The Evaluation of Early Exercise Exotic Options

A Thesis Submitted for the Degree of
Doctor of Philosophy

by

Jonathan Ziveyi

B.Sc.(Hons) (National University of Science and Technology, Zimbabwe)
Jonathan.Ziveyi@student.uts.edu.au

in

School of Finance and Economics
University of Technology, Sydney
PO Box 123 Broadway
NSW 2007, Australia

January 20, 2011

CERTIFICATE OF AUTHORSHIP/ORIGINALITY

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Signature of Candidate ....

Date ....

Acknowledgements

Firstly, I would like to I offer my sincerest gratitude to my supervisor Carl Chiarella whose encouragement, supervision and support during my doctoral studies has triggered and nourished my intellectual maturity.

I would also like to thank the following people in the School of Finance and Economics at UTS; Nadima El-Hassan, Tony Hall, Tony He, Boda Kang, Min Zheng, Samuel Chege for various assistance and helpful discussions during my studies. Needless to say, I am extremely grateful for financial assistance from the university which kept me going during my candidature. Many thanks to Ji-Won (Stephanie) Ough for her help with type-setting most of the equations and diagrams.

Words fail to express my appreciation to wife Nomah whose dedication, love and persistent confidence in me, has taken the load off my shoulders. Smiles from my daughter, Shalom kept me thriving to reach the mountain top. My special thanks goes to my parents, Elson and Loisi for their never dying support and encouragement.

Contents

Abstract

Research on the pricing of multifactor American options has been growing at a slow pace due to the curse of dimensionality. If we start to consider the pricing of American option contracts written on more than one underlying asset or relax the constant volatility assumption of the Black and Scholes (1973) model, the computational burden increases as more computing power is required to handle the increasing number of dimensions.

This thesis deals with the problem of pricing multifactor American options under both constant and stochastic volatility. The main focus of the thesis is to extend the representation results of Kim (1990) and Carr, Jarrow and Myneni (1992) and to devise higher dimensional numerical techniques for pricing multifactor American options. We present numerical examples for two and three factor models. The pricing problems are formulated using the well known hedging arguments. We adopt two main approaches; the first involves deriving integral expressions for the American option prices with the aid of Jamshidian's (1992) transformation of the associated partial differential equation from a homogeneous problem on a restricted domain to an inhomogeneous problem on an unrestricted domain, Duhamel's principle and integral transform methods. The second technique involves implementing the method of lines algorithm for American exotic options, with the spread call option under stochastic volatility being the main example – this approach tackles directly the pricing partial differential equation. Chapter 1 contains an overview of the American option pricing problem from the viewpoint of the applications in this thesis. The chapter concludes with some technical results used in the rest of the thesis. The main contributions of the thesis are contained in the subsequent chapters.

Chapter 2 extends the integral transform approach of McKean (1965) and Chiarella and Ziogas (2005) to the pricing of American options written on two underlying assets under Geometric Brownian motion. A bivariate transition density function of the two underlying stochastic processes is derived by solving the associated backward Kolmogorov partial differential equation. Fourier transform techniques are used to transform the partial differential equation to a corresponding ordinary differential equation whose solution can be readily found by using the integrating factor method. An integral expression of the American option written on any two assets is then obtained by applying Duhamel's principle. A numerical algorithm for calculating American spread call option prices is given as an example, with the corresponding early exercise boundaries approximated by linear functions. Numerical results are presented and comparisons made with other alternative approaches.

Chapter 3 considers the pricing of an American call option whose underlying asset evolves under the influence of two independent stochastic variance processes of the Heston (1993) type. We derive the associated partial differential equation (PDE) for the option price using standard hedging arguments. An integral expression for the general solution of the PDE is derived using Duhamel's principle, which is expressed in terms of the yet to be determined trivariate transition density function for the driving stochastic processes. We solve the backward Kolmogorov PDE satisfied by the transition density function by first transforming it to the corresponding characteristic PDE using a combination of Fourier and Laplace transforms. The characteristic PDE is solved by the method of characteristics. Having determined the density function, we provide a full representation of the American call option price. By approximating the early exercise surface with a bivariate log-linear function, we develop a numerical algorithm to calculate the pricing function. Numerical results are compared with those from the method of lines algorithm. The approach is generalised in Chapter 4 to the case when the underlying asset evolves under the influence of more than two stochastic variance processes by using a combination of induction proofs and some lengthy derivations.

A numerical technique for the evaluation of American exotic options is developed in Chapter 5, with the American spread call option whose underlying assets dynamics evolve under the influence of a single stochastic variance process being presented as an example. The numerical algorithm involves extending the method of lines approach first presented in Meyer and van der Hoek (1997) when pricing the standard American put option to the multi-dimensional setting. We transform the pricing partial differential equation to a corresponding system of ordinary differential equations with the aid of the Riccati transformation. We use the implicit trapezoidal rule to solve the resulting Riccati equations. Numerical results are presented outlining the effectiveness of the algorithm. The effects of stochastic volatility are explored by making comparisons with the geometric Brownian motion results.

We summarise all thesis findings in Chapter 6. Concluding remarks and directions for future work are also presented.

CHAPTER 1

Introduction

1.1. Early Literature Survey

A derivative security is an instrument which derives its value from an underlying asset. There are many different types of derivative securities, the most popular ones being forwards, futures and option contracts. A forward contract is an over-the-counter (OTC) agreement to buy or sell an asset at a specified price in the future. A futures contract shares similar features with forward contracts, except that a futures contract is standardised and exchange-traded. While both parties to either forward or futures contracts are obliged to fulfill their obligations as per the contract terms, the holder of an option contract has the right but not the obligation to buy or sell an underlying asset at an agreed date and at a pre-specified price. The option contract holder will only exercise if it is profitable to do so.

There are two main types of option contracts, namely European and American options. European options can only be exercised at maturity while American options can be exercised any time prior to maturity. European and American options can further be broken down into calls and puts. A call(put) option gives the holder the right to buy(sell) the underlying asset on or before maturity. Options can be written on single or multiple underlying assets. In this thesis we focus on multifactor American options.

Much work on the evaluation of options has been undertaken since the inception of the Black and Scholes (1973) framework, which is premised on hedging arguments involving the construction of a hedged portfolio consisting of a stock and an option written on the stock. Application of Ito's Lemma yields the dynamics of the option price and application of the condition of no riskless arbitrage to the hedged portfolio yields the associated option pricing partial differential equation (PDE) for European call and put options. This technique has been extended to American option pricing by adding suitable early exercise boundary conditions.

Whilst closed-form solutions have been derived in the 1970's for European calls and puts under Geometric Brownian Motion (GBM) dynamics by Black and Scholes (1973), no such solution exists for their American counterparts. A vast range of research has since occurred on how to best represent American option prices written on a single underlying asset. In so doing, four major pathways have been adopted in the literature namely, approximate solution techniques, compound option theory, discrete numerical methods and the free-boundary value problem approach.

With the quest to represent American option prices with analytical expressions, approximation techniques have been the first to be used with Roll (1977) creating a portfolio of three European call options which duplicates the cash flows of an American call option on a stock that pays a single dividend before the expiry date. This approach has been refined in Geske (1979a), Whaley (1981) and Geske (1981) who all demonstrate that valuation by duplication does not yield a unique solution. Johnson (1983) also proposes an approximate analytical expression for the price and hedge ratios of an American put option written on a non-dividend paying stock.

The compound option theory of Geske (1979b) has been used to derive an analytical expression for an American put option price in Geske and Johnson (1984). Their analytical formula satisfies the associated Black and Scholes PDE, as well as the boundary conditions of the American put valuation problem. Geske and Johnson (1984) show that the American put option is equivalent to an infinite sequence of options on options that can be exercised at discrete points in time. This infinite sequence of options on options is then solved with the aid of a polynomial expression based upon some extrapolation techniques resulting in an American option price. The compound option approach is further adopted by Whaley (1986), who considers the valuation of American futures contracts using compound option theory.

While approximation methods and compound option theory have played an important role in exposing the complexity of valuing American options, more compact numerical results can be found by the use of discrete numerical methods. Brennan and Schwartz (1977) use finite-difference techniques to solve the Black and Scholes (1973) PDE for the American put option subject to appropriate boundary and terminal conditions. Finite-difference techniques can handle any American option payoff structure and this is one of their major strengths. Cox, Ross and Rubinstein (1979) develop the binomial tree method premised on the risk-neutral pricing framework of Cox and Ross (1976). Under this framework, the continuous stock price distribution is replaced by a two-point discrete distribution over successive time intervals. Convergence to the true option value is achieved by increasing the number of time steps. This approach proves to be very fast and efficient in generating option prices.

Another numerical technique which has proved to be very powerful in terms of computational speed and accuracy is the method of lines (MOL) technique applied to option pricing in Meyer and van der Hoek (1997) and Carr and Faguet (1996). A particularly convenient feature of this approach is that it computes the price, delta and gamma of the option, as well as the free-boundary simultaneously at no additional computational cost. This approach can easily handle more complex payoff functions as will be demonstrated later in this thesis. Another novel approach is that of Chiarella, El-Hassan and Kucera (1999) who present the path integral formulation for the value of the American put option as an expansion of the value function in a series of orthogonal Fourier-Hermite expansions. A backward recursive algorithm for the American put option price together with the associated early exercise boundary is developed using the orthogonal polynomials. The valuation problem is discretised with respect to time but remains continuous in the underlying asset domain. This method proves to be particularly effective in generating the early exercise boundary.

The American option pricing problem has also been formulated as a free-boundary value problem. An initial attempt along these lines is the work of McKean (1965) who formulates the valuation problem as a free-boundary problem. As the free-boundary value problem is defined on a restricted domain, McKean uses incomplete Fourier transform techniques to reduce the partial differential equation (PDE) to a corresponding ordinary differential equation (ODE). The solution of the resulting ordinary differential equation is presented in integral form with the aid of the integrating factor method. A refinement of McKean's approach is presented in Kim (1990) who develops an equivalent but different expression for the American put option price. This representation is very convenient as it does not involve the derivative of the early exercise boundary, which becomes infinite at maturity and is difficult to handle numerically. The resulting expression can be readily implemented numerically. The American option price is shown to be the sum of two parts namely, a European option component and an early exercise premium component. This result has been authenticated by many researchers during the early 90's with Jacka (1991), Carr et al. (1992) and Jamshidian (1992) who all show that the American option consists of a European option component and an early exercise premium component. Kim (1990) also reconciles the convergence of Geske and Johnson (1984) to McKean (1965) when the number of possible exercise times approach infinity. A derivation of the early exercise boundary at expiration is also presented in Kim (1990).

Armed with the Kim (1990) integral expression, Huang, Subrahmanyam and Yu (1996) provide a unified analytical valuation formula for American option prices that is based on the recursive computation of the early exercise boundary. This method uses the Geske-Johnson extrapolation techniques to compute option prices and hedge ratios simultaneously. Like the Geske and Johnson (1984) approach, the optimal-exercise boundary is calculated at a few points and the Richardson Extrapolation scheme incorporated to approximate the entire boundary. Ju (1998) takes Kim's (1990) integral equation and discretises it with respect to time into small subintervals. The early exercise boundary is then treated as an exponential function during each subinterval. This technique yields a closed-form integral formula for the entire early exercise premium term. Kallast and Kivinukk (2003) provide a numerical approximation technique for the Kim (1990) option pricing equation together with its delta and gamma using quadrature methods. The early exercise boundary is generated by making use of the Newton-Raphson iterative procedure. The method proves to be very efficient in terms of computational speed and accuracy. A survey of the integral representation of American option is presented in Chiarella and Ziogas (2004) where numerical comparisons of such methods are also highlighted.

Extensions to pricing options written on more than one underlying asset have also been considered in the literature. Magrabe (1978) derives a closed-form integral expression for the price of the European exchange call option[1] by taking one of the underlying assets as a numeraire and applying the Black and Scholes (1973) hedging arguments. Magrabe (1978) also shows that the closed-form solution for the European exchange option can be used to price the special form of the corresponding American exchange option written on the two underlying assets as it is never optimal to exercise such an option before maturity. Stulz (1982) provides a closed-form formula for pricing European puts and calls on the maximum or the minimum of two assets. The solution is expressed in terms of bivariate cumulative standard normal distributions. When the strike price is zero the formula reduces to the Magrabe (1978) formula. Much of the work on multifactor options that followed has mainly focused on European options with Johnson (1987) deriving explicit formulas for the price of European options written on the maximum or minimum of several assets. When only two underlying securities are considered, the formulas reduce to the results of Stulz (1982). Boyle, Evnine and

[1]This is an option to exchange one asset for another asset.

Gibbs (1989) also develop some numerical approximation schemes for pricing European options on multiple assets by extending the lattice binomial techniques of Cox et al. (1979). A survey of different forms of European spread option pricing models is also presented in Carmona and Durrleman (2003).

Whilst the pricing of multifactor European options has attracted a great deal of attention, much less has been done on pricing the corresponding American options. Geltner, Riddiough and Stojanovic (1996) derive expressions for option prices and their associated early exercise boundaries on real estate investments. Broadie and Detemple (1997) consider different payoff functions for American options written on two underlying assets and provide integral representations of the corresponding prices. Jackson, Jaimungal and Surkov (2008) and Jaimungal and Surkov (2008) develop a dynamic programming algorithm for pricing American options written on two underlying assets using the Fourier space time-stepping method.

All the work considered above is premised on the restrictive Black and Scholes (1973) assumptions, in particular that the volatility of the underlying asset returns is constant. This assumption implies that asset returns are normally distributed. However, a vast range of empirical findings even before the Black and Scholes (1973) model reveals that asset returns are not normally distributed. Mandelbrot (1963) postulates that the empirical distributions of price changes are usually too peaked to be viewed as samples from Gaussian populations and suggests symmetric-stable distributions as a possible candidate to model such changes. Press (1967) devises a distribution that is a Poisson mixture of normal distributions. This distribution is able to capture well the skewness, leptokurtosis and high peaks exhibited by asset returns. Praetz (1972) modifies the Brownian motion theory to account for changing variance of the share market leading to a scaled t-distribution. Officer (1972) shows that the distribution of asset returns has some characteristics of a non-normal generating process. The return distribution is shown to be fat-tailed relative to a normal distribution. Officer (1972) also presents evidence illustrating that there is a tendency for longitudinal sums of daily stock returns to become thinner-tailed for larger sums but not to the extent that a normal distribution approximates the distribution.

Clark (1973) proposes a subordinated stochastic process to model asset price series after noting the leptokurtic feature of their return distribution. It is also shown in Clark (1973) that conditions for the Central Limit Theorem which guarantee normality for the return distribution are not satisfied. Blattberg and Gonedes (1974) also show that the student-t distribution can very well account for the fat-tails observed in return distributions. Comparisons between student-t and the symmetric-stable distributions of Mandelbrot (1963) are also presented in this paper. Much work on empirical distribution of asset returns followed, with the most recent findings by Platen and Rendek (2008) showing that returns of world stock indices are non-normal and can also be modelled by a student-t distribution. As pointed out earlier, many empirical studies demonstrate that volatility of asset returns is not constant with Rosenberg (1972), Latané and Rendleman (1976) among others coming to the same conclusion through the study of implied volatility.

Much work has followed along this line of research with Scott (1987) also providing empirical evidence showing that volatility changes with time and that the changes are unpredictable. Though persistent in nature, Scott notes that volatility has a tendency to revert to a long-run average. These findings triggered interest in incorporating mean-reverting stochastic volatility into option pricing models with much earlier work focusing on European type options. Scott derives the pricing function of a European option written on an underlying asset whose dynamics evolve under the influence of instantaneous volatility following an Ornstein-Uhlenbeck process. Risk neutral arguments are used to derive the European pricing function which can then be solved by using standard Monte Carlo simulation techniques. Monte Carlo techniques have also been applied in Johnson and Shanno (1987) to simulate European call options for varying drift and diffusion parameters of the stochastic volatility process influencing the dynamics of the underlying asset.

Hull and White (1987) assume that both the underlying asset and the corresponding instantaneous variance are both driven by the Geometric Brownian Motion (GBM) type of processes. They derive a Taylor series solution for the European option price by solving the associated option pricing PDE under the assumption that the underlying asset price is instantaneously uncorrelated with the volatility. Wiggins (1987) formulates and solves the pricing PDE of a European call option under stochastic volatility using finite difference techniques. An explicit closed-form solution is derived in Stein and Stein (1991) by considering an underlying asset influenced by stochastic volatility whose evolution follows an arithmetic Ornstein-Uhlenbeck process. The most widely used formula for pricing European options under stochastic volatility is that of Heston (1993) who derives an analytical expression by modelling the dynamics of instantaneous variance with the square-root process, originally proposed by Feller (1951) and used by Cox, Ingersoll and Ross (1985) in interest rate modelling. Many practitioners and researchers frequently use the Heston (1993) model because of the integral form of the resulting solution. Zhu (2000) also provides an integral formula for European options where the underlying asset evolves under the influence of a stochastic variance process following a double square root process.

Extensions have been made to American option pricing with Touzi (1999) generalising the Black and Scholes (1973) model by allowing volatility to vary stochastically. Touzi also describes the dependence of the early exercise boundary of the American put option on the volatility parameter and proves that such a boundary is a decreasing function of volatility implying that for a fixed underlying asset price, as the volatility increases, the early exercise boundary gets lower. Clarke and Parrott (1999) develop an implicit finite-difference scheme for pricing American options written on underlying assets whose dynamics evolve under the influence of stochastic volatility. A multigrid algorithm is described for the fast iterative solution of the resulting discrete linear complementarity problems. Computational efficiency is also enhanced by a strike price related analytical transformation of the asset price and adaptive time-stepping.

Detemple and Tian (2002) provide analytical integral formulas for the early exercise boundary and the option price when the asset price follows a Constant Elasticity of Variance (CEV) process. The characteristic functions of the formulas are expressed in terms of $\mathcal{X}^2$ distribution functions. Tzavalis and Wang (2003) derive the integral representation of an American call option price when the volatility process evolves according to the square-root process proposed by Heston (1993). They derive the integral expressions using optimal stopping theory along the lines of Karatzas (1988). By appealing to the empirical findings by Broadie, Detemple, Ghysels and Torres (2000) who show that the early exercise boundary when variance evolves stochastically is a log-linear function of both time and instantaneous variance, a Taylor series expansion is applied to the resulting early exercise surface around the long-run variance. The unknown functions resulting from the Taylor series expansion are then approximated by fitting Chebyshev polynomials. Ikonen and Toivanen (2004) formulate and solve the linear complementarity problem of the American call option under stochastic volatility using componentwise splitting methods. The resulting subproblems from componentwise splitting are solved by using standard partial differential equation methods.

Adolfsson, Chiarella, Ziogas and Ziveyi (2009) also derive the integral representation of the American call option under stochastic volatility by formulating the pricing PDE as an inhomogeneous problem and then using Duhamel's principle to represent the corresponding solution which is in terms of the joint transition density function. The joint density function solves the associated backward Kolmogorov PDE and a systematic approach for solving such a PDE is developed. A combination of Fourier and Laplace transforms is used to transform the homogeneous PDE for the density function to a characteristic PDE. The resulting system is then solved using ideas first presented by Feller (1951). The early exercise boundary is approximated by a log-linear function as proposed in Tzavalis and Wang (2003). Instead of using approximating polynomials as in Tzavalis and Wang (2003) they derive an explicit characteristic function for the early exercise premium component and then incorporate numerical root finding techniques to find the unknown functions from the log-linear approximation.

There have also been attempts to generalise the Heston (1993) model to a multifactor specification for the volatility process in a single asset framework with da Fonseca, Grasselli and Tebaldi (2008) considering the pricing of European type options written on a single underlying asset whose dynamics evolve under the influence of the matrix Wishart volatility process. As demonstrated in da Fonseca et al. (2008) the main advantages of a multiple volatility system is that it calibrates short-term and long-term volatility levels better than a single process.

1.2. Thesis Structure

There are three major contributions in this thesis. The first, which is presented in Chapter 2, focuses on the pricing of American options written on more than one underlying asset whose dynamics evolve according to the geometric Brownian motion processes proposed by Black and Scholes (1973). This chapter first handles the case of American options written on two underlying assets and the results are then extended to the multi-asset setting. The second contribution is contained in Chapters 3 and 4 and involves the derivation of the American call option price for a contract written on a single underlying asset whose dynamics evolve under the influence of multiple stochastic variance processes of the Heston (1993) type. Chapter 3 covers the case when the underlying asset dynamics evolve under the influence of two stochastic variance processes. Chapter 4 extends this model to the case where the underlying asset dynamics are influenced by multiple stochastic variance processes. Chapter 5 contains the third contribution which extends the model of Dempster and Hong (2000) for pricing European spread options when the two underlying assets dynamics evolve under the influence of a single stochastic variance process to the American spread option case. The method of lines approach is adopted to develop a very convenient numerical scheme.

1.2.1. Fourier Transform Approach for American Options on Two Assets. Fourier transform techniques are widely used to solve partial differential equations in the physical sciences and have proved to be very powerful tools in option pricing. McKean (1965) is the first to our knowledge to represent the general solution of the standard American put option by formulating the pricing problem as a free-boundary problem. Chiarella and Ziogas (2004,2005) operationalise McKean's (1965) approach by providing numerical algorithms and numerical examples for American calls and strangles respectively.

Chapter 2 extends these ideas to American options written on two underlying assets and we provide as an example the American spread call option. The difference between our approach and that based on McKean is that we derive the general solution of the pricing PDE by exploiting the techniques of Jamshidian (1992) which involves the transition density function. We solve the transition density PDE with the aid of complete Fourier transforms resulting in an ODE that is easier to handle unlike the incomplete Fourier transforms adopted by McKean (1965). The most powerful feature of transform methods is that the resulting pricing function is applicable to any continuous payoff function involving two underlying assets whose dynamics are governed by some processes. We present a systematic approach for solving the pricing PDE for American options written on two underlying assets in Chapter 2. A numerical algorithm for the American spread call option example is provided followed by a section on numerical results. We also compare our numerical results with other alternative approaches to verify their accuracy.

1.2.2. American Option Pricing Under Two Stochastic Volatility Processes. As discussed in earlier sections, a vast range of empirical studies (see Mandelbrot (1963) through to the most recent work by Platen and Rendek (2008)) reveal that asset returns are non-normal, exhibit fat tails and possess some leptokurtic features. It has also been proven empirically that volatility of asset returns changes with time and the changes are unpredictable (see Scott (1987)). These observations contradict one of the basic Black and Scholes (1973) assumptions namely that volatility of asset returns is constant. Rather volatility has a tendency to revert back to a long-run average. Based on these empirical findings, we are motivated to study models that accommodate stochastic volatility with mean reversion.

In Chapter 3 we develop an option pricing model for the case when the underlying asset evolves under the influence of two independent stochastic variance processes of the Heston (1993) type. Such processes have been considered in da Fonseca, Grasselli and Tebaldi (2005) and da Fonseca et al. (2008) when pricing European style options. The main advantage cited in da Fonseca et al. (2008) is that such processes fit accurately market data for short or long maturities. While this has been empirically tested in their work, we would also suggest that the dynamics of the underlying asset may be influenced by many independent stochastic factors which can be modelled as mean reverting processes. Motivated by these considerations, we derive the free-boundary pricing PDE and represent its general solution by exploiting Duhamel's principle.

The general solution is a function of the transition density function which in turn is the solution of the Kolomogorov PDE for the driving stochastic processes. A systematic approach for solving the transition density function is presented and involves a combination of Fourier and Laplace transforms. We also provide a numerical algorithm for the evaluation of the American call option. Numerical results follow and comparisons are made with the method of lines algorithm. A natural extension of the model to the case when the underlying asset evolves under the influence of more than two stochastic variance processes is provided in Chapter 4.

1.2.3. American Spread Option Evaluation under Stochastic Volatility.

Spread option contracts are frequently written on correlated underlying assets implying that the fundamentals that affect one asset also directly influence the other asset. In Chapter 5 we propose an American spread option model where the two underlying asset dynamics are influenced by a single stochastic variance process. The European option case is handled in Dempster and Hong (2000) who develop a fast Fourier transform algorithm for European spread options. An extension of the fast Fourier transform method to the pricing of American option written on two underlying assets whose dynamics are governed by Levy processes is presented in Jackson et al. (2008) and Jaimungal and Surkov (2008). Broadie and Detemple (1997) provide integral expressions for American option written on two underlying assets under the geometric Brownian motion framework. To date not much work has been done on American spread options under stochastic volatility.

In Chapter 5 we extend the method of lines, first applied to option pricing by Meyer and van der Hoek (1997), to the pricing of American spread options under stochastic volatility. The method of lines has also been considered in Chiarella, Kang, Meyer and Ziogas (2009) when evaluating an American option written on an underlying asset whose dynamics evolve under the influence of both stochastic volatility and jump diffusion processes. The most powerful feature of the method of lines is that the price, delta, gamma and the early exercise boundary are all generated simultaneously as part of the solution at no additional computational cost.

1.3. Key Definitions and Assumptions

In this section we state some key definitions which we will be using in most parts of the thesis. In Chapters 2-4 we will consider transition probability density functions[2], $U(\tau, \mathbf{x}, \mathbf{v}; \mathbf{x_0}, \mathbf{v_0})$ which denotes the transition from the state $(\mathbf{x}, \mathbf{v})$ at time-to-maturity, τ to $(\mathbf{x_0}, \mathbf{v_0})$ at maturity of an option contract. The underlying state variables are $\mathbf{x}$ and $\mathbf{v}$. Such density functions are solutions of the backward Kolmogorov partial differential equations (PDEs) associated with the $\mathbf{x}, \mathbf{v}$ processes. For convenience unless otherwise specified, we simply write $U(\tau, \mathbf{x}, \mathbf{v})$ to denote the transition probability density function.

The derivation of solutions of these Kolmogorov PDEs will involve using a combination of Fourier and Laplace transform techniques to transform the original PDE to corresponding system which can be solved using methods we outline. We make the following assumptions about this density function:

$$\lim_{\mathbf{x} \to \pm\infty} U(\tau, \mathbf{x}, \mathbf{v}) = \lim_{\mathbf{x} \to \pm\infty} \frac{\partial U}{\partial \mathbf{x}} = 0, \tag{1.3.1}$$

$$\lim_{\mathbf{v} \to \infty} U(\tau, \mathbf{x}, \mathbf{v}) = \lim_{\mathbf{v} \to \infty} \frac{\partial U}{\partial \mathbf{v}} = 0, \tag{1.3.2}$$

$$\lim_{\mathbf{v} \to 0} U(\tau, \mathbf{x}, \mathbf{v}) = 0. \tag{1.3.3}$$

Such assumptions are made since by their nature, transition density functions and their derivatives converge to zero as their underlying processes assume larger values. We present the definitions of Fourier and Laplace transforms below. We further assume that the density function, $U(\tau, \mathbf{x}, \mathbf{v})$ is twice differentiable with respect to both $\mathbf{x}$ and $\mathbf{v}$.

In Chapter 2 we operate under the GBM assumption implying that the $\mathbf{v}$ – vector will be constant. When considering American option pricing under multiple stochastic

[2]Here, $\mathbf{x} = (x_1, \cdots, x_n)$ is a vector of log underlying asset prices and $\mathbf{v} = (v_1, \cdots, v_n)$ is a vector of instantaneous variances on the respective assets.

volatility processes in Chapters 3 and 4 the underlying asset vector, $\mathbf{x}$ will be a scalar as we consider only one asset.

1.3.1. The Fourier and Laplace Transforms.

DEFINITION 1.3.1. *The Fourier transform of a function $U(\tau, \mathbf{x}, \mathbf{v})$ with respect to the vector $\mathbf{x}$ is defined as*

$$\mathcal{F}\{U(\tau, \mathbf{x}, \mathbf{v})\} = \int_{\mathbb{R}^n} e^{i\boldsymbol{\eta}\cdot\mathbf{x}} U(\tau, \mathbf{x}, \mathbf{v}) d\mathbf{x} =: \hat{U}(\tau, \boldsymbol{\eta}, \mathbf{v}), \tag{1.3.4}$$

where i is the complex number and $\boldsymbol{\eta}$ is the transform argument.

DEFINITION 1.3.2. *The inverse Fourier transform of $\hat{U}(\tau, \boldsymbol{\eta}, \mathbf{v})$ is represented as*

$$\mathcal{F}^{-1}\{\hat{U}(\tau, \boldsymbol{\eta}, \mathbf{v})\} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\boldsymbol{\eta}\cdot\mathbf{x}} \hat{U}(\tau, \boldsymbol{\eta}, \mathbf{v}) d\boldsymbol{\eta} = U(\tau, \mathbf{x}, \mathbf{v}). \tag{1.3.5}$$

DEFINITION 1.3.3. *The Laplace transform of a function $\hat{U}(\tau, \boldsymbol{\eta}, \mathbf{v})$ with respect to the vector $\mathbf{v}$ is defined as*

$$\mathcal{L}\{\hat{U}(\tau, \boldsymbol{\eta}, \mathbf{v})\} = \int_{\mathbb{R}^n_+} e^{-\mathbf{s}\cdot\mathbf{v}} \hat{U}(\tau, \boldsymbol{\eta}, \mathbf{v}) d\mathbf{v} =: \tilde{U}(\tau, \boldsymbol{\eta}, \mathbf{s}), \tag{1.3.6}$$

where $\mathbf{s}$ is a vector of complex variables whenever the improper integral exists.

The necessary conditions for Definition 1.3.3 to hold are that for every fixed $\mathbf{s} > 0$ and $\tau > 0$ the functions $e^{-\mathbf{s}\cdot\mathbf{v}} \hat{U}(\tau, \boldsymbol{\eta}, \mathbf{v})$ and $e^{-\mathbf{s}\cdot\mathbf{v}} \frac{\partial \hat{U}}{\partial \tau}(\tau, \boldsymbol{\eta}, \mathbf{v})$ should be integrable over $0 < \mathbf{s} < \infty$, and this uniformly in every interval $0 < \tau < T$, which we will assume. We also assume that the Laplace transform of $\frac{\partial \hat{U}}{\partial \tau}(\tau, \boldsymbol{\eta}, \mathbf{v})$ exists and can be obtained by formal differentiation of equation (1.3.6).

DEFINITION 1.3.4. *The inverse Laplace transform*[3] *of the function* $\tilde{U}(\tau, \boldsymbol{\eta}, \mathbf{s})$ *with respect to* $\mathbf{s}$ *is defined as*

$$\mathcal{L}^{-1}\{\tilde{U}(\tau, \boldsymbol{\eta}, \mathbf{s})\} = \frac{1}{(2\pi i)^n} \int_{\mathbf{c}-i\infty}^{\mathbf{c}+i\infty} e^{\mathbf{s}\cdot\mathbf{v}} \tilde{U}(\tau, \boldsymbol{\eta}, \mathbf{s}) d\mathbf{s} =: \hat{U}(\tau, \boldsymbol{\eta}, \mathbf{v}), \tag{1.3.7}$$

where the integral may be multidimensional and $\mathbf{c} > \mathbf{d}$ *for some real vector* $\mathbf{d}$ *and* $\tilde{U}(\tau, \boldsymbol{\eta}, \mathbf{s})$ *is analytic for*[4] $Re(\mathbf{s}) > \mathbf{d}$.

1.3.2. Duhamel's Principle. In most cases we will be making use of an important result for the representation of the general solution of inhomogeneous PDEs. This result is stated below.

PROPOSITION 1.3.1. *(Duhamel's principle) Consider the one dimensional inhomogeneous parabolic PDE of the form*

$$\frac{\partial C}{\partial \tau} = \mathcal{L}C + f(\tau, x), \tag{1.3.8}$$

where $\mathcal{L}$ *is a parabolic partial differential operator (which will be made explicit in particular applications) and solved subject to the initial condition*

$$C(0, x) = \phi(x). \tag{1.3.9}$$

Let $U(\tau, x)$ *be the transition probability density function which is the solution to*

$$\frac{\partial U}{\partial \tau} = \mathcal{L}U, \tag{1.3.10}$$

subject to the initial condition

$$U(0, x) = \delta(x - x_0), \tag{1.3.11}$$

[3] In this thesis we will not evaluate directly the inverse Laplace transform from this definition as we will make use of those tabulated in Abramowitz and Stegun (1964).
[4] $Re(\mathbf{s}) > \mathbf{d}$ specifies the region in which $\tilde{U}(\tau, \boldsymbol{\eta}, \mathbf{s})$ is analytic.

then the solution of equation (1.3.8) *can be represented as*

$$C(\tau, x) = \int_{-\infty}^{\infty} \phi(y) U(\tau, x - y) dy + \int_{0}^{\tau} \int_{-\infty}^{\infty} f(\xi, y) U(\tau - \xi, x - y) dy d\xi.$$

Proof: The proof of this proposition can be found in Logan (2004) . □

The multi-dimensional version of Duhamel's principle can be readily derived as shown in Appendix 2.2. More details about Duhamel's principle can be found in Logan (2004).

1.4. Properties of American Spread Call Options

An American spread call option, $V(t, S_1, S_2)$, is a contract written on two underlying assets, S_1 and S_2 whose payoff can be represented as $v(S_1, S_2) = \max(S_1 - S_2 - K, 0)$, where K is the strike price and t is the current time. As highlighted before, American style option contracts can be exercised anytime prior to maturity. Due to this early exercise feature, the price $V(t, S_1, S_2)$ of the American spread call option should be at least the same as the payoff function $v(S_1, S_2)$. This leads to the early exercise constraint $V(t, S_1, S_2) \geq v(S_1, S_2)$. It is not known a priori where this constraint is active thus making derivation of analytical formulas for the price and corresponding hedge ratios generally intractable. The point at which this constraint becomes active is termed the early/immediate exercise boundary. We will denote this early exercise boundary of the American spread call option as $S_1 = b(t, S_2)$. By setting $\tau = T - t$ where T is the expiry time, the American spread call option satisfies the linear complementarity problem

$$\mathcal{L}V - rV - \frac{\partial V}{\partial \tau} \geq 0, \quad \text{for all} \quad \tau \in [0, T], \quad S_1 \geq 0, \quad S_2 \geq 0, \tag{1.4.1}$$

$$V(\tau, S_1, S_2) \geq v(S_1, S_2), \quad \text{for all} \quad \tau \in [0, T], \quad S_1 \geq 0, \quad S_2 \geq 0, \tag{1.4.2}$$

where

$$\mathcal{L} = (r - q_1)S_1\frac{\partial}{\partial S_1} + (r - q_2)S_2\frac{\partial}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2}{\partial S_2^2}. \tag{1.4.3}$$

is the Black-Scholes partial differential operator. Here, r is the risk-free rate of interest, q_1 and q_2 are the continuously compounded dividend yields of S_1 and S_2 respectively, ρ is the correlation between the two underlying assets and should be chosen such that $|\rho| < 1$, and σ_1 and σ_2 are the constant volatilities associated with S_1 and S_2 respectively. Boundary conditions are imposed on equations (1.4.1) and (1.4.2) such that

$$V(\tau, 0, S_2) = 0 \quad \text{and} \quad V(\tau, S_1, 0) = \max(S_1 - K, 0). \tag{1.4.4}$$

On the far-field, the asymptotic behaviour of $V(\tau, S_1, S_2)$ is given by

$$\lim_{S_1 \to \infty} \frac{\partial V}{\partial S_1}(\tau, S_1, S_2) = 0 \quad \text{and} \quad \lim_{S_2 \to \infty} \frac{\partial V}{\partial S_2}(\tau, S_1, S_2) = 0. \tag{1.4.5}$$

Due to the early exercise constraint we introduced earlier, we can then divide the set $\{(\tau, S_1, S_2); 0 \leq \tau \leq T, S_1 \geq 0, S_2 \geq 0\}$ into two regions, namely; the stopping set

$$\mathcal{S} = \left\{(\tau, S_1, S_2); V(\tau, S_1, S_2) = (S_1 - S_2 - K)^+\right\}, \tag{1.4.6}$$

and the continuation set

$$\mathcal{C} = \left\{(\tau, S_1, S_2); V(\tau, S_1, S_2) > (S_1 - S_2 - K)^+\right\}. \tag{1.4.7}$$

The surface of $S_1 = b(\tau, S_2)$ forms the boundary between $\mathcal{C}$ and $\mathcal{S}$ and belongs to the set of stopping times $\mathcal{S}$. Shreve (2000) shows that equality holds in equation (1.4.1) for $(\tau, S_1, S_2) \in \mathcal{C}$ provided that $\tau \neq T$. For $(\tau, S_1, S_2) \in \mathcal{S}$ a strict inequality holds in equation (1.4.1) except on the surface $S_1 = b(\tau, S_2)$ where the equality holds in (1.4.1).

Now, for $S_1 \geq b(\tau, S_2)$ we have

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V - rV + q_1 S_1 - q_2 S_2 - rK. \tag{1.4.8}$$

Also, for $S_1 \geq b(\tau, S_2)$ we have $V(\tau, S_1, S_2) = S_1 - S_2 - K$ such that[5]

$$\lim_{S_1 \to b(\tau,S_2)} \frac{\partial V}{\partial S_1}(\tau, S_1, S_2) = 1 \quad \text{and} \quad \lim_{S_1 \to b(\tau,S_2)} \frac{\partial V}{\partial S_2}(\tau, S_1, S_2) = -1, \quad 0 < \tau \leq T. \tag{1.4.9}$$

The two equations in (1.4.9) are popularly termed smooth pasting conditions as they guarantee continuity of the pricing function at the early exercise boundary and thus eliminate the possibility of arbitrage opportunities. Given the above arguments, the American spread call option price solves the following partial differential equation

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V - rV, \tag{1.4.10}$$

where $0 < S_1 < b(\tau, S_2)$, and $0 < S_2 < \infty$. The PDE (1.4.10) is solved subject to the initial and boundary conditions

$$V(0, S_1, S_2) = (S_1 - S_2 - K)^+, \quad 0 < S_1, S_2 < \infty, \tag{1.4.11}$$

$$V(\tau, 0, S_2) = 0, \quad , \tau \geq 0, \tag{1.4.12}$$

$$V(\tau, S_1, 0) = (S_1 - K)^+, \quad \tau \geq 0, \tag{1.4.13}$$

$$V(\tau, b(\tau, S_2), S_2) = b(\tau, S_2) - S_2 - K, \quad \tau \geq 0, \tag{1.4.14}$$

$$\lim_{S_1 \to b(\tau,S_2)} \frac{\partial V}{\partial S_1}(\tau, S_1, S_2) = 1, \quad \lim_{S_1 \to b(\tau,S_2)} \frac{\partial V}{\partial S_2}(\tau, S_1, S_2) = -1, \quad \tau \geq 0, \tag{1.4.15}$$

$$\lim_{S_1 \to \infty} \frac{\partial V}{\partial S_1}(\tau, S_1, S_2) = 0 \quad \text{and} \quad \lim_{S_2 \to \infty} \frac{\partial V}{\partial S_2}(\tau, S_1, S_2) = 0. \tag{1.4.16}$$

In Chapter 2, we will present a technique for solving the PDE (1.4.10) using Fourier transform techniques.

[5]Note that $\tau = T - t$ is the time to maturity.

CHAPTER 2

Fourier Transform Approach for American Options on Two Assets

2.1. Introduction

The initial attempt at the pricing of American options is presented in McKean (1965) where the valuation problem is presented as a free-boundary value problem. Due to the fact that the free boundary causes a consideration of the pricing partial differential equation (PDE) on a bounded interval, McKean uses incomplete Fourier transforms to transform the pricing partial differential equation (PDE) to a corresponding ordinary differential equation (ODE). No numerical results are presented in his paper but an exposition on how to tackle the problem is highlighted. A refinement of McKean's approach is presented in Chiarella and Ziogas (2004,2005) who provide numerical results for the American call and strangle options respectively. The transform techniques prove to be very effective in transforming the PDEs to solvable ODEs. However, there has not been much focus on multifactor American options due to the high computational complexity and the curse of dimensionality.

In this chapter we attempt to extend the Fourier transform approach of McKean (1965) to the pricing of American options written on two underlying assets. We differ from McKean's approach in that we first transform the homogeneous pricing PDE to a corresponding inhomogeneous PDE by using the techniques of Jamshidian (1992). By making use of Duhamel's principle, we then present the general solution of the inhomogeneous PDE which depends on the yet to be determined transition density function for the two stochastic processes driving the dynamics of the underlying assets. It is well known that the transition density function is the solution of the associated backward Kolmogorov PDE. With this knowledge, we derive and transform this PDE to a corresponding ODE by using complete Fourier transforms. The ODE is then solved by the integrating factor method, a familiar technique for solving such ODEs.

This chapter is organized as follows, Section 2.2 outlines the problem statement. It is in this section where we derive the general solution for the price of an American option written on two underlying assets. The associated PDE for the density function is also presented. We apply the Fourier transform to the PDE for the density function in Section 2.3. The resulting ODE is then solved using the integrating factor method. Once the solution is found, we then apply an inverse Fourier transform as detailed in Section 2.4. The inverse Fourier transform generates the explicit representation of the transition density function. In Section 2.5 we provide the full price representation. Having found the general pricing function, we present the American spread option example in Section 2.6. Details of numerical implementation of the American spread option are presented in Section 2.8. We compare our results with three existing methods, namely the Monte Carlo algorithm of Ibáñez and Zapatero (2004), the Fourier space time-stepping method of Jackson et al. (2008) and the method of lines approach first applied to option pricing in Meyer and van der Hoek (1997). Details for the implementation of the method of lines algorithm are discussed in Chapter 5. The numerical algorithm for the Monte Carlo method is outlined in Section 2.9. We provide numerical examples for the spread option in Section 2.10. Concluding remarks are then presented in Section 2.12.

2.2. Problem Statement of the American Option written on Two Assets

Let $V(t, S_1, S_2)$ be the price of an American option written on two underlying assets, S_1 and S_2 at current time, t. The two underlying assets pay continuously compounded dividend yields q_1 and q_2 per unit time respectively in a market with constant rate of interest, denoted as r. The payoff at maturity of the option contract is denoted as $v(S_1, S_2)$[1]. We assume that under the real world probability measure, $\mathbb{P}$, the underlying assets are driven by the geometric Brownian motion processes

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dZ_1, \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dZ_2, \end{aligned} \tag{2.2.1}$$

where t denotes the current time, μ_1 and μ_2 are the instantaneous returns per unit time on S_1 and S_2 respectively, σ_1 and σ_2 are the corresponding constant volatilities, and Z_1 and Z_2 are correlated Wiener processes such that, $\mathbb{E}(dZ_1 dZ_2) = \rho dt$. To avoid trivial cases, we assume that $|\rho| < 1$.

It is more convenient to work with independent rather than correlated Wiener processes, so we apply the Cholesky decomposition to the joint SDE system in (2.2.1) which allow us to express it in terms of independent Wiener processes[2]. Effecting this transformation to the SDE system we obtain

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dW_1 \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 \rho S_2 dW_1 + \sigma_2 \sqrt{1-\rho^2} S_2 dW_2, \end{aligned} \tag{2.2.2}$$

where W_1 and W_2 are two independent Wiener processes. The SDE system (2.2.2) represents the real world evolution of S_1 and S_2 and we denote the associated real world probability measure as $\mathbb{P}$.

[1] This is the general payoff function for any option contract which is not path dependent written on two underlying assets. By path dependency we mean exotic payoffs such as barrier, lookback and Asian options among others. We will provide the spread option example in this chapter.

[2] This is accomplished by setting

$$\begin{aligned} dZ_1 &= dW_1 \\ dZ_2 &= \rho dW_1 + \sqrt{1-\rho^2} dW_2. \end{aligned}$$

PROPOSITION 2.2.1. *By setting $\tau = T - t$ where T is the maturity time, the value of an American option contract, $V(\tau, S_1, S_2)$, written on S_1 and S_2 satisfies the PDE*

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V - rV, \tag{2.2.3}$$

where $\mathcal{L}$ is a differential operator associated with the processes deriving S_1 and S_2 and is defined as

$$\mathcal{L} = (r - q_1)S_1\frac{\partial}{\partial S_1} + (r - q_2)S_2\frac{\partial}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2}{\partial S_2^2}. \tag{2.2.4}$$

Equation (2.2.3) *is solved subject to the terminal and boundary conditions*

$$V(0, S_1, S_2) = v(S_1, S_2), \quad 0 < S_1, S_2 < \infty, \tag{2.2.5}$$

$$V(\tau, 0, 0) = 0, \quad \tau \geq 0. \tag{2.2.6}$$

The pricing partial differential equation (2.2.3) *has to satisfy smooth-pasting and value matching conditions which are enforced to eliminate arbitrage opportunities. Such conditions are payoff specific. The smooth pasting conditions guarantee continuity of the first derivatives of the American option prices with respect to the underlying assets at the free-boundary.*

Proof: Refer to Appendix 2.1. □

REMARK 2.2.1. *Strictly speaking we should use different symbols to denote $V(T - \tau, S_1, S_2)$ and $V(\tau, S_1, S_2)$, but for convenience we use the same symbol.*

REMARK 2.2.2. *The smooth pasting conditions are payoff specific. For instance, when considering the American spread call option, we have demonstrated in Section 1.4 that smooth pasting conditions are imposed such that*

$$\lim_{S_1 \to b(\tau, S_2)} \frac{\partial V}{\partial S_1} = 1, \tag{2.2.7}$$

and

$$\lim_{S_1 \to b(\tau, S_2)} \frac{\partial V}{\partial S_2} = -1, \tag{2.2.8}$$

where $S_1 = b(\tau, S_2)$ *is the early exercise boundary which needs to be determined as part of the solution and this will be handled later in the chapter when we consider the American spread call option example.*

The early exercise and continuation regions will be payoff specific as we will demonstrate in later sections when we present the American spread option example. Condition (2.2.5) is the payoff at maturity of the option contract while equation (2.2.6) specifies that the price of the American option is zero if the underlying asset values are zero. The underlying asset domains of the PDE (2.2.3) are usually defined in restricted domains which are payoff specific. Such restricted domains pose formidable challenges when trying to solve PDEs like (2.2.3) subject to the relevant initial and boundary conditions.

One alternative for bypassing the challenges associated with the restricted underlying asset domains is to apply the techniques of Jamshidian (1992) which transforms equation (2.2.3) to a corresponding inhomogeneous PDE which is defined on an unbounded underlying asset domain.

Proposition 2.2.2. *The PDE* (2.2.3) *can be alternatively represented as*

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V - rV + g(\tau, S_1, S_2), \tag{2.2.9}$$

where

$$g(\tau, S_1, S_2) = \mathbb{1}_{\{\mathcal{Y} \geq 0\}} \left[rv - \mathcal{L}v(S_1, S_2) \right]. \tag{2.2.10}$$

Here as before $v(S_1, S_2)$ *is the general payoff function,* $0 < S_1, S_2 < \infty$*, and* $\mathcal{Y}$ *is the event that* S_1 *and* S_2 *fall in the early exercise region. The formula* $\mathbb{1}_{\{\mathcal{Y}\geq 0\}}$ *is an indicator function whose value is one if the event* $\mathcal{Y} \geq 0$ *or zero otherwise and if* $\mathcal{Y} \geq 0$*, it is then optimal to exercise the American option early. Equation* (2.2.9) *is to be solved subject to the conditions in equations* (2.2.5)-(2.2.6)*. Smooth pasting and value matching conditions are also imposed as highlighted in Section 1.4 and Proposition 2.2.1 to guarantee continuity of the pricing function at the early exercise boundary.*

Proof: At each time instant, τ, there exists an optimal early exercise boundary point above[3] which the option can be exercised, that is $V(\tau, S_1, S_2) = v(S_1, S_2)$. When it is optimal to exercise the American option Jamshidian (1992) has shown that

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V - rV + rv - \mathcal{L}v(S_1, S_2). \tag{2.2.11}$$

If it is not optimal to exercise, the option will be held and satisfies the PDE

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V - rV. \tag{2.2.12}$$

Combing equations (2.2.11) and (2.2.12) and also using the fact that the pricing function $V(\tau, S_1, S_2)$ is continuously differentiable at the early exercise boundary we obtain the result in equation (2.2.9). □

Remark 2.2.3. *In the American spread call option case, the inequality* $\mathcal{Y} \geq 0$ *is equivalent to* $S_1 \geq b(\tau, S_2)$.

The bounds for the underlying assets are now defined in the entire positive real axis. Instead of solving the homogeneous PDE (2.2.3) on a restricted asset domain, we

[3] The early exercise boundaries are payoff specific. For example, if we are dealing with the American spread call option case, we have shown that the early exercise boundary point above which the call option can be exercised is represented as $S_1 = b(\tau, S_2)$. By making use of the value matching condition we end up with the equation $V(\tau, b(\tau, S_2), S_2) = v(b(\tau, S_2), S_2)$.

are now faced with solving the inhomogeneous PDE (2.2.9) on an unrestricted asset domain.

Also associated with the SDE system (2.2.2) is the bivariate transition density function[4] which we denote here as[5], $G(\tau, S_1, S_2; S_{1,0}, S_{2,0})$ and this represents the transition probability density function for the passage from the state (S_1, S_2) at time-to-maturity τ to $(S_{1,0}, S_{2,0})$ at maturity. The transition density function is a solution of the backward Kolmogorov PDE

$$\frac{\partial G}{\partial \tau} = \mathcal{L}G. \tag{2.2.13}$$

Equation (2.2.13) is solved subject to the initial condition

$$G(0, S_1, S_2; S_{1,0}, S_{2,0}) = \delta(S_1 - S_{1,0})\delta(S_2 - S_{2,0}),$$

where $\delta(\cdot)$ is the Dirac delta function.

We can nicely represent the general solution of the inhomogeneous PDE (2.2.9) by first switching to log-underlying asset space variables. This is accomplished by letting $S_j = e^{x_j}$ for $j = 1, 2$ and setting

$$C(\tau, x_1, x_2) \equiv V(\tau, e^{x_1}, e^{x_2}), \quad c(x_1, x_2) \equiv v(e^{x_1}, e^{x_2}), \quad f(\tau, x_1, x_2) \equiv g(\tau, e^{x_1}, e^{x_2}), \tag{2.2.14}$$

such that equation (2.2.9) becomes

$$\frac{\partial C}{\partial \tau} = \mathcal{M}C - rC + f(\tau, x_1, x_2), \tag{2.2.15}$$

where $\mathcal{M}$ is a partial differential operator defined as

[4]Sometimes called the Green's function.

[5]Here $S_{1,0}$ and $S_{2,0}$ are the values of S_1 and S_2 at maturity.

$$\mathcal{M} = \left(r - q_1 - \frac{1}{2}\sigma_1^2\right)\frac{\partial}{\partial x_1} + \left(r - q_2 - \frac{1}{2}\sigma_2^2\right)\frac{\partial}{\partial x_2} + \frac{1}{2}\sigma_1^2\frac{\partial^2}{\partial x_1^2} + \rho\sigma_1\sigma_2\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2\frac{\partial^2}{\partial x_2^2}. \tag{2.2.16}$$

The corresponding initial and boundary conditions become

$$C(0, x_1, x_2) = c(x_1, x_2), \quad -\infty < x_1, x_2 < \infty, \tag{2.2.17}$$

$$C(\tau, -\infty, -\infty) = 0, \quad \tau \geq 0. \tag{2.2.18}$$

Applying similar transformations to the transition density partial differential equation and setting $U(\tau, x_1, x_2; x_{1,0}, x_{2,0}) \equiv G(\tau, e^{x_1}, e^{x_2}; e^{x_{1,0}}, e^{x_{2,0}})$ we obtain

$$\frac{\partial U}{\partial \tau} = \mathcal{M}U. \tag{2.2.19}$$

Equation (2.2.19) is solved subject to the initial condition

$$U(0, x_1, x_2) = \delta(x_1 - x_{1,0})\delta(x_2 - x_{2,0}). \tag{2.2.20}$$

For convenience we will write $U(\tau, x_1, x_2)$ to denote the transition density function unless otherwise explicitly specified. We are now in a position to present the general solution of the PDE (2.2.15) as stated in the proposition below.

Proposition 2.2.3. *By considering the American spread call option payoff, the solution (see Proposition 1.3.1) of the inhomogeneous PDE* (2.2.15) *can be represented as*

$$C(\tau, x_1, x_2) = C_E(\tau, x_1, x_2) + C_P(\tau, x_1, x_2), \tag{2.2.21}$$

where

$$C_E(\tau, x_1, x_2) = e^{-r\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{u_1} - e^{u_2} - K)^+ U(\tau, x_1, x_2; u_1, u_2)du_1 du_2,$$

and

$$C_P(\tau, x_1, x_2) = \int_0^\tau e^{-r(\tau-\xi)} \int_{-\infty}^{\infty} \int_{\ln b(\tau, e^{u_2})}^{\infty} (q_1 e^{u_1} - q_2 e^{u_2} - rK) U(\tau - \xi, x_1, x_2; u_1, u_2) du_1 du_2 d\xi.$$

Here, K is the strike price of the option contract and $b(\tau, e^{x_2})$ is the optimal early exercise boundary which need to be implicitly determined as part of the solution.

The first component of equation (2.2.21) is the European option component whilst the second component is the early exercise premium.

Proof: Refer to Appendix 2.2. □

Corollary 2.2.4. *Following the results of Proposition 2.2.3, the general solution (see Proposition 1.3.1) of the inhomogeneous PDE* (2.2.15) *applicable to any continuous payoff function can be represented as*

$$C(\tau, x_1, x_2) = C_E(\tau, x_1, x_2) + C_P(\tau, x_1, x_2), \tag{2.2.22}$$

where

$$C_E(\tau, x_1, x_2) = e^{-r\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(u_1, u_2) U(\tau, x_1, x_2; u_1, u_2) du_1 du_2,$$

and

$$C_P(\tau, x_1, x_2) = \int_0^\tau e^{-r(\tau-\xi)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, u_1, u_2) U(\tau - \xi, x_1, x_2; u_1, u_2) du_1 du_2 d\xi.$$

Equations (2.2.21) and (2.2.22) are expressed in terms of the transition density function, $U(\tau, x_1, x_2; u_1, u_2)$ which is the solution of the PDE (2.2.19) subject to the initial condition in equation (2.2.20). This PDE needs to be solved in order for us to have an explicit representation of the option price. Note that by presenting the solution in Proposition 2.2.3 we have managed to transform a free-boundary value problem to one of solving the homogeneous PDE for the density function which is a relatively simpler task to handle as we will demonstrate in the coming sections. In solving (2.2.19) we will first make use of Fourier transform techniques to reduce the PDE to a corresponding ODE whose solution can be readily found by using the integrating factor method. We will present a step-by-step procedure for solving the homogeneous PDE.

2.3. Applying the Fourier Transform

Given Definition 1.3.1 of the Fourier transform, we can apply this to the transition density PDE (2.2.19) so as to obtain the corresponding ODE for the Fourier transform.

Proposition 2.3.1. *By applying the Fourier transform definition given in equation* (1.3.4) *to the PDE* (2.2.19) *we find that* $\hat{U}(\tau, \eta_1, \eta_2)$ *satisfies the ODE*

$$\frac{\partial \hat{U}}{\partial \tau}(\tau, \eta_1, \eta_2) + (i\eta_1\kappa_1 + i\eta_2\kappa_2 + \frac{1}{2}\sigma_1^2\eta_1^2 + \eta_1\eta_2\rho\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2\eta_2^2)\hat{U}(\tau, \eta_1, \eta_2) = 0, \tag{2.3.1}$$

where

$$\kappa_1 = r - q_1 - \frac{1}{2}\sigma_1^2 \quad \textit{and} \quad \kappa_2 = r - q_2 - \frac{1}{2}\sigma_2^2. \tag{2.3.2}$$

Equation (2.3.1) *is to be solved subject to the initial condition*

$$\hat{U}(0, \eta_1, \eta_2) = e^{i\eta_1 x_{1,0} + i\eta_2 x_{2,0}}. \tag{2.3.3}$$

Proof: Refer to Appendix 2.3. ☐ The ODE (2.3.1) can be solved by applying the integrating factor method. We present the solution in the proposition below.

Proposition 2.3.2. *The solution of the ODE* (2.3.1) *can be represented as*

$$\hat{U}(\tau, \eta_1, \eta_2) = \hat{U}(0, \eta_1, \eta_2)\hat{K}(\tau, \eta_1, \eta_2), \tag{2.3.4}$$

where

$$\hat{K}(\tau,\eta_1,\eta_2) = \exp\left\{-\left[i\eta_1\kappa_1 + i\eta_2\kappa_2 + \frac{1}{2}\sigma_1^2\eta_1^2 + \eta_1\eta_2\rho\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2\eta_2^2\right]\tau\right\}.$$

Proof: Refer to Appendix 2.4. □

2.4. Inverting the Fourier Transform

We have managed to solve the ODE for $\hat{U}(\tau,\eta_1,\eta_2)$, the next step is to recover the original quantity $U(\tau,x_1,x_2)$. This is accomplished by applying the Fourier inversion theorem to equation (2.3.4). We present this result in the proposition below.

PROPOSITION 2.4.1. *By using Definition* 1.3.2*, the inverse Fourier transform of equation* (2.3.4) *can be represented as*

$$U(\tau,x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2\tau(1-\rho^2)}\left[\left(\frac{x_1-x_{1,0}+\kappa_1\tau}{\sigma_1}\right)^2\right.\right.$$
$$\left.\left. -2\rho\left(\frac{x_1-x_{1,0}+\kappa_1\tau}{\sigma_1}\right)\left(\frac{x_2-x_{2,0}+\kappa_2\tau}{\sigma_2}\right)+\left(\frac{x_2-x_{2,0}+\kappa_2\tau}{\sigma_2}\right)^2\right]\right\}. \quad (2.4.1)$$

Proof: Refer to Appendix 2.5. □

Equation (2.4.1) is the explicit form of the bivariate transition density function of the two underlying stochastic processes. Of course it could have been written down as a well known result (see for instance Wilmott (2006) or Wystup (2009)), however we believe it is useful to give full details as they illustrate the power of the Fourier transform technique. We will use Fourier transform techniques in later chapters of this thesis for situations in which the transition density function is not known. Given this density function we can now price option contracts written on these two underlying assets whose dynamics evolve according to the system of equations in (2.2.2).

2.5. The American Option Price

In this section we re-state the results presented in Proposition 2.2.3 for the American option price in terms of the explicit bivariate density function as provided in equation (2.4.1). The American option price can be represented as

$$C(\tau, x_1, x_2) = C_E(\tau, x_1, x_2) + C_P(\tau, x_1, x_2), \tag{2.5.1}$$

with

$$\begin{aligned} C_E(\tau, x_1, x_2) = {} & \frac{e^{-r\tau}}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(u_1, u_2) \exp\left\{ \frac{-1}{2\tau(1-\rho^2)} \left[\left(\frac{x_1 - u_1 + \kappa_1\tau}{\sigma_1} \right)^2 \right.\right. \\ & \left.\left. - 2\rho \left(\frac{x_1 - u_1 + \kappa_1\tau}{\sigma_1} \right) \left(\frac{x_2 - u_2 + \kappa_2\tau}{\sigma_2} \right) + \left(\frac{x_2 - u_2 + \kappa_2\tau}{\sigma_2} \right)^2 \right] \right\} du_1 du_2, \end{aligned} \tag{2.5.2}$$

and

$$\begin{aligned} C_P(\tau, x_1, x_2) = {} & \int_0^\tau \frac{e^{-r(\tau-\xi)}}{2\pi\sigma_1\sigma_2(\tau-\xi)\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, u_1, u_2) \\ & \times \exp\left\{ \frac{-1}{2(\tau-\xi)(1-\rho^2)} \left[\left(\frac{x_1 - u_1 + \kappa_1(\tau-\xi)}{\sigma_1} \right)^2 - 2\rho \left(\frac{x_1 - u_1 + \kappa_1(\tau-\xi)}{\sigma_1} \right) \right.\right. \\ & \left.\left. \times \left(\frac{x_2 - u_2 + \kappa_2(\tau-\xi)}{\sigma_2} \right) + \left(\frac{x_2 - u_2 + \kappa_2(\tau-\xi)}{\sigma_2} \right)^2 \right] \right\} du_1 du_2 d\xi. \end{aligned} \tag{2.5.3}$$

As highlighted in Proposition 2.2.3, the first component of equation (2.5.1) is the European option component whilst the second component is the early exercise premium component. The option will be exercised early only if it is optimal to do so. We can use the representation in equation (2.5.1) to specify any payoff function involving the two underlying assets to obtain the corresponding option price. As stated before, we will implement the American spread call option example, details of which are given in the next section.

2.6. Price of the American Spread Option

In this section we consider the pricing of an American spread call option whose payoff is known to be $c(x_1, x_2) = \max(0, e^{x_1} - e^{x_2} - K)$, where K is the strike price. By substituting this payoff function into equation (2.5.1) we can represent the American spread call option price as

$$
\begin{aligned}
C^S(\tau, x_1, x_2) = & \frac{e^{-r\tau}}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{u_1} - e^{u_2} - K)^+ \exp\left\{\frac{-1}{2\tau(1-\rho^2)}\left[\left(\frac{x_1 - u_1 + \kappa_1\tau}{\sigma_1}\right)^2\right.\right. \\
& \left.\left. - 2\rho\left(\frac{x_1 - u_1 + \kappa_1\tau}{\sigma_1}\right)\left(\frac{x_2 - u_2 + \kappa_2\tau}{\sigma_2}\right) + \left(\frac{x_2 - u_2 + \kappa_2\tau}{\sigma_2}\right)^2\right]\right\} du_1 du_2 \\
& + \int_0^\tau \frac{e^{-r(\tau-\xi)}}{2\pi\sigma_1\sigma_2(\tau-\xi)\sqrt{(1-\rho^2)}} \int_{-\infty}^{\infty}\int_{\ln B(\xi, u_2)}^{\infty} (q_1 e^{u_1} - q_2 e^{u_2} - rK) \exp\left\{\frac{-1}{2(1-\rho^2)(\tau-\xi)}\right. \\
& \times \left[\left(\frac{x_1 - u_1 + \kappa_1(\tau-\xi)}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1 - u_1 + \kappa_1(\tau-\xi)}{\sigma_1}\right)\left(\frac{x_2 - u_2 + \kappa_2(\tau-\xi)}{\sigma_2}\right)\right. \\
& \left.\left. + \left(\frac{x_2 - u_2 + \kappa_2(\tau-\xi)}{\sigma_2}\right)^2\right]\right\} du_1 du_2 d\xi. \qquad (2.6.1)
\end{aligned}
$$

Here, $x_1 = \ln B(\tau, x_2) \equiv \ln b(\tau, e^{x_2})$ is the early exercise surface of the American spread call option which needs to be determined as part of the solution. It is the calculation of this component which makes the pricing of American options a formidable task. The two dimensional nature of $B(\tau, x_2)$ makes the valuation problem even harder compared to options written on a single underlying asset. The above pricing equation can be simplified further by evaluating the integral with respect to u_1. We present the result in the proposition below.

Proposition 2.6.1. *The value of the American spread call option can be expressed as*

$$
C^S(\tau, x_1, x_2) = C_E^S(\tau, x_1, x_2) + C_P^S(\tau, x_1, x_2), \qquad (2.6.2)
$$

where

$$
\begin{aligned}
C_E^S(\tau,x_1,x_2) = & \frac{e^{x_1-q_1\tau}}{\sigma_2\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \exp\left\{-\frac{[x_2-u_2+\kappa_2\tau+\rho\sigma_1\sigma_2\tau]^2}{2\sigma_2^2\tau}\right\} \mathcal{N}\big(d_1[\tau,x_1,K_1(u_2)]\big)du_2 \\
& - \frac{e^{x_2-q_2\tau}}{\sigma_2\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \exp\left\{-\frac{[x_2-u_2+\kappa_2\tau+\sigma_2^2\tau]^2}{2\sigma_2^2\tau}\right\} \mathcal{N}\big(d_2[\tau,x_1,K_1(u_2)]\big)du_2 \\
& - \frac{Ke^{-r\tau}}{\sigma_2\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x_2-u_2+\kappa_2\tau)^2}{2\sigma_2^2\tau}\right\} \mathcal{N}\big(d_2[\tau,x_1,K_1(u_2)]\big)du_2, \qquad (2.6.3)
\end{aligned}
$$

with

$$
K_1(u_2) = e^{u_2} + K,
$$

and

$$
\begin{aligned}
C_P^S(\tau,x_1,x_2) = & \int_0^{\tau} \frac{q_1 e^{x_1-q_1(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{[x_2-u_2+\kappa_2(\tau-\xi)+\rho\sigma_1\sigma_2(\tau-\xi)]^2}{2\sigma_2^2(\tau-\xi)}\right\} \\
& \times \mathcal{N}\big(d_1[(\tau-\xi),x_1,B(\xi,u_2)]\big)du_2 d\xi \\
& - \int_0^{\tau} \frac{q_2 e^{x_2-q_2(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{[x_2-u_2+\kappa_2(\tau-\xi)+\sigma_2^2(\tau-\xi)]^2}{2\sigma_2^2(\tau-\xi)}\right\} \\
& \times \mathcal{N}\big(d_2[(\tau-\xi),x_1,B(\xi,u_2)]\big)du_2 d\xi \\
& - \int_0^{\tau} \frac{rKe^{-r(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x_2-u_2+\kappa_2(\tau-\xi))^2}{2\sigma_2^2(\tau-\xi)}\right\} \\
& \times \mathcal{N}\big(d_2[(\tau-\xi),x_1,B(\xi,u_2)]\big)du_2 d\xi. \qquad (2.6.4)
\end{aligned}
$$

Here

$$
d_1[\tau,x_1,K_1(u_2)] = \frac{x_1+\kappa_1\tau-\rho\frac{\sigma_1}{\sigma_2}(x_2-u_2+\kappa_2\tau)-\ln K_1(u_2)+\sigma_1^2(1-\rho^2)\tau}{\sigma_1\sqrt{(1-\rho^2)\tau}},
$$

$$
d_2[\tau,x_1,K_1(u_2)] = d_1[\tau,x_1,K_1(u_2)] - \sigma_1\sqrt{(1-\rho^2)\tau},
$$

and

$$\mathcal{N}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx.$$

Proof: Refer to Appendix 2.6. □

Equation (2.6.2) involves the function $B(\xi, u_2)$ which is the unknown early exercise boundary. This function needs to be determined at each time instant for us to be able to calculate the corresponding option price. To find the early exercise surface we make use of the value-matching condition which ensures continuity of the option price at the early exercise boundary. We present this in the proposition below.

PROPOSITION 2.6.2. *The early exercise boundary of the American spread call option satisfies the non-linear Volterra integral equation*

$$B(\tau, x_2) - e^{x_2} - K = C^S(\tau, B(\tau, x_2), x_2). \tag{2.6.5}$$

Proof: Continuity at the free-boundary implies that the option's continuation value should equal the early exercise value. This amounts to setting $x_1 = \ln B(\tau, x_2)$ in equation (2.6.2) which leads to equation (2.6.5). □

Once the early exercise boundary is determined, the corresponding American spread option price can then be evaluated using equation (2.6.2). We have managed to simplify the integral with respect to u_1 on both the European option component and the early exercise premium component which is a major step especially when it comes to numerical implementation. The European option component can be readily solved using in-built quadrature routines which come with a variety of software applications. However, such routines cannot be readily applied to the early exercise premium component as it involves the entire history of the early exercise boundary, $x_1 = \ln B(\tau, x_2)$, which is a function of two continuous variables, τ and x_2. In implementing the early exercise premium component and equation (2.6.5), we resort to numerical approximation techniques of the early exercise boundary as discussed in the next section.

2.7. Approximating the Early Exercise Boundary

As pointed out in the above section, to successfully implement the American spread option pricing equation (2.6.2), we need to first numerically solve equation (2.6.5) to obtain the early exercise boundary, $x_1 = \ln B(\tau, x_2)$. The most immediate approach would involve the discretisation of both the running time-to-maturity, ξ, and the integral with respect to u_2 using suitable quadrature techniques such as the Gaussian quadrature. However, such schemes are not computationally efficient due to the two-dimensional nature of the problem. For instance, if we discretise the running time-to-maturity domain into N subintervals and the u_2 domain into M subinterval, this would imply that at each time step we need to solve a system of M equations, this becomes cumbersome for an increasing number of time steps as we would require the entire history of the previously calculated early exercise boundary mesh points.

One suitable approach to lessen this computational burden is to approximate the early exercise boundary with a suitable approximating function. Early exercise boundary approximation techniques have found greater application in the pricing of American options written on a single underlying asset. Carr (1998) uses a randomization approximation technique to value American put and call options on dividend paying underlying assets. Ait-Sahlia and Lai (2001) have used exponential linear functions to approximate Kim's (1990) early exercise boundary. A variety of other approximation techniques have also been utilised to approximate Kim's (1990) pricing function.

By implementing the method of lines algorithm for the American spread option pricing PDE (2.2.3), we have found that the resulting early exercise boundary is approximately linear in $e^{x_2} = S_2$. This observation leads us to approximate the early exercise boundary with a linear function such that

$$B(\tau, x_2) \approx b_0(\tau) + b_1(\tau)e^{x_2}, \tag{2.7.1}$$

where $b_0(\tau)$ and $b_1(\tau)$ are functions of time which need to be determined numerically. By incorporating this approximation into equations (2.6.2) and (2.6.5) we obtain a simplified system for $C^S(\tau, x_1, x_2)$ and $B(\tau, x_2)$ respectively as demonstrated in the proposition below.

PROPOSITION 2.7.1. *By approximating linearly the early exercise boundary, $B(\tau, x_2)$ as given by equation (2.7.1), the value of the American spread call option may be approximated by*

$$C^S(\tau, x_1, x_2) \approx C^S_E(\tau, x_1, x_2) + C^A_P(\tau, x_1, x_2), \tag{2.7.2}$$

where $C^S_E(\tau, x_1, x_2)$ is given in Proposition 2.6.1 and the approximate early exercise premium is

$$\begin{aligned} C^A_P(\tau,x_1, x_2) = &\int_0^\tau \frac{q_1 e^{x_1} e^{-q_1(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{[x_2 - u_2 + \kappa_2(\tau-\xi) + \rho\sigma_1\sigma_2(\tau-\xi)]^2}{2\sigma_2^2(\tau-\xi)}\right\} \\ &\times \mathcal{N}\big[d_1[(\tau-\xi), x_1, b_0(\xi) + b_1(\xi)e^{u_2}]\big] du_2 d\xi \\ &- \int_0^\tau \frac{q_2 e^{x_2} e^{-q_2(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{[x_2 - u_2 + \kappa_2(\tau-\xi) + \sigma_2^2(\tau-\xi)]^2}{2\sigma_2^2(\tau-\xi)}\right\} \\ &\times \mathcal{N}\big[d_2[(\tau-\xi), x_1, b_0(\xi) + b_1(\xi)e^{u_2}]\big] du_2 d\xi \\ &- \int_0^\tau \frac{rKe^{-r(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x_2 - u_2 + \kappa_2(\tau-\xi))^2}{2\sigma_2^2(\tau-\xi)}\right\} \\ &\times \mathcal{N}\big[d_2[(\tau-\xi), x_1, b_0(\xi) + b_1(\xi)e^{u_2}]\big] du_2 d\xi. \end{aligned}$$

The approximate early exercise boundary solves the integral equation

$$b_0(\tau) + b_1(\tau)e^{x_2} - e^{x_2} - K = C^S(\tau, b_0(\tau) + b_1(\tau)e^{x_2}, x_2). \tag{2.7.3}$$

Proof: By substituting (2.7.1) into the pricing equation (2.6.2) and using the value matching condition (2.6.5) we obtain the results in the above proposition. □

REMARK 2.7.1. *One might first suspect that the approximate early exercise boundary should exhibit a sub-optimality feature. Whilst this is certainly observed from the numerical simulations in Section 2.10, it does not seem possible to prove this result for the multiasset situation being considered in this thesis.*

REMARK 2.7.2. *We use the subscripts on C_E^S as a convenient way to denote the European spread option component while C_P^A denotes the approximate early exercise premium component.*

To successfully solve the resulting equations in Proposition 2.7.1, we need to determine the functions, $b_0(\tau)$ and $b_1(\tau)$. In order to develop an iterative scheme we write equation (2.7.3) in terms of the two functions

$$b_0(\tau) = C^S(\tau, b_0(\tau) + b_1(\tau)e^{x_2}, x_2) - b_1(\tau)e^{x_2} + e^{x_2} + K, \tag{2.7.4}$$

$$b_1(\tau) = \frac{1}{e^{x_2}}\left[C^S(\tau, b_0(\tau) + b_1(\tau)e^{x_2}, x_2) - b_0(\tau) + e^{x_2} + K\right]. \tag{2.7.5}$$

Here $x_2 = \ln S_2$ so for computational purposes, any price level of x_2 in the domain, $(-\infty, \infty)$ can be chosen. We have chosen S_2 to be in the range $[0, 200]$ and calculated $b_0(\tau)$ and $b_1(\tau)$ at each of these values as presented in Section 2.10 on numerical results. As we shall see in the next section, when we discretise in the τ direction we use equations (2.7.4) and (2.7.5) as the basis of an iterative scheme to calculate $b_0(\tau)$ and $b_1(\tau)$ respectively at successive time steps.

2.8. Numerical Implementation of the American Spread Option

Having approximated the American spread call option price as presented in equation (2.7.2) and the corresponding system of equations (2.7.4)-(2.7.5) for tracking the approximate early exercise boundary, we now present an algorithm that implements these equations. We adopt the numerical integration techniques developed in Huang et al. (1996) where they implement Kim's (1990) American put option pricing equation. Similar techniques have also been applied in Kallast and Kivinukk (2003) for approximating both the American call option price and the option delta, gamma and vega.

As pointed out earlier, the European option component of equation (2.7.2) involves a single integral with respect to u_2, and a cumulative normal distribution function both of which can be easily handled by a variety of in-built software applications. We have used Gaussian quadrature techniques to handle the integral with respect to u_2. However, as stated previously such software applications cannot be readily applied to the early exercise premium component as this term involves the entire history of the early exercise boundary, $B(\tau, x_2)$ which we are approximating by a linear function in (2.7.3). The early-exercise premium component also involves an integral with respect to the running time-to-maturity, ξ, which also makes use of the whole history of the early exercise boundary at each point in time.

In implementing equations (2.7.2), (2.7.4) and (2.7.5), we treat the option as a Bermudan, that is an option that can be exercised at discrete points in time. We discretise the time variable, τ, into M equally spaced subintervals of length $h = T/M$ and apply Simpson's rule. As with the European option component, we apply Gaussian quadrature techniques to the integral with respect to u_2. The numerical algorithm is initiated at maturity, $\tau_0 = 0$ where we know the option value as the payoff of the spread option. It has been shown in Broadie and Detemple (1997) that the early exercise boundary of the American spread call option at maturity takes the form

$$B(0, x_2) = \max\left(K + e^{x_2}, \frac{r}{q_1}K + \frac{q_2}{q_1}e^{x_2}\right). \tag{2.8.1}$$

By comparing coefficients, we can readily deduce that

$$b_0(0) = \max\left(K, \frac{r}{q_1}K\right), \quad \text{and} \quad b_1(0) = \max\left(1, \frac{q_2}{q_1}\right). \tag{2.8.2}$$

The two conditions in equation (2.8.2) serve as the starting values for tracking the early exercise boundary backwards in time. We denote the time steps as $\tau_m = mh$, for $m = 1, 2, \cdots, M$. The discretised version of the approximate American spread call option price is then represented as

$$C^S(mh, x_1, x_2) \approx C^S_E(mh, x_1, x_2) + C^A_P(mh, x_1, x_2), \tag{2.8.3}$$

where $C^S_E(mh, x_1, x_2)$ is found by setting $\tau = mh$ in equation (2.6.3) and

$$\begin{aligned}
C^A_P(mh,x_1, x_2) = h\sum_{j=0}^{m} w_j \frac{q_1 e^{x_1} e^{-q_1(m-j)h}}{\sigma_2\sqrt{2\pi(m-j)h}} \int_{-\infty}^{\infty} \exp\left\{-\frac{[x_2 - u_2 + \kappa_2(m-j)h + \rho\sigma_1\sigma_2(m-j)h]^2}{2\sigma_2^2(m-j)h}\right\} \\
\times \mathcal{N}\big[d_1[(\tau-\xi), x_1, b_0(mh) + b_1(mh)e^{u_2}]\big]\, du_2 \\
- h\sum_{j=0}^{m} w_j \frac{q_2 e^{x_2} e^{-q_2(m-j)h}}{\sigma_2\sqrt{2\pi(m-j)h}} \int_{-\infty}^{\infty} \exp\left\{-\frac{[x_2 - u_2 + \kappa_2(m-j)h + \sigma_2^2(m-j)h]^2}{2\sigma_2^2(m-j)h}\right\} \\
\times \mathcal{N}\big[d_2[(m-j)h, x_1, b_0(mh) + b_1(mh)e^{u_2}]\big]\, du_2 \\
- h\sum_{j=0}^{m} w_j \frac{rK e^{-r(m-j)h}}{\sigma_2\sqrt{2\pi(m-j)h}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x_2 - u_2 + \kappa_2(m-j)h)^2}{2\sigma_2^2(m-j)h}\right\} \\
\times \mathcal{N}\big[d_2[(m-j)h, x_1, b_0(mh) + b_1(mh)e^{u_2}]\big]\, du_2.
\end{aligned}$$

Here, w_j are the weights of Simpson's rule for integration in the ξ direction while h is the corresponding step size. At each time step, τ_m, we need to determine the two unknown boundary functions, $b_0^m = b_0(mh)$ and $b_1^m = b_1(mh)$ each of which depends on its entire history up to the current time step. Root finding techniques are employed to find these unknown functions at each time step.

When iterating for b_0^m and b_1^m we use as initial guesses[6] $b_{0,0}^m = b_0^{m-1}$ and $b_{1,0}^m = b_1^{m-1}$, and then solve the following system of linked equations

$$b_{0,k}^m = C(mh, b_{0,k}^m + b_{1,k-1}^m e^{x_2}, x_2) - b_{1,k-1}^m e^{x_2} + e^{x_2} + K, \tag{2.8.4}$$

$$b_{1,k}^m = \frac{1}{e^{x_2}} \left[C(mh, b_{0,k}^m + b_{1,k}^m e^{x_2}, x_2) - b_{0,k}^m + e^{x_2} + K \right]. \tag{2.8.5}$$

We continuously repeat the iterative process at each time step for $k = 1, 2, \cdots$ until $|b_{0,k}^m - b_{0,k-1}^m| < \epsilon_0$ and $|b_{1,k}^m - b_{1,k-1}^m| < \epsilon_1$, where ϵ_0 and ϵ_1 are pre-set tolerance values. We present the numerical implementation of the Monte Carlo algorithm in the next section.

2.9. Numerical Implementation of the Monte Carlo Method

In validating our numerical integration approach one of the methods we will use for comparison purposes is the Monte Carlo algorithm of Ibáñez and Zapatero (2004). This algorithm treats the American option problem as a Bermudan option and the early exercise boundary is tracked by using a dynamic programming algorithm. The valuation method is heavily dependent on the value-matching condition which asserts that[7]

$$V(t, b(t, S_2), S_2) = v(b(t, S_2), S_2), \tag{2.9.1}$$

where as before $v(b(t, S_2), S_2)$ is the payoff of the option and $S_1 = b(t, S_2)$ is the early exercise surface at time t. We use the bisection method as a root finding method for the early exercise surface. The time domain is discretised into M equally spaced subintervals and we set $h = T/M$. Each time step is denoted as $t_m = hm$ for $m = 0, 1, \cdots, M$ with t_0 being the initial time. In computing the American option price

[6]The subscript k in the two functions $b_{0,k}^m$ and $b_{1,k}^m$ represents the number of iterations required for convergence of the iterative process at time step m.
[7]In this section we use t to denote the current time.

at each time step, we take advantage of the knowledge of both the price and exercise surface at the previous time steps. The value of the American option is then treated like a European option which allows us to implement the valuation problem using the plain vanilla Monte Carlo method. The intrinsic value of the American spread option at any given time step, t_m, is denoted as $(S_{1,t_m} - S_{2,t_m} - K)^+$. The value of the American spread option is then represented as

$$V(t_m, S_{1,t_m}, S_{2,t_m}) = \tilde{\mathbb{E}}_{t_m}[e^{-r(\tau - t_m)}(S_{1,\tau} - S_{2,\tau} - K)^+ | (S_{1,t_m}, S_{2,t_m})], \tag{2.9.2}$$

where τ is given by

$$\tau = \begin{cases} t_{m+i} & \text{if } S_{1,t_{m+i}} > b(t_{m+i}, S_{2,t_{m+i}}) \text{ for } t_{m+i} = t_{m+1}, \cdots, t_M \\ \infty & \text{otherwise.} \end{cases} \tag{2.9.3}$$

The relationship (2.9.3) implies that the option can either be exercised at t_{m+1} or t_{m+2} or $\cdots$ or t_M else it expires worthless. This process is recursively repeated at each time step yielding both the early exercise surface and pricing surface simultaneously. The algorithm can be schematised as in Figure 2.1.

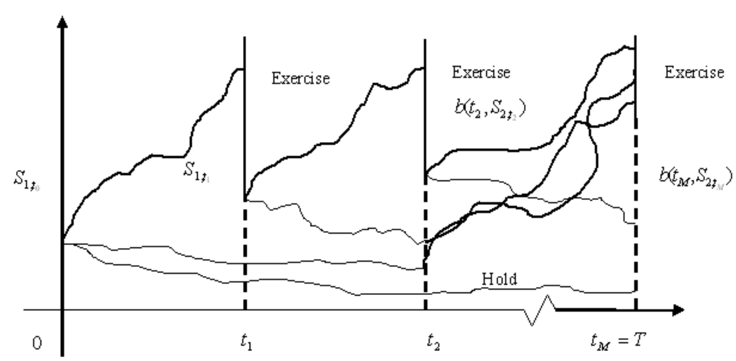

FIGURE 2.1. Possible price path of S_1 for the Monte Carlo Approach

2.10. Numerical Results of the American Spread Option

In this section we present numerical results obtained using the method developed in Section 2.8, which we dub the numerical integration method. We assess the effectiveness of the numerical integration method by making comparisons with three other methods namely the Monte Carlo approach of Ibáñez and Zapatero (2004), the Fourier space time-stepping (FST) approach of Surkov (2008) and the method of lines (MOL) approach which has been used to price American options written on a single underlying asset under different frameworks in Meyer and van der Hoek (1997), Meyer (1998) and Chiarella et al. (2009). The source codes[8] for the numerical integration method, the FST and Monte Carlo approaches have been implemented in MATLAB whilst a FORTRAN code was developed for the MOL algorithm.

We have used the FST method for price comparison purposes only as it does not explicitly generate the early exercise boundary as part of the solution but instead uses a dynamic programming algorithm to check the early exercise condition at each time step. Details on how to implement the FST method can be found in Surkov (2008). For the early exercise boundary, comparisons are made between the numerical integration method and the MOL. Monte Carlo results are used to assess confidence bounds for the option prices. We consider a six-month American spread call option whose parameters are provided in Table 2.1.

For all Monte Carlo calculations we have used 10,000 simulations as presented in Ibáñez and Zapatero (2004). Details of the numerical schemes for both the numerical integration method and Monte Carlo method have been outlined in Sections 2.8 and 2.9 respectively. The numerical algorithm for the MOL approach is explained later in Chapter 5 where we consider the pricing of an American spread option under stochastic volatility using this method. In all our numerical experiments unless otherwise

[8]All the source codes have been run on the UTS HPC Linux Cluster with a graphics user interface consisting of 8 nodes running Red Hat Enterprise Linux 4.0 (64 bit) with 2×3 GHz 4MB cache Xeon 5160 dual core Processors, 8GB 667 MHz DDR2-RAM.

TABLE 2.1. Parameters for the six-month American spread call option.

Parameter	Value	FST Parameters	Value
K	100	N	1,024
r	3%	M	50
q_1	6%	T	0.5
q_2	2%		
σ_1	25%		
σ_2	30%		
ρ	0.5		

stated, we have used $M = 50$ time steps implying that $\triangle\tau = 1 \times 10^{-2}$. For the FST method $N = 1,024$ is the total number of mesh points for the two underlying asset domains. For convenience, the two underlying assets are discretised into a square mesh implying an equal number of grid points in both asset domains.

For the method of lines we have used cubic splines for any interpolation and we considered the underlying asset domains, $0 \leq S_1 \leq 500$ and $0 \leq S_2 \leq 200$. The S_1 domain has been discretised into 1,438 non-uniformly spaced points. The large number of points in the S_1 direction is required to help improve the smoothness of the early exercise boundary estimates. The S_2 domain has been discretised into 200 equally spaced nodes.

We first present the early exercise surface of the American spread call option in Figure 2.2. As highlighted in earlier sections, the early exercise boundary needs to be determined first at each time step in order to obtain the corresponding option price except for the method of lines algorithm which generates the prices, delta, gamma and the free-boundary simultaneously as part of the solution. Figure 2.3 shows the nature of the early exercise boundaries for varying maturities; these have been calculated using the MOL. The linear relationship we proposed on Section 2.7 between S_1 and S_2 is clear from this figure. We have also indicated the early exercise and continuation regions on this graph.

We also present the early exercise boundaries we found from the two methods by fitting seventh degree polynomials in Figure 2.4 when $S_2 = 100$. As can be seen from the graphs, the numerical integration method and the MOL yield comparable boundaries which are not significantly different from each other implying a high level of accuracy for the two methods. By their nature, early exercise boundaries are very sensitive to the way in which the state variables are discretised. However, option prices are not as sensitive as their associated early exercise boundaries. The effects for varying correlation on the early exercise surface are presented in Figure 2.5. We note that by varying correlation from positive to negative, the early exercise boundary becomes steeper.

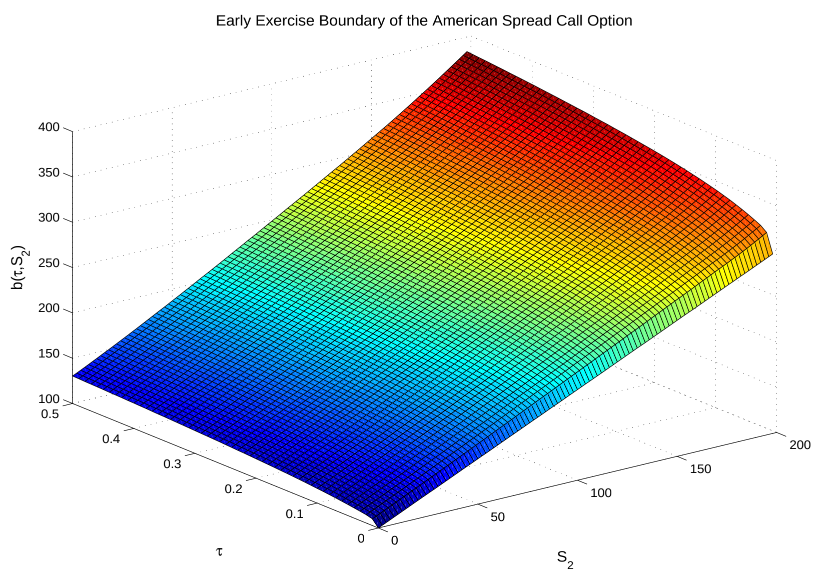

FIGURE 2.2. Free surface of the American spread option generated by the method of lines.

Having presented the early exercise boundaries, the next task involves price comparisons for the approaches under consideration. Figure 2.6 shows the price surface of the American spread call option generated using the numerical integration method. A similar surface can be generated by using the MOL. We present option prices from the four different pricing techniques in Tables 2.2-2.4. In these tables, we present American spread call option prices for increasing S_1 and S_2. From the three tables, we

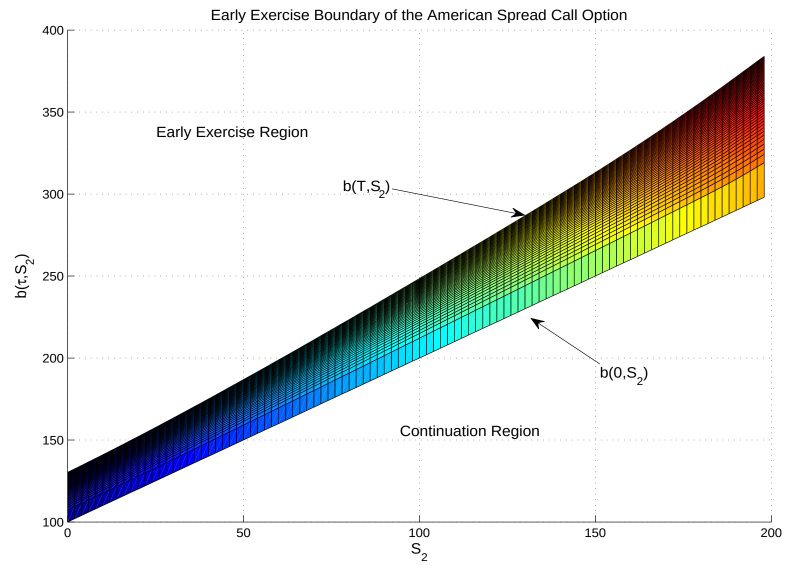

FIGURE 2.3. Early exercise boundaries of the American spread option of varying maturities using the method of lines.

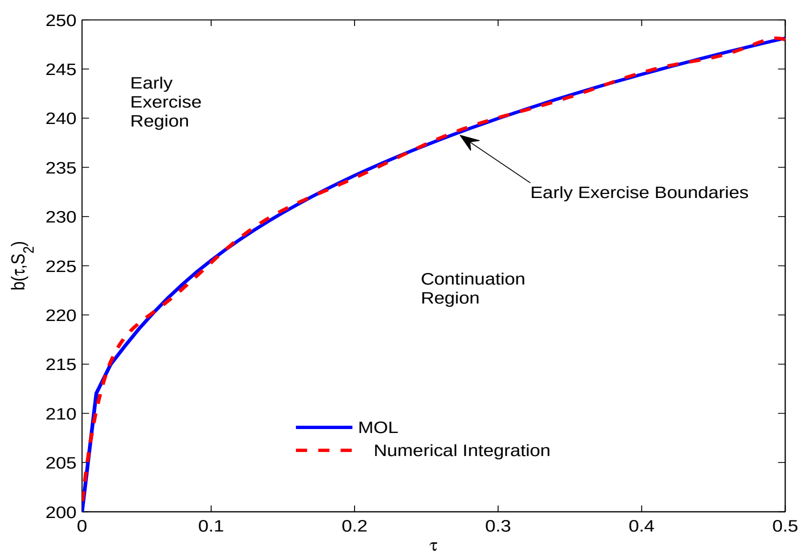

FIGURE 2.4. Comparison of early exercise boundaries when $S_2 = 100$.

note that all the four approaches considered provide quite comparable results. Most of the computed prices lie within the 95% Monte Carlo confidence interval except for a few instances when the three approaches generate prices which do not lie within the confidence interval. This maybe due to the weak convergence properties of the Monte

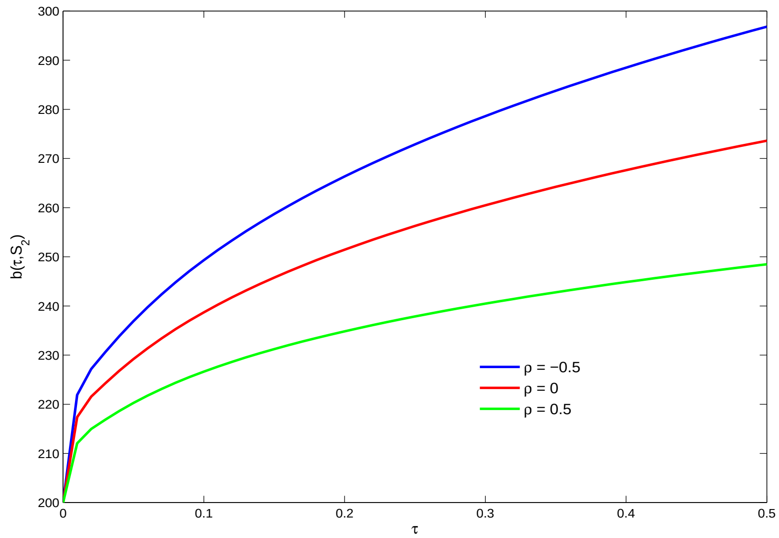

FIGURE 2.5. Early Exercise boundaries for varying correlation when $S_2 = 100$ using the method of lines.

Carlo method as will be demonstrated shortly when assessing time convergence among the four approaches.

Figure 2.7 shows price differences for varying correlation where we make prices generated when $\rho = 0$ the reference. For instance, the negative(positive) correlation option price differences are calculated by subtracting prices generated when correlation is negative(positive) from prices generated when correlation is zero. From this figure, we note high price differences for at-the-money options with option prices increasing as correlation between the two underlying assets is varied from positive to negative. These price differences have been calculated when $S_2 = 100$.

It is worthwhile to study the time convergence rates of the methods under consideration for us to have an idea of which is computationally faster and more efficient. We have taken the FST with $N = 4,096$ and $M = 300$ as a reference method for the time convergence comparisons. Here, we used large values of M and N to eliminate time discretisation errors and the errors arising from the discretisation of the two underlying asset domains. It took 919.04 seconds to compute the reference price.

We present the absolute price differences for both the numerical integration method and Monte Carlo approach in Table 2.5. We have used 50,000 Monte Carlo simulations in this table. We have also included the computation time elapsed by each method at an increasing number of time steps. As revealed from this table, the numerical integration method and the MOL quickly converge to the reference solution. However, this is not the case with Monte Carlo approach which seems not to change much as we increase both time steps and the number of simulations. It requires only 16 time steps to achieve 3-digit convergence with the numerical integration method. The calculation of both the early exercise boundary and the corresponding option prices takes 33.49 seconds. The results in Table 2.5 can also be summarised graphically as shown in Figure 2.8. This figure shows the high rate of convergence for both the numerical integration approach and MOL as highlighted earlier.

We have highlighted that the most powerful feature of the MOL algorithm is that the prices, free boundaries and the option deltas are all generated simultaneously as part of the solution at no additional computational cost. We present the American spread call option delta in Figure 2.9 calculated using the method of lines. As is well known and implied by the figure, delta of a call option is defined in the interval, $0 \leq \Delta \leq 1$. A deep out-of-the-money call option has a delta of zero, while that of a deep in-the-money is one. At-the-money options have a delta of 0.5.

TABLE 2.2. American spread call option price comparisons when $S_1 = 160$.

S_2	Numerical Integration	FST	MOL	Monte Carlo (95% CI)
40	20.4409	21.1262	21.1379	[19.9529, 20.7732]
60	8.4127	8.5788	8.5847	[8.1833, 8.7477]
80	2.6740	2.7057	2.7075	[2.5709, 2.8994]
100	0.6927	0.6977	0.6985	[0.5919, 0.7454]
120	0.1555	0.1561	0.1611	[0.1276, 0.1933]
140	0.0318	0.0319	0.0604	[0.0160, 0.0469]
160	0.0061	0.0061	0.00974	[0.000755, 0.00877]

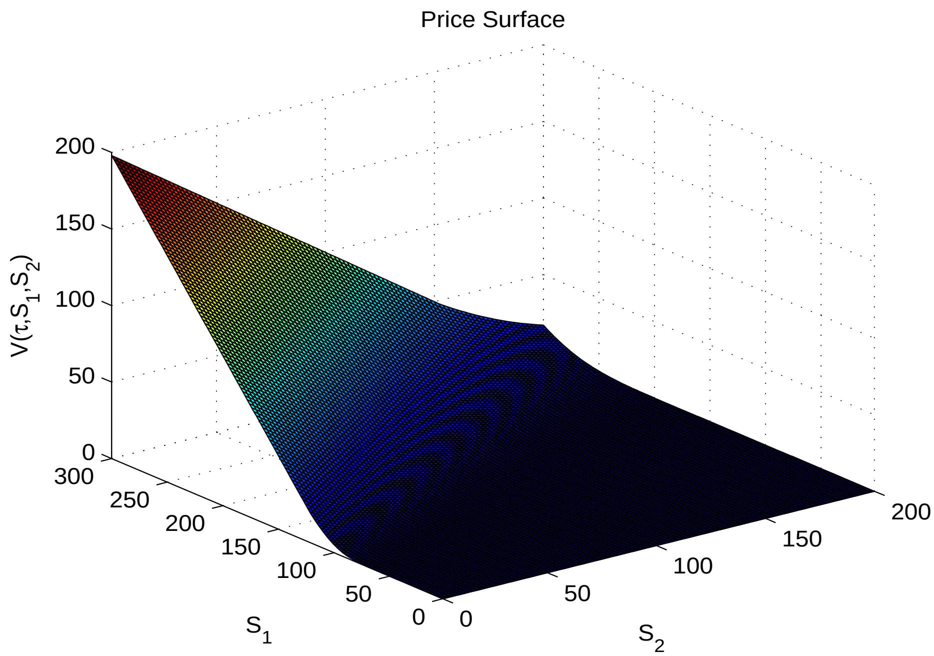

FIGURE 2.6. Price surface of the American spread option using the numerical integration method.

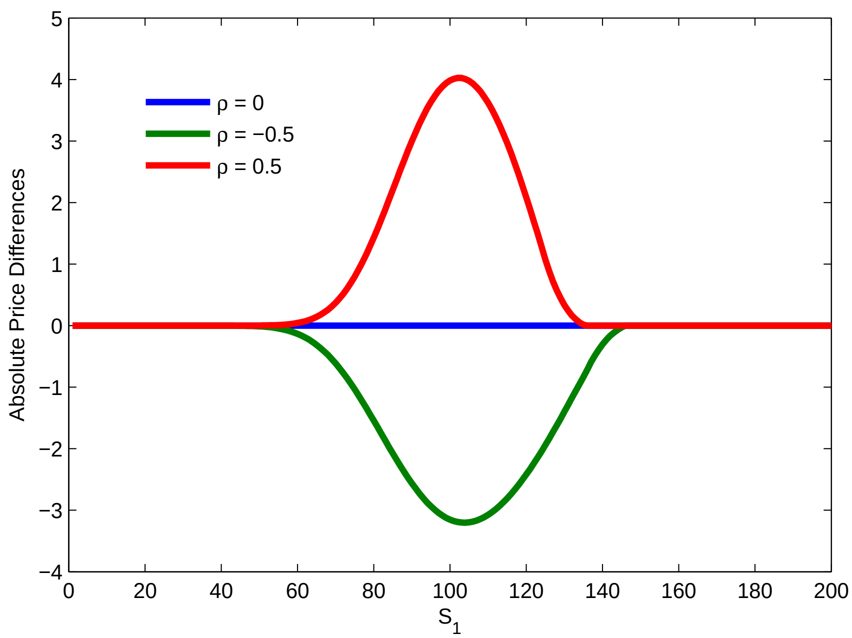

FIGURE 2.7. Price differences of the American spread call option for varying correlation $S_2 = 100$.

TABLE 2.3. American spread call option price comparisons when $S_1 = 200$.

S_2	Numerical Integration	FST	MOL	Monte Carlo (95% CI)
40	60	60	60	[59.3957, 60.6043]
60	40	40.0004	40	[39.4294, 40.5706]
80	21.8236	22.3875	22.4014	[20.9267, 21.8765]
100	10.7548	10.7441	10.7508	[10.2896, 10.9940]
120	4.4190	4.4596	4.4589	[4.2123, 4.6768]
140	1.6316	1.6405	1.6415	[1.4577, 1.7261]
160	0.5480	0.5496	0.6020	[0.5357, 0.6981]

TABLE 2.4. American spread call option price comparisons when $S_1 = 300$.

S_2	Numerical Integration	FST	MOL	Monte Carlo (95% CI)
40	160	160	160	[159.0530, 160.9470]
60	140	140	140	[139.0780, 140.9220]
80	120	120	120	[119.0982, 120.9018]
100	100	100	100	[99.1346, 100.8654]
120	80	80	80	[79.1207, 80.8793]
140	60	60.0015	60	[59.1645, 60.8355]
160	41.3035	41.7167	41.0932	[39.2412, 40.7660]

TABLE 2.5. Price differences for increasing time-steps $S_1 = 200$ and $S_2 = 100$.

	Numerical Integration		MOL		Monte Carlo	
M	Absolute Difference	CPU Time(Sec)	Absolute Difference	CPU Time(Sec)	Absolute Difference	CPU Time(Sec)
4	0.0364	4.117	0.2512	45.47	0.1351	355.75
8	0.0105	10.360	0.074	49.01	0.1316	1,350
16	0.0012	33.493	0.0236	51.718	0.132	5,149
32	0.0035	127.069	0.006	55.12	0.1148	19,811
50	0.003	296.598	0.001	59.20	0.1735	47,970

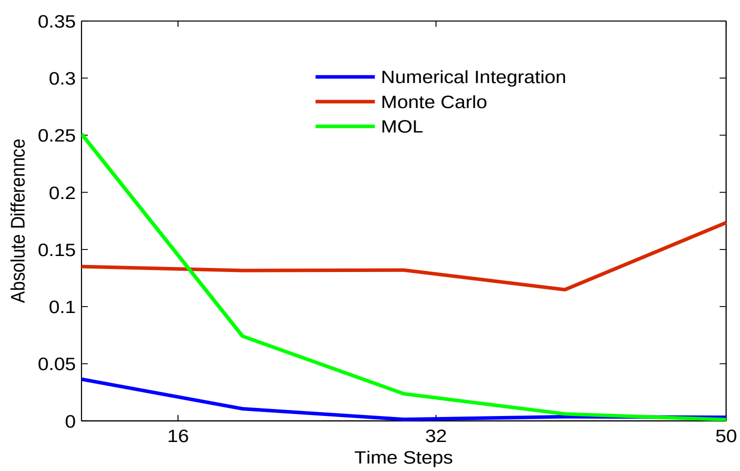

FIGURE 2.8. Time Step Convergence Comparisons

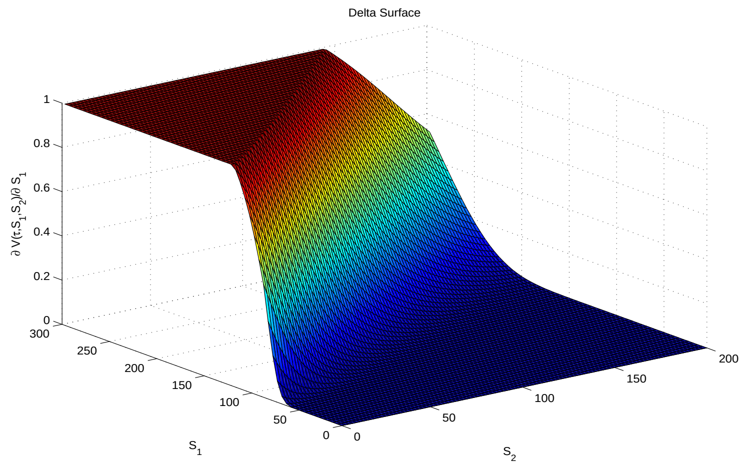

FIGURE 2.9. Delta Surface of the American Spread Call Option

2.11. Generalization of the Fourier Transform Approach

2.11.1. Problem Statement. Let $V(t, S_1, \cdots, S_n)$ be the price at time, t of the American option written on n underlying assets, $S_1, \cdots, S_n$. The underlying assets pay continuously compounded dividend yields $q_1, \cdots, q_n$ respectively in a market offering a risk-free rate of interest, r. The payoff at maturity of the option contract is represented as $v(S_1, \cdots, S_n)$, this may resemble the payoff of a portfolio of basket options. Under the real world probability measure, $\mathbb{P}$, we assume that the underlying assets are driven by the geometric Brownian motion processes

$$dS_i = \mu_i S_i dt + \sigma_i S_i dZ_i, \quad i = 1, \cdots, n \tag{2.11.1}$$

where μ_i is the instantaneous return of the i^{th} asset per unit time, Z_i, for $i = 1, \cdots, n$ are correlated Wiener processes under the real world probability measure $\mathbb{P}$. We denote the correlation between any two Wiener processes by $\rho_{i,j}$. To avoid trivial cases, we assume that $|\rho_{i,j}| \leq 1$, $\rho_{i,i} = 1$ and $\rho_{i,j} = \rho_{j,i}$ for $i, j = 1, \cdots, n$. Using standard hedging arguments it can be shown that an American option contract written on the n underlying assets satisfies the PDE

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V - rV, \tag{2.11.2}$$

where $\mathcal{L}$ is a Dynkin operator defined as

$$\mathcal{L} = \sum_{i=1}^{n}(r-q_i)S_i\frac{\partial}{\partial S_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j S_i S_j\frac{\partial^2}{\partial S_i \partial S_j}. \tag{2.11.3}$$

The PDE (2.11.2) is solved subject to the initial and boundary conditions

$$V(0, S_1, \cdots, S_n) = v(S_1, \cdots, S_n), \quad 0 < S_1, \cdots, S_n < \infty, \tag{2.11.4}$$

$$V(\tau, 0, \cdots, 0) = 0, \quad \tau \geq 0. \tag{2.11.5}$$

Equation (2.11.4) is the payoff at maturity and (2.11.5) is the absorbing state ensuring that the value of the option is zero if the underlying assets prices are zero. The PDE (2.11.2) is also solved subject to other boundary and smooth-pasting conditions which are payoff specific. The early exercise feature of American options coupled with the dimension of the contract make their pricing more demanding as compared to their European counterparts.

Also associated with the n stochastic differential equations in the system of equations in (2.11.1) is the multivariate transition probability density function which we denote as $G(\tau, S_1, \cdots, S_n; S_{1,0}, \cdots, S_{n,0})$ and represents the transition probability density for the passage from the state $(S_1, \cdots, S_n)$ at time-to-maturity τ to $S_{1,0}, \cdots, S_{n,0}$ at maturity. The transition density function is the solution of the backward Kolmogorov PDE

$$\frac{\partial G}{\partial \tau} = \sum_{i=1}^{n}(r-q_i)S_i\frac{\partial G}{\partial S_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j S_i S_j\frac{\partial^2 G}{\partial S_i \partial S_j}. \tag{2.11.6}$$

Equation (2.11.6) is solved subject to the initial condition

$$G(0, S_1, \cdots, S_n; S_{1,0}, \cdots, S_{n,0}) = \delta(S_1 - S_{1,0}) \cdots \delta(S_n - S_{n,0}),$$

where $\delta(\cdot)$ is the Dirac delta function. For convenience, we will simply write $G(\tau, S_1, \cdots, S_n)$ to denote the transition density function unless the context requires that we make explicit $S_{1,0}, \cdots, S_{n,0}$ dependence.

Depending on the payoff under consideration, the underlying assets of the American option pricing problem will have finite bounds which will be the associated early exercise boundaries. Such bounds pose challenges when trying to find the corresponding solution of the pricing PDE. To bypass this challenge, we again employ the technique of Jamshidian (1992) and transform the homogeneous PDE (2.11.2) to a corresponding inhomogeneous PDE by introducing an indicator function such that

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V - rV + y(\tau, S_1, \cdots, S_n), \tag{2.11.7}$$

where

$$y(\tau, S_1, \cdots, S_n) = \mathbb{1}_{\mathcal{Y}>0}\left[rv - \mathcal{L}v(S_1, \cdots, S_n)\right]. \tag{2.11.8}$$

Here, $\mathbb{1}_{\{\mathcal{Y}>0\}}$ is an indicator function which is one if the event $\mathcal{Y} > 0$ implying that it is optimal to exercise the American option early, otherwise it is zero. The underlying asset bounds are now defined in the unbounded domains, $0 < S_1, \cdots, S_n < \infty$. This transformation ensures that all the underlying assets have an unbounded domain which is a desirable property for the solution procedure we wish to adopt. The PDE (2.11.9) is solved subject to the initial and terminal conditions in equations (2.11.4) and (2.11.5) respectively.

In solving the inhomogeneous PDE (2.11.7), we first switch to the log of the underlying asset space by setting $S_j = e^{x_j}$ for $j = 1, \cdots, n$ and letting $C(\tau, x_1, \cdots, x_n) \equiv V(\tau, e^{x_1}, \cdots, e^{x_n})$, $f(\tau, x_1, \cdots, x_n) \equiv y(\tau, e^{x_1}, \cdots, e^{x_n})$. Applying these changes to the PDE we obtain

$$\frac{\partial C}{\partial \tau} = \sum_{i=1}^{n}(r - q_i - \frac{1}{2}\sigma_i^2)\frac{\partial C}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j\frac{\partial^2 C}{\partial x_i \partial x_j} - rC + f(\tau, x_1, \cdots, x_n). \tag{2.11.9}$$

The initial and boundary conditions transform to

$$C(0, x_1, \cdots, x_n) = c(x_1, \cdots, x_n), \quad -\infty < x_1, \cdots, x_n < \infty, \tag{2.11.10}$$

$$C(\tau, -\infty, \cdots, -\infty) = 0, \quad \tau \geq 0. \tag{2.11.11}$$

We can also apply similar log underlying asset transformations to the transition density function and letting $U(\tau, x_1, \cdots, x_n) \equiv G(\tau, e^{x_1}, \cdots, e^{x_n})$ to obtain

$$\frac{\partial U}{\partial \tau} = \sum_{i=1}^{n}(r - q_i - \frac{1}{2}\sigma_i^2)\frac{\partial U}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j\frac{\partial^2 U}{\partial x_i \partial x_j}. \tag{2.11.12}$$

This is solved subject to the initial condition

$$U(0, x_1, \cdots, x_n; x_{1,0}, \cdots, x_{n,0}) = \delta(x_1 - x_{1,0}) \cdots \delta(x_n - x_{n,0}). \tag{2.11.13}$$

We are now in a position to represent the the general solution for the PDE (2.11.9) by using Duhamel's principle. The solution is presented in the following proposition.

PROPOSITION 2.11.1. *The general solution of the PDE* (2.11.9) *can be represented as*

$$C(\tau, x_1, \cdots, x_n) = C_E(\tau, x_1, \cdots, x_n) + C_P(\tau, x_1, \cdots, x_n), \tag{2.11.14}$$

where

$$C_E(\tau, x_1, \cdots, x_n) = e^{-r\tau}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} c(u_1, \cdots, u_n)U(\tau, x_1, \cdots, x_n; u_1, \cdots, u_n)du_1 \cdots du_n,$$

and

$$\begin{aligned} C_P(\tau, x_1, \cdots, x_n) = \int_0^\tau e^{-r(\tau-\xi)}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} y(\xi, u_1, \cdots, u_n) \\ \times U(\tau - \xi, x_1, \cdots, x_n; u_1, \cdots, u_n)du_1 \cdots du_n d\xi. \end{aligned}$$

Proof: Refer to Appendix 2.7. □

The first component on the RHS of equation (2.11.14) is the European option component whilst the second component is the early exercise premium. The early exercise premium constitutes the difference between total dividends receivable on the underlying assets and the total interest payments on the strike. The pricing function in

Proposition 2.11.1 is a function of the multivariate density function, $U(\tau, x_1, \cdots, x_n)$ which is not known in explicit form but is the solution to the PDE (2.11.12). Once this density function is determined, we will be in a position to represent the price of any American option written on the n underlying assets.

We have managed to reduce an inhomogeneous free-boundary value problem to that of solving a homogeneous PDE (2.11.12) for the density function subject to initial condition (2.11.13) which needs to be satisfied. This is much easier to handle than solving the free-boundary problem directly. This is the major advantage of using Jamshidian's (1992) transformation. In solving equation (2.11.12) we first convert this to a corresponding ODE by applying Fourier transforms. The resulting ODE is then solved by the integrating factor method. These steps are outlined below.

2.11.2. Applying the Fourier Transform. We apply a Fourier transform to the PDE (2.11.12) by using the result in Definition 1.3.1. We present the resulting ODE in the proposition below.

PROPOSITION 2.11.2. *The Fourier transform of the PDE* (2.11.12) *satisfies the ODE*

$$\frac{\partial \hat{U}}{\partial \tau}(\tau, \eta_1, \cdots, \eta_n) + \left[\sum_{j=1}^{n} i\eta_j \kappa_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \rho_{j,k} \sigma_j \sigma_k \eta_j \eta_k \right] \hat{U}(\tau, \eta_1, \cdots, \eta_n) = 0, \quad (2.11.15)$$

where

$$\kappa_j = r - q_j - \frac{1}{2}\sigma_j^2, \quad j = 1, \cdots, n. \quad (2.11.16)$$

Equation (2.11.15) *is solved subject to the initial condition*

$$\hat{U}(0, \eta_1, \cdots, \eta_n) = e^{i\eta_1 x_{1,0} + \cdots + i\eta_n x_{n,0}}. \quad (2.11.17)$$

Proof: Application of the Fourier transform operator in equation (1.3.4) to the partial differential equation (2.11.12) subject to the initial condition (2.11.13) yields the result in the above proposition. □

We have managed to transform the homogeneous PDE to a corresponding homogeneous ODE which we can readily solve by using the integrating factor method.

PROPOSITION 2.11.3. *The solution of the ODE* (2.11.15) *is represented as*

$$\hat{U}(\tau,\eta_1,\cdots,\eta_n)=\hat{U}(0,\eta_1,\cdots,\eta_n)\hat{K}(\tau,\eta_1,\cdots,\eta_n), \tag{2.11.18}$$

where

$$\hat{K}(\tau,\eta_1,\cdots,\eta_n)=\exp\left\{-\left(\sum_{j=1}^{n}i\eta_j\kappa_j+\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\rho_{j,k}\sigma_j\sigma_k\eta_j\eta_k\right)\tau\right\}, \tag{2.11.19}$$

and $\hat{U}(0,\eta_1,\cdots,\eta_n)$ *is given by equation* (2.11.17).

Proof: Refer to Appendix 2.8. □

2.11.3. Inverting the Fourier Transform. Having solved the ODE, the next step involves recovering the original transition density function, $U(\tau,x_1,\cdots,x_n)$ which is expressed in terms of the time-to-maturity τ, and the vector $x_1,\cdots,x_n$. This is accomplished by applying the Fourier inversion theorem to equation (2.11.18) and we give the result in the proposition below.

PROPOSITION 2.11.4. *The inverse Fourier transform of* (2.11.18) *is represented as*

$$U(\tau,x_1,\cdots,x_n)=\frac{e^{-r\tau}}{(2\pi\tau)^{\frac{n}{2}}\sigma_1\cdots\sigma_n\sqrt{\det(\Sigma)}}\exp\left\{-\frac{1}{2}\alpha^T\Sigma^{-1}\alpha\right\}, \tag{2.11.20}$$

where Σ *is the variance-covariance matrix for the vector of underlying assets and*

$$\alpha_j=\frac{1}{\sigma_i\sqrt{\tau}}\left(x_j-x_{j,0}+\kappa_j\tau\right),$$

for $j=1,\cdots,n$.

Proof: By applying the results of Definition 1.3.2 for the inverse Fourier transform to equation (2.11.18) and subsequently generalising the arguments in Proposition 2.4.1 we arrive at the results stated in Proposition 2.11.4. □

Having found the explicit representation of the multivariate transition density function, the next step involves substituting this back into equation (2.11.14) to obtain the full representation of our pricing function.

2.11.4. The American Option Value. With the density function in Proposition 2.11.4, the American option price can be re-expressed as

$$C(\tau, x_1, \cdots, x_n) = C_E(\tau, x_1, \cdots, x_n) + C_P(\tau, x_1, \cdots, x_n), \tag{2.11.21}$$

where

$$\begin{aligned}C_E(\tau, x_1, \cdots, x_n) = & \frac{e^{-r\tau}}{(2\pi\tau)^{\frac{n}{2}}\sigma_1 \cdots \sigma_n \sqrt{\det(\Sigma)}} \int_{-\infty}^{\infty} \cdots \int_{\infty}^{\infty} c(u_1, \cdots, u_n) \\ & \times \exp\left\{-\frac{1}{2}\alpha^T \Sigma^{-1} \alpha\right\} du_1 \cdots du_n,\end{aligned}$$

and

$$\begin{aligned}C_P(\tau, x_1, \cdots, x_n) = & \int_0^{\tau} \frac{e^{-r(\tau-\xi)}}{(2\pi(\tau-\xi))^{\frac{n}{2}}\sigma_1 \cdots \sigma_n \sqrt{\det(\Sigma)}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\tau, u_1, \cdots, u_n) \\ & \times \exp\left\{-\frac{1}{2}\alpha^T \Sigma^{-1} \alpha\right\} du_1 \cdots du_n d\xi.\end{aligned}$$

Equation (2.11.21) forms the general solution of any American option price written on n underlying assets. The only thing needed is the specification of the payoff function and the corresponding early exercise premium component. As is well known, the early exercise component of any American option will be a function of the early exercise boundary which needs to be determined as part of the solution at every instant. The process of finding the early exercise boundary is heavily dependent on the value matching condition which ensures that at this boundary the continuation value should equal the immediate exercise value of the option. Many techniques have been employed for lower dimensional problems such as numerical integration as in Kallast and Kivinukk

(2003) where the option time domain is discretised into smaller intervals. The option is treated like a Bermudan and then root finding techniques employed to find the corresponding early exercise boundary at each time step.

Similar techniques have been applied in Chiarella and Ziogas (2004) where they value an American strangle option. Approximation techniques can also be used to determine the functional form of the early exercise boundary. With the knowledge of the shape of the early exercise boundary, suitable approximating functions can be used to generate the entire early exercise boundary. Ait-Sahlia and Lai (2001) use some piecewise linear functions to track the early exercise boundary of the standard American put option while we have used a linear approximating function for the American spread option free-boundary. As we progress to higher dimensions such techniques may be computationally cumbersome due to the curse of dimensionality and lack of knowledge for the general shape of such early exercise surfaces.

2.12. Conclusion

In this chapter we have derived the American option price of a contract written on more than one underlying asset under the geometric Brownian motion framework of Black and Scholes (1973). Unlike McKean (1965) and Chiarella and Ziogas (2004) where the pricing partial differential equation (PDE) is converted to the corresponding ordinary differential equation (ODE) using the incomplete Fourier transform, we have exploited the transformation proposed by Jamshidian (1992) to represent the American option pricing PDE as an inhomogeneous PDE in an unrestricted domain thus paving the way to the application of the complete Fourier transform. By using Duhamel's principle, we have managed to derive the general solution of the inhomogeneous PDE whose solution involves the transition density function of the two underlying stochastic processes.

The transition density function satisfies the backward Kolmogorov equation associated with the underlying stochastic processes. By presenting the general solution of the inhomogeneous PDE, we have managed to transform our problem from that of solving the free boundary problem to that of solving the PDE for the transition density function which is a comparatively simpler task to handle. In solving the PDE for the density function, we have transformed the PDE to a corresponding ODE by using Fourier transforms. The ODE has been solved using the well known integrating factor approach. After finding the solution of the ODE, we revert back to the original variables by applying the Fourier inversion theorem to recover the full representation of the density function which holds for any payoff function associated with these two assets.

For illustrative purposes, we have presented a two-dimensional example of the American spread call option. In implementing the pricing function, we first reduced the dimensions of integration by simplifying the integral with respect to one asset variable and then performed numerical integration on the second asset variable. Details on how to tackle the pricing components have been presented with an approximation technique for the free-boundary introduced. To carry out the implementation process, the time integral has been discretised into equally spaced intervals followed by the application of Simpson's rule. We have compared the accuracy of our results with the Monte Carlo simulation method of Ibáñez and Zapatero (2004), the Fourier space time-stepping method of Surkov (2008) and the MOL algorithm we developed for the spread option PDE.

It has been left for future research to develop computational schemes for the integral expressions of contracts written on more than two underlying assets. Another line of research is to factor stochastic volatility into the pricing model for the American spread option as will be demonstrated in Chapter 5.

Appendix 2.1. Proof of Proposition 2.2.1

In this appendix we derive the pricing PDE for an American option written on the two assets S_1 and S_2 using hedging arguments. The underlying assets, S_1 and S_2 pay continuously compounded dividend yields at rates q_1 and q_2 respectively. We set $V(t, S_1, S_2)$ to be the option value. Application of Ito's Lemma yields

$$dV = \mu_V V dt + \sigma_V^1 V dW_1 + \sigma_V^2 V dW_2, \tag{A2.1.1}$$

where expected return of the option contract is given by

$$\mu_V V = \frac{\partial V}{\partial t} + \mu_1 S_1 \frac{\partial V}{\partial S_1} + \mu_2 S_2 \frac{\partial V}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}.$$

and the volatilities associated with each Wiener process are given by

$$\sigma_V^1 V = \sigma_1 S_1 \frac{\partial V}{\partial S_1} + \rho\sigma_2 S_2 \frac{\partial V}{\partial S_2},$$
$$\sigma_V^2 V = \sigma_2 \sqrt{1-\rho^2} S_2 \frac{\partial V}{\partial S_2}.$$

We now construct a portfolio consisting of a long position in an option, and short positions in Q_1 units of S_1 and Q_2 units of S_2. At current time, t, the value of this portfolio is represented as

$$H = Q_1 S_1 + Q_2 S_2 + V. \tag{A2.1.2}$$

Over a small change in time, dt, the corresponding change in portfolio value, given that the underlying assets pay continuously compounded dividend yields at the rates q_1 and q_2 respectively, is

$$dH = Q_1(dS_1 + q_1 S_1 dt) + Q_2(dS_2 + q_2 S_2 dt) + dV. \tag{A2.1.3}$$

Substituting equation (A2.1.1) and the system (2.2.2) into equation (A2.1.3) yields

$$\begin{aligned} dH &= [Q_1(\mu_1 + q_1)S_1 + Q_2(\mu_2 + q_2)S_2 + \mu_V V]dt \\ &+ [Q_1\sigma_1 S_1 + Q_2\rho\sigma_2 S_2 + \sigma_V^1 V]dW_1 \\ &+ [Q_2\sigma_2\sqrt{1-\rho^2}S_2 + \sigma_V^2 V]dW_2. \end{aligned} \tag{A2.1.4}$$

The quantities, Q_1 and Q_2 are chosen so as to eliminate risk and this is achieved by setting

$$\begin{aligned} Q_1\sigma_1 S_1 + Q_2\rho\sigma_2 S_2 + \sigma_V^1 V &= 0, \\ Q_2\sigma_2\sqrt{1-\rho^2}S_2 + \sigma_V^2 V &= 0. \end{aligned} \tag{A2.1.5}$$

Solving for Q_1 and Q_2 yields

$$Q_1 = -\frac{\partial V}{\partial S_1} \quad \text{and} \quad Q_2 = -\frac{\partial V}{\partial S_2}. \tag{A2.1.6}$$

This implies that the portfolio consists of short positions on the underlying assets, S_1 and S_2 and a long option position. By substituting (A2.1.6) into equations (A2.1.4) and (A2.1.2) we find that in a risk free market H is given by

$$H = -S_1\frac{\partial V}{\partial S_1} - S_2\frac{\partial V}{\partial S_2} + V, \tag{A2.1.7}$$

and H evolves according to

$$dH = \left[-\frac{\partial V}{\partial S_1}(\mu_1 + q_1)S_1 - \frac{\partial V}{\partial S_2}(\mu_2 + q_2)S_2 + \mu_V V\right] dt. \tag{A2.1.8}$$

Given that the change in the portfolio value is now riskless, it must earn the risk free rate of interest r to avoid riskless arbitrage opportunities, that is

$$dH = rHdt.$$

The last two equations imply that

$$-\frac{\partial V}{\partial S_1}(\mu_1+q_1)S_1-\frac{\partial V}{\partial S_2}(\mu_2+q_2)S_2+\mu_V V=r\left(-S_1\frac{\partial V}{\partial S_1}-S_2\frac{\partial V}{\partial S_2}+V\right). \quad \text{(A2.1.9)}$$

Substituting for $\mu_V V$ yields

$$\frac{\partial V}{\partial t}+(r-q_1)S_1\frac{\partial V}{\partial S_1}+(r-q_2)S_2\frac{\partial V}{\partial S_2}+\frac{1}{2}\sigma_1^2S_1^2\frac{\partial^2 V}{\partial S_1^2}$$
$$+\rho\sigma_1\sigma_2S_1S_2\frac{\partial^2 V}{\partial S_1\partial S_2}+\frac{1}{2}\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_2^2}-rV=0. \quad \text{(A2.1.10)}$$

By letting $\tau=T-t$ the above equation becomes

$$\frac{\partial V}{\partial \tau}=(r-q_1)S_1\frac{\partial V}{\partial S_1}+(r-q_2)S_2\frac{\partial V}{\partial S_2}+\frac{1}{2}\sigma_1^2S_1^2\frac{\partial^2 V}{\partial S_1^2}$$
$$+\rho\sigma_1\sigma_2S_1S_2\frac{\partial^2 V}{\partial S_1\partial S_2}+\frac{1}{2}\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_2^2}-rV, \quad \text{(A2.1.11)}$$

which is the pricing PDE in Proposition 2.2.1. The solution of the PDE depends on different payoff functions and their associated boundary conditions.

Appendix 2.2. Proof of Proposition 2.2.3

In this appendix we want to show that Duhamel's principle holds by demonstrating that the result in Proposition 2.2.3 is consistent with the PDE (2.2.15). The variables x_1, x_2 and $C(\tau,x_1,x_2)$, the PDE (2.2.15) can be written as

$$\frac{\partial C}{\partial \tau}=\mathcal{M}C-rC+f(\tau,x_1,x_2), \quad \text{(A2.2.1)}$$

where

$$\mathcal{M}C=(r-q_1-\frac{1}{2}\sigma_1^2)\frac{\partial C}{\partial x_1}+(r-q_2-\frac{1}{2}\sigma_2^2)\frac{\partial C}{\partial x_2}+\frac{1}{2}\sigma_1^2\frac{\partial^2 C}{\partial x_1^2}+\rho\sigma_1\sigma_2\frac{\partial^2 C}{\partial x_1\partial x_2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 C}{\partial x_2^2}.$$

Since we are considering the American spread call option payoff, $c(x_1,x_2)=(e^{x_1}-e^{x_2}-K)^+$ and $f(\tau,x_1,x_2)=q_1e^{x_1}-q_2e^{x_2}-rK$. We now want to show that (2.2.21) satisfies (A2.2.1). Substituting $C(\tau,x_1,x_2)=C_E(\tau,x_1,x_2)+C_P(\tau,x_1,x_2)$ into (A2.2.1) and making use of the initial condition in equation (2.2.20) we find that

$$
\begin{aligned}
\frac{\partial C}{\partial \tau} - \mathcal{M}C + rC - f(\tau, x_1, x_2) &= e^{-r\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(u_1, u_2) \left\{ \frac{\partial U}{\partial \tau} - \mathcal{M}U \right\} du_1 du_2 \\
&+ \int_0^{\tau} e^{-r(\tau-\xi)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, u_1, u_2) \left\{ \frac{\partial U}{\partial \tau} - \mathcal{M}U \right\} du_1 du_2 d\xi + rC - rC \\
&+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau, u_1, u_2) U(0, x_1, x_2) du_1 du_2 - f(\tau, x_1, x_2) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau, u_1, u_2) \delta(x_1 - u_1) \delta(x_2 - u_2) du_1 du_2 - f(\tau, x_1, x_2) \\
&= f(\tau, x_1, x_2) - f(\tau, x_1, x_2) = 0. \qquad \text{(A2.2.2)}
\end{aligned}
$$

This implies that the pricing function (2.2.21) satisfies the PDE (A2.2.1).

Appendix 2.3. Proof of Proposition 2.3.1

We consider first the partial derivative of $U(\tau, x_1, x_2)$ with respect to time. By using the results in Definition 1.3.1 and interchanging the order of differentiation and integration we note that

$$
\mathcal{F}\left\{ \frac{\partial U}{\partial \tau} \right\} = \frac{\partial}{\partial \tau} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2} U(\tau, x_1, x_2) dx_1 dx_2 \right] = \frac{\partial \hat{U}}{\partial \tau}(\tau, \eta_1, \eta_2). \qquad \text{(A2.3.1)}
$$

We will make the following assumptions about the density function:

$$
\lim_{x_1, x_2 \to \pm\infty} U(\tau, x_1, x_2) = \lim_{x_1 \to \pm\infty} \frac{\partial U}{\partial x_1} = \lim_{x_2 \to \pm\infty} \frac{\partial U}{\partial x_2} = 0. \qquad \text{(A2.3.2)}
$$

The Fourier transform of the first derivative of $U(\tau, x_1, x_2)$ with respect to x_1 is

$$
\mathcal{F}\left\{ \frac{\partial U}{\partial x_1} \right\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2} \frac{\partial U}{\partial x_1} dx_1 dx_2. \qquad \text{(A2.3.3)}
$$

Integrating by parts and then applying the assumptions in equation (A2.3.2) we obtain

$$
\mathcal{F}\left\{ \frac{\partial U}{\partial x_1} \right\} = -i\eta_1 \hat{U}(\tau, \eta_1, \eta_2). \qquad \text{(A2.3.4)}
$$

Likewise

$$\mathcal{F}\Big\{\frac{\partial U}{\partial x_2}\Big\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2}\frac{\partial U}{\partial x_2}dx_1 dx_2 = -i\eta_2\hat{U}(\tau,\eta_1,\eta_2). \qquad \text{(A2.3.5)}$$

The Fourier transform of the second derivative with respect to x_1 is

$$\mathcal{F}\Big\{\frac{\partial^2 U}{\partial x_1^2}\Big\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2}\frac{\partial^2 U}{\partial x_1^2}dx_1 dx_2 = -\eta_1^2\hat{U}(\tau,\eta_1,\eta_2). \qquad \text{(A2.3.6)}$$

Similarly by using the assumptions in (A2.3.2) we find that

$$\mathcal{F}\left\{\frac{\partial^2 U}{\partial x_2^2}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2}\frac{\partial^2 U}{\partial x_2^2}dx_1 dx_2 = -\eta_2^2\hat{U}(\tau,\eta_1,\eta_2). \qquad \text{(A2.3.7)}$$

Lastly, the Fourier transform of the mixed partial derivative term becomes

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial^2 U}{\partial x_1 \partial x_2}\right\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2}\frac{\partial^2 U}{\partial x_1 \partial x_2}dx_1 dx_2 \\
&= \int_{-\infty}^{\infty}\frac{\partial}{\partial x_2}\left[\int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2}\frac{\partial U}{\partial x_1}dx_1\right]dx_2 \\
&= \int_{-\infty}^{\infty}\frac{\partial}{\partial x_2}\left[-i\eta_1\int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2}U(\tau,x_1,x_2)dx_1\right]dx_2 \\
&= -i\eta_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2}\frac{\partial U}{\partial x_2}dx_1 dx_2 \\
&= -\eta_1\eta_2\hat{U}(\tau,\eta_1,\eta_2). \qquad \text{(A2.3.8)}
\end{aligned}$$

By substituting equations (A2.3.1)-(A2.3.8) into equation (2.2.19) we obtain the result in Proposition 2.3.1.

The Fourier transform of the initial condition (2.2.20) with respect to x_1 and x_2 is

$$\begin{aligned}
\mathcal{F}\left\{U(0,x_1,x_2)\right\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2}\delta(x_1 - x_{1,0})\delta(x_2 - x_{2,0})dx_1 dx_2 \\
&= e^{i\eta_1 x_{1,0} + i\eta_2 x_{2,0}}, \qquad \text{(A2.3.9)}
\end{aligned}$$

which is the result presented in equation (2.3.3) of Proposition 2.3.1.

Appendix 2.4. Proof of Proposition 2.3.2

For convenience, we first set

$$\alpha(\eta_1, \eta_2) = i\eta_1\kappa_1 + i\eta_2\kappa_2 + \frac{1}{2}\sigma_1^2\eta_1^2 + \sigma_1\sigma_2\eta_1\eta_2 + \frac{1}{2}\sigma_2^2\eta_2^2,$$

such that equation (2.3.1) can be rewritten as

$$\frac{\partial \hat{U}}{\partial \tau}(\tau, \eta_1, \eta_2) + \alpha(\eta_1, \eta_2)\hat{U}(\tau, \eta_1, \eta_2) = 0. \tag{A2.4.1}$$

Taking $e^{\int_0^\tau \alpha(\eta_1,\eta_2)ds}$ as the integrating factor yields the solution

$$\hat{U}(\tau, \eta_1, \eta_2) = e^{-\alpha(\eta_1,\eta_2)\tau}\hat{U}(0, \eta_1, \eta_2), \tag{A2.4.2}$$

which is the result presented in Proposition 2.3.2.

Appendix 2.5. Proof of Proposition 2.4.1

In this appendix we want to find the inverse Fourier transform of the transform given in Proposition 2.3.2. From definition 1.3.2, we note that

$$\begin{aligned} U(\tau,x_1, x_2) &= \mathcal{F}^{-1}\left\{\hat{U}(0, \eta_1, \eta_2)\hat{K}(\tau, \eta_1, \eta_2)\right\} \qquad \text{(A2.5.1)} \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\eta_1 x_1 - i\eta_2 x_2} e^{i\eta_1 x_{1,0} + i\eta_2 x_{2,0}} \\ &\quad \times \exp\left\{-\left[i\eta_1\kappa_1 + i\eta_2\kappa_2 + \frac{1}{2}\sigma_1^2\eta_1^2 + \rho\sigma_1\sigma_2\eta_1\eta_2 + \frac{1}{2}\sigma_2^2\eta_2^2\right]\tau\right\} d\eta_1 d\eta_2. \end{aligned}$$

By rearranging the components of equation (A2.5.1) we obtain[9]

$$\begin{aligned} U(\tau, x_1, x_2) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\sigma_2^2\tau\eta_2^2 + (x_{2,0} - x_2 - \kappa_2\tau)i\eta_2\right\} \qquad \text{(A2.5.3)} \\ &\times \left[\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\sigma_1^2\tau\eta_1^2 - i(x_1 - x_{1,0} + \kappa_1\tau - \rho\sigma_1\sigma_2 i\eta_2)i\eta_1\right\} d\eta_1\right] d\eta_2. \end{aligned}$$

By letting $p = \frac{1}{2}\sigma_1^2\tau$, $q = i(x_1 - x_{1,0} + \kappa_1\tau - \rho\sigma_1\sigma_2 i\eta_2)$ and making use of the result in Footnote 9 with $n = 0$ in the above equation we obtain

$$\begin{aligned} U(\tau,x_1, x_2) &= \frac{\sqrt{2\pi}e^{-r\tau}}{4\pi^2\sigma_1\sqrt{\tau}}\int_{-\infty}^{\infty} \exp\Big\{ -\frac{1}{2}\sigma_2^2\tau\eta_2^2 + (x_{2,0} - x_2 - \kappa_2\tau)i\eta_2\Big\} \\ &\times \exp\Big\{ -\frac{\{x_1 - x_{1,0} + (\kappa_1 - \rho\sigma_1\sigma_2 i\eta_2)\tau\}^2}{2\sigma_1^2\tau}\Big\}d\eta_2 \qquad \text{(A2.5.4)} \\ &= \frac{\sqrt{2\pi}e^{-r\tau}}{4\pi^2\sigma_1\sqrt{\tau}} \exp\Big\{ -\frac{(x_1 - x_{1,0} + \kappa_1\tau)^2}{2\sigma_1^2\tau}\Big\} \\ &\times \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\sigma_2^2\tau(1-\rho^2)\eta_2^2 - \left[x_2 - x_{2,0} + \kappa_2\tau - \frac{\rho\sigma_2(x_1 - x_{1,0} + \kappa_1\tau)}{\sigma_1}\right] i\eta_2\right\} d\eta_2. \end{aligned}$$

We apply similar techniques to the integral with respect to η_2 by letting $p = \frac{1}{2}\sigma_2^2\tau(1-\rho^2)$ and $q = i\big[x_2 - x_{2,0} + \kappa_2\tau - \frac{\rho\sigma_2(x_1 - x_{1,0}+\kappa_1\tau)}{\sigma_1}\big]$. Application of the result in Footnote 9 to equation (A2.5.4) and rearranging terms yields

$$\begin{aligned} U(\tau, x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2\tau(1-\rho^2)}\Big[\Big(\frac{x_1 - x_{1,0} + \kappa_1\tau}{\sigma_1}\Big)^2\right. \\ &\left. -2\rho\Big(\frac{x_1 - x_{1,0} + \kappa_1\tau}{\sigma_1}\Big)\Big(\frac{x_2 - x_{2,0} + \kappa_2\tau}{\sigma_2}\Big) + \Big(\frac{x_2 - x_{2,0} + \kappa_2\tau}{\sigma_2}\Big)^2\Big]\right\}, \qquad \text{(A2.5.5)} \end{aligned}$$

which is the result presented in Proposition 2.4.1.

[9]We make use of the following key result for complex functions (See for example Abramowitz and Stegun (1964)). Let p and q be complex variables independent of the integration variable η, with $\text{Re}(p) \geq 0$. Also let n be a positive integer. Then the integral of the exponential quadratic function with respect to η is given by

$$\int_{-\infty}^{\infty} e^{-p\eta^2 - q\eta}\eta^n d\eta = (-1)^n\sqrt{\frac{\pi}{p}}\frac{\partial^n}{\partial q^n}\left(e^{\frac{q^2}{4p}}\right). \qquad \text{(A2.5.2)}$$

Appendix 2.6. Proof of Proposition 2.6.1

We break this proof into a number of components as it is quite lengthy. We start by deriving equation (2.6.3) for the European option component followed by equation (2.6.4) for the early exercise premium component.

(1) **Derivation of the European Spread Option Component**

In order to derive equation (2.6.3), we know from the spread call option payoff that $c(x_1, x_2) = (e^{x_1} - e^{x_2} - K)^+$. Substituting this into equation (2.5.2) we obtain

$$C_E^S(\tau,x_1,x_2) = \frac{e^{-r\tau}}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{u_1} - e^{u_2} - K)^+ \exp\left\{\frac{-1}{2\tau(1-\rho^2)}\left[\left(\frac{x_1-u_1+\kappa_1\tau}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-u_1+\kappa_1\tau}{\sigma_1}\right)\left(\frac{x_2-u_2+\kappa_2\tau}{\sigma_2}\right) + \left(\frac{x_2-u_2+\kappa_2\tau}{\sigma_2}\right)^2\right]\right\} du_1 du_2. \tag{A2.6.1}$$

We note first of all that $e^{u_1} - e^{u_2} - K = 0$ at $e^{u_1} = e^{u_2} + K \equiv K_1(u_2)$. We break the above integral into three parts such that

$$C_E^S(\tau,x_1,x_2) = A_1(\tau,x_1,x_2) - A_2(\tau,x_1,x_2) - A_3(\tau,x_1,x_2), \tag{A2.6.2}$$

where

$$A_1(\tau,x_1,x_2) = \frac{e^{-r\tau}}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{\ln K_1(u_2)}^{\infty} e^{u_1} \exp\left\{\frac{-1}{2\tau(1-\rho^2)}\left[\left(\frac{x_1-u_1+\kappa_1\tau}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-u_1+\kappa_1\tau}{\sigma_1}\right)\left(\frac{x_2-u_2+\kappa_2\tau}{\sigma_2}\right) + \left(\frac{x_2-u_2+\kappa_2\tau}{\sigma_2}\right)^2\right]\right\} du_1 du_2, \tag{A2.6.3}$$

$$A_2(\tau,x_1,x_2) = \frac{e^{-r\tau}}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{\ln K_1(u_2)}^{\infty} e^{u_2} \exp\left\{\frac{-1}{2\tau(1-\rho^2)}\left[\left(\frac{x_1-u_1+\kappa_1\tau}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-u_1+\kappa_1\tau}{\sigma_1}\right)\left(\frac{x_2-u_2+\kappa_2\tau}{\sigma_2}\right) + \left(\frac{x_2-u_2+\kappa_2\tau}{\sigma_2}\right)^2\right]\right\} du_1 du_2, \tag{A2.6.4}$$

and

$$A_3(\tau,x_1,x_2) = \frac{e^{-r\tau}K}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{\ln K_1(u_2)}^{\infty} \exp\left\{\frac{-1}{2\tau(1-\rho^2)}\left[\left(\frac{x_1-u_1+\kappa_1\tau}{\sigma_1}\right)^2 \right.\right.$$
$$\left.\left. -2\rho\left(\frac{x_1-u_1+\kappa_1\tau}{\sigma_1}\right)\left(\frac{x_2-u_2+\kappa_2\tau}{\sigma_2}\right)+\left(\frac{x_2-u_2+\kappa_2\tau}{\sigma_2}\right)^2\right]\right\}du_1du_2. \tag{A2.6.5}$$

<u>Simplifying the $A_1(\tau,x_1,x_2)$ term:</u>

We first rearrange equation (A2.6.3) such that

$$A_1(\tau,x_1,x_2) = \frac{e^{-r\tau}}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2-u_2+\kappa_2\tau)^2}{-2\sigma_2^2(1-\rho^2)\tau}\right\} \int_{\ln K_1(u_2)}^{\infty} \exp\left\{\frac{(x_1-u_1+\kappa_1\tau)^2}{-2\sigma_1^2(1-\rho^2)\tau}\right\}$$
$$\times\exp\left\{\frac{\rho(x_2-u_2+\kappa_2\tau)(x_1-u_1+\kappa_1\tau)}{\sigma_1\sigma_2(1-\rho^2)\tau}+u_1\right\}du_1du_2. \tag{A2.6.6}$$

Equation (A2.6.6) simplifies to

$$A_1(\tau,x_1,x_2) = \frac{e^{-r\tau}}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2-u_2+\kappa_2\tau)^2}{-2\sigma_2^2(1-\rho^2)\tau}\right\}$$
$$\times\int_{\ln K_1(u_2)}^{\infty} \exp\left\{\frac{(x_1-u_1+\kappa_1\tau)^2-2\rho\frac{\sigma_1}{\sigma_2}(x_2-u_2+\kappa_2\tau)(x_1-u_1+\kappa_1\tau)-2\sigma_1^2\tau(1-\rho^2)u_1}{-2\sigma_1^2\tau(1-\rho^2)}\right\}du_1du_2.$$

By collecting like terms and completing the square in the above equation we obtain

$$A_1(\tau,x_1,x_2) = \frac{e^{x_1}e^{-q_1\tau}}{\sigma_2\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \exp\left\{\frac{[x_2-u_2+\kappa_2\tau+\rho\sigma_1\sigma_2\tau]^2}{-2\sigma_2^2\tau}\right\} \tag{A2.6.7}$$
$$\times\frac{1}{\sigma_1\sqrt{2\pi\tau(1-\rho^2)}}\int_{\ln K_1(u_2)}^{\infty} \exp\left\{\frac{[u_1-\{x_1+\kappa_1\tau-\rho\frac{\sigma_1}{\sigma_2}(x_2-u_2+\kappa_2\tau)+\sigma_1^2\tau(1-\rho^2)\}]^2}{-2\sigma_1^2\tau(1-\rho^2)}\right\}du_1du_2.$$

We make the change of variables

$$y_1 = \frac{u_1-\{x_1+\kappa_1\tau-\rho\frac{\sigma_1}{\sigma_2}(x_2-u_2+\kappa_2\tau)+\sigma_1^2\tau(1-\rho^2)\}}{\sigma_1\sqrt{\tau(1-\rho^2)}},$$

from which $du_1 = \sigma_1\sqrt{(1-\rho^2)\tau}dy_1$.

Also,

$$u_1 = \ln K_1(u_2) \implies y_1 = \frac{\ln K_1(u_2) - \{x_1 + \kappa_1\tau - \rho\frac{\sigma_1}{\sigma_2}(x_2 - u_2 + \kappa_2\tau) + \sigma_1^2\tau(1-\rho^2)\}}{\sigma_1\sqrt{\tau(1-\rho^2)}}.$$

Substituting this into equation (A2.6.7) yields

$$A_1(\tau, x_1, x_2) = \frac{e^{x_1}e^{-q_1\tau}}{\sigma_2\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}\exp\left\{\frac{[x_2 - u_2 + \kappa_2\tau + \rho\sigma_1\sigma_2\tau]^2}{-2\sigma_2^2\tau}\right\} \tag{A2.6.8}$$

$$\times \frac{1}{\sigma_1\sqrt{2\pi\tau(1-\rho^2)}}\int_{-d_1[\tau,x_1,K_1(u_2)]}^{\infty} e^{-\frac{y_1^2}{2}}dy_1du_2$$

$$= \frac{e^{x_1}e^{-q_1\tau}}{\sigma_2\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}\exp\left\{\frac{[x_2 - u_2 + \kappa_2\tau + \rho\sigma_1\sigma_2\tau]^2}{-2\sigma_2^2\tau}\right\}\frac{1}{\sigma_1\sqrt{2\pi\tau(1-\rho^2)}}\int_{-\infty}^{d_1[\tau,x_1,K_1(u_2)]} e^{-\frac{y_1^2}{2}}dy_1du_2.$$

where

$$d_1[\tau, x_1, K_1(u_2)] = \frac{x_1 + \kappa_1\tau - \rho\frac{\sigma_1}{\sigma_2}(x_2 - u_2 + \kappa_2\tau) - \ln K_1(u_2) + \sigma_1^2(1-\rho^2)\tau}{\sigma_1\sqrt{(1-\rho^2)\tau}}. \tag{A2.6.9}$$

Equation (A2.6.8) can be more simply represented as

$$A_1(\tau, x_1, x_2) = \frac{e^{x_1}e^{-q_1\tau}}{\sigma_2\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}\exp\left\{\frac{[x_2 - u_2 + \kappa_2\tau + \rho\sigma_1\sigma_2\tau]^2}{-2\sigma_2^2\tau}\right\}\mathcal{N}\big(d_1[\tau, x_1, K_1(u_2)]\big)du_2, \tag{A2.6.10}$$

where $\mathcal{N}(x)$ is the cumulative normal distribution function.

Simplifying the $A_2(\tau, x_1, x_2)$ term:

The second term, equation (A2.6.4) can can be rearranged to

$$A_2(\tau, x_1, x_2) = \frac{e^{-r\tau}}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left\{\frac{(x_2 - u_2 + \kappa_2\tau)^2}{-2\sigma_2^2(1-\rho^2)\tau} + u_2\right\}\int_{\ln K_1(u_2)}^{\infty}\exp\left\{\frac{(x_1 - u_1 + \kappa_1\tau)^2}{-2\sigma_1^2(1-\rho^2)\tau}\right\}$$

$$\times \exp\left\{\frac{\rho(x_2 - u_2 + \kappa_2\tau)(x_1 - u_1 + \kappa_1\tau)}{\sigma_1\sigma_2(1-\rho^2)\tau}\right\}du_1du_2. \tag{A2.6.11}$$

After completing the square, the above equation simplifies to

$$A_2(\tau,x_1,x_2) = \frac{e^{x_2}e^{-q_2\tau}}{\sigma_2\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}\exp\left\{\frac{[u_2-(x_2+\kappa_2\tau+\sigma_2^2\tau)]^2}{-2\sigma_2^2\tau}\right\} \quad \text{(A2.6.12)}$$
$$\times\frac{1}{\sigma_1\sqrt{2\pi\tau(1-\rho^2)}}\int_{\ln K_1(u_2)}^{\infty}\exp\left\{\frac{[x_1-u_1+\kappa_1\tau-\rho\frac{\sigma_1}{\sigma_2}(x_2-u_2+\kappa_2\tau)]^2}{-2\sigma_1^2(1-\rho^2)\tau}\right\}du_1du_2.$$

We set

$$y_2 = \frac{x_1-u_1+\kappa_1\tau-\rho\frac{\sigma_1}{\sigma_2}(x_2-u_2+\kappa_2\tau)}{\sigma_1\sqrt{(1-\rho^2)\tau}},$$

so that $du_1 = \sigma_1\sqrt{(1-\rho^2)\tau}dy_2$. Also note that

$$u_1 = \ln K_1(u_2) \implies y_2 = \frac{x_1+\kappa_1\tau-\rho\frac{\sigma_1}{\sigma_2}(x_2-u_2+\kappa_2\tau)-\ln K_1(u_2)}{\sigma_1\sqrt{(1-\rho^2)\tau}}.$$

Substituting these expressions into equation (A2.6.12) yields

$$A_2(\tau,x_1,x_2) = \frac{e^{x_2}e^{-q_2\tau}}{\sigma_2\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}\exp\left\{\frac{[u_2-(x_2+\kappa_2\tau+\sigma_2^2\tau)]^2}{-2\sigma_2^2\tau}\right\}$$
$$\times\frac{1}{\sigma_1\sqrt{2\pi\tau(1-\rho^2)}}\int_{d_2[\tau,x_1,K_1(u_1)]}^{\infty}e^{-\frac{y_2^2}{2}}dy_2du_2, \quad \text{(A2.6.13)}$$

where

$$d_2[\tau,x_1,K_1(u_2)] = \frac{x_1+\kappa_1\tau-\rho\frac{\sigma_1}{\sigma_2}(x_2-u_2+\kappa_2\tau)-\ln K_1(u_2)}{\sigma_1\sqrt{(1-\rho^2)\tau}}$$
$$= d_1[\tau,x_1,K_1(u_2)]-\sigma_1^2\sqrt{(1-\rho^2)\tau}.$$

Equation (A2.6.13) simplifies to

$$A_2(\tau,x_1,x_2) = \frac{e^{x_2}e^{-q_2\tau}}{\sigma_2\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}\exp\left\{\frac{[u_2-(x_2+\kappa_2\tau+\sigma_2^2\tau)]^2}{-2\sigma_2^2\tau}\right\}\mathcal{N}\big(d_2[\tau,x_1,K_1(u_2)]\big)du_2. \quad \text{(A2.6.14)}$$

Simplifying the $A_3(\tau,x_1,x_2)$ term:

By rearranging equation (A2.6.5) we obtain

$$A_3(\tau, x_1, x_2) = \frac{e^{-r\tau}K}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2 - u_2 + \kappa_2\tau)^2}{-2\sigma_2^2(1-\rho^2)\tau}\right\} \int_{\ln K_1(u_2)}^{\infty} \exp\left\{\frac{(x_1 - u_1 + \kappa_1\tau)^2}{-2\sigma_1^2(1-\rho^2)\tau}\right\}$$
$$\times \exp\left\{\frac{\rho(x_2 - u_2 + \kappa_2\tau)(x_1 - u_1 + \kappa_1\tau)}{\sigma_1\sigma_2(1-\rho^2)\tau}\right\} du_1 du_2. \quad \text{(A2.6.15)}$$

After completing the square, the above equation simplifies to

$$A_3(\tau, x_1, x_2) = \frac{e^{-r\tau}K}{\sigma_2\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2 - u_2 + \kappa_2\tau)^2}{-2\sigma_2^2\tau}\right\} \quad \text{(A2.6.16)}$$
$$\times \frac{1}{\sigma_1\sqrt{2\pi\tau(1-\rho^2)}} \int_{\ln K_1(u_2)}^{\infty} \exp\left\{\frac{[x_1 - u_1 + \kappa_1\tau - \rho\frac{\sigma_1}{\sigma_2}(x_2 - u_2 + \kappa_2\tau)]^2}{-2\sigma_1^2(1-\rho^2)\tau}\right\} du_1 du_2.$$

If we let

$$y_3 = \frac{x_1 - u_1 + \kappa_1\tau - \rho\frac{\sigma_1}{\sigma_2}(x_2 - u_2 + \kappa_2\tau)}{\sigma_1\sqrt{(1-\rho^2)\tau}},$$

then $du_1 = \sigma_1\sqrt{(1-\rho^2)\tau}\,dy_3$. Also

$$u_1 = \ln K_1(u_2) \implies y_3 = \frac{x_1 + \kappa_1\tau - \rho\frac{\sigma_1}{\sigma_2}(x_2 - u_2 + \kappa_2\tau) - \ln K_1(u_2)}{\sigma_1\sqrt{(1-\rho^2)\tau}}.$$

Substituting this into equation (A2.6.16) we obtain

$$A_3(\tau, x_1, x_2) = \frac{e^{-r\tau}K}{\sigma_2\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2 - u_2 + \kappa_2\tau)^2}{-2\sigma_2^2\tau}\right\}$$
$$\times \frac{1}{\sigma_1\sqrt{2\pi\tau(1-\rho^2)}} \int_{d_2[\tau, x_1, K_1(u_2)]}^{\infty} e^{-\frac{y_3^2}{2}} dy_3 du_2. \quad \text{(A2.6.17)}$$

Equation (A2.6.17) simplifies to

$$A_3(\tau, x_1, x_2) = \frac{e^{-r\tau}K}{\sigma_2\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2 - u_2 + \kappa_2\tau)^2}{-2\sigma_2^2\tau}\right\} \mathcal{N}\big(d_2[\tau, x_1, K_1(u_2)]\big) du_2.$$
$$\text{(A2.6.18)}$$

By combining equations (A2.6.10), (A2.6.14) and (A2.6.18) together we can represent the European spread call option component as

$$\begin{aligned}C_E^S(\tau,x_1,x_2) = &\frac{e^{x_1}e^{-q_1\tau}}{\sigma_2\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}\exp\left\{-\frac{[x_2-u_2+\kappa_2\tau+\rho\sigma_1\sigma_2\tau]^2}{2\sigma_2^2\tau}\right\}\mathcal{N}\big(d_1[\tau,x_1,K_1(u_2)]\big)du_2\\ &-\frac{e^{x_2}e^{-q_2\tau}}{\sigma_2\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}\exp\left\{-\frac{[x_2-u_2+\kappa_2\tau+\sigma_2^2\tau]^2}{2\sigma_2^2\tau}\right\}\mathcal{N}\big(d_2[\tau,x_1,K_1(u_2)]\big)du_2\\ &-\frac{e^{-r\tau}K}{\sigma_2\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}\exp\left\{-\frac{(x_2-u_2+\kappa_2\tau)^2}{2\sigma_2^2\tau}\right\}\mathcal{N}\big(d_2[\tau,x_1,K_1(u_2)]\big)du_2, \qquad \text{(A2.6.19)}\end{aligned}$$

which is the result in equation (2.6.3) of Proposition 2.6.2.

(2) <u>Derivation of the Early Exercise Premium Component</u>

We now turn to the derivation equation (2.6.4) for the early exercise premium component. The derivation is very similar to that of $C_E^S(\tau,x_1,x_2)$ the main difference being in the limits of the u_1 integral and also the early exercise premium component involves a time integral. By specifying the spread option payoff, it can be shown that $f(\tau,x_1,x_2)=q_1e^{x_1}-q_2e^{x_2}-rK$ so that equation (2.5.3) becomes

$$\begin{aligned}C_P^S(\tau,x_1,x_2) = &\int_0^{\tau}\frac{e^{-r(\tau-\xi)}}{2\pi\sigma_1\sigma_2(\tau-\xi)\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{\ln B(\xi,u_2)}^{\infty}(q_1e^{u_1}-q_2e^{u_2}-rK)\\ &\times\exp\left\{\frac{-1}{2(\tau-\xi)(1-\rho^2)}\left[\left(\frac{x_1-u_1+\kappa_1(\tau-\xi)}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-u_1+\kappa_1(\tau-\xi)}{\sigma_1}\right)\right.\right.\\ &\left.\left.\times\left(\frac{x_2-u_2+\kappa_2(\tau-\xi)}{\sigma_2}\right)+\left(\frac{x_2-u_2+\kappa_2(\tau-\xi)}{\sigma_2}\right)^2\right]\right\}du_1du_2d\xi. \qquad \text{(A2.6.20)}\end{aligned}$$

In simplifying equation (A2.6.20), we first split it into three parts such that

$$C_P^S(\tau,x_1,x_2)=I_1(\tau,x_1,x_2)-I_2(\tau,x_1,x_2)-I_3(\tau,x_1,x_2), \qquad \text{(A2.6.21)}$$

where

$$I_1(\tau,x_1,x_2) = \int_0^\tau \frac{e^{-r(\tau-\xi)}}{2\pi\sigma_1\sigma_2(\tau-\xi)\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{\ln B(\xi,u_2)}^{\infty} \tag{A2.6.22}$$
$$\times\, q_1 e^{u_1} \exp\left\{\frac{-1}{2(\tau-\xi)(1-\rho^2)}\left[\left(\frac{x_1-u_1+\kappa_1(\tau-\xi)}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-u_1+\kappa_1(\tau-\xi)}{\sigma_1}\right)\right.\right.$$
$$\left.\left.\times\left(\frac{x_2-u_2+\kappa_2(\tau-\xi)}{\sigma_2}\right)+\left(\frac{x_2-u_2+\kappa_2(\tau-\xi)}{\sigma_2}\right)^2\right]\right\}du_1du_2d\xi,$$

$$I_2(\tau,x_1,x_2) = \int_0^\tau \frac{e^{-r(\tau-\xi)}}{2\pi\sigma_1\sigma_2(\tau-\xi)\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{\ln B(\xi,u_2)}^{\infty} \tag{A2.6.23}$$
$$\times\, q_2 e^{u_2} \exp\left\{\frac{-1}{2(\tau-\xi)(1-\rho^2)}\left[\left(\frac{x_1-u_1+\kappa_1(\tau-\xi)}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-u_1+\kappa_1(\tau-\xi)}{\sigma_1}\right)\right.\right.$$
$$\left.\left.\times\left(\frac{x_2-u_2+\kappa_2(\tau-\xi)}{\sigma_2}\right)+\left(\frac{x_2-u_2+\kappa_2(\tau-\xi)}{\sigma_2}\right)^2\right]\right\}du_1du_2d\xi,$$

and

$$I_3(\tau,x_1,x_2) = \int_0^\tau \frac{rKe^{-r(\tau-\xi)}}{2\pi\sigma_1\sigma_2(\tau-\xi)\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{\ln B(\xi,u_2)}^{\infty} \tag{A2.6.24}$$
$$\times \exp\left\{\frac{-1}{2(\tau-\xi)(1-\rho^2)}\left[\left(\frac{x_1-u_1+\kappa_1(\tau-\xi)}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-u_1+\kappa_1(\tau-\xi)}{\sigma_1}\right)\right.\right.$$
$$\left.\left.\times\left(\frac{x_2-u_2+\kappa_2(\tau-\xi)}{\sigma_2}\right)+\left(\frac{x_2-u_2+\kappa_2(\tau-\xi)}{\sigma_2}\right)^2\right]\right\}du_1du_2d\xi.$$

Simplifying the $I_1(\tau, x_1, x_2)$ term:

We first rearrange equation (A2.6.22) such that

$$
\begin{aligned}
I_1(\tau, x_1, x_2) = &\int_0^\tau \frac{q_1 e^{-r(\tau-\xi)}}{2\pi\sigma_1\sigma_2(\tau-\xi)\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2 - u_2 + \kappa_2(\tau-\xi))^2}{-2\sigma_2^2(1-\rho^2)(\tau-\xi)}\right\} \\
&\times \int_{\ln B(\xi,u_2)}^{\infty} \exp\left\{\frac{(x_1 - u_1 + \kappa_1(\tau-\xi))^2}{-2\sigma_1^2(1-\rho^2)(\tau-\xi)}\right\} \\
&\times \exp\left\{\frac{\rho(x_2 - u_2 + \kappa_2(\tau-\xi))(x_1 - u_1 + \kappa_1(\tau-\xi))}{\sigma_1\sigma_2(1-\rho^2)(\tau-\xi)} + u_1\right\} du_1 du_2 d\xi.
\end{aligned}
\tag{A2.6.25}
$$

Equation (A2.6.25) can be rearranged to read

$$
\begin{aligned}
I_1(\tau, x_1, x_2) = &\int_0^\tau \frac{q_1 e^{-r(\tau-\xi)}}{2\pi\sigma_1\sigma_2(\tau-\xi)\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2 - u_2 + \kappa_2(\tau-\xi))^2}{-2\sigma_2^2(1-\rho^2)(\tau-\xi)}\right\} \\
&\times \int_{B(\xi,u_2)}^{\infty} \exp\left\{\frac{(x_1 - u_1 + \kappa_1(\tau-\xi))^2 - 2\sigma_1^2(\tau-\xi)(1-\rho^2)u_1}{-2\sigma_1^2(\tau-\xi)(1-\rho^2)}\right\} \\
&\times \exp\left\{\frac{-2\rho\frac{\sigma_1}{\sigma_2}(x_2 - u_2 + \kappa_2(\tau-\xi))(x_1 - u_1 + \kappa_1(\tau-\xi))}{-2\sigma_1^2(\tau-\xi)(1-\rho^2)}\right\} du_1 du_2 d\xi.
\end{aligned}
$$

By proceeding as we did when handling the A_1 term for the European option component in equation (A2.6.6) it can be shown that the above equation reduces to

$$
\begin{aligned}
I_1(\tau, x_1, x_2) = &\int_0^\tau \frac{q_1 e^{x_1} e^{-q_1(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{\frac{[x_2 - u_2 + \kappa_2(\tau-\xi) + \rho\sigma_1\sigma_2(\tau-\xi)]^2}{-2\sigma_2^2(\tau-\xi)}\right\} \\
&\times \frac{1}{\sigma_1\sqrt{2\pi(\tau-\xi)(1-\rho^2)}} \int_{-d_1[\tau,x_1,B(\xi,u_2)]}^{\infty} e^{-\frac{y_1^2}{2}} dy_1 du_2 d\xi.
\end{aligned}
\tag{A2.6.26}
$$

The above equation can further be simplified to

$$
\begin{aligned}
I_1(\tau, x_1, x_2) = &\int_0^\tau \frac{q_1 e^{x_1} e^{-q_1(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{\frac{[x_2 - u_2 + \kappa_2(\tau-\xi) + \rho\sigma_1\sigma_2(\tau-\xi)]^2}{-2\sigma_2^2(\tau-\xi)}\right\} \\
&\times \frac{1}{\sigma_1\sqrt{2\pi(\tau-\xi)(1-\rho^2)}} \int_{-\infty}^{d_1[\tau,x_1,B(\xi,u_2)]} e^{-\frac{y_1^2}{2}} dy_1 du_2 d\xi,
\end{aligned}
\tag{A2.6.27}
$$

where $d_1(\tau, x, y)$ is defined in equation (A2.6.9).

Equation (A2.6.27) thus simplifies to

$$I_1(\tau, x_1, x_2) = \int_0^\tau \frac{q_1 e^{x_1} e^{-q_1(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{\frac{[x_2 - u_2 + \kappa_2(\tau-\xi) + \rho\sigma_1\sigma_2(\tau-\xi)]^2}{-2\sigma_2^2(\tau-\xi)}\right\}$$
$$\times \mathcal{N}\big(d_1[(\tau-\xi), x_1, B(\xi, u_2)]\big) du_2 d\xi. \tag{A2.6.28}$$

Simplifying the $I_2(\tau, x_1, x_2)$ term:

First we rearrange equation (A2.6.23) to the form

$$I_2(\tau, x_1, x_2) = \int_0^\tau \frac{q_2 e^{-r(\tau-\xi)}}{2\pi\sigma_1\sigma_2(\tau-\xi)\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2 - u_2 + \kappa_2(\tau-\xi))^2}{-2\sigma_2^2(1-\rho^2)(\tau-\xi)} + u_2\right\}$$
$$\times \int_{\ln B(\xi,u_2)}^{\infty} \exp\left\{\frac{(x_1 - u_1 + \kappa_1(\tau-\xi))^2}{-2\sigma_1^2(1-\rho^2)(\tau-\xi)}\right\}$$
$$\times \exp\left\{\frac{\rho(x_2 - u_2 + \kappa_2(\tau-\xi))(x_1 - u_1 + \kappa_1(\tau-\xi))}{\sigma_1\sigma_2(1-\rho^2)(\tau-\xi)}\right\} du_1 du_2 d\xi. \tag{A2.6.29}$$

By proceeding as we did for the A_2 term in equation (A2.6.11) it can be shown that

$$I_2(\tau, x_1, x_2) = \int_0^\tau \frac{q_2 e^{x_2} e^{-q_2(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{\frac{[u_2 - (x_2 + \kappa_2(\tau-\xi) + \sigma_2^2(\tau-\xi))]^2}{-2\sigma_2^2(\tau-\xi)}\right\}$$
$$\times \frac{1}{\sigma_1\sqrt{2\pi(\tau-\xi)(1-\rho^2)}} \int_{d_2[\tau, x_1, B(\xi,u_1)]}^{\infty} e^{-\frac{y_2^2}{2}} dy_2 du_2 d\xi. \tag{A2.6.30}$$

Equation (A2.6.30) thus simplifies to

$$I_2(\tau, x_1, x_2) = \int_0^\tau \frac{q_2 e^{x_2} e^{-q_2(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{\frac{[x_2 - u_2 + \kappa_2(\tau-\xi) + \sigma_2^2(\tau-\xi)]^2}{-2\sigma_2^2(\tau-\xi)}\right\}$$
$$\times \mathcal{N}\big(d_2[(\tau-\xi), x_1, B(\xi, u_2)]\big) du_2 d\xi. \tag{A2.6.31}$$

Simplifying the $I_3(\tau, x_1, x_2)$ term:

By rearranging equation (A2.6.24) we obtain

$$\begin{aligned} I_3(\tau, x_1, x_2) = &\int_0^\tau \frac{rKe^{-r(\tau-\xi)}}{2\pi\sigma_1\sigma_2(\tau-\xi)\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2-u_2+\kappa_2(\tau-\xi))^2}{-2\sigma_2^2(1-\rho^2)(\tau-\xi)}\right\} \\ &\times \int_{\ln B(\xi,u_2)}^{\infty} \exp\left\{\frac{(x_1-u_1+\kappa_1(\tau-\xi))^2}{-2\sigma_1^2(1-\rho^2)(\tau-\xi)}\right\} \\ &\times \exp\left\{\frac{\rho(x_2-u_2+\kappa_2(\tau-\xi))(x_1-u_1+\kappa_1(\tau-\xi))}{\sigma_1\sigma_2(1-\rho^2)(\tau-\xi)}\right\} du_1 du_2 d\xi. \end{aligned} \tag{A2.6.32}$$

By repeating the steps involved in simplifying the A_3 term in equation (A2.6.15) to the above equation we find that

$$\begin{aligned} I_3(\tau, x_1, x_2) = &\int_0^\tau \frac{rKe^{-r(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2-u_2+\kappa_2(\tau-\xi))^2}{-2\sigma_2^2(\tau-\xi)}\right\} \\ &\times \mathcal{N}\big(d_2[(\tau-\xi), x_1, B(\xi, u_2)]\big) du_2 d\xi. \end{aligned} \tag{A2.6.33}$$

Combining equations (A2.6.28), (A2.6.31) and (A2.6.33) together we can represent the early exercise premium component as

$$\begin{aligned} C_P(\tau, x_1, x_2) = &\int_0^\tau \frac{q_1 e^{x_1} e^{-q_1(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{\frac{[x_2-u_2+\kappa_2(\tau-\xi)+\rho\sigma_1\sigma_2(\tau-\xi)]^2}{-2\sigma_2^2(\tau-\xi)}\right\} \\ &\times \mathcal{N}\big(d_1[(\tau-\xi), x_1, B(\xi, u_2)]\big) du_2 d\xi \\ - &\int_0^\tau \frac{q_2 e^{x_2} e^{-q_2(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{\frac{[x_2-u_2+\kappa_2(\tau-\xi)+\sigma_2^2(\tau-\xi)]^2}{-2\sigma_2^2(\tau-\xi)}\right\} \\ &\times \mathcal{N}\big(d_2[(\tau-\xi), x_1, B(\xi, u_2)]\big) du_2 d\xi \\ - &\int_0^\tau \frac{rKe^{-r(\tau-\xi)}}{\sigma_2\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x_2-u_2+\kappa_2(\tau-\xi))^2}{-2\sigma_2^2(\tau-\xi)}\right\} \\ &\times \mathcal{N}\big(d_2[(\tau-\xi), x_1, B(\xi, u_2)]\big) du_2 d\xi, \end{aligned} \tag{A2.6.34}$$

which is the result in Proposition 2.6.1.

Appendix 2.7. Proof of Proposition 2.11.1

In this appendix we want to show that Duhamel's principle holds, that is we want to show that the function in equation (2.11.14) is a solution of the PDE (2.11.9). The PDE (2.11.9) can be written as

$$\frac{\partial C}{\partial \tau} = \mathcal{M}C - rC + f(\tau, x_1, \cdots, x_n), \tag{A2.7.1}$$

where

$$\mathcal{M}C = \sum_{i=1}^{n}(r - q_i - \frac{1}{2}\sigma_i^2)\frac{\partial C}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j\frac{\partial^2 C}{\partial x_i \partial x_j},$$

and $f(\tau, x_1, \cdots, x_n)$ is the inhomogeneous term which is the difference between total dividends receivable and total interest payments on the strike in the case of a call option. We now want to show that (2.11.14) satisfies (A2.7.1). Substituting $C(\tau, x_1, \cdots, x_n) = C_E(\tau, x_1, \cdots, x_n) + C_P(\tau, x_1, \cdots, x_n)$ into (A2.7.1) we proceed as follows

$$\begin{aligned}
&\frac{\partial C}{\partial \tau} - \mathcal{M}C + rC - f(\tau, x_1, \cdots, x_n) = e^{-r\tau}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} c(u_1, \cdots, u_n)\left\{\frac{\partial U}{\partial \tau} - \mathcal{M}U\right\} du_1 \cdots du_n \\
&\quad + \int_0^{\tau} e^{-r(\tau-\xi)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\xi, u_1, \cdots, u_n)\left\{\frac{\partial U}{\partial \tau} - \mathcal{M}U\right\} du_1 \cdots du_n d\xi + rC - rC \\
&\quad + \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\tau, u_1, \cdots, u_n)U(0, x_1, \cdots, x_n)du_1 \cdots du_n - f(\tau, x_1, \cdots, x_n) \\
&\quad = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\tau, u_1, \cdots, u_n)\delta(x_1 - u_1)\cdots\delta(x_n - u_n)du_1 \cdots du_n - f(\tau, x_1, \cdots, x_n) \\
&\qquad = f(\tau, x_1, \cdots, x_n) - f(\tau, x_1, \cdots, x_n) = 0.
\end{aligned} \tag{A2.7.2}$$

This implies that the pricing function in equation (2.11.14) satisfies the PDE (A2.7.1).

Appendix 2.8. Proof of Proposition 2.11.3

First let

$$\alpha(\eta_1, \cdots, \eta_n) = \sum_{j=1}^{n} i\eta_j\kappa_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\rho_{j,k}\sigma_j\sigma_k\eta_j\eta_k,$$

so that equation (2.11.15) can be rewritten as

$$\frac{\partial \hat{U}}{\partial \tau}(\tau, \eta_1, \cdots, \eta_n) + \alpha(\eta_1, \cdots, \eta_n)\hat{U}(\tau, \eta_1, \cdots, \eta_n) = 0.$$

Taking $e^{\int_0^\tau \alpha(\eta_1, \cdots, \eta_n) ds}$ as the integrating factor and rearranging yields,

$$\hat{U}(\tau, \eta_1, \cdots, \eta_n) = \hat{U}(0, \eta_1, \cdots, \eta_n) e^{-\alpha(\eta_1, \cdots, \eta_n)\tau},$$

which is the result presented in Proposition 2.11.3.

CHAPTER 3

American Option Pricing Under Two Stochastic Volatility Processes

3.1. Introduction

As reviewed in Chapter 1, volatility of asset returns changes with time and the changes are unpredictable. Volatility has the tendency of reverting to a long-run average. This has led to a great deal of research on the pricing of European options under a single stochastic volatility process such as the contributions of Scott (1987), Wiggins (1987), Hull and White (1987), Stein and Stein (1991) and Heston (1993).

Of all these models, that of Heston (1993) leads to a semi closed-form solution for the European option price involving the evaluation of complex integrals. However, Cont and da Fonseca (2002) use principal component analysis to show that the smile features on implied volatility surfaces can be well captured by at least two stochastic factors. Furthermore Bergomi (2004) reveals that there is a structural limitation that prevents single-factor stochastic volatility models from generating consistent option prices. Given such evidence, da Fonseca et al. (2005) and (2008) consider the pricing of European options when the underlying asset evolves under the influence of multifactor stochastic variance processes. The multifactor stochastic variance processes are taken to be instantaneous variances processes effective over different maturity periods. Christoffersen, Heston and Jacobs (2009) have also used principal componentwise analysis to investigate the need to model Black-Scholes implied variances with multifactor stochastic volatility models and note that such models capture a lot of the variation in empirical European style options data.

Whilst a great deal of effort has been dedicated to European options under stochastic volatility, little research has been devoted to American option pricing. Ikonen and Toivanen (2004) use operator splitting methods to solve numerically the PDE for the American put option under stochastic volatility. Tzavalis and Wang (2003) derive the integral representation of the American put option and approximate the early exercise surface with a log-linear function. The unknown functions of the log-linear representation are then approximated by Chebyshev polynomials. Adolfsson et al. (2009) use the same log-linear approximation but however, instead of using Chebyshev polynomials, the integral equation is explicitly solved resulting in a Heston (1993) type characteristic function for the early exercise premium component.

Motivated by the multifactor volatility feature, we seek to extend the American option pricing model of Adolfsson et al. (2009) to the multifactor stochastic volatility case. As a starting point we will assume that the underlying asset is driven by two independent stochastic variance processes of the Heston (1993) type. These Whilst da Fonseca et al. (2005) and (2008) specify the two stochastic variance processes to be effective during different periods of the maturity domain, in this work we model the variance processes as independent risk factors influencing the dynamics of the underlying asset over the entire maturity domain.

By first applying the Girsanov theorem for Wiener processes to the driving stochastic processes, we derive the corresponding pricing PDE using Ito's Lemma and some hedging arguments. The PDE is solved subject to initial and boundary conditions that specify the type of option under consideration. As is well known, the underlying asset of the American call option is bounded above by the early exercise boundary and below by zero. As in Chapter 2 of this thesis, we convert the upper bound of the underlying asset to an unbounded domain by using the approach of Jamshidian (1992). The three stochastic processes; one for the underlying asset and the two variance processes can also be used to derive the corresponding PDE for their joint transition probability density function which satisfies a backward Kolmogorov PDE. Coupled with this and the unbounded PDE for the option price, we derive the general solution for the American option price by using Duhamel's principle. The only unknown term in the general solution is the transition density function which is the solution of the backward Kolmogorov PDE for the three driving processes.

In solving the Kolmogorov PDE, we first reduce it to a characteristic PDE by using a combination of Fourier and Laplace transforms. The resulting equation is then solved by the method of characteristics. Once the solution is found, we revert back to the original variables by applying the Fourier and Laplace inversion theorems. With the transition density in place, we can readily obtain the full integral representation of the American option price. As implied by Duhamel's principle, the American option price is the sum of two components namely the European and early exercise premium components. The European option component can be readily reduced to the Heston (1993) form by using similar techniques to those in Adolfsson et al. (2009). In dealing with the early exercise premium component, we extend the idea of Tzavalis and Wang (2003) and approximate the early exercise boundary as a bivariate log-linear function. This approximation allows us to reduce the integral dimensions of the early exercise premium by simplifying the integrals with respect to the two variance processes. The reduction of the dimensionality has the net effect of enhancing computational efficiency by reducing the computational time of the early exercise premium component.

This chapter is organized as follows, we present the problem statement and the corresponding general solution of the American call option price in Section 3.2. We apply a Fourier transform to the underlying asset variable in the PDE for the density function in Section 3.3 followed by application of a bivariate Laplace transform to the variance variables in Section 3.4. Application of the Laplace transform yields the PDE which we solve by the method of characteristics, details of which are given in Section 3.5. Once this PDE is solved the next step involves reverting back to the original underlying asset and variance variables. This is accomplished by applying Laplace and Fourier inversion theorems as detailed in Sections 3.6 and 3.7 respectively. The resulting function is the explicit representation of the transition density function. Section 3.8 nicely represents the integral form of the American call option price. Having found a representation of the American option price we then present details of how to implement the pricing relationship in Section 3.10. Numerical results are finally presented in Section 3.11 followed by concluding remarks in Section 3.12. Lengthy derivations have been relegated to appendices.

3.2. Problem Statement

In this chapter we consider the evaluation of the American call option written on an underlying asset whose dynamics evolve under the influence of two stochastic variance processes of the Heston (1993) type. We represent the value of this option at the current time, t as $V(t, S, v_1, v_2)$ where S is the price of the underlying asset paying a continuously compounded dividend yield at a rate q in a market offering a risk-free rate of interest denoted here as r, and v_1 and v_2 are the two variance processes driving S. Under the real world probability measure, $\mathbb{P}$, the underlying asset dynamics are governed by the stochastic differential equation (SDE) system

$$dS = \mu S dt + \sqrt{v_1} S dZ_1 + \sqrt{v_2} S dZ_2, \tag{3.2.1}$$

$$dv_1 = \kappa_1(\theta_1 - v_1)dt + \sigma_1\sqrt{v_1} dZ_3, \tag{3.2.2}$$

$$dv_2 = \kappa_2(\theta_2 - v_2)dt + \sigma_2\sqrt{v_2} dZ_4, \tag{3.2.3}$$

where μ is the instantaneous return per unit time of the underlying asset, θ_1 and θ_2 are the long-run means of v_1 and v_2 respectively, κ_1 and κ_2 are the speeds of mean-reversion, while σ_1 and σ_2 are the instantaneous volatilities of v_1 and v_2 per unit time respectively. The processes, Z_1, Z_2, Z_3 and Z_4 are correlated Wiener processes with a special correlation structure such that $\mathbb{E}^{\mathbb{P}}(dZ_1 dZ_3) = \rho_{13}dt$, $\mathbb{E}^{\mathbb{P}}(dZ_2 dZ_4) = \rho_{24}dt$ and all other correlations are zero.

In the next few paragraphs we will apply Girsanov's Theorem for multiple Wiener processes. As this theorem is usually stated in terms of independent Wiener processes, it is convenient to transform the Wiener processes in the SDE system (3.2.1) - (3.2.3) to a corresponding system which is expressed in terms of independent Wiener processes whose increments we denote as dW_j for $j = 1, \cdots, 4$. This transformation is accomplished by performing the Cholesky decomposition such that

$$\begin{bmatrix} dZ_1 \\ dZ_2 \\ dZ_3 \\ dZ_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \rho_{13} & 0 & \sqrt{1-\rho_{13}^2} & 0 \\ 0 & \rho_{24} & 0 & \sqrt{1-\rho_{24}^2} \end{bmatrix} \begin{bmatrix} dW_1 \\ dW_2 \\ dW_3 \\ dW_4 \end{bmatrix}. \tag{3.2.4}$$

As highlighted in the correlation matrix above, we assume that correlation exists between the pairs, (Z_1, Z_3) and (Z_2, Z_4) such that all other correlation terms except ρ_{13} and ρ_{24} are zero. These assumptions about the correlation structure allow us to apply transform methods as we avoid the product term $\sqrt{v_1}\sqrt{v_2}$ which is impossible to handle using the transform based methods that we propose. By incorporating the transformation (3.2.4) into equations (3.2.1)-(3.2.3) we obtain the system of SDEs

$$dS = \mu S dt + \sqrt{v_1} S dW_1 + \sqrt{v_2} S dW_2, \tag{3.2.5}$$

$$dv_1 = \kappa_1(\theta_1 - v_1)dt + \rho_{13}\sigma_1\sqrt{v_1}dW_1 + \sqrt{1-\rho_{13}^2}\sigma_1\sqrt{v_1}dW_3, \tag{3.2.6}$$

$$dv_2 = \kappa_2(\theta_2 - v_2)dt + \rho_{24}\sigma_2\sqrt{v_2}dW_2 + \sqrt{1-\rho_{24}^2}\sigma_2\sqrt{v_2}dW_4. \tag{3.2.7}$$

It has been documented in Feller (1951) that for equations like (3.2.6) and (3.2.7) to be positive processes, the following conditions need to be satisfied:

$$2\kappa_1\theta_1 \geq \sigma_1^2 \quad \text{and} \quad 2\kappa_2\theta_2 \geq \sigma_2^2. \tag{3.2.8}$$

Cheang, Chiarella and Ziogas (2009) also show that in addition to the two conditions in (3.2.8) the following conditions need to be satisfied for the two variances to be finite:

$$-1 < \rho_{13} < \min\left(\frac{\kappa_1}{\sigma_1}, 1\right) \quad \text{and} \quad -1 < \rho_{24} < \min\left(\frac{\kappa_2}{\sigma_2}, 1\right). \tag{3.2.9}$$

By following similar arguments to those in Cheang et al. (2009), it can be shown that the two conditions in equation (3.2.9) together with (3.2.8) also ensure that the solution of the underlying asset pricing process takes the form

$$S_t = S_0 \exp\left\{\mu t - \frac{1}{2}\int_0^t v_1 du - \frac{1}{2}\int_0^t v_2 du + \int_0^t \sqrt{v_1} dW_1 + \int_0^t \sqrt{v_2} dW_2\right\}, \tag{3.2.10}$$

where

$$\exp\left\{-\frac{1}{2}\int_0^t v_1 du - \frac{1}{2}\int_0^t v_2 du + \int_0^t \sqrt{v_1} dW_1 + \int_0^t \sqrt{v_2} dW_2\right\}, \tag{3.2.11}$$

is a martingale under the real world probability measure, $\mathbb{P}$.

The system (3.2.5)-(3.2.7) contains four Wiener processes but only one traded asset S as the two variance processes are non-tradable. This single asset is insufficient to hedge away these four risk factors when combined in a portfolio with an option dependent on the underlying asset, S. This situation leads to market incompleteness. In order to hedge away these risk sources, the market needs to be completed in some way. The process of completing the market is usually done by placing a sufficient number of options of different maturities in the hedging portfolio[1].

The hedging technique usually results in the triplet of underlying processes , (S, v_1, v_2) having different drift coefficients from those specified in the system, (3.2.5)-(3.2.7) thus resulting in different processes. We would however prefer to keep the original underlying asset price dynamics, a process achieved by switching from the real world

[1]After applying these hedging arguments, it turns out that the resulting option pricing PDE is a function of two market prices of risk corresponding to the number of non-traded factors under consideration.

probability measure, $\mathbb{P}$ to the risk-neutral probability measure, $\mathbb{Q}$. The change of measure is accomplished by making use of the Girsanov's Theorem for Wiener processes. Girsanov's Theorem[2] uses the so-called Radon-Nikodym derivative, (R_N) which takes the form (see for instance Cheang et al. (2009))

$$R_N = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{ -\frac{1}{2}\int_0^t \mathbf{\Lambda}_u^T \mathbf{\Sigma}^{-1}\mathbf{\Lambda}_u du - \int_0^t (\mathbf{\Sigma}^{-1}\mathbf{\Lambda}_u)^T d\mathbf{W} \right\}, \tag{3.2.12}$$

where $\mathbf{\Sigma}$ is the correlation matrix in (3.2.4) and $\mathbf{\Lambda}_t$ is the vector of market prices of risk associated with the vector of Wiener processes, $\mathbf{W}$. Market prices of risk associated with shocks on traded assets can be diversified away, however, for non-traded assets investors will always require a positive risk premium to compensate them for bearing such risk. Once the market prices of risk vector is specified, then by Girsanov's Theorem for Wiener processes there exist

$$d\tilde{W}_j = \lambda_j(t)dt + dW_j, \tag{3.2.13}$$

where $\tilde{W}_j$, for $j = 1, \cdots, 4$ are Wiener processes under the risk neutral measure $\mathbb{Q}$. From the vector, $\mathbf{\Lambda}_t$, we denote the constituent parameters as $\lambda_1(t)$ and $\lambda_2(t)$ to represent the market prices of risk associated with the shocks, dW_1 and dW_2, on the underlying asset price dynamics, and $\lambda_3(t)$ and $\lambda_4(t)$ to be the market prices of risk associated with bearing the dW_3 and dW_4 risks on the non-traded variance factors, v_1 and v_2 respectively. As highlighted above, $\lambda_3(t)$ and $\lambda_4(t)$ cannot be diversified away as variance cannot be traded. Application of Girsanov's Theorem to the system (3.2.5)-(3.2.7) yields

$$dS = (r-q)Sdt + \sqrt{v_1}Sd\tilde{W}_1 + \sqrt{v_2}Sd\tilde{W}_2, \tag{3.2.14}$$

$$dv_1 = \kappa_1(\theta_1 - v_1)dt - \lambda_3(t)\sqrt{1-\rho_{13}^2}\sigma_1\sqrt{v_1}dt + \rho_{13}\sigma_1\sqrt{v_1}d\tilde{W}_1 + \sqrt{1-\rho_{13}^2}\sigma_1\sqrt{v_1}d\tilde{W}_3,$$

$$dv_2 = \kappa_2(\theta_2 - v_2)dt - \lambda_4(t)\sqrt{1-\rho_{24}^2}\sigma_2\sqrt{v_2}dt + \rho_{24}\sigma_2\sqrt{v_2}d\tilde{W}_2 + \sqrt{1-\rho_{24}^2}\sigma_2\sqrt{v_2}d\tilde{W}_4,$$

[2]For a detailed discussion see Harrison (1990).

where r is the risk-free interest rate and q is the continuously compounded dividend yield on the underlying asset, S. The key assumption we make on $\lambda_3(t)$ and $\lambda_4(t)$ is that both quantities are strictly positive to guarantee an investor a positive risk premium for holding the non-traded variance factors. In determining the market prices of the two variance risks, we use the same reasoning as in Heston (1993) with a slight modification such that

$$\lambda_3(t) = \frac{\lambda_1\sqrt{v_1}}{\sigma_1\sqrt{1-\rho_{13}^2}}, \quad \text{and} \quad \lambda_4(t) = \frac{\lambda_2\sqrt{v_2}}{\sigma_2\sqrt{1-\rho_{24}^2}}, \tag{3.2.15}$$

where λ_1 and λ_2 are constants. By substituting these into the system (3.2.14) we obtain

$$dS = (r-q)Sdt + \sqrt{v_1}Sd\tilde{W}_1 + \sqrt{v_2}Sd\tilde{W}_2, \tag{3.2.16}$$

$$dv_1 = [\kappa_1\theta_1 - (\kappa_1+\lambda_1)v_1]dt + \rho_{13}\sigma_1\sqrt{v_1}d\tilde{W}_1 + \sqrt{1-\rho_{13}^2}\sigma_1\sqrt{v_1}d\tilde{W}_3, \tag{3.2.17}$$

$$dv_2 = [\kappa_2\theta_2 - (\kappa_2+\lambda_2)v_2]dt + \rho_{24}\sigma_2\sqrt{v_2}d\tilde{W}_2 + \sqrt{1-\rho_{24}^2}\sigma_2\sqrt{v_2}d\tilde{W}_4. \tag{3.2.18}$$

The conditions in equations (3.2.8) and (3.2.9) also ensure that the explicit solution of the asset price process, (3.2.16) can be represented as

$$S_t = S_0\exp\left\{(r-q)t - \frac{1}{2}\int_0^t v_1du - \frac{1}{2}\int_0^t v_2du + \int_0^t \sqrt{v_1}d\tilde{W}_1 + \int_0^t \sqrt{v_2}d\tilde{W}_2\right\}, \tag{3.2.19}$$

where

$$\exp\left\{-\frac{1}{2}\int_0^t v_1du - \frac{1}{2}\int_0^t v_2du + \int_0^t \sqrt{v_1}d\tilde{W}_1 + \int_0^t \sqrt{v_2}d\tilde{W}_2\right\}, \tag{3.2.20}$$

is a positive martingale under the risk-neutral probability measure, $\mathbb{Q}$. Now with the system of equations (3.2.16)-(3.2.18), the next step involves the derivation of the corresponding American call option pricing PDE for the option written on the underlying asset, S. The pricing PDE can be shown to be[3]

$$\frac{\partial V}{\partial \tau}(\tau, S, v_1, v_2) = \mathcal{L}V(\tau, S, v_1, v_2) - rV, \tag{3.2.21}$$

where

$$\begin{aligned}\mathcal{L} = {} & (r-q)S\frac{\partial}{\partial S} + [\kappa_1(\theta_1 - v_1) - \lambda_1 v_1]\frac{\partial}{\partial v_1} + [\kappa_2(\theta_2 - v_2) - \lambda_2 v_2]\frac{\partial}{\partial v_2} \\ & + \frac{1}{2}v_1 S^2\frac{\partial^2}{\partial S^2} + \frac{1}{2}\sigma_1^2 v_1\frac{\partial^2}{\partial v_1^2} + \frac{1}{2}v_2 S^2\frac{\partial^2}{\partial S^2} + \frac{1}{2}\sigma_2^2 v_2\frac{\partial^2}{\partial v_2^2} \\ & + \rho_{13}\sigma_1 v_1 S\frac{\partial^2}{\partial S\partial v_1} + \rho_{14}\sigma_2 v_2 S\frac{\partial^2}{\partial S\partial v_2}.\end{aligned} \tag{3.2.22}$$

Here, $\mathcal{L}$ is the Dynkin operator associated with the SDE system (3.2.16)-(3.2.18). The state variables are defined in the domains $0 < v_1, v_2 < \infty$ and $0 \leq S < b(\tau, v_1, v_2)$ where $S = b(\tau, v_1, v_2)$, is the early exercise boundary of the American call option at time-to-maturity, τ when the instantaneous variances are v_1 and v_2 respectively. The PDE (3.2.21) is to be solved subject to the initial and boundary conditions

$$V(0, S, v_1, v_2) = (S-K)^+, \quad 0 < S < \infty, \tag{3.2.23}$$

$$V(\tau, 0, v_1, v_2) = 0, \quad \tau \geq 0, \tag{3.2.24}$$

$$V(\tau, b(\tau, v_1, v_2), v_1, v_2) = b(\tau, v_1, v_2) - K, \quad \tau \geq 0, \tag{3.2.25}$$

$$\lim_{S \to b(\tau, v_1, v_2)} \frac{\partial V}{\partial S}(\tau, S, v_1, v_2) = 1 \quad \tau \geq 0. \tag{3.2.26}$$

Condition (3.2.23) is the payoff of the option contract if it is held to maturity, while equation (3.2.24) is the absorbing state condition which ensures that the option ceases to exist once the underlying asset price hits zero. Equation (3.2.25) is the value matching condition which guarantees continuity of the option value function at the early exercise boundary, $b(\tau, v_1, v_2)$. Equation (3.2.26) is the smooth pasting condition which together with the value matching condition are imposed to eliminate arbitrage opportunities. Boundary conditions at $v_1 = 0$ and $v_2 = 0$ are found by extrapolation

[3]The definition of τ has been provided in Chapter 2.

techniques when numerically implementing the resulting American call option pricing equation.

Also associated with the system of stochastic differential equations in (3.2.16)-(3.2.18) is the transition density function which we denote here as $G(\tau, S, v_1, v_2; S_0, v_{1,0}, v_{2,0})$. The transition density function represents the transition probability of passage from S, v_1, v_2 at time-to-maturity τ to $S_0, v_{1,0}, v_{2,0}$ at maturity. It is well known that the transition density function satisfies the backward Kolmogorov PDE associated with the stochastic differential equations in the system (3.2.16)-(3.2.18) (see for example Chiarella (2010)). The Kolmogorov equation in the current situation can be shown to be of the form

$$\frac{\partial G}{\partial \tau} = \mathcal{L}G, \tag{3.2.27}$$

where $0 \leq S < \infty$ and $0 \leq v_1, v_2 < \infty$. Equation (3.2.27) is solved subject to the initial condition

$$G(0, S, v_1, v_2; S_0, v_{1,0}, v_{2,0}) = \delta(S - S_0)\delta(v_1 - v_{1,0})\delta(v_2 - v_{2,0}), \tag{3.2.28}$$

where $\delta(\cdot)$ is the Dirac delta function.

As noted in the PDE (3.2.21), the underlying asset domain is bounded above by the early exercise boundary, $b(\tau, v_1, v_2)$. Jamshidian (1992) shows that one can consider an unbounded domain for the underlying asset by introducing an indicator function and transforming the PDE to

$$\frac{\partial V}{\partial \tau}(\tau, S, v_1, v_2) = \mathcal{L}V(\tau, S, v_1, v_2) - rV + \mathbb{1}_{S \geq b(\tau, v_1, v_2)}(qS - rK). \tag{3.2.29}$$

Here, $0 \leq S < \infty$, $0 < v_1, v_2 < \infty$ and $\mathbb{1}_{S \geq b(\tau, v_1, v_2)}$ is an indicator function which is equal to one if $S \geq b(\tau, v_1, v_2)$ and zero otherwise. Now the PDE (3.2.29) is defined on an unbounded domain for the underlying asset.

As an initial step to solving the PDE (3.2.29), we switch to log asset variables by letting $S = e^x$ and setting

$$C(\tau, x, v_1, v_2) \equiv V(\tau, e^x, v_1, v_2), \tag{3.2.30}$$

$$U(\tau, x, v_1, v_2; x_0, v_{1,0}, v_{2,0}) \equiv G(\tau, e^x, v_1, v_2; e^{x_0}, v_{1,0}, v_{2,0}), \tag{3.2.31}$$

to obtain

$$\frac{\partial C}{\partial \tau} = \mathcal{M}C - rC + \mathbb{1}_{x \geq \ln b(\tau, v_1, v_2)}(qe^x - rK), \tag{3.2.32}$$

where

$$\begin{aligned}\mathcal{M} = &\left(r - q - \frac{1}{2}v_1 - \frac{1}{2}v_2\right)\frac{\partial}{\partial x} + \Phi_1\frac{\partial}{\partial v_1} - \beta_1 v_1\frac{\partial}{\partial v_1} + \Phi_2\frac{\partial}{\partial v_2} - \beta_2 v_2\frac{\partial}{\partial v_2} + \frac{1}{2}v_1\frac{\partial^2}{\partial x^2} \\ &+ \frac{1}{2}v_2\frac{\partial^2}{\partial x^2} + \rho_{13}\sigma_1 v_1\frac{\partial^2}{\partial x \partial v_1} + \rho_{14}\sigma_2 v_2\frac{\partial^2}{\partial x \partial v_2} + \frac{1}{2}\sigma_1^2 v_1\frac{\partial^2}{\partial v_1^2} + \frac{1}{2}\sigma_2^2 v_2\frac{\partial^2}{\partial v_2^2},\end{aligned} \tag{3.2.33}$$

$$\Phi_1 = \kappa_1\theta_1, \quad \Phi_2 = \kappa_2\theta_2, \quad \beta_1 = \kappa_1 + \lambda_1 \quad \text{and} \quad \beta_2 = \kappa_2 + \lambda_2. \tag{3.2.34}$$

Equation (3.2.32) is solved subject to the initial condition

$$C(0, x, v_1, v_2) = (e^x - K)^+, \quad -\infty < x < \infty. \tag{3.2.35}$$

Likewise, the transition density PDE (3.2.27) is transformed to

$$\frac{\partial U}{\partial \tau} = \mathcal{M}U. \tag{3.2.36}$$

Equation (3.2.36) is to be solved subject to the initial condition

$$U(0, x, v_1, v_2; x_0, v_{1,0}, v_{2,0}) = \delta(x - x_0)\delta(v_1 - v_{1,0})\delta(v_2 - v_{2,0}). \tag{3.2.37}$$

The inhomogeneous PDE (3.2.32) is in a form whose general solution can be represented by use of Duhamel's principle. We present the general solution of (3.2.32) in the proposition below.

PROPOSITION 3.2.1. *The solution of the American call option pricing PDE* (3.2.32) *can be represented as*

$$C(\tau,x,v_1,v_2)=C_E(\tau,x,v_1,v_2)+C_P(\tau,x,v_1,v_2), \tag{3.2.38}$$

where

$$C_E(\tau,x,v_1,v_2)=e^{-r\tau}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (e^u-K)^+U(\tau,x,v_1,v_2;u,w_1,w_2)dudw_1dw_2, \tag{3.2.39}$$

and

$$\begin{aligned}C_P(\tau,x,v_1,v_2)=\int_0^\tau e^{-r(\tau-\xi)}\int_0^\infty\int_0^\infty\int_{\ln b(\xi,w_1,w_2)}^\infty (qe^u-rK)\\ \times U(\tau-\xi,x,v_1,v_2;u,w_1,w_2)dudw_1dw_2d\xi.\end{aligned} \tag{3.2.40}$$

Proof: Refer to Appendix 3.1. □

The first component of equation (3.2.38) is the European option component whilst the second component is the early exercise premium. For us to operationalise the representation of equation (3.2.38), we need an explicit form of the density function, $U(\tau,x,v_1,v_2)$ which is the solution of the PDE in equation (3.2.36). In order to solve equation (3.2.36), we first reduce it to a corresponding system of characteristic PDEs which can then be readily solved using a vast array of methods for tackling such problems. In this chapter, we will apply a combination of Fourier and Laplace transforms to this PDE resulting in a system of characteristic PDEs. We will apply Fourier transforms to the underlying log asset variable as its domain matches that of the transform. A bivariate Laplace transform will then be applied to the stochastic variance variables.

3.3. Applying Fourier Transforms

We first apply the Fourier transform to the log asset price variable in the PDE (3.2.36).

PROPOSITION 3.3.1. *The Fourier transform,* $\hat{U}(\tau, \eta, v_1, v_2)$ *of the function* $U(\tau, x, v_1, v_2)$ *which is the solution of the homogeneous PDE* (3.2.36) *satisfies the PDE*

$$\begin{aligned}\frac{\partial \hat{U}}{\partial \tau} = &-i\eta(r-q)\hat{U} + \frac{1}{2}\Lambda(\eta)v_1\hat{U} + \frac{1}{2}\Lambda(\eta)v_2\hat{U} + \Phi_1\frac{\partial \hat{U}}{\partial v_1} + \Phi_2\frac{\partial \hat{U}}{\partial v_2}\\ &- \Theta_1 v_1\frac{\partial \hat{U}}{\partial v_1} - \Theta_2 v_2\frac{\partial \hat{U}}{\partial v_2} + \frac{1}{2}\sigma_1^2 v_1\frac{\partial^2 \hat{U}}{\partial v_1^2} + \frac{1}{2}\sigma_2^2 v_2\frac{\partial^2 \hat{U}}{\partial v_2^2},\end{aligned} \tag{3.3.1}$$

where

$$\Theta_1 = \Theta_1(\eta) \equiv \beta_1 + i\eta\rho_{13}\sigma_1, \quad \Theta_2 = \Theta_2(\eta) \equiv \beta_2 + i\eta\rho_{24}\sigma_2, \quad \textit{and} \quad \Lambda(\eta) = i\eta - \eta^2. \tag{3.3.2}$$

Equation (3.3.1) *is to be solved subject to the initial condition*

$$\hat{U}(0, \eta, v_1, v_2) = e^{i\eta x_0}\delta(v_1 - v_{1,0})\delta(v_2 - v_{2,0}). \tag{3.3.3}$$

Proof: Refer to Appendix 3.2. □

3.4. Applying Laplace Transforms

We have applied the Fourier transform to the log asset price variable. To successfully solve the resulting PDE (3.3.1), we apply a bivariate Laplace transform to the variance variables and this is accomplished by the proposition below.

Proposition 3.4.1. *By applying Definition 1.3.3 to the PDE* (3.3.1) *the Laplace transform,* $\tilde{U}(\tau, \eta, s_1, s_2)$ *is found to satisfy the first order PDE*

$$\frac{\partial \tilde{U}}{\partial \tau} + \left\{\frac{1}{2}\sigma_1^2 s_1^2 - \Theta_1 s_1 + \frac{1}{2}\Lambda(\eta)\right\}\frac{\partial \tilde{U}}{\partial s_1} + \left\{\frac{1}{2}\sigma_2^2 s_2^2 - \Theta_2 s_2 + \frac{1}{2}\Lambda(\eta)\right\}\frac{\partial \tilde{U}}{\partial s_2}$$
$$= \left\{(\Phi_1 - \sigma_1^2)s_1 + (\Phi_2 - \sigma_2^2)s_2 - i\eta(r-q) + \Theta_1 + \Theta_2\right\}\tilde{U} + f_1(\tau, s_2) + f_2(\tau, s_1). \tag{3.4.1}$$

Equation (3.4.1) *is to be solved subject to the initial condition*

$$\tilde{U}(0, \eta, s_1, s_2) = e^{i\eta x_0 - s_1 v_{1,0} - s_2 v_{2,0}}. \tag{3.4.2}$$

The two functions, f_1 *and* f_2 *are given by*

$$f_1(\tau, s_2) = \left(\frac{1}{2}\sigma_1^2 - \Phi_1\right)\tilde{U}(\tau, \eta, 0, s_2), \quad \textit{and} \quad f_2(\tau, s_1) = \left(\frac{1}{2}\sigma_2^2 - \Phi_2\right)\tilde{U}(\tau, \eta, s_1, 0)$$

and are determined such that

$$\lim_{s_1 \to \infty} \tilde{U}(\tau, \eta, s_1, s_2) = 0, \quad \textit{and} \quad \lim_{s_2 \to \infty} \tilde{U}(\tau, \eta, s_1, s_2) = 0, \tag{3.4.3}$$

respectively.

Proof: Refer to Appendix 3.3. □

3.5. Solution of the Characteristic Equation

Equation (3.4.1) is a first order PDE known as a characteristic equation. We are going to solve this equation using the method of characteristics. Similar techniques have been successfully used in Feller (1951) and Adolfsson et al. (2009) to solve PDEs like equation (3.4.1). Cheang et al. (2009) use similar techniques when solving the American call option where the underlying asset is being driven by both stochastic volatility and jumps. We present the solution of this PDE in the proposition below.

Proposition 3.5.1. *The solution of equation* (3.4.1) *subject to the initial and boundary conditions in equations* (3.4.2) *and* (3.4.3) *can be represented as*

$$\begin{aligned}
\tilde{U}(\tau,\eta,s_1,s_2) &= \left(\frac{2\Omega_1}{(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{\Omega_1\tau}-1)+2\Omega_1}\right)^{2-\frac{2\Phi_1}{\sigma_1^2}}\left(\frac{2\Omega_2}{(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{\Omega_2\tau}-1)+2\Omega_2}\right)^{2-\frac{2\Phi_2}{\sigma_2^2}}\\
&\times\exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_{1,0}-\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)v_{2,0}+i\eta x_0\right\}\\
&\times\exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right]\tau\right\}\\
&\times\exp\left\{\frac{-2\Omega_1 v_{1,0}(\sigma_1^2 s_1-\Theta_1+\Omega_1)e^{\Omega_1\tau}}{\sigma_1^2[(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{\Omega_1\tau}-1)+2\Omega_1]}\right\}\exp\left\{\frac{-2\Omega_2 v_{2,0}(\sigma_2^2 s_2-\Theta_2+\Omega_2)e^{\Omega_2\tau}}{\sigma_2^2[(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{\Omega_2\tau}-1)+2\Omega_2]}\right\}\\
&\times\left[\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1;\frac{2\Omega_1 v_{1,0}e^{\Omega_1\tau}}{\sigma_1^2(e^{\Omega_1\tau}-1)}\times\frac{2\Omega_1}{(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{\Omega_1\tau}-1)+2\Omega_1}\right)\right.\\
&\left.+\Gamma\left(\frac{2\Phi_2}{\sigma_2^2}-1;\frac{2\Omega_2 v_{2,0}e^{\Omega_2\tau}}{\sigma_2^2(e^{\Omega_2\tau}-1)}\times\frac{2\Omega_2}{(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{\Omega_2\tau}-1)+2\Omega_2}\right)-1\right], \qquad (3.5.1)
\end{aligned}$$

where

$$\Omega_1=\sqrt{\Theta_1^2-\Lambda(\eta)\sigma_1^2} \quad \textit{and} \quad \Omega_2=\sqrt{\Theta_2^2-\Lambda(\eta)\sigma_2^2}. \qquad (3.5.2)$$

The function $\Gamma(n;z)$ *is the incomplete gamma function defined as*

$$\Gamma(n;z)=\frac{1}{\Gamma(n)}\int_0^z e^{-\xi}\xi^{n-1}d\xi, \qquad (3.5.3)$$

and

$$\Gamma(n)=\int_0^\infty e^{-\xi}\xi^{n-1}d\xi. \qquad (3.5.4)$$

Proof: Refer to Appendix 3.4. □

3.6. Inverting the Laplace Transform

Now that we have solved equation (3.4.1), the next step is to recover the original function, $U(\tau,x,v_1,v_2)$ which is expressed in terms of the original state variables. This process is accomplished by applying the Laplace and Fourier inversion theorems respectively. In Proposition 3.6.1 below, we will first present the inverse Laplace transform of the function given in Proposition 3.5.1.

Proposition 3.6.1. *The inverse Laplace transform of* $\tilde{U}(\tau,\eta,s_1,s_2)$ *in equation* (3.5.1) *is*

$$\begin{aligned}\hat{U}(\tau,\eta,v_1,v_2) &= \exp\left\{\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)(v_1-v_{1,0}+\Phi_1\tau)+\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)(v_2-v_{2,0}+\Phi_2\tau)\right\}\\ &\times\exp\left\{-\left(\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\right)\left(v_{1,0}e^{\Omega_1\tau}+v_1\right)-\left(\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}\right)\left(v_{2,0}e^{\Omega_2\tau}+v_2\right)\right\}\\ &\times e^{i\eta x_0-i\eta(r-q)\tau}\frac{2\Omega_1e^{\Omega_1\tau}}{\sigma_1^2(e^{\Omega_1\tau}-1)}\frac{2\Omega_2e^{\Omega_2\tau}}{\sigma_2^2(e^{\Omega_2\tau}-1)}\left(\frac{v_{1,0}e^{\Omega_1\tau}}{v_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\left(\frac{v_{2,0}e^{\Omega_2\tau}}{v_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}\\ &\times I_{\frac{2\Phi_1}{\sigma_1^2}-1}\left(\frac{4\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\left(v_1v_{1,0}e^{\Omega_1\tau}\right)^{\frac{1}{2}}\right)I_{\frac{2\Phi_2}{\sigma_2^2}-1}\left(\frac{4\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}\left(v_2v_{2,0}e^{\Omega_2\tau}\right)^{\frac{1}{2}}\right),\end{aligned} \tag{3.6.1}$$

where $I_k(z)$ *is the modified Bessel function of the first kind defined as*

$$I_k(z)=\sum_{n=0}^{\infty}\frac{(\frac{z}{2})^{2n+k}}{\Gamma(n+k+1)n!}. \tag{3.6.2}$$

Proof: Refer to Appendix 3.5. □

3.7. Inverting the Fourier Transform

The next task is to find the inverse Fourier transform of equation (3.6.1), this is accomplished by applying Definition 1.3.2 to Proposition 3.6.1 and we present the result in the proposition below.

Proposition 3.7.1. *The inverse Fourier transform,* $U(\tau,x,v_1,v_2;x_0,v_{1,0},v_{2,0})$ *of equation* (3.6.1) *can be expressed as*

$$U(\tau,x,v_1,v_2;x_0,v_{1,0},v_{2,0})=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\eta x_0}H(\tau,x,v_1,v_2;-\eta,v_{1,0},v_{2,0})d\eta \tag{3.7.1}$$

where

$$
\begin{aligned}
H(\tau, x, v_1, v_2; \eta, v_{1,0}, v_{2,0}) &= \exp\left\{ \left(\frac{\Theta_1 - \Omega_1}{\sigma_1^2}\right)(v_1 - v_{1,0} + \Phi_1\tau) + \left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2}\right)(v_2 - v_{2,0} + \Phi_2\tau) \right\} \\
&\times \exp\left\{ -\left(\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau} - 1)}\right)\left(v_{1,0}e^{\Omega_1\tau} + v_1\right) - \left(\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau} - 1)}\right)\left(v_{2,0}e^{\Omega_2\tau} + v_2\right) \right\} \\
&\times e^{i\eta x + i\eta(r-q)\tau} \frac{2\Omega_1 e^{\Omega_1\tau}}{\sigma_1^2(e^{\Omega_1\tau} - 1)} \frac{2\Omega_2 e^{\Omega_2\tau}}{\sigma_2^2(e^{\Omega_2\tau} - 1)} \left(\frac{v_{1,0}e^{\Omega_1\tau}}{v_1}\right)^{\frac{\Phi_1}{\sigma_1^2} - \frac{1}{2}} \left(\frac{v_{2,0}e^{\Omega_2\tau}}{v_2}\right)^{\frac{\Phi_2}{\sigma_2^2} - \frac{1}{2}} \\
&\times I_{\frac{2\Phi_1}{\sigma_1^2} - 1}\left(\frac{4\Omega_1}{\sigma_1^2(e^{\Omega_1\tau} - 1)}(v_1 v_{1,0} e^{\Omega_1\tau})^{\frac{1}{2}}\right) I_{\frac{2\Phi_2}{\sigma_2^2} - 1}\left(\frac{4\Omega_2}{\sigma_2^2(e^{\Omega_2\tau} - 1)}(v_2 v_{2,0} e^{\Omega_2\tau})^{\frac{1}{2}}\right). \qquad (3.7.2)
\end{aligned}
$$

Proof: Refer to Appendix 3.6. □

Now that we have managed to obtain $U(\tau, x, v_1, v_2)$, we revert back to the original density function $G(\tau, S, v_1, v_2)$ which is expressed in terms of the underlying asset variable, S.

PROPOSITION 3.7.2. *The transition density function expressed in terms of the original state variables can be represented as*

$$
G(\tau, S, v_1, v_2; S_0, v_{1,0}, v_{2,0}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta \ln S_0} H(\tau, \ln S, v_1, v_2; -\eta, v_{1,0}, v_{2,0}) d\eta. \qquad (3.7.3)
$$

Proof: Recall that $S \equiv e^x$ and $U(\tau, x, v_1, v_2; S_0, v_{1,0}, v_{2,0}) \equiv G(\tau, e^x, v_1, v_2; e^{x_0}, v_{1,0}, v_{2,0})$. Substituting these into equation (3.7.1) we obtain the result in the above proposition.□

Having found the explicit form of the trivariate transition density function, we can now obtain the full representation of the American call option presented in Proposition 3.2.1. As demonstrated in that proposition, the American option price is expressed in terms of the unknown early exercise boundary $S = b(\tau, v_1, v_2)$. We use the value matching condition to find the integral equation that determines this function, details of which are given in the next section.

3.8. The American Option Price

As stated in the previous section, given the explicit transition probability density function in Proposition 3.7.2, we can derive the simplified version of the American call option written on the underlying asset, S whose dynamics evolve according to the SDE system (3.2.16)-(3.2.18). By using the relationship $C(\tau, x, v_1, v_2) \equiv V(\tau, e^x, v_1, v_2)$, the value of the American call option can be represented as

$$V(\tau, S, v_1, v_2) = V_E(\tau, S, v_1, v_2) + V_P(\tau, S, v_1, v_2). \tag{3.8.1}$$

The two terms on the RHS of equation (3.8.1) are presented in the two propositions below. As pointed out earlier, the first component on the RHS of equation (3.8.1) is the European option component whilst the second is the early exercise premium.

PROPOSITION 3.8.1. *The European option component of the American call option can be written as*

$$V_E(\tau, S, v_1, v_2) = e^{-q\tau} S P_1(\tau, S, v_1, v_2; K) - e^{-r\tau} K P_2(\tau, S, v_1, v_2; K), \tag{3.8.2}$$

where

$$P_j(\tau, S, v_1, v_2; K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{g_j(\tau, S, v_1, v_2; \eta) e^{-i\eta \ln K}}{i\eta} \right) d\eta, \tag{3.8.3}$$

for $j = 1, 2$ *with*

$$\begin{aligned}
g_j(\tau, S, v_1, v_2; \eta) &= \exp\Big\{ i\eta \ln S + B_j(\tau, \eta) + D_{1,j}(\tau, \eta)v_1 + D_{2,j}(\tau, \eta)v_2 \Big\}, \\
B_j(\tau, \eta) &= i\eta(r - q)\tau + \frac{\Phi_1}{\sigma_1^2}\Big\{ (\Theta_{1,j} + \Omega_{1,j})\tau - 2\ln\Big(\frac{1 - Q_{1,j}e^{\Omega_{1,j}\tau}}{1 - Q_{1,j}}\Big)\Big\} \\
&\qquad + \frac{\Phi_2}{\sigma_2^2}\Big\{ (\Theta_{2,j} + \Omega_{2,j})\tau - 2\ln\Big(\frac{1 - Q_{2,j}e^{\Omega_{2,j}\tau}}{1 - Q_{2,j}}\Big)\Big\}, \qquad (3.8.4) \\
D_{1,j}(\tau, \eta) &= \frac{(\Theta_{1,j} + \Omega_{1,j})}{\sigma_1^2}\Big[\frac{1 - e^{\Omega_{1,j}\tau}}{1 - Q_{1,j}e^{\Omega_{1,j}\tau}}\Big], \\
D_{2,j}(\tau, \eta) &= \frac{(\Theta_{2,j} + \Omega_{2,j})}{\sigma_2^2}\Big[\frac{1 - e^{\Omega_{2,j}\tau}}{1 - Q_{2,j}e^{\Omega_{2,j}\tau}}\Big].
\end{aligned}$$

Here, $Q_{m,j} = (\Theta_{m,j} + \Omega_{m,j})/(\Theta_{m,j} - \Omega_{m,j})$ *for* $m = 1, 2$ *and* $j = 1, 2$ *where* $\Theta_{1,1} = \Theta_1(i - \eta)$, $\Theta_{1,2} = \Theta_1(-\eta)$, $\Theta_{2,1} = \Theta_2(i - \eta)$, $\Theta_{2,2} = \Theta_2(-\eta)$, $\Omega_{1,1} = \Omega_1(i - \eta)$, $\Omega_{1,2} = \Omega_1(-\eta)$, $\Omega_{2,1} = \Omega_2(i - \eta)$ *and* $\Omega_{2,2} = \Omega_2(-\eta)$.

Proof: Refer to Appendix 3.8. □

Remark 3.8.1. *We recall that the definitions of* Θ_1, Θ_2, Ω_1 *and* Ω_2 *have been provided in equation* (3.3.2). *Also* Φ_1 *and* Φ_2 *have been defined in equation* (3.2.34).

When numerically implementing this pricing function in Proposition 3.8.2, it is desirable to adopt the ideas proposed in Kahl and Jäckel (2005) and Albrecher, Mayer, Schoutens and Tistaert (2007). Such techniques prevent the possibilities of branch cuts[4] and ensure that the density function is continuous in the complex plane. Discontinuities are frequently more pronounced when pricing long maturity options. An example to this effect can be found on Table 1 of Albrecher et al. (2007).

[4]A branch cut is a curve in the complex plane across which a function is discontinuous.

PROPOSITION 3.8.2. *The early exercise premium component of the American call option can be represented as*

$$V_P(\tau,S,v_1,v_2) = \int_0^\tau \int_0^\infty \int_0^\infty [qe^{-q(\tau-\xi)} SP_1^A(\tau-\xi,S,v_1,v_2;w_1,w_2,b(\xi,w_1,w_2)) - re^{-r(\tau-\xi)} KP_2^A(\tau-\xi,S,v_1,v_2;w_1,w_2,b(\xi,w_1,w_2))] dw_1 dw_2 d\xi, \qquad (3.8.5)$$

where

$$P_j^A(\tau-\xi,S,v_1,v_2;w_1,w_2,b(\xi,v_1,v_2)) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{g_j^A(\tau-\xi,S,v_1,v_2;\eta,w_1,w_2)e^{-i\eta\ln b(\xi,w_1,w_2)}}{i\eta}\right)d\eta, \qquad (3.8.6)$$

for $j = 1,2$ *with*

$$\begin{aligned} g_j^A(\tau-\xi,S,v_1,v_2;\eta,w_1,w_2) = \exp\Bigg\{ &\left(\frac{\Theta_{1,j}-\Omega_{1,j}}{\sigma_1^2}\right)(v_1-w_1+\Phi_1(\tau-\xi)) \qquad (3.8.7)\\ &+ \left(\frac{\Theta_{2,j}-\Omega_{2,j}}{\sigma_2^2}\right)(v_2-w_2+\Phi_2(\tau-\xi))\Bigg\} \\ \times \exp\Bigg\{-&\left(\frac{2\Omega_{1,j}}{\sigma_1^2(e^{\Omega_{1,j}(\tau-\xi)}-1)}\right)\left(w_1e^{\Omega_{1,j}(\tau-\xi)}+v_1\right) - \left(\frac{2\Omega_{2,j}}{\sigma_2^2(e^{\Omega_{2,j}(\tau-\xi)}-1)}\right)\left(w_2e^{\Omega_{2,j}(\tau-\xi)}+v_2\right)\Bigg\} \\ \times e^{i\eta\ln S+i\eta(r-q)(\tau-\xi)} &\frac{2\Omega_{1,j}e^{\Omega_{1,j}(\tau-\xi)}}{\sigma_1^2(e^{\Omega_1(\tau-\xi)}-1)} \frac{2\Omega_{2,j}e^{\Omega_{2,j}(\tau-\xi)}}{\sigma_2^2(e^{\Omega_{2,j}(\tau-\xi)}-1)} \left(\frac{w_1e^{\Omega_{1,j}(\tau-\xi)}}{v_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}} \left(\frac{w_2e^{\Omega_{2,j}(\tau-\xi)}}{v_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}} \\ \times I_{\frac{2\Phi_1}{\sigma_1^2}-1} &\left(\frac{4\Omega_{1,j}}{\sigma_1^2(e^{\Omega_{1,j}(\tau-\xi)}-1)}(v_1w_1e^{\Omega_{1,j}(\tau-\xi)})^{\frac{1}{2}}\right) I_{\frac{2\Phi_2}{\sigma_2^2}-1}\left(\frac{4\Omega_{2,j}}{\sigma_2^2(e^{\Omega_{2,j}(\tau-\xi)}-1)}(v_2w_2e^{\Omega_{2,j}(\tau-\xi)})^{\frac{1}{2}}\right). \end{aligned}$$

The expressions for $\Theta_{m,j}$ *and* $\Omega_{m,j}$ *are given in Proposition* 3.8.1 *above.*

Proof: Refer to Appendix 3.9. □

Equation (3.8.1) is in terms of the early exercise boundary, $b(\tau,v_1,v_2)$ which is still unknown. This function needs to be determined for us to have the corresponding option price at each point in time. Also, the three integrals of the early exercise premium component cannot be integrated out as we do not know the functional form of this early exercise boundary. The only knowledge we have is that it is a function of time and the two instantaneous variances. The early exercise boundary also satisfies the value matching condition

$$b(\tau, v_1, v_2) - K = V(\tau, b(\tau, v_1, v_2), v_1, v_2), \tag{3.8.8}$$

which, given the integral expression for $V(\tau, b(\tau, v_1, v_2), v_1, v_2)$ is a non-linear Volterra integral equation. This can be solved directly for the free-boundary but we seek some approximation techniques in order to reduce the computational burden. We present one approximation technique for the early exercise boundary in the next section.

3.9. Approximating the Early Exercise Surface

The idea of approximating early exercise boundaries has gained popularity in pricing standard[5] American options; Ju (1998) uses multi-piece exponential functions to approximate the early exercise boundary of the American put option. Chiarella et al. (1999) use Fourier-Hermite series expansions to represent the underlying asset price evolution and then present a systematic approach for evaluating the corresponding options written on a particular underlying asset. Ait-Sahlia and Lai (2001) approximate Kim's (1990) early exercise boundary with a piecewise linear function. They first discretise the time domain into equally spaced sub-intervals. Linear interpolation is then incorporated to fit the early exercise boundary between two successive subintervals thereby generating the entire free-boundary. Mallier (2002) consider series solutions for the location of the early exercise boundary close to expiry.

Approximation techniques have also been generalised to American options under stochastic volatility. Broadie et al. (2000) have shown empirically in the single stochastic volatility case that $\ln b(\tau, v)$ can well be approximated by a function that is linear in v. Based on these empirical findings, Tzavalis and Wang (2003) have expanded the logarithm of the early exercise boundary using Taylor series around the long-run volatility.

[5]Here, the term "standard" means an option pricing model that satisfies all Black and Scholes (1973) assumptions.

This expansion yields two unknown functions of time which they later determine using Chebyshev polynomial expansion techniques. Instead of applying a Chebyshev approximation, Adolfsson et al. (2009) use numerical integration techniques and root finding methods to find these unknown functions of time. This method proves to be adequate enough in terms of accuracy and computational speed as compared to other valuation methods that they consider. It is this approach that we employ in this chapter.

We first use a Taylor series expansion to expand the logarithm of the early exercise boundary, $\ln b(\tau, v_1, v_2)$ around the corresponding long-run variances such that

$$\ln b(\tau, v_1, v_2) \approx b_0(\tau) + b_1(\tau)v_1 + b_2(\tau)v_2, \tag{3.9.1}$$

where $b_0(\tau)$, $b_1(\tau)$ and $b_2(\tau)$ are functions of time to be determined. This approach allow us to simplify the two integrals with respect to w_1 and w_2 in equation (3.8.1) before applying the numerical algorithm. Incorporating the expansion (3.9.1) into equation (3.8.5) we obtain the results in the proposition below.

Proposition 3.9.1. *By approximating the early exercise boundary with the expression*

$$\ln b(\tau, v_1, v_2) \approx b_0(\tau) + b_1(\tau)v_1 + b_2(\tau)v_2, \tag{3.9.2}$$

the value of the American call option can be re-expressed as

$$V(\tau, S, v_1, v_2) \approx V_E(\tau, S, v_1, v_2) + V_P^A(\tau, S, v_1, v_2), \tag{3.9.3}$$

where $V_E(\tau, S, v_1, v_2)$ is as presented in Proposition 3.8.1 and the approximation to the early exercise premium is given by

$$\begin{aligned} V_P^A(\tau, S, v_1, v_2) = \int_0^\tau &[qe^{-q(\tau-\xi)} S\hat{P}_1^A(\tau-\xi, S, v_1, v_2; b_0(\xi), b_1(\xi), b_2(\xi)) \\ &- re^{-r(\tau-\xi)} K\hat{P}_2^A(\tau-\xi, S, v_1, v_2; b_0(\xi), b_1(\xi), b_2(\xi))]d\xi, \end{aligned} \tag{3.9.4}$$

where

$$\hat{P}_j^A(\tau-\xi, S, v_1, v_2; b_0(\xi), b_1(\xi), b_2(\xi)) = \frac{1}{2} \tag{3.9.5}$$
$$+\frac{1}{\pi}\int_0^\infty Re\left(\frac{\hat{g}_j^A(\tau-\xi, S, v_1, v_2; \eta, b_1(\xi), b_2(\xi))e^{-i\eta b_0(\xi)}}{i\eta}\right)d\eta,$$

for $j = 1,2$ *with*

$$\hat{g}_j^A(\tau, S, v_1, v_2; \eta, b_1, b_2) = \exp\left\{i\eta \ln S + B_j^A(\tau, \eta, b_1, b_2) + D_{1,j}^A(\tau, \eta, b_1)v_1 + D_{2,j}^A(\tau, \eta, b_2)v_2\right\},$$

$$B_j^A(\tau, \eta, b_1, b_2) = i\eta(r-q)\tau + \frac{\Phi_1}{\sigma_1^2}\left\{(\Theta_{1,j}+\Omega_{1,j})\tau - 2\ln\left(\frac{1-Q_{1,j}^A e^{\Omega_{1,j}\tau}}{1-Q_{1,j}^A}\right)\right\}$$
$$+\frac{\Phi_2}{\sigma_2^2}\left\{(\Theta_{2,j}+\Omega_{2,j})\tau - 2\ln\left(\frac{1-Q_{2,j}^A e^{\Omega_{2,j}\tau}}{1-Q_{2,j}^A}\right)\right\},$$

$$D_{1,j}^A(\tau, \eta, b_1) = -i\eta b_1 + \frac{(\Theta_{1,j}+\Omega_{1,j})}{\sigma_1^2}\left[\frac{1-e^{\Omega_{1,j}\tau}}{1-Q_{1,j}^A e^{\Omega_{1,j}\tau}}\right],$$

$$D_{2,j}^A(\tau, \eta, b_2) = -i\eta b_2 + \frac{(\Theta_{2,j}+\Omega_{2,j})}{\sigma_2^2}\left[\frac{1-e^{\Omega_{2,j}\tau}}{1-Q_{2,j}^A e^{\Omega_{2,j}\tau}}\right].$$

Here

$$Q_{m,j}^A = \frac{\Theta_{m,j} + i\eta\sigma_m^2 b_m + \Omega_{m,j}}{\Theta_{m,j} + i\eta\sigma_m^2 b_m - \Omega_{m,j}}, \quad \Theta_{m,j} = \Theta_j(i-\eta), \quad \textit{and} \quad \Omega_{m,j} = \Omega_j(i-\eta),$$

for $m = 1,2$ *and* $j = 1,2$.

Proof: Refer to Appendix 3.10. □

By using the approximation (3.9.1), we have managed to reduce the number of integral dimensions from four to two as we have simplified the two integrals with respect to w_1 and w_2 in equation (3.8.5). The simplified version of the early exercise premium component enhances computational speed of our numerical scheme for finding both the early exercise boundary and the corresponding option price as the resulting equation is now independent of the modified Bessel functions which tends to consume much computational time. Given the approximation in equation (3.9.1), the value-matching condition can also be expressed as

$$e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2} - K = V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2). \tag{3.9.6}$$

Equation (3.9.6) is implicit in $b_0(\tau)$, $b_1(\tau)$ and $b_2(\tau)$, hence root finding techniques need to be employed for us to obtain explicit forms of these functions. In determining these functions, we formulate three equations such that

$$\begin{aligned} b_0(\tau) &= \ln[V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2) + K] - b_1(\tau)v_1 - b_2(\tau)v_2, \\ b_1(\tau) &= \frac{1}{v_1}\Big(\ln[V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2) + K] - b_0(\tau) - b_2(\tau)v_2\Big), \\ b_2(\tau) &= \frac{1}{v_2}\Big(\ln[V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2) + K] - b_0(\tau) - b_1(\tau)v_1\Big). \end{aligned} \tag{3.9.7}$$

These equations need to be solved iteratively at each instant, details of which are outlined in the next section.

3.10. Numerical Implementation

Having derived the integral expression for the American call option price in equation (3.9.3) and the corresponding system of equations (3.9.7) for approximating the early exercise boundary, we now present the numerical algorithm for the implementation of these equations. A variety of techniques have been proposed in the literature for numerically solving equations like (3.9.7). Huang et al. (1996) use a numerical integration scheme to solve the Kim (1990) American put integral equation. Kallast and Kivinukk (2003) also use quadrature methods to approximate the price, delta, gamma and vega of both American call and put options. Adolfsson et al. (2009) use similar techniques to implement the integral expression for the American call option price when the dynamics of the underlying asset evolve under the influence of a stochastic variance process of the Heston (1993) type. In implementing our pricing algorithm, we shall use quadrature techniques as applied in Kallast and Kivinukk (2003).

The European option component of equation (3.9.3) involves only one integral with respect to the Fourier transform variable, this integration is easily handled by many software applications with in-built functions to perform such complex integrations. However, the early exercise premium component has two integrals, one with respect to the Fourier transform variable and the other with respect to running time-to-maturity, ξ. The integral with respect to the Fourier transform variable is handled in a similar way as in the European component case. However, the integral with respect to ξ requires the entire history of the three functions, $b_0(\tau)$, $b_1(\tau)$, and $b_2(\tau)$ up to and including the current time. To our knowledge, there are no in – built packages that can handle such an integral. We therefore need to devise an algorithm to determine these three functions iteratively at each point in time.

In implementing equation (3.9.3) and the system (3.9.7), we treat the American option as a Bermudan option. The time interval is partitioned into M-equally spaced subintervals of length $h = T/M$. The algorithm is initiated at maturity and we then progress backwards in time. We denote the starting point as, $\tau_0 = 0$ which corresponds to maturity time. At maturity, it has been shown in Kim (1990) that the early exercise boundary of the American call option takes the form

$$b(0, v_1, v_2) = \max\left(\frac{r}{q}K, K\right). \tag{3.10.1}$$

By comparing coefficients, we can readily deduce that

$$b_0(0) = \max\left(\ln K, \ln\left[\frac{r}{q}K\right]\right), \quad b_1(0) = 0, \quad \text{and} \quad b_2(0) = 0. \tag{3.10.2}$$

All other time steps are denoted as $\tau_m = mh$, for $m = 1, 2, \cdots, M$. The discretised version of the American call option price is thus

$$V(mh, S, v_1, v_2) \approx V_E(mh, S, v_1, v_2) + V_P^A(mh, S, v_1, v_2). \tag{3.10.3}$$

At each subsequent time step we need to determine the three unknown boundary terms, $b_0^m = b_0(mh)$, $b_1^m = b_1(mh)$ and $b_2^m = b_2(mh)$ each of which depends on the entire early exercise boundary history. We use iterative techniques to find the values of these three unknown functions at each time step, that is, when iterating for b_0^m, b_1^m and b_2^m, we use as initial guesses[6] $b_{0,0}^m = b_0^{m-1}$, $b_{1,0}^m = b_1^{m-1}$ and $b_{2,0}^m = b_2^{m-1}$ followed by solving the system of linked equations

$$\begin{aligned} b_{0,k}^m &= \ln[V(mh, e^{b_{0,k}^m + b_{1,k-1}^m v_1 + b_{2,k-1}^m v_2}, v_1, v_2) + K] - b_{1,k-1}^m v_1 - b_{2,k-1}^m v_2, \\ b_{1,k}^m &= \frac{1}{v_1}\Big(\ln[V(mh, e^{b_{0,k}^m + b_{1,k}^m v_1 + b_{2,k-1}^m v_2}, v_1, v_2) + K] - b_{0,k}^m - b_{2,k-1}^m v_2 \Big), \\ b_{2,k}^m &= \frac{1}{v_2}\Big(\ln[V(mh, e^{b_{0,k}^m + b_{1,k}^m v_1 + b_{2,k}^m v_2}, v_1, v_2) + K] - b_{0,k}^m - b_{1,k}^m v_1 \Big). \end{aligned} \tag{3.10.4}$$

We find the value of k such that $|b_{0,k}^m - b_{0,k-1}^m| < \epsilon_0$, $|b_{1,k}^m - b_{1,k-1}^m| < \epsilon_1$ and $|b_{2,k}^m - b_{2,k-1}^m| < \epsilon_2$, where ϵ_0, ϵ_1 and ϵ_2 are tolerance values. Once the tolerance values are attained, we then proceed to the next time step. This algorithm is applicable to any root finding method for determining the triplet, b_0, b_1 and b_2. Adolfsson et al. (2009) use Newton's method while we prefer to use the bisection method in this chapter as it does not involve the computation of the first derivative of the pricing function.

In the next section we present numerical and graphical results for the early exercise boundary and the corresponding American call option prices obtained using the above approach. We also provide graphs for the joint probability density functions of the state variables. Such density functions are crucial as they give us a clue on how to handle the unbounded integral domains of the state variables and address the convergence property of density functions.

[6]The subscript k in the three functions $b_{0,k}^m$, $b_{1,k}^m$ and $b_{2,k}^m$ represents the number of iterations required for convergence of the iterative process at time step m.

3.11. Numerical Results

Having presented the numerical algorithm as outlined above, we now provide some numerical examples in this section. In what follows, we will dub our method the numerical integration scheme. We have also implemented the Method of Lines (MOL) algorithm for the PDE (3.2.21) for comparison purposes. As highlighted in Chapter 2, details on the implementation of the MOL algorithm will be presented in Chapter 5. In all the numerical experiments that follow unless otherwise stated, we will use the parameters provided in Table 3.1 where, $v_1^{\max}$ and $v_2^{\max}$ are the maximum levels of the two instantaneous variances under consideration. We have discretised the two variance domains into 30 equally spaced sub intervals and $M = 200$ time steps. For the MOL algorithm a non-uniform grid is applied to the underlying asset domain and a total number of 1,438 grid points has been used. The large number of grid points in the underlying asset domain helps in stabilising the numerical scheme and enhancing the smoothness of the early exercise boundary. For the numerical scheme to be stable, we have used 40 grid points in the interval $0 \leq S \leq 1$, 198 points in the interval $1 < S \leq 100$ and 1200 points within the interval $100 < S \leq 500$.

Parameter	Value	v_1 – Parameter	Value	v_2 – Parameter	Value
K	100	θ_1	6%	θ_2	8%
r	3%	κ_1	3	κ_2	4
q	5%	σ_1	10%	σ_2	11%
T	0.5	ρ_{13}	± 0.5	ρ_{24}	± 0.5
M	200	λ_1	0	λ_2	0
		$v_1^{\max}$	20%	$v_2^{\max}$	20%

TABLE 3.1. Parameters used for the American call option. The v_1 column contains parameters for the first variance process whilst the v_2 column contains parameters for the second variance process.

We start by presenting the joint probability density function of S and v_1 when v_2 is fixed and that of S and v_2 when v_1 is fixed in Figures 3.1 and 3.2 respectively. These surfaces are generated by implementing equation (3.7.3). The nature of these probability density functions guide us on how to truncate the infinite domains of the state variables when performing numerical integration experiments. From these figures we note that density functions are zero everywhere except near the origins of the state variables. For instance, instead of integrating the underlying asset domain from zero to infinity in our case we have simply integrated from zero to 50 since beyond this point the density function is extremely close to zero. Such diagrams also provide a natural way of analysing the distribution of asset returns under stochastic volatility.

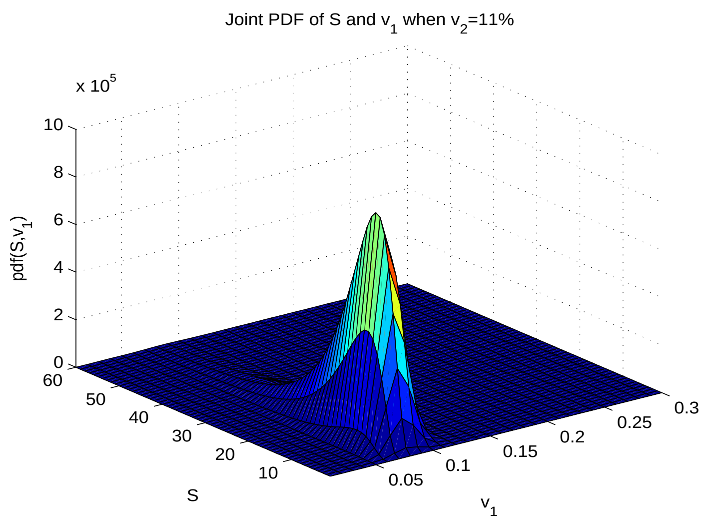

FIGURE 3.1. The probability density function of S and v_1 when v_2 is fixed. We considered the case when $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$ with all other parameters as provided in Table 3.1.

Having established the integration domains for the state variables, we present in Figure 3.3 the early exercise surface for the American call option when v_2 if fixed. This surface shows how an increase in v_1 affects the free-boundary of the American call option. We note from this figure that the early exercise surface is an increasing function of v_1 and is of the form typical for that of an American call option written on a single underlying asset whose dynamics evolve under the influence of a single stochastic variance process as presented in Chiarella et al. (2009). A similar surface can be obtained by fixing v_1 and allowing v_2 to vary.

We can also compare the early exercise boundaries for the American call option when both v_1 and v_2 are fixed. Figure 3.4 shows these comparisons for varying correlation

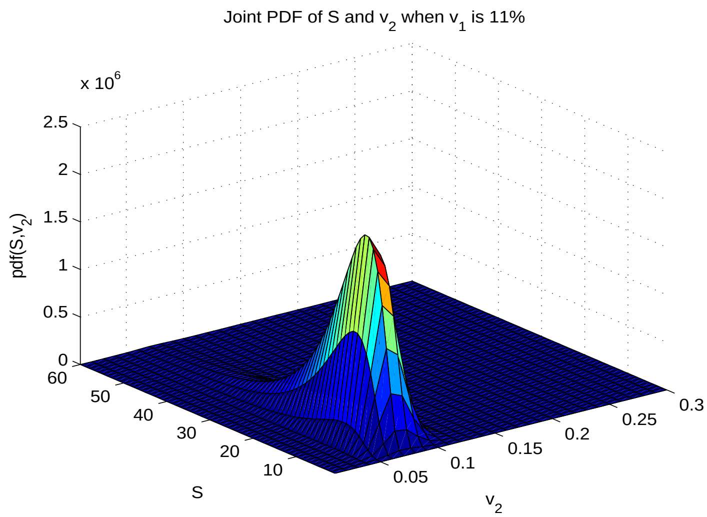

FIGURE 3.2. The probability density function of S and v_2 when v_1 is fixed. We considered the case when $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$ with all other parameters as provided in Table 3.1.

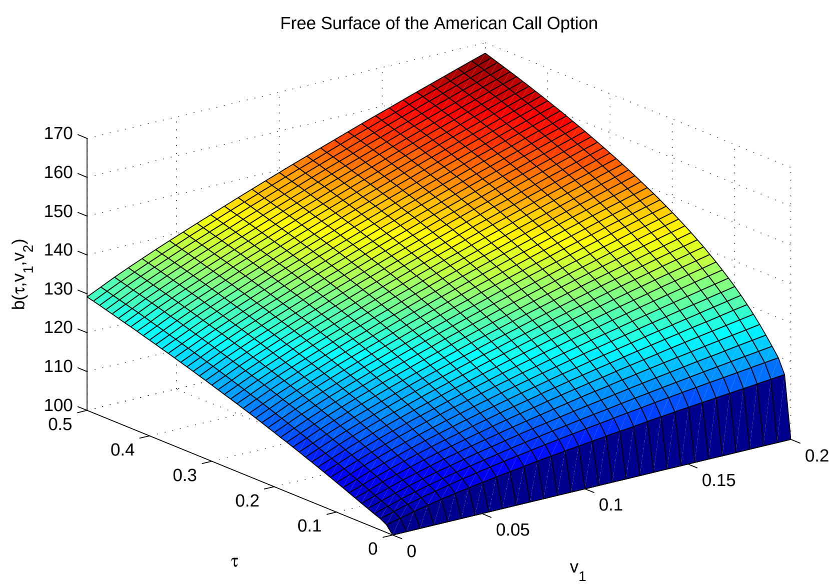

FIGURE 3.3. Early exercise surface of the American Call option when $v_2 = 0.67\%$, $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$. All other parameters are as presented in Table 3.1.

coefficients. Note from this figure that for fixed v_1 and v_2, early exercise boundaries typical for standard American call options are generated. We have also included the free-boundary generated from the geometric Brownian motion (GBM) model to highlight the impact of stochastic volatility on the American call option free-boundary. Since the two instantaneous variance processes under consideration are mean reverting, we calculate the corresponding GBM constant standard deviation as

$$\sigma_{GBM} = \sqrt{\theta_1 + \theta_2}, \tag{3.11.1}$$

where θ_1 and θ_2 are the long run variances of v_1 and v_2 respectively. From Figure 3.4 we note that zero correlations almost correspond to the GBM case. The early exercise boundary generated when the correlations are negative lies above that of the GBM model whilst that for positive correlations lies below as revealed in Figure 3.4.

Figure 3.5 shows the effects of varying volatilities of v_1 and v_2 to the early exercise boundary. We note that increasing both σ_1 and σ_2 has the effect of lowering the exercise boundary. We have considered the case when both ρ_{13} and ρ_{24} are equal to 0.5 and the instantaneous variances equal to their long run means.

To justify the effectiveness of our approach in valuing American call options, we need to compare the results with other pricing methods. In Figure 3.6 we present the early exercise boundaries from the method of lines (MOL) algorithm and numerical integration respectively. From this diagram we note that the early exercise boundary generated by the numerical integration method is slightly lower than that from the MOL. This might be attributed to approximation and discretisation errors from the numerical integration method. Discretisation errors can be reduced by making the grids finer. Errors from early exercise boundary approximation can be reduced by devising better approximating functions empirically or by any other suitable approach. Similar comparisons can be made for different parameter combinations. We also present Figure 3.7 which compares the effects of different correlation coefficients on the same process. For example when $\rho_{13} = -0.5$ and $\rho_{24} = 0.5$ we see that the corresponding early exercise boundary is slightly below the $\rho_{13} = 0$ and $\rho_{24} = 0$ boundary. This might be due to cancelation effect of the influential stochastic terms of the variance processes.

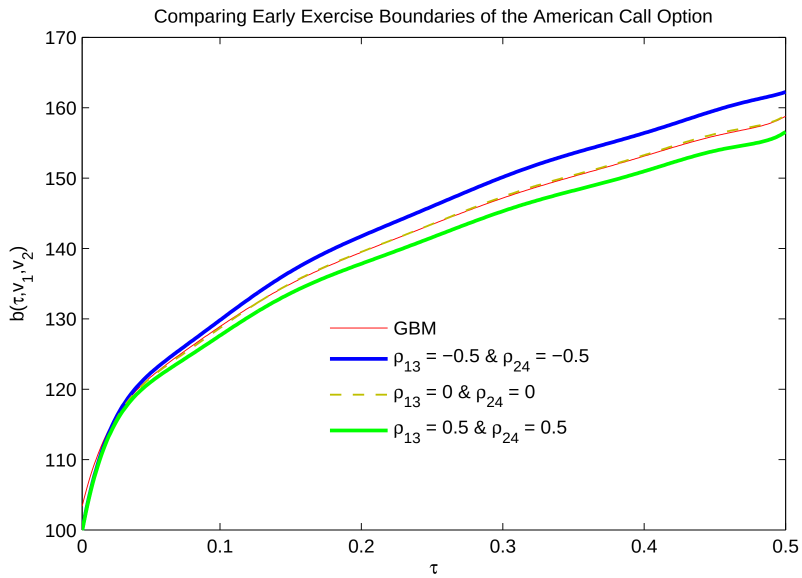

FIGURE 3.4. Exploring the effects of stochastic volatility on the early exercise boundary of the American call option for varying correlation coefficients when $\sigma_{GBM} = 0.3742$, $v_1 = 6\%$ and $v_2 = 8\%$. All other parameters are provided in Table 3.1.

We now turn to an analysis of option prices using the two approaches. Figure 3.8 shows the general American call option price surface at a fixed level of v_2. A similar surface can be generated by fixing v_1 and allowing S and v_2 to vary. We can also assess the effects of stochastic volatility on the option prices for different correlation coefficients by making comparisons with GBM prices where we calculate the corresponding constant volatility using equation (3.11.1) which is the square-root of the average of the two long run variances. Figure 3.9 shows option price differences found by subtracting option prices from the numerical integration method from the corresponding GBM prices. As with the early exercise boundary comparisons, the zero correlation price differences are not significantly different from GBM prices. As documented in Heston (1993) and Chiarella et al. (2009), higher price differences are noted for far out-and in-the-money options. Positive correlations yield option prices which are lower than GBM prices for in-the-money options while generating prices which are higher for out-of-money

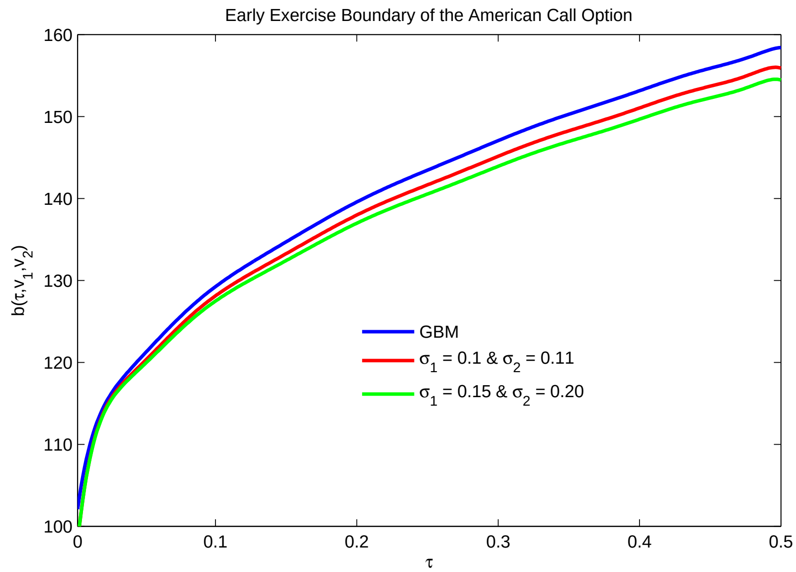

FIGURE 3.5. Exploring the effects of varying the volatilities of v_1 and v_2 on the early exercise boundary of the American call option. We have used the following parameters, $\sigma_{GBM} = 0.3742$, $v_1 = 6\%$, $v_2 = 8\%$, $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$ with all other parameters as given in Table 3.1.

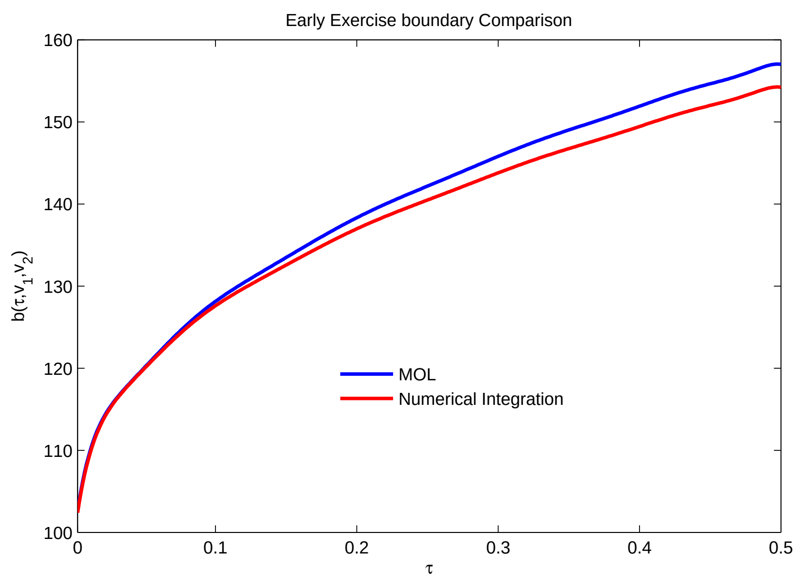

FIGURE 3.6. Comparing early exercise boundaries from the MOL and numerical integration approach when the two instantaneous variances are fixed. Here, $v_1 = 0.67\%$, $v_2 = 13.33\%$, $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$ with all other parameters as given in Table 3.1.

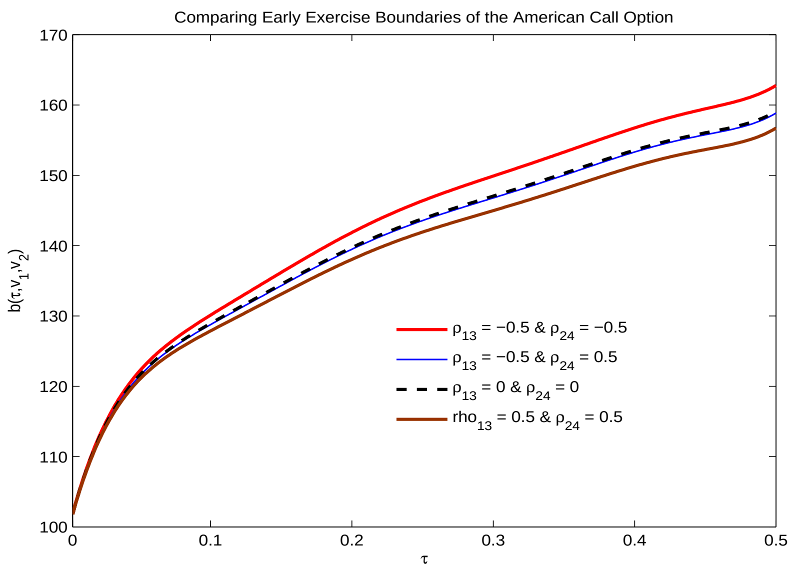

FIGURE 3.7. Exploring the effects of mixed correlation coefficients on the early exercise boundary of the American call option. We have used the following parameters, $v_1 = 6\%$, $v_2 = 8\%$ with all other parameters as given in Table 3.1.

options. The reverse effect holds for negative correlations. Higher price differences of up to 0.1 are noted for both positive and negative correlations.

We also present option prices obtained from the MOL and numerical integration methods together with the associated GBM prices in Table 3.2 when $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$. From this table we note that option prices obtained from the MOL and numerical integration methods are not significantly different from each other which shows that both methods are suitable for practical purposes in valuing American call options under stochastic volatility. We have included GBM prices to highlight the impact of stochastic volatility on option prices. When we presented numerical results for the early exercise boundaries we highlighted the effects of changes in the volatilities of v_1 and v_2, we also provide graphical results on how such changes affect option prices in Figure 3.10. In this figure, we have used the case when $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$. We can readily see that higher price differences occur for higher σ_1 and σ_2 with all other parameters as provided in Table 3.1. This implies that higher volatilities of v_1 and v_2 have the effect of increasing the variances which then results in higher price differences for in-and out-of-the-money options relative to GBM prices. Similar conclusions have been derived in Heston (1993) when considering the European call option under stochastic volatility.

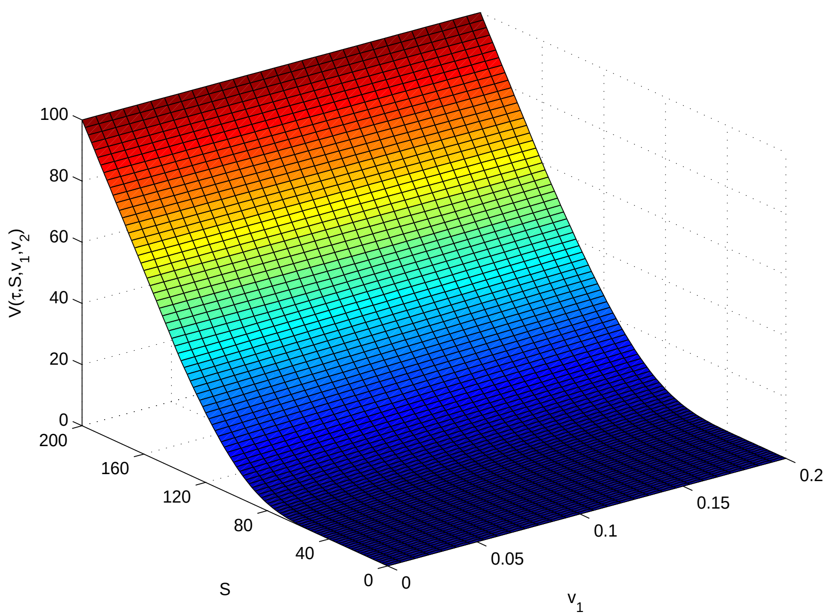

FIGURE 3.8. American call option price surface when $v_2 = 13.33\%$, $\rho_{13} = 0$ and $\rho_{24} = 0$ with all other parameters provided in Table 3.1.

S	Numerical Integration	MOL	GBM
60	0.2036	0.2029	0.1850
80	2.4088	2.400	2.4154
100	9.8082	9.7918	9.9452
120	23.1069	23.0920	23.3006
140	40.4756	40.4686	40.5922
160	60	60	60
180	80	80	80
200	100	100	100

TABLE 3.2. American call option price comparisons when $v_1 = 0.67\%$, $v_2 = 13.33\%$, $\rho_{13} = 0.5$, $\rho_{24} = 0.5$. We have taken GBM volatility to be $\sigma_{GBM} = 0.3741657$ and this is found by using equation (3.11.1).

As will be highlighted in Chapter 5 the most important feature of the MOL is that the option price, delta and the free-boundary are all generated simultaneously as part of the solution process at no added computational cost. Given such a tremendous

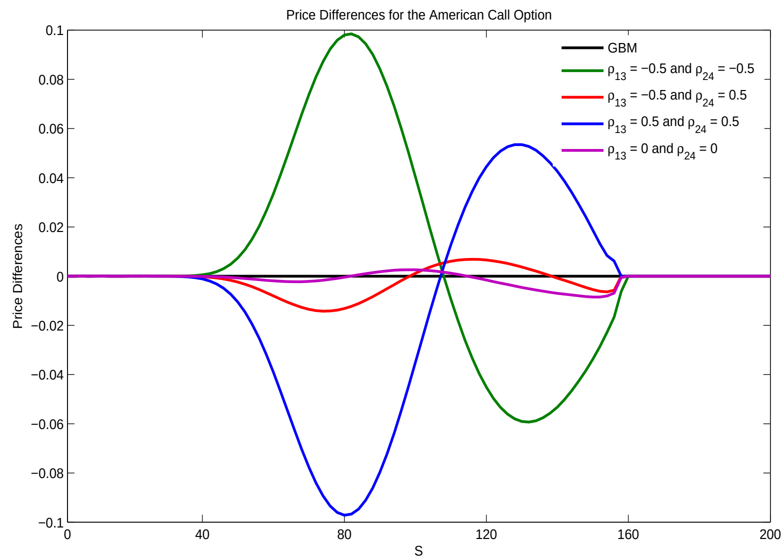

FIGURE 3.9. Option prices from the geometric Brownian motion minus option prices from the Stochastic volatility model for varying correlation coefficients. Here, $\sigma_{GBM} = 0.3742$, $v_1 = 6\%$ and $v_2 = 8\%$ with all other parameters provided in Table 3.1.

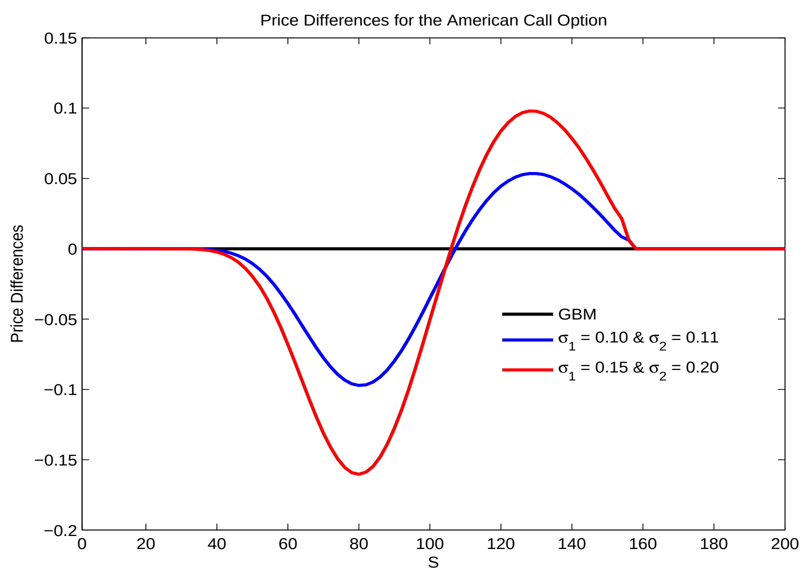

FIGURE 3.10. Option prices from the geometric Brownian motion minus option prices from the Stochastic volatility model for varying volatilities of volatility. Here, $\sigma_{GBM} = 0.3742$, $v_1 = 6\%$, $v_2 = 8\%$, $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$. All other parameters are provided in Table 3.1.

convenience, we wrap up this section by presenting the delta surface of the American call option in Figure 3.11 for fixed v_2. A similar surface can be obtained by holding v_1 constant. We also explore the effects of stochastic volatility on the delta by making comparisons with the GBM delta in Figure 3.12. From this figure, we note that the option delta is very sensitive to the changes in the variance.

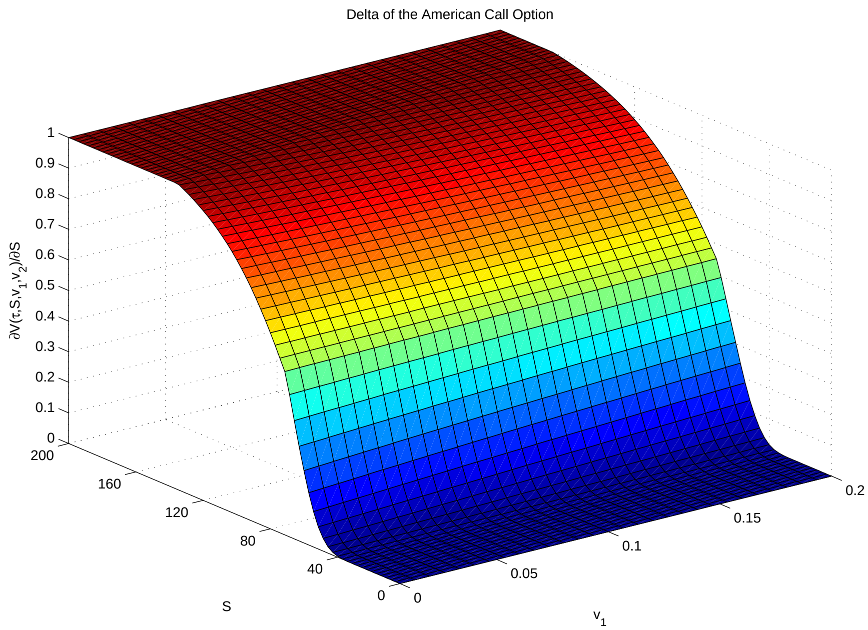

FIGURE 3.11. Delta surface of the American call option when $v_2 = 0.67\%$, $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$. All other parameters are as provided in Table 3.1.

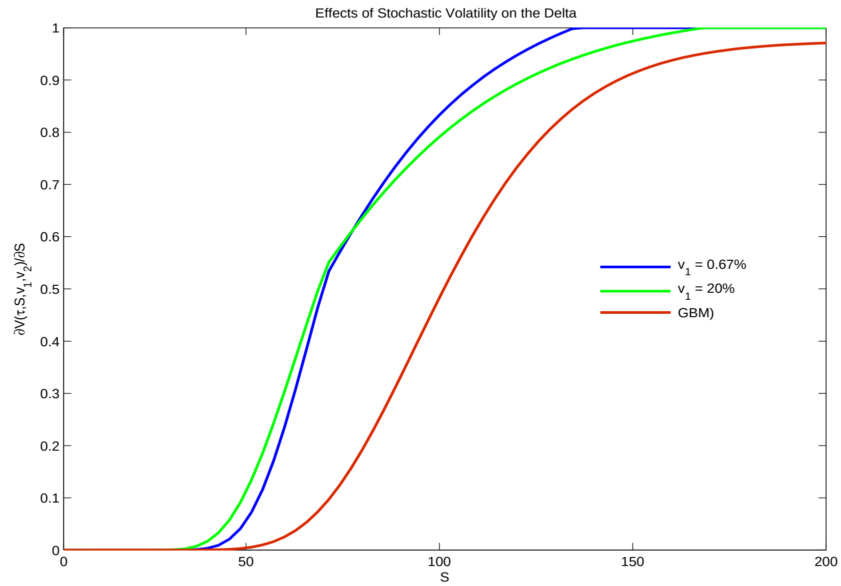

FIGURE 3.12. Exploring the effects of Stochastic volatility on the Delta of the American call option when $v_2 = 13.33\%$, $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$. All other parameters are as provided in Table 3.1.

3.12. Conclusion

In this chapter we have presented a numerical integration technique for pricing an American call option written on an underlying asset whose dynamics evolve under the influence of two stochastic variance processes of the Heston (1993) type. The approach involves the transformation of the pricing partial differential equation (PDE) to an inhomogeneous form by exploiting Jamshidian's (1992) techniques. An integral expression has been presented as the general solution of the inhomogeneous PDE with the aid of Duhamel's principle and this is a function of the transition density function.

The transition density function is a solution of the associated Kolmogorov backward PDE for the three stochastic processes under consideration. A systematic approach for solving the Kolmogorov PDE using a combination of Fourier and Laplace transforms has been presented. A means for numerically implementing the integral equation for the American call option has been provided based on the numerical techniques discussed in Chapter 2 and a log-linear approximation of the associated early exercise boundary. The early exercise boundary approximation has allowed a simplification of the double integrals with respect to the running variance variables. This reduces the computational burden when one proceeds to numerical implementation.

Numerical results exploring the impact of stochastic volatility on both option prices and the free-boundary have been provided and we have discovered that the correlations between the underlying asset and the two variance processes have a significant effect on in-and out-of-the money options. The numerical results presented yield similar findings of Heston (1993) and Chiarella et al. (2009) on the impact of stochastic volatility on option prices where they consider European and American option pricing under stochastic volatility respectively. We have also analysed the effects of varying the volatilities of instantaneous variances on both the early exercise boundary and the corresponding option prices. We note that an increase in the volatility of volatility increases the corresponding variance levels resulting in higher price differences for in- and out-of-the-money options when compared with geometric Brownian motion prices.

We have assessed the accuracy of the numerical integration approach by making comparisons with numerical results from the method of lines (MOL) algorithm. Both approaches provide comparable results though there are slight differences on the early exercise boundary plots. Such differences are mainly due to early exercise boundary approximation and discretisation errors associated with the numerical integration method. As the MOL has an additional advantage of generating the option delta as part of the solution, we have exploited this feature and explored the impact of stochastic volatility on the American spread option delta generated by the Black and Scholes (1973) model.

As also highlighted in Chapter 2 the integral expression derived in this chapter is applicable to any continuous payoff function which is a powerful feature of Fourier and Laplace transform based methods.

Appendix 3.1. Proof of Proposition 3.2.1

Consider the PDE

$$\frac{\partial C}{\partial \tau} = \mathcal{D}_{x,v_1,v_2}C - rC + f(\tau, x, v_1, v_2), \tag{A3.1.1}$$

whose initial condition is the payoff at maturity, $C(0, x, v_1, v_2) = (e^x - K)^+$. The PDE (A3.1.1) is to be solved in the region $0 \leq \tau \leq T$, $-\infty \leq x < \infty$ and $0 \leq v_1, v_2 < \infty$, and where we define the Dynkin operator $\mathcal{D}_{x,v_1,v_2}$ as

$$\begin{aligned}\mathcal{D}_{x,v_1,v_2} &= (r - q - \frac{1}{2}v_1 - \frac{1}{2}v_2)\frac{\partial}{\partial x} + \Phi_1\frac{\partial}{\partial v_1} - \beta_1 v_1\frac{\partial}{\partial v_1} + \Phi_2\frac{\partial}{\partial v_2} - \beta_2 v_2\frac{\partial}{\partial v_2} + \frac{1}{2}v_1\frac{\partial^2}{\partial x^2} \\ &+ \frac{1}{2}v_2\frac{\partial^2}{\partial x^2} + \rho_{13}\sigma_1 v_1\frac{\partial^2}{\partial x \partial v_1} + \rho_{14}\sigma_2 v_2\frac{\partial^2}{\partial x \partial v_2} + \frac{1}{2}\sigma_1^2 v_1\frac{\partial^2}{\partial v_1^2} + \frac{1}{2}\sigma_2^2 v_2\frac{\partial^2}{\partial v_2^2},\end{aligned}$$

with

$$\Phi_j = \kappa_j\theta_j \quad \text{and} \quad \beta_j = \kappa_j + \lambda_j, \quad \text{for} \quad j = 1, 2. \tag{A3.1.2}$$

By use of Duhamel principle, the solution of the PDE (A3.1.1) is given by

$$\begin{aligned}C(\tau, x, v_1, v_2) &= e^{-r\tau}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (e^u - K)^+ U(\tau, x, v_1, v_2; u, w_1, w_2)du dw_1 dw_2 \\ &+ \int_0^\tau e^{-r(\tau-\xi)}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f(\xi, u, w_1, w_2)U(\tau - \xi, x, v_1, v_2; u, w_1, w_2)du dw_1 dw_2 d\xi. \\ &\equiv C_E(\tau, x, v_1, v_2) + C_P(\tau, x, v_1, v_2).\end{aligned} \tag{A3.1.3}$$

To verify that this is the correct solution, we will show that (A3.1.3) satisfies the PDE (A3.1.1). Substituting $C(\tau, x, v_1, v_2) = C_E(\tau, x, v_1, v_2) + C_P(\tau, x, v_1, v_2)$ into (A3.1.1) we proceed as follows:

$$
\begin{aligned}
&\frac{\partial C}{\partial \tau} + rC - \mathcal{D}_{x,v_1,v_2}C - f(\tau, x, v_1, v_2) \\
&\qquad = e^{-r\tau} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty (e^u - K) \left\{ \frac{\partial U}{\partial \tau} - \mathcal{D}_{x,v_1,v_2} U \right\} du dw_1 dw_2 - rC_E + rC_E \\
&\qquad\quad + \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\tau, u, w_1, w_2) U(0, x, v_1, v_2; u, w_1, w_2) du dw_1 dw_2 \\
&\qquad\quad + \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\xi, u, w_1, w_2) \frac{\partial U}{\partial \tau} du dw_1 dw_2 d\xi - rC_P + rC_P \\
&\qquad\quad - \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\xi, u, w_1, w_2) \mathcal{D}_{x,v_1,v_2} U du dw_1 dw_2 d\xi - f(\tau, x, v_1, v_2) \\
&\quad = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\tau, x, v_1, v_2) \delta(e^x - e^u) \delta(v_1 - w_1) \delta(v_2 - w_2) du dw_1 dw_2 \\
&\qquad + \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\xi, x, v_1, v_2) \left[\frac{\partial U}{\partial \tau} - \mathcal{D}_{x,v_1,v_2} U \right] du dw_1 dw_2 d\xi - f(\tau, x, v_1, v_2) \\
&\qquad\qquad = f(\tau, x, v_1, v_2) + 0 - f(\tau, x, v_1, v_2) = 0
\end{aligned}
$$

Hence $C(\tau, x, v_1, v_2)$ satisfies the PDE (A3.1.1).

Appendix 3.2. Proof of Proposition 3.3.1

By use of equation (1.3.4) and the assumptions in (1.3.1), we note that

$$
\begin{aligned}
&\mathcal{F}\Big\{ \frac{\partial U}{\partial x}(\tau, x, v_1, v_2) \Big\} = -i\eta \hat{U}(\tau, \eta, v_1, v_2), \quad \mathcal{F}\Big\{ \frac{\partial^2 U}{\partial x^2}(\tau, x, v_1, v_2) \Big\} = -\eta^2 \hat{U}(\tau, \eta, v_1, v_2), \\
&\mathcal{F}\Big\{ \frac{\partial^2 U}{\partial x \partial v_1}(\tau, x, v_1, v_2) \Big\} = -i\eta \frac{\partial \hat{U}}{\partial v_1}(\tau, \eta, v_1, v_2), \quad \mathcal{F}\Big\{ \frac{\partial^2 U}{\partial x \partial v_2}(\tau, x, v_1, v_2) \Big\} = -i\eta \frac{\partial \hat{U}}{\partial v_2}(\tau, \eta, v_1, v_2), \\
&\mathcal{F}\Big\{ \frac{\partial U}{\partial v_1}(\tau, x, v_1, v_2) \Big\} = \frac{\partial \hat{U}}{\partial v_1}(\tau, \eta, v_1, v_2), \quad \mathcal{F}\Big\{ \frac{\partial U}{\partial v_2}(\tau, x, v_1, v_2) \Big\} = \frac{\partial \hat{U}}{\partial v_2}(\tau, \eta, v_1, v_2), \\
&\mathcal{F}\Big\{ \frac{\partial^2 U}{\partial v_1^2}(\tau, x, v_1, v_2) \Big\} = \frac{\partial^2 \hat{U}}{\partial v_1^2}(\tau, \eta, v_1, v_2), \quad \mathcal{F}\Big\{ \frac{\partial^2 U}{\partial v_2^2}(\tau, x, v_1, v_2) \Big\} = \frac{\partial^2 \hat{U}}{\partial v_2^2}(\tau, \eta, v_1, v_2), \\
&\mathcal{F}\Big\{ \frac{\partial U}{\partial \tau}(\tau, x, v_1, v_2) \Big\} = \frac{\partial \hat{U}}{\partial \tau}(\tau, \eta, v_1, v_2). \qquad \text{(A3.2.1)}
\end{aligned}
$$

Substituting all these expressions into equation (3.2.36) we obtain the PDE in Proposition 3.3.1.

The Fourier transform of the initial condition in equation (3.2.37) is simplified as follows:

$$\begin{aligned}\mathcal{F}\left\{U(0,x,v_1,v_2)\right\} &= \int_{-\infty}^{\infty} e^{i\eta x}U(0,x,v_1,v_2)dx \\ &= \int_{-\infty}^{\infty} e^{i\eta x}\delta(x-x_0)\delta(v_1-v_{1,0})\delta(v_2-v_{2,0})dx \\ &= e^{i\eta x_0}\delta(v_1-v_{1,0})\delta(v_2-v_{2,0}),\end{aligned} \tag{A3.2.2}$$

which is the result presented in equation (3.3.3) of Proposition 3.3.1.

Appendix 3.3. Proof of Proposition 3.4.1

By applying equation (1.3.6) and the assumptions in (1.3.2) and (1.3.3) to the respective components of equation (3.3.1) we obtain

$$\mathcal{L}\Big\{v_1\hat{U}(\tau,\eta,v_1,v_2)\Big\} = -\frac{\partial}{\partial s_1}\int_0^{\infty}\int_0^{\infty} e^{-s_1v_1-s_2v_2}\hat{U}dv_1dv_2 = -\frac{\partial\tilde{U}}{\partial s_1}(\tau,\eta,s_1,s_2),$$

$$\mathcal{L}\Big\{v_2\hat{U}(\tau,\eta,v_1,v_2)\Big\} = -\frac{\partial}{\partial s_2}\int_0^{\infty}\int_0^{\infty} e^{-s_1v_1-s_2v_2}\hat{U}dv_1dv_2 = -\frac{\partial\tilde{U}}{\partial s_2}(\tau,\eta,s_1,s_2),$$

$$\mathcal{L}\Big\{\frac{\partial\hat{U}}{\partial v_1}(\tau,\eta,v_1,v_2)\Big\} = \int_0^{\infty}\int_0^{\infty} e^{-s_1v_1-s_2v_2}\frac{\partial\hat{U}}{\partial v_1}dv_1dv_2 = -\tilde{U}(\tau,\eta,0,s_2)+s_1\tilde{U}(\tau,\eta,s_1,s_2),$$

$$\mathcal{L}\Big\{\frac{\partial\hat{U}}{\partial v_2}(\tau,\eta,v_1,v_2)\Big\} = \int_0^{\infty}\int_0^{\infty} e^{-s_1v_1-s_2v_2}\frac{\partial\hat{U}}{\partial v_2}dv_1dv_2 = -\tilde{U}(\tau,\eta,s_1,0)+s_2\tilde{U}(\tau,\eta,s_1,s_2),$$

$$\begin{aligned}\mathcal{L}\Big\{v_1\frac{\partial\hat{U}}{\partial v_1}(\tau,\eta,v_1,v_2)\Big\} &= \int_0^{\infty}\int_0^{\infty} v_1e^{-s_1v_1-s_2v_2}\frac{\partial\hat{U}}{\partial v_1}dv_1dv_2 \\ &= -\frac{\partial}{\partial s_1}\int_0^{\infty}\int_0^{\infty} e^{-s_1v_1-s_2v_2}\frac{\partial\hat{U}}{\partial v_1}dv_1dv_2 \\ &= -\frac{\partial}{\partial s_1}\big[-\tilde{U}(\tau,\eta,0,s_2)+s_1\tilde{U}(\tau,\eta,s_1,s_2)\big] \\ &= -\tilde{U}(\tau,\eta,s_1,s_2)-s_1\frac{\partial\tilde{U}}{\partial s_1}(\tau,\eta,s_1,s_2),\end{aligned}$$

$$\mathcal{L}\Big\{v_2\frac{\partial \hat{U}}{\partial v_2}(\tau,\eta,v_1,v_2)\Big\} = \int_0^\infty\int_0^\infty v_2 e^{-s_1v_1-s_2v_2}\frac{\partial \hat{U}}{\partial v_1}dv_1dv_2$$
$$= -\frac{\partial}{\partial s_2}\int_0^\infty\int_0^\infty e^{-s_1v_1-s_2v_2}\frac{\partial \hat{U}}{\partial v_2}dv_1dv_2$$
$$= -\frac{\partial}{\partial s_2}\big[-\tilde{U}(\tau,\eta,s_1,0)+s_2\tilde{U}(\tau,\eta,s_1,s_2)\big]$$
$$= -\tilde{U}(\tau,\eta,s_1,s_2) - s_2\frac{\partial \tilde{U}}{\partial s_2}(\tau,\eta,s_1,s_2),$$
$$\mathcal{L}\Big\{v_1\frac{\partial^2 \hat{U}}{\partial v_1^2}(\tau,\eta,v_1,v_2)\Big\} = \tilde{U}(\tau,\eta,0,s_2) - 2s_1\tilde{U}(\tau,\eta,s_1,s_2) - s_1^2\frac{\partial \tilde{U}}{\partial s_1}(\tau,\eta,s_1,s_2),$$
$$\mathcal{L}\Big\{v_2\frac{\partial^2 \hat{U}}{\partial v_2^2}(\tau,\eta,v_1,v_2)\Big\} = \tilde{U}(\tau,\eta,s_1,0) - 2s_2\tilde{U}(\tau,\eta,s_1,s_2) - s_2^2\frac{\partial \tilde{U}}{\partial s_2}(\tau,\eta,s_1,s_2). \tag{A3.3.1}$$

Substituting these expressions into equation (3.3.1) and noting that $f_1(\tau,s_2)$ and $f_2(\tau,s_1)$ are terms involving the Laplace transforms of $\tilde{U}(\tau,\eta,0,s_2)$ and $\tilde{U}(\tau,\eta,s_1,0)$ we obtain the result in Proposition 3.4.1. Feller (1951) has demonstrated that assumptions like those in the first equation of (1.3.3) imply that

$$\lim_{s_1\to\infty}\tilde{U}(\tau,\eta,s_1,s_2)=0 \quad \text{and} \quad \lim_{s_2\to\infty}\tilde{U}(\tau,\eta,s_1,s_2)=0, \tag{A3.3.2}$$

which is equation (3.4.3) of Proposition 3.4.1.

Appendix 3.4. Proof of Proposition 3.5.1

This appendix contains lengthy derivations for generating the solution of the partial differential equation system (3.4.1). Because of the nature of this PDE, we use the method of characteristics to find its solution. We break the appendix into three major parts where the first involves derivation of the general solution of the characteristic equations. The second part involves determination of the two functions, $f_1(\tau,S_2)$ and $f_2(\tau,S_1)$ appearing in equation (3.4.1). Once these two functions are determined, we then present the explicit form of $\tilde{U}(\tau,\eta,s_1,s_2)$ in the third part.

(1) **Solving the Characteristic equation in terms of $f_1(\tau, S_2)$ and $f_2(\tau, S_1)$:**

Here we attempt to solve equation (3.4.1) subject to the initial condition (3.4.2) by using the method of characteristics. Equation (3.4.1) can be re-expressed in characteristic form as

$$d\tau = \frac{ds_1}{\frac{1}{2}\sigma_1^2 s_1^2 - \Theta_1 s_1 + \frac{1}{2}\Lambda} = \frac{ds_2}{\frac{1}{2}\sigma_2^2 s_2^2 - \Theta_2 s_2 + \frac{1}{2}\Lambda} \tag{A3.4.1}$$

$$= \frac{d\tilde{U}}{\big\{(\Phi_1 - \sigma_1^2)s_1 - i\eta(r-q) + (\Phi_2 - \sigma_2^2)s_2 + \Theta_1 + \Theta_2\big\}\tilde{U} + f_1(\tau, s_2) + f_2(\tau, s_1)}.$$

Simplifying the first characteristic pair

By adopting the method of characteristics, we solve the first pair of equation (A3.4.1) by integration to obtain

$$\int d\tau = \frac{2}{\sigma_1^2}\int \frac{ds_1}{s_1^2 - \frac{2\Theta_1}{\sigma_1^2}s_1 + \frac{\Lambda}{\sigma_1^2}}.$$

By factorising the RHS of the above equation we obtain

$$\int d\tau = \frac{1}{\Omega_1}\int \left(\frac{1}{s_1 - \left(\frac{\Theta_1+\Omega_1}{\sigma_1^2}\right)} - \frac{1}{s_1 - \left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)}\right) ds_1. \tag{A3.4.2}$$

where we set

$$\Omega_1 = \sqrt{\Theta_1^2 - \Lambda(\eta)\sigma_1^2}. \tag{A3.4.3}$$

Equation (A3.4.2) implies that

$$\tau + c_1 = \frac{1}{\Omega_1}\int \left(\frac{1}{s_1 - \left(\frac{\Theta_1+\Omega_1}{\sigma_1^2}\right)} - \frac{1}{s_1 - \left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)}\right) ds_1,$$

where c_1 is an integration constant[7]. Integrating the RHS yields

$$\Omega_1\tau + c_2 = \ln\left(\frac{\sigma_1^2 s_1 - \Theta_1 - \Omega_1}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1}\right), \tag{A3.4.4}$$

which implies that

$$e^{\Omega_1\tau}e^{c_2} = \frac{\sigma_1^2 s_1 - \Theta_1 - \Omega_1}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1},$$

hence

$$e^{c_2} = \frac{(\sigma_1^2 s_1 - \Theta_1 - \Omega_1)e^{-\Omega_1\tau}}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1}. \tag{A3.4.5}$$

[7]In what follows we use c_j and d_j, $j = 1, 2, 3$ to denote integration constants.

The exponent of an integration constant is another constant, so that equation (A3.4.5) can be represented as

$$c_3 = \frac{(\sigma_1^2 s_1 - \Theta_1 - \Omega_1)e^{-\Omega_1 \tau}}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1}. \tag{A3.4.6}$$

Now, given equation (A3.4.6), we can obtain an expression for s_1 by making it the subject of the formula such that

$$c_3(\sigma_1^2 s_1 - \Theta_1 + \Omega_1) = (\sigma_1^2 s_1 - \Theta_1 - \Omega_1)e^{-\Omega_1 \tau},$$

that is

$$c_3 \sigma_1^2 s_1 - \sigma_1^2 s_1 e^{-\Omega_1 \tau} = (\Theta_1 - \Omega_1)c_3 - (\Theta_1 + \Omega_1)e^{-\Omega_1 \tau},$$

which becomes

$$\sigma_1^2(c_3 - e^{-\Omega_1 \tau})s_1 = \Theta_1(c_3 - e^{-\Omega_1 \tau}) - \Omega_1(c_3 + e^{-\Omega_1 \tau} + e^{-\Omega_1 \tau} - e^{-\Omega_1 \tau}),$$

which then implies that

$$s_1 = \frac{\Theta_1 - \Omega_1}{\sigma_1^2} - \frac{2\Omega_1 e^{-\Omega_1 \tau}}{\sigma_1^2(c_3 - e^{-\Omega_1 \tau})}. \tag{A3.4.7}$$

Solving the second characteristic pair

The characteristic equation of the second pair can be represented as

$$\int d\tau = \frac{2}{\sigma_2^2} \int \frac{ds_2}{s_2^2 - \frac{2\Theta_2}{\sigma_2^2} s_2 + \frac{\Lambda}{\sigma_2^2}}.$$

By factorising the RHS we obtain

$$\int d\tau = \frac{1}{\Omega_2} \int \left(\frac{1}{s_2 - \left(\frac{\Theta_2 + \Omega_2}{\sigma_2^2}\right)} - \frac{1}{s_2 - \left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2}\right)} \right) ds_2,$$

where we set

$$\Omega_2 = \sqrt{\Theta_2^2 - \Lambda(\eta)\sigma_2^2}. \tag{A3.4.8}$$

The above equation simplifies to

$$\tau + d_1 = \frac{1}{\Omega_2}\int\left(\frac{1}{s_2 - \left(\frac{\Theta_2+\Omega_2}{\sigma_2^2}\right)} - \frac{1}{s_2 - \left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)}\right)ds_2.$$

Solving the RHS yields

$$\Omega_2\tau + d_2 = \ln\left(\frac{\sigma_2^2 s_2 - \Theta_2 - \Omega_2}{\sigma_2^2 s_2 - \Theta_2 + \Omega_2}\right), \tag{A3.4.9}$$

which implies that

$$e^{\Omega_2\tau}e^{d_2} = \frac{\sigma_2^2 s_2 - \Theta_2 - \Omega_2}{\sigma_2^2 s_2 - \Theta_2 + \Omega_2},$$

hence

$$e^{d_2} = \frac{(\sigma_2^2 s_2 - \Theta_2 - \Omega_2)e^{-\Omega_2\tau}}{\sigma_2^2 s_2 - \Theta_2 + \Omega_2}. \tag{A3.4.10}$$

which can be written as

$$d_3 = \frac{(\sigma_2^2 s_2 - \Theta_2 - \Omega_2)e^{-\Omega_2\tau}}{\sigma_2^2 s_2 - \Theta_2 + \Omega_2}. \tag{A3.4.11}$$

Given equation (A3.4.11), we can show that

$$s_2 = \frac{\Theta_2 - \Omega_2}{\sigma_2^2} - \frac{2\Omega_2 e^{-\Omega_2\tau}}{\sigma_2^2(d_3 - e^{-\Omega_2\tau})}. \tag{A3.4.12}$$

Solving the third characteristic pair

We now turn to the last pair in equation (A3.4.1) which we represent as

$$\frac{d\tilde{U}}{d\tau} + \left\{(\sigma_1^2 - \Phi_1)s_1 + (\sigma_2^2 - \Phi_2)s_2 + i\eta(r-q) - \Theta_1 - \Theta_2\right\}\tilde{U}$$
$$= f_1(\tau, s_2) + f_2(\tau, s_1), \tag{A3.4.13}$$

where s_1 and s_2 are given by equations (A3.4.7) and (A3.4.12) respectively. The integrating factor of equation (A3.4.13) is

$$R(\tau) = \exp\left(\int \Big\{(\sigma_1^2 - \Phi_1)s_1 + (\sigma_2^2 - \Phi_2)s_2 + i\eta(r-q) - \Theta_1 - \Theta_2\Big\} d\tau\right). \quad \text{(A3.4.14)}$$

The integral inside the exponent can be simplified as

$$\begin{aligned}
&\int\Bigg((\sigma_1^2 - \Phi_1)\left[\frac{\Theta_1 - \Omega_1}{\sigma_1^2} - \frac{2\Omega_1 e^{-\Omega_1\tau}}{\sigma_1^2(c_3 - e^{-\Omega_1\tau})}\right] + (\sigma_2^2 - \Phi_2)\left[\frac{\Theta_2 - \Omega_2}{\sigma_2^2} - \frac{2\Omega_2 e^{-\Omega_2\tau}}{\sigma_2^2(d_3 - e^{-\Omega_2\tau})}\right] \\
&\quad + i\eta(r-q) - \Theta_1 - \Theta_2\Bigg) d\tau \\
&= \left\{\frac{(\sigma_1^2 - \Phi_1)(\Theta_1 - \Omega_1)}{\sigma_1^2} + \frac{(\sigma_2^2 - \Phi_2)(\Theta_2 - \Omega_2)}{\sigma_2^2} + i\eta(r-q) - \Theta_1 - \Theta_2\right\}\tau \\
&\quad - (\sigma_1^2 - \Phi_1)\int \frac{2\Omega_1 e^{-\Omega_1\tau}}{\sigma_1^2(c_3 - e^{-\Omega_1\tau})} d\tau - (\sigma_2^2 - \Phi_2)\int \frac{2\Omega_2 e^{-\Omega_2\tau}}{\sigma_2^2(d_3 - e^{-\Omega_2\tau})} d\tau \\
&= \left\{\frac{(\sigma_1^2 - \Phi_1)(\Theta_1 - \Omega_1)}{\sigma_1^2} + \frac{(\sigma_2^2 - \Phi_2)(\Theta_2 - \Omega_2)}{\sigma_2^2} + i\eta(r-q) - \Theta_1 - \Theta_2\right\}\tau \\
&\quad - \frac{2\Omega_1(\sigma_1^2 - \Phi_1)}{\sigma_1^2}\int \frac{e^{-\Omega_1\tau}}{c_3 - e^{-\Omega_1\tau}} d\tau - \frac{2\Omega_2(\sigma_2^2 - \Phi_2)}{\sigma_2^2}\int \frac{e^{-\Omega_2\tau}}{d_3 - e^{-\Omega_2\tau}} d\tau.
\end{aligned}$$

$$\text{(A3.4.15)}$$

In the first integral on the last line set $u_1 = c_3 - e^{-\Omega_1\tau}$, and in the second integral set $u_2 = d_3 - e^{-\Omega_2\tau}$. The two integral components of (A3.4.15) then simplify to

$$\begin{aligned}
\int \frac{e^{-\Omega_1\tau}}{c_3 - e^{-\Omega_1\tau}} d\tau &= \frac{1}{\Omega_1}\int \frac{du_1}{u_1} = \frac{1}{\Omega_1}\ln|u_1| \\
&= \frac{1}{\Omega_1}\ln|c_3 - e^{-\Omega_1\tau}|,
\end{aligned} \quad \text{(A3.4.16)}$$

and

$$\int \frac{e^{-\Omega_2\tau}}{d_3 - e^{-\Omega_2\tau}} d\tau = \frac{1}{\Omega_2}\ln|d_3 - e^{-\Omega_2\tau}|. \quad \text{(A3.4.17)}$$

Thus the integrating factor of (A3.4.13) can be represented as

$$R(\tau) = \left|\frac{1}{c_3 - e^{-\Omega_1\tau}}\right|^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)} \left|\frac{1}{d_3 - e^{-\Omega_2\tau}}\right|^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)} \tag{A3.4.18}$$
$$\times \exp\left\{\left[\frac{(\sigma_1^2-\Phi_1)(\Theta_1-\Omega_1)}{\sigma_1^2} + \frac{(\sigma_2^2-\Phi_2)(\Theta_2-\Omega_2)}{\sigma_2^2} + i\eta(r-q) - \Theta_1 - \Theta_2\right]\tau\right\}.$$

Now that we have the integrating factor of equation (A3.4.13), we can solve this equation by writing it as

$$\frac{d}{d\tau}\Big(R(\tau)\tilde{U}(\tau,\eta,s_1,s_2)\Big) = R(\tau)[f_1(\tau,s_2) + f_2(\tau,s_1)].$$

Integrating the above system yields

$$R(\tau)\tilde{U}(\tau,\eta,s_1,s_2) = \int_0^\tau R(t)[f_1(t,s_2) + f_2(t,s_1)]dt + c_4,$$

which implies that

$$\tilde{U}(\tau,\eta,s_1,s_2) = \frac{1}{R(\tau)}\left\{\int_0^\tau R(t)[f_1(t,s_2) + f_2(t,s_1)]dt + c_4\right\}. \tag{A3.4.19}$$

The above equation can be explicitly represented as

$$\tilde{U}(\tau,\eta,s_1,s_2) = \exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2} + \frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2} - i\eta(r-q) + \Theta_1 + \Theta_2\right]\tau\right\}$$
$$\times \left|c_3 - e^{-\Omega_1\tau}\right|^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)} \left|d_3 - e^{-\Omega_2\tau}\right|^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)}$$
$$\times \left\{\int_0^\tau \Big[f_1(t,s_2) + f_2(t,s_1)\Big]\right.$$
$$\times \exp\left(\left[\frac{(\sigma_1^2-\Phi_1)(\Theta_1-\Omega_1)}{\sigma_1^2} + \frac{(\sigma_2^2-\Phi_2)(\Theta_2-\Omega_2)}{\sigma_2^2} + i\eta(r-q) - \Theta_1 - \Theta_2\right]t\right)$$
$$\left.\times \left|\frac{1}{c_3 - e^{-\Omega_1 t}}\right|^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)} \left|\frac{1}{d_3 - e^{-\Omega_2 t}}\right|^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)} dt + c_4\right\}. \tag{A3.4.20}$$

Here, c_4 is a constant of integration whose value is determined by use of the initial condition, that is when $\tau = 0$.

Determining the integration constant, c_4.

The constant, c_4, is a function of two constants namely c_3 and d_3 which are given by (A3.4.6) and (A3.4.11) respectively. By letting $c_4 = A(c_3, d_3)$ at $\tau = 0$, it can be readily shown that equation (A3.4.20) becomes

$$\begin{aligned} \tilde{U}(0, \eta, s_1, s_2) &= A(c_3, d_3) \\ &\times \left| \frac{\sigma_1^2 s_1 - \Theta_1 - \Omega_1}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1} - 1 \right|^{\frac{2}{\sigma_1^2}(\sigma_1^2 - \Phi_1)} \left| \frac{\sigma_2^2 s_2 - \Theta_2 - \Omega_2}{\sigma_2^2 s_2 - \Theta_2 + \Omega_2} - 1 \right|^{\frac{2}{\sigma_2^2}(\sigma_2^2 - \Phi_2)} . \end{aligned} \tag{A3.4.21}$$

By substituting the values of s_1 and s_2 from equations (A3.4.7) and (A3.4.12) at $\tau = 0$ and making $A(c_3, d_3)$ the subject of formula we obtain

$$\begin{aligned} A(c_3, d_3) &= \left| c_3 - 1 \right|^{-\frac{2}{\sigma_1^2}(\sigma_1^2 - \Phi_1)} \left| d_3 - 1 \right|^{-\frac{2}{\sigma_2^2}(\sigma_2^2 - \Phi_2)} \\ &\times \tilde{U}\left(0, \eta, \frac{\Theta_1 - \Omega_1}{\sigma_1^2} - \frac{2\Omega_1}{\sigma_1^2(c_3 - 1)}, \frac{\Theta_2 - \Omega_2}{\sigma_2^2} - \frac{2\Omega_2}{\sigma_2^2(d_3 - 1)}\right). \end{aligned} \tag{A3.4.22}$$

Having determined $A(c_3, d_3)$, the expression involving the constant term in equation (A3.4.20) can be written as

$$\begin{aligned} &\left| c_3 - e^{-\Omega_1 \tau} \right|^{\frac{2}{\sigma_1^2}(\sigma_1^2 - \Phi_1)} \left| d_3 - e^{-\Omega_2 \tau} \right|^{\frac{2}{\sigma_2^2}(\sigma_2^2 - \Phi_2)} A(c_3, d_3) \\ &\quad = \left| \frac{2\Omega_1 e^{-\Omega_1 \tau}}{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)(1 - e^{-\Omega_1 \tau}) + 2\Omega_1 e^{-\Omega_1 \tau}} \right|^{\frac{2}{\sigma_1^2}(\sigma_1^2 - \Phi_1)} \\ &\quad\quad \times \left| \frac{2\Omega_2 e^{-\Omega_2 \tau}}{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)(1 - e^{-\Omega_2 \tau}) + 2\Omega_2 e^{-\Omega_2 \tau}} \right|^{\frac{2}{\sigma_2^2}(\sigma_2^2 - \Phi_2)} \\ &\quad\quad \times \tilde{U}\left(0, \eta, \frac{\Theta_1 - \Omega_1}{\sigma_1^2} - \frac{2\Omega_1}{\sigma_1^2(c_3 - 1)}, \frac{\Theta_2 - \Omega_2}{\sigma_2^2} - \frac{2\Omega_2}{\sigma_2^2(d_3 - 1)}\right). \end{aligned} \tag{A3.4.23}$$

With the knowledge of the two constants, c_3 and d_3 as in equations (A3.4.6) and (A3.4.11), it can be shown that the expressions occurring in the arguments of $\tilde{U}$ in (A3.4.23) can be expressed as

$$\frac{2\Omega_1}{\sigma_1^2(c_3-1)} = \frac{2\Omega_1(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)}{\sigma_1^2[(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)(e^{-\Omega_1\tau}-1) - 2\Omega_1 e^{-\Omega_1\tau}]}, \tag{A3.4.24}$$

and

$$\frac{2\Omega_2}{\sigma_2^2(d_3-1)} = \frac{2\Omega_2(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)}{\sigma_2^2[(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)(e^{-\Omega_2\tau}-1) - 2\Omega_2 e^{-\Omega_2\tau}]}. \tag{A3.4.25}$$

Also by use of equations (A3.4.6) and (A3.4.11), for $0 \le t \le \tau$ we have

$$\begin{aligned}
&\left|\frac{c_3 - e^{-\Omega_1\tau}}{c_3 - e^{-\Omega_1 t}}\right|^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)} \left|\frac{d_3 - e^{-\Omega_2\tau}}{d_3 - e^{-\Omega_2 t}}\right|^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)} \\
&\qquad = \left|\frac{2\Omega_1 e^{-\Omega_1\tau}}{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)(e^{-\Omega_1 t} - e^{-\Omega_1\tau}) + 2\Omega_1 e^{-\Omega_1\tau}}\right|^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)} \\
&\qquad \times \left|\frac{2\Omega_2 e^{-\Omega_2\tau}}{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)(e^{-\Omega_2 t} - e^{-\Omega_2\tau}) + 2\Omega_2 e^{-\Omega_2\tau}}\right|^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)},
\end{aligned} \tag{A3.4.26}$$

and it turns out that all real arguments in $|\cdot|$ are all positive. Substituting (A3.4.23) and (A3.4.26) into (A3.4.20) we obtain the expression for the transform as

$$
\begin{aligned}
\tilde{U}(\tau,\eta,s_1,s_2) = \exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right]\tau\right\} \\
\times\left(\frac{2\Omega_1 e^{-\Omega_1\tau}}{(\sigma_1^2 s_1-\Theta_1+\Omega_1)(1-e^{-\Omega_1\tau})+2\Omega_1 e^{-\Omega_1\tau}}\right)^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)} \\
\times\left(\frac{2\Omega_2 e^{-\Omega_2\tau}}{(\sigma_2^2 s_2-\Theta_2+\Omega_2)(1-e^{-\Omega_2\tau})+2\Omega_2 e^{-\Omega_2\tau}}\right)^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)} \\
\times\tilde{U}\Bigg(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2}-\frac{2\Omega_1(\sigma_1^2 s_1-\Theta_1+\Omega_1)}{\sigma_1^2[(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{-\Omega_1\tau}-1)-2\Omega_1 e^{-\Omega_1\tau}]}, \\
\frac{\Theta_2-\Omega_2}{\sigma_2^2}-\frac{2\Omega_2(\sigma_2^2 s_2-\Theta_2+\Omega_2)}{\sigma_2^2[(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{-\Omega_2\tau}-1)-2\Omega_2 e^{-\Omega_2\tau}]}\Bigg) \\
+\int_0^\tau\Big[f_1(t,s_2)+f_2(t,s_1)\Big] \\
\times\exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right](\tau-t)\right\} \\
\times\left(\frac{2\Omega_1 e^{-\Omega_1\tau}}{(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{-\Omega_1 t}-e^{-\Omega_1\tau})+2\Omega_1 e^{-\Omega_1\tau}}\right)^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)} \\
\times\left(\frac{2\Omega_2 e^{-\Omega_2\tau}}{(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{-\Omega_2 t}-e^{-\Omega_2\tau})+2\Omega_2 e^{-\Omega_2\tau}}\right)^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)} dt. \qquad \text{(A3.4.27)}
\end{aligned}
$$

This expression for the transform still involves the yet unknown functions $f_1(t,s_2)$ and $f_2(t,s_1)$. We next discuss how to obtain these functions.

(2) **Determining the functional forms of $f_1(\tau,s_2)$ and $f_2(\tau,s_1)$:**

The task of finding the functional forms of $f_1(\tau,s_2)$ and $f_2(\tau,s_1)$ is accomplished by using the conditions in equation (3.4.3). We first tackle the $f_1(\tau,s_2)$ component. As $s_1\to\infty$ and making use of l'Hôpital's rule, equation (A3.4.27) simplifies to

$$-\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2}-\frac{2\Omega_1}{\sigma_1^2(e^{-\Omega_1\tau}-1)},\frac{\Theta_2-\Omega_2}{\sigma_2^2}-\frac{2\Omega_2(\sigma_2^2s_2-\Theta_2+\Omega_2)}{\sigma_2^2[(\sigma_2^2s_2-\Theta_2+\Omega_2)(e^{-\Omega_2\tau}-1)-2\Omega_2e^{-\Omega_2\tau}]}\right)$$
$$=\int_0^\tau f_1(t,s_2)\exp\left\{-\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right]t\right\}$$
$$\times\left(\frac{1-e^{-\Omega_1\tau}}{e^{-\Omega_1t}-e^{-\Omega_1\tau}}\right)^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)}\left(\frac{(\sigma_2^2s_2-\theta_2+\Omega_2)(1-e^{-\Omega_2\tau})+2\Omega_2e^{-\Omega_2\tau}}{(\sigma_2^2s_2-\theta_2+\Omega_2)(e^{-\Omega_2t}-e^{-\Omega_2\tau})+2\Omega_2e^{-\Omega_2\tau}}\right)^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)}dt. \tag{A3.4.28}$$

Now let

$$\zeta_1^{-1}=1-e^{-\Omega_1t},\quad \zeta_2^{-1}=1-e^{-\Omega_2t}, \tag{A3.4.29}$$

$$z_1^{-1}=1-e^{-\Omega_1\tau},\quad z_2^{-1}=1-e^{-\Omega_2\tau} \tag{A3.4.30}$$

We substitute these arguments into equation (A3.4.28) and defining the function

$$g_1(\zeta_1)=f_1(t,s_2)\exp\left\{-\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right]t\right\}$$
$$\times\left(\frac{\zeta_2[(\sigma_2^2s_2-\Theta_2+\Omega_2)+2\Omega_2(z_2-1)]}{(\sigma_2^2s_2-\Theta_2+\Omega_2)(\zeta_2-z_2)+2\Omega_2\zeta_2(z_2-1)}\right)^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)}\frac{\zeta_1^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)}}{\zeta_1(\zeta_1-1)}. \tag{A3.4.31}$$

which constitutes the terms inside the integrand of equation (A3.4.28) after factoring the substitutions in equations (A3.4.29) and (A3.4.30). Equation (A3.4.28) becomes after rearranging

$$\int_{z_1}^\infty g_1(\zeta_1)(\zeta_1-z_1)^{\frac{2}{\sigma_1^2}(\Phi_1-\sigma_1^2)}d\zeta_1$$
$$=-\Omega_1\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1z_1}{\sigma_1^2},\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2s_2-\Theta_2+\Omega_2)2\Omega_2z_2}{[\sigma_2^2s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right). \tag{A3.4.32}$$

We can obtain another expression for

$$\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2},\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right)$$

appearing in equation (A3.4.32) from the definition of the Laplace Transform. From Definition 1.3.3 we can write the Laplace transform on the RHS of equation (A3.4.32) as

$$\begin{aligned}&\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2},\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right)\\&=\int_0^\infty\int_0^\infty \exp\left\{-\left[\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2}\right]v_1-\left[\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right]v_2\right\}\\&\times\hat{U}(0,\eta,v_1,v_2)dv_1dv_2. \qquad \text{(A3.4.33)}\end{aligned}$$

For convenience, we introduce a gamma function such that[8]

$$\begin{aligned}&\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2},\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right)\\&=\frac{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)}\int_0^\infty\int_0^\infty \hat{U}(0,\eta,v_1,v_2)\exp\left\{-\left[\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2}\right]v_1\right\}\\&\times\exp\left\{-\left[\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right]v_2\right\}dv_1dv_2. \qquad \text{(A3.4.34)}\end{aligned}$$

[8]The choice of the gamma function $\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)$ may seem arbitrary as it seems we could have chosen $\Gamma(\beta_1)$, for any β_1. However it turns out that to make equation (A3.4.34) match with (A3.4.32) we need to take $\beta_1=\frac{2\Phi_1}{\sigma_1^2}-1$.

Further manipulations yield

$$\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2},\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right)$$

$$=\frac{1}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-a_1}a_1^{\frac{2\Phi_1}{\sigma_1^2}-2}\hat{U}(0,\eta,v_1,v_2)\exp\left\{-\left[\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2}\right]v_1\right\}$$

$$\times\exp\left\{-\left[\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right]v_2\right\}da_1dv_1dv_2. \tag{A3.4.35}$$

Now, we make the substitution $a_1=\left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right)y_1$ in equation (A3.4.35) and obtain

$$\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2},\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right)$$

$$=\frac{1}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)}\int_0^\infty\int_0^\infty\int_0^\infty\exp\left\{-\left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right)y_1\right\}\left(\frac{2\Omega_1 v_1}{\sigma_1^2}y_1\right)^{\frac{2\Phi_1}{\sigma_1^2}-2}\hat{U}(0,\eta,v_1,v_2)$$

$$\times\exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2}\right)v_1\right\}\left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right)$$

$$\times\exp\left\{-\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right)v_2\right\}dy_1dv_1dv_2. \tag{A3.4.36}$$

Rearranging equation (A3.4.36) yields

$$\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2},\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right)$$
$$=\frac{1}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)}\int_0^\infty\int_0^\infty \hat{U}(0,\eta,v_1,v_2)\exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_1\right\}$$
$$\times\exp\left\{-\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right)v_2\right\}$$
$$\times\left[\int_0^\infty \exp\left\{-\left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right)y_1\right\}\left(\frac{2\Omega_1 v_1}{\sigma_1^2}y_1\right)^{\frac{2\Phi_1}{\sigma_1^2}-2}\exp\left\{-\left(\frac{2\Omega_1 z_1}{\sigma_1^2}\right)v_1\right\}dy_1\right]dv_1 dv_2. \tag{A3.4.37}$$

The terms inside the third integral component of equation (A3.4.37) can further be rearranged to yield

$$\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{2\Omega_1 z_1}{\sigma_1^2},\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right)$$
$$=\frac{1}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)}\int_0^\infty\int_0^\infty \hat{U}(0,\eta,v_1,v_2)\exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_1\right\}$$
$$\times\exp\left\{-\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}+\frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right)v_2\right\}\left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right)^{\frac{2\Phi_1}{\sigma_1^2}-1}$$
$$\times\left[\int_0^\infty \exp\left\{-\left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right)(y_1+z_1)\right\}y_1^{\frac{2\Phi_1}{\sigma_1^2}-2}dy_1\right]dv_1 dv_2. \tag{A3.4.38}$$

We make one further transformation by letting $\varrho_1 = y_1 + z_1$. Incorporating this in equation (A3.4.38) we obtain

$$\tilde{U}\left(0, \eta, \frac{\Theta_1 - \Omega_1}{\sigma_1^2} + \frac{2\Omega_1 z_1}{\sigma_1^2}, \frac{\Theta_2 - \Omega_2}{\sigma_2^2} + \frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)]\sigma_2^2}\right)$$
$$= \frac{1}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2} - 1\right)} \int_0^\infty \int_0^\infty \hat{U}(0, \eta, v_1, v_2) \exp\left\{ -\left(\frac{\Theta_1 - \Omega_1}{\sigma_1^2}\right) v_1 \right\} \left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right)^{\frac{2\Phi_1}{\sigma_1^2} - 1}$$
$$\times \exp\left\{ -\left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2} + \frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)]\sigma_2^2}\right) v_2 \right\}$$
$$\times \left[\int_{z_1}^\infty (\varrho_1 - z_1)^{\frac{2}{\sigma_1^2}(\Phi_1 - \sigma_1^2)} \exp\left\{ -\left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right) \varrho_1 \right\} \right] d\varrho_1 dv_1 dv_2. \tag{A3.4.39}$$

This last equation can be represented as

$$\tilde{U}\left(0, \eta, \frac{\Theta_1 - \Omega_1}{\sigma_1^2} + \frac{2\Omega_1 z_1}{\sigma_1^2}, \frac{\Theta_2 - \Omega_2}{\sigma_2^2} + \frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)]\sigma_2^2}\right)$$
$$= \int_{z_1}^\infty (\varrho_1 - z_1)^{\frac{2}{\sigma_1^2}(\Phi_1 - \sigma_1^2)} \left[\int_0^\infty \int_0^\infty \frac{\hat{U}(0, \eta, v_1, v_2)}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2} - 1\right)} \left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right)^{\frac{2\Phi_1}{\sigma_1^2} - 1} \exp\left\{ -\left(\frac{\Theta_1 - \Omega_1}{\sigma_1^2} + \frac{2\Omega_1 \varrho_1}{\sigma_1^2}\right) v_1 \right\}\right.$$
$$\left.\times \exp\left\{ -\left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2} + \frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)]\sigma_2^2}\right) v_2 \right\} dv_1 dv_2 \right] d\varrho_1. \tag{A3.4.40}$$

By comparing equations (A3.4.32) and (A3.4.40) we have in fact shown that

$$g_1(\zeta_1) = -\Omega_1 \int_0^\infty \int_0^\infty \frac{\hat{U}(0, \eta, v_1, v_2)}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2} - 1\right)} \left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right)^{\frac{2\Phi_1}{\sigma_1^2} - 1} \exp\left\{ -\left(\frac{\Theta_1 - \Omega_1}{\sigma_1^2} + \frac{2\Omega_1 \zeta_1}{\sigma_1^2}\right) v_1 \right\}$$
$$\times \exp\left\{ -\left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2} + \frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)]\sigma_2^2}\right) v_2 \right\} dv_1 dv_2 \tag{A3.4.41}$$

We recall from equation (3.3.3) that the initial condition is expressed as

$$\hat{U}(0, \eta, v_1, v_2) = e^{i\eta x_0} \delta(v_1 - v_{1,0}) \delta(v_2 - v_{2,0}). \tag{A3.4.42}$$

Substituting this into equation (A3.4.41) we obtain

$$g_1(\zeta_1) = -\Omega_1 \int_0^\infty \int_0^\infty \frac{\delta(v_1 - v_{1,0})\delta(v_2 - v_{2,0})}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2} - 1\right)} \left(\frac{2\Omega_1 v_1}{\sigma_1^2}\right)^{\frac{2\Phi_1}{\sigma_1^2}-1} \exp\left\{ - \left(\frac{\Theta_1 - \Omega_1}{\sigma_1^2} + \frac{2\Omega_1\zeta_1}{\sigma_1^2}\right) v_1 + i\eta x_0 \right\}$$
$$\times \exp\left\{ - \left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2} + \frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)]\sigma_2^2}\right) v_2 \right\} dv_1 dv_2. \qquad \text{(A3.4.43)}$$

Using properties of the delta functions the above expression simplifies to

$$g_1(\zeta_1) = \frac{-\Omega_1}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2} - 1\right)} \left(\frac{2\Omega_1 v_{1,0}}{\sigma_1^2}\right)^{\frac{2\Phi_1}{\sigma_1^2}-1} \exp\left\{ - \left(\frac{\Theta_1 - \Omega_1}{\sigma_1^2} + \frac{2\Omega_1\zeta_1}{\sigma_1^2}\right) v_{1,0} + i\eta x_0 \right\}$$
$$\times \exp\left\{ - \left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2} + \frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)]\sigma_2^2}\right) v_{2,0} \right\}. \qquad \text{(A3.4.44)}$$

Given the explicit representation of $g_1(\zeta_1)$ we can now find the explicit form of the function $f_1(t, s_2)$ by comparing equations (A3.4.31) and (A3.4.44) such that[9]

$$f_1(t, s_2) = \frac{-\Omega_1}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2} - 1\right)} \left(\frac{2\Omega_1 v_{1,0}}{\sigma_1^2}\right)^{\frac{2\Phi_1}{\sigma_1^2}-1} \exp\left\{ - \left(\frac{\Theta_1 - \Omega_1}{\sigma_1^2} + \frac{2\Omega_1\zeta_1}{\sigma_1^2}\right) v_{1,0} + i\eta x_0 \right\}$$
$$\times \exp\left\{ - \left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2} + \frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)]\sigma_2^2}\right) v_{2,0} \right\}$$
$$\times \exp\left\{ \left[\frac{(\Phi_1 - \sigma_1^2)(\Theta_1 - \Omega_1)}{\sigma_1^2} + \frac{(\Phi_2 - \sigma_2^2)(\Theta_2 - \Omega_2)}{\sigma_2^2} - i\eta(r - q) + \Theta_1 + \Theta_2 \right] t \right\}$$
$$\times \left(\frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)(\zeta_2 - z_2) + 2\Omega_2\zeta_2(z_2 - 1)}{\zeta_2[(\sigma_2^2 s_2 - \Theta_2 + \Omega_2) + 2\Omega_2(z_2 - 1)]} \right)^{\frac{2}{\sigma_2^2}(\sigma_2^2 - \Phi_2)} \frac{\zeta_1(\zeta_1 - 1)}{\zeta_1^{\frac{2}{\sigma_1^2}(\sigma_1^2 - \Phi_1)}}. \qquad \text{(A3.4.45)}$$

[9]In actual fact, from equations (A3.4.7) and (A3.4.12) $s_j = s_j(\tau)$ and from equation (A3.4.29) $\zeta_j = \zeta_j(t)$ for $j = 1, 2$. We suppress the dependence on time for convenience.

By performing similar operations it can be shown that

$$
\begin{aligned}
f_2(t,s_1) = {} & \frac{-\Omega_2}{\Gamma\left(\frac{2\Phi_2}{\sigma_2^2}-1\right)}\left(\frac{2\Omega_2 v_{2,0}}{\sigma_2^2}\right)^{\frac{2\Phi_2}{\sigma_2^2}-1} \exp\left\{-\left(\frac{\Theta_1-\Omega_2}{\sigma_2^2}+\frac{2\Omega_2\zeta_2}{\sigma_2^2}\right)v_{2,0}+i\eta x_0\right\} \\
& \times \exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}+\frac{(\sigma_1^2 s_1-\Theta_1+\Omega_1)2\Omega_1 z_1}{[\sigma_1^2 s_1-\Theta_1+\Omega_1+2\Omega_1(z_1-1)]\sigma_1^2}\right)v_{1,0}\right\} \\
& \times \exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right]t\right\} \\
& \times \left(\frac{(\sigma_1^2 s_1-\Theta_1+\Omega_1)(\zeta_1-z_1)+2\Omega_1\zeta_1(z_1-1)}{\zeta_1[(\sigma_1^2 s_1-\Theta_1+\Omega_1)+2\Omega_1(z_1-1)]}\right)^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)} \frac{\zeta_2(\zeta_2-1)}{\zeta_2^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)}}.
\end{aligned}
\tag{A3.4.46}
$$

(3) <u>**Deriving the explicit representation of $\tilde{U}(\tau,\eta,s_1,s_2)$:**</u>

Now that we have found the two unknown functions namely $f_1(t,s_2)$ and $f_2(t,s_1)$, the next step is to substitute these two functions into equation (A3.4.27) in order for us to finally obtain the representation of the transform. We are going to do this in three steps. We break equation (A3.4.27) into three parts. The first part being the first term on the RHS of (A3.4.27), the second part is the term involving $f_1(t,s_2)$ and the third part is the one involving the $f_2(t,s_1)$ term.

The ***first component*** on the RHS of (A3.4.27) can be represented as

$$J_1 = \exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2} + \frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2} - i\eta(r-q) + \Theta_1 + \Theta_2\right]\tau\right\}$$
$$\times\left(\frac{2\Omega_1 e^{-\Omega_1\tau}}{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)(1-e^{-\Omega_1\tau}) + 2\Omega_1 e^{-\Omega_1\tau}}\right)^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)}$$
$$\times\left(\frac{2\Omega_2 e^{-\Omega_2\tau}}{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)(1-e^{-\Omega_2\tau}) + 2\Omega_2 e^{-\Omega_2\tau}}\right)^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)}$$
$$\times\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2} - \frac{2\Omega_1(\sigma_1^2 s_1 - \Theta_1+\Omega_1)}{\sigma_1^2[(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{-\Omega_1\tau}-1) - 2\Omega_1 e^{-\Omega_1\tau}]},\right.$$
$$\left.\frac{\Theta_2-\Omega_2}{\sigma_2^2} - \frac{2\Omega_2(\sigma_2^2 s_2 - \Theta_2+\Omega_2)}{\sigma_2^2[(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{-\Omega_2\tau}-1) - 2\Omega_2 e^{-\Omega_2\tau}]}\right). \quad \text{(A3.4.47)}$$

Making use of equation (A3.4.30) we obtain

$$J_1 = \exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2} + \frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2} - i\eta(r-q) + \Theta_1 + \Theta_2\right]\tau\right\}$$
$$\times\left(\frac{2\Omega_1(z_1-1)}{(\sigma_1^2 s_1-\Theta_1+\Omega_1)+2\Omega_1(z_1-1)}\right)^{2-\frac{2\Phi_1}{\sigma_1^2}}\left(\frac{2\Omega_2(z_2-1)}{(\sigma_2^2 s_2-\Theta_2+\Omega_2)+2\Omega_2(z_2-1)}\right)^{2-\frac{2\Phi_2}{\sigma_2^2}}$$
$$\times\tilde{U}\left(0,\eta,\frac{\Theta_1-\Omega_1}{\sigma_1^2} + \frac{2\Omega_1(\sigma_1^2 s_1-\Theta_1+\Omega_1)z_1}{\sigma_1^2[(\sigma_1^2 s_1-\Theta_1+\Omega_1)+2\Omega_1(z_1-1)]}, \frac{\Theta_2-\Omega_2}{\sigma_2^2} + \frac{2\Omega_2(\sigma_2^2 s_2-\Theta_2+\Omega_2)z_2}{\sigma_2^2[(\sigma_2^2 s_2-\Theta_2+\Omega_2)+2\Omega_2(z_2-1)]}\right).$$
$$\text{(A3.4.48)}$$

Applying the initial condition (3.4.2) to equation (A3.4.48) yields

$$
\begin{aligned}
J_1 = & \left(\frac{2\Omega_1(z_1-1)}{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1) + 2\Omega_1(z_1-1)} \right)^{2-\frac{2\Phi_1}{\sigma_1^2}} \left(\frac{2\Omega_2(z_2-1)}{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2) + 2\Omega_2(z_2-1)} \right)^{2-\frac{2\Phi_2}{\sigma_2^2}} \\
& \times \exp\left\{ \left[\frac{(\Phi_1 - \sigma_1^2)(\Theta_1 - \Omega_1)}{\sigma_1^2} + \frac{(\Phi_2 - \sigma_2^2)(\Theta_2 - \Omega_2)}{\sigma_2^2} - i\eta(r-q) + \Theta_1 + \Theta_2 \right] \tau \right\} \\
& \times \exp\left\{ - \left(\frac{\Theta_1 - \Omega_1}{\sigma_1^2} \right) v_{1,0} - \left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2} \right) v_{2,0} + i\eta x_0 \right\} \\
& \times \exp\left\{ \frac{-2\Omega_1 v_{1,0}(\sigma_1^2 s_1 - \Theta_1 + \Omega_1) z_1}{\sigma_1^2[(\sigma_1^2 s_1 - \Theta_1 + \Omega_1) + 2\Omega_1(z_1-1)]} \right\} \exp\left\{ \frac{-2\Omega_2 v_{2,0}(\sigma_2^2 s_2 - \Theta_2 + \Omega_2) z_2}{\sigma_2^2[(\sigma_2^2 s_2 - \Theta_2 + \Omega_2) + 2\Omega_2(z_2-1)]} \right\}
\end{aligned}
\tag{A3.4.49}
$$

The ***second component*** is here represented as[10]

$$
\begin{aligned}
J_2 = & \frac{1}{\Omega_1} \int_{z_1}^{\infty} f_1(t, s_2) \exp\left\{ \left[\frac{(\Phi_1 - \sigma_1^2)(\Theta_1 - \Omega_1)}{\sigma_1^2} + \frac{(\Phi_2 - \sigma_2^2)(\Theta_2 - \Omega_2)}{\sigma_2^2} - i\eta(r-q) + \Theta_1 + \Theta_2 \right] (\tau - t) \right\} \\
& \times \left(\frac{2\Omega_1 \zeta_1 (z_1 - 1)}{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)(\zeta_1 - z_1) + 2\Omega_1 \zeta_1 (z_1 - 1)} \right)^{\frac{2}{\sigma_1^2}(\sigma_1^2 - \Phi_1)} \\
& \times \left(\frac{2\Omega_2 \zeta_2 (z_2 - 1)}{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)(\zeta_2 - z_2) + 2\Omega_2 \zeta_2 (z_2 - 1)} \right)^{\frac{2}{\sigma_2^2}(\sigma_2^2 - \Phi_2)} \frac{d\zeta_1}{\zeta_1(\zeta_1 - 1)}.
\end{aligned}
\tag{A3.4.50}
$$

[10]We recall the link between ζ_1 and t from (A3.4.29) and that between z_1 and τ from (A3.4.30).

Now by substituting the value of $f_1(t, s_2)$ in equation (A3.4.45) into equation (A3.4.50) we obtain

$$J_2 = \frac{-1}{\Gamma\big(\frac{2\Phi_1}{\sigma_1^2} - 1\big)} \int_{z_1}^{\infty} \Big(\frac{2\Omega_1 v_{1,0}}{\sigma_1^2}\Big)^{\frac{2\Phi_1}{\sigma_1^2}-1} \exp\left\{ -\left(\frac{\Theta_1 - \Omega_1}{\sigma_1^2} + \frac{2\Omega_1\zeta_1}{\sigma_1^2}\right) v_{1,0} + i\eta x_0 \right\}$$
$$\times \exp\left\{ -\left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2} + \frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)]\sigma_2^2}\right) v_{2,0} \right\}$$
$$\times \exp\left\{ \left[\frac{(\Phi_1 - \sigma_1^2)(\Theta_1 - \Omega_1)}{\sigma_1^2} + \frac{(\Phi_2 - \sigma_2^2)(\Theta_2 - \Omega_2)}{\sigma_2^2} - i\eta(r - q) + \Theta_1 + \Theta_2 \right] \tau \right\}$$
$$\times \left(\frac{2\Omega_1(z_1 - 1)}{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)(\zeta_1 - z_1) + 2\Omega_1\zeta_1(z_1 - 1)} \right)^{\frac{2}{\sigma_1^2}(\sigma_1^2 - \Phi_1)}$$
$$\times \left(\frac{2\Omega_2(z_2 - 1)}{\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)} \right)^{\frac{2}{\sigma_2^2}(\sigma_2^2 - \Phi_2)} d\zeta_1. \tag{A3.4.51}$$

By rearranging the respective components of equation (A3.4.51) we obtain

$$J_2 = \frac{-\big[2\Omega_1(z_1 - 1)\big]^{2 - \frac{2\Phi_1}{\sigma_1^2}}}{\Gamma\big(\frac{2\Phi_1}{\sigma_1^2} - 1\big)} \Big(\frac{2\Omega_1 v_{1,0}}{\sigma_1^2}\Big)^{\frac{2\Phi_1}{\sigma_1^2}-1} \exp\Big\{ -\Big(\frac{\Theta_1 - \Omega_1}{\sigma_1^2}\Big) v_{1,0} + i\eta x_0 \Big\}$$
$$\times \exp\left\{ -\left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2} + \frac{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)2\Omega_2 z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)]\sigma_2^2}\right) v_{2,0} \right\}$$
$$\times \exp\left\{ \left[\frac{(\Phi_1 - \sigma_1^2)(\Theta_1 - \Omega_1)}{\sigma_1^2} + \frac{(\Phi_2 - \sigma_2^2)(\Theta_2 - \Omega_2)}{\sigma_2^2} - i\eta(r - q) + \Theta_1 + \Theta_2 \right] \tau \right\}$$
$$\times \left(\frac{2\Omega_2(z_2 - 1)}{\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2 - 1)} \right)^{\frac{2}{\sigma_2^2}(\sigma_2^2 - \Phi_2)} G_1(v_{1,0}), \tag{A3.4.52}$$

where

$$G_1(v_1) = \int_{z_1}^{\infty} e^{-\frac{2\Omega_1 v_1}{\sigma_1^2}\zeta_1} [(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)(\zeta_1 - z_1) + 2\Omega_1\zeta_1(z_1 - 1)]^{\frac{2\Phi_1}{\sigma_1^2} - 2} d\zeta_1. \tag{A3.4.53}$$

As a way of simplifying equation (A3.4.53), we let $y_1 = (\sigma_1^2 s_1 - \Theta_1 + \Omega_1)(\zeta_1 - z_1) + 2\Omega_1\zeta_1(z_1 - 1)$ so that

$$d\zeta_1 = \frac{dy_1}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)}.$$

Substituting this into equation (A3.4.53) and rearranging terms we obtain

$$G_1(v_1) = \frac{1}{\sigma_1^2 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)} \exp\left\{ - \frac{2\Omega_1 v_1}{\sigma_1^2} \left(\frac{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1) z_1}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)} \right) \right\}$$
$$\times \int_{2\Omega_1 z_1(z_1 - 1)}^{\infty} \exp\left\{ \frac{-2\Omega_1 v_1 y_1}{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)]} \right\} y_1^{\frac{2\Phi_1}{\sigma_1^2} - 2} dy_1. \qquad \text{(A3.4.54)}$$

Now let

$$\xi_1 = \frac{2\Omega_1 v_1 y_1}{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)]},$$

which implies that

$$dy_1 = \frac{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)]}{2\Omega_1 v_1} d\xi_1.$$

Substituting these into equation (A3.4.54) yields

$$G_1(v_1) = \frac{1}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)} \exp\left\{ - \frac{2\Omega_1 v_1}{\sigma_1^2} \left(\frac{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1) z_1}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)} \right) \right\}$$
$$\times \int_{\frac{4\Omega_1^2 v_1 z_1(z_1-1)}{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)]}}^{\infty} e^{-\xi_1} \left(\frac{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)]\xi_1}{2\Omega_1 v_1} \right)^{\frac{2\Phi_1}{\sigma_1^2} - 2}$$
$$\times \frac{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)]}{2\Omega_1 v_1} d\xi_1$$
$$= \frac{\sigma_1^2}{2\Omega_1 v_1} \exp\left\{ - \frac{2\Omega_1 v_1}{\sigma_1^2} \left(\frac{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1) z_1}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)} \right) \right\}$$
$$\times \left(\frac{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1 - 1)]}{2\Omega_1 v_1} \right)^{\frac{2\Phi_1}{\sigma_1^2} - 2} \int_{\frac{4\Omega_1^2 v_1 z_1(z_1-1)}{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)]}}^{\infty} e^{-\xi_1} \xi_1^{\left(\frac{2\Omega_1}{\sigma_1^2} - 1\right) - 1} d\xi_1.$$

$$\text{(A3.4.55)}$$

Rearranging and recalling the definition of the gamma function (see equation (3.5.4)) equation (A3.4.55) can be expressed as

$$G_1(v_1) = [\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)]^{\frac{2\Phi_1}{\sigma_1^2}-2}\left(\frac{\sigma_1^2}{2\Omega_1 v_1}\right)^{\frac{2\Phi_1}{\sigma_1^2}-1} \exp\left\{\frac{-2\Omega_1 v_1(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)z_1}{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)]}\right\}$$
$$\times \left[\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right) - \int_0^{\frac{4\Omega_1^2 v_1 z_1(z_1-1)}{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)]}} e^{-\xi_1}\xi_1^{\left(\frac{2\Omega_1}{\sigma_1^2}-1\right)-1} d\xi_1\right]. \tag{A3.4.56}$$

Substituting equation (A3.4.56) into (A3.4.52) and making use of equation (3.5.3) for the incomplete gamma function we obtain

$$J_2 = \frac{-1}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)}\left(\frac{2\Omega_1(z_1-1)}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)}\right)^{2-\frac{2\Phi_1}{\sigma_1^2}}\left(\frac{2\Omega_2(z_2-1)}{\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2-1)}\right)^{2-\frac{2\Phi_2}{\sigma_2^2}}$$
$$\times \exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_{1,0} - \left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)v_{2,0} + i\eta x_0\right\}$$
$$\times \exp\left\{\frac{-2\Omega_1 v_{1,0}(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)z_1}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)}\right\}\exp\left\{\frac{-2\Omega_2 v_{2,0}(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2-1)]\sigma_2^2}\right\}$$
$$\times \exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2} + \frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2} - i\eta(r-q) + \Theta_1 + \Theta_2\right]\tau\right\}$$
$$\times \Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)\left[1 - \Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1; \frac{4\Omega_1^2 v_{1,0} z_1(z_1-1)}{\sigma_1^2[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)]}\right)\right]. \tag{A3.4.57}$$

The ***third component*** may be represented as

$$J_3 = \frac{1}{\Omega_2}\int_{z_2}^{\infty} f_2(t, s_1)\exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2} + \frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2} - i\eta(r-q) + \Theta_1 + \Theta_2\right](\tau - t)\right\} \tag{A3.4.58}$$
$$\times \left(\frac{2\Omega_1\zeta_1(z_1-1)}{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)(\zeta_1 - z_1) + 2\Omega_1\zeta_1(z_1-1)}\right)^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)}$$
$$\times \left(\frac{2\Omega_2\zeta_2(z_2-1)}{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)(\zeta_2 - z_2) + 2\Omega_2\zeta_2(z_2-1)}\right)^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)}\frac{d\zeta_2}{\zeta_2(\zeta_2-1)}.$$

Now by substituting the value of $f_2(t,s_1)$ in equation (A3.4.46) into equation (A3.4.58) we obtain

$$J_3 = \frac{-1}{\Gamma\big(\frac{2\Phi_2}{\sigma_2^2}-1\big)} \int_{z_2}^{\infty} \Big(\frac{2\Omega_2 v_{2,0}}{\sigma_2^2}\Big)^{\frac{2\Phi_2}{\sigma_2^2}-1} \exp\left\{ -\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2} + \frac{2\Omega_2\zeta_2}{\sigma_2^2}\right) v_{2,0} + i\eta x_0 \right\}$$
$$\times \exp\left\{ -\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2} + \frac{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)2\Omega_1 z_1}{[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)]\sigma_1^2}\right) v_{1,0} \right\}$$
$$\times \exp\left\{ \left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2} + \frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2} - i\eta(r-q) + \Theta_1 + \Theta_2\right]\tau \right\}$$
$$\times \left(\frac{2\Omega_1(z_1-1)}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)}\right)^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)}$$
$$\times \left(\frac{2\Omega_2(z_2-1)}{(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)(\zeta_2 - z_2) + 2\Omega_2\zeta_2(z_2-1)}\right)^{\frac{2}{\sigma_2^2}(\sigma_2^2-\Phi_2)} d\zeta_2. \qquad \text{(A3.4.59)}$$

By proceeding as we did when handling the J_2 term in equation (A3.4.51) it can be shown that

$$J_3 = \frac{-\big[2\Omega_2(z_2-1)\big]^{2-\frac{2\Phi_2}{\sigma_2^2}}}{\Gamma\big(\frac{2\Phi_2}{\sigma_2^2}-1\big)} \Big(\frac{2\Omega_2 v_{2,0}}{\sigma_2^2}\Big)^{\frac{2\Phi_2}{\sigma_2^2}-1} \exp\left\{ -\Big(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\Big) v_{2,0} + i\eta x_0 \right\}$$
$$\times \exp\left\{ -\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2} + \frac{(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)2\Omega_1 z_1}{[\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)]\sigma_1^2}\right) v_{1,0} \right\}$$
$$\times \exp\left\{ \left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2} + \frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2} - i\eta(r-q) + \Theta_1 + \Theta_2\right]\tau \right\}$$
$$\times \left(\frac{2\Omega_1(z_1-1)}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)}\right)^{\frac{2}{\sigma_1^2}(\sigma_1^2-\Phi_1)} G_2(v_{2,0}), \qquad \text{(A3.4.60)}$$

where

$$G_2(v_2) = \int_{z_2}^{\infty} e^{-\frac{2\Omega_2 v_2}{\sigma_2^2}\zeta_2} \big[(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)(\zeta_2 - z_2) + 2\Omega_2\zeta_2(z_2-1)\big]^{\frac{2\Phi_2}{\sigma_2^2}-2} d\zeta_2. \qquad \text{(A3.4.61)}$$

Simplifying $G_2(v_2)$ in an analogous fashion to the way $G_1(v_1)$ was simplified we obtain

$$G_2(v_2) = [\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2-1)]^{\frac{2\Phi_2}{\sigma_2^2}-2}\left(\frac{\sigma_2^2}{2\Omega_2 v_2}\right)^{\frac{2\Phi_2}{\sigma_2^2}-1}\exp\left\{\frac{-2\Omega_2 v_2(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)z_2}{\sigma_2^2[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2-1)]}\right\}$$
$$\times\left[\Gamma\Big(\frac{2\Phi_2}{\sigma_2^2}-1\Big) - \int_0^{\frac{4\Omega_2^2 v_2 z_2(z_2-1)}{\sigma_2^2[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2-1)]}} e^{-\xi_2}\xi_2^{\left(\frac{2\Omega_2}{\sigma_2^2}-1\right)-1}d\xi_2\right]. \tag{A3.4.62}$$

Substituting equation (A3.4.62) into equation (A3.4.60) we obtain

$$J_3 = \frac{-1}{\Gamma\Big(\frac{2\Phi_2}{\sigma_2^2}-1\Big)}\left(\frac{2\Omega_1(z_1-1)}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)}\right)^{2-\frac{2\Phi_1}{\sigma_1^2}}\left(\frac{2\Omega_2(z_2-1)}{\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2-1)}\right)^{2-\frac{2\Phi_2}{\sigma_2^2}}$$
$$\times\exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_{1,0} - \left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)v_{2,0} + i\eta x_0\right\}$$
$$\times\exp\left\{\frac{-2\Omega_1 v_{1,0}(\sigma_1^2 s_1 - \Theta_1 + \Omega_1)z_1}{\sigma_1^2 s_1 - \Theta_1 + \Omega_1 + 2\Omega_1(z_1-1)}\right\}\exp\left\{\frac{-2\Omega_2 v_{2,0}(\sigma_2^2 s_2 - \Theta_2 + \Omega_2)z_2}{[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2-1)]\sigma_2^2}\right\}$$
$$\times\exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2} + \frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2} - i\eta(r-q) + \Theta_1 + \Theta_2\right]\tau\right\}$$
$$\times\Gamma\left(\frac{2\Phi_2}{\sigma_2^2}-1\right)\left[1-\Gamma\left(\frac{2\Phi_2}{\sigma_2^2}-1;\frac{4\Omega_2^2 v_{2,0}z_2(z_2-1)}{\sigma_2^2[\sigma_2^2 s_2 - \Theta_2 + \Omega_2 + 2\Omega_2(z_2-1)]}\right)\right]. \tag{A3.4.63}$$

By combining J_1, J_2 and J_3 equation (A3.4.27) becomes

$$\begin{aligned}\tilde{U}(\tau,\eta,s_1,s_2) &= \left(\frac{2\Omega_1(z_1-1)}{\sigma_1^2 s_1-\Theta_1+\Omega_1+2\Omega_1(z_1-1)}\right)^{2-\frac{2\Phi_1}{\sigma_1^2}}\left(\frac{2\Omega_2(z_2-1)}{\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)}\right)^{2-\frac{2\Phi_2}{\sigma_2^2}}\\
&\times\exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_{1,0}-\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)v_{2,0}+i\eta x_0\right\}\\
&\times\exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right]\tau\right\}\\
&\times\exp\left\{\frac{-2\Omega_1 v_{1,0}(\sigma_1^2 s_1-\Theta_1+\Omega_1)z_1}{\sigma_1^2[\sigma_1^2 s_1-\Theta_1+\Omega_1+2\Omega_1(z_1-1)]}\right\}\exp\left\{\frac{-2\Omega_2 v_{2,0}(\sigma_2^2 s_2-\Theta_2+\Omega_2)z_2}{[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]\sigma_2^2}\right\}\\
&\times\left[\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1;\frac{4\Omega_1^2 v_{1,0}z_1(z_1-1)}{\sigma_1^2[\sigma_1^2 s_1-\Theta_1+\Omega_1+2\Omega_1(z_1-1)]}\right)\right.\\
&\qquad\left.+\Gamma\left(\frac{2\Phi_2}{\sigma_2^2}-1;\frac{4\Omega_2^2 v_{2,0}z_2(z_2-1)}{\sigma_2^2[\sigma_2^2 s_2-\Theta_2+\Omega_2+2\Omega_2(z_2-1)]}\right)-1\right].
\end{aligned}\tag{A3.4.64}$$

We recall from equation (A3.4.30) that $z_1^{-1}=1-e^{-\Omega_1\tau}$ and $z_2^{-1}=1-e^{-\Omega_2\tau}$ where Ω_1 and Ω_2 have been defined in equations (A3.4.3) and (A3.4.8) respectively. Substituting these expressions into the above equation we finally obtain

$$\begin{aligned}\tilde{U}(\tau,\eta,s_1,s_2) &= \left(\frac{2\Omega_1}{(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{\Omega_1\tau}-1)+2\Omega_1}\right)^{2-\frac{2\Phi_1}{\sigma_1^2}}\left(\frac{2\Omega_2}{(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{\Omega_2\tau}-1)+2\Omega_2}\right)^{2-\frac{2\Phi_2}{\sigma_2^2}}\\
&\times\exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_{1,0}-\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)v_{2,0}+i\eta x_0\right\}\\
&\times\exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right]\tau\right\}\\
&\times\exp\left\{\frac{-2\Omega_1 v_{1,0}(\sigma_1^2 s_1-\Theta_1+\Omega_1)e^{\Omega_1\tau}}{\sigma_1^2[(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{\Omega_1\tau}-1)+2\Omega_1]}\right\}\exp\left\{\frac{-2\Omega_2 v_{2,0}(\sigma_2^2 s_2-\Theta_2+\Omega_2)e^{\Omega_2\tau}}{\sigma_2^2[(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{\Omega_2\tau}-1)+2\Omega_2]}\right\}\\
&\times\left[\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1;\frac{2\Omega_1 v_{1,0}e^{\Omega_1\tau}}{\sigma_1^2(e^{\Omega_1\tau}-1)}\times\frac{2\Omega_1}{(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{\Omega_1\tau}-1)+2\Omega_1}\right)\right.\\
&\qquad\left.+\Gamma\left(\frac{2\Phi_2}{\sigma_2^2}-1;\frac{2\Omega_2 v_{2,0}e^{\Omega_2\tau}}{\sigma_2^2(e^{\Omega_2\tau}-1)}\times\frac{2\Omega_2}{(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{\Omega_2\tau}-1)+2\Omega_2}\right)-1\right],
\end{aligned}\tag{A3.4.65}$$

which is the result presented in Proposition 3.5.1.

Appendix 3.5. Proof of Proposition 3.6.1

Our calculations are facilitated by carrying out the transformations

$$\begin{cases} A_1 &= \frac{2\Omega_1 v_{1,0}}{\sigma_1^2(1-e^{-\Omega_1\tau})}, \\ A_2 &= \frac{2\Omega_2 v_{2,0}}{\sigma_2^2(1-e^{-\Omega_2\tau})}, \end{cases} \tag{A3.5.1}$$

$$\begin{cases} z_1 &= \frac{(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{\Omega_1\tau}-1)+2\Omega_1}{2\Omega_1}, \\ z_2 &= \frac{(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{\Omega_2\tau}-1)+2\Omega_2}{2\Omega_2}, \end{cases} \tag{A3.5.2}$$

and

$$\begin{aligned} h(\tau,\eta,v_{1,0},v_{2,0}) = \exp&\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right]\tau\right\} \\ \times \exp&\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_{1,0}-\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)v_{2,0}+i\eta x_0\right\}. \end{aligned} \tag{A3.5.3}$$

Substituting these into equation (3.5.1) we obtain[11]

[11]Note that the system (A3.5.2) defines s_1 as a function of z_1 and s_2 as a function of z_2.

$$\begin{aligned}\tilde{U}(\tau,\eta,s_1(z_1),s_2(z_2)) &= h(\tau,\eta,v_{1,0},v_{2,0}) z_1^{\frac{2\Phi_1}{\sigma_1^2}-2} z_2^{\frac{2\Phi_2}{\sigma_2^2}-2} \exp\left\{-\frac{A_1}{z_1}(z_1-1)\right\}\exp\left\{-\frac{A_2}{z_2}(z_2-1)\right\} \\ &\quad\times\left[\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1;\frac{A_1}{z_1}\right)+\Gamma\left(\frac{2\Phi_2}{\sigma_2^2}-1;\frac{A_2}{z_2}\right)-1\right] \\ &= h(\tau,\eta,v_{1,0},v_{2,0}) z_1^{\frac{2\Phi_1}{\sigma_1^2}-2} z_2^{\frac{2\Phi_2}{\sigma_2^2}-2} \exp\left\{-\frac{A_1}{z_1}(z_1-1)\right\}\exp\left\{-\frac{A_2}{z_2}(z_2-1)\right\} \\ &\quad\times\left[\frac{1}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)}\int_0^{\frac{A_1}{z_1}} e^{-\beta_1}\beta_1^{\frac{2\Phi_1}{\sigma_1^2}-2}d\beta_1+\frac{1}{\Gamma\left(\frac{2\Phi_2}{\sigma_2^2}-1\right)}\int_0^{\frac{A_2}{z_2}} e^{-\beta_2}\beta_2^{\frac{2\Phi_2}{\sigma_2^2}-2}d\beta_2-1\right].\end{aligned} \tag{A3.5.4}$$

In order to evaluate equation (A3.5.4), we break it into three parts such that

$$\tilde{U}(\tau,\eta,s_1(z_1),s_2(z_2)) = \tilde{F}_1(\tau,\eta,s_1(z_1),s_2(z_2))+\tilde{F}_2(\tau,\eta,s_1(z_1),s_2(z_2))+\tilde{F}_3(\tau,\eta,s_1(z_1),s_2(z_2)), \tag{A3.5.5}$$

where

$$\begin{aligned}\tilde{F}_1(\tau,\eta,s_1(z_1),s_2(z_2)) &= h(\tau,\eta,v_{1,0},v_{2,0}) z_1^{\frac{2\Phi_1}{\sigma_1^2}-2} z_2^{\frac{2\Phi_2}{\sigma_2^2}-2} \exp\left\{-\frac{A_1}{z_1}(z_1-1)\right\}\exp\left\{-\frac{A_2}{z_2}(z_2-1)\right\} \\ &\quad\times\frac{1}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)}\int_0^{\frac{A_1}{z_1}} e^{-\beta_1}\beta_1^{\frac{2\Phi_1}{\sigma_1^2}-2}d\beta_1,\end{aligned} \tag{A3.5.6}$$

$$\begin{aligned}\tilde{F}_2(\tau,\eta,s_1(z_1),s_2(z_2)) &= h(\tau,\eta,v_{1,0},v_{2,0}) z_1^{\frac{2\Phi_1}{\sigma_1^2}-2} z_2^{\frac{2\Phi_2}{\sigma_2^2}-2} \exp\left\{-\frac{A_1}{z_1}(z_1-1)\right\}\exp\left\{-\frac{A_2}{z_2}(z_2-1)\right\} \\ &\quad\times\frac{1}{\Gamma\left(\frac{2\Phi_2}{\sigma_2^2}-1\right)}\int_0^{\frac{A_2}{z_2}} e^{-\beta_2}\beta_2^{\frac{2\Phi_2}{\sigma_2^2}-2}d\beta_2,\end{aligned} \tag{A3.5.7}$$

and

$$\tilde{F}_3(\tau,\eta,s_1(z_1),s_2(z_2)) = -h(\tau,\eta,v_{1,0},v_{2,0}) z_1^{\frac{2\Phi_1}{\sigma_1^2}-2} z_2^{\frac{2\Phi_2}{\sigma_2^2}-2} \exp\left\{-\frac{A_1}{z_1}(z_1-1)\right\}\exp\left\{-\frac{A_2}{z_2}(z_2-1)\right\}. \tag{A3.5.8}$$

Simplifying the $\tilde{F}_1(\tau, \eta, s_1(z_1), s_2(z_2))$ term:

The first term is simplified by first setting $\xi_1 = 1 - \frac{z_1}{A_1}\beta_1$ in equation (A3.5.6) to obtain

$$\tilde{F}_1(\tau, \eta, s_1(z_1), s_2(z_2)) = h(\tau, \eta, v_{1,0}, v_{2,0})e^{-(A_1+A_2)} \frac{A_1^{\frac{2\Phi_1}{\sigma_1^2}-1}}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)} e^{\frac{A_2}{z_2}} z_2^{\frac{2\Phi_2}{\sigma_2^2}-2}$$
$$\times \int_0^1 (1-\xi_1)^{\frac{2\Phi_1}{\sigma_1^2}-2} z_1^{-1} e^{\frac{A_1}{z_1}\xi_1} d\xi_1. \tag{A3.5.9}$$

From the transformation (A3.5.2) we can express the Laplace transform variables as

$$s_1 = \frac{2\Omega_1(z_1-1)}{\sigma_1^2(e^{\Omega_1\tau}-1)} + \frac{\Theta_1-\Omega_1}{\sigma_1^2}, \quad s_2 = \frac{2\Omega_2(z_2-1)}{\sigma_2^2(e^{\Omega_2\tau}-1)} + \frac{\Theta_2-\Omega_2}{\sigma_2^2},$$

Using the definition of Laplace transforms provided in equation (1.3.6), we can represent the above transform as

$$\mathcal{L}\{\hat{F}_1(\tau,\eta,v_1,v_2)\} = \int_0^\infty \int_0^\infty \exp\left\{ -\left[\frac{2\Omega_1(z_1-1)}{\sigma_1^2(e^{\Omega_1\tau}-1)} + \frac{\Theta_1-\Omega_1}{\sigma_1^2}\right]v_1 - \left[\frac{2\Omega_2(z_2-1)}{\sigma_2^2(e^{\Omega_2\tau}-1)} + \frac{\Theta_2-\Omega_2}{\sigma_2^2}\right]v_2\right\}$$
$$\times \hat{F}_1(\tau,\eta,v_1,v_2)dv_1dv_2. \tag{A3.5.10}$$

Making the change of variables

$$y_1 = \frac{2\Omega_1 v_1}{\sigma_1^2(e^{\Omega_1\tau}-1)} \quad \text{and} \quad y_2 = \frac{2\Omega_2 v_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}, \tag{A3.5.11}$$

in equation (A3.5.10) and rearranging we obtain

$$\mathcal{L}\{\hat{F}_1(\tau,\eta,v_1(y_1),v_2(y_2))\} = \frac{\sigma_1^2(e^{\Omega_1\tau}-1)}{2\Omega_1}\frac{\sigma_2^2(e^{\Omega_2\tau}-1)}{2\Omega_2}\int_0^\infty\int_0^\infty e^{-z_1y_1-z_2y_2}$$
$$\times \exp\left(-\left[\frac{(\Theta_1-\Omega_1)(e^{\Omega_1\tau}-1)}{2\Omega_1}-1\right]y_1\right)\exp\left(-\left[\frac{(\Theta_2-\Omega_2)(e^{\Omega_2\tau}-1)}{2\Omega_2}-1\right]y_2\right)$$
$$\times \hat{F}_1(\tau,\eta,v_1(y_1),v_2(y_2))dy_1dy_2. \tag{A3.5.12}$$

The Laplace transform of the RHS of equation (A3.5.12) can also be represented as

$$\mathcal{L}\Big\{\hat{F}_1(\tau,\eta,v_1(y_1),v_2(y_2))\Big\} = \frac{\sigma_1^2(e^{\Omega_1\tau}-1)}{2\Omega_1}\frac{\sigma_2^2(e^{\Omega_2\tau}-1)}{2\Omega_2}\mathcal{L}\Bigg\{\exp\Bigg(-\Bigg[\frac{(\Theta_1-\Omega_1)(e^{\Omega_1\tau}-1)}{2\Omega_1}-1\Bigg]y_1\Bigg) \times \exp\Bigg(-\Bigg[\frac{(\Theta_2-\Omega_2)(e^{\Omega_2\tau}-1)}{2\Omega_2}-1\Bigg]y_2\Bigg)\hat{F}_1(\tau,\eta,v_1(y_1),v_2(y_2))\Bigg\}. \quad \text{(A3.5.13)}$$

Recalling that

$$\mathcal{L}\Big\{\hat{F}_1(\tau,\eta,y_1,y_2)\Big\} = \int_0^\infty\int_0^\infty e^{-z_1y_1-z_2y_2}\hat{F}_1(\tau,\eta,y_1,y_2)dy_1dy_2 = \tilde{F}(\tau,\eta,z_1,z_2), \quad \text{(A3.5.14)}$$

equation (A3.5.13) can be written in terms of the inverse Laplace transform as

$$\mathcal{L}^{-1}\Big\{\tilde{F}_1(\tau,\eta,s_1(z_1),s_2(z_2))\Big\} = \frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}\exp\Bigg(\Bigg[\frac{(\Theta_1-\Omega_1)(e^{\Omega_1\tau}-1)}{2\Omega_1}-1\Bigg]y_1\Bigg) \times \exp\Bigg(\Bigg[\frac{(\Theta_2-\Omega_2)(e^{\Omega_2\tau}-1)}{2\Omega_2}-1\Bigg]y_2\Bigg)\mathcal{L}^{-1}\Big\{\tilde{F}_1(\tau,\eta,s_1(z_1),s_2(z_2))\Big\}. \quad \text{(A3.5.15)}$$

Applying the inverse transform (A3.5.15) to equation (A3.5.9) we obtain

$$\hat{F}_1(\tau,\eta,v_1(y_1),v_2(y_2)) = h(\tau,\eta,v_{1,0},v_{2,0})e^{-(A_1+A_2)}\frac{A_1^{\frac{2\Phi_1}{\sigma_1^2}-1}}{\Gamma\big(\frac{2\Phi_1}{\sigma_1^2}-1\big)}\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)} \times \exp\Bigg\{\Bigg[\frac{(\Theta_1-\Omega_1)(e^{\Omega_1\tau}-1)}{2\Omega_1}-1\Bigg]y_1\Bigg\}\exp\Bigg\{\Bigg[\frac{(\Theta_2-\Omega_2)(e^{\Omega_2\tau}-1)}{2\Omega_2}-1\Bigg]y_2\Bigg\} \times \int_0^1(1-\xi_1)^{\frac{2\Phi_1}{\sigma_1^2}-2}\mathcal{L}^{-1}\Big\{z_2^{\frac{2\Phi_2}{\sigma_2^2}-2}e^{\frac{A_2}{z_2}}z_1^{-1}e^{\frac{A_1}{z_1}\xi_1}\Big\}d\xi_1. \quad \text{(A3.5.16)}$$

From Abramowitz and Stegun (1964)[12] we find that

$$\mathcal{L}\Big\{\Big(\frac{y_1}{A_1}\Big)^{\frac{u_1-1}{2}}I_{u_1-1}(2\sqrt{A_1y_1})\Big(\frac{y_2}{A_2}\Big)^{\frac{u_2-1}{2}}I_{u_2-1}(2\sqrt{A_2y_2})\Big\} = \frac{1}{z_1^{u_1}}e^{\frac{A_1}{z_1}}\frac{1}{z_2^{u_2}}e^{\frac{A_2}{z_2}}, \quad \text{(A3.5.17)}$$

[12]This result is tabulated on page 1026 of the referenced book.

where, $I_k(z)$ is the modified Bessel function of the first kind defined as

$$I_k(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{2n+k}}{\Gamma(n+k+1)n!}. \tag{A3.5.18}$$

Application of the result in equation (A3.5.17) to equation (A3.5.16) yields

$$\begin{aligned}\hat{F}_1(\tau,\eta,&v_1(y_1),v_2(y_2)) = h(\tau,\eta,v_{1,0},v_{2,0})e^{-(A_1+A_2)}\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}\\ &\times\exp\left\{\left[\frac{(\Theta_1-\Omega_1)(e^{\Omega_1\tau}-1)}{2\Omega_1}-1\right]y_1\right\}\exp\left\{\left[\frac{(\Theta_2-\Omega_2)(e^{\Omega_2\tau}-1)}{2\Omega_2}-1\right]y_2\right\}\\ &\times\left(\frac{y_2}{A_2}\right)^{\frac{1}{2}-\frac{\Phi_2}{\sigma_2^2}}I_{1-\frac{2\Phi_2}{\sigma_2^2}}(2\sqrt{A_2y_2})\frac{A_1^{\frac{2\Phi_1}{\sigma_1^2}-1}}{\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)}\int_0^1(1-\xi_1)^{\frac{2\Phi_1}{\sigma_1^2}-2}I_0(2\sqrt{A_1y_1\xi_1})d\xi_1.\end{aligned} \tag{A3.5.19}$$

By expanding both terms inside the integral in power series followed by integration we find that

$$\int_0^1(1-\xi_1)^{\frac{2\Phi_1}{\sigma_1^2}-2}I_0(2\sqrt{A_1y_1\xi_1})d\xi_1 = \Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1\right)(A_1y_1)^{\frac{1}{2}-\frac{\Phi_1}{\sigma_1^2}}I_{\frac{2\Phi_1}{\sigma_1^2}-1}(2\sqrt{A_1y_1}). \tag{A3.5.20}$$

Substituting this into equation (A3.5.19) we obtain the inverse Laplace transform of the first component of (A3.5.4) as[13]

$$\begin{aligned}\hat{F}_1(\tau,\eta,&v_1(y_1),v_2(y_2)) = h(\tau,\eta,v_{1,0},v_{2,0})e^{-(A_1+A_2)}\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}\\ &\times\exp\left\{\left[\frac{(\Theta_1-\Omega_1)(e^{\Omega_1\tau}-1)}{2\Omega_1}-1\right]y_1\right\}\exp\left\{\left[\frac{(\Theta_2-\Omega_2)(e^{\Omega_2\tau}-1)}{2\Omega_2}-1\right]y_2\right\}\\ &\times\left(\frac{A_1}{y_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\left(\frac{A_2}{y_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}I_{\frac{2\Phi_1}{\sigma_1^2}-1}(2\sqrt{A_1y_1})I_{\frac{2\Phi_2}{\sigma_2^2}-1}(2\sqrt{A_2y_2})\end{aligned} \tag{A3.5.21}$$

[13]We make use of the symmetry relation

$$I_{1-a}(x) = I_{a-1}(x).$$

Simplifying the $\tilde{F}_2(\tau, \eta, s_1(z_1), s_2(z_2))$ term:

By performing similar calculations to those outlined from (A3.5.9) - (A3.5.19) we find that

$$\begin{aligned}\hat{F}_2(\tau, \eta, &v_1(y_1), v_2(y_2)) = h(\tau, \eta, v_{1,0}, v_{2,0}) e^{-(A_1+A_2)} \frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1 \tau}-1)} \frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2 \tau}-1)} \\ &\times \exp\left\{\left[\frac{(\Theta_1-\Omega_1)(e^{\Omega_1\tau}-1)}{2\Omega_1}-1\right]y_1\right\} \exp\left\{\left[\frac{(\Theta_2-\Omega_2)(e^{\Omega_2\tau}-1)}{2\Omega_2}-1\right]y_2\right\} \\ &\times \left(\frac{A_1}{y_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}} \left(\frac{A_2}{y_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}} I_{\frac{2\Phi_1}{\sigma_1^2}-1}(2\sqrt{A_1 y_1}) I_{\frac{2\Phi_2}{\sigma_2^2}-1}(2\sqrt{A_2 y_2}). \qquad \text{(A3.5.22)}\end{aligned}$$

Simplifying the $\tilde{F}_3(\tau, \eta, s_1(z_1), s_2(z_2))$ term:

By using similar steps to those presented from (A3.5.9) - (A3.5.17) we obtain

$$\begin{aligned}\hat{F}_3(\tau, \eta, &v_1(y_1), v_2(y_2)) = -h(\tau, \eta, v_{1,0}, v_{2,0}) e^{-(A_1+A_2)} \frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1 \tau}-1)} \frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2 \tau}-1)} \\ &\times \exp\left\{\left[\frac{(\Theta_1-\Omega_1)(e^{\Omega_1\tau}-1)}{2\Omega_1}-1\right]y_1\right\} \exp\left\{\left[\frac{(\Theta_2-\Omega_2)(e^{\Omega_2\tau}-1)}{2\Omega_2}-1\right]y_2\right\} \\ &\times \left(\frac{A_1}{y_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}} \left(\frac{A_2}{y_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}} I_{1-\frac{2\Phi_1}{\sigma_1^2}}(2\sqrt{A_1 y_1}) I_{1-\frac{2\Phi_2}{\sigma_2^2}}(2\sqrt{A_2 y_2}). \qquad \text{(A3.5.23)}\end{aligned}$$

Explicit form of the inverse Laplace transform:

Combining (A3.5.21), (A3.5.22) and (A3.5.23) we conclude that

$$\hat{U}(\tau, \eta, v_1(y_1), v_2(y_2)) = \hat{F}_1(\tau, \eta, v_1(y_1), v_2(y_2)) + \hat{F}_2(\tau, \eta, v_1(y_1), v_2(y_2)) + \hat{F}_3(\tau, \eta, v_1(y_1), v_2(y_2)),$$

which implies that

$$\hat{U}(\tau,\eta,v_1(y_1),v_2(y_2)) = h(\tau,\eta,v_{1,0},v_{2,0})e^{-(A_1+y_1)-(A_2+y_2)}\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}$$
$$\times\exp\left\{\left[\frac{(\Theta_1-\Omega_1)(e^{\Omega_1\tau}-1)}{2\Omega_1}\right]y_1\right\}\exp\left\{\left[\frac{(\Theta_2-\Omega_2)(e^{\Omega_2\tau}-1)}{2\Omega_2}\right]y_2\right\}$$
$$\times\Bigg[\left(\frac{A_1}{y_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\left(\frac{A_2}{y_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}I_{\frac{2\Phi_1}{\sigma_1^2}-1}(2\sqrt{A_1y_1})I_{1-\frac{2\Phi_2}{\sigma_2^2}}(2\sqrt{A_2y_2})$$
$$+\left(\frac{A_1}{y_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\left(\frac{A_2}{y_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}I_{1-\frac{2\Phi_1}{\sigma_1^2}}(2\sqrt{A_1y_1})I_{\frac{2\Phi_2}{\sigma_2^2}-1}(2\sqrt{A_2y_2})$$
$$-\left(\frac{A_1}{y_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\left(\frac{A_2}{y_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}I_{1-\frac{2\Phi_1}{\sigma_1^2}}(2\sqrt{A_1y_1})I_{1-\frac{2\Phi_2}{\sigma_2^2}}(2\sqrt{A_2y_2})\Bigg]. \tag{A3.5.24}$$

Now, substituting for A_1, A_2, $h(\tau,\eta,v_{1,0},v_{2,0})$, y_1 and y_2 from equations (A3.5.1), (A3.5.3) and (A3.5.11) we obtain

$$\hat{U}(\tau,\eta,v_1,v_2) = \exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right]\tau\right\}$$
$$\times\exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_{1,0}-\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)v_{2,0}+i\eta x_0\right\}\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}$$
$$\times\exp\left\{-\left(\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\right)(v_{1,0}e^{\Omega_1\tau}+v_1)-\left(\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}\right)(v_{2,0}e^{\Omega_2\tau}+v_2)\right\}$$
$$\times\exp\left\{\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_1\right\}\exp\left\{\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)v_2\right\}$$
$$\times\Bigg[\left(\frac{v_{1,0}e^{\Omega_1\tau}}{v_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\left(\frac{v_{2,0}e^{\Omega_2\tau}}{v_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}I_{\frac{2\Phi_1}{\sigma_1^2}-1}\left(\frac{4\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}(v_1v_{1,0}e^{\Omega_1\tau})^{\frac{1}{2}}\right)$$
$$\times I_{1-\frac{2\Phi_2}{\sigma_2^2}}\left(\frac{4\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}(v_2v_{2,0}e^{\Omega_2\tau})^{\frac{1}{2}}\right)$$
$$+\left(\frac{v_{1,0}e^{\Omega_1\tau}}{v_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\left(\frac{v_{2,0}e^{\Omega_2\tau}}{v_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}I_{1-\frac{2\Phi_1}{\sigma_1^2}}\left(\frac{4\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}(v_1v_{1,0}e^{\Omega_1\tau})^{\frac{1}{2}}\right)$$
$$\times I_{\frac{2\Phi_2}{\sigma_2^2}-1}\left(\frac{4\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}(v_2v_{2,0}e^{\Omega_2\tau})^{\frac{1}{2}}\right)$$
$$-\left(\frac{v_{1,0}e^{\Omega_1\tau}}{v_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\left(\frac{v_{2,0}e^{\Omega_2\tau}}{v_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}I_{1-\frac{2\Phi_1}{\sigma_1^2}}\left(\frac{4\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}(v_1v_{1,0}e^{\Omega_1\tau})^{\frac{1}{2}}\right)$$
$$\times I_{1-\frac{2\Phi_2}{\sigma_2^2}}\left(\frac{4\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}(v_2v_{2,0}e^{\Omega_2\tau})^{\frac{1}{2}}\right)\Bigg]. \tag{A3.5.25}$$

After making further simplifications to the above equation and noting again that $I_{\phi-1}(x) = I_{1-\phi}(x)$ we obtain

$$\hat{U}(\tau,\eta,v_1,v_2) = \exp\left\{\Big(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\Big)\big(v_1 - v_{1,0} + \Phi_1\tau\big) + \Big(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\Big)\big(v_2 - v_{2,0} + \Phi_2\tau\big)\right\}$$
$$\times \exp\left\{-\Big(\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\Big)\big(v_{1,0}e^{\Omega_1\tau} + v_1\big) - \Big(\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}\Big)\big(v_{2,0}e^{\Omega_2\tau} + v_2\big)\right\}$$
$$\times e^{i\eta x_0 - i\eta(r-q)\tau}\frac{2\Omega_1 e^{\Omega_1\tau}}{\sigma_1^2(e^{\Omega_1\tau}-1)}\frac{2\Omega_2 e^{\Omega_2\tau}}{\sigma_2^2(e^{\Omega_2\tau}-1)}\Big(\frac{v_{1,0}e^{\Omega_1\tau}}{v_1}\Big)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\Big(\frac{v_{2,0}e^{\Omega_2\tau}}{v_2}\Big)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}$$
$$\times I_{\frac{2\Phi_1}{\sigma_1^2}-1}\Big(\frac{4\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\big(v_1 v_{1,0}e^{\Omega_1\tau}\big)^{\frac{1}{2}}\Big) I_{\frac{2\Phi_2}{\sigma_2^2}-1}\Big(\frac{4\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}\big(v_2 v_{2,0}e^{\Omega_2\tau}\big)^{\frac{1}{2}}\Big), \tag{A3.5.26}$$

which is equation (3.6.1). This concludes the proof.

Appendix 3.6. Proof of Proposition 3.7.1

By applying equation (1.3.5) to equation (3.6.1), the inverse Fourier Transform of the density function can be represented as

$$\mathcal{F}^{-1}\big\{\hat{U}(\tau,\eta,v_1,v_2)\big\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\eta x}\hat{U}(\tau,\eta,v_1,v_2)d\eta = U(\tau,x,v_1,v_2). \tag{A3.6.1}$$

Substituting for $\hat{U}(\tau,\eta,v_1,v_2)$ from equation (3.6.1) we obtain

$$U(\tau,x,v_1,v_2;x_0,v_{1,0},v_{2,0}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta x_0} H(\tau,x,v_1,v_2;-\eta,v_{1,0},v_{2,0})d\eta \tag{A3.6.2}$$

where

$$H(\tau,x,v_1,v_2;\eta,v_{1,0},v_{2,0}) = \exp\left\{\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)(v_1-v_{1,0}+\Phi_1\tau)+\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)(v_2-v_{2,0}+\Phi_2\tau)\right\}$$
$$\times\exp\left\{-\left(\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}\right)\left(v_{1,0}e^{\Omega_1\tau}+v_1\right)-\left(\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}\right)\left(v_{2,0}e^{\Omega_2\tau}+v_2\right)\right\}$$
$$\times\, e^{i\eta x+i\eta(r-q)\tau}\frac{2\Omega_1 e^{\Omega_1\tau}}{\sigma_1^2(e^{\Omega_1\tau}-1)}\frac{2\Omega_2 e^{\Omega_2\tau}}{\sigma_2^2(e^{\Omega_2\tau}-1)}\left(\frac{v_{1,0}e^{\Omega_1\tau}}{v_1}\right)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\left(\frac{v_{2,0}e^{\Omega_2\tau}}{v_2}\right)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}$$
$$\times\, I_{\frac{2\Phi_1}{\sigma_1^2}-1}\left(\frac{4\Omega_1}{\sigma_1^2(e^{\Omega_1\tau}-1)}(v_1v_{1,0}e^{\Omega_1\tau})^{\frac{1}{2}}\right)I_{\frac{2\Phi_2}{\sigma_2^2}-1}\left(\frac{4\Omega_2}{\sigma_2^2(e^{\Omega_2\tau}-1)}(v_2v_{2,0}e^{\Omega_2\tau})^{\frac{1}{2}}\right). \quad \text{(A3.6.3)}$$

which is the result given in Proposition 3.7.1.

Appendix 3.7. Useful Complex Integrals

In this appendix we reproduce the integral representation of complex functions given in Adolfsson et al. (2009) and Shephard (1991) as they are required for the calculations in Appendices 3.8 and 3.9 . We seek complex integral representations of expressions involving the function g which satisfies the following two conditions:

- $h(\phi) \equiv g(\phi - i)$,
- $h(-\phi) = \overline{h(\phi)}$.

We need to consider integrals of the form

$$Q_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(-\phi)\int_a^{\infty} e^y e^{i\phi y}dyd\phi, \quad \text{(A3.7.1)}$$

and

$$Q_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(-\phi)\int_a^{\infty} e^{i\phi y}dyd\phi. \quad \text{(A3.7.2)}$$

Equation (A3.7.1) is simplified by letting $\xi = \phi - i$. Substituting this into (A3.7.1) gives

$$Q_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(-\xi - i) \int_{a}^{\infty} e^{i\xi y} dy d\xi. \tag{A3.7.3}$$

Making a further change of variable $\eta = -\xi$ yields

$$\begin{aligned} Q_1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\eta - i) \int_{a}^{\infty} e^{-i\eta y} dy d\eta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\eta - i) \Big[\lim_{b\to\infty} \frac{e^{-i\eta a} - e^{-i\eta b}}{i\eta} \Big] d\eta. \end{aligned} \tag{A3.7.4}$$

Equation (A3.7.4) can be expressed as

$$\begin{aligned} Q_1 &= \frac{1}{2\pi} \lim_{b\to\infty} \left[\int_{0}^{\infty} g(\eta - i) \Big(\frac{e^{-i\eta a} - e^{-i\eta b}}{i\eta} \Big) d\eta + \int_{0}^{\infty} g(-\eta - i) \Big(\frac{e^{i\eta a} - e^{i\eta b}}{-i\eta} \Big) d\eta \right] \\ &= \frac{1}{2\pi} \int_{0}^{\infty} \frac{g(\eta - i) e^{-i\eta a} - g(-\eta - i) e^{i\eta a}}{i\eta} d\eta \\ &- \frac{1}{2\pi} \lim_{b\to\infty} \left[\int_{0}^{\infty} \frac{g(\eta - i) e^{-i\eta b} - g(-\eta - i) e^{i\eta b}}{i\eta} \right] d\eta. \end{aligned} \tag{A3.7.5}$$

Now using the result in Shephard (1991) that[14]

$$F(x) = \frac{1}{2} - \frac{1}{2\pi} \int_{0}^{\infty} \frac{g(\eta - i) e^{-i\eta x} - g(-\eta - i) e^{i\eta x}}{i\eta} d\eta, \tag{A3.7.6}$$

where $F(x)$ is a cumulative density function we can show that

$$\begin{aligned} &\lim_{b\to\infty} \frac{1}{2\pi} \int_{0}^{\infty} \frac{g(\eta - i) e^{-i\eta b} - g(-\eta - i) e^{i\eta b}}{i\eta} d\eta \\ &\qquad = \lim_{b\to\infty} \frac{1}{2\pi} \int_{0}^{\infty} \frac{h(\eta) e^{-i\eta b} - h(-\eta) e^{i\eta b}}{i\eta} d\eta \\ &\qquad = \lim_{b\to\infty} \left[\frac{1}{2} - F(b) \right] = -\frac{1}{2}. \end{aligned} \tag{A3.7.7}$$

Using this result, equation (A3.7.5) can be represented as

$$Q_1 = \frac{1}{2} + \frac{1}{2\pi} \int_{0}^{\infty} \frac{h(\eta) e^{-i\eta a} - h(-\eta) e^{i\eta a}}{i\eta} d\eta. \tag{A3.7.8}$$

[14] The function F is defined by $F(x) = \int_{-\infty}^{x} g(\eta) d\eta$.

Also, if $h(\eta)$ and $h(-\eta)$ are complex conjugates[15] then

$$\frac{h(-i\eta)e^{i\eta a}}{-i\eta} = \overline{\left(\frac{h(i\eta)e^{-i\eta a}}{i\eta}\right)}. \tag{A3.7.9}$$

Using this result (A3.7.8) simplifies to

$$Q_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{h(\eta)e^{-i\eta a}}{i\eta}\right) d\eta. \tag{A3.7.10}$$

Performing similar operations on equation (A3.7.2) we obtain

$$Q_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{g(\eta)e^{-i\eta a}}{i\eta}\right) d\eta. \tag{A3.7.11}$$

Appendix 3.8. Proof of Proposition 3.8.1

By first letting $x = \log(S)$ and making use of the relation $C_E(\tau, \log(S), v_1, v_2) = V_E(\tau, S, v_1, v_2)$ which we introduced in equation (3.2.30), followed by substituting the explicit density function presented in Proposition 3.7.1 into the European option component in equation (3.2.39) we obtain after rearranging

$$\begin{aligned} V_E(\tau, S, v_1, v_2) = e^{-r\tau} \int_0^\infty \int_0^\infty \int_{\ln K}^\infty (e^x - K) \Bigg[\frac{1}{2\pi} \int_{-\infty}^\infty e^{i\eta x} \\ \times u_2(\tau, S, v_1, v_2; -\eta, w_1, w_2) d\eta \Bigg] dx dw_1 dw_2, \end{aligned} \tag{A3.8.1}$$

[15]Note that $h(-\eta) = \overline{h(\eta)}$ represents the complex conjugate.

where

$$
\begin{aligned}
u_2(\tau,S,v_1,v_2;\eta,w_1,w_2) &= \exp\Big\{\Big(\frac{\Theta_{1,2}-\Omega_{1,2}}{\sigma_1^2}\Big)(v_1-w_1+\Phi_1\tau)+\Big(\frac{\Theta_{2,2}-\Omega_{2,2}}{\sigma_2^2}\Big)(v_2-w_2+\Phi_2\tau)\Big\}\\
&\times\exp\Big\{-\Big(\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}\Big)(w_1e^{\Omega_{1,2}\tau}+v_1)-\Big(\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}\Big)(w_2e^{\Omega_{2,2}\tau}+v_2)\Big\}\\
&\times e^{i\eta\ln S+i\eta(r-q)\tau}\frac{2\Omega_{1,2}e^{\Omega_{1,2}\tau}}{\sigma_1^2(e^{\Omega_1\tau}-1)}\frac{2\Omega_{2,2}e^{\Omega_{2,2}\tau}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}\Big(\frac{w_1e^{\Omega_{1,2}\tau}}{v_1}\Big)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\Big(\frac{w_2e^{\Omega_{2,2}\tau}}{v_2}\Big)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}\\
&\times I_{\frac{2\Phi_1}{\sigma_1^2}-1}\Big(\frac{4\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}(v_1w_1e^{\Omega_{1,2}\tau})^{\frac{1}{2}}\Big)I_{\frac{2\Phi_2}{\sigma_2^2}-1}\Big(\frac{4\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}(v_2w_2e^{\Omega_{2,2}\tau})^{\frac{1}{2}}\Big),
\end{aligned}
\tag{A3.8.2}
$$

with $\Theta_{1,1}=\Theta_1(i-\eta)$, $\Theta_{1,2}=\Theta_1(-\eta)$, $\Theta_{2,1}=\Theta_2(i-\eta)$, $\Theta_{2,2}=\Theta_2(-\eta)$, $\Omega_{1,1}=\Omega_1(i-\eta)$, $\Omega_{1,2}=\Omega_1(-\eta)$, $\Omega_{2,1}=\Omega_2(i-\eta)$ and $\Omega_{2,2}=\Omega_2(-\eta)$.

From the above equation, we note that the payoff of the European call option is independent of the running variance variables namely w_1 and w_2. This gives us the flexibility to calculate the integrals with respect to w_1 and w_2 first thus equation (A3.8.1) can be written as

$$
\begin{aligned}
V_E(\tau,S,v_1,v_2) &= \frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty}\int_{\ln K}^{\infty}e^{i\eta x}(e^x-K)\\
&\times\Big[\int_0^\infty\int_0^\infty u_2(\tau,S,v_1,v_2;-\eta,w_1,w_2)dw_1dw_2\Big]d\eta dx.
\end{aligned}
\tag{A3.8.3}
$$

In evaluating the double integral with respect to the running variance variables, we first let[16]

$$
\begin{cases}
A_1=\frac{2\Omega_{1,2}w_1}{\sigma_1^2(1-e^{-\Omega_{1,2}\tau})},\\
A_2=\frac{2\Omega_{2,2}w_2}{\sigma_2^2(1-e^{-\Omega_{2,2}\tau})},
\end{cases}
\tag{A3.8.4}
$$

[16]Note that we have introduced these functions before in the systems (A3.5.1) and (A3.5.11).

and

$$\begin{cases} y_1 = \frac{2\Omega_{1,2} v_1}{\sigma_1^2 (e^{\Omega_{1,2}\tau} - 1)}, \\ y_2 = \frac{2\Omega_{2,2} v_2}{\sigma_2^2 (e^{\Omega_{2,2}\tau} - 1)}. \end{cases} \tag{A3.8.5}$$

Using $g_2(\tau, S, v_1, v_2)$ to denote the inner double integral in equation (A3.8.3) we find that it can be written as

$$\begin{aligned} g_2(\tau, S, v_1, v_2) = \exp\Big\{ \Big(\frac{\Theta_{1,2} - \Omega_{1,2}}{\sigma_1^2} \Big) (v_1 + \Phi_1 \tau) + \Big(\frac{\Theta_{2,2} - \Omega_{2,2}}{\sigma_2^2} \Big) (v_2 + \Phi_2 \tau) \Big\} e^{i\eta \ln S + i\eta (r-q)\tau} \\ \times \exp\Big\{ - \Big(\frac{2\Omega_{1,2}}{\sigma_1^2 (e^{\Omega_{1,2}\tau} - 1)} \Big) v_1 - \Big(\frac{2\Omega_{2,2}}{\sigma_2^2 (e^{\Omega_{2,2}\tau} - 1)} \Big) v_2 \Big\} \\ \times \int_0^\infty \int_0^\infty \exp\Big\{ - \Big[\frac{(\Theta_{1,2} - \Omega_{1,2})(1 - e^{-\Omega_{1,2}\tau})}{2\Omega_{1,2}} + 1 \Big] A_1 \Big\} \Big(\frac{A_1}{y_1} \Big)^{\frac{\Phi_1}{\sigma_1^2} - \frac{1}{2}} \\ \times \exp\Big\{ - \Big[\frac{(\Theta_{2,2} - \Omega_{2,2})(1 - e^{-\Omega_{2,2}\tau})}{2\Omega_{2,2}} + 1 \Big] A_2 \Big\} \Big(\frac{A_2}{y_2} \Big)^{\frac{\Phi_2}{\sigma_2^2} - \frac{1}{2}} \\ \times I_{\frac{2\Phi_1}{\sigma_1^2} - 1}(2\sqrt{A_1 y_1}) I_{\frac{2\Phi_2}{\sigma_2^2} - 1}(2\sqrt{A_2 y_2}) dA_1 dA_2, \end{aligned} \tag{A3.8.6}$$

Now using the definition of the modified Bessel function, we can further simplify the above equation to

$$\begin{aligned} g_2(\tau, S, v_1, v_2) = \exp\Big\{ \Big(\frac{\Theta_{1,2} - \Omega_{1,2}}{\sigma_1^2} \Big) (v_1 + \Phi_1 \tau) + \Big(\frac{\Theta_{2,2} - \Omega_{2,2}}{\sigma_2^2} \Big) (v_2 + \Phi_2 \tau) \Big\} e^{i\eta \ln S + i\eta (r-q)\tau} \\ \times \exp\Big\{ - \Big(\frac{2\Omega_{1,2}}{\sigma_1^2 (e^{\Omega_{1,2}\tau} - 1)} \Big) v_1 - \Big(\frac{2\Omega_{2,2}}{\sigma_2^2 (e^{\Omega_{2,2}\tau} - 1)} \Big) v_2 \Big\} \\ \times \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \frac{y_1^{n_1}}{n_1!} \frac{y_2^{n_2}}{n_2!} \int_0^\infty \int_0^\infty \exp\Big\{ - \Big[\frac{(\Theta_{1,2} - \Omega_{1,2})(1 - e^{-\Omega_{1,2}\tau}) + 2\Omega_{1,2}}{2\Omega_{1,2}} \Big] A_1 \Big\} \\ \times \exp\Big\{ - \Big[\frac{(\Theta_{2,2} - \Omega_{2,2})(1 - e^{-\Omega_{2,2}\tau}) + 2\Omega_{2,2}}{2\Omega_{2,2}} \Big] A_2 \Big\} \\ \times \frac{A_1^{n_1 + \frac{2\Phi_1}{\sigma_1^2} - 1}}{\Gamma\Big(n_1 + \frac{2\Phi_1}{\sigma_1^2}\Big)} \frac{A_2^{n_2 + \frac{2\Phi_2}{\sigma_2^2} - 1}}{\Gamma\Big(n_2 + \frac{2\Phi_2}{\sigma_2^2}\Big)} dA_1 dA_2. \end{aligned} \tag{A3.8.7}$$

Let

$$\xi_1 = \Big[\frac{(\Theta_{1,2} - \Omega_{1,2})(1 - e^{-\Omega_{1,2}\tau}) + 2\Omega_{1,2}}{2\Omega_{1,2}} \Big] A_1, \quad \xi_2 = \Big[\frac{(\Theta_{2,2} - \Omega_{2,2})(1 - e^{-\Omega_{2,2}\tau}) + 2\Omega_{2,2}}{2\Omega_{2,2}} \Big] A_2.$$

Substituting these into (A3.8.7) we obtain

$$
\begin{aligned}
g_2(\tau,S,v_1,v_2) &= \exp\Big\{\Big(\frac{\Theta_{1,2}-\Omega_{1,2}}{\sigma_1^2}\Big)\big(v_1+\Phi_1\tau\big)+\Big(\frac{\Theta_{2,2}-\Omega_{2,2}}{\sigma_2^2}\Big)\big(v_2+\Phi_2\tau\big)\Big\}e^{i\eta\ln S+i\eta(r-q)\tau}\\
&\times \exp\Big\{-\Big(\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}\Big)v_1-\Big(\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}\Big)v_2\Big\}\\
&\times \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{y_1^{n_1}}{n_1!\Gamma\Big(n_1+\frac{2\Phi_1}{\sigma_1^2}\Big)}\frac{y_2^{n_2}}{n_2!\Gamma\Big(n_2+\frac{2\Phi_2}{\sigma_2^2}\Big)}\Big(\frac{2\Omega_{1,2}}{(\Theta_{1,2}-\Omega_{1,2})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}\Big)^{n_1+\frac{2\Phi_1}{\sigma_1^2}}\\
&\times \Big(\frac{2\Omega_{2,2}}{(\Theta_{2,2}-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}\Big)^{n_2+\frac{2\Phi_2}{\sigma_2^2}}\int_0^\infty\int_0^\infty e^{-\xi_1-\xi_2}\xi_1^{n_1+\frac{2\Phi_1}{\sigma_1^2}-1}\xi_2^{n_2+\frac{2\Phi_2}{\sigma_2^2}-1}d\xi_1 d\xi_2.
\end{aligned}
\tag{A3.8.8}
$$

By noting that

$$
\int_0^\infty\int_0^\infty e^{-\xi_1-\xi_2}\xi_1^{n_1+\frac{2\Phi_1}{\sigma_1^2}-1}\xi_2^{n_2+\frac{2\Phi_2}{\sigma_2^2}-1}d\xi_1 d\xi_2=\Gamma\left(n_1+\frac{2\Phi_1}{\sigma_1^2}\right)\Gamma\left(n_2+\frac{2\Phi_2}{\sigma_2^2}\right), \tag{A3.8.9}
$$

we obtain

$$
\begin{aligned}
g_2(\tau,S,v_1,v_2) &= \exp\Big\{\Big(\frac{\Theta_{1,2}-\Omega_{1,2}}{\sigma_1^2}\Big)\big(v_1+\Phi_1\tau\big)+\Big(\frac{\Theta_{2,2}-\Omega_{2,2}}{\sigma_2^2}\Big)\big(v_2+\Phi_2\tau\big)\Big\}e^{i\eta\ln S+i\eta(r-q)\tau}\\
&\times \exp\Big\{-\Big(\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}\Big)v_1-\Big(\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}\Big)v_2\Big\}\\
&\times \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{y_1^{n_1}}{n_1!}\frac{y_2^{n_2}}{n_2!}\Big(\frac{2\Omega_{1,2}}{(\Theta_{1,2}-\Omega_{1,2})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}\Big)^{n_1+\frac{2\Phi_1}{\sigma_1^2}}\\
&\times \Big(\frac{2\Omega_{2,2}}{(\Theta_{2,2}-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}\Big)^{n_2+\frac{2\Phi_2}{\sigma_2^2}}\frac{\Gamma\Big(n_1+\frac{2\Phi_1}{\sigma_1^2}\Big)}{\Gamma\Big(n_1+\frac{2\Phi_1}{\sigma_1^2}\Big)}\frac{\Gamma\Big(n_2+\frac{2\Phi_2}{\sigma_2^2}\Big)}{\Gamma\Big(n_2+\frac{2\Phi_2}{\sigma_2^2}\Big)}.
\end{aligned}
\tag{A3.8.10}
$$

The above equation simplifies to

$$
\begin{aligned}
g_2(\tau,S,v_1,v_2) &= \exp\Big\{\Big(\frac{\Theta_{1,2}-\Omega_{1,2}}{\sigma_1^2}\Big)(v_1+\Phi_1\tau)+\Big(\frac{\Theta_{2,2}-\Omega_{2,2}}{\sigma_2^2}\Big)(v_2+\Phi_2\tau)\Big\}e^{i\eta\ln S+i\eta(r-q)\tau}\\
&\times\exp\Big\{-\Big(\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}\Big)v_1-\Big(\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}\Big)v_2\Big\}\\
&\times\Big(\frac{2\Omega_{1,2}}{(\Theta_{1,2}-\Omega_{1,2})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}\Big)^{\frac{2\Phi_1}{\sigma_1^2}}\Big(\frac{2\Omega_{2,2}}{(\Theta_{2,2}-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}\Big)^{\frac{2\Phi_2}{\sigma_2^2}}\\
&\times\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{y_1^{n_1}}{n_1!}\frac{y_2^{n_2}}{n_2!}\Big(\frac{2\Omega_{1,2}}{(\Theta_{1,2}-\Omega_{1,2})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}\Big)^{n_1}\\
&\times\Big(\frac{2\Omega_{2,2}}{(\Theta_{2,2}-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}\Big)^{n_2}. \qquad \text{(A3.8.11)}
\end{aligned}
$$

Now applying Taylor series expansion of the exponential function to the double summation we obtain

$$
\begin{aligned}
g_2(\tau,S,v_1,v_2) &= \exp\Big\{\Big(\frac{\Theta_{1,2}-\Omega_{1,2}}{\sigma_1^2}\Big)(v_1+\Phi_1\tau)+\Big(\frac{\Theta_{2,2}-\Omega_{2,2}}{\sigma_2^2}\Big)(v_2+\Phi_2\tau)\Big\}e^{i\eta\ln S+i\eta(r-q)\tau}\\
&\times\exp\Big\{-\Big(\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}\Big)v_1-\Big(\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}\Big)v_2\Big\}\\
&\times\Big(\frac{2\Omega_{1,2}}{(\Theta_{1,2}-\Omega_{1,2})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}\Big)^{\frac{2\Phi_1}{\sigma_1^2}}\Big(\frac{2\Omega_{2,2}}{(\Theta_{2,2}-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}\Big)^{\frac{2\Phi_2}{\sigma_2^2}}\\
&\times\exp\Big\{\frac{2\Omega_{1,2}y_1}{(\Theta_{1,2}-\Omega_{1,2})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}+\frac{2\Omega_{2,2}y_2}{(\Theta_{2,2}-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}\Big\}.
\end{aligned}
$$
(A3.8.12)

Reverting to the v_1 and v_2 variables from the system (A3.8.5) we obtain

$$
\begin{aligned}
g_2(\tau,S,v_1,v_2) &= \exp\Big\{\Big(\frac{\Theta_{1,2}-\Omega_{1,2}}{\sigma_1^2}\Big)(v_1+\Phi_1\tau)+\Big(\frac{\Theta_{2,2}-\Omega_{2,2}}{\sigma_2^2}\Big)(v_2+\Phi_2\tau)\Big\}e^{i\eta\ln S+i\eta(r-q)\tau}\\
&\times\exp\Big\{-\Big(\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}\Big)v_1-\Big(\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}\Big)v_2\Big\}\\
&\times\Big(\frac{2\Omega_{1,2}}{(\Theta_{1,2}-\Omega_{1,2})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}\Big)^{\frac{2\Phi_1}{\sigma_1^2}}\Big(\frac{2\Omega_{2,2}}{(\Theta_{2,2}-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}\Big)^{\frac{2\Phi_2}{\sigma_2^2}}\\
&\times\exp\Big\{\frac{2\Omega_{1,2}}{(\Theta_{1,2}-\Omega_{1,2})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}\times\frac{2\Omega_{1,2}v_1}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}\Big\}\\
&\times\exp\Big\{\frac{2\Omega_{2,2}}{(\Theta_{2,2}-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}\times\frac{2\Omega_{2,2}v_2}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}\Big\}. \qquad \text{(A3.8.13)}
\end{aligned}
$$

For convenience, we now attempt to represent this density in the form presented in Heston (1993). This is accomplished by adopting the representation

$$g_2(\tau,S,v_1,v_2;-\eta)=\exp\Big(-i\eta\ln S+B_2(\tau,-\eta)+D_{1,2}(\tau,-\eta)v_1+D_{2,2}(\tau,-\eta)v_2\Big), \tag{A3.8.14}$$

where

$$\begin{aligned}
B_2(\tau,\eta)&=i\eta(r-q)\tau+\frac{\Phi_1}{\sigma_1^2}\Big\{(\Theta_{1,2}-\Omega_{1,2})\tau-2\ln\Big(\frac{(\Theta_{1,2}-\Omega_{1,2})(1-e^{\Omega_{1,2}\tau})+2\Omega_{1,2}}{2\Omega_{1,2}}\Big)\Big\}\\
&\quad+\frac{\Phi_2}{\sigma_2^2}\Big\{(\Theta_{2,2}-\Omega_{2,2})\tau-2\ln\Big(\frac{(\Theta_{2,2}-\Omega_{2,2})(1-e^{\Omega_{2,2}\tau})+2\Omega_{2,2}}{2\Omega_{2,2}}\Big)\Big\},\\
D_{1,2}(\tau,\eta)&=\frac{(\Theta_{1,2}-\Omega_{1,2})}{\sigma_1^2}-\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}+\frac{2\Omega_{1,2}}{(\Theta_{1,2}-\Omega_{1,2})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}\times\frac{2\Omega_{1,2}v_1}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)},\\
D_{2,2}(\tau,\eta)&=\frac{(\Theta_{2,2}-\Omega_{2,2})}{\sigma_2^2}-\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}+\frac{2\Omega_{2,2}}{(\Theta_{2,2}-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}\times\frac{2\Omega_{2,2}v_2}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}.
\end{aligned}$$

By letting

$$Q_{1,2}=\frac{\Theta_{1,2}+\Omega_{1,2}}{\Theta_{1,2}-\Omega_{1,2}}\quad\text{and}\quad Q_{2,2}=\frac{\Theta_{2,2}+\Omega_{2,2}}{\Theta_{2,2}-\Omega_{2,2}},$$

the above three functions reduce to

$$\begin{aligned}
B_2(\tau,\eta)&=i\eta(r-q)\tau+\frac{\Phi_1}{\sigma_1^2}\Big\{(\Theta_{1,2}+\Omega_{1,2})\tau-2\ln\Big(\frac{1-Q_{1,2}e^{\Omega_{1,2}\tau}}{1-Q_{1,2}}\Big)\Big\}\\
&\quad+\frac{\Phi_2}{\sigma_2^2}\Big\{(\Theta_{2,2}+\Omega_{2,2})\tau-2\ln\Big(\frac{1-Q_{2,2}e^{\Omega_{2,2}\tau}}{1-Q_{2,2}}\Big)\Big\},
\end{aligned}\tag{A3.8.15}$$

$$D_{1,2}(\tau,\eta)=\frac{(\Theta_{1,2}+\Omega_{1,2})}{\sigma_1^2}\Big[\frac{1-e^{\Omega_{1,2}\tau}}{1-Q_{1,2}e^{\Omega_{1,2}\tau}}\Big], \tag{A3.8.16}$$

$$D_{2,2}(\tau,\eta)=\frac{(\Theta_{2,2}+\Omega_{2,2})}{\sigma_2^2}\Big[\frac{1-e^{\Omega_{2,2}\tau}}{1-Q_{2,2}e^{\Omega_{2,2}\tau}}\Big]. \tag{A3.8.17}$$

Substituting equation (A3.8.14) into equation (A3.8.3) we obtain

$$\begin{aligned}
V_E(\tau,S,v_1,v_2)=\frac{e^{-r\tau}}{2\pi}&\int_{-\infty}^{\infty}g_2(\tau,S,v_1,v_2;-\eta)\int_{\ln K}^{\infty}e^xe^{i\eta x}dxd\eta\\
&-K\int_{-\infty}^{\infty}g_2(\tau,S,v_1,v_2;-\eta)\int_{\ln K}^{\infty}e^{i\eta x}dxd\eta.
\end{aligned}\tag{A3.8.18}$$

The two components on the RHS of the above equation have similar properties to equations (A3.7.1) and (A3.7.2) respectively described in Appendix 3.7. We can evaluate the integrals in equation (A3.8.18) using equations (A3.7.10) and (A3.7.11) provided that $g_2(\tau, S, v_1, v_2; \eta - i)$ satisfies appropriate assumptions. The first assumption we we must verify is that $g_2(\tau, S, v_1, v_2; \eta - i)$ can be expressed as a function of η. This assumption is satisfied since

$$g_2(\tau, S, v_1, v_2; \eta - i) = Se^{(r-q)\tau} g_1(\tau, S, v_1, v_2; \eta),$$

where

$$g_1(\tau, S, v_1, v_2; \eta) = \exp\Big(i\eta \ln S + B_2(\tau, \eta) + D_{1,1}(\tau, \eta)v_1 + D_{2,1}(\tau, -\eta)v_2\Big), \quad \text{(A3.8.19)}$$

with

$$\begin{aligned} B_1(\tau, \eta) = i\eta(r-q)\tau &+ \frac{\Phi_1}{\sigma_1^2}\Big\{(\Theta_{1,1} + \Omega_{1,1})\tau - 2\ln\Big(\frac{1 - Q_{1,2}e^{\Omega_{1,1}\tau}}{1 - Q_{1,1}}\Big)\Big\} \\ &+ \frac{\Phi_2}{\sigma_2^2}\Big\{(\Theta_{2,1} + \Omega_{2,1})\tau - 2\ln\Big(\frac{1 - Q_{2,1}e^{\Omega_{2,1}\tau}}{1 - Q_{2,1}}\Big)\Big\}, \qquad \text{(A3.8.20)} \\ D_{1,1}(\tau, \eta) &= \frac{(\Theta_{1,1} + \Omega_{1,1})}{\sigma_1^2}\Big[\frac{1 - e^{\Omega_{1,1}\tau}}{1 - Q_{1,1}e^{\Omega_{1,1}\tau}}\Big], \qquad \text{(A3.8.21)} \\ D_{2,1}(\tau, \eta) &= \frac{(\Theta_{2,1} + \Omega_{2,1})}{\sigma_2^2}\Big[\frac{1 - e^{\Omega_{2,1}\tau}}{1 - Q_{2,1}e^{\Omega_{2,1}\tau}}\Big]. \qquad \text{(A3.8.22)} \end{aligned}$$

Furthermore, by using the same reasoning as in equation (A3.7.9) it can also be shown that

$$g_j(\tau, S, v_1, v_2; -\eta) = \overline{g_j(\tau, S, v_1, v_2; \eta,)}, \quad \text{for} \quad j = 1, 2, \qquad \text{(A3.8.23)}$$

and hence all the assumptions required to carry out the calculations yielding (A3.7.10) and (A3.7.11) are satisfied. Thus equation (A3.8.18) becomes

$$V_E(\tau, S, v_1, v_2) = e^{-q\tau} SP_1(\tau, S, v_1, v_2; K) - e^{-r\tau} KP_2(\tau, S, v_1, v_2; K), \qquad \text{(A3.8.24)}$$

where

$$P_j(\tau,S,v_1,v_2;K)=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{g_j(\tau,S,v_1,v_2;\eta)e^{-i\eta\ln K}}{i\eta}\right)d\eta, \qquad \text{(A3.8.25)}$$

for $j=1,2$ which is the result in Proposition 3.8.1.

Appendix 3.9. Proof of Proposition 3.8.2

We proceed as we did in Appendix 3.8 by first letting $x=\log(S)$ and making use of the relation $C_P(\tau,\log(S),v_1,v_2)=V_P(\tau,S,v_1,v_2)$ introduced in equation (3.2.30). Substituting the density function presented in Proposition 3.7.1 to the early exercise premium component in equation (3.2.40) we obtain after rearranging

$$\begin{aligned}V_P(\tau,S,v_1,v_2)=&\int_0^\tau e^{-r(\tau-\xi)}\int_0^\infty\int_0^\infty\int_{\ln b(\xi,w_1,w_2)}^\infty[qe^y-rK] \qquad \text{(A3.9.1)}\\&\times\left[\frac{1}{2\pi}\int_{-\infty}^\infty e^{i\eta y}g_2^A(\tau-\xi,S,v_1,v_2;-\eta,w_1,w_2)d\eta\right]dydw_1dw_2d\xi,\end{aligned}$$

where

$$\begin{aligned}g_2^A(\tau-\xi,S,v_1,v_2;\eta,w_1,w_2)=&\exp\Big\{\Big(\frac{\Theta_{1,2}-\Omega_{1,2}}{\sigma_1^2}\Big)\big(v_1-w_1+\Phi_1(\tau-\xi)\big) \qquad \text{(A3.9.2)}\\&+\Big(\frac{\Theta_{2,2}-\Omega_{2,2}}{\sigma_2^2}\Big)\big(v_2-w_2+\Phi_2(\tau-\xi)\big)\Big\}\\\times\exp\Big\{-\Big(\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}(\tau-\xi)}-1)}\Big)&\big(w_1e^{\Omega_{1,2}(\tau-\xi)}+v_1\big)-\Big(\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}(\tau-\xi)}-1)}\Big)\big(w_2e^{\Omega_{2,2}(\tau-\xi)}+v_2\big)\Big\}\\\times e^{i\eta\ln S+i\eta(r-q)(\tau-\xi)}&\frac{2\Omega_{1,2}e^{\Omega_{1,2}(\tau-\xi)}}{\sigma_1^2(e^{\Omega_1(\tau-\xi)}-1)}\frac{2\Omega_{2,2}e^{\Omega_{2,2}(\tau-\xi)}}{\sigma_2^2(e^{\Omega_{2,2}(\tau-\xi)}-1)}\Big(\frac{w_1e^{\Omega_{1,2}(\tau-\xi)}}{v_1}\Big)^{\frac{\Phi_1}{\sigma_1^2}-\frac{1}{2}}\Big(\frac{w_2e^{\Omega_{2,2}(\tau-\xi)}}{v_2}\Big)^{\frac{\Phi_2}{\sigma_2^2}-\frac{1}{2}}\\\times I_{\frac{2\Phi_1}{\sigma_1^2}-1}&\Big(\frac{4\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}(\tau-\xi)}-1)}(v_1w_1e^{\Omega_{1,2}(\tau-\xi)})^{\frac{1}{2}}\Big)I_{\frac{2\Phi_2}{\sigma_2^2}-1}\Big(\frac{4\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}(\tau-\xi)}-1)}(v_2w_2e^{\Omega_{2,2}(\tau-\xi)})^{\frac{1}{2}}\Big).\end{aligned}$$

Equation (A3.9.1) is equivalent to

$$\begin{aligned}V_P(\tau,S,v_1,v_2)=&\int_0^\tau e^{-r(\tau-\xi)}\int_0^\infty\int_0^\infty\frac{1}{2\pi}\Bigg[q\int_{-\infty}^\infty g_2^A(\tau-\xi,S,v_1,v_2;-\eta,w_1,w_2)\int_{\ln b(\xi,w_1,w_2)}^\infty e^ye^{i\eta y}dyd\eta\\&-rK\int_{-\infty}^\infty g_2^A(\tau-\xi,S,v_1,v_2;-\eta,w_1,w_2)\int_{\ln b(\xi,w_1,w_2)}^\infty e^{i\eta y}dyd\eta\Bigg]dw_1dw_2d\xi. \qquad \text{(A3.9.3)}\end{aligned}$$

By proceeding in the same way that we handled equation (A3.8.18) when simplifying the complex integrals we can write

$$g_2^A(\tau-\xi,S,v_1,v_2;\eta-i,w_1,w_2)=e^{(r-q)(\tau-\xi)}Sg_1^A(\tau-\xi,S,v_1,v_2;\eta,w_1,w_2). \quad \text{(A3.9.4)}$$

so that equation (A3.9.3) reduces to

$$\begin{aligned}V_P(\tau,S,v_1,v_2)=\int_0^{\tau}\int_0^{\infty}\int_0^{\infty}\Big[&qe^{-q(\tau-\xi)}SP_1^A\left[\tau-\xi,S,v_1,v_2;w_1,w_2,b(\xi,w_1,w_2)\right]\\&-re^{-r(\tau-\xi)}KP_2^A\left[\tau-\xi,S,v_1,v_2;w_1,w_2,b(\xi,w_1,w_2)\right]\Big]dw_1dw_2d\xi, \quad \text{(A3.9.5)}\end{aligned}$$

where

$$P_j^A(\tau-\xi,S,v_1,v_2;w_1,w_2,b(\xi,w_1,w_2))=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}Re\left(\frac{g_j^A(\tau-\xi,S,v_1,v_2;\eta,w_1,w_2)e^{-i\eta\ln b(\xi,w_1,w_2)}}{i\eta}\right)d\eta,$$

for $j=1,2$ which is the result given in Proposition 3.8.2.

Appendix 3.10. Proof of Proposition 3.9.1

Going back to equation (A3.9.1), we have expressed the early exercise premium value as

$$\begin{aligned}V_P^A(\tau,S,v_1,v_2)=\int_0^{\tau}e^{-r(\tau-\xi)}\int_0^{\infty}\int_0^{\infty}\frac{1}{2\pi}\Big[&q\int_{-\infty}^{\infty}g_2^A(\tau-\xi,S,v_1,v_2;-\eta,w_1,w_2)\\&\times\int_{\ln b(\xi,w_1,w_2)}^{\infty}e^{y}e^{i\eta y}dyd\eta\\-rK\int_{-\infty}^{\infty}g_2^A(\tau-\xi,S,v_1,v_2;-\eta,w_1,w_2)&\int_{\ln b(\xi,w_1,w_2)}^{\infty}e^{i\eta y}dyd\eta\Big]dw_1dw_2d\xi.\end{aligned}$$

With the approximation, $\ln b(\tau, v_1, v_2) = b_0(\tau) + b_1(\tau)v_1 + b_2(\tau)v_2$, the above equation is transformed to

$$V_P^A(\tau, S, v_1, v_2) = \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_0^\infty \frac{1}{2\pi}\Big[q \int_{-\infty}^\infty g_2^A(\tau-\xi, S, v_1, v_2; -\eta, w_1, w_2) \times \int_{b_0(\xi)+b_1(\xi)w_1+b_2(\xi)w_2}^\infty e^y e^{i\eta y} dy d\eta - rK \int_{-\infty}^\infty g_2^A(\tau-\xi, S, v_1, v_2; -\eta, w_1, w_2) \int_{b_0(\xi)+b_1(\xi)w_1+b_2(\xi)w_2}^\infty e^{i\eta y} dy d\eta\Big] dw_1 dw_2 d\xi. \tag{A3.10.1}$$

By letting $z = y - b_1(\xi)w_1 - b_2(\xi)w_2$ and substituting this into equation (A3.10.1) we obtain

$$V_P^A(\tau, S, v_1, v_2) = \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_0^\infty \frac{1}{2\pi}\Big[q \int_{-\infty}^\infty g_2^A(\tau-\xi, S, v_1, v_2; -\eta, w_1, w_2) \times \int_{b_0(\xi)}^\infty e^{(1+i\eta)(z+b_1(\xi)w_1+b_2(\xi)w_2)} dz d\eta - rK \int_{-\infty}^\infty g_2^A(\tau-\xi, S, v_1, v_2; -\eta, w_1, w_2) \int_{b_0(\xi)}^\infty e^{i\eta(z+b_1(\xi)w_1+b_2(\xi)w_2)} dz d\eta\Big] dw_1 dw_2 d\xi. \tag{A3.10.2}$$

This can be further simplified to

$$V_P^A(\tau, S, v_1, v_2) = \int_0^\tau e^{-r(\tau-\xi)} \int_{b_0(\xi)}^\infty [qe^z J_1(\xi, z) - rK J_2(\xi, z)] dz d\xi, \tag{A3.10.3}$$

where

$$J_1(\xi, z) = \int_0^\infty \int_0^\infty e^{b_1(\xi)w_1+b_2(\xi)w_2+i\eta(z+b_1(\xi)w_1+b_2(\xi)w_2)} \times \frac{1}{2\pi} \int_{-\infty}^\infty g_2^A(\tau-\xi, S, v_1, v_2; -\eta, w_1, w_2) d\eta dw_1 dw_2, \tag{A3.10.4}$$

and

$$J_2(\xi, z) = \int_0^\infty \int_0^\infty e^{i\eta[b_1(\xi)w_1+b_2(\xi)w_2]} \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\eta z} g_2^A(\tau-\xi, S, v_1, v_2; -\eta, w_1, w_2) d\eta dw_1 dw_2. \tag{A3.10.5}$$

The two expressions for $J_1(\xi, z)$ and $J_2(\xi, z)$ are now in a form that can allow us to simplify the integrals with respect to w_1 and w_2. By making use of equations (A3.8.4) and (A3.8.5) followed by repeating similar steps to those from equations (A3.8.6)-(A3.8.13) in Appendix 3.8 we note that $J_2(\xi, z)$ becomes

$$\begin{aligned}
J_2(\xi,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Bigg[&\exp\Big\{\Big(\frac{\Theta_{1,2}-\Omega_{1,2}}{\sigma_1^2}\Big)\big(v_1+\Phi_1(\tau-\xi)\big)+\Big(\frac{\Theta_{2,2}-\Omega_{2,2}}{\sigma_2^2}\Big)\big(v_2+\Phi_2(\tau-\xi)\big)\Big\}\\
&\times\exp\Big\{-\Big(\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}(\tau-\xi)}-1)}\Big)v_1-\Big(\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}(\tau-\xi)}-1)}\Big)v_2\Big\}e^{i\eta(z-\ln S)-i\eta(r-q)(\tau-\xi)}\\
&\times\Big(\frac{2\Omega_{1,2}}{(\Theta_{1,2}-i\eta\sigma_1^2b_1(\xi)-\Omega_{1,2})(1-e^{-\Omega_{1,2}(\tau-\xi)})+2\Omega_{1,2}}\Big)^{\frac{2\Phi_1}{\sigma_1^2}}\\
&\times\Big(\frac{2\Omega_{2,2}}{(\Theta_{2,2}-i\eta\sigma_2^2b_2(\xi)-\Omega_{2,2})(1-e^{-\Omega_{2,2}(\tau-\xi)})+2\Omega_{2,2}}\Big)^{\frac{2\Phi_2}{\sigma_2^2}}\\
&\times\exp\Big\{\frac{2\Omega_{1,2}}{(\Theta_{1,2}-i\eta\sigma_1^2b_1(\xi)-\Omega_{1,2})(1-e^{-\Omega_{1,2}(\tau-\xi)})+2\Omega_{1,2}}\times\frac{2\Omega_{1,2}v_1}{\sigma_1^2(e^{\Omega_{1,2}(\tau-\xi)}-1)}\Big\}\\
&\times\exp\Big\{\frac{2\Omega_{2,2}}{(\Theta_{2,2}-i\eta\sigma_2^2b_2(\xi)-\Omega_{2,2})(1-e^{-\Omega_{2,2}(\tau-\xi)})+2\Omega_{2,2}}\times\frac{2\Omega_{2,2}v_2}{\sigma_2^2(e^{\Omega_{2,2}(\tau-\xi)}-1)}\Big\}\Bigg]d\eta.
\end{aligned} \tag{A3.10.6}$$

Proceeding the way we handled equation (A3.8.14) the above equation simplifies to

$$J_2(\xi,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta z}\hat{g}_2^A(\tau-\xi, S, v_1, v_2; -\eta, b_1, b_2)d\eta, \tag{A3.10.7}$$

where

$$\begin{aligned}
\hat{g}_2^A(\tau,S,v_1,v_2;\eta,b_1,b_2) &= \exp\Big\{i\eta\ln S + B_2^A(\tau,\eta,b_1,b_2)+D_{1,2}^A(\tau,\eta,b_1)v_1+D_{2,2}^A(\tau,\eta,b_2)v_2\Big\},\\
B_2^A(\tau,\eta) &= i\eta(r-q)\tau+\frac{\Phi_1}{\sigma_1^2}\Big\{(\Theta_{1,2}-\Omega_{1,2})\tau-2\ln\Big(\frac{(\Theta_{1,2}+i\eta\sigma_1^2b_1-\Omega_{12})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}{2\Omega_{1,2}}\Big)\Big\}\\
&\quad+\frac{\Phi_2}{\sigma_2^2}\Big\{(\Theta_{2,2}-\Omega_{2,2})\tau-2\ln\Big(\frac{(\Theta_{2,2}+i\eta\sigma_2^2b_2-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}{2\Omega_{2,2}}\Big)\Big\},\\
D_{1,2}^A(\tau,\eta) &= \frac{\Theta_{1,2}-\Omega_{1,2}}{\sigma_1^2}-\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}+\frac{2\Omega_{1,2}}{(\Theta_{1,2}+i\eta\sigma_1^2b_1-\Omega_{1,2})(1-e^{-\Omega_{1,2}\tau})+2\Omega_{1,2}}\times\frac{2\Omega_{1,2}}{\sigma_1^2(e^{\Omega_{1,2}\tau}-1)}\\
D_{2,2}^A(\tau,\eta) &= \frac{\Theta_{2,2}-\Omega_{2,2}}{\sigma_2^2}-\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}+\frac{2\Omega_{2,2}}{(\Theta_{2,2}+i\eta\sigma_2^2b_2-\Omega_{2,2})(1-e^{-\Omega_{2,2}\tau})+2\Omega_{2,2}}\times\frac{2\Omega_{2,2}}{\sigma_2^2(e^{\Omega_{2,2}\tau}-1)}.
\end{aligned} \tag{A3.10.8}$$

The functions $B_2^A(\tau,\eta)$, $D_{1,2}^A(\tau,\eta)$ and $D_{2,2}^A(\tau,\eta)$ can be simplified further by letting

$$Q_{1,2}^A = \frac{\Theta_{1,2} + i\eta\sigma_1^2 b_1 + \Omega_{1,2}}{\Theta_{1,2} + i\eta\sigma_1^2 b_1 - \Omega_{1,2}}, \quad \text{and} \quad Q_{2,2}^A = \frac{\Theta_{2,2} + i\eta\sigma_2^2 b_2 + \Omega_{2,2}}{\Theta_{2,2} + i\eta\sigma_2^2 b_2 - \Omega_{2,2}}, \tag{A3.10.9}$$

such that

$$\begin{aligned}
B_2^A(\tau,\eta,b_1,b_2) &= i\eta(r-q)\tau + \frac{\Phi_1}{\sigma_1^2}\left\{(\Theta_{1,2}+\Omega_{1,2})\tau - 2\ln\left(\frac{1-Q_{1,2}^A e^{\Omega_{1,2}\tau}}{1-Q_{1,2}^A}\right)\right\} \\
&\quad + \frac{\Phi_2}{\sigma_2^2}\left\{(\Theta_{2,2}+\Omega_{2,2})\tau - 2\ln\left(\frac{1-Q_{2,2}^A e^{\Omega_{2,2}\tau}}{1-Q_{2,2}^A}\right)\right\}, \\
D_{1,2}^A(\tau,\eta,b_1) &= -i\eta b_1 + \frac{(\Theta_{1,2}+\Omega_{1,2})}{\sigma_1^2}\left[\frac{1-e^{\Omega_{1,2}\tau}}{1-Q_{1,2}^A e^{\Omega_{1,2}\tau}}\right], \\
D_{2,2}^A(\tau,\eta,b_2) &= -i\eta b_2 + \frac{(\Theta_{2,2}+\Omega_{2,2})}{\sigma_2^2}\left[\frac{1-e^{\Omega_{2,2}\tau}}{1-Q_{2,2}^A e^{\Omega_{2,2}\tau}}\right].
\end{aligned} \tag{A3.10.10}$$

By using a similar transformation to that between Q_1 and Q_2 in Appendix 3.7 we can write

$$\hat{g}_2^A(\tau-\xi,S,v_1,v_2;\eta-i,b_1,b_2) = e^{(r-q)(\tau-\xi)}S\hat{g}_1^A(\tau-\xi,S,v_1,v_2;\eta,b_1,b_2), \tag{A3.10.11}$$

where

$$\begin{aligned}
\hat{g}_1^A(\tau,S,v_1,v_2;\eta,b_1,b_2) &= \exp\left\{i\eta\ln S + B_1^A(\tau,\eta,b_1,b_2) + D_{1,1}^A(\tau,\eta,b_1)v_1 + D_{2,1}^A(\tau,\eta,b_2)v_2\right\}, \\
B_1^A(\tau,\eta,b_1,b_2) &= i\eta(r-q)\tau + \frac{\Phi_1}{\sigma_1^2}\left\{(\Theta_{1,1}+\Omega_{1,1})\tau - 2\ln\left(\frac{1-Q_{1,1}^A e^{\Omega_{1,2}\tau}}{1-Q_{1,1}^A}\right)\right\} \\
&\quad + \frac{\Phi_2}{\sigma_2^2}\left\{(\Theta_{2,1}+\Omega_{2,1})\tau - 2\ln\left(\frac{1-Q_{2,1}^A e^{\Omega_{2,1}\tau}}{1-Q_{2,1}^A}\right)\right\}, \\
D_{1,1}^A(\tau,\eta,b_1) &= -i\eta b_1 + \frac{(\Theta_{1,1}+\Omega_{1,1})}{\sigma_1^2}\left[\frac{1-e^{\Omega_{1,1}\tau}}{1-Q_{1,1}^A e^{\Omega_{1,1}\tau}}\right], \\
D_{2,1}^A(\tau,\eta,b_2) &= -i\eta b_2 + \frac{(\Theta_{2,1}+\Omega_{2,1})}{\sigma_2^2}\left[\frac{1-e^{\Omega_{2,1}\tau}}{1-Q_{2,1}^A e^{\Omega_{2,1}\tau}}\right].
\end{aligned}$$

Incorporating these into equation (A3.10.3) we obtain

$$V_P^A(\tau,S,v_1,v_2)=\int_0^\tau e^{-r(\tau-\xi)}\int_{b_0(\xi)}^\infty qe^z\frac{1}{2\pi}\int_{-\infty}^\infty e^{i\eta z}e^{(r-q)(\tau-\xi)}S\hat{g}_1^A(\tau-\xi,S,v_1,v_2;-\eta,b_1(\xi),b_2(\xi))d\eta dz d\xi$$
$$-\int_0^\tau e^{-r(\tau-\xi)}\int_{b_0(\xi)}^\infty rK\frac{1}{2\pi}\int_{-\infty}^\infty e^{i\eta z}\hat{g}_2^A(\tau-\xi,S,v_1,v_2;-\eta,b_1(\xi),b_2(\xi))d\eta dz d\xi,$$

which implies that

$$V_P^A(\tau,S,v_1,v_2)=\int_0^\tau e^{-r(\tau-\xi)}qSe^{(r-q)(\tau-\xi)}\frac{1}{2\pi}\int_{-\infty}^\infty \hat{g}_1^A(\tau-\xi,S,v_1,v_2;-\eta,b_1(\xi),b_2(\xi))\int_{b_0(\xi)}^\infty e^z e^{i\eta z}dz\eta d\xi$$
$$-\int_0^\tau e^{-r(\tau-\xi)}rK\frac{1}{2\pi}\int_{-\infty}^\infty \hat{g}_2^A(\tau-\xi,S,v_1,v_2;-\eta,b_1(\xi),b_2(\xi))\int_{b_0(\xi)}^\infty e^{i\eta z}dz d\eta d\xi. \tag{A3.10.12}$$

Proceeding as we did in equation (A3.8.18) we obtain the result that

$$V_P(\tau,S,v_1,v_2)=\int_0^\tau [qe^{-q(\tau-\xi)}S\hat{P}_1^A(\tau-\xi,S,v_1,v_2;b_0(\xi),b_1(\xi),b_2(\xi))$$
$$-re^{-r(\tau-\xi)}K\hat{P}_2^A(\tau-\xi,S,v_1,v_2;b_0(\xi),b_1(\xi),b_2(\xi))]d\xi, \tag{A3.10.13}$$

where

$$\hat{P}_j^A(\tau-\xi,S,v_1,v_2;b_0(\xi),b_1(\xi),b_2(\xi))=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{\hat{g}_j^A(\tau-\xi,S,v_1,v_2;\eta,b_1(\xi),b_2(\xi))e^{-i\eta b_0(\xi)}}{i\eta}\right)d\eta, \tag{A3.10.14}$$

for $j=1,2$, which is the result presented in Proposition 3.9.1.

CHAPTER 4

Option Pricing Under Multiple Stochastic Volatility Processes

4.1. Introduction

Whilst much work has been centred on single stochastic volatility processes, little has been done on generalizing to a situation where the underlying asset is driven by more than one stochastic variance process. As highlighted in earlier chapters, da Fonseca et al. (2005), (2008) have considered the pricing of European options written on an underlying asset whose dynamics evolve under the influence of multiple stochastic variance processes. American option pricing under multiple stochastic volatility processes has not been pursued mainly because of the additional challenges associated with the multi dimensional early exercise boundary.

In this chapter we generalise the model in Chapter 3 to a situation where the underlying asset evolves under the influence of multiple stochastic variance processes of the Heston (1993) type. We model the variance processes as independent stochastic factors that can influence the dynamics of the return process. We use hedging arguments to derive the corresponding pricing partial differential equation (PDE). We follow almost similar steps to those presented in Chapter 3 to come up with the integral representation of the American call option price. This integral representation is a function of the corresponding transition density function for the driving stochastic processes. We find the transition density functions by solving the associated backward Kolmogorov PDE.

Based on results in Chapter 3 and Cheang et al. (2009) we use induction proofs to show that all the results we present hold. This chapter is organized as follows, the problem statement is outlined in Section 4.2. The pricing PDE is derived using traditional hedging arguments. We also derive the integral representation of the American option price in this section together with the corresponding transition density PDE. As in Chapter 3 the solution procedure involves application of a combination of Fourier and Laplace transforms to this PDE. In Sections 4.3 and 4.4 we transform the PDE to a characteristic PDE which we solve by the method of characteristics in Section 4.5. Once we obtain the solution of the characteristic equations, we then present the explicit density function in terms of the original state variables by applying Laplace and Fourier inversion Theorems. This is outlined in Sections 4.6 and 4.7 respectively. We then present the full representation of the American option price in Section 4.8. By suitable manipulations, we provide a multi-dimensional version of the Heston (1993) characteristic function for the European option component.

4.2. Problem Statement

Let $V(t, S, v_1, \cdots, v_n)$ be the price of an American call option written on the underlying asset, S which pays a continuously compounded dividend yield at a rate of q in a market offering a constant interest rate, r. The payoff of the American call option at maturity is represented as $(S-K)^+$, where K is the strike price. Under the real world probability measure, $\mathbb{P}$, the dynamics of S is governed by the stochastic differential equation (SDE) system

$$
\begin{aligned}
dS &= \mu S dt + \sum_{j=1}^{n} \sqrt{v_j} S dZ_j, \\
dv_j &= \kappa_j(\theta_j - v_j)dt + \sigma_j \sqrt{v_j} dZ_{n+j}, \quad j = 1, \cdots, n.
\end{aligned} \tag{4.2.1}
$$

where μ is the instantaneous return per unit time, $v_1, \cdots, v_n$ is a vector of instantaneous variances per unit time, $\theta_1, \cdots, \theta_n$ are the long run variances for the n stochastic variance processes respectively while[1] $\sigma_1, \cdots, \sigma_n$ are the volatilities of $v_1, \cdots, v_n$ respectively. The speeds of mean reversion for the n variance processes are denoted as

[1]Strictly speaking, the volatilities are $\sigma_1\sqrt{v_1}, \cdots, \sigma_n\sqrt{v_n}$ but we shall loosely refer to σ_j as the volatility of volatility as this has become common usage.

$\kappa_1, \cdots, \kappa_n$ respectively while $Z_1, \cdots, Z_{2n}$ are correlated Wiener processes. Correlation exists among the pairs, $(Z_1, Z_{n+1}), \cdots, (Z_n, Z_{2n})$ with all other cross correlations being equal to zero. We denote the corresponding correlation coefficients by $\rho_{1,n+1}, \cdots, \rho_{n,2n}$.

By following similar arguments to those used in Section 3.2 and Cheang et al. (2009), the system (4.2.1) can be re-expressed in terms of independent Wiener processes such that

$$dS = \mu S dt + \sum_{j=1}^{n} \sqrt{v_j} S dW_j, \tag{4.2.2}$$

$$dv_j = \kappa_j(\theta_j - v_j)dt + \rho_{j,n+j}\sigma_j\sqrt{v_j}dW_j + \sqrt{1-\rho_{j,n+j}^2}\sigma_j\sqrt{v_j}dW_{n+j}, \quad j = 1, \cdots, n.$$

The n variance processes are of the Cox et al. (1985) type and for such processes to be positive definite, the arguments in Feller (1951) can be extended to show that the conditions

$$2\kappa_j\theta_j \geq \sigma_j^2, \quad j = 1\cdots, n, \tag{4.2.3}$$

need to be satisfied. We have also argued in Section 3.2 that the correlation coefficients must satisfy the inequalities

$$-1 < \rho_{j,n+j} < \min\left(\frac{\kappa_j}{\sigma_j}, 1\right), \quad j = 1, \cdots, n, \tag{4.2.4}$$

for the variance processes to be finite. Cheang et al. (2009) have shown that the conditions like those in equations (4.2.3) and (4.2.4) also ensure that the general solution of the underlying asset process takes the form

$$S_t = S_0 \exp\left\{\mu t - \frac{1}{2}\sum_{j=1}^{n}\int_0^t v_j du + \sum_{j=1}^{n}\int_0^t \sqrt{v_j}dW_j\right\}, \tag{4.2.5}$$

where

$$\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\int_0^t v_j du + \sum_{j=1}^{n}\int_0^t \sqrt{v_j}dW_j\right\}, \tag{4.2.6}$$

is a martingale[2] under the measure, $\mathbb{P}$. The traded underlying asset in the system (4.2.2) is insufficient to hedge away all sources of risk when combined in a portfolio with an option written on S. As discussed in Chapter 3, this leads to market incompleteness, a situation whereby different option prices can be derived from the same input parameters. A sufficient number of options needs to be placed in the hedging portfolio in order to eliminate arbitrage opportunities. To facilitate the pricing procedure, we need to switch from the measure, $\mathbb{P}$ to a corresponding risk neutral probability measure, $\mathbb{Q}$. This change of measure is accomplished by invoking Girsanov's theorem for Wiener processes. As demonstrated in Chapter 3, we need to determine first the associated Radon-Nikodym derivative (R_N) such that

$$R_N = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\int_0^t \mathbf{\Lambda}_u^T \mathbf{\Sigma}^{-1}\mathbf{\Lambda}_u du - \int_0^t (\mathbf{\Sigma}^{-1}\mathbf{\Lambda}_u)^T d\mathbf{W}\right\}, \tag{4.2.7}$$

where, $\mathbf{\Sigma}$ is the correlation matrix for the Wiener processes and $\mathbf{\Lambda}_t = [\lambda_1(t), \cdots, \lambda_{2n}(t)]$ is the vector of market prices of risk associated with the vector of Wiener processes, $\mathbf{W}$. Once the vector of market prices of risk is specified, Girsanov's Theorem for Wiener processes then asserts that there exist some Wiener processes under $\mathbb{Q}$ such that

$$d\tilde{W}_j = \lambda_j(t)dt + dW_j, \quad j = 1, \cdots, n. \tag{4.2.8}$$

From the vector, $\mathbf{\Lambda}_t$, the elements $\lambda_1(t), \cdots, \lambda_n(t)$ are the market prices of risk associated with shocks impinging on the traded underlying asset. Such risk is usually diversified away if the hedging portfolio is large enough. The other n elements, $\lambda_{n+1}(t), \cdots, \lambda_{2n}(t)$, are the market prices of variance risk associated with shocks from the instantaneous variance processes. Application of Girsanov's Theorem to the SDE system in equation (4.2.2) yields

[2]Details of the proof are discussed in Cheang et al. (2009).

$$\begin{cases} dS = (r-q)Sdt + \sum_{j=1}^{n} \sqrt{v_j} S d\tilde{W}_j, \\ dv_j = \left[\kappa_j(\theta_j - v_j) - \lambda_{n+j}(t)\sqrt{1-\rho_{j,n+j}^2}\sigma_j\sqrt{v_j}\right] dt \\ \qquad +\rho_{j,n+j}\sigma_j\sqrt{v_j}d\tilde{W}_j + \sqrt{1-\rho_{j,n+j}^2}\sigma_j\sqrt{v_j}d\tilde{W}_{n+j}, \quad j = 1, \cdots, n. \end{cases} \tag{4.2.9}$$

In determining the market prices of variance risk, we use the same reasoning as we did in Chapter 3 and set

$$\lambda_{n+j}(t) = \frac{\lambda_j\sqrt{v_j}}{\sigma_j\sqrt{1-\rho_{j,n+j}^2}}, \quad j = 1, \cdots, n, \tag{4.2.10}$$

where the λ_j's are constants. By substituting this transformation into the SDE system (4.2.9) we obtain

$$\begin{cases} dS = (r-q)Sdt + \sum_{j=1}^{n} \sqrt{v_j} S d\tilde{W}_j, \\ dv_j = \left[\kappa_j\theta_j - (\kappa_j + \lambda_j)v_j\right] dt + \rho_{j,n+j}\sigma_j\sqrt{v_j}d\tilde{W}_j \\ \qquad + \sqrt{1-\rho_{j,n+j}^2}\sigma_j\sqrt{v_j}d\tilde{W}_{n+j}, \quad j = 1, \cdots, n. \end{cases} \tag{4.2.11}$$

Given these dynamics, we can use standard hedging arguments to derive the American call option pricing PDE, namely

$$\frac{\partial V}{\partial \tau}(\tau, S, v_1, \cdots, v_n) = \mathcal{L}V(\tau, S, v_1, \cdots, v_n) - rV(\tau, S, v_1, \cdots, v_n), \tag{4.2.12}$$

where

$$\begin{aligned} \mathcal{L} = (r-q)S\frac{\partial}{\partial S} &+ \sum_{j=1}^{n}[\kappa_j(\theta_j - v_j) - \lambda_j v_j]\frac{\partial}{\partial v_j} + \frac{1}{2}S^2\sum_{j=1}^{n} v_j \frac{\partial^2}{\partial S^2} \\ &+ \frac{1}{2}\sum_{j=1}^{n}\sigma_j^2 v_j \frac{\partial^2}{\partial v_j^2} + \sum_{j=1}^{n}\rho_{j,n+j}\sigma_j v_j S\frac{\partial^2}{\partial S \partial v_j}, \end{aligned} \tag{4.2.13}$$

is the Dynkin operator associated with the SDE system (4.2.11). The PDE holds in the region $0 < S < b(\tau, v_1, \cdots, v_n)$ and $0 < v_1, \cdots, v_n < \infty$. Here, $S = b(\tau, v_1, \cdots, v_n)$ is the early exercise boundary of the American call option which makes the PDE (4.2.12) a free-boundary problem. The PDE (4.2.12) is solved subject to the initial and boundary conditions

$$V(0,S,v_1,\cdots,v_n) = (S-K)^+, \quad 0 < S < \infty, \tag{4.2.14}$$

$$V(\tau,0,v_1,\cdots,v_n) = 0, \quad \tau \geq 0, \tag{4.2.15}$$

$$V(\tau,b(\tau,v_1,\cdots,v_n),v_1,\cdots,v_n) = b(\tau,v_1,\cdots,v_n) - K, \quad \tau \geq 0, \tag{4.2.16}$$

$$\lim_{S\to b(\tau,v_1,\cdots,v_n)} \frac{\partial V}{\partial S}(\tau,S,v_1,\cdots,v_n) = 1, \quad \tau \geq 0. \tag{4.2.17}$$

Condition (4.2.14) is the payoff at maturity of the option contract while equation (4.2.15) is the absorbing state which ensures that the option ceases to exist once the underlying asset price hit zero. Equation (4.2.16) is the value matching condition which guarantees continuity of the option value function at the early exercise boundary, $b(\tau,v_1,\cdots,v_n)$. Equation (4.2.17) is called the smooth pasting condition which together with the value matching condition are imposed to avoid arbitrage opportunities.

Also associated with the system of equations in (4.2.11) is the multivariate transition density function which we denote as $G(\tau,S,v_1,\cdots,v_n;S_0,v_{1,0},\cdots,v_{n,0})$. The transition density function represents the transition probability of moving from the state $(S,v_1,\cdots,v_n)$ at time-to-maturity τ to the state $(S_0,v_{1,0},\cdots,v_{n,0})$ at maturity. This transition density function is a solution of the backward Kolmogorov PDE

$$\frac{\partial G}{\partial \tau} = \mathcal{L}G, \tag{4.2.18}$$

where $0 \leq S < \infty$ and $0 \leq v_1,\cdots,v_n < \infty$. Equation (4.2.18) is solved subject to the initial condition

$$G(0,S,v_1,\cdots,v_n;S_0,v_{1,0},\cdots,v_{n,0}) = \delta(S-S_0)\delta(v_1-v_{1,0})\cdots\delta(v_n-v_{n,0}), \tag{4.2.19}$$

where as in the previous chapters, $\delta(\cdot)$ is the Dirac delta function. For ease of notation we will simply write $G(\tau, S, v_1, \cdots, v_n)$ to denote the density function unless there is a need to make the full dependence explicit.

The underlying asset domain of the pricing PDE (4.2.12) is bounded above by the early exercise boundary, $b(\tau, v_1, \cdots, v_n)$. By using again Jamshidian's (1992) techniques, we can transform this into an unbounded domain resulting in the inhomogeneous PDE

$$\begin{aligned}\frac{\partial V}{\partial \tau}(\tau, S, v_1, \cdots, v_n) &= \mathcal{L}V(\tau, S, v_1, \cdots, v_n) - rV(\tau, S, v_1, \cdots, v_n) \\ &\quad + \mathbb{1}_{S \geq b(\tau, v_1, \cdots, v_n)}(qS - rK),\end{aligned} \tag{4.2.20}$$

where, $0 < S < \infty$, $0 < v_1, \cdots, v_n < \infty$ and $\mathbb{1}_{S \geq b(\tau, v_1, \cdots, v_n)}$ is an indicator function which is equal to one if $S \geq b(\tau, v_1, \cdots, v_n)$ or zero otherwise. Equation (4.2.20) is an inhomogeneous PDE whose solution can be readily found by applying Duhamel's principle. This is facilitated by switching to the log underlying asset space by setting $S = e^x$ and letting

$$\begin{aligned}C(\tau, x, v_1, \cdots, v_n) &\equiv V(\tau, e^x, v_1, \cdots, v_n), \\ U(\tau, x, v_1, \cdots, v_n) &\equiv G(\tau, e^x, v_1, \cdots, v_n).\end{aligned} \tag{4.2.21}$$

With these transformations the pricing PDE (4.2.20) becomes

$$\frac{\partial C}{\partial \tau} = \mathcal{M}C - rC + \mathbb{1}_{x \geq \ln b(\tau, v_1, \cdots, v_n)}(qe^x - rK), \tag{4.2.22}$$

where $\mathcal{M}$ is a differential operator defined as

$$\begin{aligned}\mathcal{M} = \Big(r - q - \frac{1}{2}\sum_{j=1}^{n} v_j\Big)\frac{\partial}{\partial x} + \sum_{j=1}^{n} \Phi_j \frac{\partial}{\partial v_j} - \sum_{j=1}^{n} \beta_j v_j \frac{\partial}{\partial v_j} + \sum_{j=1}^{n} v_j \frac{\partial^2}{\partial v_j^2} \\ + \sum_{j=1}^{n} \sigma_j^2 v_j \frac{\partial^2}{\partial v_j^2} + \sum_{j=1}^{n} \rho_{j,n+j}\sigma_j v_j \frac{\partial^2}{\partial x \partial v_j},\end{aligned} \tag{4.2.23}$$

with

$$\Phi_j = \kappa_j\theta_j \quad \beta_j = \kappa_j + \lambda_j, \quad j = 1, \cdots, n. \tag{4.2.24}$$

Equation (4.2.22) is solved subject to the initial and boundary condition

$$C(0, x, v_1, \cdots, v_n) = (e^x - K)^+, \quad -\infty < x < \infty, \tag{4.2.25}$$

$$b(\tau, v_1, \cdots, v_n) - K = C(\tau, b(\tau, v_1, \cdots, v_n), v_1, \cdots, v_n). \tag{4.2.26}$$

The smooth pasting condition can also be imposed depending on the valuation method under consideration.

The transition density PDE (4.2.18) also transforms to

$$\frac{\partial U}{\partial \tau} = \mathcal{M}U, \tag{4.2.27}$$

which is solved subject to the initial condition

$$U(0, x, v_1, \cdots, v_n) = \delta(x - x_0)\delta(v_1 - v_{1,0}) \cdots \delta(v_n - v_{n,0}). \tag{4.2.28}$$

We present the general solution of the PDE (4.2.22) in the proposition below.

Proposition 4.2.1. *The general solution of equation* (4.2.22) *can be represented as*

$$C(\tau, S, v_1, \cdots, v_n) = C_E(\tau, S, v_1, \cdots, v_n) + C_P(\tau, S, v_1, \cdots, v_n), \tag{4.2.29}$$

where

$$C_E(\tau,S, v_1, \cdots, v_n) = e^{-r\tau} \int_0^\infty \cdots \int_0^\infty \int_{-\infty}^\infty (e^u - K)^+ U(\tau, x, v_1, \cdots, v_n; u, w_1, \cdots, w_n) du dw_1 \cdots dw_n,$$

and

$$C_P(\tau, x, v_1, \cdots, v_n) = \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \cdots \int_0^\infty \int_{b(\xi,w_1,\cdots,w_n)}^\infty (qe^u - rK)$$
$$\times U(\tau - \xi, x, v_1, \cdots, v_n; u, w_1, \cdots, w_n) du dw_1 \cdots dw_n d\xi,$$

Proof: Refer to Appendix 4.1. □

The first component of equation (4.2.29) is the European option component as it is a function of the call option payoff function while the second term is the early exercise premium. To obtain a full representation of equation (4.2.29) we need an explicit form of the transition density function, $U(\tau, x, v_1, \cdots, v_n)$ which is the solution of the homogeneous PDE (4.2.27). Instead of solving the free-boundary problem (4.2.12), we are now faced with the relatively simpler simpler task of solving the homogeneous PDE for the density function. As we have seen in earlier chapters, this is the main strength of using the Jamshidian (1992) approach. In the following sections we outline the solution procedure for solving the PDE (4.2.27). As we have done in Chapter 3, we use a combination of Fourier and Laplace transforms to reduce the PDE to a form that we can readily solve. We will again apply a Fourier transform to the underlying asset variable and Laplace transforms to the instantaneous variance variables. We then solve the resulting system of PDEs by the method of characteristics.

4.3. Applying Fourier Transforms

As an initial step of finding the joint transition density function, we apply Definition 1.3.1 to the PDE (4.2.27) the results of which are presented in the proposition below.

Proposition 4.3.1. *The Fourier transform $\hat{U}(\tau, \eta, v_1, \cdots, v_n)$ of the PDE* (4.2.27) *is the solution of the PDE*

$$\frac{\partial \hat{U}}{\partial \tau} = -i\eta(r-q)\hat{U} + \frac{1}{2}\Lambda(\eta)\sum_{j=1}^{n} v_j \hat{U} + \sum_{j=1}^{n} \Phi_j \frac{\partial \hat{U}}{\partial v_j} + \frac{1}{2}\sum_{j=1}^{n} \sigma_j^2 v_j \frac{\partial^2 \hat{U}}{\partial v_j^2} - \sum_{j=1}^{n} \Theta_j v_j \frac{\partial \hat{U}}{\partial v_j}, \tag{4.3.1}$$

where

$$\Theta_j = \kappa_j + \lambda_j + i\eta\rho_{j,n+j}\sigma_j, \quad \Omega_j = \sqrt{\Theta_j^2 - \Lambda(\eta)\sigma_j^2} \quad \textit{and} \quad \Lambda(\eta) = i\eta - \eta^2, \quad j = 1, \cdots, n. \tag{4.3.2}$$

Equation (4.3.1) *is solved subject to the initial condition*

$$\hat{U}(0,\eta,v_1,\cdots,v_n) = e^{i\eta x_0}\delta(v_1 - v_{1,0})\cdots\delta(v_n - v_{n,0}) \tag{4.3.3}$$

Proof: Refer to Appendix 4.2. □

4.4. Applying Laplace Transforms

We have managed to apply a Fourier transform to the log underlying asset variable yielding the results of Proposition 4.3.1, the next step involves applying the Laplace transform to the instantaneous variance variables of this PDE. We present the resulting characteristic PDE in the proposition below.

PROPOSITION 4.4.1. *The Laplace transform* $\tilde{U}(\tau,\eta,s_1,\cdots,s_n)$ *of the PDE* (4.3.1) *satisfies the first order PDE*

$$\begin{aligned}\frac{\partial\tilde{U}}{\partial\tau} + \sum_{j=1}^{n}\Big\{\frac{1}{2}\sigma_j^2 s_j^2 - \Theta_j s_j + \frac{1}{2}\Lambda(\eta)\Big\}\frac{\partial\tilde{U}}{\partial s_j} &= \sum_{j=1}^{n}\Big\{(\Phi_j - \sigma_j^2)s_j - i\eta(r-q) + \Theta_j\Big\}\tilde{U} \\ &+ f_1(\tau,s_2,\cdots,s_n) + f_2(\tau,s_1,s_3,\cdots,s_n) + \cdots + f_n(\tau,s_1,\cdots,s_{n-1}).\end{aligned} \tag{4.4.1}$$

Equation (4.4.1) *is solved subject to the initial condition*

$$\tilde{U}(0,\eta,s_1,\cdots,s_n) = e^{i\eta x_0 - s_1 v_{1,0} - \cdots - s_n v_{n,0}}. \tag{4.4.2}$$

The inhomogeneous functions on the RHS of equation (4.4.1) *are given as*

$$\begin{aligned} f_1(\tau,s_2,\cdots,s_n) &= \Big(\frac{1}{2}\sigma_1^2 - \Phi_1\Big)\tilde{U}(\tau,\eta,0,s_2,\cdots,s_n) \\ &\vdots \\ f_n(\tau,s_1,\cdots,s_{n-1}) &= \Big(\frac{1}{2}\sigma_n^2 - \Phi_n\Big)\tilde{U}(\tau,\eta,s_1,\cdots,s_{n-1},0), \end{aligned} \tag{4.4.3}$$

and are determined such that

$$\lim_{s_1\to\infty}\tilde{U}(\tau,\eta,s_1,\cdots,s_n)=0,\quad \cdots \quad \lim_{s_n\to\infty}\tilde{U}(\tau,\eta,s_1,\cdots,s_n)=0, \tag{4.4.4}$$

respectively.

Proof: Refer to Appendix 4.3. □

We have managed to reduce the homogeneous PDE (4.2.27) for the density function to the PDE (4.4.1) which can be solved by using the method of characteristics. As highlighted before, most of our proofs in the coming sections are heavily dependant on the results of Chapter 3.

4.5. Solution of the Characteristic Equation

We are now in a position to solve the first – order PDE in (4.4.1). We present the corresponding solution in the proposition below.

PROPOSITION 4.5.1. *The solution of the PDE in equation* (4.4.1) *subject to the initial and boundary conditions* (4.4.2) *and* (4.4.4) *can be represented as*

$$\begin{aligned}\tilde{U}(\tau,\eta,s_1,\cdots,s_n) &= e^{i\eta x_0-i\eta(r-q)}\prod_{j=1}^{n}\Bigg[\left(\frac{2\Omega_j}{(\sigma_j^2 s_j-\Theta_j+\Omega_j)(e^{\Omega_j\tau}-1)+2\Omega_j}\right)^{2-\frac{2\Phi_j}{\sigma_j^2}}\\ &\times\exp\left\{-\left(\frac{\Theta_j-\Omega_j}{\sigma_j^2}\right)v_{j,0}\right\}\exp\left\{\left[\frac{(\Phi_j-\sigma_j^2)(\Theta_j-\Omega_j)}{\sigma_j^2}+\Theta_j\right]\tau\right\}\\ &\times\exp\left\{\frac{-2\Omega_j v_{j,0}(\sigma_j^2 s_j-\Theta_j+\Omega_j)e^{\Omega_j\tau}}{\sigma_j^2[(\sigma_j^2 s_j-\Theta_j+\Omega_j)(e^{\Omega_j\tau}-1)+2\Omega_j]}\right\}\Bigg]\\ &\times\left[\sum_{l=1}^{n}\Gamma\left(\frac{2\Phi_l}{\sigma_l^2}-1;\frac{2\Omega_l v_{l,0}e^{\Omega_l\tau}}{\sigma_l^2(e^{\Omega_l\tau}-1)}\times\frac{2\Omega_l}{(\sigma_l^2 s_l-\Theta_l+\Omega_l)(e^{\Omega_l\tau}-1)+2\Omega_l}\right)-(n-1)\right].\end{aligned} \tag{4.5.1}$$

The function $\Gamma(n;z)$ *is the incomplete gamma function defined in equation* (3.5.3).

Proof: Refer to Appendix 4.4. □

Now that we have managed to solve the characteristic PDE (4.4.1), the next step involves recovering the original instantaneous variance and underlying asset variables by applying Laplace and Fourier inversion theorems to the result in Proposition 4.5.1. We start by presenting the inverse Laplace transform in Section 4.6 followed by the inverse Fourier transform in Section 4.7.

4.6. Inverting the Laplace Transform

PROPOSITION 4.6.1. *The inverse Laplace transform of equation* (4.5.1) *can be represented as*

$$\begin{aligned}\hat{U}(\tau,\eta,v_1,\cdots,v_n) = e^{i\eta x_0 - i\eta(r-q)}\prod_{j=1}^{n}\Bigg[&\exp\left\{\left(\frac{\Theta_j-\Omega_j}{\sigma_j^2}\right)(v_j - v_{j,0}+\Phi_j\tau)\right\}\\ &\times\exp\left\{-\left(\frac{2\Omega_j}{\sigma_j^2(e^{\Omega_j\tau}-1)}\right)\left(v_{j,0}e^{\Omega_j\tau}+v_j\right)\right\}\frac{2\Omega_j e^{\Omega_j\tau}}{\sigma_j^2(e^{\Omega_j\tau}-1)}\left(\frac{v_{j,0}e^{\Omega_j\tau}}{v_j}\right)^{\frac{\Phi_j}{\sigma_j^2}-\frac{1}{2}}\\ &\times I_{\frac{2\Phi_j}{\sigma_j^2}-1}\left(\frac{4\Omega_j}{\sigma_j^2(e^{\Omega_j\tau}-1)}(v_jv_{j,0}e^{\Omega_j\tau})^{\frac{1}{2}}\right)\Bigg],\end{aligned} \tag{4.6.1}$$

where $I_k(z)$ is the modified Bessel function of the first kind defined as

$$I_k(z) = \sum_{n=0}^{\infty}\frac{(\frac{z}{2})^{2n+k}}{\Gamma(n+k+1)n!}. \tag{4.6.2}$$

Proof: Refer to Appendix 4.5. □

4.7. Inverting the Fourier Transform

PROPOSITION 4.7.1. *The inverse Fourier transform* $U(\tau, x, v_1, \cdots, v_n)$ *of equation* (4.6.1) *evaluates to*

$$U(\tau, x, v_1, \cdots, v_n; x_0, v_{1,0}, \cdots, v_{n,0}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta x_0} H(\tau, x, v_1, \cdots, v_n; -\eta, v_{1,0}, \cdots, v_{n,0}) d\eta, \tag{4.7.1}$$

where

$$\begin{aligned} H(\tau, x, v_1, \cdots, v_n; \eta, v_{1,0}, \cdots, v_{n,0}) = e^{i\eta x + i\eta(r-q)} \prod_{j=1}^{n} \Bigg[\frac{2\Omega_j e^{\Omega_j \tau}}{\sigma_j^2 (e^{\Omega_j \tau} - 1)} \left(\frac{v_{j,0} e^{\Omega_j \tau}}{v_j} \right)^{\frac{\Phi_j}{\sigma_j^2} - \frac{1}{2}} \\ \times \exp\left\{ \left(\frac{\Theta_j - \Omega_j}{\sigma_j^2} \right) (v_j - v_{j,0} + \Phi_j \tau) - \left(\frac{2\Omega_j}{\sigma_j^2 (e^{\Omega_j \tau} - 1)} \right) \left(v_{j,0} e^{\Omega_j \tau} + v_j \right) \right\} \\ \times I_{\frac{2\Phi_j}{\sigma_j^2} - 1} \left(\frac{4\Omega_j}{\sigma_j^2 (e^{\Omega_j \tau} - 1)} (v_j v_{j,0} e^{\Omega_j \tau})^{\frac{1}{2}} \right) \Bigg]. \end{aligned} \tag{4.7.2}$$

Proof: Refer to Appendix 4.6. □

Equation (4.7.2) is the explicit representation of the density function which was the only unknown function in Proposition 4.2.1. Having found this function, we are now in a position to present the full representation of the American call option price, which we do in the next section.

4.8. The American Option Price

In this section we provide the integral representation of the American call option price whose characteristic function we present in Heston (1993) form. In this chapter we are dealing with the call option case. It should be noted that the transition density function in Proposition 4.7.1 holds for any option contract with a continuous payoff written on the underlying asset, S. The American call option price is here represented as

$$V(\tau, S, v_1, \cdots, v_n) = V_E(\tau, S, v_1, \cdots, v_n) + V_P(\tau, S, v_1, \cdots, v_n), \tag{4.8.1}$$

where the two terms on the RHS of equation (4.8.1) are given by the two propositions below.

PROPOSITION 4.8.1. *The European option component of the American call option is represented as*

$$V_E(\tau, S, v_1, \cdots, v_n) = e^{-q\tau} S P_1(\tau, S, v_1, \cdots, v_n; K) - e^{-r\tau} K P_2(\tau, S, v_1, \cdots, v_n; K), \tag{4.8.2}$$

where

$$P_m(\tau, S, v_1, \cdots, v_n; K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{g_m(\tau, S, v_1, \cdots, v_n; \eta) e^{-i\eta \ln K}}{i\eta} \right) d\eta, \tag{4.8.3}$$

for $m = 1, 2$ with

$$\begin{aligned} g_m(\tau, S, v_1, \cdots, v_n; \eta) &= \exp\left\{ i\eta S + B_m(\tau, \eta) + \sum_{j=1}^n D_{j,m}(\tau, \eta) v_j \right\}, \\ B_m(\tau, \eta) &= i\eta(r-q)\tau + \sum_{j=1}^n \frac{\Phi_j}{\sigma_j^2} \left\{ (\Theta_{j,m} + \Omega_{j,m})\tau - 2\ln\left(\frac{1 - Q_{j,m} e^{\Omega_{j,m}\tau}}{1 - Q_{j,m}} \right) \right\}, \\ D_{j,m}(\tau, \eta) &= \frac{(\Theta_{j,m} + \Omega_{j,m})}{\sigma_j^2} \left[\frac{1 - e^{\Omega_{j,m}\tau}}{1 - Q_{j,m} e^{\Omega_{j,m}\tau}} \right]. \end{aligned} \tag{4.8.4}$$

Here[3], $\Theta_{j,m} = \Theta_j(i - \eta)$, $\Omega_{j,m} = \Omega_j(i - \eta)$ *and*

$$Q_{j,m} = \frac{\Theta_{j,m} + \Omega_{j,m}}{\Theta_{j,m} - \Omega_{j,m}}, \quad j = 1, \cdots, n \quad \text{and} \quad m = 1, 2. \tag{4.8.5}$$

Proof: Refer to Appendix 4.7. □

[3] Θ_j and Ω_j for $j = 1, \cdots, n$ have been defined in equation (4.3.2).

PROPOSITION 4.8.2. *The early exercise premium component of the American call option is represented as*

$$V_P(\tau,S,v_1,\cdots,v_n)=\int_0^\tau\int_0^\infty\cdots\int_0^\infty\Bigg[qe^{-q(\tau-\xi)}SP_1^A(\tau,S,v_1,\cdots,v_n;w_1,\cdots,w_n,b(\xi,w_1,\cdots,w_n))$$
$$-re^{-r(\tau-\xi)}KP_2^A(\tau-\xi,S,v_1,\cdots,v_n;w_1,\cdots,w_n,b(\xi,w_1,\cdots,w_n))\Bigg]dw_1\cdots dw_n d\xi, \tag{4.8.6}$$

where

$$P_m^A(\tau-\xi,S,v_1,\cdots,v_n;w_1,\cdots,w_n,b(\xi,w_1,\cdots,w_n))$$
$$=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{g_m^A(\tau-\xi,S,v_1,\cdots,v_n;\eta,w_1,\cdots,w_n)e^{-i\eta\ln b(\xi,w_1,\cdots,w_n)}}{i\eta}\right)d\eta,$$

for $m=1,2$ with

$$g_m^A(\tau-\xi,S,v_1,\cdots,v_n;\eta)=\prod_{j=1}^n\Bigg[\exp\left\{\left(\frac{\Theta_{j,m}-\Omega_{j,m}}{\sigma_j^2}\right)(v_j-w_j+\Phi_j\tau)\right\}$$
$$\times\exp\left\{-\left(\frac{2\Omega_{j,m}}{\sigma_j^2(e^{\Omega_{j,m}(\tau-\xi)}-1)}\right)\left(w_je^{\Omega_j(\tau-\xi)}+v_j\right)\right\}\frac{2\Omega_{j,m}e^{\Omega_{j,m}(\tau-\xi)}}{\sigma_j^2(e^{\Omega_{j,m}(\tau-\xi)}-1)}\left(\frac{w_je^{\Omega_{j,m}(\tau-\xi)}}{v_j}\right)^{\frac{\Phi_j}{\sigma_j^2}-\frac{1}{2}}$$
$$\times e^{i\eta S+i\eta(r-q)(\tau-\xi)}I_{\frac{2\Phi_j}{\sigma_j^2}-1}\left(\frac{4\Omega_{j,m}}{\sigma_j^2(e^{\Omega_{j,m}(\tau-\xi)}-1)}\left(v_jw_je^{\Omega_{j,m}(\tau-\xi)}\right)^{\frac{1}{2}}\right)\Bigg], \tag{4.8.7}$$

where, $\Theta_{j,m}$, $\Omega_{j,m}$ and $Q_{j,m}$ as given in Proposition 4.8.1.

Proof: Refer to Appendix 4.8. □

The American call option price is a function of the early exercise boundary, $b(\tau,v_1,\cdots,v_n)$ which is an unknown function. The early exercise boundary needs to be determined first as part of the solution process. This is achieved by solving the value matching condition

$$b(\tau,v_1,\cdots,v_n)-K=V(\tau,b(\tau,v_1,\cdots,v_n),v_1,\cdots,v_n). \tag{4.8.8}$$

Unlike in valuing American options under the GBM where the early exercise boundary is only a function of time, here it is a function of time and the n instantaneous variances. Different techniques can be invoked to solve equation (4.8.8), one of which would involve approximating the early exercise boundary first followed by applying root finding techniques as detailed in Chapters 2 and 3. To find a better approximating function one would first implement a finite difference algorithm of the pricing PDE and have an idea of the functional form of the early exercise boundaries. Another approach would also involve performing some empirical studies along the lines of Broadie et al. (2000). Once the early exercise boundary is determined, it becomes a simple matter of substituting this into equation (4.8.6) to find the corresponding option price. We have left numerical computations for future work in the area.

4.9. Conclusion

We have managed to generalise the results in Chapter 3 to the case when the underlying asset is being driven by multiple stochastic variance processes. The integral representation of the American call option has been derived using Duhamel's principle. This integral expression involves the transition density function for the underlying stochastic processes. It is well known that this transition density function satisfies the corresponding backward Kolmogorov PDE. With this knowledge, we have derived and presented a systematic approach for solving such a PDE. Instead of solving the free-boundary value problem for the American option, we have reduced this to a corresponding simpler problem of finding the transition density function.

We have used induction proofs to show how the propositions in Chapter 3 can be generalised. The price of the American call option is presented as the sum of the European option component and the early exercise premium component. Using similar arguments as in Cheang et al. (2009) and in Chapter 3, the European option component has been reduced to the corresponding Heston (1993) form. The early exercise premium component is a function of multiple integrals with respect to the variance variables, multiple modified Bessel functions and an integral with respect to time-to-maturity. The approach used to simplify the European option component cannot be directly applied to the early exercise premium component as the variance integrals of this component are functions of $b(\tau, v_1, \cdots, v_n)$ whose functional form is unknown. However, such simplifications can only be done after suitable early exercise boundary approximations. Such approximations have been considered for lower dimensional problems as detailed in Chapter 1. We leave it for future research to explore appropriate numerical schemes to handle this valuation problem.

Appendix 4.1. Proof of Proposition 4.2.1

Consider the PDE

$$\frac{\partial C}{\partial \tau} = \mathcal{M}C - rC + f(\tau, x, v_1, \cdots, v_n), \tag{A4.1.1}$$

where $\mathcal{M}$ is the Dynkins' operator is defined as

$$\begin{aligned}\mathcal{M} = (r-q)S\frac{\partial}{\partial S} + \frac{1}{2}S^2\sum_{j=1}^{n} v_j \frac{\partial^2}{\partial S^2} + \sum_{j=1}^{n}[\kappa_j(\theta_j - v_j) - \lambda_j v_j]\frac{\partial}{\partial v_j} \\ + \frac{1}{2}\sum_{j=1}^{n}\sigma_j^2 v_j \frac{\partial^2}{\partial v_j^2} + \sum_{j=1}^{n}\rho_{j,n+j}\sigma_j v_j S\frac{\partial^2}{\partial S \partial v_j}.\end{aligned}$$

The payoff at maturity of the American call option is denoted as $C(0, x, v_1, \cdots, v_n) = (e^x - K)^+$. The PDE (A4.1.1) is to be solved in the region $\tau \geq 0$, $-\infty \leq x < \infty$ and $0 \leq v_1, \cdots, v_n < \infty$. By use of Duhamel's principle[4], the solution of the PDE (A4.1.1)

[4]A natural extension of Duhamel's principle to the multi-dimensional setting yields this result.

is given by

$$
\begin{aligned}
C(\tau, x, v_1, \cdots, v_n) = e^{-r\tau} \int_0^\infty \cdots \int_0^\infty \int_{-\infty}^\infty (e^u - K)^+ \\
\times U(\tau, x, v_1, \cdots, v_n; u, w_1, \cdots, w_n) du dw_1 \cdots dw_n \\
+ \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\tau, u, w_1, \cdots, w_n) \\
\times U(\tau - \xi, x, v_1, \cdots, v_n; u, w_1, \cdots, w_n) du dw_1 \cdots dw_n d\xi \\
\equiv C_E(\tau, x, v_1, \cdots, v_n) + C_P(\tau, x, v_1, \cdots, v_n), \qquad \text{(A4.1.2)}
\end{aligned}
$$

where $C_E(\tau, x, v_1, \cdots, v_n)$ is the European call option component as it involves the payoff of a call option and $C_P(\tau, x, v_1, \cdots, v_n)$ is the corresponding early exercise premium component. To verify that this is the correct solution, we will show that (A4.1.2) satisfies the PDE (A4.1.1). Substituting $C(\tau, x, v_1, \cdots, v_n) = C_E(\tau, x, v_1, \cdots, v_n) + C_P(\tau, x, v_1, \cdots, v_n)$ into (A4.1.1), we obtain

$$
\begin{aligned}
&\frac{\partial C}{\partial \tau} + rC - \mathcal{M}C - f(\tau, x, v_1, \cdots, v_n) \\
&\quad = e^{-r\tau} \int_0^\infty \cdots \int_0^\infty \int_{-\infty}^\infty (e^u - K)^+ \left\{ \frac{\partial U}{\partial \tau} - \mathcal{M}U \right\} du dw_1, \cdots dw_n - rC_E + rC_E \\
&\quad + \int_0^\infty \cdots \int_0^\infty \int_{-\infty}^\infty f(\tau, u, w_1, \cdots, w_n) U(\tau, x, v_1, \cdots, v_n; u, w_1, \cdots, w_n) du dw_1 \cdots dw_n \\
&\quad + \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \cdots \int_0^\infty \int_{-\infty}^\infty f(\xi, u, w_1, \cdots, w_n) \frac{\partial U}{\partial \tau} du dw_1 \cdots dw_n d\xi - rC_P + rC_P \\
&\quad - \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \cdots \int_0^\infty \int_{-\infty}^\infty f(\xi, u, w_1, \cdots, w_n) \mathcal{M}U du dw_1 \cdots dw_n d\xi - f(\tau, x, v_1, \cdots, v_n) \\
&\quad = \int_0^\infty \cdots \int_0^\infty \int_{-\infty}^\infty f(\tau, x, v_1, \cdots, v_n) \delta(x - u) \delta(v_1 - w_1) \cdots \delta(v_n - w_n) du dw_1 \cdots dw_n \\
&+ \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\xi, x, v_1, \cdots, v_n) \left[\frac{\partial U}{\partial \tau} - \mathcal{M}U \right] du dw_1 \cdots dw_n d\xi - f(\tau, x, v_1, \cdots, v_n) \\
&\quad = f(\tau, x, v_1, \cdots, v_n) + 0 - f(\tau, x, v_1, \cdots, v_n) = 0
\end{aligned}
$$

Hence $C(\tau, x, v_1, \cdots, v_n)$ satisfies the PDE (A4.1.1) implying that the results of Proposition 4.2.1 hold.

Appendix 4.2. Proof of Proposition 4.3.1

In all the proofs from Appendices 4.2-4.8 we will use mathematical induction to show that the corresponding results hold. By induction we first show that the results hold for $n = 1$ or $n = 2$. If any of these two cases hold, we move on to assume that the result is true for $n = k$ and prove that it holds for $n = k + 1$. In this appendix we prove the result in Proposition 4.3.1.

When $n = 2$, equation (4.3.1) reduces to

$$\begin{aligned}\frac{\partial \hat{U}}{\partial \tau} &= -i\eta(r-q)\hat{U} + \frac{1}{2}\Lambda(\eta)v_1\hat{U} + \frac{1}{2}\Lambda(\eta)v_2\hat{U} + \Phi_1\frac{\partial \hat{U}}{\partial v_1} + \Phi_2\frac{\partial \hat{U}}{\partial v_2} \\ &\quad - \Theta_1 v_1\frac{\partial \hat{U}}{\partial v_1} - \Theta_2 v_2\frac{\partial \hat{U}}{\partial v_2} + \frac{1}{2}\sigma_1^2 v_1\frac{\partial^2 \hat{U}}{\partial v_1^2} + \frac{1}{2}\sigma_2^2 v_2\frac{\partial^2 \hat{U}}{\partial v_2^2}, \qquad \text{(A4.2.1)}\end{aligned}$$

which is the result in equation (3.3.1) of Chapter 3 hence the result holds for $n = 2$.

Having demonstrated that the result holds for $n = 2$, we proceed to the inductive step. This is achieved by assuming that the results in Proposition 4.3.1 holds for $n = k$, and then prove that it holds for $n = k + 1$. We proceed as follows

$$\begin{aligned}\frac{\partial \hat{U}}{\partial \tau} &= -i\eta(r-q)\hat{U} + \frac{1}{2}\Lambda(\eta)\sum_{j=1}^{k} v_j\hat{U} + \frac{1}{2}\Lambda(\eta)v_{k+1}\hat{U} + \sum_{j=1}^{k}\Phi_j\frac{\partial \hat{U}}{\partial v_j} + \Phi_{k+1}\frac{\partial \hat{U}}{\partial v_{k+1}} + \frac{1}{2}\sum_{j=1}^{k}\sigma_j^2 v_j\frac{\partial^2 \hat{U}}{\partial v_j^2} \\ &\quad - \sum_{j=1}^{k}\Theta_j v_j\frac{\partial \hat{U}}{\partial v_j} + \frac{1}{2}\sigma_{k+1}^2 v_{k+1}\frac{\partial^2 \hat{U}}{\partial v_{k+1}^2} - \Theta_{k+1}v_{k+1}\frac{\partial \hat{U}}{\partial v_{k+1}} \qquad \text{(A4.2.2)} \\ &= -i\eta(r-q)\hat{U} + \frac{1}{2}\Lambda(\eta)\sum_{j=1}^{k+1} v_j\hat{U} + \sum_{j=1}^{k+1}\Phi_j\frac{\partial \hat{U}}{\partial v_j} + \frac{1}{2}\sum_{j=1}^{k+1}\sigma_j^2 v_j\frac{\partial^2 \hat{U}}{\partial v_j^2} - \sum_{j=1}^{k+1}\Theta_j v_j\frac{\partial \hat{U}}{\partial v_j}.\end{aligned}$$

This implies that the result holds for $n = k + 1$ which means that equation (4.3.1) of Proposition 4.3.1 holds.

The initial condition in equation (4.3.3) is determined as follows

$$\begin{aligned}
\mathcal{F}\left\{U(0,x,v_1,\cdots,v_n)\right\} &= \int_{-\infty}^{\infty} e^{i\eta x} U(0,x,v_1,\cdots,v_n)dx \\
&= \int_{-\infty}^{\infty} e^{i\eta x} \delta(x-x_0)\delta(v_1-v_{1,0})\cdots\delta(v_2-v_{n,0})dx \\
&= e^{i\eta x_0}\delta(v_1-v_{1,0})\cdots\delta(v_n-v_{n,0}). \qquad \text{(A4.2.3)}
\end{aligned}$$

Appendix 4.3. Proof of Proposition 4.4.1

When $n=2$, equation (4.4.1) simplifies to

$$\begin{aligned}
&\frac{\partial\tilde{U}}{\partial\tau} + \left\{\frac{1}{2}\sigma_1^2 s_1^2 - \Theta_1 s_1 + \frac{1}{2}\Lambda(\eta)\right\}\frac{\partial\tilde{U}}{\partial s_1} + \left\{\frac{1}{2}\sigma_2^2 s_2^2 - \Theta_2 s_2 + \frac{1}{2}\Lambda(\eta)\right\}\frac{\partial\tilde{U}}{\partial s_2} \qquad \text{(A4.3.1)} \\
&\quad = \Big\{(\Phi_1-\sigma_1^2)s_1 + (\Phi_2-\sigma_2^2)s_2 - i\eta(r-q) + \Theta_1 + \Theta_2\Big\}\tilde{U} + f_1(\tau,s_2) + f_2(\tau,s_1),
\end{aligned}$$

which is the result in equation (3.4.1) of Chapter 3. It can easily be shown that when $n=1$ the corresponding results in[5] Cheang et al. (2009) are recovered. We now proceed to the inductive step. We assume that equation (4.4.1) holds for $n=k$ such that when $n=k+1$

$$\begin{aligned}
&\frac{\partial\tilde{U}}{\partial\tau} + \sum_{j=1}^{k}\left\{\frac{1}{2}\sigma_j^2 s_j^2 - \Theta_j s_j + \frac{1}{2}\Lambda(\eta)\right\}\frac{\partial\tilde{U}}{\partial s_j} + \left\{\frac{1}{2}\sigma_{k+1}^2 s_{k+1}^2 - \Theta_{k+1} s_{k+1} + \frac{1}{2}\Lambda(\eta)\right\}\frac{\partial\tilde{U}}{\partial s_{k+1}} \\
&\quad = \sum_{j=1}^{k}\left\{(\Phi_j-\sigma_j^2)s_j - i\eta(r-q) + \Theta_j\right\}\tilde{U} + \left\{(\Phi_{k+1}-\sigma_{k+1}^2)s_{k+1} - i\eta(r-q) + \Theta_{k+1}\right\}\tilde{U} \\
&\quad + f_1(\tau,s_2,\cdots,s_k) + f_2(\tau,s_1,s_3,\cdots,s_k) + \cdots + f_k(\tau,s_1,\cdots,s_{k-1}) \\
&\quad + f_{k+1}(\tau,s_1,\cdots,s_k). \qquad \text{(A4.3.2)}
\end{aligned}$$

[5]These authors consider the valuation of the American call option written on an underlying asset whose dynamics evolve under the influence of a single stochastic variance process and jump diffusion processes.

The above equation simplifies to

$$\frac{\partial \tilde{U}}{\partial \tau} + \sum_{j=1}^{k+1} \left\{ \frac{1}{2}\sigma_j^2 s_j^2 - \Theta_j s_j + \frac{1}{2}\Lambda(\eta) \right\} \frac{\partial \tilde{U}}{\partial s_j} = \sum_{j=1}^{k+1} \left\{ (\Phi_j - \sigma_j^2) s_j - i\eta(r-q) + \Theta_j \right\} \tilde{U}$$

$$+ f_1(\tau, s_2, \cdots, s_{k+1}) + f_2(\tau, s_1, s_3, \cdots, s_{k+1}) + \cdots + f_{k+1}(\tau, s_1, \cdots, s_k), \quad \text{(A4.3.3)}$$

which implies that equation (4.4.1) of Proposition 4.4.1 holds.

The Laplace transform of the initial condition in equation (4.3.3) is determined as follows:

$$\begin{aligned} \tilde{U}(0, \eta, s_1, \cdots, s_n) &= \int_0^\infty \cdots \int_0^\infty e^{-s_1 v_1 \cdots - s_n v_n} \hat{U}(0, \eta, v_1, \cdots, v_n) dv_1 \cdots dv_n \\ &= \int_0^\infty \cdots \int_0^\infty e^{-s_1 v_1 \cdots - s_n v_n} e^{i\eta x_0} \delta(v_1 - v_{1,0}) \cdots \delta(v_n - v_{n,0}) dv_1 \cdots dv_n \\ &= e^{i\eta x_0 - s_1 v_{1,0} - \cdots - s_n v_{n,0}}, \qquad \text{(A4.3.4)} \end{aligned}$$

which is the result in equation (4.4.2) of Proposition 4.4.1. The results in equation (4.4.3) are determined by using the same approach used in Appendix 3.3 of Chapter 3.

Appendix 4.4. Proof of Proposition 4.5.1

When $n = 2$, equation (4.5.1) reduces to

$$
\begin{aligned}
\tilde{U}(\tau,\eta,s_1,s_2) &= \left(\frac{2\Omega_1}{(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{\Omega_1\tau}-1)+2\Omega_1}\right)^{2-\frac{2\Phi_1}{\sigma_1^2}}\left(\frac{2\Omega_2}{(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{\Omega_2\tau}-1)+2\Omega_2}\right)^{2-\frac{2\Phi_2}{\sigma_2^2}}\\
&\times\exp\left\{-\left(\frac{\Theta_1-\Omega_1}{\sigma_1^2}\right)v_{1,0}-\left(\frac{\Theta_2-\Omega_2}{\sigma_2^2}\right)v_{2,0}+i\eta x_0\right\}\\
&\times\exp\left\{\left[\frac{(\Phi_1-\sigma_1^2)(\Theta_1-\Omega_1)}{\sigma_1^2}+\frac{(\Phi_2-\sigma_2^2)(\Theta_2-\Omega_2)}{\sigma_2^2}-i\eta(r-q)+\Theta_1+\Theta_2\right]\tau\right\}\\
&\times\exp\left\{\frac{-2\Omega_1 v_{1,0}(\sigma_1^2 s_1-\Theta_1+\Omega_1)e^{\Omega_1\tau}}{\sigma_1^2[(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{\Omega_1\tau}-1)+2\Omega_1]}\right\}\exp\left\{\frac{-2\Omega_2 v_{2,0}(\sigma_2^2 s_2-\Theta_2+\Omega_2)e^{\Omega_2\tau}}{\sigma_2^2[(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{\Omega_2\tau}-1)+2\Omega_2]}\right\}\\
&\quad\times\left[\Gamma\left(\frac{2\Phi_1}{\sigma_1^2}-1;\frac{2\Omega_1 v_{1,0}e^{\Omega_1\tau}}{\sigma_1^2(e^{\Omega_1\tau}-1)}\times\frac{2\Omega_1}{(\sigma_1^2 s_1-\Theta_1+\Omega_1)(e^{\Omega_1\tau}-1)+2\Omega_1}\right)\right.\\
&\qquad\left.+\Gamma\left(\frac{2\Phi_2}{\sigma_2^2}-1;\frac{2\Omega_2 v_{2,0}e^{\Omega_2\tau}}{\sigma_2^2(e^{\Omega_2\tau}-1)}\times\frac{2\Omega_2}{(\sigma_2^2 s_2-\Theta_2+\Omega_2)(e^{\Omega_2\tau}-1)+2\Omega_2}\right)-1\right],
\end{aligned}
$$

which is the result in equation (3.5.1) of Chapter 3. Now assume that the proposition holds for $n = k$, we need to show that the same holds for $n = k + 1$. We proceed as follows, when $n = k+1$ we have

$$\tilde{U}(\tau,\eta,s_1,\cdots,s_{k+1}) = e^{i\eta x_0 - i\eta(r-q)} \prod_{j=1}^{k} \Bigg[\left(\frac{2\Omega_j}{(\sigma_j^2 s_j - \Theta_j + \Omega_j)(e^{\Omega_j\tau}-1) + 2\Omega_j} \right)^{2-\frac{2\Phi_j}{\sigma_j^2}}$$
$$\times \exp\left\{ -\left(\frac{\Theta_j - \Omega_j}{\sigma_j^2}\right) v_{j,0} \right\} \exp\left\{ \left[\frac{(\Phi_j - \sigma_j^2)(\Theta_j - \Omega_j)}{\sigma_j^2} + \Theta_j \right]\tau \right\}$$
$$\times \exp\left\{ \frac{-2\Omega_j v_{j,0}(\sigma_j^2 s_j - \Theta_j + \Omega_j)e^{\Omega_j\tau}}{\sigma_j^2[(\sigma_j^2 s_j - \Theta_j + \Omega_j)(e^{\Omega_j\tau}-1) + 2\Omega_j]} \right\} \Bigg]$$
$$\times \left(\frac{2\Omega_{k+1}}{(\sigma_{k+1}^2 s_{k+1} - \Theta_{k+1} + \Omega_{k+1})(e^{\Omega_{k+1}\tau}-1) + 2\Omega_{k+1}} \right)^{2-\frac{2\Phi_{k+1}}{\sigma_{k+1}^2}}$$
$$\times \exp\left\{ -\left(\frac{\Theta_{k+1} - \Omega_{k+1}}{\sigma_{k+1}^2}\right) v_{k+1,0} \right\} \exp\left\{ \left[\frac{(\Phi_{k+1} - \sigma_{k+1}^2)(\Theta_{k+1} - \Omega_{k+1})}{\sigma_{k+1}^2} + \Theta_{k+1} \right]\tau \right\}$$
$$\times \exp\left\{ \frac{-2\Omega_{k+1} v_{k+1,0}(\sigma_{k+1}^2 s_{k+1} - \Theta_{k+1} + \Omega_{k+1})e^{\Omega_{k+1}\tau}}{\sigma_{k+1}^2[(\sigma_{k+1}^2 s_{k+1} - \Theta_{k+1} + \Omega_{k+1})(e^{\Omega_{k+1}\tau}-1) + 2\Omega_{k+1}]} \right\}$$
$$\times \Bigg[\sum_{l=1}^{k} \Gamma\left(\frac{2\Phi_l}{\sigma_l^2} - 1; \frac{2\Omega_l v_{l,0} e^{\Omega_l\tau}}{\sigma_l^2(e^{\Omega_l\tau}-1)} \times \frac{2\Omega_l}{(\sigma_l^2 s_l - \Theta_l + \Omega_l)(e^{\Omega_l\tau}-1) + 2\Omega_l} \right)$$
$$+ \Gamma\left(\frac{2\Phi_{k+1}}{\sigma_{k+1}^2} - 1; \frac{2\Omega_{k+1} v_{k+1,0} e^{\Omega_{k+1}\tau}}{\sigma_{k+1}^2(e^{\Omega_{k+1}\tau}-1)} \times \frac{2\Omega_{k+1}}{(\sigma_{k+1}^2 s_{k+1} - \Theta_{k+1} + \Omega_{k+1})(e^{\Omega_{k+1}\tau}-1) + 2\Omega_{k+1}} \right)$$
$$- (k-1) - 1 \Bigg],$$

The above equation reduces to

$$\tilde{U}(\tau,\eta,s_1,\cdots,s_{k+1}) = e^{i\eta x_0 - i\eta(r-q)} \prod_{j=1}^{k+1} \Bigg[\left(\frac{2\Omega_j}{(\sigma_j^2 s_j - \Theta_j + \Omega_j)(e^{\Omega_j\tau}-1) + 2\Omega_j} \right)^{2-\frac{2\Phi_j}{\sigma_j^2}}$$
$$\times \exp\left\{ -\left(\frac{\Theta_j - \Omega_j}{\sigma_j^2}\right) v_{j,0} \right\} \exp\left\{ \left[\frac{(\Phi_j - \sigma_j^2)(\Theta_j - \Omega_j)}{\sigma_j^2} + \Theta_j \right]\tau \right\}$$
$$\times \exp\left\{ \frac{-2\Omega_j v_{j,0}(\sigma_j^2 s_j - \Theta_j + \Omega_j)e^{\Omega_j\tau}}{\sigma_j^2[(\sigma_j^2 s_j - \Theta_j + \Omega_j)(e^{\Omega_j\tau}-1) + 2\Omega_j]} \right\} \Bigg]$$
$$\times \left[\sum_{l=1}^{k+1} \Gamma\left(\frac{2\Phi_l}{\sigma_l^2} - 1; \frac{2\Omega_l v_{l,0} e^{\Omega_l\tau}}{\sigma_l^2(e^{\Omega_l\tau}-1)} \times \frac{2\Omega_l}{(\sigma_l^2 s_l - \Theta_l + \Omega_l)(e^{\Omega_l\tau}-1) + 2\Omega_l} \right) - k \right],$$

implying that the results of Proposition 4.5.1 holds.

Appendix 4.5. Proof of Proposition 4.6.1

When $n = 2$, equation (4.6.1) reduces to

$$\hat{U}(\tau, \eta, v_1, v_2) = \exp\left\{\left(\frac{\Theta_1 - \Omega_1}{\sigma_1^2}\right)(v_1 - v_{1,0} + \Phi_1\tau) + \left(\frac{\Theta_2 - \Omega_2}{\sigma_2^2}\right)(v_2 - v_{2,0} + \Phi_2\tau)\right\}$$
$$\times \exp\left\{-\left(\frac{2\Omega_1}{\sigma_1^2(e^{\Omega_1\tau} - 1)}\right)\left(v_{1,0}e^{\Omega_1\tau} + v_1\right) - \left(\frac{2\Omega_2}{\sigma_2^2(e^{\Omega_2\tau} - 1)}\right)\left(v_{2,0}e^{\Omega_2\tau} + v_2\right)\right\}$$
$$\times e^{i\eta x_0 - i\eta(r-q)\tau}\frac{2\Omega_1 e^{\Omega_1\tau}}{\sigma_1^2(e^{\Omega_1\tau} - 1)}\frac{2\Omega_2 e^{\Omega_2\tau}}{\sigma_2^2(e^{\Omega_2\tau} - 1)}\left(\frac{v_{1,0}e^{\Omega_1\tau}}{v_1}\right)^{\frac{\Phi_1}{\sigma_1^2} - \frac{1}{2}}\left(\frac{v_{2,0}e^{\Omega_2\tau}}{v_2}\right)^{\frac{\Phi_2}{\sigma_2^2} - \frac{1}{2}}$$
$$\times I_{\frac{2\Phi_1}{\sigma_1^2} - 1}\left(\frac{4\Omega_1}{\sigma_1^2(e^{\Omega_1\tau} - 1)}(v_1 v_{1,0} e^{\Omega_1\tau})^{\frac{1}{2}}\right) I_{\frac{2\Phi_2}{\sigma_2^2} - 1}\left(\frac{4\Omega_2}{\sigma_2^2(e^{\Omega_2\tau} - 1)}(v_2 v_{2,0} e^{\Omega_2\tau})^{\frac{1}{2}}\right),$$

which is the result in equation (3.6.1) of Chapter 3. Assume that the result hold for $n = k$. Then, for $n = k + 1$ we have

$$\hat{U}(\tau, \eta, v_1, \cdots, v_{k+1}) = e^{i\eta x_0 - i\eta(r-q)}\prod_{j=1}^{k}\left[\exp\left\{\left(\frac{\Theta_j - \Omega_j}{\sigma_j^2}\right)(v_j - v_{j,0} + \Phi_j\tau)\right\}\right.$$
$$\times \exp\left\{-\left(\frac{2\Omega_j}{\sigma_j^2(e^{\Omega_j\tau} - 1)}\right)\left(v_{j,0}e^{\Omega_j\tau} + v_j\right)\right\}\frac{2\Omega_j e^{\Omega_j\tau}}{\sigma_j^2(e^{\Omega_j\tau} - 1)}\left(\frac{v_{j,0}e^{\Omega_j\tau}}{v_j}\right)^{\frac{\Phi_j}{\sigma_j^2} - \frac{1}{2}}$$
$$\left.\times I_{\frac{2\Phi_j}{\sigma_j^2} - 1}\left(\frac{4\Omega_j}{\sigma_j^2(e^{\Omega_j\tau} - 1)}(v_j v_{j,0} e^{\Omega_j\tau})^{\frac{1}{2}}\right)\right]$$
$$\times \exp\left\{\left(\frac{\Theta_{k+1} - \Omega_{k+1}}{\sigma_{k+1}^2}\right)(v_{k+1} - v_{k+1,0} + \Phi_{k+1}\tau)\right\}\exp\left\{-\left(\frac{2\Omega_{k+1}}{\sigma_{k+1}^2(e^{\Omega_{k+1}\tau} - 1)}\right)\left(v_{k+1,0}e^{\Omega_{k+1}\tau} + v_{k+1}\right)\right\}$$
$$\times \frac{2\Omega_{k+1}e^{\Omega_{k+1}\tau}}{\sigma_{k+1}^2(e^{\Omega_{k+1}\tau} - 1)}\left(\frac{v_{k+1,0}e^{\Omega_{k+1}\tau}}{v_{k+1}}\right)^{\frac{\Phi_{k+1}}{\sigma_{k+1}^2} - \frac{1}{2}} I_{\frac{2\Phi_{k+1}}{\sigma_{k+1}^2} - 1}\left(\frac{4\Omega_{k+1}}{\sigma_{k+1}^2(e^{\Omega_{k+1}\tau} - 1)}(v_{k+1}v_{k+1,0}e^{\Omega_{k+1}\tau})^{\frac{1}{2}}\right). \tag{A4.5.1}$$

Equation (A4.5.1) simplifies to

$$\hat{U}(\tau,\eta,v_1,\cdots,v_{k+1}) = e^{i\eta x_0 - i\eta(r-q)}\prod_{j=1}^{k+1}\left[\exp\left\{\left(\frac{\Theta_j-\Omega_j}{\sigma_j^2}\right)(v_j - v_{j,0}+\Phi_j\tau)\right\}\right.$$
$$\times\exp\left\{-\left(\frac{2\Omega_j}{\sigma_j^2(e^{\Omega_j\tau}-1)}\right)\left(v_{j,0}e^{\Omega_j\tau}+v_j\right)\right\}\frac{2\Omega_j e^{\Omega_j\tau}}{\sigma_j^2(e^{\Omega_j\tau}-1)}\left(\frac{v_{j,0}e^{\Omega_j\tau}}{v_j}\right)^{\frac{\Phi_j}{\sigma_j^2}-\frac{1}{2}}$$
$$\left.\times I_{\frac{2\Phi_j}{\sigma_j^2}-1}\left(\frac{4\Omega_j}{\sigma_j^2(e^{\Omega_j\tau}-1)}(v_jv_{j,0}e^{\Omega_j\tau})^{\frac{1}{2}}\right)\right],$$

which completes the proof.

Appendix 4.6. Proof of Proposition 4.7.1

By applying Definition 1.3.2 to equation (4.6.1) the inverse Fourier transform of the transition density function is represented as

$$\begin{aligned}\mathcal{F}^{-1}\left\{\hat{U}(\tau,x,v_1,\cdots,v_n)\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{U}(\tau,x,v_1,\cdots,v_n)d\eta\\ &= U(\tau,x,v_1,\cdots,v_n). \qquad \text{(A4.6.1)}\end{aligned}$$

Substituting $\hat{U}(\tau,x,v_1,\cdots,v_n)$ from equation (4.6.1) we obtain

$$U(\tau,x,v_1,\cdots,v_n;x_0,v_{1,0},\cdots,v_{n,0}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\eta x_0}H(\tau,x,v_1,\cdots,v_n;-\eta,v_{1,0},\cdots,v_{n,0})d\eta \qquad \text{(A4.6.2)}$$

where

$$H(\tau,x,v_1,\cdots,v_n;\eta,v_{1,0},\cdots,v_{n,0}) = e^{i\eta x+i\eta(r-q)}\prod_{j=1}^{n}\left[\frac{2\Omega_j e^{\Omega_j\tau}}{\sigma_j^2(e^{\Omega_j\tau}-1)}\left(\frac{v_{j,0}e^{\Omega_j\tau}}{v_j}\right)^{\frac{\Phi_j}{\sigma_j^2}-\frac{1}{2}}\right.$$
$$\times\exp\left\{\left(\frac{\Theta_j-\Omega_j}{\sigma_j^2}\right)(v_j-v_{j,0}+\Phi_j\tau)-\left(\frac{2\Omega_j}{\sigma_j^2(e^{\Omega_j\tau}-1)}\right)\left(v_{j,0}e^{\Omega_j\tau}+v_j\right)\right\}$$
$$\left.\times I_{\frac{2\Phi_j}{\sigma_j^2}-1}\left(\frac{4\Omega_j}{\sigma_j^2(e^{\Omega_j\tau}-1)}(v_jv_{j,0}e^{\Omega_j\tau})^{\frac{1}{2}}\right)\right], \qquad \text{(A4.6.3)}$$

which is the result presented in Proposition 4.7.1.

Appendix 4.7. Proof of Proposition 4.8.1

We perform proof by induction, when $n = 2$, equation (4.8.2) reduces to

$$V_E(\tau, S, v_1, v_2) = e^{-q\tau} S P_1(\tau, S, v_1, v_2; K) - e^{-r\tau} K P_2(\tau, S, v_1, v_2; K), \tag{A4.7.1}$$

where

$$P_m(\tau, S, v_1, v_2; K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{g_m(\tau, S, v_1, v_2; \eta) e^{-i\eta \ln K}}{i\eta} \right) d\eta, \tag{A4.7.2}$$

for $m = 1, 2$ with

$$\begin{aligned}
g_m(\tau, S, v_1, v_2; \eta) &= \exp\left\{ i\eta \ln S + B_m(\tau, \eta) + D_{1,m}(\tau, \eta) v_1 + D_{2,m}(\tau, \eta) v_2 \right\}, \\
B_m(\tau, \eta) &= i\eta(r - q)\tau + \frac{\Phi_1}{\sigma_1^2} \left\{ (\Theta_{1,m} + \Omega_{1,m})\tau - 2 \ln\left(\frac{1 - Q_{1,j} e^{\Omega_{1,m}\tau}}{1 - Q_{1,m}} \right) \right\} \\
&\quad + \frac{\Phi_2}{\sigma_2^2} \left\{ (\Theta_{2,m} + \Omega_{2,m})\tau - 2 \ln\left(\frac{1 - Q_{2,m} e^{\Omega_{2,m}\tau}}{1 - Q_{2,m}} \right) \right\}, \\
D_{1,m}(\tau, \eta) &= \frac{(\Theta_{1,m} + \Omega_{1,m})}{\sigma_1^2} \left[\frac{1 - e^{\Omega_{1,m}\tau}}{1 - Q_{1,m} e^{\Omega_{1,m}\tau}} \right], \\
D_{2,m}(\tau, \eta) &= \frac{(\Theta_{2,m} + \Omega_{2,m})}{\sigma_2^2} \left[\frac{1 - e^{\Omega_{2,m}\tau}}{1 - Q_{2,m} e^{\Omega_{2,m}\tau}} \right].
\end{aligned} \tag{A4.7.3}$$

Here, $\Theta_{j,m} = \Theta_j(i - \eta)$, $\Omega_{j,m} = \Omega_j(i - \eta)$, $Q_{j,m} = (\Theta_{j,m} + \Omega_{j,m})/(\Theta_{j,m} - \Omega_{j,m})$ for $j = 1, 2$ and $m = 1, 2$. This is the European option component presented in Chapter 3.

For the inductive step, we assume that the result holds for $n = k$ and then prove that it also holds for $n = k + 1$. When $n = k + 1$ we have

$$V_E(\tau, S, v_1, \cdots, v_k, v_{k+1}) = e^{-q\tau} S P_1(\tau, S, v_1, \cdots, v_k, v_{k+1}; K) - e^{-r\tau} K P_2(\tau, S, v_1, \cdots, v_k, v_{k+1}; K), \tag{A4.7.4}$$

where

$$P_m(\tau, S, v_1, \cdots, v_k, v_{k+1}; K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{g_m(\tau, S, v_1, \cdots, v_k, v_{k+1}; \eta)e^{-i\eta \ln K}}{i\eta}\right)d\eta, \tag{A4.7.5}$$

for $m = 1, 2$ with

$$g_m(\tau, S, v_1, \cdots, v_k, v_{k+1}; \eta) = \exp\left\{i\eta S + B_m(\tau, \eta) + \sum_{j=1}^{k} D_{j,m}(\tau, \eta)v_j + D_{k+1,m}(\tau, \eta)v_{k+1}\right\}, \tag{A4.7.6}$$

$$\begin{aligned} B_m(\tau, \eta) = i\eta(r-q)\tau &+ \sum_{j=1}^{k} \frac{\Phi_j}{\sigma_j^2}\left\{(\Theta_{j,m} + \Omega_{j,m})\tau - 2\ln\left(\frac{1 - Q_{j,m}e^{\Omega_{j,m}\tau}}{1 - Q_{j,m}}\right)\right\} \\ &+ \frac{\phi_{k+1}}{\sigma_{k+1}^2}\left\{(\Theta_{k+1,m} + \Omega_{k+1,m})\tau - 2\ln\left(\frac{1 - Q_{k+1,m}e^{\Omega_{k+1,m}\tau}}{1 - Q_{k+1,m}}\right)\right\}, \end{aligned} \tag{A4.7.7}$$

$$D_{j,m}(\tau, \eta) = \frac{(\Theta_{j,m} + \Omega_{j,m})}{\sigma_j^2}\left[\frac{1 - e^{\Omega_{j,m}\tau}}{1 - Q_{j,m}e^{\Omega_{j,m}\tau}}\right], \tag{A4.7.8}$$

$$D_{k+1,m}(\tau, \eta) = \frac{(\Theta_{k+1,m} + \Omega_{k+1,m})}{\sigma_{k+1}^2}\left[\frac{1 - e^{\Omega_{k+1,m}\tau}}{1 - Q_{k+1,m}e^{\Omega_{k+1,m}\tau}}\right]. \tag{A4.7.9}$$

The above system simplifies to

$$V_E(\tau, S, v_1, \cdots, v_{k+1}) = e^{-q\tau}SP_1(\tau, S, v_1, \cdots, v_{k+1}; K) - e^{-r\tau}KP_2(\tau, S, v_1, \cdots, v_{k+1}; K), \tag{A4.7.10}$$

where

$$P_m(\tau, S, v_1, \cdots, v_{k+1}; K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{g_m(\tau, S, v_1, \cdots, v_{k+1}; \eta)e^{-i\eta \ln K}}{i\eta}\right)d\eta, \tag{A4.7.11}$$

with

$$g_m(\tau, S, v_1, \cdots, v_{k+1}; \eta) = \exp\left\{ i\eta S + B_m(\tau, \eta) + \sum_{j=1}^{k+1} D_{j,m}(\tau, \eta) v_j \right\}, \qquad \text{(A4.7.12)}$$

$$B_m(\tau, \eta) = i\eta(r-q)\tau + \sum_{j=1}^{k+1} \frac{\Phi_j}{\sigma_j^2} \left\{ (\Theta_{j,m} + \Omega_{j,m})\tau - 2\ln\left(\frac{1 - Q_{j,m} e^{\Omega_{j,m}\tau}}{1 - Q_{j,m}} \right) \right\}, \qquad \text{(A4.7.13)}$$

$$D_{j,m}(\tau, \eta) = \frac{(\Theta_{j,m} + \Omega_{j,m})}{\sigma_j^2} \left[\frac{1 - e^{\Omega_{j,m}\tau}}{1 - Q_{j,m} e^{\Omega_{j,m}\tau}} \right], \qquad \text{(A4.7.14)}$$

Hence the result holds for $n = k+1$.

Appendix 4.8. Proof of Proposition 4.8.2

By first letting $V_P(\tau, S, v_1, \cdots, v_n) \equiv C_P(\tau, \log(S), v_1, \cdots, v_n)$ the early exercise premium component in Proposition 4.2.1 is simplified as follows

$$V_P(\tau, S, v_1, \cdots, v_n) = \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_0^\infty \int_{\ln b(\xi, w_1, \cdots, w_n)}^\infty [qe^y - rK] \times \left[\frac{1}{2\pi} \int_{-\infty}^\infty e^{i\eta y} g_2^A(\tau-\xi, S, v_1, \cdots, v_n; -\eta, w_1, \cdots, w_n) \right] dw_1 \cdots dw_n d\xi, \qquad \text{(A4.8.1)}$$

where

$$g_2^A(\tau-\xi, S, v_1, \cdots, v_n; \eta, w_1, \cdots, w_n) = \prod_{j=1}^n \left[\exp\left\{ \left(\frac{\Theta_{j,2} - \Omega_{j,2}}{\sigma_j^2} \right) (v_j - w_j + \Phi_j \tau) \right\} \times \exp\left\{ -\left(\frac{2\Omega_{j,2}}{\sigma_j^2 (e^{\Omega_{j,2}(\tau-\xi)} - 1)} \right) \left(w_j e^{\Omega_j(\tau-\xi)} + v_j \right) \right\} \frac{2\Omega_{j,2} e^{\Omega_{j,2}(\tau-\xi)}}{\sigma_j^2 (e^{\Omega_{j,2}(\tau-\xi)} - 1)} \left(\frac{w_j e^{\Omega_{j,2}(\tau-\xi)}}{v_j} \right)^{\frac{\Phi_j}{\sigma_j^2} - \frac{1}{2}} \times e^{i\eta S + i\eta(r-q)(\tau-\xi)} I_{\frac{2\Phi_j}{\sigma_j^2} - 1} \left(\frac{4\Omega_{j,2}}{\sigma_j^2 (e^{\Omega_{j,2}(\tau-\xi)} - 1)} \left(v_j w_j e^{\Omega_{j,2}(\tau-\xi)} \right)^{\frac{1}{2}} \right) \right]. \qquad \text{(A4.8.2)}$$

Equation (A4.8.1) is equivalent to

$$V_P(\tau,S,v_1,\cdots,v_n)=\int_0^\tau e^{-r(\tau-\xi)}\int_0^\infty\int_0^\infty\frac{1}{2\pi}\Bigg[q\int_{-\infty}^\infty g_2^A(\tau-\xi,S,v_1,\cdots,v_n;-\eta,w_1,\cdots,w_n)$$
$$\times\int_{\ln b(\xi,w_1,\cdots,w_n)}^\infty e^y e^{i\eta y}dyd\eta \qquad \text{(A4.8.3)}$$
$$-rK\int_{-\infty}^\infty g_2^A(\tau-\xi,S,v_1,\cdots,v_n;-\eta,w_1,\cdots,w_n)\int_{\ln b(\xi,w_1,\cdots,w_n)}^\infty e^{i\eta y}dyd\eta\Bigg]dw_1,\cdots,dw_nd\xi.$$

We know from Appendix 3.7 that

$$g_2^A(\tau-\xi,S,v_1,\cdots,v_n;\eta-i,w_1,\cdots,w_n)=e^{(r-q)(\tau-\xi)}Sg_1^A(\tau-\xi,S,v_1,\cdots,v_n;\eta,w_1,\cdots,w_n). \qquad \text{(A4.8.4)}$$

Using this result and following similar arguments as in Appendix 3.8 it can be shown that (A4.8.3) reduces to

$$V_P(\tau,S,v_1,\cdots,v_n)=\int_0^\tau\int_0^\infty\int_0^\infty\Bigg[qe^{-q(\tau-\xi)}SP_1^A(\tau-\xi,S,v_1,\cdots,v_n;b(\xi,w_1,\cdots,w_n))$$
$$-re^{-r(\tau-\xi)}KP_2^A(\tau-\xi,S,v_1,\cdots,v_n;b(\xi,w_1,\cdots,w_n))\Bigg]dw_1\cdots dw_nd\xi, \qquad \text{(A4.8.5)}$$

where

$$P_m^A(\tau-\xi,S,v_1,\cdots,v_n;w_1,\cdots,w_n,b(\xi,v_1,\cdots,v_n))$$
$$=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{g_m^A(\tau-\xi,S,v_1,\cdots,v_n;\eta,w_1,\cdots,w_n)e^{-i\eta\ln b(\xi,w_1,\cdots,w_n)}}{i\eta}\right)d\eta,$$

for $m=1,2$ which is the result given in Proposition 4.8.2.

CHAPTER 5

Method of Lines Approach for Pricing American Exotic Options

5.1. Introduction

Whilst a significant amount of work has been done on European style options as reviewed in Chapter 1, research on the pricing of American options written on more than one underlying asset suffers from the difficulties associated with handling the multi-dimensional early exercise boundary.

We have considered the pricing of American options written on two underlying assets whose dynamics evolve according to geometric Brownian motion in Chapter 2. In the current chapter we consider the evaluation of the American spread call option when the two underlying asset dynamics evolve under the influence of a single stochastic variance process. The same model has been used in Bakshi and Madan (2000), Dempster and Hong (2000) and Hurd and Zhou (2009) when evaluating European spread options under stochastic volatility. As highlighted in Dempster and Hong (2000), such a model is useful in modelling correlated assets commonly encountered in the energy markets. A detailed review of a variety of European option contracts derived from such correlated assets can be found in Carmona and Durrleman (2003).

In this chapter we present a numerical algorithm for solving the pricing partial differential equation (PDE) of the American spread call option using the method of lines. Similar techniques have been used in Meyer and van der Hoek (1997), Meyer (1998) and Chiarella et al. (2009) who all consider American option pricing models under different settings as reviewed in Chapter 1.

The remainder of this chapter is organised as follows, we outline the problem statement in Section 5.2 where we derive the pricing PDE of the American spread call option. An algorithm for solving the pricing PDE using the method of lines is outlined in Section 5.3. Numerical results are then presented in Section 5.4. We also make price comparisons with results from the Monte Carlo algorithm of Ibáñez and Zapatero (2004). It is in this section where we explore the impact of stochastic volatility on the prices, free boundaries and the Deltas of the American spread call option by making comparisons with geometric Brownian motion results considered in Chapter 2. Section 5.5 concludes the chapter.

5.2. Problem Statement

In this chapter we consider pricing of the American spread call option, $C(t, S_1, S_2, v)$ where S_1 and S_2 are prices of the two underlying assets whose dynamics evolve under the influence of a single stochastic variance process, v. The two underlying assets pay continuous compounded dividend yields at the rate of q_1 and q_2 respectively in a market offering a constant risk-free rate of interest, r. The payoff at maturity of the American spread option contract is denoted as[1] $(S_1 - S_2 - X)^+$ where X is the strike price. Under the real world probability measure, $\mathbb{P}^v$, the three stochastic processes are characterised by the stochastic differential equations (SDEs)

$$dS_1 = \mu_1 S_1 dt + \sigma_1 \sqrt{v} S_1 dZ_1, \tag{5.2.1}$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 \sqrt{v} S_2 dZ_2, \tag{5.2.2}$$

$$dv = \kappa_v(\theta - v)dt + \sigma_v \sqrt{v} dZ_3, \tag{5.2.3}$$

where μ_i for $i = 1, 2$ are the instantaneous returns per unit time of S_1 and S_2 respectively, v is the instantaneous variance per unit time, σ_1 and σ_2 are the constant volatilities associated with the returns for S_1 and S_2 respectively and, Z_i for $i = 1, 2, 3$

[1] Unlike in the previous chapters we are using X to denote the strike price as we will be using K to denote the total number of time steps for the method of lines algorithm outlined in the next section.

are correlated Wiener processes with the the following properties:

$$\mathbb{E}(dZ_1 dZ_2) = \rho_{12} dt,$$

$$\mathbb{E}(dZ_1 dZ_3) = \rho_{13} dt,$$

$$\mathbb{E}(dZ_2 dZ_3) = \rho_{23} dt.$$

Here, ρ_{12} denotes the correlation between Z_1 and Z_2, ρ_{13} is the correlation between Z_1 and Z_3, while ρ_{23} denotes the correlation between Z_2 and Z_3. The parameter θ is the long-run mean of v, κ_v is the speed of mean-reversion, and σ_v is the instantaneous volatility of v per unit time.

As demonstrated in earlier chapters, applying a Cholesky decomposition to the Wiener processes yields the following system of independent Wiener processes

$$\begin{bmatrix} dZ_1 \\ dZ_2 \\ dZ_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \sqrt{\frac{1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}}{1-\rho_{12}^2}} \end{bmatrix} \begin{bmatrix} dW_1 \\ dW_2 \\ dW_3 \end{bmatrix}. \tag{5.2.4}$$

To avoid trivial cases, we assume that $|\rho_{12}| < 1$. By incorporating the transformation (5.2.4) into the system (5.2.1)-(5.2.3) we obtain

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 \sqrt{v} S_1 dW_1, \\ dS_2 &= \mu_2 S_2 dt + \rho_{12}\sigma_2 \sqrt{v} S_2 dW_1 + \hat{\rho}_{12}\sigma_2\sqrt{v} S_2 dW_2, \\ dv &= \kappa_v(\theta - v)dt + \rho_{13}\sigma\sqrt{v} dW_1 + \hat{\rho}_{23}\sigma\sqrt{v} dW_2 + \hat{\rho}_{33}\sigma\sqrt{v} dW_3, \end{aligned} \tag{5.2.5}$$

where

$$\hat{\rho}_{12} = \sqrt{1-\rho_{12}^2}, \quad \hat{\rho}_{23} = \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}, \quad \text{and} \quad \hat{\rho}_{33} = \sqrt{\frac{1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}}{1-\rho_{12}^2}} \tag{5.2.6}$$

Using arguments introduced in Chapter 3, the constant parameters in the SDE like (5.2.3) must be chosen such that the condition

$$2\kappa\theta \geq \sigma^2, \tag{5.2.7}$$

is satisfied. Cheang et al. (2009) also show that in addition to the above condition, the correlation terms must be chosen such that

$$-1 < \rho_{13}, \rho_{23} < \min\left(\frac{\kappa}{\sigma}, 1\right). \tag{5.2.8}$$

The two conditions in equations (5.2.7) and (5.2.8) ensure that the initially nonnegative variance process can neither become zero nor explode under the real world probability measure, $\mathbb{P}^v$ and any other probability measure that is equivalent to $\mathbb{P}^v$. The SDE system (5.2.5) has two traded assets but three sources of risk. This mismatch implies that the two traded securities are insufficient to hedge away all of the risk factors; a situation that leads to market incompleteness. One feasible approach of completing the market when faced with such a situation is to place a sufficient number of options with different maturity periods in the hedging portfolio.

Such a hedging strategy usually results in the underlying assets having different drift coefficients to those presented in the system (5.2.5). To preserve our original underlying processes, we switch from the real world probability measure, $\mathbb{P}^v$, to the risk neutral probability measure which we denote as $\mathbb{Q}^v$. The process of switching from $\mathbb{P}^v$ to $\mathbb{Q}^v$ is accomplished by applying Girsanov's Theorem for Wiener processes[2] to the system

[2]As stated in Chapter 3, Girsanov's Theorem of Wiener processes states that

$$d\tilde{W}_j = \lambda dt + dW_j, \tag{5.2.9}$$

where $\tilde{W}_j$ for $j = 1, 2, 3$ are independent Wiener processes under the risk neutral probability measure, $\mathbb{Q}$.

(5.2.5) which yields

$$
\begin{aligned}
dS_1 &= (r-q_1)S_1dt + \sigma_1\sqrt{v}S_1d\tilde{W}_1,\\
dS_2 &= (r-q_2)S_2dt + \rho_{12}\sigma_2\sqrt{v}S_2d\tilde{W}_1 + \hat{\rho}_{12}\sigma_2\sqrt{v}S_2d\tilde{W}_2,\\
dv &= \kappa_v(\theta-v)dt - \lambda(t,S_1,S_2,v)\hat{\rho}_{33}\sigma\sqrt{v}dt + \rho_{13}\sigma\sqrt{v}d\tilde{W}_1\\
&\qquad + \hat{\rho}_{23}\sigma\sqrt{v}d\tilde{W}_2 + \hat{\rho}_{33}\sigma\sqrt{v}d\tilde{W}_3.
\end{aligned}
\tag{5.2.10}
$$

Here, $\lambda(t, S_1, S_2, v)$ is the market price of risk associated with bearing the risk of the non-traded variance factor. The key assumption to make on $\lambda(t, S_1, S_2, v)$ is that this parameter must be strictly positive as investors will always demand positive risk premium in order for them to bear the associated risk. In determining the market price of the variance risk, we use the same reasoning as in Heston (1993) and Cheang et al. (2009) such that [3]

$$
\lambda(t, S_1, S_2, v) = \frac{\lambda\sqrt{v}}{\sigma\hat{\rho}_{33}}. \tag{5.2.11}
$$

Substituting this into (5.2.10) we obtain

$$
\begin{aligned}
dS_1 &= (r-q_1)S_1dt + \sigma_1\sqrt{v}S_1d\tilde{W}_1,\\
dS_2 &= (r-q_2)S_2dt + \rho_{12}\sigma_2\sqrt{v}S_2d\tilde{W}_1 + \hat{\rho}_{12}\sigma_2\sqrt{v}S_2d\tilde{W}_2,\\
dv &= (\alpha-\beta v)dt + \rho_{13}\sigma\sqrt{v}d\tilde{W}_1 + \hat{\rho}_{23}\sigma\sqrt{v}d\tilde{W}_2 + \hat{\rho}_{33}\sigma\sqrt{v}d\tilde{W}_3,
\end{aligned}
\tag{5.2.12}
$$

where

$$
\alpha = \kappa_v\theta \quad \text{and} \quad \beta = \kappa_v + \lambda. \tag{5.2.13}
$$

[3]While the method of lines approach can handle any form of the market price of risk, the assumption (5.2.11) is crucial when using other valuation techniques such as integral transform methods as presented in Chapter 3 as it guarantees linearity in the coefficient of v.

With the system (5.2.12), we can derive the corresponding American call option pricing PDE which can then be solved for the price and hedge ratios such as the delta and gamma. It can be shown that the PDE satisfying this system is

$$\begin{aligned}\frac{\partial C}{\partial \tau}(\tau,S_1,S_2,v) &= (r-q_1)S_1\frac{\partial C}{\partial S_1} + (r-q_2)S_2\frac{\partial C}{\partial S_2} + (\alpha-\beta v)\frac{\partial C}{\partial v} + \frac{1}{2}\sigma_1^2 v S_1^2\frac{\partial^2 C}{\partial S_1^2}\\ &+ \frac{1}{2}\sigma_2^2 v S_2^2\frac{\partial^2 C}{\partial S_2^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 C}{\partial v^2} + \rho_{12}\sigma_1\sigma_2 v S_1 S_2\frac{\partial^2 C}{\partial S_1\partial S_2}\\ &+ \rho_{13}\sigma\sigma_1 v S_1\frac{\partial^2 C}{\partial S_1\partial v} + \rho_{23}\sigma\sigma_2 v S_2\frac{\partial^2 C}{\partial S_2\partial v} - rC, \end{aligned} \tag{5.2.14}$$

where $0 < S_1 < b(\tau, S_2, v)$, and $0 < S_2, v < \infty$. Here, $S_1 = b(\tau, S_2, v)$ is the early exercise boundary of the American spread call option. It is the calculation of this component that complicates pricing of American options. The PDE (5.2.14) is solved subject to the initial and boundary conditions

$$C(0,S_1,S_2,v) = (S_1 - S_2 - X)^+, \quad 0 < S_1, S_2 < \infty, \tag{5.2.15}$$

$$C(\tau,0,S_2,v) = 0, \quad ,\tau \geq 0, \tag{5.2.16}$$

$$C(\tau,S_1,0,v) = C(\tau,S_1,v), \quad ,\tau \geq 0, \tag{5.2.17}$$

$$C(\tau,b(\tau,S_2,v),S_2,v) = b(\tau,S_2,v) - S_2 - X, \quad \tau \geq 0, \tag{5.2.18}$$

$$\lim_{S_1\to b(\tau,S_2,v)}\frac{\partial C}{\partial S_1}(\tau,S_1,S_2,v) = 1, \qquad \lim_{S_1\to b(\tau,S_2,v)}\frac{\partial C}{\partial S_2}(\tau,S_1,S_2,v) = -1, \quad \tau \geq 0. \tag{5.2.19}$$

Condition (5.2.15) is the payoff at maturity of the spread call option contract while equation (5.2.16) is the absorbing state which ensures that the option ceases to exist if the value of S_1 becomes zero. The condition in equation (5.2.17) stipulates that when $S_2 = 0$, the pricing PDE breaks down to that of the American option written on a single underlying asset, S_1 with strike price, X. Such a situation can then be handled in a similar fashion as in Chiarella et al. (2009) but without incorporating jumps. Equation (5.2.18) is the value matching condition which guarantees continuity of the option value function at the early exercise boundary, $S_1 = b(\tau, S_2, v)$. The two equations in (5.2.19) are called the smooth pasting conditions which are imposed to eliminate arbitrage opportunities that may arise. In choosing boundary conditions in the limit as $v \to 0$, we use extrapolation techniques. It has been shown in Chiarella et al. (2009) that such extrapolation methods help in stabilising the numerical techniques for solving PDEs like (5.2.14). Also since the payoff function is independent of v, it is straight forward to impose the condition that

$$\lim_{S_1 \to b(\tau, S_2, v)} \frac{\partial C}{\partial v} = 0. \tag{5.2.20}$$

This condition together with the two equations in (5.2.19) guarantee continuity of the pricing function at the early exercise boundary which is vital in eliminating certain arbitrage opportunities that may arise. The option price, free boundary and hedge ratios at $v = 0$ are found extrapolation techniques when numerically implementing problem.

Figure 5.1 shows the graphical representation of the early exercise boundary of the American spread call option at a given level of variance. As illustrated on this graph, the early exercise boundary will always be above the strike, X which resembles the minimum amount the holder should pay in order to exercise the option.

We have managed to derive the pricing PDE of the American spread call option. A variety of approaches have been applied to solve American option problems written on a single underlying asset as discussed in Chapter 1. However, with the increased number of dimensions as in the PDE (5.2.14), not all of these methods can easily be generalised to handle this particular situation. Even if it may happen that the corresponding integral representation of the solution to equation (5.2.14) is found, it will still be hard to handle the three – dimensional integral equation for the early exercise surface, $b(\tau, S_2, v)$. As in Chapters 2 and 3 it maybe possible to find suitable approximations of such a surface, but it will first be necessary to investigate the behavior of it with respect to the three variables, τ, S_2 and v. Here we bypass this complexity and directly

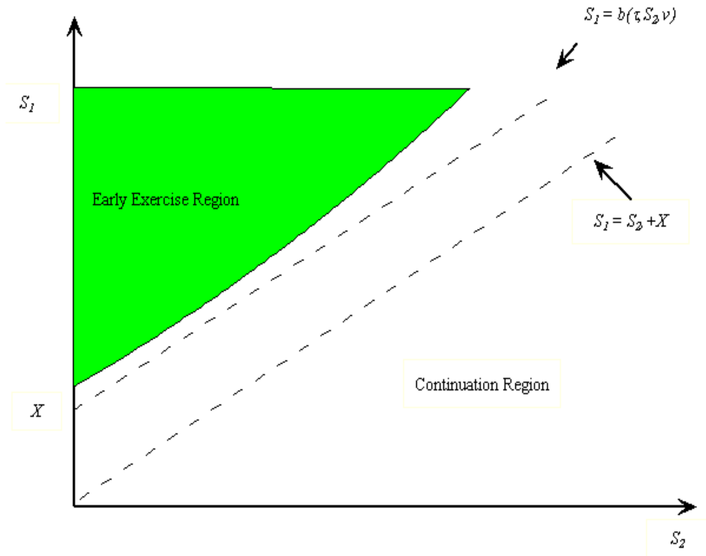

FIGURE 5.1. The early exercise region of the American spread call option at a given level of variance

apply the method of lines algorithm to the PDE (5.2.14) which seems not to be greatly affected by the increasing number of dimensions. We provide details of this procedure in the next section.

5.3. Numerical Solution Algorithm

The method of lines approach is a technique of transforming a multi-dimensional PDE to a corresponding system of one-dimensional ODEs whose solution can then be readily found by using a variety of numerical methods. The method of lines techniques have found greater application in the pricing of American options. Meyer and van der Hoek (1997) consider the valuation of the standard American put option when the underlying asset is being driven by the geometric Brownian motion process. Extension to the jump diffusion setting has been handled in Meyer (1998). Chiarella et al. (2009) consider the evaluation of the American call option when the underlying asset evolve under the influence of both jumps and stochastic volatility. In all these cited papers, the method of lines approach proves to be computationally efficient in terms of speed and accuracy.

One major advantage of this approach is that the option price, delta, gamma and the early exercise boundary are all found simultaneously as part of the solution procedure at no additional computation cost. This is an advantage since practitioners are more interested in calculating hedge ratios.

As an initial step, we discretise the partial derivative terms with respect to S_2, v and τ, and retain continuity in the S_1 direction. In discretising S_2, we set $S_{2,m} = m\triangle S_2$, for $m = 0, 1, \cdots, M$. The variance domain is discetised such that, $v_n = n\triangle v$ for $n = 0, 1, \cdots, N$ while the time interval is partitioned into K equally spaced sub-intervals by letting $\tau_k = k\triangle\tau$ for $k = 0, 1, \cdots, K$. At any given time step, the option price can then be represented as, $C(\tau_k, S_1, S_{2,m}, v_n) \equiv C^k_{m,n}(S_1)$. With this discretisation, the delta of the American spread call option with respect to S_1 is here represented as

$$V(\tau_k, S_1, S_{2,m}, v_n) = \frac{\partial C^k_{m,n}(S_1)}{\partial S_1} \equiv V^k_{m,n}(S_1). \tag{5.3.1}$$

We now present finite difference approximations for the derivatives with respect to S_2 and v. We use central difference approximations for the second order terms such that

$$\frac{\partial^2 C}{\partial S_2^2} = \frac{C^k_{m+1,n} - 2C^k_{m,n} + C^k_{m-1,n}}{(\Delta S_2)^2}, \quad \text{and} \quad \frac{\partial^2 C}{\partial v^2} = \frac{C^k_{m,n+1} - 2C^k_{m,n} + C^k_{m,n-1}}{(\Delta v)^2}. \tag{5.3.2}$$

We also use central difference approximation for the mixed partial derivative terms such that

$$\frac{\partial^2 C}{\partial S_1 \partial S_2} = \frac{V^k_{m+1,n} - V^k_{m-1,n}}{2\Delta S_2}, \quad \text{and} \quad \frac{\partial^2 C}{\partial S_1 \partial v} = \frac{V^k_{m,n+1} - V^k_{m,n-1}}{2\Delta v}. \tag{5.3.3}$$

We use upwinding[4] finite difference approximation schemes in the discretisation of the first-order derivative terms with respect to S_2 and v such that

[4]An upwind finite-difference scheme attempts to discretise a partial differential equation by using differencing biased in the direction determined by the sign of the associated coefficients of the partial derivatives.

$$\frac{\partial C}{\partial S_2} = \begin{cases} \frac{C^k_{m+1,n} - C^k_{m,n}}{\Delta S_2} & \text{if } r - q_2 > 0 \\ \\ \frac{C^k_{m,n} - C^k_{m-1,n}}{\Delta S_2} & \text{if } r - q_2 \leq 0, \end{cases} \tag{5.3.4}$$

and

$$\frac{\partial C}{\partial v} = \begin{cases} \frac{C^k_{m,n+1} - C^k_{m,n}}{\Delta v} & \text{if } v \leq \frac{\alpha}{\beta} \\ \\ \frac{C^k_{m,n} - C^k_{m,n-1}}{\Delta v} & \text{if } v > \frac{\alpha}{\beta}, \end{cases} \tag{5.3.5}$$

where, α and β are given in equation (5.2.13). Since the second order derivative terms vanish as either $S_2 \to 0$ or $v \to 0$, the upwinding schemes help in stabilising the finite difference schemes with respect to both S_2 and v.

The cross derivative term with respect to S_2 and v is discretised as

$$\frac{\partial^2 C}{\partial S_2 \partial v} = \frac{C^k_{m+1,n+1} - C^k_{m-1,n+1} - C^k_{m+1,n-1} + C^k_{m-1,n-1}}{4\Delta S_2 \Delta v}. \tag{5.3.6}$$

For the discretisation with respect to time, we use a first-order backward finite difference scheme for the first two time steps so that

$$\frac{\partial C}{\partial \tau} = \frac{C^k_{m,n} - C^{k-1}_{m,n}}{\Delta \tau}. \tag{5.3.7}$$

Equation (5.3.7) is only first-order accurate with respect to time, however, Meyer and van der Hoek (1997) show that the accuracy can be enhanced by considering a second-order approximation scheme. From the third time step onwards, this is achieved by using the scheme

$$\frac{\partial C}{\partial \tau} = \frac{3}{2}\frac{C^k_{m,n} - C^{k-1}_{m,n}}{\Delta \tau} - \frac{1}{2}\frac{C^{k-1}_{m,n} - C^{k-2}_{m,n}}{\Delta \tau}. \tag{5.3.8}$$

We substitute the finite difference approximations in equations (5.3.1)-(5.3.8) into the PDE (5.2.14) and obtain the corresponding system of ODEs for the option delta, $V^k_{m,n}$ for $k = 0, 1, \cdots, K$, $m = 0, 1, \cdots, M$ and $n = 0, 1, \cdots, N$. For the first two time steps, the PDE is transformed to[5]

$$\begin{aligned}
&\frac{1}{2}\sigma_1^2 v_n S_1^2 \frac{d^2 C_{m,n}^k}{dS_1^2} + (r-q_1)S_1 V_{m,n}^k + (r-q_2)S_{2,m}\frac{C_{m+1,n}^k - C_{m-1,n}^k}{\Delta S_2} + \frac{\alpha-\beta v_n}{2}\frac{C_{m,n+1}^k - C_{m,n-1}^k}{\Delta v} \\
&\quad + \frac{|\alpha-\beta v_n|}{2}\frac{C_{m,n+1}^k - 2C_{m,n}^k + C_{m,n-1}^k}{\Delta v} + \frac{1}{2}\sigma^2 v_n \frac{C_{m,n+1}^k - 2C_{m,n}^k + C_{m,n-1}^k}{(\Delta v)^2} \\
&\quad + |r-q_2|S_{2,m}\frac{C_{m+1,n}^k - 2C_{m,n}^k + C_{m-1,n}^k}{\Delta S_2} + \rho_{12}\sigma_1\sigma_2 v_n S_1 S_{2,m}\frac{V_{m+1,n}^k - V_{m-1,n}^k}{2\Delta S_2} \\
&\quad + \rho_{23}\sigma\sigma_2 v_n S_{2,m}\frac{C_{m+1,n+1}^k - C_{m-1,n+1}^k - C_{m+1,n-1}^k + C_{m-1,n-1}^k}{4\Delta S_2 \Delta v} \\
&\quad + \rho_{13}\sigma\sigma_1 v_n S_1 \frac{V_{m,n+1}^k - V_{m,n-1}^k}{2\Delta v} + \frac{1}{2}\sigma_2^2 v_n S_{2,m}^2 \frac{C_{m+1,n}^k - 2C_{m,n}^k + C_{m-1,n}^k}{(\Delta S_2)^2} \\
&\quad - rC_{m,n}^k - \frac{C_{m,n}^k - C_{m,n}^{k-1}}{\Delta \tau} = 0.
\end{aligned} \tag{5.3.9}$$

The ODE for all subsequent time steps can be shown to be

$$\begin{aligned}
&\frac{1}{2}\sigma_1^2 v_n S_1^2 \frac{d^2 C_{m,n}^k}{dS_1^2} + (r-q_1)S_1 V_{m,n}^k + (r-q_2)S_{2,m}\frac{C_{m+1,n}^k - C_{m-1,n}^k}{\Delta S_2} + \frac{\alpha-\beta v_n}{2}\frac{C_{m,n+1}^k - C_{m,n-1}^k}{\Delta v} \\
&\quad + \frac{|\alpha-\beta v_n|}{2}\frac{C_{m,n+1}^k - 2C_{m,n}^k + C_{m,n-1}^k}{\Delta v} + \frac{1}{2}\sigma^2 v_n \frac{C_{m,n+1}^k - 2C_{m,n}^k + C_{m,n-1}^k}{(\Delta v)^2} \\
&\quad + |r-q_2|S_{2,m}\frac{C_{m+1,n}^k - 2C_{m,n}^k + C_{m-1,n}^k}{\Delta S_2} + \rho_{12}\sigma_1\sigma_2 v_n S_1 S_{2,m}\frac{V_{m+1,n}^k - V_{m-1,n}^k}{2\Delta S_2} \\
&\quad + \rho_{23}\sigma\sigma_2 v_n S_{2,m}\frac{C_{m+1,n+1}^k - C_{m-1,n+1}^k - C_{m+1,n-1}^k + C_{m-1,n-1}^k}{4\Delta S_2 \Delta v} \\
&\quad + \rho_{13}\sigma\sigma_1 v_n S_1 \frac{V_{m,n+1}^k - V_{m,n-1}^k}{2\Delta v} + \frac{1}{2}\sigma_2^2 v_n S_{2,m}^2 \frac{C_{m+1,n}^k - 2C_{m,n}^k + C_{m-1,n}^k}{(\Delta S_2)^2} \\
&\quad - rC_{m,n}^k - \frac{3}{2}\frac{C_{m,n}^k - C_{m,n}^{k-1}}{\Delta \tau} - \frac{1}{2}\frac{C_{m,n}^{k-1} - C_{m,n}^{k-2}}{\Delta \tau} = 0.
\end{aligned} \tag{5.3.10}$$

Having derived equations (5.3.9) and (5.3.10) the next step is to impose boundary conditions at v_0, v_N, $S_{2,0}$ and $S_{2,M}$. Chiarella et al. (2009) fit a quadratic polynomial through v_1, v_2 and v_2 to approximate the option price at v_0. For large values of v, the rate of change of the option price with respect to the variance process diminishes

[5]Here the modulus sign on $|r - q_2|$ and $|\alpha - \beta v_n|$ takes care of the upwind conditions we introduced in equations (5.3.4) and (5.3.5) respectively.

leading to the condition that $\partial C/\partial v = 0$ along the line v_N. We will adopt the same approach when handling the variance boundary conditions.

From the PDE (5.2.14), we note that when $S_2 = 0$, the PDE reduces to that of the American call option pricing PDE for a contract written on a single underlying asset whose dynamics evolve under the influence of a stochastic variance process as considered in Adolfsson et al. (2009). The corresponding valuation procedure will be similar to the algorithm of Chiarella et al. (2009), but without incorporating jumps on the underlying process. Along the line $S_2 = S_{2,M}$, option prices are found by cubic spline interpolation approximation. After taking boundary conditions into consideration we must solve the $[M-1] \times [N-1]$ matrix of ODEs at each time step, τ_k. This process is accomplished in two steps. The first step involves re-writing the second order ODEs in equations (5.3.9) and (5.3.10) as a system of first order ODEs in the form

$$\frac{dC^k_{m,n}}{dS_1} = V^k_{m,n}, \tag{5.3.11}$$

$$\frac{dV^k_{m,n}}{dS_1} = A_{m,n}(S_1)C^k_{m,n} + B_{m,n}(S_1)V^k_{m,n} + P^k_{m,n}(S_1). \tag{5.3.12}$$

When a second-order finite difference scheme is applied with respect to time it can be shown that

$$A_{m,n}(S_1) = \frac{2}{\sigma_1^2 v_n S_1^2}\left[\frac{|\alpha-\beta v_n|}{\Delta v} + \frac{\sigma^2 v_n}{(\Delta v)^2} + 2|r-q_2|\frac{S_{2,m}}{\Delta S_2} + \frac{\sigma_2^2}{(\Delta S_2)^2} v_n S_{2,m}^2 + r + \frac{3}{2\Delta\tau}\right], \tag{5.3.13}$$

$$B_{m,n}(S_1) = \frac{-2(r-q_1)}{\sigma_1^2 v_n S_1}, \tag{5.3.14}$$

and

$$\begin{aligned} P_{m,n}^k(S_1) = \frac{-2}{\sigma_1^2 v_n S_1^2}\Bigg[&(r-q_2)S_{2,m}\frac{C_{m+1,n}^k - C_{m-1,n}^k}{\Delta S_2} + \frac{\alpha-\beta v_n}{2}\frac{C_{m,n+1}^k - C_{m,n-1}^k}{\Delta v} \\ &+ \frac{|\alpha-\beta v_n|}{2}\frac{C_{m,n+1}^k + C_{m,n-1}^k}{\Delta v} + \frac{1}{2}\sigma^2 v_n \frac{C_{m,n+1}^k + C_{m,n-1}^k}{(\Delta v)^2} \\ &+ |r-q_2|S_{2,m}\frac{C_{m+1,n}^k + C_{m-1,n}^k}{\Delta S_2} + \rho_{12}\sigma_1\sigma_2 v_n S_1 S_{2,m}\frac{V_{m+1,n}^k - V_{m-1,n}^k}{2\Delta S_2} \\ &+ \rho_{23}\sigma\sigma_2 v_n S_{2,m}\frac{C_{m+1,n+1}^k - C_{m-1,n+1}^k - C_{m+1,n-1}^k + C_{m-1,n-1}^k}{4\Delta S_2 \Delta v} \\ &+ \rho_{13}\sigma\sigma_1 v_n S_1 \frac{V_{m,n+1}^k - V_{m,n-1}^k}{2\Delta v} + \frac{1}{2}\sigma_2^2 v_n S_{2,m}^2 \frac{C_{m+1,n}^k + C_{m-1,n}^k}{(\Delta S_2)^2} \\ &- rC_{m,n}^k + \frac{3}{2}\frac{C_{m,n}^{k-1}}{\Delta\tau} - \frac{1}{2}\frac{C_{m,n}^{k-1} - C_{m,n}^{k-2}}{\Delta\tau}\Bigg]. \end{aligned} \tag{5.3.15}$$

The coefficients $A_{m,n}(S_1)$, $B_{m,n}(S_1)$ and $P_{m,n}^k(S_1)$ contain the points involved in determining $C_{m,n}^k$ at each grid point in Figure 5.2. This figure shows that to obtain the price at node $(\tau_k, S_{2,m}, v_n)$, we need option prices at ten surrounding nodes.

The second step involves applying the Riccati transformation to equations (5.3.11) and (5.3.12). By using similar arguments as in Meyer and van der Hoek (1997) and Chiarella et al. (2009), the solution of the system (5.3.11)-(5.3.12) can be represented by the Riccati transformation

$$C_{m,n}^k(S_1) = R_{m,n}(S_1)V_{m,n}^k(S_1) + W_{m,n}^k(S_1), \tag{5.3.16}$$

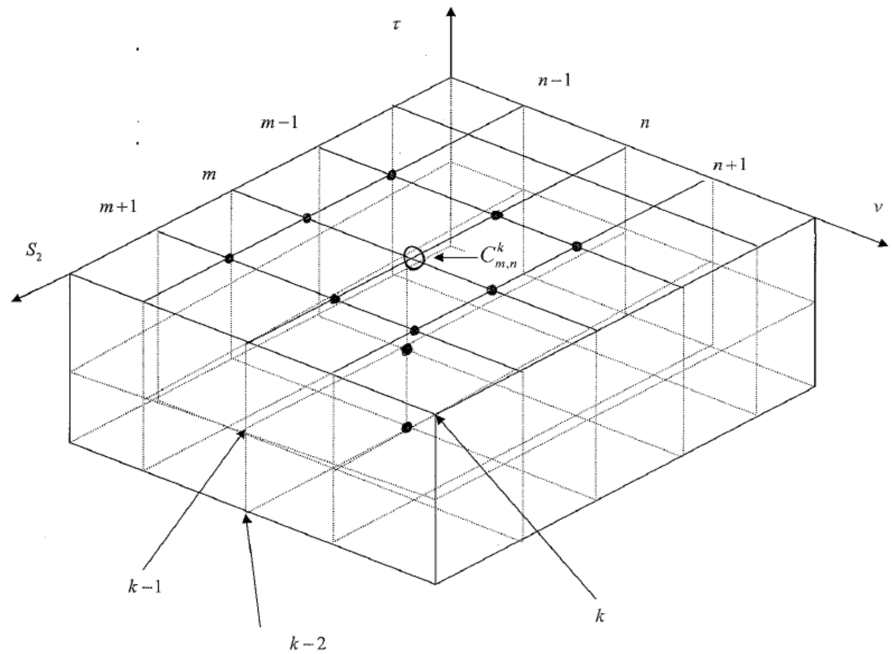

FIGURE 5.2. Three Dimensional view of grid points involved in the calculation of $C^k_{m,n}$ (The Circled Node).

where $R_{m,n}(S_1)$ and $W^k_{m,n}(S_1)$ are solutions of the initial value problems

$$\frac{dR_{m,n}}{dS_1} = 1 - B_{m,n}(S_1)R_{m,n}(S_1) - A_{m,n}(S_1)(R_{m,n}(S_1))^2, \quad R_{m,n}(0) = 0, \tag{5.3.17}$$

$$\frac{dW^k_{m,n}}{dS_1} = -A_{m,n}(S_1)R_{m,n}(S_1)W^k_{m,n}(S_1) - R_{m,n}(S_1)P^k_{m,n}(S_1), \quad W^k_{m,n}(0) = 0. \tag{5.3.18}$$

The option delta, $V^k_{m,n}(S_1)$ solves the ordinary differential equation

$$\frac{dV^k_{m,n}}{dS_1} = A_{m,n}(S_1)[R_{m,n}(S_1)V^k_{m,n} + W^k_{m,n}(S_1)] + B_{m,n}(S_1)V^k_{m,n} + P^k_{m,n}(S_1). \tag{5.3.19}$$

Equation (5.3.19) is solved subject to the boundary condition

$$V^k_{m,n}(b^k_{m,n}) = 1, \tag{5.3.20}$$

where $S_1 = b^k_{m,n}$ is the early exercise boundary at the grid point $(\tau_k, S_{2,m}, v_n)$. In solving the above system, we first apply the implicit trapezoidal rule[6] to equation (5.3.17) on

[6]Full details on how to implement the implicit trapezoidal rule have been documented in Meyer (2010).

a non-uniform grid for the S_1 domain from $S_1 = S_{1,\min}$ to $S_1 = S_{1,\max}$ where $S_{1,\min}$ is chosen to be very small and $S_{1,\max}$ is large enough to cover the early exercise boundary. The non-uniform grid is partitioned such that $S_{1,\min} < \cdots < S_{1,\max}$. Meyer and van der Hoek (1997) take $S_{1,\max}$ to be three times the strike price. In our case since we are dealing with the spread call option case, the early exercise boundary is also a function of $S_{2,m}$ so we will take $S_{1,\max}$ to be at least $3X + S_{2,M}$, where $S_{2,M}$ is the maximum price level of second underlying asset. Once equation (5.3.17) is solved, we store the results offline as this is independent of time. Having determined $R_{m,n}(S_1)$, we proceed to solve equation (5.3.18) for increasing S_1 from $S_{1,\min}$ to $S_{1,\max}$ again using the implicit trapezoidal rule. This step requires the previously calculated values of $R_{m,n}(S_1)$. Once $R_{m,n}(S_1)$ and $W^k_{m,n}(S_1)$ have been found, it then follows from (5.3.16) and the condition (5.3.20) that the early exercise boundary satisfies

$$b^k_{m,n} - m\Delta S_2 - K = R_{m,n}(b^k_{m,n}) \cdot 1 + W^k_{m,n}(b^k_{m,n}). \tag{5.3.21}$$

As equation (5.3.21) is implicit in $b^k_{m,n}$, we need to employ root – finding algorithms to find the early exercise boundary at each grid point, $(\tau_k, S_{2,m}, v_n)$. Once the early exercise boundary has been determined, we then solve equation (5.3.19) by sweeping backwards from $S_1 = b^k_{m,n}$ down to the initial value, $S_{1,\min}$. Having solved equations (5.3.17)-(5.3.19) for $R_{m,n}(S_1)$, $W^k_{m,n}(S_1)$ and $V^k_{m,n}(S_1)$ at each grid point $(\tau_k, S_{2,m}, v_n)$, we can then substitute the resulting solutions into equation (5.3.16) to obtain the corresponding option price, $C^k_{m,n}(S_1)$.

Figure 5.3 shows how the algorithm evolves by sweeping along the $\tau - S_2$ plane. The same sweeping process holds for the $\tau - v$ plane.

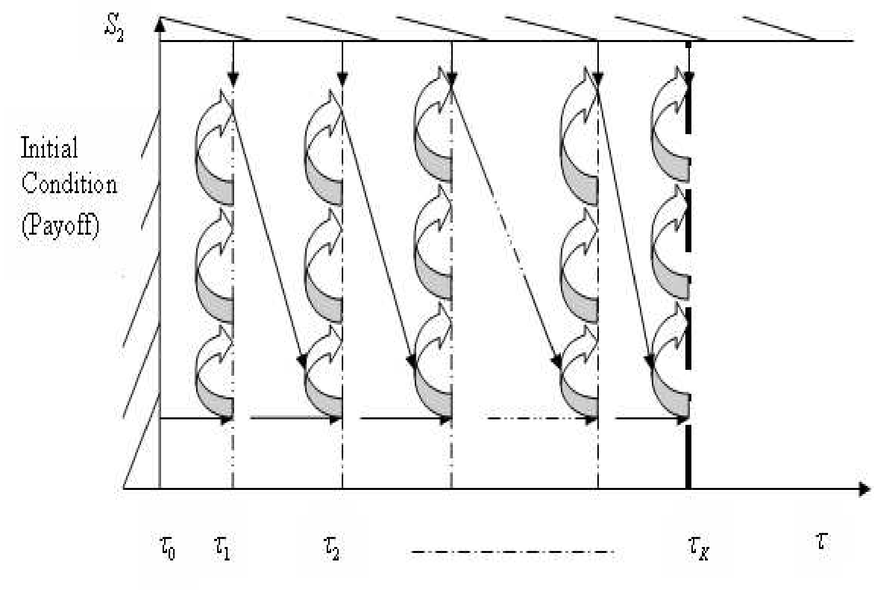

FIGURE 5.3. Representation of how the algorithm sweeps through the $\tau - S_2$ grid for a fixed level of v.

5.4. Numerical Results

Having outlined the numerical implementation of the method of lines in Section 5.3, we now present the numerical results from implemention of the algorithm. Since existing literature does not offer any numerical method for the problem at hand, we make price comparisons with the Monte Carlo algorithm of Ibáñez and Zapatero (2004). Details on how to implement the Monte Carlo algorithm have been presented in Chapter 2 when we considered the American spread option under geometric Brownian motion. The only difference is that we are now faced with simulating three stochastic processes and the resulting early exercise boundary is three-dimensional unlike the two-dimensional one considered in Chapter 2. The set of parameters used for most of the numerical experiments in this section are provided in Table 5.1. We select these parameters to best illustrate the impact of stochastic volatility on the price, free surface and the delta of the American spread option. In exploring the impact of stochastic volatility, we make comparisons with those from the geometric Brownian motion framework considered in Chapter 2.

We consider a six month maturity American spread call option. For the method of lines algorithm we have used $K = 100$ time steps, $M = 100$-S_2 points with $S_{2,M} = 200$ and $N = 50$ variance steps. We have taken the maximum variance to be $v_N = 50\%$ and $S_{1,\max} = 500$. A non-uniform grid has been applied in the S_1 direction to avoid oscillations of the numerical solution with the corresponding grid points being equal to 1438. These points are distributed such that there are 40 points in the interval $0 \leq S_1 \leq 1$, 198 points in the interval $1 < S_1 \leq 100$ and 1200 points in the interval $100 < S_1 \leq 500$.

Parameter	Value	Variance Parameters	Value
T	0.5	θ	0.04
r	0.03	κ_v	2
q_1	0.06	σ_v	0.1
q_2	0.02	λ	0
σ_1	1.25	ρ_{13}	0.5
σ_2	1.5	ρ_{23}	-0.5
ρ_{12}	0.5		
X	100		

TABLE 5.1. Parameters used in the calculation of the American spread option prices, free surfaces and deltas.

We start by presenting the early exercise surfaces of the American spread call option generated by the method of lines algorithm outlined in Section 5.3 with the parameters provided in Table 5.1. Figure 5.4 shows the free surface obtained at a fixed variance level. We note that for a given variance level, the early exercise surface of the form typical for American spread options under geometric Brownian motion (GBM) is generated. As expected, this surface is an increasing function of both S_2 and time-to-maturity. We also present the early exercise surface generated at a fixed S_2-level in Figure 5.5. We note that for a given level of S_2 the early exercise surface is also an increasing function of v and this is of the form typical for American options written on a single underlying asset whose dynamics evolve according to the Heston (1993) model as presented in Chiarella et al. (2009). Similar results have been obtained in Chapter 3 when we considered a contract whose underlying asset evolves under the influence of two stochastic variance processes.

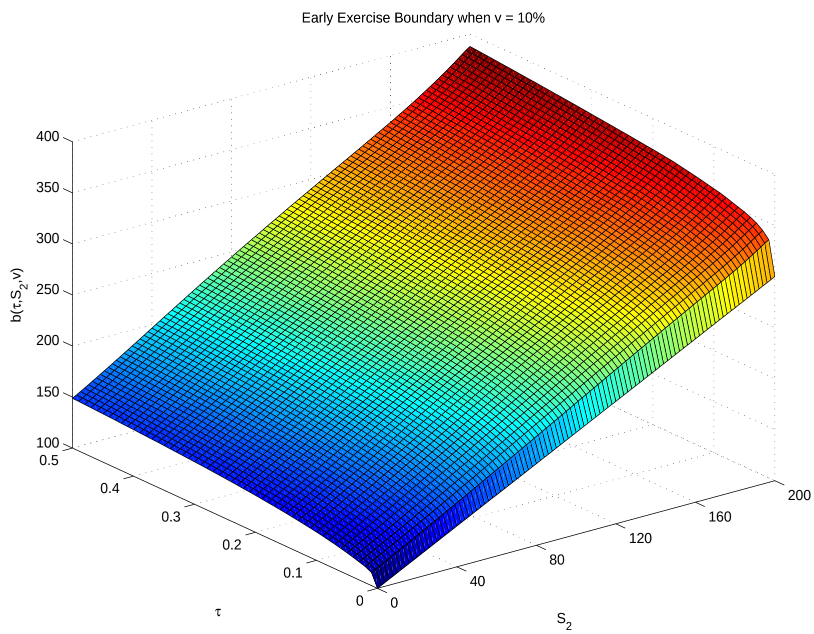

FIGURE 5.4. Early exercise surface of the American spread option when $v = 10\%$. All other parameters are given in Table 5.1.

To see the impact of stochastic volatility on the early exercise surface, we fix both S_2 and v and make comparisons with GBM results. In making such comparisons, we match the GBM volatilities by fixing the variance at its long run mean level so that

$$\sigma_{1,GBM} = \sigma_1\sqrt{\theta} \quad \text{and} \quad \sigma_{2,GBM} = \sigma_2\sqrt{\theta}, \tag{5.4.1}$$

where $\sigma_{1,GBM}$ and $\sigma_{2,GBM}$ are the constant volatilities of S_1 and S_2 respectively. Figure 5.6 shows the respective early exercise boundaries when $S_2 = 100$ and $v = 10\%$. Given the parameters in Table 5.1 we note that the early exercise boundaries for the stochastic variance case are higher than that of GBM. Similar results have been observed in Chiarella et al. (2009) when considering positive correlation between the Wiener processes of a single underlying asset and that of the instantaneous variance process.

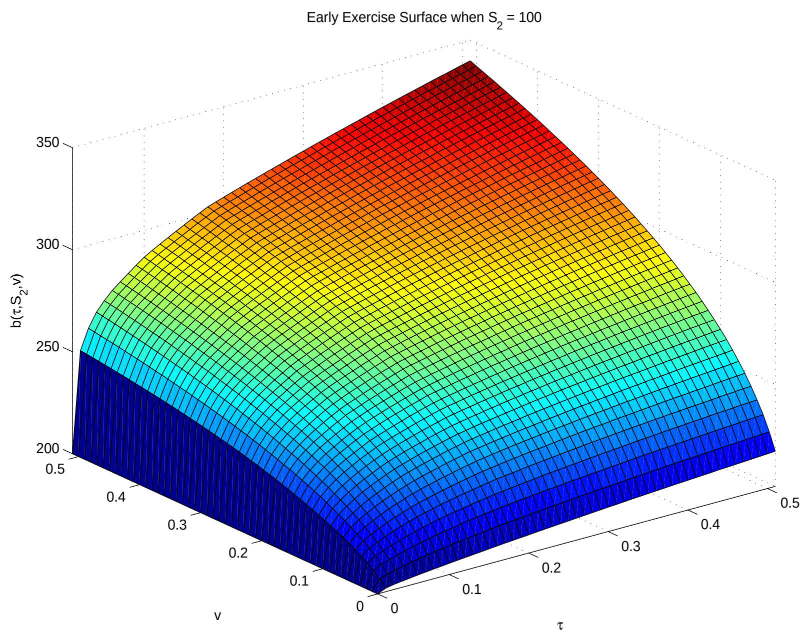

FIGURE 5.5. Early exercise surface of the American spread option when $S_2 = 100$. All other parameters are provided in Table 5.1.

The only difference is that here we are now considering a three factor model with three correlation terms. Also Chiarella et al. (2009) considered an American call option contract written on a single underlying asset, whilst we are here considering the American spread option.

Having presented the early exercise surfaces, we now turn to the presentation of the corresponding option prices for the American spread call. A sample price surface for fixed v is provided in Figure 5.7. This surface is of the form typical for spread call options under the Black and Scholes (1973) framework as demonstrated in Chapter 2. To assess the effectiveness of our approach we compare option prices with those generated by the Monte Carlo (MC) algorithm of Ibáñez and Zapatero (2004) at the 95% confidence level in Table 5.2. We have used 100,000 simulations for the MC method with 20 time steps in implementing the MC algorithm. Increasing the number of simulations improves the accuracy at the cost of computational speed. The bracketed numbers in this table are the standard errors of the calculated option prices. We

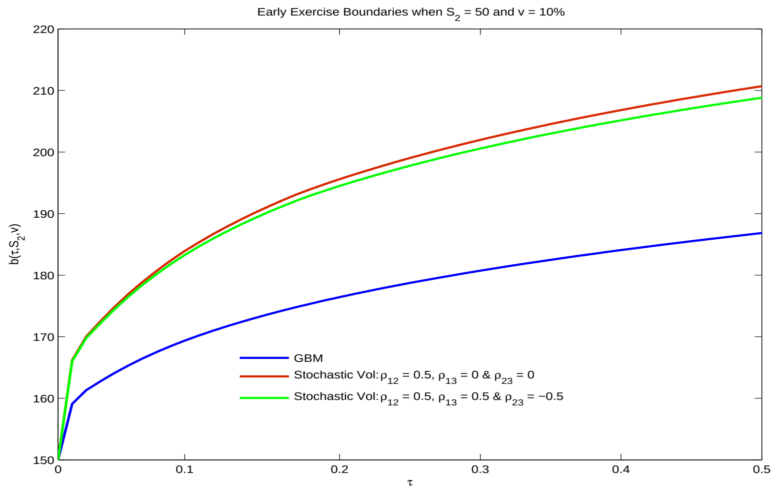

FIGURE 5.6. The early exercise boundaries of the American spread option when $v = 10\%$, $S_2 = 50$, $\sigma_{1,GBM} = 25\%$ and $\sigma_{2,GBM} = 30\%$. All other stochastic variance parameters are as presented in Table 5.1.

note that the two approaches generate option prices that are comparable in terms of accuracy. In terms of computational speed, it takes only 120 seconds to generate the option prices, free-boundaries and the hedge ratios using the MOL approach with parameters provided in Table 5.1 as compared to 3.1164×10^4 seconds elapsed by the Monte Carlo algorithm to generate option prices presented in Table 5.2.

We now explore the impact of stochastic volatility on option prices by making comparisons with results from the geometric Brownian motion (GBM) model. A range of American spread call option prices are reported in Tables 5.3-5.5 for different levels of S_1 and S_2. In these tables we report option prices when $S_1 = 50, 100, 150, \cdots, 400$. GBM prices are presented on the second column of each table followed by prices from the stochastic variance model when $v = 10\%$, 20%, 30% and 40% respectively. We note a gradual increase in option prices as the instantaneous variance level is increased.

We have also included Figures 5.8-5.10 so as to illustrate the behaviour of option prices when volatility is allowed to vary stochastically. In these graphs, we calculate price differences by subtracting option prices generated by the stochastic volatility model from those generated by the GBM model. We make comparisons when all correlations have the same sign; for instance positive correlation implies that option prices from the stochastic volatility model when $\rho_{12} = 0.5$, $\rho_{13} = 0.5$ and $\rho_{23} = 0.5$ are being subtracted from the GBM prices when the correlation between the two underlying assets is 0.5. From Figures 5.8 and 5.9 we note that GBM prices for near in-and out-of-the-money options are always lower than corresponding prices from the stochastic volatility model. As the level of S_2 increases to 150, the behaviour of option prices vary as shown in Figure 5.10 where we note instances when GBM prices are greater than corresponding prices from the stochastic volatility model when correlation is positive. A change in behaviour of option price differences is also noted for negative and zero correlations though option prices from the stochastic volatility model are always greater than GBM prices.

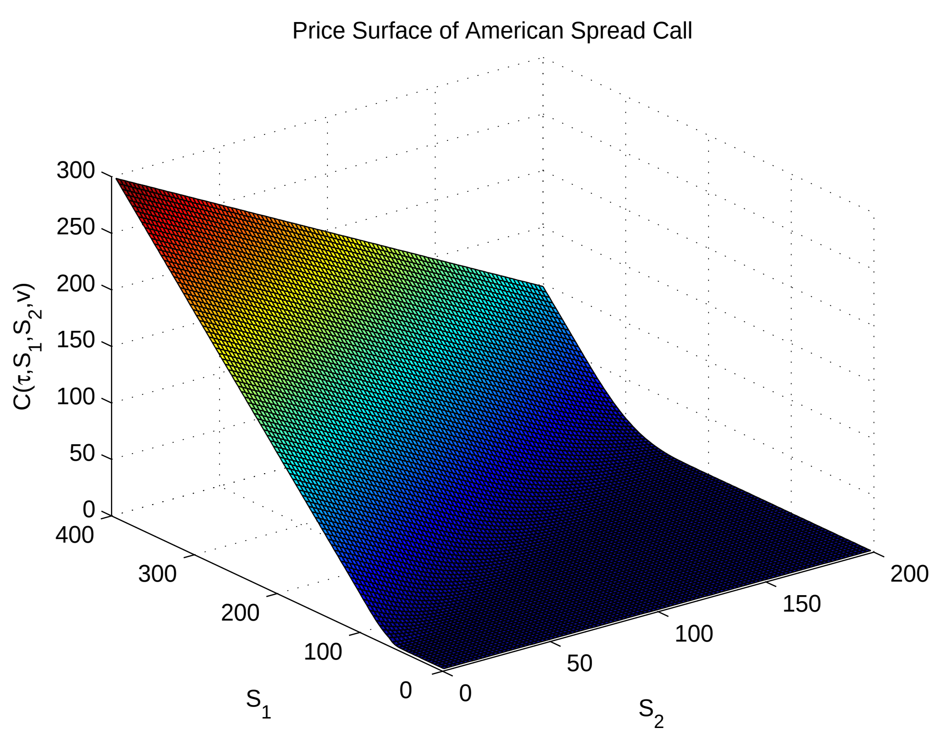

FIGURE 5.7. Price surface of the American spread option when v is fixed.

To complete the analysis we assess the impact of stochastic volatility on the delta profile of the American spread call option. A sample delta surface when $v = 10\%$ is shown in Figure 5.11 which is also of the form typical for American spread options

S_1	$v = 10\%$ and $S_2 = 50$ MOL	Monte Carlo Approach
50	0.000001	0
100	0.2253	0.2997(0.0168)
150	11.2652	11.2471(0.093)
200	50.1399	50(0.0689)
250	100	100
300	150	150

TABLE 5.2. Price comparisons between the method of lines algorithm and Monte Carlo results. The numbers in the brackets represent standard errors for the Monte Carlo approach at the 95% confidence interval. We have used the following parameters; $\rho_{12} = 0.5$, $\rho_{13} = 0.5$ and $\rho_{23} = -0.5$ with all other parameters as presented in Table 5.1.

		Stochastic Vol Case			
S_1	GBM	$v = 10\%$	$v = 20\%$	$v = 30\%$	$v = 40\%$
50	0	0	0.000095	0.001	0.004
100	0.0171	0.1939	0.7626	1.3963	1.9810
150	8.1052	11.2371	14.7805	17.1665	18.9178
200	50	50.1917	51.8193	53.4853	54.9050
250	100	100	100	100.1579	100.6599
300	150	150	150	150	150
350	200	200	200	200	200
400	250	250	250	250	250

TABLE 5.3. Price comparisons of the American spread call option when $S_2 = 50$, $\rho_{12} = 0.5$, $\rho_{13} = 0$ and $\rho_{23} = 0$ with all other parameters as presented in Table 5.1. We have taken GBM constant volatilities to be $\sigma_{1,GBM} = 25\%$ and $\sigma_{2,GBM} = 30\%$.

		Stochastic Vol Case			
S_1	GBM	$v = 10\%$	$v = 20\%$	$v = 30\%$	$v = 40\%$
50	0	0	0	0	0.000002
100	0.000009	0.000368	0.0051	0.0151	0.0277
150	0.2370	0.5630	1.2424	1.7279	2.0751
200	10.7570	12.8237	15.4553	16.8597	17.7405
250	50	50.5650	51.99940	52.9710	53.6426
300	100	100	100	100.0797	100.2699
350	150	150	150	150	150
400	200	200	200	200	200

TABLE 5.4. Price comparisons of the American spread call option when $S_2 = 100$, $\rho_{12} = 0.5$, $\rho_{13} = 0$ and $\rho_{23} = 0$ with all other parameters as presented in Table 5.1. We have taken GBM constant volatilities to be $\sigma_{1,GBM} = 25\%$ and $\sigma_{2,GBM} = 30\%$.

		Stochastic Vol Case			
S_1	GBM	$v = 10\%$	$v = 20\%$	$v = 30\%$	$v = 40\%$
50	0	0	0	0	0
100	0	0	0.000004	0.000021	0.000054
150	0.0032	0.0034	0.0155	0.0285	0.0395
200	0.9802	0.8556	1.4073	1.7103	1.8978
250	13.8717	13.6119	15.3846	16.1873	16.6444
300	50.4489	50.8026	51.8334	52.3819	52.7132
350	100	100	100	100.0214	100.0847
400	150	150	150	150	150

TABLE 5.5. Price comparisons of the American spread call option when $S_2 = 150$, $\rho_{12} = 0.5$, $\rho_{13} = 0$ and $\rho_{23} = 0$ with all other parameters as presented in Table 5.1. We have taken GBM constant volatilities to be $\sigma_{1,GBM} = 25\%$ and $\sigma_{2,GBM} = 30\%$.

under GBM. In Figure 5.12 we provide delta profiles as generated from GBM and the stochastic volatility model. We note that delta is very sensitive to the introduction of stochastic volatility. To elaborate on the extent of the impact we show in Figure 5.13 the absolute differences of the delta from the GBM minus that of the stochastic volatility model when $v = 10\%$. Various comparisons can be made for different levels of variance and S_2.

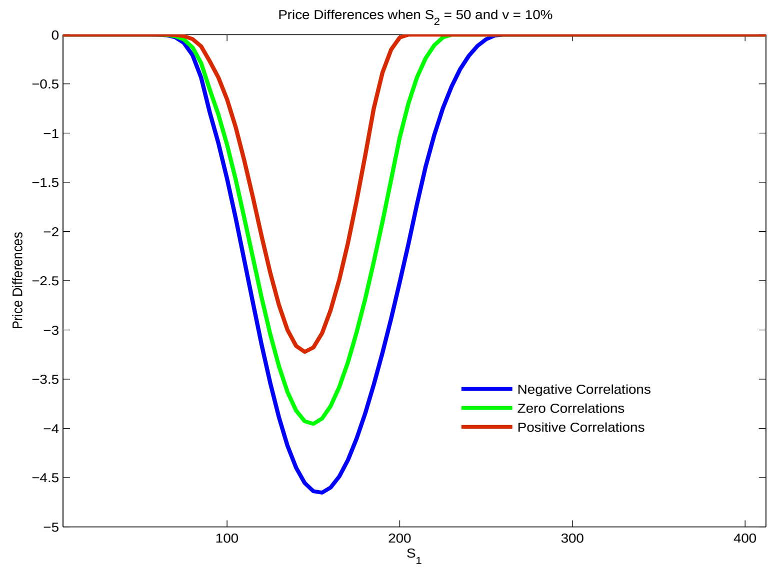

FIGURE 5.8. Option prices from GBM minus option prices from the stochastic volatility model for different correlation coefficients when $S_2 = 50$, $v = 10\%$, $\sigma_{1,GBM} = 25\%$ and $\sigma_{2,GBM} = 30\%$ with all other parameters as given in Table 5.1.

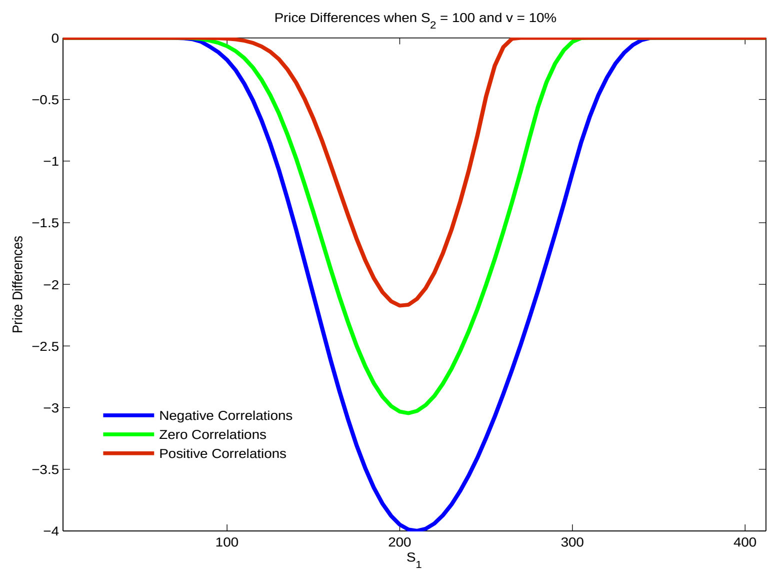

FIGURE 5.9. Option prices from GBM minus option prices from the stochastic volatility model for different correlation coefficients when $S_2 = 100$, $v = 10\%$, $\sigma_{1,GBM} = 25\%$ and $\sigma_{2,GBM} = 30\%$ with all other parameters as given in Table 5.1.

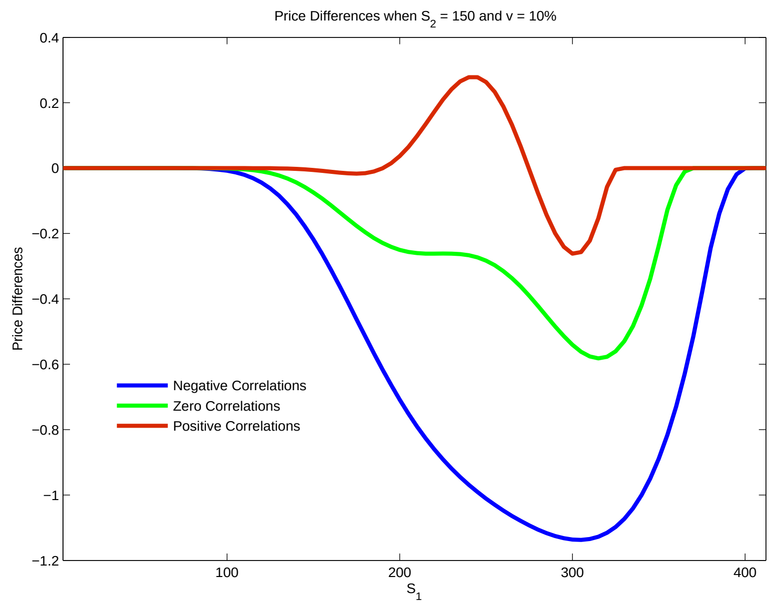

FIGURE 5.10. Option prices from GBM minus option prices from the stochastic volatility model for different correlation coefficients when $S_2 = 150$, $v = 10\%$, $\sigma_{1,GBM} = 25\%$ and $\sigma_{2,GBM} = 30\%$ with all other parameters as given in Table 5.1.

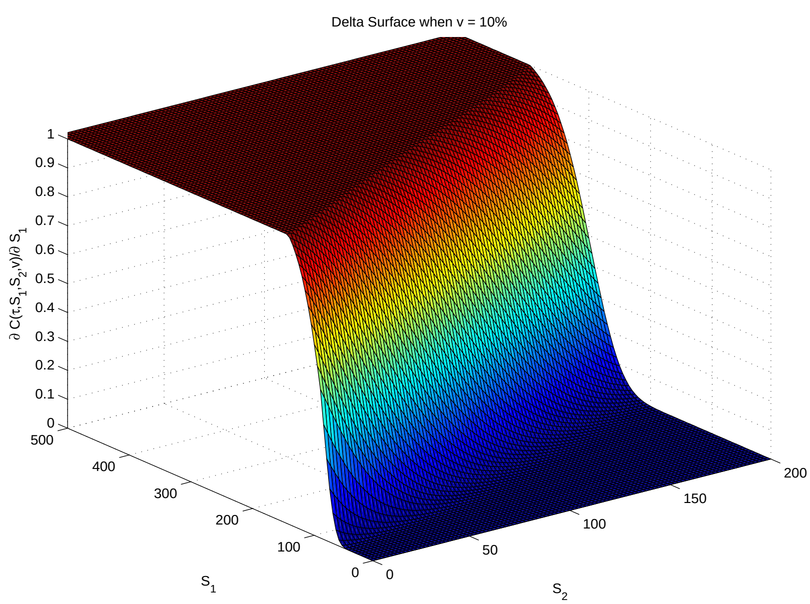

FIGURE 5.11. Delta surface of the American spread option when v is fixed with all other parameters as given in Table 5.1.

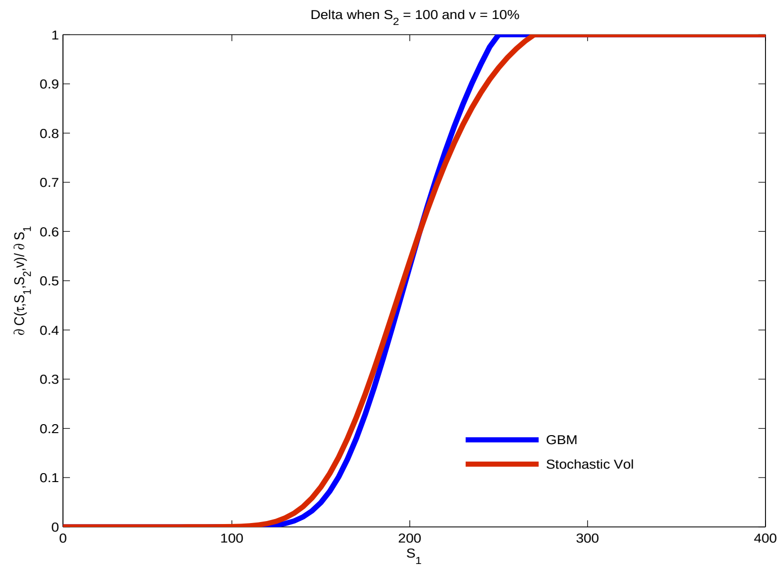

FIGURE 5.12. Exploring the effects of stochastic volatility on the Delta of the American spread call option when $\sigma_{1,GBM} = 25\%$, $\sigma_{2,GBM} = 30\%$, $\rho_{12} = 0.5$, $\rho_{13} = 0.5$ and $\rho_{23} = -0.5$ with all other parameters as given in Table 5.1.

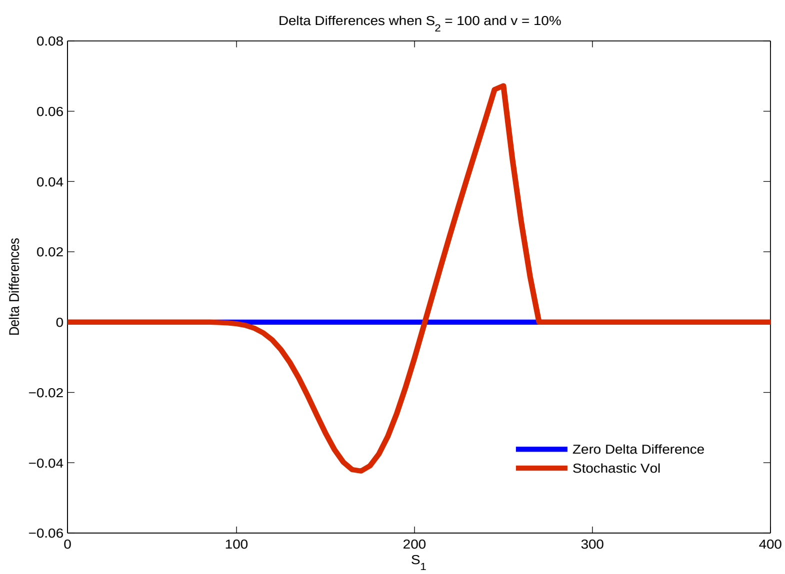

FIGURE 5.13. Spread option Deltas from geometric Brownian motion minus Spread option Deltas from the stochastic volatility model. Here, $\sigma_{1,GBM} = 25\%$, $\sigma_{2,GBM} = 30\%$, $\rho_{12} = 0.5$, $\rho_{13} = 0.5$ and $\rho_{23} = -0.5$ with all other parameters as given in Table 5.1.

5.5. Conclusion

In this chapter we have presented the method of lines algorithm for solving the pricing partial differential equation (PDE) for the American spread call option written on two underlying assets whose dynamics evolve under the influence of a single stochastic variance process. A systematic approach for solving the PDE has been outlined by extending the work of Meyer and van der Hoek (1997) to the multi-dimensional setting. Due to the four dimensional nature of the pricing PDE, we have discretised three state variables and allowed continuity with respect to one underlying state variable resulting in a system of ordinary differential equations (ODEs) in terms of one of the spatial variables. Upwinding finite difference schemes have been incorporated to stabilise the numerical scheme. Such schemes have been successfully applied in Chiarella et al. (2009) along the variance domain when pricing an American call option written on a single underlying asset under stochastic volatility and jumps.

By first applying a Riccati transformation to the ODE system, a systematic approach for solving the resulting one-dimensional ODEs has been presented by using the implicit trapezoidal rule. As is common when solving American option problems, the value matching condition has been used to obtain the early exercise boundaries.

The computational speed and accuracy of the algorithm has been assessed by making numerical comparisons with results generated from the Monte Carlo approach of Ibáñez and Zapatero (2004). We have inferred the impact of stochastic volatility on option prices, free boundaries and deltas by making parallels with geometric Brownian motion results; the algorithm for which has been presented in Chapter 2. Efficient plots and tabular results have been presented to emphasize the impact of stochastic volatility. From these results we have generally noted that by introducing stochastic volatility option prices tend to change in line with the magnitude of the variance level.

CHAPTER 6

Conclusions

6.1. Thesis Summary

American option pricing has remained an important problem since the ground breaking work by Black and Scholes (1973). Whilst much focus has been devoted to pricing contracts written on a single underlying asset, there has been less work on multifactor American option pricing. This is mainly due to challenges associated with handling the associated multi-dimensional early exercise boundaries whose functional forms are not known explicitly. The curse of dimensionality becomes a very influential factor as most of the numerical techniques applied to evaluate American option contracts written on a single underlying asset do not readily generalise to handling multifactor contracts.

This thesis has addressed some of these challenges by exploring methods for evaluating multifactor American option pricing problems under three different frameworks. The first part of the thesis explored how to value American options written on two underlying assets whose dynamics evolve according to geometric Brownian motion processes. We managed to extend in some way the Fourier transform techniques of McKean (1965) and Chiarella and Ziogas (2005) to problems involving two underlying assets. Whilst McKean (1965) used incomplete Fourier transforms in transforming the associated American option pricing partial differential equation (PDE) to the corresponding ordinary differential equation, we have used Jamshidian's (1992) techniques to transform the PDE to its corresponding inhomogeneous form that is solved on the entire domain, the general solution of which we have presented with the aid of Duhamel's principle. A bivariate Fourier transform has been applied to the Kolomogorov PDE – a process which led to the explicit solution of the bivariate transition density function.

In the second part we have explored the use of Fourier and Laplace transforms in deriving the integral expression for the price of an American call option written on an underlying asset whose dynamics evolve under the influence of two independent stochastic variance processes. This idea has been generalised to the case when the underlying asset is driven by multifactor stochastic variance processes. The final contribution of the thesis explored application of the method of lines algorithm to the evaluation of an American spread call option whose underlying assets evolve under the influence of a single stochastic variance process.

6.2. Fourier Transform Approach for American Options on Two Assets

In Chapter 2 we have derived the integral representation for the price of an American option contract written on any two underlying assets under the Black and Scholes (1973) framework with the aid of Fourier transform techniques first suggested in McKean (1965). Whilst for most valuation approaches considered in the literature the payoff function needs to be explicitly specified from the beginning, Fourier transform techniques can be used to derive the general pricing function for any continuous payoff structure as presented in Chapter 2. A numerical example of the American spread call option has been provided. As noted from this example, a non-linear Volterra integral equation satisfied by the early exercise boundary has to be solved as part of the solution.

To demonstrate the accuracy and efficiency of our approach, we have compared the numerical results from our numerical integration algorithm with those from three existing methods; these are the method of lines (MOL) approach which we implemented along the lines of Meyer and van der Hoek (1997) and Chiarella et al. (2009); the Fourier space time-stepping (FST) method proposed by Jackson et al. (2008); and the Monte Carlo algorithm of Ibáñez and Zapatero (2004). All four methods provide comparable results in terms of accuracy. Since the FST method does not explicitly generate the early exercise boundary as part of the solution, we have compared such boundaries from the numerical integration approach with those from the MOL. The two approaches generate early exercise boundaries which are not significantly different from each other. Monte Carlo confidence bounds have been used for price comparisons.

All the considered approaches generate prices which in most of the cases are comparable up to 3 significant figures. Prices and early exercise boundary comparisons have also been explored for varying correlation between the two underlying assets. From these comparisons, we have noted that the early exercise boundary corresponding to negative correlation is higher than that for zero and positive correlation.

We have also assessed time convergence properties among the four methods by taking the FST approach with a sufficient number of discretisation points as the benchmark. We noted the high rate of convergence of the numerical integration technique and the method of lines but unfortunately this is not the case with the Monte Carlo scheme which does not change much as we increase the time discretisation. This may explain the reason why some of the Monte Carlo bounds established do not capture prices generated by the other three approaches. The computational time used by the MOL increases linearly with the increasing number of time discretisations which makes the method a suitable candidate for extension to higher dimensional settings. This has been the major motivation in considering the MOL in Chapter 5. Though the numerical integration method has a higher rate of convergence to the benchmark solution, the computational time required to achieve this accuracy increases exponentially which is a drawback for possible applications to higher dimensional problems. It will however be worth devising higher dimensional numerical integration routines which can be used to generate benchmark solutions for comparisons with other efficient approaches such as the method of lines. To this end, we have concluded Chapter 2 by deriving the integral expression for an American pricing contract written on more than two underlying assets.

6.3. American Option Pricing Under Stochastic Volatility

We have presented an integral expression for the price of an American call option written on an underlying asset whose dynamics evolve under the influence of two stochastic variance processes in Chapter 3. A numerical algorithm for implementing the integral expression for the option price together with the associated implicit non-linear Volterra integral equation satisfied by the early exercise boundary has been outlined.

One of the key concepts in implementing numerical integration routines is the establishment of efficient approaches on how to handle the infinite integral domains associated with transition density functions for the underlying stochastic processes. We have noted that such domains are zero everywhere except near the origins of the state variables thereby giving us flexibility in imposing finite bounds. The establishment of such finite integral bounds increase the efficiency of the numerical algorithm as less quadrature points will be required to achieve a high level of accuracy. Numerical results for option prices and free boundaries have been presented and comparisons made with those generated by the method of lines algorithm. The two approaches generally yield comparable results except for the early exercise boundaries where we note that the one generated by the numerical integration method is slightly lower than that from the method of lines. The difference may be due to discretisation and early exercise boundary approximation errors in the numerical integration approach. However, such differences are not passed on to option prices as in most of the cases prices are comparable up to three significant figures.

As initially noted in Heston (1993), we have also discovered that by incorporating stochastic volatility into the underlying asset dynamics, the Black and Scholes (1973) model overprices in-the-money call option prices when there is positive correlation among the Wiener processes of the underlying stochastic processes while at the same time underpricing out-of-the-money options. The reverse holds for negative correlations among the Wiener processes. We have also analysed the case when one correlation is positive and the other negative and noted that the resulting prices are not significantly different from those generated by the Black and Scholes (1973) model. Similar results have been drawn for the case when the two correlation terms are zero.

We have also explored the impact of stochastic volatility on the early exercise boundary by making comparisons with geometric Brownian motion (GBM) results. Negative correlations yield free boundaries which are always above GMB boundaries whilst positive correlations yield free boundaries which are always lower than those from GBM. When the correlation terms are zero, no significant differences are noted with respect to GBM. Effects of varying the volatilities of the instantaneous variances have been explored and we have generally noted that an increase in these parameters gives rise to higher price differences when compared with GBM prices.

The integral expression of the American call option has been generalised to the case when the underlying asset dynamics evolve under the influence of multifactor stochastic variance processes in Chapter 4. Lengthy derivations and induction proofs have been used to facilitate the derivations. We did not provide numerical results for the general case but we highlight that in order to successfully implement the resulting integral expression for the option price, the functional form of the free-boundary needs to be known in advance. The multivariate log-linear approximation incorporated in Chapter 3 can be extended but however the computational algorithm will be slow due to many functional evaluations associated with generating the early exercise boundary. While we have presented the American call option case, the approaches presented in Chapters 3 and 4 can be applied to any continuous payoff function such as American put and strangle options.

6.4. Method of Lines Approach for Pricing American Exotic Options

The method of lines (MOL) approach has emerged to be one of the most convenient techniques capable for handling early exercise higher dimensional problems encountered in option pricing. The time convergence properties of the MOL as the number of dimension increases makes it a better method to consider as compared to other pricing approaches such as Monte Carlo and numerical integration whose computational speeds increase dramatically. In addition to time convergence properties, the MOL approach has the advantage of generating option prices, free-boundaries and the associated hedge ratios at the same time as part of the solution at no additional computational cost. Given such tremendous advantages, we have presented the method of lines algorithm for solving the American spread call option pricing partial differential equation when the dynamics of the two underlying assets evolve under the influence of a single stochastic variance process in Chapter 5.

A consistent way of discretising the state variable domains has been outlined with upwinding finite difference techniques introduced where necessary to stabilise the numerical scheme. We have assessed the accuracy of the MOL by making comparisons with option prices from the Monte Carlo algorithm of Ibáñez and Zapatero (2004) and noted quite comparable results. We explored the impact of stochastic volatility on American spread call option prices by making comparisons with those generated under the Black and Scholes (1973) framework. The general findings have been that near at-the-money option prices from the Black and Scholes (1973) model are always lower than those from the stochastic volatility approach for the set of parameters analysed except in the case when the level of the second underlying asset is higher than the strike price where for positive correlations, prices from the Black and Scholes (1973) are greater for out-of-the-money options and lower for in-the-money options.

Comparisons have also been made of the effects of stochastic volatility on the free boundaries and deltas of the American spread call option. We have noted that deltas of near at- and out-of-the money option are very sensitive to changes in variance.

6.5. Directions of Future Research

While we have restricted our focus to American call options, the techniques developed in this thesis can be applied to a vast range of continuous payoff functions. The general integral expression for the American option price derived in Chapter 2 is applicable to many payoff functions involving two underlying assets such as options on the maximum or minimum of two assets as presented in Broadie and Detemple (1997). There is however a need for knowledge about the functional forms of the associated multi-dimensional early exercise boundaries to successfully generate option prices. Once such functional forms are known, early exercise boundary approximation techniques can be incorporated making it possible to develop numerical algorithms for generating option prices.

We have extended the Black and Scholes (1973) framework presented in Chapter 2 to a case when the dynamics of the two underlying assets are driven by a single stochastic variance process of the Heston (1993) type in Chapter 5 and later solved the resulting partial differential equation (PDE) using the method of lines approach. While the method of lines has proven to be efficient in terms of computational speed and accuracy, it is also worthwhile to derive the integral pricing expression for the American spread call option under stochastic volatility by solving the PDE with the aid of Fourier and Laplace transforms. The European option case has been handled in Bakshi and Madan (2000). However, when handling the American option case, knowledge about the functional form of the early exercise boundary is required to successfully solve the problem. The two underlying assets dynamics can also be allowed to evolve under the influence of independent stochastic variance processes and the associated integral option pricing expressions derived with the aid of Fourier and Laplace transforms using the concepts developed in Chapter 4 and the ideas of Bakshi and Madan (2000).

The models developed in Chapters 3 and 4 can also be applied to other payoff functions such as American put and strangle payoffs. The American strangle option under the

Black and Scholes (1973) model has been considered in Chiarella and Ziogas (2005), this model can be generalised to the stochastic volatility case by simply incorporating the corresponding payoff into the general pricing functions we have derived. As we highlighted before the most powerful feature of Fourier and Laplace transform techniques is that they can handle different payoff functions.

With regards to the method of lines approach, many different models can be handled efficiently making it a very useful method when undertaking higher dimensional American option pricing. An important advantage of the method of lines is its ability to generate option prices, early exercise boundaries and the associated hedge ratios as part of the solution at no additional computational cost. Chiarella et al. (2009) have considered an American option written on a single underlying asset whose dynamics evolve under the influence of both stochastic volatility and jump diffusion processes. It is worth to extend their approach to pricing American basket option under both stochastic volatility and jump diffusion processes. The method of lines approach can also be generalised to path dependent payoff functions like the most recent work by Chiarella, Kang and Meyer (2010) who consider the pricing of American barrier options under stochastic stochastic volatility.

Bibliography

Abramowitz, M. and Stegun, I. A. (1964), 'Handbook of Mathematical Functions', *Applied Mathematics Series, Dover, New York* **55**.

Adolfsson, T., Chiarella, C., Ziogas, A. and Ziveyi, J. (2009), 'Pricing American Options under Stochastic Volatility', *Working Paper, School of Finance and Economics, University of Technology, Sydney* .

Ait-Sahlia, F. and Lai, T. L. (2001), 'Exercise Boundaries and Efficient Approximations to American Option Prices and Hedge Parameters', *Journal of Computational Finance* **4**(4), 85–96.

Albrecher, H., Mayer, P., Schoutens, W. and Tistaert, J. (2007), ' The Little Heston Trap', *Wilmott Magazine* **January**, 83–92.

Bakshi, G. and Madan, D. (2000), 'Spanning and Derivative-Security Valuation', *Journal of Financial Economics* **55**, 205–238.

Bergomi, L. (2004), 'Smile Dynamics', *Risk* **September**, 117–123.

Black, F. and Scholes, M. (1973), 'The Pricing of Corporate Liabilities', *Journal of Political Economy* **81**, 637–659.

Blattberg, R. C. and Gonedes, N. J. (1974), ' A comparison of the Stable and Student Distributions as Statistical Models for Stock Prices', *Journal of Business* **47**(2), 244–280.

Boyle, P. P., Evnine, J. and Gibbs, S. (1989), 'Numerical Evaluation of Multivariate Contingent Claims', *Review of Financial Studies* **2**(2), 241–250.

Brennan, M. J. and Schwartz, E. S. (1977), ' The Valuation of American Put Options', *Journal of Finance* **32**, 449–462.

Broadie, M. and Detemple, J. (1997), 'Valuation of American Options on Multiple Assets', *Mathematical Finance* **7**, 241–286.

Broadie, M., Detemple, J., Ghysels, E. and Torres, O. (2000), 'American Options with Stochastic Dividends and Volatility: A Nonparametric Investigation', *Journal of Econometrics* **94**, 53–92.

Carmona, R. and Durrleman, V. (2003), 'Pricing and Hedging Spread Options', *SIAM Review* **45**, 627–685.

Carr, P. (1998), 'Randomization and the American Put', *The Review of Financial Studies* **11**(3), 597–626.

Carr, P. and Faguet, D. (1996), 'Valuing Finite-Lived Options as Perpertual', *http://ssrn.com/abstract=5415* .

Carr, P., Jarrow, J. and Myneni, R. (1992), 'Alternative Characterisation of American Put Options', *Mathematical Finance* **2**, 87–106.

Cheang, G., Chiarella, C. and Ziogas, A. (2009), ' The Representation of American Option Prices Under Stochastic Volatility and Jump Diffusion Dynamics', *Quantitative Finance Research Centre, University of Technology Sydney Research Paper No: 256* .

Chiarella, C. (2010), ' An Introduction to Derivative Security Pricing: Techniques, Methods and Applications', *Springer, Forthcoming* .

Chiarella, C., El-Hassan, N. and Kucera, A. (1999), ' Evaluation of American option prices in a path integral framework using Fourier-Hermite series expansion', *Journal of Economic Dynamics and Control* **23**, 1387–1424.

Chiarella, C., Kang, B. and Meyer, G. H. (2010), ' The Evaluation of Barrier Option Prices Under Stochastic Volatility', *Quantitative Finance Research Centre, University of Technology Sydney Research Paper No: 266* .

Chiarella, C., Kang, B., Meyer, G. H. and Ziogas, A. (2009), ' The Evaluation of American Option prices Under Stochastic Volatility and Jump-Diffusion Dynamics Using the Method of Lines', *International Journal of Theoretical and Applied Finance* **12**(3), 393–425.

Chiarella, C. and Ziogas, A. (2004), 'A survey of the Integral Representation of American Option Prices', *Quantitative Finance Research Centre, University of Technology Sydney Research Paper No: 118* .

Chiarella, C. and Ziogas, A. (2005), ' Evaluation of American Strangles', *Journal of Economic Dynamics and Control* **29**(1-2), 31–62.

Christoffersen, P., Heston, S. and Jacobs, K. (2009), ' The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work So Well', *Management Science* **55**(2), 1914–1932.

Clark, P. K. (1973), 'A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices', *Econometrica* **41**(1), 135–155.

Clarke, N. and Parrott, K. (1999), 'Multigrid for American Option Pricing with Stochastic Volatility', *Applied Mathematical Finance* **6**(3), 177–195.

Cont, R. and da Fonseca, J. (2002), 'Dynamics of Implied Volatility Surfaces', *Quantitative Finance* **2**, 45–60.

Cox, J. C., Ingersoll, J. and Ross, S. A. (1985), 'A Theory of the Term Structure of Interest Rates', *Econometrica* **53**(2), 385–407.

Cox, J. C. and Ross, S. A. (1976), 'Valuation of Options for Stochastic Processes', *Journal of Financial Economics* **3**, 145–166.

Cox, J. C., Ross, S. A. and Rubinstein, M. (1979), 'Option Pricing: A Simplified Approach', *Journal of Financial Economics* **7**, 229–263.

da Fonseca, J., Grasselli, M. and Tebaldi, C. (2005), ' Wishart Multi-Dimensional Stochastic Volatility', *Working paper ESILV, RR-31* .

da Fonseca, J., Grasselli, M. and Tebaldi, C. (2008), ' A multifactor Volatility Heston Model', *Quantitative Finance* **8**(6), 591–604.

Dempster, M. A. and Hong, S. S. G. (2000), 'Spread Option Valuation and the Fast Fourier Transform', *Research Papers in Management Studies, Cambridge* (26/2000).

Detemple, J. and Tian, W. (2002), 'The Valuation of American Options for a Class of Diffusion Processes', *Management Science* **48**(7), 917–937.

Feller, W. (1951), 'Two Singular Diffusion Problems', *The Annals of Mathematics* **54**(1), 173–182.

Geltner, D., Riddiough, T. and Stojanovic, S. (1996), 'Insights on the Effect of Land Use Choice: The Perpetual Option on the Best of Two Underlying Assets', *Journal of Urban Economics* (0002), 20–50.

Geske, G. and Johnson, H. E. (1984), 'The American Put Option Valued Analytically', *Journal of Finance* **39**, 1511–1524.

Geske, R. (1979a), 'The Valuation of Compound Options', *Journal of Financial Economics* **7**, 63–81.

Geske, R. (1979b), ' A Note on an Analytical Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends', *Journal of Financial Economics* **7**, 375–380.

Geske, R. (1981), ' Comments on Whaley's Note', *Journal of Financial Economics* **9**, 213–215.

Harrison, J. M. (1990), ' Brownian Motion and Stochastic Flow Systems ', *Krieger Publishing Company, Malabar, Florida* .

Heston, S. L. (1993), 'A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bonds and Currency Options', *The Review of Financial Studies* **6**(2), 327–343.

Huang, J., Subrahmanyam, M. G. and Yu, G. G. (1996), 'Pricing and Hedging American Options: A Recursive Integration Method', *Review of Financial Studies* **9**, 277–300.

Hull, J. and White, A. (1987), ‘ The Pricing of Options on Assets with Stochastic Volatility’, *Journal of Finance* **XLII**(2), 281–300.

Hurd, T. and Zhou, Z. (2009), ‘A Fourier Transform Method for Spread Option Pricing’, *Working Paper, Dept of Mathematics and Statistics, McMaster University, Canada* .

Ibáñez, A. and Zapatero, F. (2004), ‘Monte Carlo Valuation of American Options Through Computation of the Optimal Exercise Frontier’, *Journal of Financial and Quantitative Analysis* **39**(2), 253–275.

Ikonen, S. and Toivanen, J. (2004), ‘Operator Splitting Methods for American Options with Stochastic Volatility’, *Applied Mathematics Letters* **17**, 809–814.

Jacka, S. D. (1991), ‘Optimal Stopping and the American Put’, *Mathematical Finance* **1**, 1–14.

Jackson, K. R., Jaimungal, S. and Surkov, V. (2008), ‘ Fourier Space Time-Stepping for Option pricing with Lévy Models ’, *Journal of Computational Finance* **12**(2), 1–29.

Jaimungal, S. and Surkov, V. (2008), ‘ Stepping Through Fourier Space’, *Available at SSRN: http://ssrn.com/abstract=1281743* .

Jamshidian, F. (1992), ‘An Analysis of American Options’, *Review of Futures Markets* **11**, 72–80.

Johnson, H. (1987), ‘Options on the Maximum or the Minimum of Several Assets’, *Journal of Financial and Quantitative Analysis* **22**(3), 277–283.

Johnson, H. E. (1983), ‘ An Analytic Approximation for the American Put Price’, *Journal of Financial and Quantitative Analysis* **18**, 141–148.

Johnson, H. and Shanno, D. (1987), ‘Option Pricing when the Variance is Changing’, *Journal of Financial and Quantitative Analysis* **22**(2), 143–151.

Ju, N. (1998), ‘Pricing an American Option by Approximating its Early Exercise Boundary as a Multipiece Exponential Function’, *Review of Financial Markets* **11**, 627–646.

Kahl, C. and Jäckel, P. (2005), ‘Not-So-Complex Logarithms in the Heston Model’, *Wilmott Magazine* **September**.

Kallast, S. and Kivinukk, A. (2003), ‘Pricing and Hedging American Options using Approximations by Kim Integral Equations’, *European Finance Review* **7**, 361–383.

Karatzas, I. (1988), ‘On the Pricing of American Option’, *Applied Mathematics and Optimization* **17**, 37–60.

Kim, I. J. (1990), ‘The Analytical Valuation of American Options’, *Review of Financial Studies* **3**, 547–572.

Latané, H. A. and Rendleman, R. J. (1976), ‘Standard Deviations of Stock Price Ratios Implied in Option Prices’, *The Journal of Finance* **XXXI**(2), 369–381.

Logan, J. D. (2004), 'Applied Partial Differential Equations', *Springer* .

Magrabe, W. (1978), 'The Value of an Option to Exchange one Asset for Another', *Journal of Finance* **33**, 177–186.

Mallier, R. (2002), 'Evaluating Approximations to the Optimal Exercise Boundary for American Options', *Journal of Applied Mathematics* **2**, 71–92.

Mandelbrot, B. (1963), 'The Variation of Certain Speculative Prices', *The Journal of Business* **36**, 394–419.

McKean, H. P. J. (1965), 'Appendix: A Free Boundary Problem for the Heating Function Arising from a Problem in Mathematical Economics', *Industrial Management Review* **6**, 32–39.

Meyer, G. H. (1998), ' The Numerical Valuation of Options with Underlying Jumps', *Acta Mathematica* **47**, 69–82.

Meyer, G. H. (2010), ' Pricing Options and Bonds with the Method of Lines', *http://people.math.gatech.edu/ meyer/* .

Meyer, G. H. and van der Hoek, J. (1997), ' The Evaluation of American Options with the Method of Lines', *Advances in Futures and Options Research* **9**, 265–285.

Officer, R. R. (1972), 'The Distribution of Stock Returns', *Journal of the American Statistical Association* **67**(340), 807–812.

Platen, E. and Rendek, R. (2008), ' Empirical Evidence on Student-t Log-Returns of Diversified World Stock Indicies', *Journal of Statistical Theory and Practice* **2**(2), 233–251.

Praetz, P. D. (1972), 'The Distribution of Share Price Changes', *The Journal of Business* **45**(Jan.), 49–55.

Press, S. J. (1967), 'A Compound Events Model for Security Prices', *The Journal of Business* **40**(July), 317–335.

Roll, R. (1977), ' An Analytic Valuation Formula for Unprotected American Call Options on Stocks with known Dividends', *Journal of Financial Economics* **5**, 251–258.

Rosenberg, B. (1972), 'The Behavior of Random Variables with Nonstationary Variance and the Distribution of Security Prices', *Stochastic Volatility: Selected Readings by Neil Shephard, Oxford University Press 2005 Edition* pp. 83–103.

Scott, L. O. (1987), 'Option Pricing when the Variance Changes Randomly: Theory, Estimation and Application', *Journal of Financial and Quantitative Analysis* **22**(4), 419–438.

Shephard, N. G. (1991), 'From Characteristic Function to Distribution Function: A Simple Framework for the Theory', *Econometric Theory* **7**, 519–529.

Shreve, S. E. (2000), 'Stochastic Calculus for Finance II', *Springer Finance* .

Stein, E. M. and Stein, J. C. (1991), 'Stock Price Distributions with Stochastic Volatility: An Analytical Approach', *The Review of Financial Studies* **4**(4), 727–752.

Stulz, R. (1982), 'Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications', *Journal of Financial Economics* **10**, 161–185.

Surkov, V. (2008), ' Parallel Option Pricing with Fourier Space Time-Stepping Method on Graphics Processing ', *In Proceedings of the First Workshop on Parallel and Distributed Computing in Finance* .

Touzi, N. (1999), ' American Options Exercise Boundary when the Volatility Changes Randomly', *Applied Mathematics and Optimization* **39**, 411–422.

Tzavalis, E. and Wang, S. (2003), 'Pricing American Options under Stochastic Volatility: A New Method using Chebyshev Polynomials to Approximate the Early Exercise Boundary', *Queen Mary, University of London, School of Economics and Finance Working Paper No: 488* .

Whaley, R. E. (1981), ' On the Valuation of American Call Options on Stocks with known Dividends', *Journal of Financial Economics* **9**, 207–211.

Whaley, R. E. (1986), 'On Valuing American Futures Options', *Financial Analyst Journal* pp. 49–59.

Wiggins, J. B. (1987), 'Option Values under Stochastic Volatility: Theory and Empirical Estimates', *Journal of Financial Economics* **19**, 351–372.

Wilmott, P. (2006), 'Paul Wilmott on Quantitative Finance', *John Wiley and Sons, Ltd* .

Wystup, U. (2009), ' The Ultimate Quant Cheat Sheet', *MathFinance AG, www.mathfinance.com* .

Zhu, J. (2000), 'Modular Option Pricing', *Working Paper Available at SSRN: http://ssrn.com/abstract=224969* .