = \frac{1}{\tau_a} \int_0^1 \tau_e \ V_e^{-1} \ de, \qquad \boldsymbol{\mu}_a = \frac{V_a}{\tau_a} \int_0^1 \tau_e \ V_e^{-1} \boldsymbol{\mu}_e \ de,$$

[7]From equation (4.26), market becomes less optimistic about the future return of both assets, but more so for asset 1 since $\rho\sigma_1/\sigma_2 > 2 - \rho^2$ in Example 4.6.

where

$$\boldsymbol{\mu}_e = \begin{pmatrix} \mu_{e,1} \\ \mu_{e,2} \end{pmatrix} \qquad \text{and} \qquad V_e = \begin{pmatrix} \sigma_{e,1}^2 & \rho_e\sigma_{e,1}\sigma_{e,2} \\ \rho_e\sigma_{e,1}\sigma_{e,2} & \sigma_{e,2}^2 \end{pmatrix}. \tag{4.30}$$

We assume that investors' risk tolerance τ_e and beliefs about future asset returns $\mathcal{B}_e = (V_e, \boldsymbol{\mu}_e)$ are i.i.d random variables for each investor e. Thus we can write the consensus belief $\mathcal{B}_a$ as $V_a = \tau_a \mathbb{E}[\tilde{\tau}\tilde{V}^{-1}]^{-1}$, $\boldsymbol{\mu}_a = \tau_a^{-1} V_a \mathbb{E}[\tilde{\tau}\tilde{V}^{-1}\tilde{\boldsymbol{\mu}}]$. We will see that in some cases, it is possible to write explicitly the consensus belief in terms of the first two moments of the random variables, while in others we may require Monte-Carlo simulations.

In the spirit of the case for two agents, we make the following assumptions about the distributions of heterogeneous beliefs of the continuum agent $e \in [0, 1]$. There are two risky assets and a risk-free asset in the economy. We assume that agents agree on the expected and standard deviation of future returns for the first risky asset with $\mu_{e,1} = \mu_1$ and $\sigma_{e,1} = \sigma_1$, but disagree on that for the second risky asset. For agent e and the second risky asset, let the expected return, the standard deviation, the return correlation, and the risk tolerance be given by, respectively,

$$\mu_{e,2} = \mu_2(1 + \tilde{\alpha}_e), \qquad \sigma_{e,2} = \sigma_2(1 + \tilde{\delta}_e), \qquad \rho_e = \rho(1 + \tilde{\epsilon}_e), \qquad \tau_e = \tau_o(1 + \tilde{\Delta}_e), \tag{4.31}$$

where $\tilde{\alpha}_e \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2(\tilde{\alpha}))$, $\tilde{\delta}_e \overset{i.i.d}{\sim} \mathcal{TN}(0, \sigma^2(\tilde{\delta}); (\underline{\delta}, \bar{\delta}))$, $\tilde{\epsilon}_e \overset{i.i.d}{\sim} \mathcal{TN}(0, \sigma^2(\tilde{\epsilon}); (\underline{\epsilon}, \bar{\epsilon}))$, and $\tilde{\Delta}_e \overset{i.i.d}{\sim} \mathcal{TN}(0, \sigma^2(\tilde{\Delta}); (\underline{\Delta}, \bar{\Delta}))$. We also denote correlations between random variables $\tilde{\Delta}_e$, $\tilde{\delta}_e$, $\tilde{\epsilon}_e$ and $\tilde{\alpha}_e$ by $\rho(\tilde{\Delta}, \tilde{\alpha})$, $\rho(\tilde{\delta}, \tilde{\alpha})$, $\rho(\tilde{\epsilon}, \tilde{\alpha})$ and $\rho(\tilde{\Delta}, \tilde{\delta})$, the correlations are independent of e.

In the case where investors' beliefs about the covariance matrix are homogeneous, that is,

$$\sigma^2(\tilde{\delta}) = 0 \qquad \text{and} \qquad \sigma^2(\tilde{\epsilon}) = 0, \tag{4.32}$$

we can extend Corollary 4.2 as follows.

Corollary 4.7 *Under* (4.32), *the consensus belief is given by*

$$\tau_a = \tau_o, \qquad V_a = V_o, \qquad \boldsymbol{\mu}_a = (\mu_1,\ \mu_2(1 + Cov(\tilde{\alpha}, \tilde{\Delta}))^T, \tag{4.33}$$

where $Cov(\tilde{\alpha}, \tilde{\Delta}) = \rho(\tilde{\alpha}, \tilde{\Delta})\sigma(\tilde{\Delta})\sigma(\tilde{\alpha})$. Consequently, comparing with the benchmark homogeneous belief case,

(i) the change in market portfolio is given by

$$\boldsymbol{\pi}_m - \hat{\boldsymbol{\pi}}_m = \left(-\frac{Cov(\tilde{\alpha}, \tilde{\Delta})\ \tau_o\mu_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \ \frac{Cov(\tilde{\alpha}, \tilde{\Delta})\ \tau_o\mu_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \right)^T; \tag{4.34}$$

(ii) the change in the risk-premium is given by

$$(\mathbb{E}(\tilde{r}_m) - r_f) - (\hat{\mathbb{E}}(\tilde{r}_m) - \hat{r}_f) = Cov(\tilde{\alpha}, \tilde{\Delta})\mu_2 \frac{\sigma_1(\rho\sigma_2 - \sigma_1) + \tau_o(\mu_2 - \mu_1)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.35}$$

(iii) the change in the risk-free rate is given by

$$\hat{r}_f - r_f = Cov(\tilde{\alpha}, \tilde{\Delta})\sigma_1\mu_2 \frac{\rho\sigma_2 - \sigma_1}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.36}$$

(iv) the change in market volatility is given by

$$\sigma_m^2 - \hat{\sigma}_m^2 = Cov(\tilde{\alpha}, \tilde{\Delta})\tau_o^2\mu_2 \frac{(\mu_2 - \mu_1) + (\mu_2(1 + Cov(\tilde{\alpha}, \tilde{\Delta})) - \mu_1)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.37}$$

(v) asset betas are given by

$$\begin{aligned} \beta_1 &= \frac{\tau_o(\rho\sigma_1\sigma_2 - \sigma_1^2)(\mu_2(1 + Cov(\tilde{\alpha}, \tilde{\Delta})) - \mu_1) + (1 - \rho^2)\sigma_1^2\sigma_2^2}{\tau_o^2(\mu_2(1 + Cov(\tilde{\alpha}, \tilde{\Delta})) - \mu_1)^2 + (1 - \rho^2)\sigma_1^2\sigma_2^2}, \\ \beta_2 &= \frac{\tau_o(\sigma_2^2 - \rho\sigma_1\sigma_2)(\mu_2(1 + Cov(\tilde{\alpha}, \tilde{\Delta})) - \mu_1) + (1 - \rho^2)\sigma_1^2\sigma_2^2}{\tau_o^2(\mu_2(1 + Cov(\tilde{\alpha}, \tilde{\Delta})) - \mu_1)^2 + (1 - \rho^2)\sigma_1^2\sigma_2^2}. \end{aligned} \tag{4.38}$$

Corollary 4.7 shows that we can derive expression for the equilibrium market portfolio, market risk premium, risk-free rate and market volatility analogous to Corollary 4.2, simply replacing $\alpha\Delta$ with $Cov(\tilde{\Delta}, \tilde{\alpha})$. Hence, it can be seen that the results in the two-agent economy extend to the infinite-agent economy when we characterise investors' beliefs by i.i.d random variables. In the other case, when $\sigma(\tilde{\delta})$ or $\sigma(\tilde{\epsilon})$ is positive, it appears difficult to derive analytically tractable expressions for the endogenous variable in market equilibrium. Corresponding to the four cases in Table 4.1, we approximate the continuum of agents by Monte Carlo simulations with $100,000$ investors and summarises the results in Table 4.3[8].

We can see that the results in Table 4.3 are fairly similar to those of Table 4.1. The increase in the market risk premium and the reduction in the risk-free rate are most significant when beliefs in expected future asset returns are positively correlated with beliefs about the return correlation, see Tab. 4.3 case 3. In this case, the Sharpe Ratio is also the highest. Based on this observation, we could argue that the model with two

[8]In calculation, we take $(\mu_1, \mu_2) = (1.06, 1.09)$, $(\sigma_1, \sigma_2) = (0.08, 0.3)$ and correlation coefficient $\rho = 0.8$. Also we assume that $\tilde{\delta}_e \overset{i.i.d}{\sim} \mathcal{TN}(0, \sigma^2(\tilde{\delta}); (-1, 1))$ and $\tilde{\epsilon}_e \overset{i.i.d}{\sim} \mathcal{TN}(0, \sigma^2(\tilde{\epsilon}); (-0.25, 0.25))$. In addition, to compare with the results in Table 4.1 for two agents, we assume the beliefs in expected future asset returns are positively correlated with both risk tolerance and beliefs in return correlation, with $\rho(\tilde{\Delta}, \tilde{\alpha}) = 0.9$ and $\rho(\tilde{\epsilon}, \tilde{\alpha}) = 0.9$ and negatively correlated with beliefs in the volatility of asset returns with $\rho(\tilde{\delta}, \tilde{\alpha}) = -0.9$. Furthermore, beliefs about the volatility are negatively correlated to risk tolerance with $\rho(\tilde{\Delta}, \tilde{\delta}) = -0.9$.

Cases	$(\sigma_\Delta, \sigma_\delta, \sigma_\varepsilon, \sigma_\alpha)$	$\pi_{m,2}$	$\sigma(\tilde{r}_m)$	$\mathbb{E}(\tilde{r}_m - r_f)$	r_f	$\frac{\mathbb{E}(\tilde{r}_m)-r_f}{\sigma_m}$
Benchmark	$(0, 0, 0, 0)$	0.038	8.63%	1.49%	4.62%	0.1727
Case 1	$(0.2, 0, 0, 0.1)$	0.2108	11.99%	2.45%	4.18%	0.2044
Case 2	$(0, 0.2, 0, 0.1)$	0.8737	27.03%	4.17%	4.45%	0.1543
Case 3	$(0, 0, 0.2, 0.1)$	0.5297	19.04%	5.50%	2.09%	0.2889
Case 4	$(0.2, 0.2, 0, 0)$	0.1446	10.63%	1.85%	4.58%	0.1744

Table 4.3: Effects of heterogeneity on the market proportion of asset 2 ($\pi_{m,2}$), market volatility ($\sigma(\tilde{r}_m)$), market risk-premium ($\mathbb{E}(\tilde{r}_m - r_f)$), the risk-free rate ($r_f$) and the Sharpe ratio $(\mathbb{E}(\tilde{r}_m) - r_f)/\sigma_m$ for the four cases, compared with the benchmark homogeneous case.

agents, which is simple to analyse, can provide useful insights into the model with a continuum of agents.

4.5 Extension to a Market with Consumption

To examine the effect of allowing agents to consume at time zero and the way equilibrium prices and returns co-vary as the beliefs and risk tolerance of the agents change, agent i's consumption at both time 0 and time 1, denoted by $C_{i,0}$ and $\tilde{C}_{i,1}$ respectively, we take time-0 consumption good as the numéraire. The setup is similar to that in Chapter 2. There are two risky assets with payoffs $\tilde{x}_1$ and $\tilde{x}_2$ at time 1 which are jointly normally distributed. There is also a riskless asset that pays R_f at time 1. We denote agent i's endowment and demand (number of shares) in the risky assets by $\boldsymbol{\zeta}_i = (\zeta_{i,1} \;\; \zeta_{i,2})^T$ and $\mathbf{z}_i = (z_{i,1} \;\; z_{i,2})^T$ respectively. Agent i's endowment and demand in the riskless asset are $\zeta_{i,o}$ and $z_{i,o}$ respectively. Agent i's time-1 consumption is then given by $\tilde{C}_{i,1} = z_{i,o}R_f + \mathbf{z}_i^T\tilde{\mathbf{x}}$, where $\tilde{\mathbf{x}} = (\tilde{x}_1 \;\; \tilde{x}_2)^T$. We assume that agent i's objective is

$$\max_{\{C_{i,0}, \mathbf{z}_i, z_{i,o}\}} U_i(C_{i,0}) + \eta \mathbb{E}_i[U_i(\tilde{C}_{i,1})], \tag{4.39}$$

where $U_i(C) = -\tau_i \exp\{-C/\tau_i\}$ is agent i's utility function, τ_i measures agent i's risk tolerance and η measures his patience for time-1 consumption. Agent i's budget constraint is given by

$$\zeta_{i,o} + \boldsymbol{\zeta}_i^T \mathbf{p}_0 = z_{i,o} + \mathbf{z}_i^T \mathbf{p}_0 + C_{i,0}, \tag{4.40}$$

where $\mathbf{p}_0 = (p_{0,1} \;\; p_{0,2})^T$ is the equilibrium price vector of the risky assets. Since that $\tilde{C}_{i,1}$ is normally distributed, equation (4.39) is equivalent to

$$\max_{\{C_{i,0}, \tilde{\mathbf{z}}_i, z_{i,o}\}} U_i(C_{i,0}) + \eta U_i(Q_i(\mathbf{z}_i, z_{i,o})) \tag{4.41}$$

where $Q_i(\mathbf{z}_i, z_{i,o}) = \mathbf{z}_i^T\mathbf{y}_i + z_{i,o}R_f - \frac{1}{2\tau_i}\mathbf{z}_i^T\Omega_i\mathbf{z}_i$ measures agent i's time-1 certainty equivalent consumption and

$$\mathbf{y}_i = (y_{i,1}, y_{i,2})^T, \qquad \Omega_i = \begin{pmatrix} \sigma_{i,1}^2 & \rho_i\sigma_{i,1}\sigma_{i,2} \\ \rho_i\sigma_{i,1}\sigma_{i,2} & \sigma_{i,2}^2 \end{pmatrix}$$

with $y_{i,j} = \mathbb{E}_i[\tilde{x}_j]$, $\sigma_{i,j}^2 = Var_i(\tilde{x}_j)$, $j = 1, 2$ and $\rho_i = Correl_i(\tilde{x}_1, \tilde{x}_2)$. Therefore agent i's belief about future asset payoffs is given by $\mathcal{B}_i(y_{i,1}, y_{i,2}, \sigma_{i,1}, \sigma_{i,2}, \rho_i)$.

Lemma 4.8 *Agent i's optimal portfolio is given by*

$$\mathbf{z}_i^* = \tau_i \ \Omega_i^{-1}(\mathbf{y}_i - R_f\mathbf{p}_0), \qquad z_{i,o}^* = \zeta_{i,o} + (\boldsymbol{\zeta}_i - \mathbf{z}_i^*)^T\mathbf{p}_0 - C_{i,0}^*, \tag{4.42}$$

where the optimal time-0 consumption is given by

$$C_{i,0}^* = \frac{1}{1+R_f}\left[\mathbf{y}_i^T\mathbf{z}_i^* - \frac{1}{2\tau_i}(\mathbf{z}_i^*)^T\Omega_i\mathbf{z}_i^* - \tau_i\ln(\eta R_f) + R_f(\zeta_{i,o} + (\boldsymbol{\zeta}_i - \mathbf{z}_i^*)^T\mathbf{p}_0)\right]. \tag{4.43}$$

Now we define the market clearing conditions to determine the market equilibrium prices $\mathbf{p}_0$ and the risk-free payoff R_f endogenously.

Definition 4.9 *The market clears when the following holds,*

$$\mathbf{z}_1^* + \mathbf{z}_2^* = \mathbf{z}_m := \boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2, \tag{4.44}$$

$$C_{1,0}^* + C_{2,0}^* = C_0 := \zeta_{1,o} + \zeta_{2,o}. \tag{4.45}$$

The first condition in (4.44) means that in equilibrium, the aggregate of agents' optimal portfolios must be the market portfolio. The second condition (4.45) requires the aggregate consumption at time-0 to be equal to the total endowment of the riskless asset, which naturally follows from the first condition. Next we define a consensus belief and characterise market equilibrium using the following Proposition.

Proposition 4.10 *By the market clearing condition in Definition 4.9, define the consensus risk tolerance as*

$$\tau_a = \frac{\tau_1 + \tau_2}{2},$$

then the consensus belief $\mathcal{B}_a$ is given by

$$\Omega_a^{-1} = \frac{1}{2\tau_a}\left[\tau_1 \ \Omega_1^{-1} + \tau_2 \ \Omega_2^{-1}\right], \tag{4.46}$$

$$\mathbf{y}_a = \mathbb{E}_a(\tilde{\mathbf{x}}) = \frac{1}{2\tau_a}\left[\tau_1(\Omega_a\Omega_1^{-1})\mathbf{y}_1 + \tau_2(\Omega_a\Omega_2^{-1})\mathbf{y}_2\right]; \tag{4.47}$$

The equilibrium price vector is given by

$$\mathbf{p}_0 = \frac{1}{R_f}(\mathbf{y}_a - \frac{1}{2\tau_a}\Omega_a \mathbf{z}_m). \tag{4.48}$$

Agent i's optimal portfolio in the risky assets can be written as

$$\mathbf{z}_i^* = \tau_i \Omega_i^{-1}\Big[(\mathbf{y}_i - \mathbf{y}_a) + \frac{1}{2\tau_a}\Omega_a \mathbf{z}_m\Big]. \tag{4.49}$$

Furthermore, let the consensus patience parameter be

$$\eta_a = \eta \exp\left\{\frac{Q_a - \bar{Q}}{\tau_a}\right\}, \tag{4.50}$$

where $Q_a = (\mathbf{z}_m/2)^T \mathbf{y}_a - \frac{1}{2\tau_a}(\mathbf{z}_m/2)^T \Omega_a (\mathbf{z}_m/2)$ *is the certainty equivalent time-1 consumption of a consensus agent, and* $\bar{Q} = (Q_1^* + Q_2^*)/2$*. Then the equilibrium risk-free payoff is given by*

$$R_f = \frac{1}{\eta_a}\exp\left\{\frac{Q_a - C_0/2}{\tau_a}\right\} = \frac{1}{\eta}\exp\left\{\frac{\bar{Q} - C_0/2}{\tau_a}\right\}. \tag{4.51}$$

Proposition 4.10 shows that the consensus patience parameter η_a does not necessarily equal to the common patience parameter η. The risk-free payoff R_f is determined jointly by η and the average time-1 certainty equivalent consumption $\bar{Q}$ which is independent of equilibrium asset prices and the risk-free payoff due to the fact that $z_{1,o}^* + z_{2,o}^* = \zeta_{1,o} + \zeta_{2,o} - C_0 = 0$. Therefore, R_f is determined explicitly in our model. We can define a single consensus agent with the consensus belief $\mathcal{B}_a$, risk tolerance[9] $\tau_m = \tau_1 + \tau_2$ and patience η_a such that his optimal portfolio is $\mathbf{z}_m$ and optimal time-0 consumption is C_0.

4.5.1 Benchmark Homogeneous Belief Case

To examine the impact of heterogeneity in beliefs, we first consider a benchmark economy in which agents may have different levels of risk tolerance, but have the same beliefs about future asset payoffs denoted by $\mathcal{B}_o(y_1, y_2, \sigma_1, \sigma_2, \rho)$. We characterise the market equilibrium of the benchmark homogeneous economy in the following corollary.

Corollary 4.11 *Assume that both agents have homogeneous rational beliefs, that is* $\mathcal{B}_i = \mathcal{B}_o$ *for* $i = 1, 2$*. Then the consensus agent is defined by risk tolerance* $\tau_a = \frac{\tau_1+\tau_2}{2}$*, and the*

[9] We can rewrite $C_0 = 2Q_a - 2\tau_a \ln(\eta_a R_f) = Q_m - \tau_m \ln(\eta_a R_f)$ where $Q_m = \mathbf{z}_m^T \mathbf{y}_a - \frac{1}{2\tau_m}\mathbf{z}_m^T \Omega_a \mathbf{z}_m$ is the time-1 certainty equivalent consumption of the representative agent and $\eta_a = \eta \exp\{(Q_m - Q_1 - Q_2)/\tau_m\}$.

consensus belief is $\mathcal{B}_a = \mathcal{B}_o$. The equilibrium price vector is given by

$$\hat{\mathbf{p}}_0 = \frac{1}{R_f}(\mathbf{y} - \frac{1}{2\tau_a}\Omega \mathbf{z}_m); \tag{4.52}$$

where

$$\mathbf{y} = (y_1 \;\; y_2)^T \qquad \text{and} \qquad \Omega = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

The riskless payoff is given by

$$R_f = \frac{1}{\eta}\exp\left\{\frac{\hat{Q} - C_0/2}{\tau_a}\right\}, \tag{4.53}$$

where $\hat{Q} = (\mathbf{z}_m/2)^T\mathbf{y} - \frac{1}{2\tau_a}(\mathbf{z}_m/2)^T\Omega(\mathbf{z}_m/2)$.

From Corollary 4.11, we can write down the market risk premium, the equilibrium expected returns of the risky assets, respectively, as follows (we assume that the risky assets are in unit supply, that is $\mathbf{z}_m = (1 \;\; 1)^T$ and denote $\tilde{C}_1 = \tilde{C}_{1,1} + \tilde{C}_{1,2} = \tilde{x}_1 + \tilde{x}_2$ as the aggregate time-1 consumption),

$$\begin{aligned}
\hat{\mathbb{E}}(\tilde{r}_m - r_f) &:= \hat{\mathbb{E}}\left[\frac{\tilde{C}_1}{\hat{p}_{0,1} + \hat{p}_{0,2}}\right] - \hat{R}_f = \frac{\hat{R}_f(\sigma_1^2 + \sigma_2^2 + \rho\sigma_1\sigma_2)}{2\tau_a(y_1 + y_2) - (\sigma_1^2 + \sigma_2^2 + \rho\sigma_1\sigma_2)}, \\
\hat{\mathbb{E}}(\tilde{r}_1) &:= \hat{\mathbb{E}}\left(\frac{\tilde{x}_1}{\hat{p}_{0,1}} - 1\right) = \frac{2\tau_a\hat{R}_f \;\; y_1}{2\tau_a y_1 - (\sigma_1^2 + \rho\sigma_1\sigma_2)}, \\
\hat{\mathbb{E}}(\tilde{r}_2) &:= \hat{\mathbb{E}}\left(\frac{\tilde{x}_2}{\hat{p}_{0,2}} - 1\right) = \frac{2\tau_a\hat{R}_f \;\; y_2}{2\tau_a y_2 - (\sigma_2^2 + \rho\sigma_1\sigma_2)}.
\end{aligned}$$

Next we illustrate the market risk premium and equilibrium risk-free rate in this benchmark economy using a numerical example with parameter values for risk tolerance, belief $\mathcal{B}_o$ and aggregate time-0 consumption that results in reasonable consumption growth $\mathbb{E}(\tilde{C}_1/C_0)$ and volatility $\sigma(\tilde{C}_1/C_0)$.

Example 4.12 *Let the two risky assets in the economy have expected payoffs $(y_1, y_2) = (0.4, 1.6)$, standard deviations of payoffs $(\sigma_1, \sigma_2) = (0.0115, 0.0495)$, and correlation coefficient $\rho = 0.8$. Both agents hold the benchmark belief, that is, $\mathcal{B}_i = \mathcal{B}_o = (y_1, y_2, \sigma_1, \sigma_2, \rho)$. We assume agents also have the same risk tolerance, $\tau_i = \tau = 0.5$, for $i = 1, 2$. Furthermore, the patience parameter is given by $\eta = 0.99$ and aggregate time-0 consumption $C_0 = 1.96$. This implies that the expected consumption growth and volatility of consumption growth are given by $\mathbb{E}(\tilde{C}_1/C_0) = 0.02$ and $\sigma(\tilde{C}_1/C_0) = 0.03$.*

In Example 4.12, we obtain reasonable values for consumption growth and volatility, but the market risk premium and risk-free rate

$$\hat{\mathbb{E}}(\tilde{r}_m - r_f) = 0.36\% \qquad \text{and} \qquad \hat{r}_f = \hat{R}_f - 1 = 4.94\%$$

which are too low and too high, respectively, compared to historically observed levels. Moreover, the equilibrium expected returns for the two risky assets and the market portfolio weights are given by

$$\mathbb{E}(\tilde{r}_1) = 5.26\%, \ \ \mathbb{E}(\tilde{r}_2) = 5.33\% \ \text{ and } \ \hat{\boldsymbol{\pi}}_m = \hat{\mathbf{p}}_0/(\hat{\mathbf{p}}_0^T \mathbf{z}_m) = (0.2 \ \ 0.8)^T.$$

4.5.2 Impact of Risk-Tolerance and Optimism/Pessimism

Now, similar to subsection 4.4.1, we assume agents have homogeneous beliefs about the variance and correlation of asset payoffs and also the expected payoff of the first risky asset but differ in risk tolerance and their belief about the expected payoff of the second risky asset. We characterise heterogeneity in risk tolerance and beliefs by mean-preserving spreads as follows,

$$\tau_1 = (1+\Delta)\tau, \qquad \tau_2 = (1-\Delta)\tau, \qquad y_{1,2} = (1+\alpha)y_2, \qquad y_{2,2} = (1-\alpha)y_2. \tag{4.54}$$

In the belief setup of (4.54), agent 1 is more (less) risk tolerant when Δ is positive (negative) and agent 1 is optimistic (pessimistic) compared to agent 2, when α is positive (negative). Next we can compute the endogenous variables in market equilibrium, including the equilibrium asset prices and risk-free rate according to Proposition 4.10. In particular, we are interested in the effect of Δ and α on the market risk premium and risk-free rate. We define κ as the ratio of riskless payoff under heterogeneous risk tolerance beliefs and riskless payoff under the benchmark homogeneous economy, that is

$$\kappa := \frac{R_f}{\hat{R}_f} = \exp\left\{ \frac{\bar{Q} - \hat{Q}}{\tau_a} \right\} = \exp\left\{ \frac{1}{2}\left[\frac{\alpha^2(1-\Delta^2)\ y_2^2}{(1-\rho^2)\sigma_2^2} + \frac{\alpha\Delta\ y_2}{\tau} \right] \right\}.$$

Firstly, $\kappa = 1$ when $\alpha = 0$, secondly, when $\Delta = 0$,

$$\kappa = \exp\left\{ \frac{1}{2}\left[\frac{\alpha^2\ y_2^2}{(1-\rho^2)\sigma_2^2} \right] \right\} > 1,$$

therefore in both of these cases, heterogeneity cannot reduce the risk-free rate. Clearly $\kappa < 1$ and $R_f < \hat{R}_f$ if and only if

$$\frac{\alpha\Delta \;\; y_2}{\tau} \;\; < \;\; -\frac{\alpha^2(1-\Delta^2) \;\; y_2^2}{(1-\rho^2)\sigma_2^2} \;\; \Rightarrow \;\; -\frac{\Delta}{\alpha(1-\Delta^2)} > \frac{\tau y_2}{(1-\rho^2)\sigma_2^2} \tag{4.55}$$

which holds only if $\alpha\Delta < 0$, meaning that the more risk-tolerant agent is pessimistic. Condition (4.55) is more likely to be satisfied if $|\Delta|$ is close to 1 and α close to zero, but the effect diminishes when $\alpha = 0$.

Consider the parameter values given in Example 4.12. In fact, we choose $|\Delta|$ to be close to 1 in order to reduce the risk-free rate from its benchmark value, in Fig. 4.2, $\Delta = -0.995$ which implies that agent 2 is much more risk tolerant than agent 1. As illustrated in Fig. 4.2, the risk-free rate has a convex relationship with α, while the market risk premium monotonically increases with α. If one chooses $\alpha = 0.055$, that is $y_{1,2} = 1.688$ and $y_{2,2} = 1.512$, we obtain the risk-free rate and market risk premium,

$$\mathbb{E}(\tilde{r}_m - r_f) = 4.98\%, \;\; r_f = 0.45\%,$$

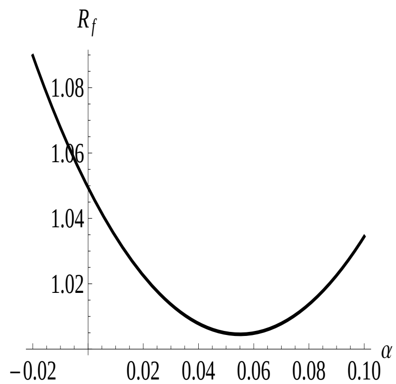
which are much more comparable to their historical observed levels. Also, the equilibrium expected returns for the risky assets are given by

$$\mathbb{E}(\tilde{r}_1) = 0.75\% \;\text{ and }\; \mathbb{E}(\tilde{r}_2) = 6.68\%$$

and the market portfolio weights are given by $\boldsymbol{\pi}_m = \mathbf{p}_0/(\mathbf{p}_0^T.\mathbf{z}_m) = (0.2093 \;\; 0.7907)^T$. Therefore we observe from this example that a negative correlation between risk tolerance and optimism can help to resolve the puzzles, and the effect is significantly strong when $|\Delta|$ is close to 1. The reason for this is that when $\alpha\Delta < 0$, the consensus agent according to Corollary 4.2 becomes less optimistic about the future payoff of asset 2, which has a negative impact on its price, thus the equilibrium expected returns on asset 2 increase significantly, while asset 1's expected return decreases purely due to the reduction in the risk-free rate. Furthermore, the market risk premium is enhanced since the market portfolio weights are rather insensitive to heterogeneity in risk tolerance and beliefs about expected payoffs and the risk-free rate is reduced by more than 4%.

4.6 Conclusion

Heterogeneity, reflecting diversity and disagreement among agents, is very common in financial markets and has significant impact on these markets. In this chapter, we have



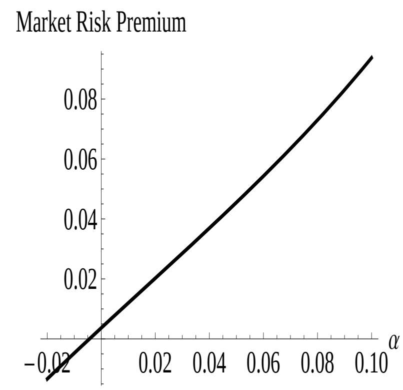


Figure 4.2: Impact of heterogeneous risk tolerance and optimism/pessimism on the equilibrium risk-free rate and market risk premium with $\Delta = -0.995$.

examined the impact of heterogeneity among investor in a market with two risky assets on the market equilibrium, in particular, the market risk premium, the risk-free rate, market volatility and the Sharpe Ratio of the market portfolio. Within a mean-variance setting, investors' heterogeneity is represented by their different risk tolerance, beliefs about the expected and variance of future asset returns and the return correlation between two risky assets. Furthermore, we assume that they agree on the expected and variance of future returns for one asset, but not the other. We show that when investors' beliefs are symmetric to the homogeneous benchmark belief characterised by mean-preserving spreads of the homogenous belief, beliefs about the market equilibrium, represented by the consensus belief, are in general different to the benchmark belief. We show that the impact of heterogeneity on a market with two risky assets is very different from that with one risky asset. For the market with only one risky asset, a negative correlation between risk tolerance and beliefs about expected returns makes the aggregate market less optimistic about the risky asset, thus increases the market risk premium while reduces the risk-free rate. However, for a market with two risky assets, we found an increase in the risk premium and a reduction in the risk-free rate when the investor who is more optimistic about future asset returns is more risk tolerant or more confident about future asset returns. More interestingly, we found that this effect becomes even stronger when the more optimistic agent also perceives a high correlation between asset returns. Therefore, we can conclude that the impact of heterogeneity on the market equilibrium is very different when there is more than one risky asset in the market. In general, depending on whether the heterogeneity is greater for the more risky asset or the less risky asset, its impact on the market can be different. We have also extended our model to a case with a continuum of investors by using i.i.d random variables to characterise heterogeneous risk tolerance and beliefs of investors. The analytical and numerical results obtained are very much in line with those in the two-agent case. Furthermore, we extend our model to include consumption, when agents have heterogeneous beliefs regarding time-1 payoff of one of the risky assets, and equilibrium asset prices and returns co-vary as agents' belief and tastes change. We study the effect of heterogeneous risk-tolerance and beliefs about the expected payoff, and we derive conditions such that the risk-free rate is reduced from its benchmark value under homogeneous beliefs. Numerically, it is shown that a negative correlation between risk tolerance and optimism can help to resolve the puzzles, which is similar to the results in Jouini and Napp (2006); although one requires a large dispersion in agents' risk-tolerance levels.

Chapter 5

Relative Consumption, Heterogeneous Beliefs and Risk Premium

5.1 Introduction

It is a common practice of modelling agents in a financial market that expected utility maximisers of their future consumption stream, agents' utility or satisfaction increases with their level of consumption. However, in reality, an investor may care more about his level of consumption relative to others (for example, the average level of consumption) than the absolute level. For example, consider a person A who is earning $\$1,000$ per week (p/w); given A's utility function $U(\cdot)$, his satisfaction is measured by $U(1,000)$. Now assume that A has a close friend B, who has a similar job to A. Consider two scenarios; (i) B is earning $\$500$ p/w and (ii) B is earning $\$5,000$ p/w. The question that follows is obvious, *Is A's level of satisfaction constant and given by U(1000) in both scenarios (i) and (ii)?* Many would tend to think that A should be more satisfied (a higher utility) in scenario (i) than in scenario (ii). The reason for may be; A's utility or satisfaction of $\$1,000$ is measured relative to B, and A would be more satisfied if B is earning a relatively lower amount and vice-versa. In fact, equilibrium asset pricing models with habit formation developed by Abel (1990), Constantinides (1990) and Sundaresan (1989) are motivated by a similar observation. An agent's habit can be internal (habit depends on one's own consumption pattern) or external (habit depends on aggregate consumption which is unaffect by any one agent's decisions). We construct a two-agent general equilibrium exchange economy where each agent maximises his intertemporal utility of consumption relative to the other agent. We assume agents have heterogeneous preferences and beliefs and agree to disagree, that is each agent knows the other agent's preference and belief in future aggregate endowment and is able to compute the optimal portfolio and consumption of the other agent. Within this setup, we use the martingale technique to solve for the agents' optimal consumption process. As expected, each agent's optimal consumption plan is a function of the consumption plan of the other agent. We call this the *benchmarking effect*. In a simple example, we show that when agent A has power utility and maximises the expected utility of terminal wealth relative to another agent B, A would go long (short) in B's portfolio when his relative risk-aversion coefficient is greater (less) than 1. When it is exactly 1, A has log-utility and the benchmarking effect disappears; that is, A no longer cares what B does. This is because log-utility is a numéraire invariant preference, as shown in Kadaras (2010). In fact, maximisation of expected utility of relative consumption is not entirely a new assumption, it is similar to the concept of habit formation (see Constantinides (1990), Sundaresan (1989) and Abel (1990, 1999)). Each agent in our model forms a habit of consumption based on the consumption process of the other agent, which is endogenously determined by the preference and belief of the other agent.

Our work is related to the extensive literature on asset pricing under heterogeneous beliefs, largely motivated by the equity risk premium and the risk-free rate puzzle discovered by Mehra and Prescott (1985) and Weil (1989). We show that when agents maximises the expected intertemporal utility of their relative consumptions, the impact of heterogeneous beliefs on equilibrium asset prices can be magnified, in particular when the sum of the agents' risk-aversion coefficients is close to 1. With reasonable assumptions on the aggregate endowment process and very small disagreement in beliefs, our model is able to generate a realistic level of equity premium and interest rate historically observed in the market. Consistent with Jouni and Napp (2006, 2007), our model shows that a positive correlation between risk aversion and an agent's optimism can increase the equity premium and reduce the interest rate from their benchmark values under homogeneous beliefs, when the sum of the agents' relative risk aversion coefficients is greater than 1. However, a negative correlation between risk aversion and optimism is required when the sum is less than 1. Our model is different from both internal and external habit models because in our case the "habit" of one agent, is the optimal consumption process of the other agent which is endogenously determined in equilibrium. Furthermore, we show that it is possible to construct a consensus consumer in our economy by first finding an equivalent optimisation problem for each agent under which they maximize utility of absolute consumption rather than relative consumption, both optimization problems are equivalent in the sense that the optimal consumption process is the same under both cases.

The long-run survival of agents with different beliefs is also examined within the framework of this chapter. We find that agents' long-run survival depends not only on the accuracy of their beliefs, but also on the sum of their risk aversions; when this sum is greater than 1, the agent with the most accurate beliefs survives, the other agent vanishes as his consumption share tends to zero. The closer the sum of risk aversions is to 1 the faster he vanishes; the time may be considerably short (less than a hundred year), even if the divergence in beliefs is small. When the sum of risk aversions is less than 1, the agent with the more accurate belief vanishes in the long run, at a speed proportional to one minus the sum of risk aversions. This is certainly a surprising result, which we explain in Section 5.6. In regards to the long-run price impact of agents' beliefs, similar to Kogan et al. (2006) though a different channel, in our model an agent's belief can have permanent impact on equilibrium prices even when his consumption share becomes sufficiently small.

This chapter is structured as follows. In Section 5.2, we present a simple example to motivate our study. In Section 5.3, we set up a pure exchange economy and solve the optimisation problem of agents with heterogeneous beliefs[1]. In Section 5.4 we derive explicitly the equilibrium perceived market price of risk and the risk-free rate in our economy. In Section 5.5, we construct a consensus consumer and demonstrate that the equilibrium results are the same as in Section 5.4. In Section 5.6, we study agents' survivability, assuming constant disagreement in beliefs. Section 5.7 presents numerical analysis of the model to address the equity premium and risk-free rate puzzles. Section 5.8 concludes.

5.2 An Example

We begin our analysis by considering a simple portfolio optimization problem in a continuous market. Assume there is one risky asset S and a riskless bond B in the market with the following dynamics,

$$dS(t) = S(t)(\mu \;\, dt + \sigma \;\, d\omega(t)) \qquad dB(t) = rB(t)dt, \tag{5.1}$$

where $\omega(t)$ is a Brownian motion, μ and σ are the instantaneous expected return and volatility of the stock, respectively, and r is the risk-free interest rate. An agent a wishes to maximise the utility of terminal wealth at time T, relative to the wealth level of another

[1] We follow Basak (2005) in setting up the economy, however the equilibrium result is very different due to the different setup in consumption.

agent b, that is, agent a's objective is to

$$\max_{\pi_a(t)} \ \mathbb{E}[u_a(Z(T))|\pi_b(t)], \tag{5.2}$$

where $Z(T) = W_a(T)/W_b(T)$ is the relative terminal wealth of agent a to agent b and $\pi_a(t)$ and $\pi_b(t)$ are the proportions of wealth at time t invested in the risky asset by agent a and b, respectively. One can write explicitly the dynamics of $Z(t)$ using Itô's lemma as follows:

$$\begin{aligned} dZ(t) = Z(t)[(\pi_a(t) - \pi_b(t))(\mu - r) + \sigma^2 \ \pi_b(t)(\pi_b(t) - \pi_a(t))]dt \\ + Z(t)[\sigma(\pi_a(t) - \pi_b(t))]d\omega(t). \end{aligned} \tag{5.3}$$

Assume the utility function of agent a is given by $u_a(x) = x^{1-\alpha}/(1-\alpha)$. From (5.2) and (5.3) we obtain the following solution,

$$\pi_a(t) = \frac{\mu - r}{\alpha \ \sigma^2} \ + \ \frac{\alpha - 1}{\alpha}\pi_b(t). \tag{5.4}$$

Based on agent a's optimal portfolio in (5.4), it is clear that agent a goes short the optimal portfolio of agent b if his CRRA coefficient α is less than one, and vice versa when $\alpha > 1$. The first term corresponds to the standard setup when agent a maximises his utility of terminal wealth, while the second term corresponds to the nature of relative consumption. Hence, we call the second term the *benchmarking effect*. The intuition of (5.4) is that when α is less than one, agent a is less risk averse to relative terminal wealth. Consequently, agent a invests more to maximize his utility (first part in (5.4)) and tries to bet agent b by shorting the portfolio of agent b. When $\alpha > 1$, agent a becomes more risk averse to relative terminal wealth in that he tries to copy to some extent agent b's portfolio to ensure that his terminal wealth is not too different from agent b's. As α approaches infinity, agent a becomes extremely risk averse by investing exactly the same portfolio as agent b and completely disregarding the optimal portfolio based on the dynamics of the stock and the bond. Interestingly, when agent a's CRRA coefficient is exactly equal to one, he becomes a log-utility maximiser and the benchmarking effect disappears. In this case, agent a is a log-utility maximiser and simply invests in the growth optimal portfolio (GOP), a well-known property of log-utility; see, for example, Platen and Heath (2006). As a log-utility maximiser, agent a ignores the information about agent b's portfolio since his own portfolio is growth optimal. This example demonstrates the change in investment behaviour when an agent cares about his relative wealth to a benchmark, in this case agent b's terminal wealth. Given the benchmark portfolio, the agent will attempt to out-grow it, but his activeness in doing so depends on his risk aversion to the relative wealth.

5.3 The Model

In this section, we will generalise the example presented in Section 5.2 to a dynamic general equilibrium model by allowing intermediate consumption of agents. We first introduce heterogeneous beliefs among two agents and then give an example of agents' possible learning. We also obtain a result on agents' relative consumption problem. In setting up the economy, we follow Basak (2005). The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ on which a one-dimensional Brownian motion $\omega(t)$ is defined. $\{\mathcal{F}_t^\omega\}$ is the information generated by the Brownian motion $\omega(t)$. $\mathcal{H}$ is a σ-field independent of $\{\mathcal{F}_t^\omega\}$ on which investor's priors are defined. The complete filtration $\{\mathcal{F}_t\}$ is given by $\mathcal{H} \vee \{\mathcal{F}_t^\omega\}$. We assume all the regularity conditions such that the process and expectations are well defined.

Consider a pure exchange economy with the cumulative aggregate endowment at time t given by $\int_0^t \varepsilon(s)ds$ and the rate of endowment follows

$$d\varepsilon(t) = \varepsilon(t)[\mu_\varepsilon(t)dt + \sigma_\varepsilon(t)d\omega(t)]. \tag{5.5}$$

Assume that there are two agents ($i = 1, 2$) who commonly observe the process $\varepsilon(t)$ but have incomplete information regarding its growth rate $\mu_\varepsilon(t)$. The volatility of aggregate endowment $\sigma_\varepsilon(t)$ is $\{\mathcal{F}_t^\varepsilon\}$ adapted and known to the agents; however, they need to make inferences on the growth rate $\mu_\varepsilon(t)$ based on the filtration $\{\mathcal{F}_t^\varepsilon\}$ generated by $\varepsilon(t)$. Investors have equivalent probability measures $\mathcal{P}^i$ ($i = 1, 2$) to $\mathcal{P}$, which may disagree on $\mathcal{H}$, so that agents have heterogeneous prior beliefs. Agent i's belief about the growth rate is given by $\mu_{i,\varepsilon}(t) = \mathbb{E}_i[\mu_\varepsilon(t)|\mathcal{F}_t^\varepsilon]$ where $\mathbb{E}_i[\cdot]$ denotes agent i's expectation ($i = 1, 2$). Following standard filtering theory (Lipster and Shiryaev (2001)), the probability space agent i lives in is given by $(\Omega, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathcal{P}^i)$, where $\mathcal{F}_t^i := \mathcal{F}_t^{\omega_i} = \mathcal{F}_t^\varepsilon$ in which the aggregate endowment process is given by

$$d\varepsilon(t) = \varepsilon(t)[\mu_{i,\varepsilon}(t)dt + \sigma_\varepsilon(t)d\omega_i(t)], \tag{5.6}$$

and $d\omega_i(t)$ is the innovation process in the standard filtering theory, that is

$$d\omega_i(t) = d\omega(t) + \frac{\mu_\varepsilon(t) - \mu_{i\varepsilon}(t)}{\sigma_\varepsilon(t)}dt = d\omega(t) + \theta_{i\varepsilon}(t)dt, \;\; \theta_{i\varepsilon}(t) = \frac{\mu_\varepsilon(t) - \mu_{i\varepsilon}(t)}{\sigma_\varepsilon(t)},$$

$$d\omega_2(t) = d\omega_1(t) + \bar{\mu}(t)dt, \qquad \bar{\mu}(t) = \frac{\mu_{1\varepsilon}(t) - \mu_{2\varepsilon}(t)}{\sigma_\varepsilon(t)} = \theta_{2\varepsilon}(t) - \theta_{1\varepsilon}(t).$$

Agent i's disagreement with the objective belief is measured by $\theta_{i\varepsilon}(t)$. Agent i is optimistic when $\theta_{i\varepsilon}(t)$ is negative and pessimistic when it is positive. Difference in beliefs is measured by $\bar{\mu}(t)$. Intuitively, agents may update their beliefs about the growth rate of the aggregate endowment process as new information becomes available. If agents are rational, then they will update their filtering beliefs according to Baye's rule. However, this is not necessary for the characterisation of market equilibrium in our model and agents can adopt any bounded rational learning schemes. For illustration, we include an example in the following.

5.3.1 Learning – Gaussian Filtering Example

Assume both μ_ε and σ_ε are constants in (5.5). Suppose both agents have normally distributed priors with mean $\mu_{i\varepsilon}(0)$ and variance $v_i(0)$ as to the growth rate of the aggregate endowment μ_ε at time $t = 0$. Then agent i's belief has the following dynamics

$$d\theta_{i\varepsilon}(t) = -\frac{v_i(t)}{\sigma_\varepsilon^2}\ d\omega_i(t),$$

where $v_i(t) = v_i(0)\sigma_\varepsilon^2/(v_i(0)t + \sigma_\varepsilon^2)$, implying

$$d\bar{\mu}(t) = -\frac{v_2(t)}{\sigma_\varepsilon^2}\bar{\mu}(t)dt + \frac{v_1(t) - v_2(t)}{\sigma_\varepsilon^2}d\omega_1(t).$$

If agents have the same prior variance, $v_1(0) = v_2(0) = v(0)$, then

$$d\bar{\mu}(t) = -\frac{v(t)}{\sigma_\varepsilon^2}\bar{\mu}(t)dt$$

and hence

$$\bar{\mu}(t) = \bar{\mu}(0)\exp\left\{\frac{1}{\sigma_\varepsilon^2}\int_0^t -v(s)ds\right\} = \bar{\mu}(0)\ \frac{\sigma_\varepsilon^2}{v(0)t + \sigma_\varepsilon^2}. \tag{5.7}$$

Equation (5.7) indicates that under Gaussian filtering, the initially optimistic (pessimistic) agent will remain optimistic (pessimistic), since the sign of $\bar{\mu}(t)$ depends only on its initial value. Agents' beliefs about the growth rate of aggregate endowment converge as t increases. The speed of convergence is negatively correlated with σ_ε and positively correlated with $v(0)$.

5.3.2 Securities Market

There are two securities in our economy; a riskless bond B and a risky security S and both are in net-zero supply. The security price dynamics satisfy

$$\begin{aligned} dB(t) &= B(t)r(t)dt, \\ dS(t) &= S(t)[\mu(t)dt + \sigma(t)d\omega(t)] \\ &= S(t)[\mu_i(t)dt + \sigma(t)d\omega_i(t)], \qquad i = 1, 2. \end{aligned} \tag{5.8}$$

We assume that $\sigma(t)$ is $\mathcal{F}_t^i$ adapted and $\mu(t)$ is $\mathcal{F}_t$ adapted. Agents observe and agree on the price of the risky security, but do not observe its drift, so they use their own inferences, $\mu_i(t)$. This implies the following "consistency" relationship:

$$\mu_1(t) - \mu_2(t) = \sigma(t)\bar{\mu}(t). \tag{5.9}$$

The market is dynamically complete in the sense that any contingent claim on the stock can be replicated. This implies that there exists a state price density (SPD) process ξ_i for each agent, with the dynamics given by

$$d\xi_i(t) = -\xi_i(t)[r(t)dt + \kappa_i(t)d\omega_i(t)], \qquad i = 1, 2 \tag{5.10}$$

with $\xi_i(0) = 1$. Here $\kappa_i(t) \equiv (\mu_i(t) - r(t))/\sigma(t)$ is the market price of risk process as perceived by agent i, and the consistency relationship implies that

$$\kappa_1(t) - \kappa_2(t) = \bar{\mu}(t). \tag{5.11}$$

5.3.3 Investors' Preferences and Optimization

Agent 1's cumulative endowment at time t is given by $\int_0^t \varepsilon_1(t)dt$ where the rate of endowment $\varepsilon_1(t) = x_1\varepsilon(t)$ is assumed to be a fixed proportion, x_1, of the aggregate endowment, $0 < x_1 < 1$. Agent 1's wealth process is given by

$$\begin{aligned} dW_1(t) = W_1(t)&[r(t) + (\varepsilon_1(t) - c_1(t)) + \pi_1(t)(\mu_1(t) - r(t))]dt \\ &+ W_1(t)[\pi_1(t)\sigma(t)]d\omega_1(t) \end{aligned}$$

with $W_1(T) \geq 0$, where $\pi_1(t)$ is the proportion of wealth invested in the risky security and $c_1(t)$ is the rate of consumption by agent 1 at time t. The initial wealth of agent 1 is $W_1(0) = 0$ (since the endowment is the only source of income). We assume that agent

1 maximises, in his own probability space $(\Omega, \mathcal{F}_t^1, \{\mathcal{F}_t^1\}, \mathcal{P}^1)$, the expected intertemporal utility of his relative consumption to agent 2 given the portfolio and the corresponding consumption process of agent 2. The usual assumptions of the utility function apply. Agent 1's optimisation problem can be described as

$$\begin{aligned} \max_{\{c_1(t),\pi_1(t)\}} \quad & \mathbb{E}_1\left[\int_0^T e^{-\beta t} u_1\left(\frac{c_1(t)}{c_2(t)}\right) dt | \{c_2(t), \pi_2(t)\}\right] \\ s.t. \quad & \mathbb{E}_1\left[\int_0^T \xi_1(t) c_1(t) dt\right] \leq \mathbb{E}_1\left[\int_0^T \xi_1(t)\varepsilon_1(t) dt\right], \end{aligned} \tag{5.12}$$

where $\beta > 0$ measures agent 1's impatience for future relative consumption. We use the martingale technique to solve the optimisation problem in (5.12) (see, for example, Cox and Huang (1989) and Karataz et al (1987), Cvitanic and Zapatero (2004) chapter 4). The first order condition

$$u_1'(c_1/c_2) = \eta_1 e^{\beta t} \xi_1 c_2$$

is both necessary and sufficient for the optimal solution of (5.12), where η_1 is the Lagrange multiplier corresponding to agent 1's budget constraint. Hence we can write agent 1's optimal consumption as

$$c_1 = I_1(\eta_1 e^{\beta t} \xi_1 c_2) c_2,$$

where I_1 is the inverse function of u_1' such that at optimum, the budget constraint holds with equality, that is

$$\mathbb{E}_1\left[\int_0^T \xi_1(t) I_1(\eta_1 e^{\beta t} \xi_1 c_2(t)) c_2(t) dt\right] = \mathbb{E}_1\left[\int_0^T \xi_1(t)\varepsilon_1(t) dt\right]. \tag{5.13}$$

Agent 2 faces a similar optimisation problem. In summary, we have the following lemma.

Lemma 5.1 *Assume that $u_i(x) = x^{1-\alpha_i}/(1-\alpha_i)$ where $\alpha_i > 0$ is agent i's relative risk-aversion coefficient of relative consumption, $i = 1, 2$. Then the optimal consumption processes of the two agents are given by*

$$c_1(t) = \eta_1^{-\frac{1}{\alpha_1}} e^{-\frac{\beta t}{\alpha_1}} \xi_1(t)^{\frac{-1}{\alpha_1}} c_2(t)^{\frac{\alpha_1 - 1}{\alpha_1}}, \qquad c_2(t) = \eta_2^{-\frac{1}{\alpha_2}} e^{-\frac{\beta t}{\alpha_2}} \xi_2(t)^{\frac{-1}{\alpha_2}} c_1(t)^{\frac{\alpha_2 - 1}{\alpha_2}}, \tag{5.14}$$

where η_1 and η_2 are the Lagrange multipliers satisfying the budget constraints of agent 1 and 2, respectively. The explicit expressions for agent 1 and 2's consumption processes are given by

$$c_1(t) = y_1 \; e^{-\beta t}(\xi_1^*(t))^{-1}, \qquad c_2(t) = y_2 \; e^{-\beta t}(\xi_2^*(t))^{-1}, \tag{5.15}$$

where

$$\xi_1^*(t) = \xi_1(t)^{\frac{\alpha_2}{\alpha_1+\alpha_2-1}} \xi_2(t)^{\frac{\alpha_1-1}{\alpha_1+\alpha_2-1}}, \qquad \xi_2^*(t) = \xi_2(t)^{\frac{\alpha_1}{\alpha_1+\alpha_2-1}} \xi_1(t)^{\frac{\alpha_2-1}{\alpha_1+\alpha_2-1}}$$

and y_1 and y_2 are positive constants that depend only on the Lagrange multipliers η_1 and η_2.

Lemma 5.1 shows that when agent i maximises the utility of relative consumption to the other agent, his optimal consumption process depends not only on his risk-aversion coefficient and SPD but also those of the other agent. In the case of homogeneous beliefs, the SPDs of the agents coincide, that is, $\xi_1 = \xi_2 = \xi$, and both agents' optimal consumption is independent of their risk aversion and becomes identical to that of a log-utility maximiser. The intuition comes from the example in Section 5.2. Suppose that agent b in the example also maximises the utility of his terminal wealth to agent a, we denote agent b's relative risk-aversion coefficient by α_b. Moreover, we assume that agent b shares the same belief as agent a that the drift of the stock price is μ. Therefore, agent b's optimal portfolio is given by

$$\pi_b(t) = \frac{\mu - r}{\alpha_b \ \sigma^2} \ + \ \frac{\alpha_b - 1}{\alpha_b} \pi_a(t),$$

substituting into (5.4) and solve for π_a we have

$$\pi_a(t) = \frac{\mu - r}{\sigma^2}$$

and by symmetry $\pi_b(t) = \pi_a(t)$. Hence agents a and b invest the same proportion into the risky asset and this proportion is identical to that of a log-utility maximiser. In other words, under homogeneous beliefs, agents a and b behave as if they both maximise the growth rate of their terminal wealth.

5.4 Equilibrium with Heterogeneous Beliefs

In this section, we determine market equilibrium under heterogeneous beliefs within the relative consumption framework in Section 5.3. Similar to Basak (2005), we define equilibrium as follows,

Definition 5.2 *An equilibrium is a price system $(r, \mu_1, \mu_2, \sigma)$ and consumption-portfolio processes (c_i, π_i) such that: (i) agents choose their optimal consumption-portfolio strategies given their perceived price processes; (ii) perceived security price processes are consistent across investors, that is,*

$$\mu_1(t) - \mu_2(t) = \sigma(t)\bar{\mu}(t);$$

and (iii) goods and security markets clear,

$$c_1(t) + c_2(t) = \varepsilon(t), \qquad \pi_1(t) + \pi_2(t) = 0, \qquad W_1(t) + W_2(t) = 0.$$

We denote $\lambda(t) = c_1(t)/\varepsilon(t)$ and $1-\lambda(t)$ as agent 1 and 2's share of aggregate consumption in the economy respectively, at time t. This definition allows us to determine equilibrium under the market clearing conditions. We can determine (r, κ_1, κ_2) by using the following fact.

$$dc_1(t) + dc_2(t) = d\varepsilon(t), \tag{5.16}$$

where

$$dc_1(t) = c_1(t)[\mu_{c_1}(t)dt + \sigma_{c_1}(t)d\omega(t)]$$

with

$$\begin{aligned}
\mu_{c_1}(t) &= r(t) - \beta + \frac{\alpha_2}{\alpha_1 + \alpha_2 - 1}\kappa_1(t)\theta_{1\varepsilon}(t) + \frac{\alpha_1 - 1}{\alpha_1 + \alpha_2 - 1}\kappa_2(t)\theta_{2\varepsilon}(t) \qquad (5.17)\\
&+ \frac{1}{2}\frac{\alpha_2(2\alpha_2 + \alpha_1 - 1)}{(\alpha_1 + \alpha_2 - 1)^2}(\kappa_1(t))^2 + \frac{1}{2}\frac{(\alpha_1 - 1)(2(\alpha_1 - 1) + \alpha_2)}{(\alpha_1 + \alpha_2 - 1)^2}(\kappa_2(t))^2\\
&+ \frac{\alpha_2(\alpha_1 - 1)}{(\alpha_1 + \alpha_2 - 1)^2}\kappa_1(t)\kappa_2(t),\\
\sigma_{c_1}(t) &= \frac{\alpha_2}{\alpha_1 + \alpha_2 - 1}\kappa_1(t) + \frac{\alpha_1 - 1}{\alpha_1 + \alpha_2 - 1}\kappa_2(t). \qquad (5.18)
\end{aligned}$$

Agent 2's consumption process follows a similar process to (5.17). By equating the drift and diffusion coefficients in (5.16) we obtain

$$\lambda(t)\mu_{c_1}(t) + (1 - \lambda(t))\mu_{c_2}(t) = \mu_\varepsilon, \tag{5.19}$$

$$\lambda(t)\sigma_{c_1}(t) + (1 - \lambda(t))\sigma_{c_2}(t) = \sigma_\varepsilon. \tag{5.20}$$

Based on equations (5.19) and (5.20), the next proposition characterises the equilibrium prices in the economy, which is the first main result of this chapter.

Proposition 5.3 *The equilibrium perceived market prices of risk as perceived by agents are given by*

$$\kappa_1(t) = \sigma_\varepsilon + \frac{\alpha_1 - \lambda(t)}{\alpha_1 + \alpha_2 - 1}\bar{\mu}(t), \qquad \kappa_2(t) = \sigma_\varepsilon - \frac{\alpha_2 - (1 - \lambda(t))}{\alpha_1 + \alpha_2 - 1}\bar{\mu}(t). \tag{5.21}$$

The equilibrium interest rate is given by

$$r(t) = \frac{\alpha_2 - (1-\lambda(t))}{\alpha_1 + \alpha_2 - 1} r_1(t) + \frac{\alpha_1 - \lambda(t)}{\alpha_1 + \alpha_2 - 1} r_2(t) + \frac{1}{2} \frac{\bar{\mu}(t)^2}{(\alpha_1 + \alpha_2 - 1)^2} \Big(\alpha_1 \alpha_2 - \lambda(t) \alpha_2 - (1-\lambda(t)) \alpha_1 \Big), \tag{5.22}$$

where $r_1(t) = \beta + \mu_\varepsilon - \sigma_\varepsilon(\theta_{1\varepsilon}(t) + \sigma_\varepsilon)$ *and* $r_2(t) = \beta + \mu_\varepsilon - \sigma_\varepsilon(\theta_{1\varepsilon}(t) + \bar{\mu}(t) + \sigma_\varepsilon)$ *denote the equilibrium interest rates that would prevail if agent* 1 *or* 2 *is the only agent in the economy. Furthermore, agent* 1*'s share of aggregate consumption is given by*

$$d\lambda(t) = \frac{\lambda(t)(1-\lambda(t))\bar{\mu}(t)}{\alpha_1 + \alpha_2 - 1} \left\{ \left[\frac{1}{2} \left(1 + \frac{1 - 2\lambda(t)}{\alpha_1 + \alpha_2 - 1} \right) \bar{\mu}(t) + \theta_{1\varepsilon}(t) \right] dt + d\omega(t) \right\}, \tag{5.23}$$

where $\lambda(0)$ *satisfies agent* 1*'s budget constraint in* (5.13)*, that is*

$$\mathbb{E}_1 \left[\int_0^T \xi_1(t)(x_1 - \lambda(t))\varepsilon(t) dt \right] = 0. \tag{5.24}$$

Proposition 5.3 shows that the perceived market price of risk (MPR) and the risk-free rate in equilibrium depend on the agents' differences in beliefs and the ratio of their risk aversions, which have a common denominator $\alpha_1 + \alpha_2 - 1$ that appears in several places in (5.21) and (5.22). Equation (5.21) shows that it is possible for both agents to perceive a high MPR than σ_ε, which is the MPR under homogeneous beliefs. This can occur when $\frac{\alpha_1 - \lambda(t)}{\alpha_1 + \alpha_2 - 1}\bar{\mu}(t) > 0$ and $\frac{\alpha_2 - (1-\lambda(t))}{\alpha_1 + \alpha_2 - 1}\bar{\mu}(t) < 0$. The expressions for perceived MPRs imply that the MPR under the objective measure is given by

$$\kappa(t) = \sigma_\varepsilon + \frac{\alpha_1 - \lambda(t)}{\alpha_1 + \alpha_2 - 1}\bar{\mu}(t) + \theta_{1\varepsilon}(t) = \sigma_\varepsilon + \frac{\alpha_1 - \lambda(t)}{\alpha_1 + \alpha_2 - 1}\theta_{2\varepsilon}(t) + \frac{\alpha_2 - (1-\lambda(t))}{\alpha_1 + \alpha_2 - 1}\theta_{1\varepsilon}(t). \tag{5.25}$$

Equation (5.25) indicates that the "true" MPR also depends on the term $\alpha_1 + \alpha_2 - 1$ and is likely to be higher than the MPR under homogeneous beliefs when agent 1 is more optimistic than agent 2, that is $\bar{\mu}(t) > 0$, and also more risk averse such that $\alpha_1 - \lambda(t) > 0$. The size of absolute bias ($\theta_{1\varepsilon}(t)$) is of secondary importance in determining the objective MPR. Equation (5.22) shows that the equilibrium interest rate depends on a weighted average of interest rates under each agent's belief plus a term that is proportional to the squared difference in beliefs $\bar{\mu}(t)^2$ and $1/(\alpha_1 + \alpha_2 - 1)^2$. The weights of $r_1(t)$ and $r_2(t)$ depend on the difference between agent 2's risk aversion and his consumption share at time t; the weights sum up to 1, but can be negative. For example, in the case of $\alpha_1 > \lambda(t)$

and $\alpha_2 < 1-\lambda(t)$, the weight of $r_1(t)$ ($r_2(t)$) is negative (positive), which means that the equilibrium risk-free rate is strongly tilted towards $r_2(t)$. Furthermore, if agent 1 is more optimistic than agent 2, which implies that $r_2(t) < r_1(t)$, then the equilibrium risk-free rate would be even lower than $r_2(t)$. Our equilibrium result also contains some special cases presented in the asset-pricing literature, which we discuss in the following.

Corollary 5.4 *When both agents are log-utility maximisers, that is $\alpha_1 = \alpha_2 = 1$. The equilibrium perceived market prices of risk by agents are given by*

$$\kappa_1(t) = \sigma_\varepsilon + (1-\lambda(t))\bar{\mu}(t), \qquad \kappa_2(t) = \sigma_\varepsilon - \lambda(t)\bar{\mu}(t). \tag{5.26}$$

The equilibrium interest rate is given by

$$r(t) = \lambda(t)r_1(t) + (1-\lambda(t))r_2(t), \tag{5.27}$$

where r_1 and r_2 are defined in Proposition 5.3. Agent 1's share of aggregate consumption is given by

$$d\lambda(t) = \lambda(t)(1-\lambda(t))\bar{\mu}(t)\{[(1-\lambda(t))\bar{\mu}(t) + \theta_{1\varepsilon}(t)]dt + d\omega(t)\} \tag{5.28}$$

with $\lambda(0) = x_1$.

We see that under log-utility the benchmarking effect disappears and we recover the result of Detemple and Murthy (1994) as a special case of our model. Corollary 5.4 shows that the perceived MPR for the optimistic (pessimistic) agent is larger (smaller) than the MPR under homogeneous beliefs. In fact, the MPR under the objective belief is given by

$$\begin{aligned}\kappa(t) &= \sigma_\varepsilon + (1-\lambda(t))\bar{\mu}(t) + \theta_{1\varepsilon}(t)\\ &= \sigma_\varepsilon + \lambda(t)\theta_{1\varepsilon}(t) + (1-\lambda(t))\theta_{2\varepsilon}(t),\end{aligned} \tag{5.29}$$

which is simply a consumption share-weighted average of the MPRs in a market with only agent 1 or agent 2. A similar expression for the equilibrium interest rate is given in (5.27). Another special case of our model is the case where agents have homogeneous beliefs, that is $\bar{\mu}(t) = 0$ and $\theta_{1\varepsilon}(t) = 0$ for all $t \in [0, T)$.

Corollary 5.5 *When agents have homogeneous beliefs, that is $\theta_{1\varepsilon}(t) = 0$, $\bar{\mu}(t) = 0$. The equilibrium market price of risk is given by*

$$\kappa(t) = \sigma_\varepsilon, \tag{5.30}$$

and the equilibrium interest rate is given by

$$r(t) = \beta + \mu_\varepsilon - \sigma_\varepsilon^2. \tag{5.31}$$

Corollary 5.5 shows that in the case of homogeneous beliefs, the equilibrium result reduces to that of a log-utility representative agent whose beliefs coincide with the objective belief. This confirms the previous claim that σ_ε corresponds to the MPR under homogeneous beliefs. With a representative agent maximising log-utility, the equilibrium interest rate and market price of risk obtained in Example 12.3 of Cvitanic and Zapatero (2004) is identical to Corollary 5.5.

5.5 Consensus Belief and Market Equilibrium

The aim of this section is to construct a consensus consumer who, when endowed with the total endowment in the economy, generates the same equilibrium prices as in the original economy. This problem was previously studied by Jouini and Napp (2007), who assume that agents with a subjective probability belief maximises the utility of their intertemporal consumption and find that the belief of the consensus consumer (consensus belief) can be decomposed into a probability belief and a stochastic discount factor. In the case where all agents have log-utility, the discount factor equals to 1 and the consensus belief is a proper belief. The problem becomes more complex in our case since agents maximise the utility of relative consumption to each other. It seems difficult to construct a representative agent. To overcome this difficulty, we take the following approach. The first step is to find an equivalent problem to (5.12), where agent 1 is a utility maximiser of his absolute level of consumption rather than relative consumption and the optimal consumption process is the same as in Lemma 5.1. The second step is to construct a consensus consumer using the method in Jouini and Napp (2007). For the first step, consider agent 1's optimal consumption process in (5.15), which can be re-written as

$$M_1(t)e^{-\beta t}\frac{1}{c_1(t)} = \frac{1}{y_1}\ \xi^*(t), \tag{5.32}$$

where $M_1 = \xi^*/\xi_1^*$ and ξ^* is the state price density process under the objective measure $\mathcal{P}$. Equation (5.32) has the same structure as in equation (2.15) in Jouini and Napp (2007) where $u_1(t,x) = e^{-\beta t}\ln(x)$, the only difference being that $M_1(t)$ in Jouini and Napp (2007) is a positive density process that corresponds the objective probability measure $\mathcal{P}$ to an equivalent probability measure $\mathcal{Q}_1$ and characterises agent 1's subjective probability belief; whereas in our case, M_1 consists of both agents' subjective beliefs and not necessarily a martingale, and hence may not be a proper belief. In fact the dynamics of M_1 are given by

$$\frac{dM_1(t)}{M_1(t)} = -\frac{1}{2}p_1q_1(\bar{\mu}(t))^2dt - [p_1\theta_{1\varepsilon}(t) + q_1\theta_{2\varepsilon}(t)]d\omega(t), \tag{5.33}$$

where[2] $p_1 = \frac{\alpha_2}{\alpha_1+\alpha_2-1}$ and $q_1 = 1 - p_1$. We denote $\mathcal{E}_1^P(\theta(t)) = p_1\theta_{1\varepsilon}(t) + q_1\theta_{2\varepsilon}(t)$ and $\mathcal{E}_1^P(\theta^2(t)) = p_1[\theta_{1\varepsilon}(t)]^2 + q_1[\theta_{2\varepsilon}(t)]^2$. It can be shown that (5.33) can be rewritten as

$$\frac{dM_1(t)}{M_1(t)} = -\frac{1}{2}Var_1^P(\theta(t))dt - \mathcal{E}_1^P(\theta(t))d\omega(t), \tag{5.34}$$

where $Var_1^P(\theta(t)) = \mathcal{E}_1^P(\theta^2(t)) - [\mathcal{E}_1^P(\theta(t))]^2$. Therefore M_1 is a martingale if and only if $Var_1^P(\theta(t)) = 0$, or equivalently $\bar{\mu}(t) = 0$ or $\alpha_1 = 1$, which correspond to the two cases where the benchmarking effect disappears. Hence, when the benchmarking effect is present, M_1 is not a proper belief, but rather consists of a proper belief and a stochastic discount factor. Under a similar argument we can find the dynamics of M_2 given by

$$\frac{dM_2(t)}{M_2(t)} = -\frac{1}{2}Var_2^P(\theta(t))dt - \mathcal{E}_2^P(\theta(t))d\omega(t), \tag{5.35}$$

where $p_2 = \frac{\alpha_2-1}{\alpha_1+\alpha_2-1}$ and $q_2 = 1 - p_2$. Next we follow Example 2.1 in Jouini and Napp (2007) and construct a consensus consumer in the following proposition (the proof is omitted since it is identical to the proof in Jouini and Napp (2007)).

Proposition 5.6 *The consensus consumer has log utility function; the first order optimality condition is given by*[3]

$$M(t)\ \ e^{-\beta t}\frac{1}{\varepsilon(t)} = \xi^*, \tag{5.36}$$

where the consensus characteristic[4] $M(t)$ *follows*

$$\frac{dM(t)}{M(t)} = \ \mu_M(t)dt + \delta_M(t)d\omega(t) \tag{5.37}$$

with $M(0) = 1$, *where*

$$\mu_M(t) = -\frac{1}{2}(\lambda(t)Var_1^P(\theta(t)) + (1-\lambda(t))Var_2^P(\theta(t)))$$

and

$$\delta_M(t) = -(\lambda(t)\mathcal{E}_1^P(\theta(t)) + (1-\lambda(t))\mathcal{E}_2^P(\theta(t))).$$

[2] The set $\{p_1, q_1\}$ is not necessarily a probability measure since p_1 can be greater than 1, in which case q_1 would be negative.

[3] As a standard practice, we normalise the Lagrange multiplier of the consensus consumer to be 1.

[4] We do not call $M(t)$ the consensus belief because it may not be a martingale, thus not a positive density process.

Proposition 5.6 indicates the consensus characteristic is a consumption share-weighted average of the individual characteristics. $M(t)$ is a martingale and thus the consensus characteristic is a proper belief if and only if $\mu_M(t) = 0$, which is true if both agents are log-utility maximisers or they have the same belief about the aggregate endowment process. In order to characterise market equilibrium under this framework, we use equation (5.36) and the SDE governing the dynamics of $M(t)$ and $\xi^*(t)$. By equating the drift and diffusion coefficient, we obtain the following expressions for the market price of risk under the objective probability measure and the equilibrium risk-free rate

$$\kappa(t) = \sigma_\varepsilon(t) - \delta_M(t) \tag{5.38}$$

and

$$r(t) = \beta + \mu_\varepsilon(t) - \sigma_\varepsilon(t)^2 - \mu_M(t) + \sigma_\varepsilon(t)\delta_M(t). \tag{5.39}$$

Equations (5.38) and (5.39) are identical to equations (5.22) and (5.25) when we make the substitutions for $\mu_M(t)$ and $\delta_M(t)$, suggesting that the results are consistent. Under homogeneous beliefs, the market price of risk and the interest rate are given by $\kappa(t) = \sigma_\varepsilon(t)$ and $r(t) = \beta + \mu_\varepsilon(t) - \sigma_\varepsilon(t)^2$. Therefore, given the agent's beliefs $(\theta_{1\varepsilon}, \theta_{2\varepsilon})$, an increase in the market price of risk and simultaneously a reduction in the interest rate can be achieved by choosing (α_1, α_2) such that $\mu_M(t) - \delta_M(t) > 0$ and $\delta_M(t) < 0$. Because weights associated with $\theta_{1\varepsilon}(t)$ and $\theta_{2\varepsilon}(t)$, p_i and q_i $(i = 1, 2)$, can be greater than 1 or negative, which is a result of our relative consumption framework, the consensus consumer can place a negative weight on the optimistic belief and place a weight greater than 1 on the pessimistic belief. Consequently, there can be a significant effect on market equilibrium at the aggregate level even if the disagreements at the individual level are small.

5.6 Long-run Survivability

This section examines the long-run survivability of the two agents. Agents 1 and 2 can have different beliefs about the growth rate of aggregate endowment and have different risk-aversion coefficients; agent 1's consumption share follows (5.23). In the following, we give a closed-form solution to (5.23) by assuming that agents have constant disagreements in their beliefs about the growth rate of aggregate endowment, that is $\theta_{1\varepsilon}(t) = \theta_{1\varepsilon}$ and $\bar{\mu}(t) = \bar{\mu}$. This is the limiting case of the example in subsection 5.3.1 when the variance of agent i's prior belief approaches zero, that is $v_i(0) \to 0$. Hence the consumption share of agent 1 can now be written as

$$d\lambda(t) = \frac{\lambda(t)(1-\lambda(t))\bar{\mu}}{\alpha_1+\alpha_2-1}\left\{\left[\frac{1}{2}\left(1+\frac{1-2\lambda(t)}{\alpha_1+\alpha_2-1}\right)\bar{\mu}+\theta_{1\varepsilon}\right]dt + d\omega(t)\right\}. \tag{5.40}$$

Equation (5.40) shows that agent 1's consumption share follows a one-dimensional SDE when agents' beliefs are constant. This SDE has an explicit solution given in Lemma 5.7.

Lemma 5.7 *Given that $\theta_{1\varepsilon}(t) = \theta_{1\varepsilon}$ and $\bar{\mu}(t) = \bar{\mu}$,*

$$\frac{1}{\lambda(t)} = 1 + \frac{1-\lambda(0)}{\lambda(0)}\exp\left\{-\frac{\bar{\mu}}{\alpha_1+\alpha_2-1}\left[\left(\frac{1}{2}\bar{\mu}+\theta_{1\varepsilon}\right)t+\omega(t)\right]\right\}.$$

Lemma 5.7 shows that agent 1's consumption share depends on the beliefs and the risk aversion of both agents and his long-run fate is determined by the term

$$-\frac{\bar{\mu}}{\alpha_1+\alpha_2-1}\left(\frac{1}{2}\bar{\mu}+\theta_{1\varepsilon}\right). \tag{5.41}$$

If this term is positive, then agent 1's consumption approaches zero as time tends to infinity, that is $\lambda(t) \overset{a.s}{\to} 0$, hence agent 1 vanishes. If this term is negative then agent 2 vanishes, since $\lambda(t) \overset{a.s}{\to} 1$. Obviously, the sign of (5.41) depends on the sign of $\alpha_1+\alpha_2-1$, but not its actual value, which means that only the sum of the agents' risk aversion is relevant to the long-run fate of agent 1, not the individual agent's risk aversion.

Definition 5.8 *Agent i ($i = 1, 2$) vanishes in the long-run when $\mathcal{P}(\lim_{t\to\infty} \lambda_i(t) = 0) = 1$ where $\lambda_1(t) = \lambda(t)$ and $\lambda_2(t) = 1-\lambda(t)$. Agent i survives in the long run if he does not vanish.*

Based on Definition 5.8, agent 1 vanishes if his consumption share approaches zero almost surely as time tends to infinity. Next we give necessary and sufficient conditions for agent 1 to vanish in the long-run.

Proposition 5.9 *Consider two situations,*

(i) When $\alpha_1+\alpha_2 > 1$, agent 1 vanishes and agent 2 survives iff $|\theta_{2\varepsilon}| < |\theta_{1\varepsilon}|$;

(ii) When $\alpha_1+\alpha_2 < 1$, agent 1 vanishes and agent 2 survives iff $|\theta_{2\varepsilon}| > |\theta_{1\varepsilon}|$.

If $|\theta_{2\varepsilon}| = |\theta_{1\varepsilon}|$, then both agents 1 and 2 survive.

Proposition 5.9 shows that when agents 1 and 2 have an aggregate risk aversion of more than 1, agent 1 vanishes if and only if his absolute bias $|\theta_{1\varepsilon}|$ is strictly larger than agent 2's bias. This is intuitive, since one would expect the agent that constantly makes less accurate predictions to be driven out of the market in the long-run. However, the condition is reversed when the aggregate risk aversion is less than 1, that is, when $\alpha_1 + \alpha_2 < 1$, agent 1 vanishes if and only if his absolute disagreement is smaller than that for the other agent. This result seems absurd. For example, if agent 1 is completely rational, that is $\theta_{1\varepsilon} = 0$, agent 1 will actually be driven out of the market instead of agent 2 unless agent 2, is also completely rational. So, how can an agent vanish when he is the one making more accurate predictions? We provide an explanation using an example described in Section 5.2. Assume that, similar to agent a, agent b maximises the expected CRRA utility of his terminal wealth relative to agent a. As a result, we obtain the following system of equations for the optimal portfolio of agents a and b,

$$\pi_a(t) = \frac{\mu_a - r}{\alpha_a \ \sigma^2} \ + \ \frac{\alpha_a - 1}{\alpha_a}\pi_b(t), \qquad \pi_b(t) = \frac{\mu_b - r}{\alpha_b \ \sigma^2} \ + \ \frac{\alpha_b - 1}{\alpha_b}\pi_a(t), \tag{5.42}$$

where π_a and π_b are the proportions of wealth invested in the risky asset, α_a and α_b are the relative risk-aversion coefficients and μ_a and μ_b are the beliefs of the expected stock return of agent a and b respectively. Solving (5.42) yields

$$\pi_a(t) = \frac{1}{\sigma^2}\left[\frac{\alpha_b}{\alpha_a + \alpha_b - 1}\mu_a + \frac{\alpha_a - 1}{\alpha_a + \alpha_b - 1}\mu_b - r\right]. \tag{5.43}$$

Equation (5.43) shows that agent b's optimal portfolio is the same as a log-utility maximiser with belief

$$\mu_a^* = \frac{\alpha_b}{\alpha_a + \alpha_b - 1}\mu_a + \frac{\alpha_a - 1}{\alpha_a + \alpha_b - 1}\mu_b. \tag{5.44}$$

In the case of homogeneous beliefs, that is $\mu_a = \mu_b = \mu$, we have $\mu_a^* = \mu$. Under heterogeneous beliefs, we are more interested in the case when $\alpha_a + \alpha_b < 1$, which also implies that both α_a and α_b are less than 1. Following (5.44), the coefficient of μ_a is negative and coefficient of μ_b is positive. This means that although agent a may have a more accurate prediction of the expected stock return, he optimally puts a negative weight on his belief and a positive weight on the belief of agent b, hence agent a vanishes in the long-run instead of agent b.

Given the necessary and sufficient conditions for an agent to vanish in the long-run, the next logical question to ask is how quickly this will happen. Given the conditions hold such that agent 1 vanishes in the long-run, we measure the vanishing time by t_l, which is the expected first time that agent 1's consumption share $\lambda(t)$ will reach the level l, with $l < \lambda(0)$. Mathematically,

$$t_l = \mathbb{E}[\inf\{t: \ \lambda(t) = l\}]. \tag{5.45}$$

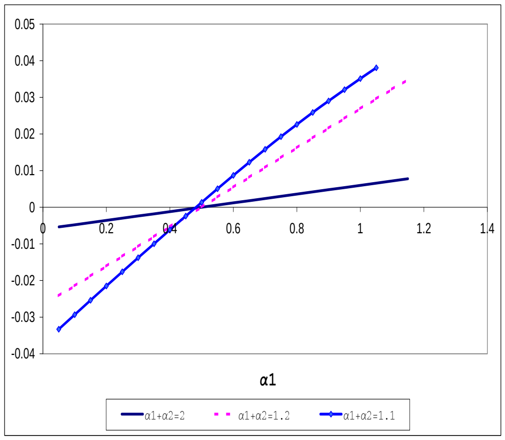
Proposition 5.10 *The expected first time that agent 1's consumption share $\lambda(t)$ will hit level l, given that agent 1 vanishes in the long run and $l < \lambda(0)$, is given by*

$$t_l = \frac{(\alpha_1 + \alpha_2 - 1)}{-\bar{\mu}(\theta_{1\varepsilon} + \frac{1}{2}\bar{\mu})} \ln\left(\frac{(1-l)\lambda(0)}{l(1-\lambda(0))}\right).$$

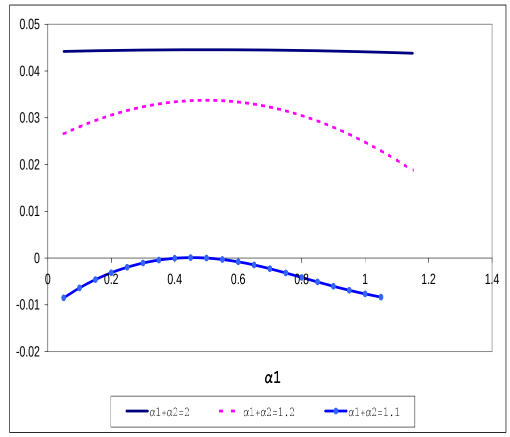
Obviously, the first hitting time t_l must be nonnegative, since $\alpha_1 + \alpha_2 \neq 1$ for an equilibrium to exist. $t_l = 0$ if and only if $l/(1-l) = \lambda(0)/(1-\lambda(0))$ implying $l = \lambda(0)$. Hence the first hitting time is zero if and only if level l is equal to agent 1's initial rate of consumption. In the other extreme cases, when $\bar{\mu} \to 0$ (homogeneous beliefs) or $\bar{\mu} \to -2\theta_{1\varepsilon}$, both are special cases of equal absolute disagreement, that is $|\theta_{1\varepsilon}| - |\theta_{2\varepsilon}| \to 0$, the vanishing time t_l approaches infinity, hence agent 1's consumption share is expected never to reach level l. In these limiting cases, agents 1 and 2 both survive in the long run.

5.7 Numerical Analysis

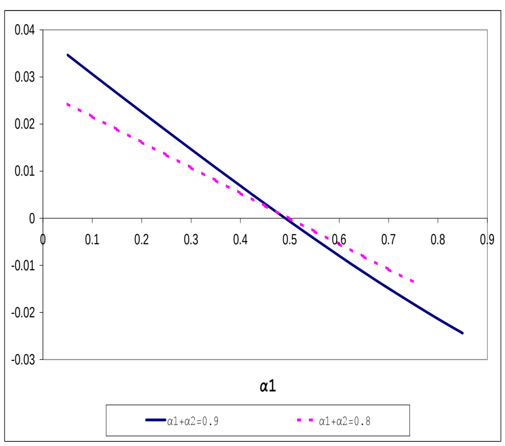
In this section, we provide some numerical results of our model to demonstrate its explanatory power of the equity risk premium puzzle and the risk-free rate puzzle. In these numerical simulations we assume that the disagreement is constant but very small. Many of the previous asset-pricing models with heterogeneous beliefs require rather large disagreement between agents in order to explain a significant part of the equity risk premium and the risk-free rate (see Detemple and Murthy (1994), Jouini and Napp (2006, 2007) and David (2008)). We demonstrate numerically that when agents maximise the expected utility of their relative consumption, a large part of the observed level of the average equity premium and risk-free rate can be explained even with very "small" heterogeneity in agents' beliefs. For the numerical analysis, we estimate the expected time average of the equity premium (EEP) $\frac{1}{T}\mathbb{E}[\int_0^T \mu(t) - r(t)dt]$ and of the risk-free rate $\frac{1}{T}\mathbb{E}[\int_0^T r(t)dt]$ (ERR) under the objective probability measure with different values of α_1 and α_2. More specifically, we assume that under the objective measure, the drift of the aggregate consumption is $\mu_\varepsilon = 0.02$ and the volatility is $\sigma_\epsilon = 0.03$ (see Campbell (2003)). Moreover, we set the constant disagreement of agent 1 to be $\theta_{1\varepsilon} = -0.03$ and difference in beliefs to be $\bar{\mu} = 0.06$, which implies that $\theta_{2\varepsilon} = 0.03$. In this scenario, agents 1 and 2 have symmetric disagreement, that is $|\theta_{1\varepsilon}| = |\theta_{2\varepsilon}|$, and agent 1 is relatively optimistic in that he overestimates the drift of the aggregate endowment by $\mu_{1\varepsilon} - \mu_\varepsilon = -\theta_{1\varepsilon}\sigma_\varepsilon = 0.0009$ or 9 basis points, whereas agent 2 is relatively pessimistic in that he underestimates the drift of the aggregate endowment by $\mu_\varepsilon - \mu_{2\varepsilon} = \theta_{2\varepsilon}\sigma_\varepsilon = 0.0009$ or 9 basis points. The patience parameter is given by $\beta = 0.025$. Assume that the volatility of the risky asset is



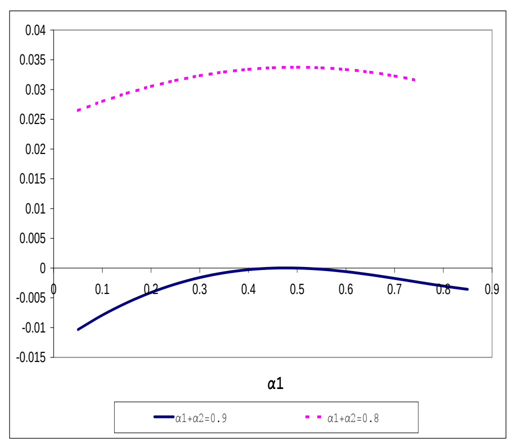
(a1) average equity premium $\alpha_1 + \alpha_2 > 1$



(a2) average interest rate $\alpha_1 + \alpha_2 > 1$



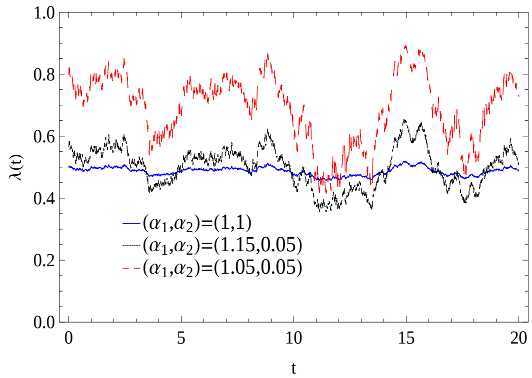
(a3) average equity premium $\alpha_1 + \alpha_2 < 1$



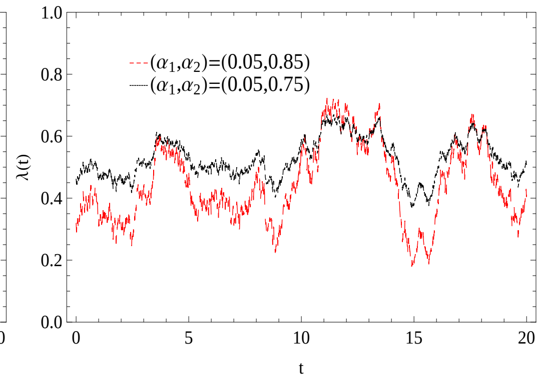
(a4) average interest rate $\alpha_1 + \alpha_2 < 1$

Figure 5.1: Impact of risk aversions and disagreements in beliefs on the time average of equity risk premium $\frac{1}{T}\mathbb{E}[\int_0^T \mu(t) - r(t)dt]$ and the risk free rate $\frac{1}{T}\mathbb{E}[\int_0^T r(t)dt]$.

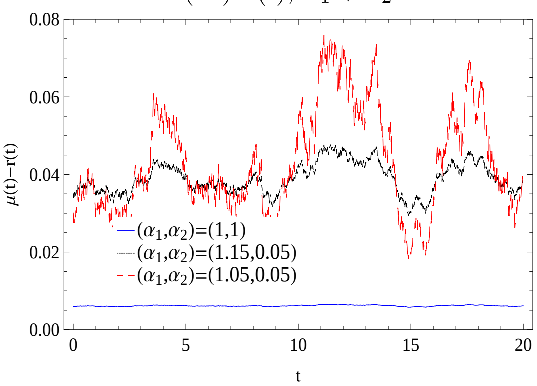
given by $\sigma(t) = 0.2$ for all $t \in [0, T)$. The planning horizon is $T = 20$ years and the time increment is $\Delta t = T/n = 0.005$ where $n = 4000$ is the number of discretisation intervals. For each pair of α_1 and α_2, we simulate $N = 30,000$ paths for $\lambda(t)$ to estimate the time average of equity premium and risk-free rate. We use the antithetic variance reduction technique to reduce the standard deviation of the Monte Carlo simulations. For each combination of α_1 and α_2, we use the bisection method and Monte Carlo to numerically



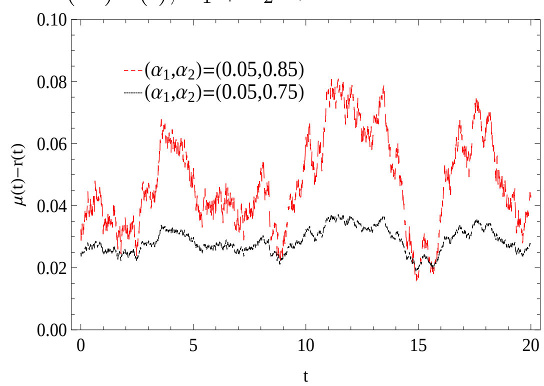
(b1) $\lambda(t)$, $\alpha_1 + \alpha_2 > 1$



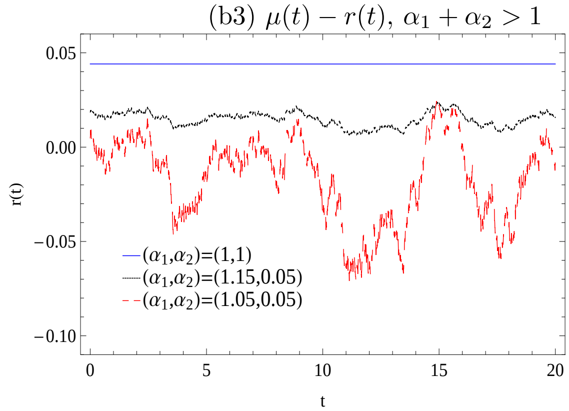
(b2) $\lambda(t)$, $\alpha_1 + \alpha_2 < 1$



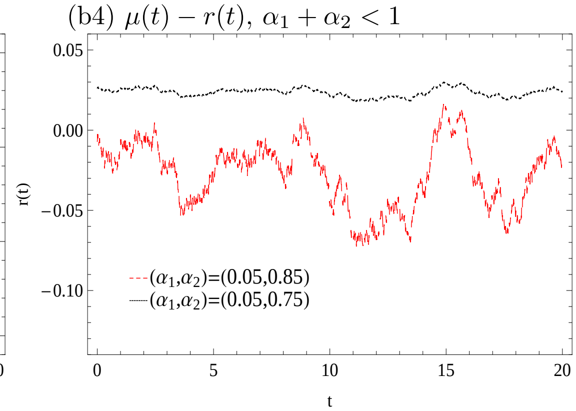
(b3) $\mu(t) - r(t)$, $\alpha_1 + \alpha_2 > 1$



(b4) $\mu(t) - r(t)$, $\alpha_1 + \alpha_2 < 1$



(b5) $r(t)$, $\alpha_1 + \alpha_2 > 1$



(b6) $r(t)$, $\alpha_1 + \alpha_2 < 1$

Figure 5.2: Impact of risk aversions and disagreements in beliefs on agent 1's consumption share $\lambda(t)$, the equity risk premium $\mu(t) - r(t)$ and the risk-free rate $r(t)$.

solve for $\lambda(0)$ in order to satisfy the budget constraint (5.24). Figure 5.1 shows that when the value of $\alpha_1 + \alpha_2$ is fixed, EEP is positively (negatively) and approximately linearly related to the risk aversion of agent 1, that is, α_1 when the sum of risk aversion $\alpha_1 + \alpha_2$ is greater (less) than 1 (the left panel in Fig. 5.1). The slope steepens as $\alpha_1 + \alpha_2$ becomes closer to 1, see Fig. 5.1 (a1) and (a3). The ERR in Fig. 5.1 (a2) and (a4) has a concave

(α_1, α_2)	$(\theta_1, \bar{\mu})$	$\frac{1}{T}\mathbb{E}[\int_0^T \mu(t) - r(t)dt]$ (EEP)	$\frac{1}{T}\mathbb{E}[\int_0^T r(t)dt]$ (ERR)
N/A	$(0.00, 0.00)$	0.60%	4.41%
$(1.00, 1.00)$	$(-0.03, 0.06)$	0.60%	4.41%
$(1.15, 0.05)$	$(-0.03, 0.06)$	3.49%	1.88%
$(1.05, 0.05)$	$(-0.03, 0.06)$	3.89%	−0.84%
$(0.05, 0.85)$	$(-0.03, 0.06)$	3.47%	−1.03%
$(0.05, 0.75)$	$(-0.03, 0.06)$	2.43%	2.65%

Table 5.1: Impact of risk aversions and disagreements in beliefs on the time average of equity risk premium $\frac{1}{T}\mathbb{E}[\int_0^T \mu(t) - r(t)dt]$ and the risk-free rate $\frac{1}{T}\mathbb{E}[\int_0^T r(t)dt]$.

relationship with α_1, the overall level decreases as the sum of risk aversion becomes closer to 1; moreover, the overall level increases and the relationship flattens when $\alpha_1 + \alpha_2 = 2$. To explain the equity premium and the risk-free rate puzzles, we refer to Table 5.1. When agents have homogeneous objective beliefs about the drift of the aggregate endowment process, the EEP is 0.60% and the ERR is 4.41%; compared to historical data, the equity premium is too low and the risk-free rate is too high[5]. When both agents have log-utility, that is $(\alpha_1, \alpha_2) = (1, 1)$, heterogeneous beliefs cannot account for the puzzles because the disagreements are too small (in fact, only 9 basis points) and mean-preserving, that is $\theta_{2\varepsilon} = -\theta_{1\varepsilon}$. However, when agents' risk aversion is different from 1, in particular when $(\alpha_1, \alpha_2) = (1.15, 0.05)$ or $(0.05, 0.75)$, we can obtain a much larger EEP with a reasonable level of ERR, hence the benchmarking effect combined with heterogeneous beliefs can help us to explain the puzzles. The reason is that when agents care not only about their absolute level of intertemporal consumption but rather consumption relative to the other agent, the effect of heterogeneous beliefs on equilibrium prices is magnified if the sum of risk aversion, that is $\alpha_1 + \alpha_2$, is close to 1. When $\alpha_1 + \alpha_2$ is greater than one, the EEP increases and ERR reduces when the optimistic agent (agent 1) is also more risk averse. When $\alpha_1 + \alpha_2$ is less than one, we can explain a large part of the puzzles if the optimistic agent is less risk averse. In contrast, Jouni and Napp (2006, 2007) find that only a positive correlation between risk tolerance and pessimism can increase the equity premium and reduce the risk-free rate; however, they do not discuss the magnitude of the increase and the reduction.

Figure 5.2 shows a time series plot of the endogenous variables, including agent 1's consumption share $\lambda(t)$, equity premium $\mu(t) - r(t)$, and the interest rate $r(t)$ given a path of the Brownian motion $\omega(t)$, $t \in [0, T)$. Agent 1's consumption share $\lambda(t)$ shows no clear time trend, see Fig. 5.2 (b1) and (b2), when agent 1's initial consumption is

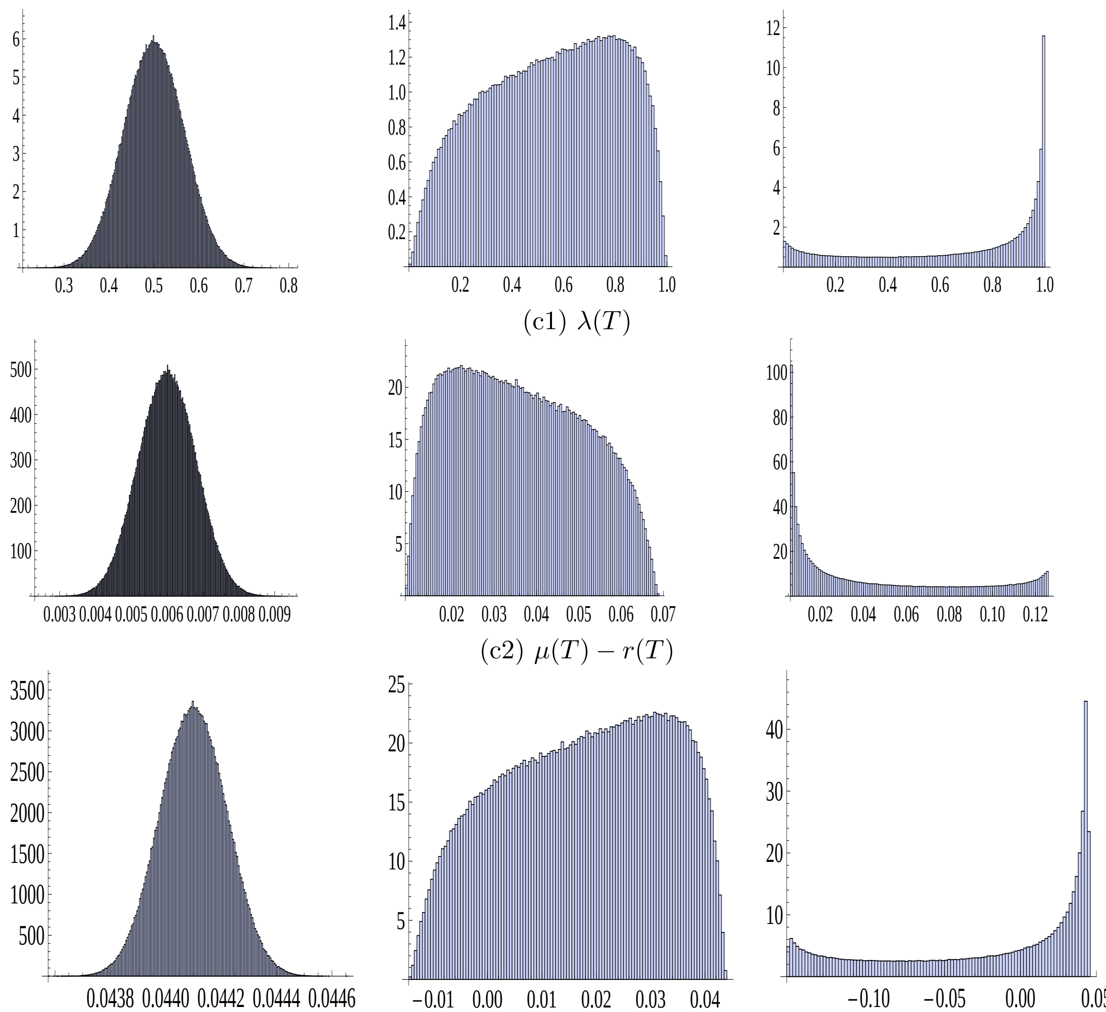
[5] According to US data, the average real stock return over the past 100 years is around 7% and bond return around 2%, this implies an average equity premium of 5%.

greater than (less than) a half and he is more (less) risk averse than agent 2. However, $\lambda(t)$ becomes more volatile as the sum of risk-aversion coefficients becomes close to one. For example, when $(\alpha_1, \alpha_2) = (1.05, 0.05)$, agent 1 can consume more than 80% of the total endowment, whereas $\lambda(t)$ fluctuates close to a half for all t under log utility. Fig. 5.2 (b3) and (b4) show that equity premium is time varying, and is on average larger and more volatile when the sum of risk-aversion coefficients gets close to one. When $(\alpha_1, \alpha_2) = (1.05, 0.05)$, the equity premium is at times more than 7% p.a., whereas it is less than 1% under log utility. Fig. 5.2 (b5) and (b6) show that the equilibrium interest rate is also time varying, and is on average lower and more volatile as the sum of risk-aversion coefficients gets close to one.

Next we examine the probability distribution of the endogenous variables at the terminal time T using Monte Carlo simulations. We draw 1,000,000 random numbers from $\omega(T) \sim \mathcal{N}(0, \sqrt{T})$, then we obtain the distribution of $\lambda(T)$, $\mu(T) - r(T)$ and $r(T)$, respectively, using Proposition 5.3 and Lemma 5.7. Results are given in Figure 5.3 for $\alpha_1 + \alpha_2 > 1$. The left panel in Fig. 5.3 shows that when both agents have log-utility, agent 1's consumption share $\lambda(T)$, the equity premium $\mu(T) - r(T)$ and the interest rate $r(T)$ all have symmetric normal-like distributions[6]. The mean of the distributions corresponds to the benchmark values under homogeneous beliefs, that is, when $\theta_{1\varepsilon} = 0$ and $\bar{\mu} = 0$, we have $\lambda(T) = 0.5$, $\mu(T) - r(T) = 0.006$ and $r(T) = 0.0441$. For $(\alpha_1, \alpha_2) = (1.15, 0.05)$, the distributions of $\lambda(T)$ and $r(T)$ are negatively skewed and positively skewed for $\mu(T) - r(T)$, see the middle panel in Fig. 5.3. Under the case where $(\alpha_1, \alpha_2) = (1.05, 0.05)$, the distributions in the right panel of Fig. 5.3 are no longer unimodal as the distributions of agent 1's consumption share $\lambda(T)$ and the interest rate $r(T)$ have large spikes on the right tail where equity premium $\mu(T) - r(T)$ has a large spike on the left tail. This suggests that when the sum of the agents' risk-aversion coefficients is close to 1 and the optimistic agent (agent 1) is more risk-averse, there is a significantly large probability that the optimistic and more risk averse agent will dominate the market leading to an equity premium close to zero and a high interest rate. Results are similar when $\alpha_1 + \alpha_2 < 1$ and agent 1 is optimistic and less risk-averse than agent 2, see Figure 5.4. In this case, there is a significant probability that agent 2 (the pessimistic and more risk-averse agent) will dominate the market, leading to an equity premium close to zero and a high interest rate. The intuition for this result can be obtained from equations (5.25) and (5.22). In the first case, where α_1 is close to 1, it is clear from (5.25) that if $\lambda(T) \to 1$, $\kappa(T) \to \sigma_\varepsilon + \theta_{1\varepsilon}$, this is the market price of risk under a representative agent with log-utility and agent 1's belief. Similarly, the interest also approaches the value under the belief of this representative agent. In the

[6]The distributions for $\lambda(T)$, $\kappa(T)$ and $r(T)$ are in fact not normal since they are all bounded.

second case, where α_1 is close to zero, the same results apply when $\lambda(T) \to 0$. Therefore, although in a different context, our results are consistent with Kogan et al (2006) that survival and price impact are two distinct concepts. In particular, the second case shows that agent 1's belief dominates the market though his consumption share approaches zero.



(c1) $\lambda(T)$

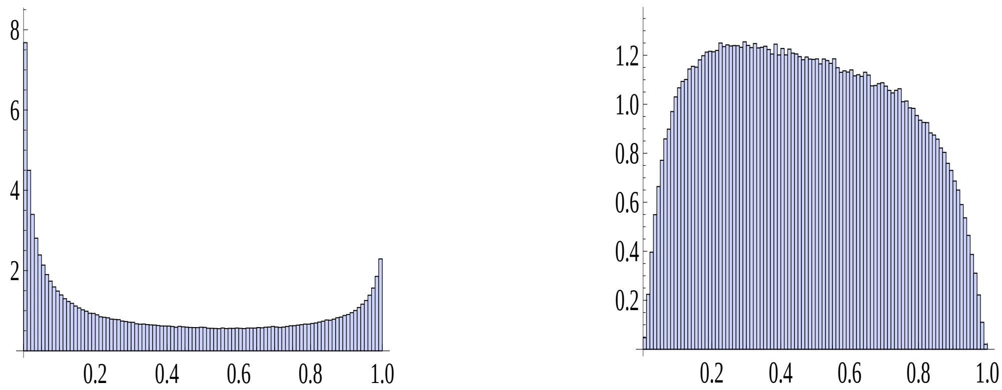
(c2) $\mu(T) - r(T)$

(c3) $r(T)$

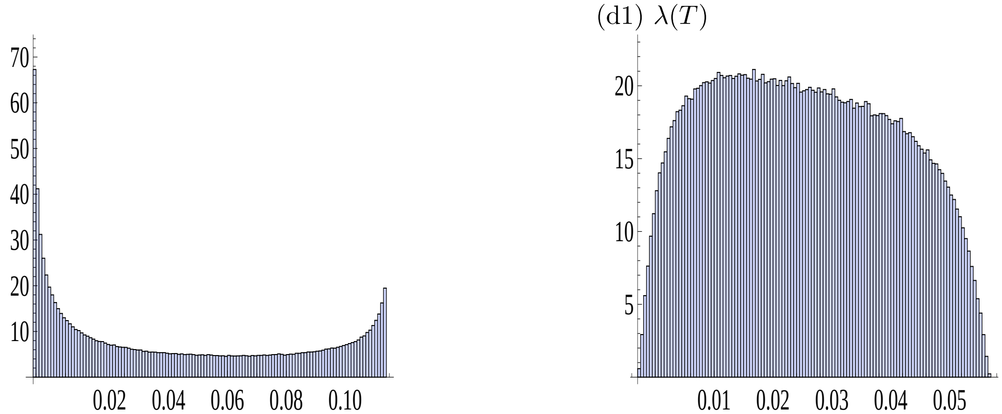
Figure 5.3: Impact of risk aversion and disagreement in beliefs on the probability distribution of agent 1's consumption share, equity premium and the interest rate in equilibrium at terminal time $T = 20$. The left panel corresponds to the case of $(\alpha_1, \alpha_2) = (1, 1)$, the middle panel to $(\alpha_1, \alpha_2) = (1.15, 0.05)$ and the right panel to $(\alpha_1, \alpha_2) = (1.05, 0.05)$.

5.8 Conclusion

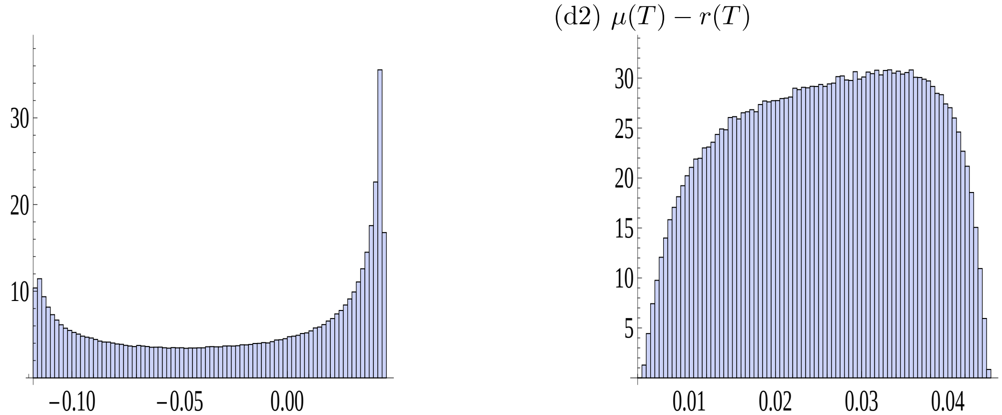
In this chapter we consider a pure exchange economy in which two agents with heterogeneous beliefs maximise the expected utility of relative consumption to the other agent.



(d1) $\lambda(T)$



(d2) $\mu(T) - r(T)$



(d3) $r(T)$

Figure 5.4: Impact of risk aversion and disagreement in beliefs on the probability distribution of agent 1's consumption share, equity premium and the interest rate in equilibrium at terminal time $T = 20$. The left panel corresponds to the case of $(\alpha_1, \alpha_2) = (0.05, 0.85)$, and the right panel to $(\alpha_1, \alpha_2) = (0.05, 0.75)$.

Within this framework, through an example we show that an agent's optimal portfolio is divided into two parts; the first part is based on his prediction of the expected stock return, and the second part is based on the optimal portfolio of the other agent, which the agent goes long (short) if his risk aversion to relative consumption is greater (less) than 1. We call the second part the *benchmarking effect*. In a general equilibrium setting where there is an exogenously given aggregate endowment process and the agents disagree on the drift of the process, we derive explicit expressions for the equilibrium market price of risk under the agents' subjective measures and under the objective measure and the risk-free rate. When beliefs are homogeneous, we obtain the equilibrium result under a representative agent with log-utility. When both agents have log-utility, the benchmarking effect vanishes and the results of Detemple and Murthy (1994) emerges as a special case. We demonstrate that it is possible to find a consensus consumer in our economy using the aggregation method in Jouini and Napp (2007). We show that under the equivalent optimisation problem, an agent has log-utility and his subjective belief consists of a proper probability belief and a stochastic discount factor. On the survivability of the agents, the long-run fate of the agents depends on the level of their absolute disagreement and also the sum of their risk aversions. When the sum is greater (less) than 1, the agent with a higher (lower) level of absolute disagreement vanishes in the long-run. The agent vanishes more quickly with larger difference in absolute disagreement and when the sum of the agents' risk aversion is closer to 1. Numerical analysis shows that only a minimal amount of mean-preserving heterogeneity in beliefs may be required to explain a significant part of the historically observed average of the equity premium and risk-free rate under the benchmarking effect. The effect is the largest when the sum of risk aversion is close to and greater than 1 and the optimistic agent is more risk averse, or when the sum of risk aversion is close to but less than 1 and the pessimistic agent is more risk averse.

Overall, we have shown that, within the relative consumption framework, a heterogeneous belief can have a significant impact on the market equilibrium risk premium, riskless rate, and the long-run survival of agents.

Chapter 6

A Binomial Model of Option Pricing with Heterogeneous Beliefs

6.1 Introduction

Since its advent in the 1970s, the binomial model has been popular and widely used in finance literature. The binomial model was first proposed by Cox, Ross and Rubinstein (1979) (CRR), which has subsequently become one of the most cited paper, in finance literature. At the time of its publication, economists were not conversant with the mathematical tools used to derive the Black-Scholes option-pricing formula, so the CRR paper re-derived the formula as a limit from the binomial model. Essentially, the binomial pricing model uses a "discrete-time binomial lattice (tree) framework" to model the dynamics of the underlying stock price. A binomial lattice can be characterised simply by the probability of an up move and the size of the move in both the upstate and the downstate. When all three parameters are constant, with an appropriate specification, the binomial lattice in a continuous time limit converges weakly to the Black-Scholes (BS) model. Due to its simplicity, the binomial lattice provides a simple framework to model stock-price dynamics and interest-rate term structure. The parameters can also be set for the binomial lattice to weakly converge to other popular diffusion models used in finance. Kloeden and Platen (1992) show that the Euler scheme converges weakly to a diffusion process if one replaces the Wiener increment in the Euler scheme with a two-point distributed random variable, hence the resulting numerical scheme is a binomial lattice with time and state dependent upward and downward moves, which are equally likely to occur. Moreover, Nelson and Ramaswamy (1990) develop methods to construct a recombined binomial lattice for the diffusion model to enhance computational efficiency. Further, Hahn and Dyer

(2008) apply the method to mean-reverting stochastic processes specifically for real option valuation. Van der Hoek and Elliott (2006) present a text-book treatment for binomial models and their application.

Binomial models have been employed to study the pricing of options in literature. Guidolin and Timmermann (2003) model the dividend growth rate as a binomial lattice with constant probability and rate changes in each state; the former is unknown whereas the latter is known. They also assume that there is a representative agent with CRRA utility who updates his belief about the probability of a positive rate change as a Bayesian learner (BL). They find that the stock price under BL with incomplete information does not converge to the Black-Scholes model, whereas the convergence occurs under complete information. Call prices under BL with certain priors exhibit implied volatility that resembles the market-implied volatility observed with S&P500 index options in the given period. In another related paper, Guidolin and Timmermann (2007) characterise equilibrium asset prices under adaptive, rational and BL schemes in a model where dividends evolve in a binomial lattice. The properties of equilibrium stock and bond prices under learning are shown to differ significantly. Our work is related to theirs in that we are also studying equilibrium asset pricing under a binomial framework with incomplete information. However, we have multiple agents with heterogeneous beliefs about the ex-dividend price of the stock, who only consumes at the maturity date. We do not impose any exogenous processes (dividend process) as inputs to our model and we do not model the agents' learning process. Instead, we consider a disagreement model and assume that agents simply form different beliefs based on the same information (currently observed stock price). We assume all agents maximise log-utility, since in a market where stock returns are assumed to be independently distributed, the optimal portfolio under log-utility is the growth optimal portfolio (GOP), which out-grows any other portfolios. Also, it is easier to obtain analytically tractable results under log-utility than the more general CRRA utility.

Our model investigates an economy with one risky asset (stock) and one riskless asset in positive and net-zero supply, respectively. The stock price follows a binomial lattice that allows time and state dependent upward and downward moves and also the probability of an up move. The agents are log-utility maximisers of their terminal wealth who form their subjective beliefs now about the probability and the size of the moves in each period from now to the terminal time or maturity date. Agents agree to disagree; the differences in opinion are due to the interpretation of the same information. Our model concerns the equilibrium pricing of options and contingent claims in an economy with disagreements about the evolution of the stock price.

This chapter is organised as follows. Section 6.2 presents the binomial model that describes our economy. Section 6.3 defines a consensus belief and shows how the consensus belief can be constructed from investors' subjective beliefs. We also define and identify a fair belief for pricing contingent claims in the market. Section 6.4 performs a static analysis and studies the impact of heterogeneous beliefs on the equilibrium price of the risky asset and the risk-free rate in a single-period setting. In section 6.5, we develop a fair option-pricing formula and use a numerical example to study the price distribution and the fair call prices. Section 6.6 concludes.

6.2 A Binomial Economy with Heterogeneous Beliefs

We consider a simple economy with one risky and one riskless asset. Let time be discrete and finite, indexed by $t = 0, 1, 2, \cdots, T$. The risky asset has one share available and the riskless asset is in zero net supply for all time t. There are I agents in the economy, indexed by $i = 1, 2, \cdots, I$. Agent i's objective at time $t = 0$ is to maximise the quantity

$$\mathbb{E}_0^i\Big(U(W_i(T))\Big), \tag{6.1}$$

where $U(\cdot)$ is investor i's utility function, $W_i(T)$ is his portfolio's terminal wealth at time T and $\mathbb{E}_0^i$ denotes agent i's expectation of the outcome of the market at time T conditional on the information available and his belief at time $t = 0$. We assume that all agents are log-utility maximisers, that is, $U_i(x) = \ln(x)$ for all i. Stock price S follows a multi-period Cox-Ross-Rubinstein model. This means, given information at time t, the cum-price of the risky asset at time $t + 1$ has the following probability distribution,

$$S(t+1) = \begin{cases} S(t)\ \ u(t, t+1), & \text{with prob. } p(t, t+1); \\ S(t)\ \ d(t, t+1), & \text{with prob. } 1 - p(t, t+1) \end{cases}$$

with $d(t, t+1) < R_f(t) < u(t, t+1)$, where $R_f(t) = 1 + r_f(t)$ is the return of the riskless asset over the period $[t, t+1]$. Note that $p(t, t+1)$, $u(t, t+1)$ and $d(t, t+1)$ can vary with time. Agents' beliefs about future asset returns are formed in the following way. Let $\mathcal{B}_i := (\mathbf{p}_i, \mathbf{u}_i, \mathbf{d}_i)$ denote agent i's belief about the probability distribution of the future asset returns conditional on available information at time $t = 0$, in which $\mathbf{p}_i := (p_i(0), p_i(1), \cdots, p_i(T-1))^T$ where $p_i(t)$ denotes the value of $p(t, t+1)$ for $t = 0, 1, \cdots, T-1$ under agent i's belief given the information at time $t = 0$. Similarly, $\mathbf{u}_i := (u_i(0), u_i(1), \cdots, u_i(T-1))^T$ and $\mathbf{d}_i := (d_i(0), d_i(1), \cdots, d_i(T-1))^T$, where $u_i(t)$ and $d_i(t)$ denote the value of $u(t, t+1)$ and $d(t, t+1)$ for $t = 0, 1, \cdots, T-1$ under agent

i's belief given information at time $t = 0$. Essentially, agents are provided with the same information at time $t = 0$; however, each of them interprets the information differently and arrives at their subjective belief about the future distribution of asset returns. Let $\omega_i(t)$ be the proportion of investor i's wealth $W_i(t)$, at time t, invested in the risky asset and define the future return of the risky asset as

$$R(t+1) = \frac{S(t+1)}{S(t)}, \qquad r(t+1) = R(t+1) - 1,$$

which is random at time t. Then agent i's objective in equation (6.1) becomes

$$\max_{\{\omega_i(0),\omega_i(1),\cdots,\omega_i(T-1)\}} \ln(W_i(0)) + \sum_{t=0}^{T-1} \mathbb{E}_0^i \left[\ln \left(R_f(t) + \omega_i(t)(R(t+1) - R_f(t)) \right) \right]. \qquad (6.2)$$

The optimisation problem in (6.2) can be solved using dynamic programming or the martingale approach and detailed solution to the problem under both methods can be found in Cvitanic and Zapatero (2004) Chapter 4. To simplify the notations, we have suppressed the time indexes and all model parameters will correspond to the time period $[t, t+1]$ unless otherwise stated.

Lemma 6.1 *Let $\bar{u}_i = u_i(t, t+1) - R_f(t)$ and $\bar{d}_i = d_i(t, t+1) - R_f(t)$ be the excess rate of return in the up and down states, respectively, over the period $[t, t+1]$ under investor i's perspective. The solution to investor i's multi-period optimisation problem in equation (6.2) is given by*

$$\omega_i = R_f \frac{\bar{u}_i \;\; p_i + \bar{d}_i \;\; (1 - p_i)}{-\bar{u}_i \;\; \bar{d}_i} \qquad (6.3)$$

for $t = 0, 1, \cdots, T-1$.

Lemma 6.1 shows that agent i is able to determine the optimal proportion of his wealth to invest in the risky asset once the risk-free rate at time t is observed. Also the optimal proportion only depends on agent i's belief about the distribution of asset return in period $[t, t+1]$. The intuition is that maximising the logarithm of a portfolio's terminal wealth is equivalent to maximising the expected growth rate $\mathbb{E}[\ln(R_p(t+1))]$ period by period, where $R_p(t+1)$ is the portfolio's rate of return from t to $t+1$. This is called the short-sighted or *myopic behaviour* of logarithmic utility, because log-utility maximisers do not consider any future investment opportunities in their portfolio selections (see Cvitanic and Zapatero (2004), (chapter 4)).

6.3 Consensus Belief and Market Equilibrium

Since we have one unit of the risky asset available in the market and zero net supply for the riskless asset, to clear the market, the agents' total dollar demand for the risky asset must equal the aggregate market wealth at all times. This means the equilibrium price of the risky asset must equal the aggregate market wealth at all times, that is,

$$\sum_{i=1}^{I} \omega_i(t)\ \ W_i(t) = W_m(t) = S(t), \qquad t = 0, 1, \cdots, T-1, \tag{6.4}$$

where $W_m(t) = \sum_{i=1}^{I} W_i(t)$ denotes the aggregate market wealth at time t. We refer to equation (6.4) as the *market clearing condition* for our economy. Substituting equation (6.3) into the market clearing condition (6.4) leads to the following expression involving the equilibrium risk-free rate over the time period $[t, t+1]$,

$$\frac{1}{R_f} + \sum_i w_i \left(\frac{p_i}{\bar{d}_i} + \frac{1-p_i}{\bar{u}_i} \right) = 0, \tag{6.5}$$

where $w_i = \frac{W_i(t)}{W_m(t)}$ is the wealth share of investor i at time t. Equation (6.5) shows how to determine the equilibrium risk-free rate given beliefs of all the agents and their wealth share. Ideally, one would like to aggregate agents' heterogeneous beliefs and construct a *consensus belief* to determine the equilibrium risk-free rate. The aggregation of heterogeneous beliefs has been studied in Chiarella et al (2010*b*, 2010*a*) under a static mean-variance setting and Jouini and Napp (2006, 2007) in an intertemporal consumption setting. Following the same idea, we introduce a consensus belief for the CRR model with heterogeneous beliefs.

Definition 6.2 *A belief $\mathcal{B}_m := (\mathbf{p}_m, \mathbf{u}_m, \mathbf{d}_m)$, defined by the probability of an up move, and returns of the risky asset in the up and down states, respectively, in period $[t, t+1]$ for $t = 0, 1, \cdots, T-1$, is called a* **consensus belief** *if the asset price and the equilibrium risk-free rate R_f under the heterogeneous beliefs is also that under the homogeneous belief $\mathcal{B}_m$.*

The introduction of a consensus belief allows the transformation of a market with heterogeneous beliefs to a market under which all agents are identical in their beliefs. If the aggregate market invests as a sole log-utility maximiser, his belief at time $t = 0$ coincides with the consensus belief $\mathcal{B}_m$ and, through the market clearing condition, the risky asset must be the growth optimal portfolio. Intuitively, this is the most informed belief about the future because it takes into account every investor's subjective belief at time $t = 0$

regarding the distribution of future asset returns over the entire period $[0, T]$. The following Proposition 6.3 provides an implicit formula to compute the consensus belief and shows that the risk-free rate and the consensus belief can be determined simultaneously.

Proposition 6.3

(i) The consensus belief $\mathcal{B}_m := (\mathbf{p}_m, \mathbf{u}_m, \mathbf{d}_m)$ *is given by*

$$\begin{aligned}\mathbf{p}_m &:= (p_m(0), p_m(1), \cdots, p_m(T-1))^T,\\ \mathbf{u}_m &:= (u_m(0), u_m(1), \cdots, u_m(T-1))^T,\\ \mathbf{d}_m &:= (d_m(0), d_m(1), \cdots, d_m(T-1))^T,\end{aligned}$$

where in the time interval $[t, t+1]$ *for* $t = 0, 1, \cdots, T-1$,

$$p_m = \sum_i w_i \ p_i, \qquad u_m = \bar{u}_m + R_f, \qquad d_m = \bar{d}_m + R_f, \tag{6.6}$$

and

$$\bar{u}_m = \left(\sum_{i=1}^{I} w_i \frac{1-p_i}{1-p_m} \ \bar{u}_i^{-1} \right)^{-1}, \tag{6.7}$$

$$\bar{d}_m = \left(\sum_{i=1}^{I} w_i \frac{p_i}{p_m} \ \bar{d}_i^{-1} \right)^{-1}. \tag{6.8}$$

(ii) The equilibrium risk-free rate is given by

$$\frac{1}{R_f} = \frac{1-p_m}{d_m} + \frac{p_m}{u_m} = \mathbb{E}_t^m \left[\frac{1}{R(t+1)} \right]. \tag{6.9}$$

(iii) State prices or the risk-neutral probabilities of the up and down states at time t *under individual subjective belief* $\mathcal{B}_i$ *and the consensus belief* $\mathcal{B}_m$ *are given by*

$$q_{i,u}(t) = \frac{-\bar{d}_i}{\bar{u}_i - \bar{d}_i}, \qquad q_{i,d}(t) = \frac{\bar{u}_i}{\bar{u}_i - \bar{d}_i}, \qquad i = 1, 2, \cdots, I, m. \tag{6.10}$$

(iv) In equilibrium, given information at time t, *the stock price at time* t *can be written as*

$$S(t) = \frac{\mathbb{E}_t^{Q_i}(S(t+1))}{R_f} = \frac{\mathbb{E}_t^i(Z \ S(t+1))}{R_f}, \qquad i = 1, 2, \cdots, I, m, \tag{6.11}$$

where Q_i is the equivalent risk-neutral measure under investor i's belief and

$$Z = \begin{cases} \frac{q_u(t)}{p(t,t+1)}, & p(t,t+1); \\ \frac{q_d(t)}{1-p(t,t+1)}, & 1-p(t,t+1) \end{cases} \tag{6.12}$$

is the Randon-Nikodym derivative that changes the probability measure from i to Q_i, often referred to as the "pricing kernel" in the asset-pricing literatures.

Proposition 6.3 shows that the consensus belief of the probability of an up move p_m is simply an arithmetic average of individual probability beliefs p_i weighted by their wealth shares w_i. This means that a wealthier investor has a stronger impact on p_m. The consensus belief of the excess return in each state $(\bar{u}_m, \bar{d}_m)$ is a harmonic mean of individual beliefs of the excess returns $(\bar{u}_i, \bar{d}_i)$ weighted by both wealth shares and probabilities. A wealthier and relatively pessimistic (optimistic) investor (in terms of return probability) would have a stronger impact on $\bar{u}_m$ $(\bar{d}_m)$. Furthermore, the consensus belief $\mathcal{B}_m$ and the risk-free rate at time t can only be determined simultaneously. Equation (6.9) indicating the relationship between the consensus belief and the risk-free rate implies that the quantity $R_f(t)/R(t+1)$ is a martingale under the consensus belief. Agents in this economy perceive different state prices, as indicated by (6.10), hence option prices implied by each agent's belief are also different. Equation (6.11) shows that agents agree on the current observed asset price though they may have their distinctive pricing kernels due to their different beliefs.

As each agent has his own set of state prices, the price of a contingent claim would differ under each agent's subjective belief $\mathcal{B}_i$, $i = 1, 2, \cdots, I$ and under the consensus belief $\mathcal{B}_m$. The question is, which belief should one use for pricing contingent claims[1]? To answer this question we introduce a concept of a "fair" belief in our economy.

Definition 6.4 *A belief $\mathcal{B}^* := (\mathbf{p}^*, \mathbf{u}^*, \mathbf{d}^*)$ given information at time $t = 0$ is called* fair *if and only if the wealth share of agent i $(i = 1, 2, \cdots, I)$ is a martingale under the belief $\mathcal{B}^*$, that is*

$$\mathbb{E}_t^*[w_i(t+1)] = w_i(t)$$

for $t = 0, 1, \cdots, T-1$.

The idea behind Definition 6.4 is that under a fair belief, agent i's wealth share is expected to remain at its current level from time t to terminal time T for all agents. It

[1] Any contingent claims other than the stock or the bond are redundant securities in our economy since the market is complete in the sense that each agent can construct their optimal portfolio by investing in the risk-free asset and the risky asset only.

is easy to see under the law of iterated expectations that $\mathbb{E}_t^*[w_i(T)/w_i(t)] = 1$. The belief $\mathcal{B}^*$ is fair in the sense that every agent will on average perform equally under this belief. In the following, we show that the consensus belief is a fair belief.

Proposition 6.5 *The consensus belief $\mathcal{B}_m$ is a fair belief, that is*

$$\mathbb{E}_t^m[w_i(t+1)] = w_i(t).$$

Proposition 6.5 shows that the consensus belief is a fair belief under which any agent i's wealth share is expected to remain the same in the next period. This is actually a very intuitive result since the consensus belief consists of beliefs from every agent, therefore, in equilibrium, the aggregate market does not expect anyone's future wealth share to be more than their current wealth share levels. Next, we show that each agent i's subjective belief $\mathcal{B}_i$ is not a fair belief in general.

Proposition 6.6 *Agent i's subjective belief $\mathcal{B}_i$ is fair if and only if*

$$\mathbb{E}_t^i[\frac{1}{R(t+1)}] = \mathbb{E}_t^m[\frac{1}{R(t+1)}] = \frac{1}{R_f}. \tag{6.13}$$

The expected wealth share of agent i satisfies

$$\mathbb{E}_t^i[w_i(t+1)] \geq w_i(t)$$

for $t = 0, 1, \cdots, T-1$. Equality will hold if and only if equation (6.13) *holds.*

Proposition 6.6 shows that agent i's subjective belief is fair if and only if the discounted value of a dollar payoff by the stock under his expectation is the same as the zero-coupon bond price. Using the law of iterated expectation, one can see that $\mathbb{E}_t^i[w_i(t+1)] \geq w_i(t)$. This indicates agent i expects his market share to grow. The above analysis indicates that the consensus belief $\mathcal{B}_m$ is a fair belief to price contingent claims in our economy and agent i's subjective belief is not fair unless the condition in Proposition 6.6 holds.

6.4 Impact of Mean-Preserving Heterogeneous Beliefs

In this section, we examine the impact of mean-preserving heterogeneous beliefs on the market consensus belief and the equilibrium risk-free rate. It is often believed that the effect of belief disagreements should cancel out if the disagreements are symmetric about the average belief. In our setting, the consensus belief $\mathcal{B}_m$ rather than the average belief[2]

[2] The average belief is defined by $\bar{\mathcal{B}} := (\frac{1}{I}\sum_{i=1}^{I}\mathbf{p}_i, \frac{1}{I}\sum_{i=1}^{I}\mathbf{u}_i, \frac{1}{I}\sum_{i=1}^{I}\mathbf{d}_i)$

$\bar{\mathcal{B}}$ in the market corresponds to the belief of the "average" agent. We aim to determine how the quantities such as the risk-free rate and risk premium differ under the consensus belief $\mathcal{B}_m$ and the average belief $\bar{\mathcal{B}}$ for an increasing level of divergence of opinions.

We first consider a static setting where agents' wealth shares are equal and their beliefs about the asset return are uniformly distributed for both the upstate and downstate. We focus on a single time period $[t, t+1]$ where information at time t is known.

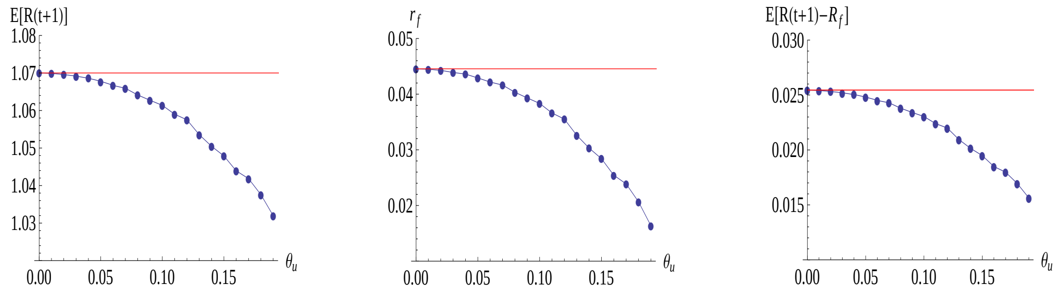
Corollary 6.7 *Let there be I investors with $w_i(t) = 1/I$ and $p_i = p$ for all i. Consider a benchmark belief $(u_o(t), d_o(t))$, and assume that investors' subjective beliefs diverge from the benchmark homogeneous belief uniformly. Let agent i's belief be given by $(u_i(t), d_i(t)) = (u_o(t) + \tilde{\epsilon}_{iu}, d_o(t) + \tilde{\epsilon}_{id})$ where $\tilde{\epsilon}_{iu}$ and $\tilde{\epsilon}_{id}$ are both i.i.d for agent i with mean zero and a bounded variance. Therefore agents' divergence of opinions regarding the stock returns in both up and down states are i.i.d. As the number of agents approaches infinity, from* (6.7) *and* (6.8) *the consensus belief can be expressed implicitly as*

$$\bar{u}_m = \left(\mathbb{E}[(u_o - R_f + \tilde{\epsilon}_u)^{-1}]\right)^{-1}, \qquad \bar{d}_m = \left(\mathbb{E}[(d_o - R_f + \tilde{\epsilon}_d)^{-1}]\right)^{-1}, \tag{6.14}$$

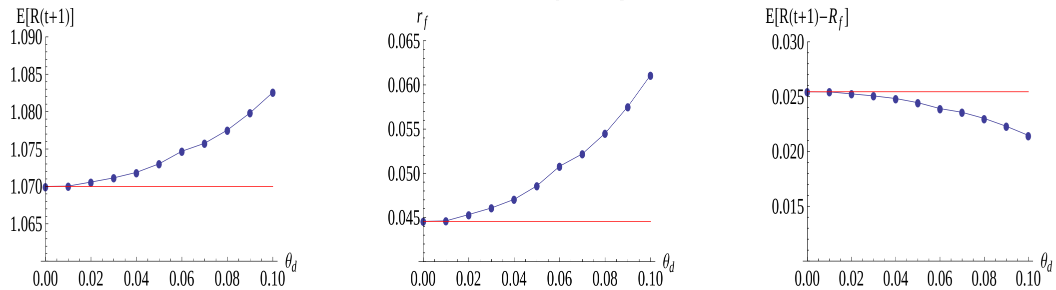
where $\tilde{\epsilon}_u$ and $\tilde{\epsilon}_d$ have the same distribution as $\tilde{\epsilon}_{iu}$ and $\tilde{\epsilon}_{id}$ respectively.

In Corollary 6.7, it is clear that agents have heterogeneous beliefs regarding the future return of the risky asset with average belief $\bar{\mathcal{B}}$ equal to the benchmark belief $\mathcal{B}_o$. Now the question is whether this form of divergence of opinions has a significant effect on the market consensus belief of asset return and the equilibrium risk-free rate. To answer this question, we let $(u_o, d_o) = (1.235, 0.905)$, $p = 0.5$ and $I = 5,000$, which means that all the agents agree on the fact that up and down states are equally likely to occur and each agent's wealth share $w_i = 0.0002$ for all i. Moreover, we assume that $\tilde{\epsilon}_u \sim Unif(-\theta_u, \theta_u)$ and $\tilde{\epsilon}_d \sim Unif(-\theta_d, \theta_d)$. Next, we approximate the consensus belief $\mathcal{B}_m$ and the risk-free rate r_f using Monte-Carlo simulations with various combinations of the parameters θ_u and θ_d. Figure 6.1 compares the consensus belief, risk-free rate and the risk premium with the benchmark for different combinations of (θ_u, θ_d). In the special case of no divergence of opinions, that is when $(\theta_u = 0, \theta_d = 0)$, the consensus belief $\mathcal{B}_m$ is the same as the benchmark belief $\mathcal{B}_o$, the risk-free rate under the benchmark belief is 0.045 and the risk premium equals 0.025.

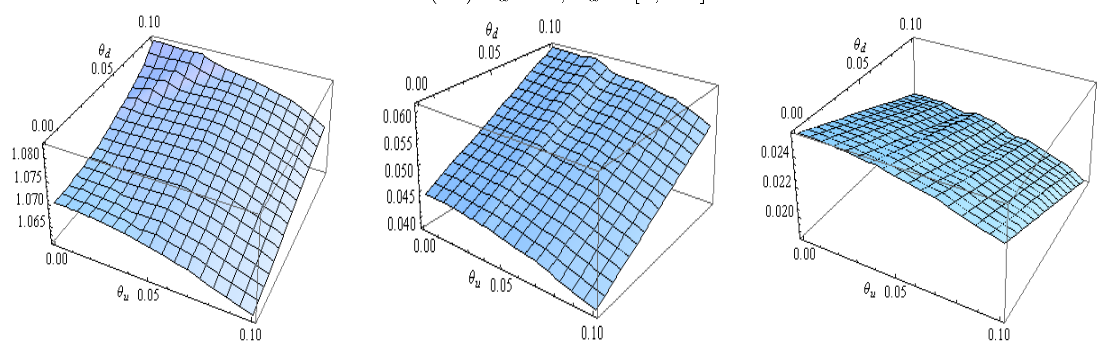
Figure 6.1 (a1) shows that the consensus belief of expected stock return decreases as the level of divergence of opinion in the upstate θ_u increases. This is because the agents who are pessimistic regarding future stock return in the upstate have a larger impact on the consensus belief in the upstate, therefore u_m decreases when θ_u increases, though



(a1) $\theta_u \in [0, 0.2]$, $\theta_d = 0$



(a2) $\theta_u = 0$, $\theta_d \in [0, 0.1]$



(a3) $\theta_u \in [0, 0.1]$, $\theta_d \in [0, 0.1]$

Figure 6.1: Impact of divergence of opinion on the consensus belief $\mathcal{B}_m = (u_m, d_m)$, expected stock return $\mathbb{E}[R(t+1)] = p\ \ u_m + (1-p)\ \ d_m$, risk-free rate $r_f = R_f - 1$ and the risk premium $\mathbb{E}[R(t+1) - R_f]$.

agents' beliefs u_i is a mean-preserving spread of the benchmark. The risk-free decreases from its benchmark value because the aggregate market is less willing to invest in the risky asset and more willing to invest in the risk-free bond. The fact that the risk premium $\mathbb{E}[R(t+1) - R_f]$ is negatively related to the divergence of opinions suggests that the reduction in the risk-free rate is less than that in the expected stock return under the consensus belief. Figure 6.1 (a2) shows that the consensus belief about stock return in the downstate d_m increases, as the level of divergence of opinion θ_d increases. This is because the optimistic agents who perceives a higher stock return in the downstate have a larger impact on the consensus belief in the downstate, therefore d_m increases though agents' beliefs d_i is a mean-preserving spread of the benchmark. The risk-free rate increases from its benchmark value because the aggregate market is more willing to invest in the risky asset and less willing to invest in the risk-free bond. The fact that the risk premium $\mathbb{E}[R(t+1) - R_f]$ is negatively related to the divergence of opinions suggests that the increase in expected stock return is not high enough compared to the increase in the risk-free rate. Finally, when we combine the divergence opinions in the upstate and downstate, Figure 6.1 (a3) shows the combined effect of θ_u and θ_d on the expected stock return, the risk-free rate and the risk premium. It is clear that θ_u and θ_d have an opposite effect on the expected stock return and risk-free rate; however, they both have a negative effect on the risk premium. We conclude this section with the following remark.

Remark 6.8 *Assuming investors' wealth shares are evenly distributed, then the consensus agent believes that divergence of opinion regarding future asset return in the upstate (downstate) is negatively related to both expected future stock return and the equilibrium risk-free rate. Higher divergence of opinions leads to a lower risk premium under the consensus belief in both the upstate and downstate.*

6.5 Option Pricing under Heterogeneous Beliefs

In this section, we focus on the pricing of options. As discussed in the previous section, agents with different beliefs disagree on the state prices, therefore agents price options differently. However, we show that the *fair* price of an option can be computed under the fair belief, the consensus belief $\mathcal{B}_m$.

Proposition 6.9 *Given information at time t, the fair price of an option V(t,S(t)) with payoff function $H(T, S(T))$ is given by*

$$V(t, S(t)) = S(t)\ \ \mathbb{E}_t^m(H(T, S(T))/S(T)).$$

In the following example, we will use the pricing formula developed in Proposition 6.9 to price European call options with different strikes written on the risky asset.

Example 6.10 *Assume there are two agents ($i = 1, 2$) whose beliefs $\mathcal{B}_i$ are characterised by*

$$\begin{aligned}
u_i(t) &= 1 + \mu_i\ \ \Delta + \sigma_i\ \ \sqrt{\Delta},\\
d_i(t) &= 1 + \mu_i\ \ \Delta - \sigma_i\ \ \sqrt{\Delta},\\
p_i(t) &= 0.5,\\
w_i(t) &= 0.5,
\end{aligned}$$

where $t = 0, 1, \cdots, T-1$, $\Delta = (T-t)/n$ *and* n *is the number of trading periods from time* 0 *to* T.

In Example 6.10, both agents agree that the stock price is always equally likely to move up or down and the relative price changes are constant from time 0 to T. However, they disagree on the size of the relative price changes, which are characterised by the parameters μ_i and σ_i. These two parameters can be interpreted as agent i's belief about the expected return and volatility of the stock per annum, compounded n periods a year. Clearly, as the number of trading periods n approaches infinity, the stock-price dynamics under agent i's belief converges weakly to the stochastic differential equation (SDE) (see Kloeden and Platen (1992) and Nelson and Ramaswamy (1990))

$$dS(t)/S(t) = \mu_i \ dt + \sigma_i \ dW(t), \tag{6.15}$$

where $W(t)$ is the Wiener process.

Remark 6.11 *In Example 6.10, when* $\mu_i = \mu$ *and* $\sigma_i = \sigma$ *for* $i = 1, 2$ *and* n *approaches infinity, the price of a call option on the risky asset is given by the Black-Scholes formula and the instantaneous risk-free rate is given by* $\mu - \sigma^2$.

The proof for Remark 6.11 is in Appendix A.27. It indicates that as the number of trading periods $n \to \infty$, the risk-free rate is constant and depends on expected return and volatility of the stock. Furthermore, call option prices are given by the Black-Scholes formula using $r_f = \mu - \sigma^2$. Next we consider the case where agents differ in their beliefs about the growth rate and the volatility of the stock. More specifically, we will assume that $(\mu_1, \mu_2) = (\mu_o + \delta_\mu, \mu_o - \delta_\mu)$ and $(\sigma_1, \sigma_2) = (\sigma_o + \delta_\sigma, \sigma_o - \delta_\sigma)$ such that μ_o and σ_o are the arithmetic average beliefs of the expected return and the volatility of future stock returns, respectively. Note that $\delta_\mu > 0$ indicates that agent 1 is relatively more optimistic than agent 2 since he believes in a higher expected return, while $\delta_\sigma > 0$ indicates that agent 1 is less confident than agent 2 in the sense that he believes in a higher volatility. We obtain the distribution of the log stock price under the consensus belief at time T and compute call option prices via Monte Carlo simulation using Proposition 6.9. The benchmark belief is given by $(\mu_o, \sigma_o) = (0.07, 0.225)$. Time to maturity is $T = 0.25$ and the time increment is set to $\Delta = 0.00025$. We use Monte Carlo simulation for the evaluation of the option prices because under the consensus belief $\mathcal{B}_m$, the binomial tree for future stock prices is non-recombining[3] but it is recombining under both agents' beliefs.

[3]Non-recombining means that an up move followed by a down move is not the same as a down move followed by an up move. For a non-recombining tree, there are 2^{1000} possible values for the stock price after 1000 steps.

Furthermore, it is computationally very expensive to generate a path for the stock price, because for each path in this example, we need to numerically solve for the risk-free rate at each step. Therefore, we use the Black-Scholes option price as a control variate to reduce the variance of the simulated option payoff. This is because the option payoff under the Black-Scholes model at time T is strongly correlated with the one under the consensus belief, correlation is estimated to be close to 1. Using this technique, we are able to reduce the standard deviation associated with the option payoff by up to ten times. After obtaining the call option prices under the consensus belief; we then calculate the implied volatilities for these prices such that

$$C_m(K) = BS(S(0), \sigma_{imp}, r_f, T),$$

where K is the strike price, $S(0)$ is the current stock price, T is the time to maturity and r_f is the current risk-free rate. $C_m(K)$ is the fair price of the call option with strike K.

Figure 6.2 shows the distribution of the log stock price at maturity ($\ln(S_T)$) under agent 1 and 2's subjective beliefs ($\mathcal{B}_1$ and $\mathcal{B}_2$) and under the consensus belief ($\mathcal{B}_m$), Table 6.1 shows the sample statistics and Figure 6.3 shows the implied volatilities for different strike prices. If we normalise the current stock price to 1, then $\ln(S_T)$ measures the continuous return in the period $[0, T]$, it is normally distributed with mean $(\mu_i - \frac{1}{2}\sigma_i^2)T$ and standard deviation $\sigma_i\sqrt{T}$ under the subjective belief of agent i. If we interpret the expected log price as the growth rate of the stock, then agents can agree on the expected stock return but perceive different growth rates. Fig. 6.2 (b1) and Tab. 6.1 (c1) demonstrate the case when agents agree on the volatility but disagree on the expected stock return. In this case, the growth rate under the consensus belief $\mathcal{B}_m$ is between that of both agents but closer to that of agent 1 (the more optimistic agent); the volatility is close to the common belief. The distribution of $\ln(S_T)$ is approximately normal under $\mathcal{B}_m$ since skewness is close to zero and kurtosis close to 3. Fig. 6.3 (d1) shows that the implied volatility is almost flat with respect to the strike, which means that the call prices are consistent with the BS formula. Fig. 6.2 (b2) and Tab. 6.1 (c2) show that when agents agree on the expected return but disagree on the volatility, $\ln(S_T)$ becomes negatively skewed under the consensus belief $\mathcal{B}_m$, and the growth rate and volatility under $\mathcal{B}_m$ are closer to agent 2's belief, who is the more confident agent. Furthermore, the market perceives a higher growth rate than both agents. Fig. 6.3 (d2) shows that the implied volatility exhibits a positive skewness consistent with the observed pattern in option markets. The intuition is that since agent 2 perceives a higher growth rate, he has a larger wealth share and

	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_m$
mean	0.024	-0.0001	0.016
std	0.1125	0.1125	0.1099
skew	0	0	0.003
kurt	3	3	2.874

(c1) $(\delta_\mu, \delta_\sigma) = (0.05, 0)$

	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_m$
mean	0.006	0.015	0.017
std	0.15	0.075	0.095
skew	0	0	-0.138
kurt	3	3	3.326

(c2) $(\delta_\mu, \delta_\sigma) = (0, 0.075)$

	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_m$
mean	0.027	-0.00625	0.019
std	0.075	0.15	0.098
skew	0	0	-0.159
kurt	3	3	3.119

(c3) $(\delta_\mu, \delta_\sigma) = (0.05, -0.075)$

	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_m$
mean	0.01875	0.002	0.003
std	0.15	0.075	0.105
skew	0	0	0.100
kurt	3	3	2.976

(c4) $(\delta_\mu, \delta_\sigma) = (0.05, 0.075)$

Table 6.1: Impact of heterogeneous beliefs in the growth rate (μ) and the volatility (σ) of future stock returns on the distribution of the log stock price at maturity ($\ln(S_T)$) for $T = 0.25$. The first and second columns in each table provide the first 4 moments of the distribution of $\ln(S_T)$ under the subjective beliefs of agents 1 and 2, respectively, and the third column of each table corresponds to that under the consensus belief $\mathcal{B}_m$.

dominates the consensus belief in the upper part of the binomial lattice, which matters more for pricing out of the money (OTM) call options (calls with strikes above the current spot price). This means that the OTM call prices would reflect more of agent 2's belief about stock volatility. Since agent 2 perceives a lower stock volatility, OTM call prices have a lower implied volatility than at the money (ATM) and in the money (ITM) call prices. As one moves gradually towards the lower part of the tree, agent 1's belief becomes more and more dominant in determining the consensus belief, and therefore the implied volatility also increases as the strike prices decreases. Fig. 6.2 (b3) and Tab. 6.1 (c3) show that when agent 1 is more optimistic and confident, the consensus belief of the growth rate and volatility of the stock is closer to those under agent 1's belief. The distribution of $\ln(S_t)$ is negatively skewed and Fig. 6.3 (d3) indicates that the call prices exhibit volatility skew. The intuition is similar to the previous case. Fig. 6.2 (b4) and Tab. 6.1 (c4) show that when agent 1 is more optimistic but less confident about future stock returns, the growth rate and volatility under consensus belief are closer to those under agent 2's belief, who is less optimistic and more confident. The distribution of $\ln(S_T)$ is positively skewed and Fig. 6.3 (d4) indicates the OTM call prices have higher implied volatilities than ATM and ITM call prices. The intuition is that agent 1, who is more optimistic, dominates the consensus belief in the upper part of the tree, as previously discussed. However, agent 1 perceives a larger volatility than agent 2, and therefore implied volatility is positively related to the strike price.

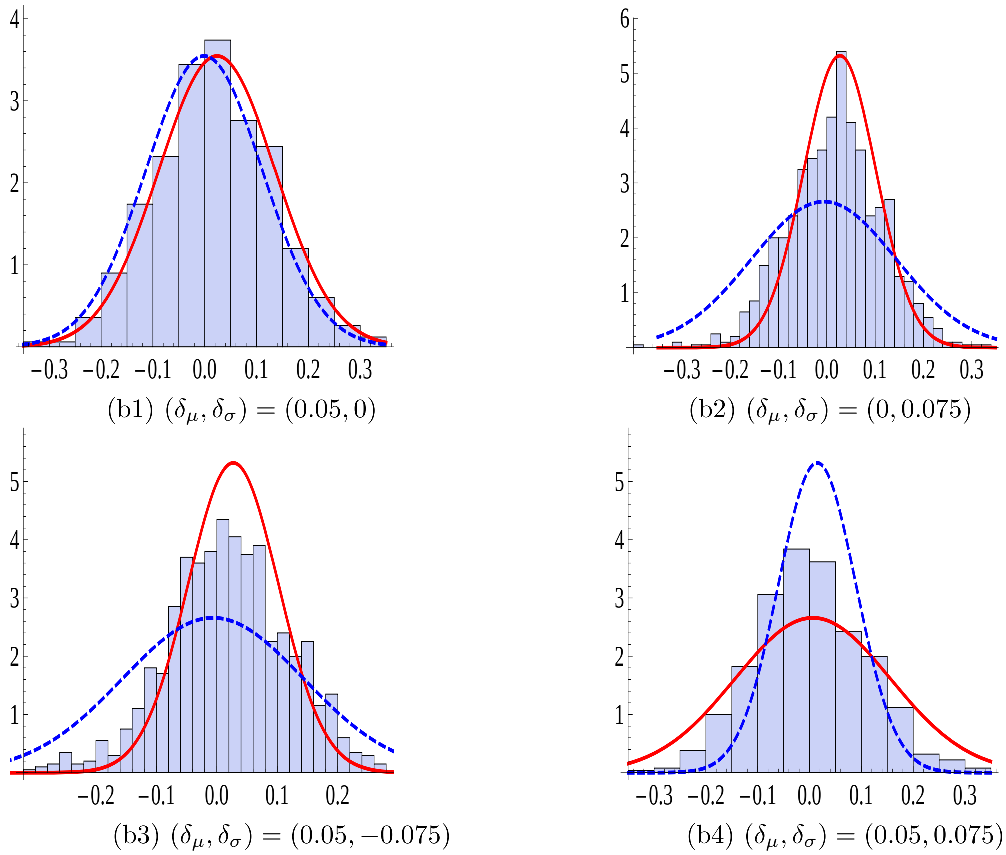
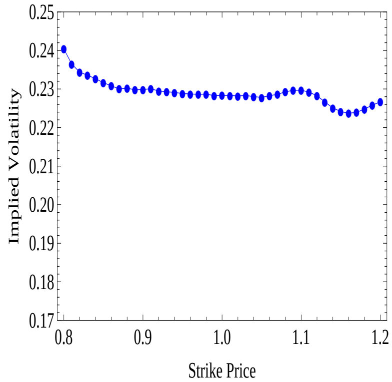
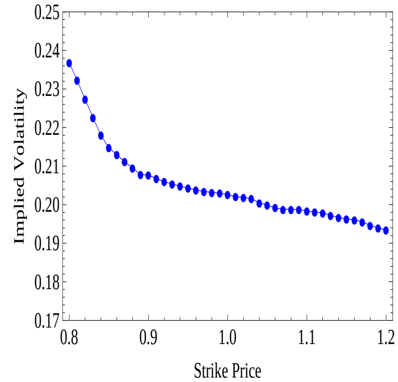


Figure 6.2: Impact of heterogeneous beliefs on the growth rate (μ) and the volatility (σ) of future stock returns on the distribution of the log stock price at maturity ($\ln(S_T)$) for $T = 0.25$. The solid and dashed lines represent the perceived distribution of $\ln(S_T)$ by agents 1 and 2, respectively, and the histogram represents the distribution of $\ln(S_T)$ under the consensus belief $\mathcal{B}_m$.

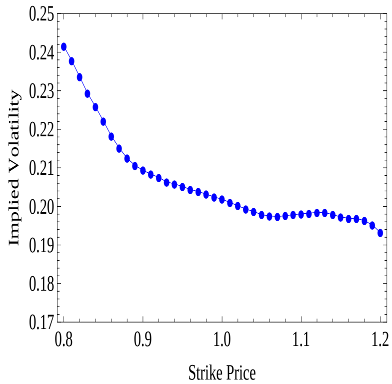
In order to examine the term structure of the implied volatilities, we calculate the implied volatilities from the fair prices of call option with time to maturity $T \in [0.25, 1.00]$ for $\delta_\mu = 0.05$ and $\delta_\sigma = -0.075$. Figure 6.4 shows that the implied volatility surface flattens out as time to maturity increases, consistent with the observed patterns from market data. This indicates that our fair option prices can mimic important features exhibited by the market data of option prices, but Example 6.10 is only a very simple specification of our model. We expect the volatility surface generated from fair option prices under the consensus belief to exhibit even richer patterns with a more general specification.



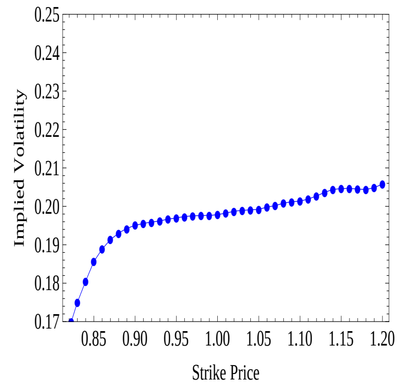
(d1) $(\delta_\mu, \delta_\sigma) = (0.05, 0)$



(d2) $(\delta_\mu, \delta_\sigma) = (0, 0.075)$



(d3) $(\delta_\mu, \delta_\sigma) = (0.05, -0.075)$



(d4) $(\delta_\mu, \delta_\sigma) = (0.05, 0.075)$

Figure 6.3: Impact of heterogeneous beliefs in the growth rate (μ) and the volatility (σ) of future stock returns on the implied volatilities of fair call prices.

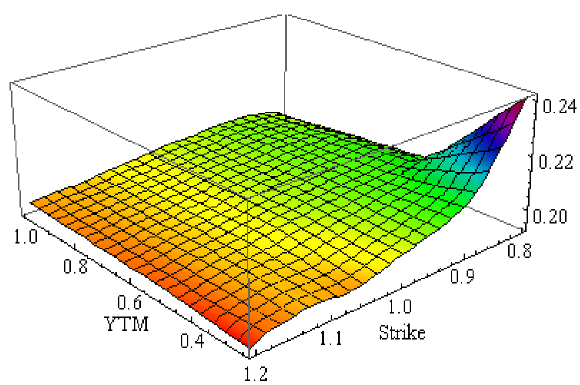


Figure 6.4: The implied volatilities calculated with fair prices of a call option with time to maturity $T \in [0.25, 1.00]$. The current stock price is $S(t) = 1$, the difference in agents' belief is characterized by $\delta_\mu = 0.05$ and $\delta_\sigma = -0.075$.

Example 6.10 can be generalised to take into account other popular stochastic processes for modelling stock-price dynamics. In general, if the belief of agent i is characterised by

$$
\begin{aligned}
u_i(t) &= 1 + \mu_i(t, Y_t)\ \ \Delta + \sigma_i(t, Y_t)\ \ \sqrt{\Delta}, \\
d_i(t) &= 1 + \mu_i(t, Y_t)\ \ \Delta - \sigma_i(t, Y_t)\ \ \sqrt{\Delta}, \\
p_i(t) &= 0.5,
\end{aligned}
$$

where $t = 0, 1, \cdots, T-1$, Y_t is the stock price at time t and $\Delta = T/n$ is the time increment, then as n approaches infinity, the above characterization implies that agent i believes that the stock-price dynamics are described by the SDE,

$$dS(t)/S(t) = \mu_i(t, S(t))\ dt + \sigma_i(t, S(t))\ dW(t).$$

Therefore, in principle our option-pricing formula can take into account not only disagreement in model parameters, but also differences in the model structure. This is an advantage, the number of option-pricing models has exploded with the recent advances in mathematical finance, so it is reasonable to assume that quantitative analysts can use quite distinctive models for modelling stock-price dynamics. Furthermore, we can also accommodate currently observed values of other explanatory variables of stock returns. If we denote $\boldsymbol{\phi}(t)$ as the value of a set of explanatory variables at time t, then agent i's belief can be written as

$$\begin{aligned} u_i(t) &= 1 + \mu_i(t, Y_t, \boldsymbol{\phi}_t)\ \Delta + \sigma_i(t, Y_t, \boldsymbol{\phi}_t)\ \sqrt{\Delta}, \\ d_i(t) &= 1 + \mu_i(t, Y_t, \boldsymbol{\phi}_t)\ \Delta - \sigma_i(t, Y_t, \boldsymbol{\phi}_t)\ \sqrt{\Delta}, \\ p_i(t) &= 0.5, \end{aligned}$$

which weakly converges to the SDE as $\Delta \to 0$ given by

$$dS(t)/S(t) = \mu_i(t, S(t), \boldsymbol{\phi}(t))\ dt + \sigma_i(t, S(t), \boldsymbol{\phi}(t))\ dW(t).$$

6.6 Conclusion

In this chapter, we provide an aggregation method of heterogeneous beliefs within a multi-period binomial lattice framework. The heterogeneity is characterised by the differences in agents' beliefs about the probability of an up move in each period and also the relative size of the price changes in each period. Agents are bounded rational in the sense that they invest in the growth-optimal portfolio based on their own subjective belief. To analyse the impact of heterogeneity, we introduce the concept of a consensus belief, which relates our heterogeneous market to an equivalent homogeneous market. The consensus belief is basically a wealth weighted average of agents' subjective beliefs and can be determined simultaneously with the risk-free rate. Through various numerical examples, we examine the impact of heterogenous beliefs on the equilibrium risk-free rate, the equity risk premium and option prices. By static analysis, given that agents' wealth shares are equal and agree on the probability, the market expects a lower (higher) future return when the divergence of opinions is greater regarding future stock return in upstate (downstate), the risk-free rate is negatively (positively) related to the divergence of opinions about future return in the upstate (downstate) and the risk premium is negatively related to the divergence of opinions in general. Dynamically, we found that the consensus belief is a fair belief in that every agent's wealth share is a martingale under the consensus belief. Agents' subjective beliefs are not fair unless their perceived present value of a dollar payoff by the stock in the next period is equal to the zero-coupon bond price. Also, agent i's wealth share process is a sub-martingale under his own subjective belief and expected to grow in the future. Options prices calculated under the consensus belief are called fair prices. We demonstrate that the implied volatilities of fair call prices can exhibit "volatility skew" observed in real data when agents disagree only on the volatility of future stock returns or when the optimistic agent is also confident about future stock returns. Finally, our binomial model can take into account agents' disagreements in both model parameters and the model structure.

Chapter 7

Conclusion and Future Research

Most popular models in finance, including the CAPM, consumption-based asset-pricing models and the Black-Scholes option-pricing model, rely on the assumption that investors have homogeneous expectations. Based on this assumption, investors choose optimal portfolios that lies on the efficient portfolio frontier or the capital market line. The cross-section of expected asset returns is explained by the covariance between future asset and market returns or aggregate consumption and the option prices are obtained by the commonly understood stochastic processes driving the price of the underlying asset. However, these models cannot explain the relationship between the divergence of opinions and expected asset returns, under-performances of certain managed funds, the level of interest rate and the equity premium in the economy, nor can they explain the positive skew in the implied volatility of option prices. In reality, investors may have different beliefs about future prospects for the economy either in terms of the joint probability distribution of asset returns or the aggregate endowment of the economy. This thesis analyses the impact of different aspects and combinations of heterogeneity on market equilibrium. We use two types of framework – the mean-variance asset pricing framework and the Arrow-Debreu general equilibrium pricing framework. The thesis consists of three parts. Firstly, we analyze the impact of heterogeneity in both investors' risk-aversion and beliefs on the relationship between divergence of opinions and expected stock returns, performance of investors' subjectively optimal portfolio and the geometric relation between portfolio frontiers with and without a riskless asset. Secondly, we studied the effect of investors' heterogeneity in risk tolerance and beliefs at the micro level on endogenous variables at the aggregate level, in particular the risk-free rate and the market risk premium. We studied this first under a static multi-asset model, then under a continuous-time general dynamic equilibrium model, with agents maximising the utility of their relative consumption to each other. The thesis finished with a binomial model of option pricing where agents' beliefs are heterogeneous in the return of the risky asset in the up and downstates and the probability of an up move. The main contributions of the three parts and related future work are summarised in the following.

7.1 Portfolio Analysis under Heterogeneous Beliefs

The geometric tangency relation and the two-fund separation theorem are the foundations underlying the *Capital Asset Pricing Model*. When investors have homogeneous beliefs regarding the joint probability distribution of future asset returns, they hold the same portfolio of risky assets in equilibrium, which is the tangency portfolio of the portfolio frontiers with and without a riskless asset. Furthermore, all investors' optimal portfolios lie on the efficient portfolio frontier, which means that they have the same mean-variance (MV) efficiency. Sharpe (1970) admits that the most casual observation would suggest that investors may disagree about the future return of risky assets, but he argues that on average, the effect from investors' heterogeneity should be cancelled out and the market should behave as if every investor has the same average belief. Chapters 2 and 3 examine the extent to which Sharpe's statement is true. In particular, by constructing a consensus belief we examine the impact of different forms of heterogeneity on the MV efficiency of the optimal portfolios, the tangency relation, and the market equilibrium in general. In Chapter 2, we assume that the heterogeneous beliefs are formed in asset payoffs. We find that divergence in beliefs can have a negative impact on future asset return, which is consistent with Miller's hypothesis, if the optimistic agent is less risk averse and the expected asset return is greater than the risk-free rate. Moreover, when there is not a riskless asset, the expected return needs to be less than the zero-beta rate adjusted by the asset's market capitalisation. In Chapter 3, we assume that investors form their beliefs about the rates of return of the risky assets and focus on the case with a continuum of investors. We find that when investors' beliefs are uncorrelated with their risk tolerance, the impact of heterogeneity is cancelled out and the heterogeneity has no impact on market equilibrium. However, the MV efficiency of investors' optimal portfolios decreases with increasing dispersion in beliefs. The geometric tangency relation holds and the market portfolio is invariant to riskless borrowing/lending if beliefs in the variance/covariances of asset returns are homogeneous. When beliefs in the variance/covariances are heterogeneous, introducing a risk-free asset can improve the MV efficiency and marginal utility of the market if the more confident investors are also more optimistic about future asset returns. For both setups in Chapters 2 and 3, we extend the standard zero-beta

CAPM under homogeneous beliefs to the one under heterogeneous beliefs, which provides a framework for further analysis in Chapter 4.

The mean-variance framework used in Chiarella et al. (2010*a*, 2010*b*) and Chapters 2 and 3 focuses on a static economy. The implication of the heterogeneity on the market under different market conditions is far more complicated than it seems and deserves further study. It would be interesting to extend the current static framework to a dynamic setting so that the intertemporal effect can be examined. Li and Ng (2000) and Basak and Chabakauri (2010) look at the dynamic optimisation problem of an individual investor and present explicit solutions of the optimal portfolio. An extension of these results from homogeneous beliefs to heterogeneous beliefs would be interesting and challenging. It is also interesting to allow investors to learn over time from the market through various learning mechanisms, such as the Bayesian updating rule and the adaptive learning mechanism, so that the expectation feedback effect can be examined. These extensions will give us a richer modelling environment and hopefully lead to a better understanding of the phenomena in our financial market.

7.2 Equilibrium Asset Pricing under Heterogeneous Beliefs

The equity premium puzzle by Mehra and Prescott (1985) and the risk-free rate puzzle by Weil (1989) motivated a strand of literature in asset pricing with heterogeneous beliefs, analysing the relationship between heterogeneity in agents' beliefs and the equilibrium asset prices. The question of interest is which combination of heterogeneity can generate a higher market risk premium and lower interest rate than the homogeneous benchmark case. In a discrete time Arrow-Debreu economy, Jouini and Napp (2006) show that a positive correlation between an agent's risk tolerance and pessimism/doubt gives a desirable results and can help to resolve the puzzles. In Chapter 4, we argue that agents may not be able to distinguish between different states of the world, but rather have heterogeneous beliefs about the joint distribution of future asset returns. Within the mean-variance asset pricing framework, similar to the one in Chiarella et al. (2010*a*), with two risky assets and two agents, we examine explicitly the impact of different combinations of heterogeneity in beliefs and risk tolerance on the market risk premium and the risk-free rate through the consensus belief. We show that depending on the correlation between asset returns, Jouini and Napp (2006)'s result does not necessarily hold; furthermore, a positive correlation between optimism and belief in the correlation coefficient generates the most desirable results and the highest Sharpe ratio. We show that, in the context of the CAPM, the correlation between risk tolerance and heterogeneous beliefs is part of the systematic risk. We obtain similar results when the model is generalised to a continuum of agents. However, the results are significantly different when consumption is introduced. In this case, due to the price effect, heterogeneity in agents' beliefs can have a significant impact on the market risk premium and the risk-free rate only when there is a large dispersion in agents' risk tolerance.

The difference in beliefs may depend on the market conditions. Intuitively, there may be more disagreement among agents on the risky stocks when markets are moving downwards. The empirical implications of the results obtained in Chapter 4, in particular when the difference in beliefs become part of the systematic risk in the CAPM, would be very interesting to explore further (such as in Anderson, Ghysels. and Juergens (2005)). The disagreement in Chapter 4 is characterised by mean-preserving spreads about a benchmark homogeneous belief. It would also be interesting to extend the analysis to situations with skewed distribution about heterogeneous beliefs, such as in Abel (2002). In addition, extension to a dynamic model to examine the profitability and survivability of agents with different beliefs and the impact on the market equilibrium and trading volume in the long run (such as in Kogan et al. (2006) and Hong and Stein (2007)) would also be interesting.

In Chapter 5, we introduce a framework of relative consumption, by assuming that two agents maximises the expected utility of their intertemporal consumption relative to each other. We show that the effect of disagreement in beliefs is magnified when the sum of risk aversions is close to 1. Based on this result, we are able to explain a significant part of the equity premium observed in the market while simultaneously keeping the interest rate at a realistic level. In terms of agents' survival, we show that the less rational agent does not necessarily vanish in the long term; even when he does vanish, his belief and risk tolerance can still impact on market equilibrium. Furthermore, a consensus consumer can be constructed using techniques introduced by Jouini and Napp (2007).

In Chapter 5, agents are assumed to have the same patience for future consumption and constant disagreement in beliefs. Future avenues of research can involve an extension of the current model to allow agents to have different patience for future consumption and analyse the impact of various learning schemes on the equilibrium asset prices.

As stated in Anderson et al. (2005), it is common to ignore heterogeneity of beliefs in empirical asset-pricing models, because it is troublesome to implement and data can be of low quality. Anderson et al. (2005) show that incorporating analyst forecasts can improve the performance of asset-pricing models, and heterogeneity is a miss factor in predicting asset returns and volatility. In reality, investors do not only differ in beliefs, but also in their preferences and endowments and jointly they affect the equilibrium outcome of the economy. Therefore, it is important that more experimental and survey data become available to empirically test the effect of heterogeneity on the financial market.

It is also interesting to study the equilibrium effect of market imperfections when beliefs are heterogeneous. It is not clear how liquidity constraints, short-sale constraints, behavioural biases, etc. jointly impact on asset prices when agents agree to disagree. Individually, their effect on the market may be insignificant; however when combined, the effect can become significant and deserves further research. Moreover, Anderson, Ghysels. and Juergens (2010) show that uncertainty/ambiguity also matters for asset pricing in the cross section, as it is likely that agents would have different levels of ambiguity and heterogeneity in beliefs and preferences. Therefore it is important to understand their joint impact on asset prices.

7.3 Option Pricing with Heterogeneous Beliefs

When agents disagree about the price dynamics of the underlying asset, the Black-Scholes option-pricing formula is no longer applicable since the option price could be different under each agent's belief if they perceive different state prices. The question is under which belief should one price options or contingent claims in general? In Chapter 6, we provide a multi-period binomial lattice framework where agents' beliefs differ in the probability of an up move in each period and also the size of the move in each period. Agents invest in their perceived growth-optimal portfolio. The advantages of our model is that it allows for disagreement in models as well as model parameters. It is well known through the work of Kloeden and Platen (1992) that the solution of a stochastic differential equation (SDE) can be approximated by a Euler scheme with the normally distributed Wiener increment by a two-point distribution random variable, thus the structure becomes identical as a binomial lattice, although not necessarily recombining. Therefore, in the limit as the number of steps goes to infinity, agents can believe in a diffusion process with different functional forms for the drift and the diffusion term. By defining and constructing a consensus belief to represent the belief of the market about the future evolution of the underlying asset price, we show that the consensus belief is a fair belief under which any agent's wealth share process is a martingale. Therefore, options can be priced under the consensus belief. In the simplest case, when agents' beliefs are constant, the asset price is a geometric Brownian motion (GBM) under each agent's belief; however, it is not a GBM under the consensus belief unless beliefs are homogeneous, due to the fact that agents' wealth shares are stochastic. Furthermore, we illustrate that different combinations of heterogeneity in the beliefs of the drift and the diffusion coefficient can generate different patterns in the implied volatilities, in particular when agents disagree about the volatility of the underlying stock, implied volatilities becomes negatively correlated with the strike price (volatility skew).

Future avenues of research include incorporating disagreements in model structures into option pricing and compared fair option prices implied by agents' beliefs[1] and those option prices observed in the market to quantify the level of mispricing. It would also be interesting to extend the current model to include multiple assets, including the bond and currency markets. Another direction of research is to study the pricing option in an incomplete market, which allows us to understand the open interest and trade volume in option markets, see Buraschi and Jiltsov (2006). Finally, research on the relationship between Knightian uncertainty/ambiguity on option prices is also interesting. Some might argue that under ambiguity agents typically consider the worst-case scenario and offer the minimum price. However, if an option can somehow be perceived as a hedging instrument for ambiguity, then one can argue the opposite case, that is, ambiguity drives up option prices. A better understanding of the relationship between ambiguity and option prices will hopefully explain the cross section and term structure of implied volatilities.

[1]Analyst forecasts can act as a proxy for agents' belief about future asset returns, the risky asset can be taken to be a stock index.

Appendix A

Proofs

A.1 Proof of Lemma 2.1

Let λ_i be the Lagrange multiplier and set

$$L(\mathbf{z}_i, \lambda_i) := \mathbf{y}_i^T \mathbf{z}_i - \frac{\theta_i}{2} \mathbf{z}_i \Omega_i \mathbf{z}_i + \lambda_i [\mathbf{p}_0^T \mathbf{z}_i - W_0^i]. \tag{A.1}$$

Then the optimal portfolio of agent i is determined by the first order condition

$$\frac{\partial L}{\partial \mathbf{z}_i} = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{z}_i = \theta_i^{-1} \Omega_i^{-1} [\mathbf{y}_i - \lambda_i \mathbf{p}_0]. \tag{A.2}$$

Substituting (A.2) into (2.3) yields (2.5). $\square$

A.2 Proof of Proposition 2.3

From Definition 2.2, if the consensus belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$ exists, then

$$\mathbf{z}_i^* = \theta_a^{-1} \Omega_a^{-1} [\mathbf{y}_a - \lambda_a^* \mathbf{p}_0]. \tag{A.3}$$

Applying the market equilibrium condition to (A.3), we must have

$$\mathbf{z}_m = \sum_{i=1}^{I} \mathbf{z}_i^* = I \left[\theta_a^{-1} \Omega_a^{-1} [\mathbf{y}_a - \lambda_a^* \mathbf{p}_0] \right]. \tag{A.4}$$

This leads to the equilibrium price (2.11). On the other hand, it follows from the individuals demand (2.4) and the market clearing condition (2.7) that, under the heterogenous beliefs,

$$\mathbf{z}_m = \sum_{i=1}^{I} \mathbf{z}_i^* = \sum_{i=1}^{I} \theta_i^{-1}\Omega_i^{-1}[\mathbf{y}_i - \lambda_i^* \mathbf{p}_0]. \tag{A.5}$$

Under the definitions in (2.9) and (2.10), we can re-write equation (A.5) as

$$\mathbf{z}_m = \sum_{i=1}^{I} \theta_i^{-1}\Omega_i^{-1}\mathbf{y}_i - \left(\sum_{i=1}^{I} \theta_i^{-1}\lambda_i^*\Omega_i^{-1}\right)\mathbf{p}_0 = I\theta_a^{-1}\Omega_a^{-1}\mathbf{y}_a - I\theta_a^{-1}\lambda_a^*\Omega_a^{-1}\mathbf{p}_0, \tag{A.6}$$

which leads to the same market equilibrium price (2.11). This shows that $\mathcal{B}_a = \{\Omega_a, \mathbf{y}_a\}$ defined in (2.9) and (2.10) is the consensus belief. Inserting (2.11) into (2.4) give the equilibrium optimal portfolio (2.12) of investor i. $\square$

A.3 Proof of Corollary 2.4

The equilibrium price vector in (2.11) can be re-written to express the price of each asset

$$p_{0,j} = \frac{1}{\lambda_a^*}(y_{a,j} - \theta_a/I \sum_{k=1}^{N} \sigma_{j,k} z_{m,k}) = \frac{1}{\lambda_a^*}[y_{a,j} - \frac{\theta_a}{I} Cov_a(\tilde{x}_j, \tilde{W}_m)]. \tag{A.7}$$

It follows from (A.7) that $y_{a,j} - \lambda_a^* p_{0,j} = \frac{\theta_a}{I} Cov_a(\tilde{x}_j, \tilde{W}_m)$ and hence

$$\frac{y_{a,j}}{p_{0,j}} - \lambda_a^* = \frac{1}{p_{0,j}} \frac{\theta_a}{I} Cov_a(\tilde{x}_j, \tilde{W}_m).$$

Therefore

$$\mathbb{E}_a(\tilde{r}_j) - (\lambda_a^* - 1) = \frac{1}{p_{0,j}} \frac{\theta_a}{I} Cov_a(\tilde{x}_j, \tilde{W}_m). \tag{A.8}$$

It follows from $W_{m0} = \frac{1}{\lambda_a^*}\mathbf{z}_m^T(\mathbf{y}_a - \theta_a\Omega_a\mathbf{z}_m/I)$ that

$$\lambda_a^* = \frac{\mathbf{z}_m^T\mathbf{y}_a - \theta_a\mathbf{z}_m^T\Omega_a\mathbf{z}_m/I}{W_{m0}}. \tag{A.9}$$

Using the definition of λ_a^* in (A.9), we obtain

$$\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1) = \frac{\mathbf{y}_a^T\mathbf{z}_m}{\mathbf{z}_m^T\mathbf{p}_0} - \lambda_a^* = \frac{\mathbf{y}_a^T\mathbf{z}_m}{W_{m0}} - \frac{\mathbf{z}_m^T\mathbf{y}_a - \theta_a\mathbf{z}_m^T\Omega_a\mathbf{z}_m/I}{W_{m0}}.$$

Thus

$$\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1) = \frac{\theta_a\mathbf{z}_m^T\Omega_a\mathbf{z}_m/I}{W_{m0}} \neq 0. \tag{A.10}$$

Dividing (A.8) by (A.10) leads to

$$\frac{\mathbb{E}_a(\tilde{r}_j)-(\lambda_a^*-1)}{\mathbb{E}_a(\tilde{r}_m)-(\lambda_a^*-1)}=\frac{\left(\frac{1}{p_{0,j}}\frac{\theta_a}{I}Cov_a(\tilde{x}_j,\tilde{W}_m)\right)}{\left(\frac{\theta_a\mathbf{z}_m^T\Omega_a\mathbf{z}_m/I}{W_{m0}}\right)}=\frac{\frac{1}{p_{0,j}}Cov_a(\tilde{x}_j,\tilde{W}_m)}{\frac{\sigma_{a,m}^2}{W_{m0}}}$$
$$=\frac{Cov_a\left(\frac{\tilde{x}_j}{p_{0,j}},\frac{\tilde{W}_m}{W_{m0}}\right)}{\frac{\sigma_{a,m}^2}{W_{m0}^2}}=\frac{Cov_a(\tilde{r}_j,\tilde{r}_m)}{\sigma_a^2(\tilde{r}_m)}=\beta_j \tag{A.11}$$

leading to the CAPM-like relation in (2.13). □

A.4 Proof of Proposition 2.9

Since $\Omega_a^{-1}\mathbf{y}_a$ and $\mathbf{z}_m/I$ are both frontier portfolios, it follow that the portfolio $\mathbf{z}$ in (2.25) is also a frontier portfolio. Furthermore, if $\mathbf{z}$ has a higher expected return than the *minimum variance portfolio* $\mathbf{z}_{MVP}$, then $\mathbf{z}$ must be on the efficient portfolio frontier. □

A.5 Proof of Lemma 3.1

The optimization problem of investor is given by

$$\max_{\boldsymbol{\pi}_i}\boldsymbol{\mu}_i^T\boldsymbol{\pi}_i-\frac{\theta_i}{2}\boldsymbol{\pi}_iV_i\boldsymbol{\pi}_i \tag{A.12}$$

subject to the wealth constraint $\boldsymbol{\pi}_i^T\mathbf{1}=W_{i,0}$. Let λ_i be the Lagrange multiplier and set

$$L(\boldsymbol{\pi}_i,\lambda_i):=\boldsymbol{\mu}_i^T\boldsymbol{\pi}_i-\frac{\theta_i}{2}\boldsymbol{\pi}_iV_i\boldsymbol{\pi}_i+\lambda_i[\boldsymbol{\pi}_i^T\mathbf{1}-W_{i,0}]. \tag{A.13}$$

Since $U_i(.)$ is concave, the optimal portfolio of agent i is determined by the first order conditions,

$$\frac{\partial L}{\partial\boldsymbol{\pi}_i}=\mathbf{0}\qquad\Rightarrow\qquad\boldsymbol{\pi}_i^*=\tau_iV_i^{-1}(\boldsymbol{\mu}_i-\lambda_i\mathbf{1}). \tag{A.14}$$

Substituting (A.14) into the wealth constraint yields $\lambda_i^*=\frac{\mathbf{1}^TV_i^{-1}\boldsymbol{\mu}_i-\theta_iW_{i,0}}{\mathbf{1}^TV_i^{-1}\mathbf{1}}$. □

A.6 Proof of Proposition 3.3

From equation (3.5) which is resulted from the aggregation condition, we know that the market equilibrium prices in terms of investors' subjective beliefs are given by

$$\mathbf{p}_0 = Z^{-1}\bigg(\sum_{i=1}^{I}\tau_i V_i^{-1}\boldsymbol{\mu}_i - \sum_{i=1}^{I}\tau_i\lambda_i^* V_i^{-1}\mathbf{1}\bigg). \tag{A.15}$$

Using equation (3.6) and (3.7), we can re-write the equilibrium prices as

$$\mathbf{p}_0 = Z^{-1}\bigg(I\tau_a V_a^{-1}\boldsymbol{\mu}_a - I\tau_i\lambda_i^* V_a^{-1}\mathbf{1}\bigg) = \bigg(\frac{Z}{I}\bigg)^{-1}\tau_a V_a^{-1}(\boldsymbol{\mu}_a - \lambda_a^*\mathbf{1}), \tag{A.16}$$

which corresponds to the equilibrium price specified in equation (3.8). This means that if every investor in the market takes on the belief $\mathcal{B}_a := (V_a, \boldsymbol{\mu}_a)$, then the equilibrium prices in the homogeneous market are identical to the ones in the heterogeneous market. Hence one can conclude that $\mathcal{B}_a$ is a consensus belief.

Next, we prove the ZHCAPM where investors form their beliefs about the future asset returns. Define the expected market return under the consensus belief as

$$\mathbb{E}_a(\tilde{r}_m) = \mathbb{E}_a(\frac{\tilde{W}_m}{W_{m0}} - 1),$$

where $W_{m0} = \boldsymbol{\pi}_m^T\mathbf{1}$ is the total initial market wealth. Let

$$\mu_{am} = \mathbb{E}_a(\frac{\tilde{W}_m}{W_{m0}}) = \mathbb{E}_a(\tilde{r}_m) + 1 \qquad \text{and} \qquad \boldsymbol{\omega}_m = \frac{\boldsymbol{\pi}_m}{W_{m0}}.$$

It follows from $\tilde{W}_m = \boldsymbol{\pi}_m^T(\tilde{\mathbf{r}} + \mathbf{1})$ and $\boldsymbol{\omega}_m^T\mathbf{1} = 1$ that

$$\mu_{am} - \lambda_a^* = \boldsymbol{\omega}_m^T(\boldsymbol{\mu}_a - \lambda_a^*\mathbf{1}) = \boldsymbol{\omega}_m^T\bigg(\frac{1}{I}\tau_a V_a\boldsymbol{\pi}_m\bigg) = \frac{W_{m0}}{I}\tau_a\sigma_{am}^2. \tag{A.17}$$

Also we can rearrange equation (3.8) to get

$$\boldsymbol{\mu}_a - \lambda_a^*\mathbf{1} = \frac{1}{I}\tau_a V_a\boldsymbol{\pi}_m, \tag{A.18}$$

which can be written for each asset j as follows

$$\mu_{a,j} - \lambda_a^* = \frac{W_{m0}}{I}\tau_a\sum_{k=1}^{N}\sigma_{a,jk}\omega_{mk} = \frac{W_{m0}}{I}\tau_a Cov_a(\tilde{r}_j, \tilde{r}_m). \tag{A.19}$$

Therefore

$$\frac{\mu_{a,j} - \lambda_a^*}{\mu_{am} - \lambda_a^*} = \frac{Cov_a(\tilde{r}_j, \tilde{r}_m)}{\sigma_{am}^2} = \beta_j, \tag{A.20}$$

leading to

$$\boldsymbol{\mu}_a - \lambda_a^* \mathbf{1} = \boldsymbol{\beta}(\mu_{am} - \lambda_a^*), \tag{A.21}$$

which in turn leads to the relation in (3.9). □

A.7 Proof of Proposition 3.5

According to equation (3.10), when beliefs in variance/covariance matrix is homogeneous, then independent of the existence of the riskless security, the consensus belief in the expected return for risky asset j is given by

$$\mu_{a,j} = \mathbb{E}[\tilde{\tau} \ \ (\mu_{o,j} + \tilde{\alpha}_j)]/\tau_a = \mu_{o,j} + \rho(\tilde{\tau}, \ \ \tilde{\alpha}_j)\sigma_\tau \sigma_\alpha / \tau_a, \tag{A.22}$$

since $\mathbb{E}[\tilde{\tau}] = \tau_a$ and $\mathbb{E}[\tilde{\alpha}_j] = 0$. Obviously, $\rho(\tilde{\tau}, \ \ \tilde{\alpha}_j) = 0$ leads to our desired result. □

A.8 Proof of Proposition 3.6

Given that δ_e is small, we have

$$V_e^{-1} \approx V_o^{-1} - \delta_e \ \ V_o^{-1} X V_o^{-1}.$$

From equation (3.10), it is clear that

$$V_a^{-1} \approx V_o^{-1} + \frac{1}{\tau_a}\mathbb{E}(\tilde{\tau}, \tilde{\delta})V_o^{-1} X V_o^{-1} = V_o^{-1} + \rho(\tilde{\tau}, \tilde{\delta})\sigma_\tau \sigma_\delta V_o^{-1} X V_o^{-1}. \tag{A.23}$$

Obviously, $\rho(\tilde{\tau}, \tilde{\delta}) = 0$ leads to $V_a = V_o$.

From equation (3.11), the consensus belief of the expected future asset returns is given by

$$\begin{aligned}
\boldsymbol{\mu}_a &\approx \frac{V_a}{\tau_a} \ \ \mathbb{E}\Big[\tilde{\tau}(V_o^{-1} - \tilde{\delta} V_o^{-1} X V_o^{-1})(\boldsymbol{\mu}_o + \tilde{\alpha}\mathbf{1})\Big] \\
&= \boldsymbol{\mu}_o + \mathbb{E}(\tilde{\tau}\tilde{\alpha})V_o^{-1}\mathbf{1} - \mathbb{E}(\tilde{\tau}\tilde{\delta})(V_o^{-1} X V_o^{-1})\boldsymbol{\mu}_o - \mathbb{E}(\tilde{\tau}\tilde{\delta}\tilde{\alpha})(V_o^{-1} X V_o^{-1})\mathbf{1} \\
&= \boldsymbol{\mu}_o + \rho(\tilde{\tau}, \tilde{\alpha})\sigma_\alpha \sigma_\tau V_o^{-1}\mathbf{1} - \rho(\tilde{\tau}, \tilde{\delta})\sigma_\tau \sigma_\delta (V_o^{-1} X V_o^{-1})\boldsymbol{\mu}_o - Cov(\tilde{\tau}, \tilde{\alpha}\tilde{\delta})(V_o^{-1} X V_o^{-1})\mathbf{1}.
\end{aligned}$$

From the above expression, the independence of δ_e, α_e and τ_e leads to the desirable result. □

A.9 Proof of Corollary 3.7

Note that the tangency relation would hold if the expected returns of the zero-beta portfolio of the market portfolio without the riskless asset is the same as the riskless rate with a riskless asset. Denote $\boldsymbol{\mu}_{a,f}$ and $\boldsymbol{\mu}_{a,z}$ as the consensus belief of expected future asset returns with and without a riskless security respectively. Given $V_i = V_o$ for all i, or $V_e = V_o$ for any $e \in [0, 1]$, it is obvious from equations (3.7) and (3.11) that

$$\boldsymbol{\mu}_{a,f} = \boldsymbol{\mu}_{a,z} = \frac{1}{I}\sum_{i=1}^{I} \frac{\tau_i}{\tau_a}\boldsymbol{\mu}_i, \tag{A.24}$$

or

$$\boldsymbol{\mu}_{a,f} = \boldsymbol{\mu}_{a,z} = \mathbb{E}[\tilde{\tau}\ \ \tilde{\boldsymbol{\mu}}]/\tau_a. \tag{A.25}$$

It follows from equation (3.8) and the fact $\mathbf{1}^T\boldsymbol{\pi}_m = W_{m0}$ that

$$\lambda_a - 1 = r_f = \frac{\mathbf{1}^T V_o^{-1}\mathbb{E}_a(\tilde{\mathbf{r}}) - \bar{W}_0/\tau_a}{\mathbf{1}^T V_o^{-1}\mathbf{1}}, \tag{A.26}$$

where $\mathbb{E}_a(\tilde{\mathbf{r}}) = \boldsymbol{\mu}_{a,f} - 1 = \boldsymbol{\mu}_{a,z} - 1$. Hence the equilibrium market prices for risky assets $\mathbf{p}_0$ is independent on the existence of a riskless security in the market and the tangency relation holds. $\square$

A.10 Proof of Corollary 4.2

Substitute (4.11) into (4.7)-(4.10) yields risk tolerance and beliefs for both agents, then the consensus belief in (4.12) can be computed by applying equation (4.2). The market portfolio $\boldsymbol{\pi}_m$ can be computed by equation (4.3) and the risk-free rate r_f by (4.4). Asset betas $\boldsymbol{\beta}$, market volatility σ_m^2 and the market premium $\mathbb{E}(\tilde{r}_m - r_f)$ can be easily calculated subsequently once the market portfolio is computed. $\square$

A.11 Proof of Corollary 4.3

Substitute $\Delta = 0$ and $\epsilon = 0$ into (4.7)-(4.10) yields risk tolerance and beliefs for both agents. Then the consensus belief $\mathcal{B}_a$ can be computed by applying equation (4.2). $\square$

A.12 Proof of Corollary 4.7

From (4.31) and (4.32), the consensus belief of expected asset returns according to equation (4.30) is given by $\boldsymbol{\mu}_a = (\mu_1, \ \mu_{a,2})^T$, where $\mu_{a,2} = \tau_a^{-1}(\mathbb{E}(\tilde{\tau})\mathbb{E}(\tilde{\mu}_2) + Cov(\tilde{\tau}, \tilde{\mu}_2))$ where $\tau_a = \mathbb{E}(\tilde{\tau}) = \tau_o$. Since $Cov(\tilde{\tau}, \tilde{\mu}_2) = Cov(\tau_o(1+\tilde{\Delta}), \mu_2(1+\tilde{\alpha}))$, we obtain $\mu_{a,2} = \mu_2(1 + Cov(\tilde{\alpha}, \tilde{\Delta})$. The rest of the proof is analogous to that of Corollary 4.2 by simply replacing $\alpha\Delta$ with $Cov(\tilde{\alpha}, \tilde{\Delta})$. $\square$

A.13 Proof of Lemma 4.8

Define $L(\mathbf{z}_i, z_{i,o}, C_{i,0}) := Q_i(\mathbf{z}_i, z_{i,o}) + \lambda_i(\zeta_{i,o} + \boldsymbol{\zeta}_i^T \mathbf{p}_0 - z_{i,o} - \mathbf{z}_i^T \mathbf{p}_0 - C_{i,0})$ where λ_i is the Lagrange multiplier. The first order condition gives

$$e^{-\frac{C_{i,0}}{\tau_i}} = \lambda_i, \tag{A.27}$$

$$\eta e^{-\frac{Q_i}{\tau_i}} R_f = \lambda_i, \tag{A.28}$$

$$\eta e^{-\frac{Q_i}{\tau_i}} \left(\mathbf{y}_i - \frac{1}{\tau_i}\Omega_i \mathbf{z}_i \right) = \lambda_i \mathbf{p}_0. \tag{A.29}$$

Equations (A.28) and (A.29) leads to agent i's optimal portfolio of risky assets in (4.42), the optimal amount invested in the riskless asset follows from the budget constraint (4.40). Equations (A.27) and (A.28) implies that $e^{-\frac{C_{i,0}}{\tau_i}} = \eta e^{-\frac{Q_i}{\tau_i}} R_f$ which leads to agent i's optimal time-0 consumption (4.43). $\square$

A.14 Proof of Proposition 4.10

The results of the consensus belief $\mathcal{B}_a$, equilibrium price vector $\mathbf{p}_0$ and agent i's optimal portfolio are the same as in Chiarella et al. (2010*b*). To obtain the consensus patience parameter η_a, rewrite the agent i's optimal time-0 consumption $C_{i,0}^* = Q_i^* - \tau_i \ln(\eta R_f)$, applying market clearing condition gives $(C_{0,1}^* + C_{0,2}^*)/2 = C_0/2 = \bar{Q} - \tau_a \ln(\eta Rf)$, which leads to (4.50). Moreover, optimality implies $C_0/2 = Q_a - \tau_a \ln(\eta_a Rf)$, which leads to (4.51). $\square$

A.15 Proof of Corollary 4.11

Given that $\mathbf{y}_i = \mathbf{y}$ and $\Omega_i = \Omega$, according to Proposition 4.10, the consensus belief is given by $\mathbf{y}_a = \mathbf{y}$ and $\Omega_a = \Omega$ and $Q_a = \bar{Q} = \hat{Q}$. This completes the proof. $\square$

A.16 Proof of Lemma 5.1

By agent 1's first-order condition for optimal consumption,

$$\frac{c_1(t)}{c_2(t)} = I_1(\eta_1 e^{\beta t}\xi_1(t)c_2(t))$$

where η_1 is the Lagrange multiplier to satisfy agent 1's budget constraint. Similarly,

$$\frac{c_2(t)}{c_1(t)} = I_2(\eta_2 e^{\beta t}\xi_1(t)c_1(t)).$$

These lead to the expressions in (5.14), solving this system of equations for $c_1(t)$ and $c_2(t)$ leads to agent 1 and 2's optimal consumption processes in (5.15). $\square$

A.17 Proof of Proposition 5.3

The dynamics of agent 1's optimal consumption is obtained in (5.17) by applying Itô's formula to (5.15). We first compute in equilibrium the market price of risk perceived by agent 1 (κ_1) and agent 2 (κ_2) by equating the diffusion coefficient of the aggregate consumption process with that of the aggregate endowment process, stated in (5.20). We write $d\omega_1(t) = d\omega(t) + \theta_{1\varepsilon}(t)dt$ and $d\omega_2(t) = d\omega(t) + \theta_{2\varepsilon}(t)dt$ to compute equilibrium under the objective probability measure. Doing so we obtain the expression for $\kappa_1(t)$ and applying the consistency relationship (5.11) yields the expression for $\kappa_2(t)$ in (5.21). By equating the drift coefficients as stated in (5.19) we obtain the expression for the equilibrium risk-free rate in terms of $\lambda(t)$ as well as $\kappa_1(t)$ and $\kappa_2(t)$, which is given by

$$\begin{aligned} r(t) &= \beta + \mu_\varepsilon - \frac{\alpha_2 - \lambda_2(t)}{\alpha_1 + \alpha_2 - 1}\theta_{1\varepsilon}(t)\kappa_1(t) - \frac{\alpha_1 - \lambda_1(t)}{\alpha_1 + \alpha_2 - 1}\theta_{2\varepsilon}(t)\kappa_2(t) \\ &- \frac{[(\alpha_2 - \lambda_2(t))^2 + \lambda_2(t)(1-\lambda_2(t))] + [\alpha_1(\alpha_2 - \lambda_2(t)) - \alpha_2(1-\lambda_2(t))]/2}{(\alpha_1 + \alpha_2 - 1)^2}(\kappa_1(t))^2 \\ &- \frac{[(\alpha_1 - \lambda_1(t))^2 + \lambda_1(t)(1-\lambda_1(t))] + [\alpha_2(\alpha_1 - \lambda_1(t)) - \alpha_1(1-\lambda_1(t))]/2}{(\alpha_1 + \alpha_2 - 1)^2}(\kappa_2(t))^2 \\ &- \frac{\alpha_1\alpha_2 - \lambda_1(t)\alpha_2 - \lambda_2(t)\alpha_1}{(\alpha_1 + \alpha_2 - 1)^2}\kappa_1(t)\kappa_2(t), \end{aligned} \tag{A.30}$$

where $\lambda_1(t) = \lambda(t)$ and $\lambda_2(t) = 1 - \lambda(t)$ are the consumption share of agent 1 and agent 2. Moreover, we obtain the dynamics for $\lambda(t)$ by applying Itô's formula to $c_1(t)/\varepsilon(t)$ which is given by

$$d\lambda(t) = \lambda(t)[\mu_\lambda(t)dt + \sigma_\lambda(t)d\omega(t)], \tag{A.31}$$

where

$$\mu_\lambda(t) = \mu_{c_1}(t) - \mu_\varepsilon(t) + \sigma_\varepsilon(t)(\sigma_\varepsilon(t) - \sigma_{c_1}(t)),$$
$$\sigma_\lambda(t) = \sigma_{c_1}(t) - \sigma_\varepsilon(t),$$

then we substitute expressions for $r(t)$, $\kappa_1(t)$ and $\kappa_2(t)$ in (A.30) and (5.21) respectively and simplify to obtain the explicit expression for $\lambda(t)$ in (5.23). By substituting (5.21) into (A.30) we obtain the expression of the equilibrium risk-free rate in terms of only $\lambda(t)$, using the fact that $r_1(t) - r_2(t) = \sigma_\varepsilon(t)\bar{\mu}(t)$. $\square$

A.18 Proof of Corollary 5.4

Substituting $\alpha_1 = \alpha_2 = 1$ into (5.21) leads to the perceived market price of risk in (5.26). Substituting $\alpha_1 = \alpha_2 = 1$ into (5.22) and (5.23) leads to the equilibrium risk-free rate and agent 1's consumption share in (5.27) and (5.28) respectively. $\lambda(0)$ is determined by the budget constraint. Substituting $\alpha_1 = \alpha_2 = 1$ into (5.15) leads to $c_1(t) = y_1 \ e^{-\beta t}\xi_1(t)^{-1}$, which we substitute into the budget constraint (5.13) and obtain

$$y_1 = x_1 \frac{\mathbb{E}_1\left[\int_0^T \xi_1(t)\varepsilon(t)dt\right]}{\int_0^T e^{-\beta t}dt}.$$

It follows that $y_1 = x_1\varepsilon(0)$ since

$$y_1 + y_2 = c_1(0) + c_2(0) = \frac{x_1\mathbb{E}_1\left[\int_0^T \xi_1(t)\varepsilon(t)dt\right] + x_2\mathbb{E}_2\left[\int_0^T \xi_2(t)\varepsilon(t)dt\right]}{\int_0^T e^{-\beta t}dt} = \varepsilon(0)$$

and

$$\mathbb{E}_1\left[\int_0^T \xi_1(t)\varepsilon(t)dt\right] = \mathbb{E}_2\left[\int_0^T \xi_2(t)\varepsilon(t)dt\right].$$

Hence $\lambda(0) = c_1(0)/\varepsilon(0) = y_1/\varepsilon(0) = x_1$. $\square$

A.19 Proof of Corollary 5.5

Substituting $\bar{\mu}(t) = (0)$ and $\theta_{1\varepsilon}(t) = 0$ into (5.21) leads to the homogeneous market price risk in (5.30), substituting the same into (5.22) leads to the equilibrium risk-free rate in (5.31).

A.20 Proof of Lemma 5.7

From (5.40), we can write the dynamics of $\lambda(t)$ as

$$d\lambda(t) = b(\lambda(t))\left\{\left[\frac{1}{2}b(\lambda(t))b^{'}(\lambda(t)) + \left(\frac{1}{2}\bar{\mu} + \theta_{1\varepsilon}\right)b(\lambda(t))\right]dt + d\omega(t)\right\}, \qquad \text{(A.32)}$$

where $b(\lambda) = \lambda(1-\lambda)\bar{\mu}/(\alpha_1+\alpha_2-1)$. The SDE in (A.32) can be solved explicitly according to Kloeden and Platen (1992) and is given by

$$\lambda(t) = h^{-1}\left[\left(\frac{1}{2}\bar{\mu} + \theta_{1\varepsilon}\right)t + \omega(t) + h(\lambda(0))\right], \qquad \text{(A.33)}$$

where

$$h(x) = \int^{x} ds/b(s) \;\; ds = \frac{\alpha_1+\alpha_2-1}{\bar{\mu}}\ln\left(\frac{x}{1-x}\right),$$

hence the inverse function is given by

$$h^{-1}(x) = \frac{1}{1+\exp\{-\frac{\bar{\mu}}{\alpha_1+\alpha_2-1} \;\; x\}}.$$

Evaluating (A.33) and simplifying leads to the desirable expression. □

A.21 Proof of Proposition 5.9

Consider the term

$$\exp\left\{-\frac{\bar{\mu}}{\alpha_1+\alpha_2-1}\left[\left(\frac{1}{2}\bar{\mu} + \theta_{1\varepsilon}\right)t + \omega(t)\right]\right\}. \qquad \text{(A.34)}$$

Using the strong Law of Large Numbers for Brownian Motion (see Karatzas and Shreve (1991), Sec.2.9.A), for any value of σ,

$$\lim_{t\to 0}\; \exp\{at + \sigma\;\; \omega(t)\} = \begin{cases} 0, & a < 0 \\ \infty, & a > 0, \end{cases}$$

where convergence takes place almost surely. In our case, $a = \frac{\bar{\mu}}{\alpha_1+\alpha_2-1}(\frac{1}{2}\bar{\mu} + \theta_{1\varepsilon})$. We consider two cases, (i) $\alpha_1+\alpha_2 > 1$ and (ii) $\alpha_1+\alpha_2 < 1$. Under case (i), a is a concave quadratic function of $\bar{\mu}$, therefore (A.34) diverges a.s to infinity, that is agent 1 vanishes and agent 2 survives iff $\theta_{1\varepsilon} > 0$ *and* $-2\theta_{1\varepsilon} < \bar{\mu} < 0$ or $\theta_{1\varepsilon} < 0$ *and* $0 < \bar{\mu} < -2\theta_{1\varepsilon}$, which is equivalent to $|\theta_{2\varepsilon}| < |\theta_{1\varepsilon}|$. Similarly, (A.34) diverges a.s to infinity, that is agent 1 survives and agent 2 vanishes iff $|\theta_{2\varepsilon}| > |\theta_{1\varepsilon}|$. Under case (ii), a is a convex function of

(i) $\alpha_1 + \alpha_2 > 1$	(ii) $\alpha_1 + \alpha_2 < 1$
$\theta_{1\varepsilon} > 0 \;\Rightarrow\; -2\theta_{1\varepsilon} < \bar{\mu} < 0$ $\Rightarrow\; a > 0,\; l^* > 0$	$\theta_{1\varepsilon} > 0 \;\Rightarrow\; \bar{\mu} > 0 \;\Rightarrow\; a > 0,\; l^* > 0$ or $\Rightarrow\; \bar{\mu} < -2\theta_{1\varepsilon} \;\Rightarrow\; a < 0,\; l^* < 0$
$\theta_{1\varepsilon} < 0 \;\Rightarrow\; 0 < \bar{\mu} < -2\theta_{1\varepsilon}$ $\Rightarrow\; a < 0,\; l^* < 0$	$\theta_{1\varepsilon} < 0 \;\Rightarrow\; \bar{\mu} < 0 \;\Rightarrow\; a < 0,\; l^* < 0$ or $\Rightarrow\; \bar{\mu} > -2\theta_{1\varepsilon} \;\Rightarrow\; a > 0,\; l^* > 0$

Table A.1: Sign of a and l^* when agent 1 vanishes in the long-run.

$\bar{\mu}$, therefore a has the opposite sign compare to case 1. Lastly, $a = 0$ iff $|\theta_{1\varepsilon}| = |\theta_{2\varepsilon}|$ under which there does not exist an stationary distribution for (A.34), hence both agent 1 and agent 2 survive in the long-run. $\square$

A.22 Proof of Proposition 5.10

An equivalent problem to computing (5.45) is to compute

$$t_l = \mathbb{E}[\inf\{t : \;\; at + \omega(t) = l^*\}], \tag{A.35}$$

where

$$a = \frac{1}{2}\bar{\mu} + \theta_{1\varepsilon} \;\text{ and }\; l^* = -\frac{\alpha_1 + \alpha_2 - 1}{\bar{\mu}} \ln\left[\frac{\lambda(0)}{1-\lambda(0)}\left(\frac{1-l}{l}\right)\right].$$

This is a well studied problem, see Karatzas and Shreve (1991) Chapter 3.5, the explicit density function of t_l is given by

$$\mathcal{P}[l \in dt] = \frac{|l^*|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(l^* - at)^2}{2t}\right\} dt, \;\; t > 0.$$

To find the expected first hitting time, we compute the integral

$$\int_0^\infty \frac{|l^*|}{\sqrt{2\pi t}} \exp\left\{-\frac{(l^* - at)^2}{2t}\right\} dt. \tag{A.36}$$

The explicit solution of (A.36) depends on the sign of a and l^*. There are two cases in Proposition 5.9 under which agent 1 vanishes in the long-run, (i) $\alpha_1 + \alpha_2 > 1$ and (ii) $\alpha_1 + \alpha_2 < 1$. In (i), the sufficient and necessary condition for agent 1 to vanish is $|\theta_{2\varepsilon}| < |\theta_{1\varepsilon}|$, $l^* < 0$. In (ii), the sufficient and necessary condition for agent 2 to vanish in the long-run is $|\theta_{2\varepsilon}| > |\theta_{1\varepsilon}|$. We analyze the sign of a and l^* under (i) and (ii) in Table A.1, which shows that a and l^* always have the same sign under which the integral in

(A.36) has the closed-form solution given by

$$t_l = \frac{l^*}{a}.$$

Substituting in the values for a and l^* completes the proof. $\square$

A.23 Proof of Proposition 6.3

We start from equation (6.5) which is given below

$$\frac{1}{R_f} + \sum_i w_i(t)\left(\frac{p_i}{\bar{d}_i} + \frac{1-p_i}{\bar{u}_i}\right) = 0.$$

On the one hand, if every investor has identical belief about the future return in the upstate and downstate respectively in the period $[t, t+1]$, i.e $p_i = p_m$, $\bar{u}_i = \bar{u}_m$ and $\bar{d}_i = \bar{d}_m$ for all i. Then it is obvious that equation (6.5) becomes

$$\frac{1}{R_f} + \frac{p_m}{\bar{d}_m} + \frac{1-p_m}{\bar{u}_m} = 0. \tag{A.37}$$

Solving equation (A.37) for R_f leads to relationship between the consensus belief and the the risk-free rate in (6.9). Next it is obvious that in order for $\mathcal{B}_m(t)$ to be the consensus belief, the following must hold in every period $[t, t+1]$ for $t = 0, 1, \cdots, T$,

$$\frac{p_m}{\bar{d}_m} = \sum_i w_i \frac{p_i}{\bar{d}_i}, \tag{A.38}$$

$$\frac{p_m}{\bar{u}_m} = \sum_i w_i \frac{1-p_i}{\bar{u}_i}, \tag{A.39}$$

which leads to equations (6.7) and (6.8). When investors agree on the future return in each state, then the consensus belief must reflect this common belief. This means that $u_i = u_o \Rightarrow u_m = u_o$ and $d_i = d_o \Rightarrow d_m = d_o$. This fact gives us the expression for the consensus probability belief $p_m = \sum_i \; w_i p_i$.

To prove (iv), we simply substitute equation (6.12) into the right hand side of equation (6.11), then we find that under the belief of a particular agent i,

$$[S(t)/R_f(t)](q_{i,u}(t) \;\; u_i(t) + q_{i,d}(t) \;\; d_i(t)) = S(t).$$

Since $q_{i,u}(t)\ u_i(t) + q_{i,d}(t)\ d_i(t) = R_f(t)$, the relation holds for all agent i, hence it must also hold for the consensus investor m.

A.24 Proof of Proposition 6.5

The wealth for individual i at time $t+1$ is given by

$$W_i(t+1) = W_i(t)(\omega_i(t)\ R(t+1) + (1-\omega_i(t))\ R_f).$$

Dividing the aggregate market wealth $W_m(t+1)$ on both side and using the fact that $W_m(t+1) = W_m(t)R(t+1)$,

$$w_i(t+1) = w_i(t)\ \left(\omega_i(t) + (1-\omega_i(t))\frac{R_f}{R(t+1)}\right).$$

Taking expectation under the consensus belief on both side leads to

$$\mathbb{E}_t^m\left[\frac{w_i(t+1)}{w_i(t)}\right] = \omega_i(t) + (1-\omega_i(t))\mathbb{E}_t^m\left[\frac{R_f}{R(t+1)}\right]. \tag{A.40}$$

By Proposition 6.3 (ii), $\mathbb{E}_t^m[R_f/R(t+1)] = 1$ and this completes the proof.

A.25 Proof of Proposition 6.6

Similar to the proof of Proposition 6.5, we have equation (A.40), however, expectation will be taken under agent i's belief,

$$\mathbb{E}_t^i\left[\frac{w_i(t+1)}{w_i(t)}\right] = \omega_i(t) + (1-\omega_i(t))\mathbb{E}_t^i\left[\frac{R_f}{R(t+1)}\right]. \tag{A.41}$$

Using Lemma 6.1 to expand above expression leads to

$$\mathbb{E}_t^i\left[\frac{w_i(t+1)}{w_i(t)}\right] - 1 = \frac{[R_f(p_i\ d_i + (1-p_i)\ u_i) - u_i\ d_i]^2}{-\bar{u}_i\ \bar{d}_i\ u_i\ d_i} \geq 0.$$

Equality holds if and only if the numerator is zero, that is

$$R_f(p_i\ d_i + (1-p_i)\ u_i) = u_i\ d_i\ \Rightarrow\ \frac{p_i}{u_i} + \frac{(1-p_i)}{d_i} = \frac{1}{R_f}$$

Hence we must have $\mathbb{E}_t^i[\frac{1}{R(t+1)}] = \mathbb{E}_t^m[\frac{1}{R(t+1)}] = \frac{1}{R_f}$.

A.26 Proof of Proposition 6.9

We can always replicate the option with a portfolio that invests $\omega(t)$ in the risky asset and $1-\omega(t)$ in the risk-free asset, this means that we can express the value of the option at time $t+1$ as

$$V(t+1, S(t+1)) = V(t, S(t))\ \ (R_f(t) + \omega(t)(R(t+1) - R_f(t))).$$

Dividing by $S(t+1)$ on both side and taking expectation under the consensus belief yields

$$\begin{aligned}&\mathbb{E}_t^m\left(\frac{V(t+1,S(t+1))}{S(t+1)}\right) = \frac{V(t,S(t))}{S(t)}\mathbb{E}_t^m\left(\omega(t) + (1-\omega(t))\frac{R_f(t)}{R(t+1)}\right)\\ &= \frac{V(t,S(t))}{S(t)}.\end{aligned}$$

Then using the law of iterated expectations, we get

$$\frac{V(t,S(t))}{S(t)} = \mathbb{E}_t^m\left(\frac{V(t+1,S(t+1))}{S(t+1)}\right) = \mathbb{E}_t^m\left(\frac{V(T,S(T))}{S(T)}\right)$$

and $V(T, S(T)) = H(T, S(T)$ and that completes the proof.

A.27 Proof of Remark 6.11

Since both agents have identical beliefs about the distribution of future asset returns, the consensus belief belief must coincide with the homogeneous belief, that is

$$\begin{aligned}u_m(t) &= 1 + \mu\ \ \Delta + \sigma\ \ \sqrt{\Delta}\\ d_m(t) &= 1 + \mu\ \ \Delta - \sigma\ \ \sqrt{\Delta}\\ p_m(t) &= 0.5.\end{aligned}$$

for $t = 0, 1, \cdots, T-1$. By Proposition 6.3 (ii), the price of a zero-coupon bond is given by

$$\frac{1}{R_f} = \frac{1}{2}\left[\frac{1}{1+\mu\ \ \Delta+\sigma\ \ \sqrt{\Delta}} + \frac{1}{1+\mu\ \ \Delta-\sigma\ \ \sqrt{\Delta}}\right] = \frac{1+\mu\ \ \Delta}{(1+\mu\ \ \Delta)^2 - \sigma^2\Delta}.$$

Let r be the continuous compounded risk-free rate, we have from above that

$$R_f = e^{r\Delta} = 1 + \frac{\mu\Delta - \sigma^2\Delta}{1+\mu\ \ \Delta}.$$

Therefore the instantaneous risk-free rate r_f is given by

$$r_f = \lim_{\Delta \to 0} \frac{1}{\Delta} \ln \left[1 + \mu\Delta - \frac{\sigma^2 \Delta}{1 + \mu \ \Delta} \right]$$

By applying $L^{'}H\hat{o}pital$'s rule we obtain,

$$r_f = \mu - \lim_{\Delta \to 0} \frac{\sigma^2}{(1+\mu\Delta)^2(1+\mu\Delta - \frac{\sigma^2\Delta}{1+\mu\Delta})} = \mu - \sigma^2.$$

Appendix B

A Numerical Example

Example B.1 *Let $I = 2$ and $N = 3$. Consider the set up in Table B.1*

Initial Wealth	Risk Aversion	Expected payoffs	Variance/Covariance of payoffs
$W_0^1 = 10$	$\theta_1 = 5$	$\mathbf{y}_1 = \begin{pmatrix} 6.60 \\ 9.35 \\ 9.78 \end{pmatrix}$	$\Omega_1 = \begin{pmatrix} 0.6292 & 0.1553 & 0.2262 \\ & 0.7692 & 0.1492 \\ & & 2.1381 \end{pmatrix}$
$W_0^2 = 10$	$\theta_2 = 1$	$\mathbf{y}_2 = \begin{pmatrix} 9.60 \\ 12.35 \\ 12.78 \end{pmatrix}$	$\Omega_2 = \begin{pmatrix} 0.4292 & -0.0447 & 0.0262 \\ & 0.5692 & -0.0508 \\ & & 1.7381 \end{pmatrix}$

Table B.1: Market specifications and heterogeneous beliefs.

Assume that there is one share available for each asset, that is, $\mathbf{z}_m = (1, 1, 1)^T$. Based on the information in Table B.1, we use equation (2.8) and Excel Solver to solve for the equilibrium price vector and obtain the market equilibrium price $\mathbf{p}_0 = (5.6436, 7.4328, 6.9236)^T$. The optimal portfolios and shadow prices of the investors are given by $\mathbf{z}_1^* = (0.380, 0.768, 0.310)^T$, $\lambda_1^* = 0.7894$ for investor 1 and $\mathbf{z}_2^* = (0.620, 0.232, 0.690)^T$ and $\lambda_2^* = 1.6520$ for investor 2. Using Proposition 2.3, we construct the consensus belief $\mathcal{B}_a$, the aggregate risk aversion coefficient θ_a, and the aggregate shadow price λ_a^*, and obtain the result in Table B.2. We then use the market

Market Initial Wealth	Shadow Price	Risk Aversions	Expected payoffs	Variance/Covariance of payoffs
$W_{m0} = 20$	$\lambda_a^* = 1.5083$	$\theta_a = 1.6667$	$\mathbf{y}_a = \begin{pmatrix} 8.88 \\ 11.63 \\ 12.06 \end{pmatrix}$	$\Omega_a = \begin{pmatrix} 0.4383 & -0.0356 & 0.0352 \\ & 0.5783 & -0.0417 \\ & & 1.9472 \end{pmatrix}$

Table B.2: The market consensus beliefs, shadow price and ARA.

equilibrium price to convert the consensus belief from payoffs to returns as follows. Let

$P_0 = diag[\mathbf{p}_0] = diag(5.6436, 7.4328, 6.9236)$ and

$$\mathbb{E}_i(\tilde{\mathbf{r}}) := P_0^{-1}\mathbf{y}_i - \mathbf{1}, \qquad V_i(\tilde{\mathbf{r}}) := P_0^{-1}\Omega_i P_0^{-1}, \qquad i = 1, 2, a;$$
$$\mathbf{w}_i^* := \frac{1}{W_{i,o}} P_0 \mathbf{z}_i^*, \qquad \mathbb{E}_i(\tilde{r}_{ip}^*) := \mathbb{E}_i(\tilde{\mathbf{r}})^T \mathbf{w}_i^*, \qquad \sigma_{ip}^* = (\mathbf{w}_i^{*T} V_i(\tilde{\mathbf{r}})\mathbf{w}_i^*)^{1/2}, \quad i = 1, 2;$$
$$\mathbb{E}_a(\tilde{r}_{ip}^*) := \mathbb{E}_a(\tilde{\mathbf{r}})^T \mathbf{w}_i^*, \qquad \sigma_{ip}^a = (\mathbf{w}_i^{*T} V_a(\tilde{\mathbf{r}})\mathbf{w}_i^*)^{1/2}, \qquad i = 1, 2;$$
$$\mathbf{w}_m := \frac{1}{W_{m,o}} P_0 \mathbf{z}_m, \qquad \mathbb{E}_a(\tilde{r}_m) := \mathbb{E}_a(\tilde{\mathbf{r}})^T \mathbf{w}_m,$$
$$\sigma_{a,m} = (\mathbf{w}_m^T V_a(\tilde{\mathbf{r}})\mathbf{w}_m)^{1/2}, \qquad \boldsymbol{\beta} := \frac{V_a(\tilde{\mathbf{r}})\mathbf{w}_m}{\sigma_{a,m}^2}.$$

We then obtain the following results.

Expected returns	Variance/Covariance of returns	portfolio weights	Portfolio Return/SD
$\mathbb{E}_1(\tilde{\mathbf{r}}) = \begin{pmatrix} .1690 \\ .2577 \\ .4126 \end{pmatrix}$	$V_1 = \begin{pmatrix} .0198 & .0037 & .0058 \\ & .0139 & .0029 \\ & & .0446 \end{pmatrix}$	$\mathbf{w}_1^* = \begin{pmatrix} .2144 \\ .5711 \\ .2145 \end{pmatrix}$	$\mathbb{E}_1(r_{1p}^*) = .2719$ $\sigma_{1p}^* = .09824$ $\mathbb{E}_a(r_{1p}^*) = .6043$ $\sigma_{1p}^a = .0748$
$\mathbb{E}_2(\tilde{\mathbf{r}}) = \begin{pmatrix} .7006 \\ .6613 \\ .8459 \end{pmatrix}$	$V_2 = \begin{pmatrix} .0135 & -.0011 & .0007 \\ & .0103 & -.0010 \\ & & .0404 \end{pmatrix}$	$\mathbf{w}_2^* = \begin{pmatrix} .3499 \\ .1722 \\ .4778 \end{pmatrix}$	$\mathbb{E}_2(r_{2p}^*) = .7633$ $\sigma_{2p}^* = .1054$ $\mathbb{E}_a(r_{2p}^*) = .6522$ $\sigma_{2p}^a = .1065$
$\mathbb{E}_a(\tilde{\mathbf{r}}) = \begin{pmatrix} .5729 \\ .5644 \\ .7418 \end{pmatrix}$	$V_a = \begin{pmatrix} .0138 & -.0008 & .0009 \\ & .0105 & -.0008 \\ & & .0406 \end{pmatrix}$	$\mathbf{w}_m = \begin{pmatrix} .2822 \\ .3716 \\ .3462 \end{pmatrix}$	$\mathbb{E}_a(r_m) = .6283$ $\sigma_{a,m} = .0848$

$$\boldsymbol{\beta} = \begin{pmatrix} 0.5390 & 0.4681 & 1.9468 \end{pmatrix}^T$$

Table B.3: Heterogeneous beliefs and the consensus belief, the individual optimal and market portfolios in equilibrium, and the means and standard deviations of these portfolios under heterogeneous and consensus belies, respectively.

In the above definitions, $\mathbb{E}_i(\mathbf{r})$ and $V_i(\tilde{\mathbf{r}})$ are the expected return vectors and covariance matrices in terms of asset returns for each investor. Subsequently, $\mathbf{w}_i^*$ are the individuals' optimal portfolio weights, $\mathbb{E}_i(\tilde{r}_{ip}^*)$ and σ_{ip}^* are the expected return and standard deviations of the optimal portfolios of investors, respectively, under their subjective beliefs, $\mathcal{B}_i = (\mathbb{E}_a(\tilde{r}_{ip}^*), \sigma_{ip}^a)$. Similarly, under the market belief $\mathcal{B}_a$, $\mathbf{w}_m$ is the market portfolio weight vector, $\mathbb{E}_a(\tilde{r}_m)$ and $\sigma_{a,m}$ are the market return and volatility under the market belief respectively. Finally $\boldsymbol{\beta}$ is the vector of beta coefficients. According to these definitions, we obtain results in Table B.3.

Bibliography

Abel, A. (1990), 'Asset prices under habit formation and catching up with the joneses', *American Economic Review Papers and Proceedings* **80**, 38–42.

Abel, A. (1999), 'Risk premia and term premia in general equilibirum', *Journal of Monetary Economics* **43**, 3–33.

Abel, A. (2002), 'An exploration of the effects of pessimism and doubt on asset returns', *Journal of Economic Dynamics and Control* **26**, 1075–1092.

Admati, A. (1985), 'A noisy rational expectations equilibrium for multi-asset security markets', *Econometrica* **53**, 629–657.

Adrian, T. and Franzoni, F. (2005), Learning about beta: Time-varying factor loadings, expected returns, and the conditional CAPM, Technical report, Federal Reserve Bank of New York. working paper.

Anderson, E., Ghysels., E. and Juergens, J. (2005), 'Do heterogeneous beliefs matter for asset pricing?', *Review of Financial Studies* **18**(3), 875–924.

Anderson, E., Ghysels., E. and Juergens, J. (2010), 'The impact of risk and uncertainty on expected returns', *Journal of Financial Economics* . forthcoming.

Bansal, R. and Yaron, A. (1996), 'A monetary explanation of the equity premium, term premium and risk free rate puzzles', *Journal of Political Economy* **104**, 1135–1171.

Barberis, N. and Thaler, R. (2003), *Handbook of the Economics of Finance*, Elsevier, chapter 18. A Survey of Behavioral Finance, pp. 1053–1128.

Basak, S. (2000), 'A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous beliefs', *Journal of Economic Dynamics and Control* **24**, 63–95.

Basak, S. (2005), 'Asset pricing with heterogeneous beliefs', *Journal of Banking and Finance* **29**, 2849–2881.

Basak, S. and Chabakauri, G. (2010), 'Dynamic mean-variance asset allocation', *Review of Financial Studies* p. forthcoming.

Basak, S. and Cuoco, D. (1998), 'An equilibrium model with restricted stock market participation', *Review of Financial Studies* **11**, 309–341.

Bates, D. (1991), 'The crash of '87: was it expected? The evidence form option markets', *Journal of Finance* **46**, 1009–1044.

Black, F. (1972), 'Capital market equilibrium with restricted borrowing', *Journal of Business* **45**, 444–454.

Blume, L. and Easley, D. (2006), 'If you are so smart, why aren't you rich? belief selection in complete and incomplete markets', *Econometrica* **74**, 929–966.

Böhm, V. and Chiarella, C. (2005), 'Mean variance preferences, expectations formation, and the dynamics of random asset prices', *Mathematical Finance* **15**, 61–97.

Böhm, V. and Wenzelburger, J. (2005), 'On the performance of efficient portfolios', *Journal of Economic Dynamics and Control* **29**, 721–740.

Bollerslev, T. (1986), 'Generalized autoregressive conditional heteroskedasticity', *Journal of Econometrics* **31**, 307–327.

Brock, W. and Hommes, C. (1997), 'A rational route to randomness', *Econometrica* **65**, 1059–1095.

Brock, W. and Hommes, C. (1998), 'Heterogeneous beliefs and routes to chaos in a simple asset pricing model', *Journal of Economic Dynamics and Control* **22**, 1235–1274.

Brown, A. and Rogers, C. (2009), *Diverse Beliefs*, Preprint, Statistical Laboratory, University of Cambrige.

Buraschi, A. and Jiltsov, A. (2006), 'Model uncertianty and option markets with heterogeneous beliefs', *Journal of Finance* **61**, 2814–2897.

Campbell, J. (2003), *Consumption-Based Asset Pricing*, Elsevier, pp. 803–887. in *Handbook of the Economics of Finance*, Constantinides, G.M., M. Harris and R. Stulz (Eds).

Campbell, J. and Vuolteenaho, T. (2004), 'Bad beta, good beta', *American Economic Review* **94**(5), 1249–1275.

Cao, H. and Ou-Yang, H. (2009), 'Differences of opinion of public information and speculative trading in stocks and options', *Review of Financial Studies* **22**, 299–335.

Carhart, M. (1997), 'On persistence in mutual fund performance', *Journal of Finance* **52**, 57–82.

Chiarella, C., Dieci, R. and Gardini, L. (2005), 'The dyanmic interaction of speculation and diversification', *Applied Mathematical Finance* **12**, 17–52.

Chiarella, C., Dieci, R. and He, X. (2007), 'Heterogeneous expectations and speculative behaviour in a dynamic multi-asset framework', *Journal of Economic Behavior and Organization* **62**, 402–427.

Chiarella, C., Dieci, R. and He, X. (2009), *Heterogeneity, market mechanisms and asset pricing dynamics*. in Hens, T. and K. R. Schenk-Hoppe (Eds), *Handbook of Financial Markets: Dynamics and Evolution*, pp. 277-344, Elsevier.

Chiarella, C., Dieci, R. and He, X. (2010*a*), *A framework for CAPM with heterogeneous beliefs.* in Bishi, G.I., C. Chiarella. and L. Gardini (Eds), *Nonlinear Dyanmaics in Economics, Finance and Social Sciences: Essays in Honour of John Barkley Rosser Jr*, pp. 353-369, Springer.

Chiarella, C., Dieci, R. and He, X. (2010*b*), 'Do heterogeneous beliefs diversity market risk?'. *European Journal of Finance*, forthcoming.

Chiarella, C., Dieci, R. and He, X. (2010*c*), Time-varying beta: A boundedly rational equilibrium approach, working paper, Quantitative Finance Research Center, University of Techonology, Sydney. No. 275.

Conlisk, J. (1996), 'Why bounded rationality?', *Journal of Economic Literature* **34**, 669–700.

Constantinides, G. (1990), 'Habit formation: a resolution of the equity premium puzzle', *Journal of Political Economy* **98**, 519–643.

Constantinides, G., Donaldson, J. and Mehra, R. (2002), 'Juniors can't borrow: a new perspective on the equity premium puzzle', *Quarterly Journal of Economics* **117**, 269–296.

Constantinides, G. and Duffie, D. (1996), 'Asset pricing with heterogenous conumers', *Journal of Political Economy* **104**, 219–240.

Cox, J. and Huang, C. F. (1989), 'Optimal consumption and portfolio policies when asset process follow a diffusion process', *Journal of Economic Theory* **49**, 33–83.

Cox, J., Ross, S. and Rubinstein, M. (1979), 'Option pricing: A Simplified Apporach', *Journal of Financial Economics* **7**, 229–263.

Cvitanic, J. and Zapatero, F. (2004), *Introduction to the Economics and Mathematics of Financial Markets*, The MIT Press, Cambridge, Massachusetts.

David, A. (2008), 'Heterogeneous beliefs, speculation, and the equity premium', *Journal of Finance* **63**(1), 41–83.

David, A. and Veronesi, P. (2002), Options under uncertain fundamentals. Working Paper, Univeristy of Chicago.

DeLong, J., Shleifer, A., Summers, L. and Waldmann, R. (1991), 'The survival of noise traders in financial markets', *Journal of Business* **64**, 1–19.

Detemple, J. and Murthy, S. (1994), 'Intertemporal asset pricing with heterogeneous beliefs', *Journal of Economic Theory* **62**, 294–320.

Detemple, J. and Serrat, A. (2003), 'Dynamic equilibrium with liquidity constraints', *Review of Financial Studies* **16**, 597–629.

Diether, K., Malloy, C. and Scherbina, A. (2002), 'Differences of opinion and cross section of stock returns', *Journal of Finance* **57**, 2113–2141.

Duan, J. C. (1995), 'The GARCH option pricing model', *Mathematical Finance* **5**, 13–32.

Easley, D. and O'Hara, M. (2004), 'Information and the cost of capital', *JF* **59**, 1553–1583.

Engle, R. (1982), 'Autoregressive conditional heteroscedasticity with estimates of the variance of uk inflation', *Econometrica* **50**, 987–1008.

Epstein, L. and Zin, S. (1989), 'Substitution, risk aversion, and the temporal behavior of consumption growth and asset returns i: A theoretical framework', *Econometrica* **57**, 937–969.

Epstein, L. and Zin, S. (1991), 'Substitution, risk aversion, and the temporal behavior of consumption growth and asset returns: an empirial investigation', *Journal of Political Economy* **99**, 263–286.

Fama, E. and French, K. (1992), 'The cross section of expected stock returns', *Journal of Finance* **47**, 427–465.

Fama, E. and French, K. (2007), 'Disagreement, tastes, and asset prices', *Journal of Financial Economics* **83**, 667–689.

Hara, C. (2009), Heterogeneous impatience in a continuous-time model. Preprint, Institute of Economic Research, Kyoto University.

Freeman, M. (2002), 'Asset pricing with jump/diffusion permanent income shocks', *Economic Letters* **77**, 1–8.

Friedman, M. (1953), *The case of flexible exchange rates*, Univ. Chicago Press. in Essays in Positive Economics.

Gallmeyer, M. and Hollifield, B. (2007), An examination of heterogeneous beliefs with a short sale constraint, working paper, Presented at the EFA 2002 Berlin Meetings. Available at SSRN: http://ssrn.com/abstract=302809.

Giordani, P. and Söderlind, P. (2006), 'Is there evidence of pessimism and doubt in subjective distribution? implications for the equity premium puzzle', *Journal of Economic Dynamics and Control* **30**, 1027–1043.

Guidolin, M. and Timmermann, A. (2003), 'Option prices under Bayesian learning: implied volatility dynamics and predictive densities', *Journal of Economic Dynamics and Control* **27**, 717–769.

Guidolin, M. and Timmermann, A. (2007), 'Properties of equilibrium asset prices under alternative learning schemes', *Journal of Economic Dynamics and Control* **31**, 161–217.

Hahn, W. and Dyer, J. (2008), 'Discrete time modelling of mean-reverting stochastic processes for real option valuation', *European Journal of Operational Research* **184**(184), 534–548.

Hara, C. (2009), Heterogeneous impatience in a continuous-time model. Preprint, Institute of Economic Research, Kyoto University.

Heath, D. and Platen, E. (2006), *A benchmark approach to finance*, Springer.

Heckman, J. (2001), 'Micro data, heterogeneity, and evaluation of public policy: Nobel Lecture', *Journal of Political Economy* **109**, 673–748.

Heston, S. (1993), 'A closed form solution for options with stocahstic volatility with appllciation to bond and currency options', *Review of Financial Studies* **6**, 327–343.

Heston, S. and Nandi, S. (2000), 'A closed form GARCH option valuation model', *Review of Financial Studies* **13**, 585–625.

Hong, H. and Stein, J. (2007), 'Disagreement and the stock market', *Journal of Economic Prospectives* **21**, 109–128.

Horst, U. and Wenzelburger, J. (2008), 'On no-ergodic asset prices', *Economic Theory* **34**, 207–234.

Huang, C.-F. and Litzenberger, R. (1988), *Foundations for Financial Economics*, Elsevier, North-Holland.

Hull, W. and White, A. (1987), 'The pricing of options on assets with stochastic volatlities', *Journal of Finance* **42**, 281–300.

Jagannathan, R. and Wang, Z. (1996), 'The conditional CAPM and cross-section of expected returns', *Journal of Finance* **51**, 3–53.

Jarrow, R. (1980), 'Heterogeneous expectations, restrictions on short-sales and equilibrium prices', *Journal of Finance* **35**, 1105–1111.

Johnson, T. (2004), 'Forecast dispersion and the cross section of expected returns', *Journal of Finance* **59**, 1957–1978.

Jouini, E. and Napp, C. (2006), 'Heterogeneous beliefs and asset pricing in discrete time: An analysis of pessimism and doubt', *Journal of Economic Dynamics and Control* **30**, 1233–1260.

Jouini, E. and Napp, C. (2007), 'Consensus consumer and intertemporal asset pricing with heterogeneous beliefs', *Review of Economic Studies* **74**, 1149–1174.

Kadaras, C. (2010), 'Numéraire-invariant preference in finanical modelling'. *Annals in Applied Probability*, forthcoming.

Karatzas, I., Lehoczky, J. P. and Shreve, S. (1987), 'Optimal portfolio and consumption decisions for a small investor on a finite horizon', *SIAM Journal of Control and Optimization* **25**, 1557–1586.

Karatzas, I. and Shreve, S. (1991), *Brownian Motion and Stochastic Calculus (2nd Ed)*, Springer, New York.

Kloeden, P. and Platen, E. (1992), *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag.

Kogan, L., Ross, S., Wang, J. and Westerfield, M. (2006), 'The price impact and survival of irrational traders', *Journal of Finance* **61**, 195–229.

Kothari, S., Shanken, J. and Sloan, R. (1995), 'Another look at the cross section of expected stock returns', *Journal of Finance* **50**, 185–224.

Kurz, M. (2009), *Rational Diverse Beliefs and Economic Volatility.* Hens, T. and K.R. Schenk-Hoppe (Eds), *Handbook of Financial Markets: Dynamics and Evoluation*, pp. 439-506, Elsevier.

Levy, M., Levy., H. and Benita, G. (2006), 'Capital asset prices with heterogeneous beliefs', *Journal of Business* **79**, 1317–1353.

Li, D. and Ng, W. (2000), 'Optimal dynamic portfolio selection: Multiperiod mean-variance formulation', *Mathematical Finance* **10**(3), 387–406.

Li, T. (2007), Heterogeneous beliefs, option prices, and volatlity smiles. SSRN Working Paper, http://ssrn.com/abstract=890277.

Lintner, J. (1965), 'The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets', *Review of Economics and Statistics* **47**, 13–37.

Lintner, J. (1969), 'The aggregation of investor's diverse judgements and preferences in purely competitive security markets', *Journal of Financial and Quantitative Analysis* **4**, 347–400.

Lintner, J. (1970), 'The market price of risk, size of market and investor's risk aversion', *Review of Economics and Statistics* **52**, 87–99.

Liptser, R. S. and Shiryaev, A. N. (2001), *Statistics of Random Processes*, Springer-Verlag, London.

Lux, T. (1995), 'Herd behaviour, bubbles and crashes', *Economic Journal* **105**, 881–896.

Lux, T. (2009), *Stochastic Behavioral Asset-Pricing Models and the Stylized Facts.* in Hens, T. and K. R. Schenk-Hoppe (Eds), *Handbook of Financial Markets: Dynamics and Evolution*, pp. 277-344, Elsevier.

Mehra, R. and Prescott, E. (1985), 'The equity premium: a puzzle', *Journal of Monetary Economics* **15**, 145–161.

Mehra, R. and Prescott, E. (2003), *The equity premium in retrospect*, Elsevier, pp. 887–936. in *Handbook of the Economics of Finance*, Constantinides, G.M., M. Harris and R. Stulz (Eds).

Melino, A. and Turnbull, S. (1990), 'Pricing foreign currency options with stochastic volatility', *Journal of Econometrics* **45**, 239–265.

Merton, R. (1976), 'Option pricing when underlying stock returns are discontinuous', *Journal of Financial Economics* **3**, 125–144.

Miller, E. (1977), 'Risk, uncertainity, and divergence of opinion', *Journal of Finance* **32**, 1151–1168.

Mossin, J. (1966), 'Equilibrium in a capital asset market', *Econometrica* **35**, 768–783.

Nelson, D. and Ramaswamy, K. (1990), 'Simple binomial process as diffusion approximations in finanical models', *Review of Financial Studies* **3**, 393–430.

Rubinstein, M. (1974), 'An aggregation theorem for securities markets', *Journal of Financial Economics* **1**, 225–244.

Rubinstein, M. (1975), 'Security market efficiency in an arrow-debreu economy', *American Economic Review* **65**, 812–824.

Rubinstein, M. (1976), 'The strong case for the generalized logarithmic utility model as the premier model of financial markets', *Journal of Finance* **31**, 551–571.

Sandroni, A. (2000), 'Do markets favor agents able to make accurate predictions?', *Econometrica* **68**, 1303–1342.

Sharpe, W. (1964), 'Capital asset prices: A theory of market equilibrium under conditions of risk', *Journal of Finance* **19**, 425–442.

Sharpe, W. (1970), *Portfolio Theory and Capital Markets*, McGraw-Hill Series in Finance.

Sharpe, W. (2007), *Investors and Markets, Portfolio Choices, Asset Prices, and Investment Advice*, Princeton University Press.

Shefrin, H. (2005), *A Behavioral Approach to Asset Pricing*, Elsevier.

Sun, N. and Yang, Z. (2003), 'Existence of equilibrium and zero-beta pricing forumla in the capitial asset pricing model with heterogeneous beliefs', *Annals of Economics and Finance* **4**, 51–71.

Sundaresan, S. (1989), 'Intertemporally dependent preferences and the volatility of consumption and wealth', *Review of Financial Studies* **2**, 73–88.

Van der Hoek, J. and Elliott, R. (2006), *Binomial Models in Finance*, Springer.

Weil, P. (1989), 'The equity premium puzzle and the risk-free rate puzzle', *Journal of Monetary Economics* **24**, 401–421.

Wenzelburger, J. (2004), 'Learning to predict rationally when beliefs are heterogeneous', *Journal of Economic Dynamics and Control* **28**, 2075–2104.

Wiggins, J. (1987), 'Option values under stochastic volatility: theory and empirical estimates', *Journal of Financial Economics* **19**, 351–372.

Williams, J. (1977), 'Capital asset prices with heterogeneous beliefs', *Journal of Financial Economics* **5**, 219–239.

Zapatero, F. (1998), 'Effects of financial innovations on market volatility when beliefs are heterogeneous', *Journal of Economic Dynamics and Control* **22**, 597–626.