Credit Risk Modelling in Markovian HJM Term Structure Class of Models with Stochastic Volatility

A Thesis Submitted for the Degree of

Doctor of Philosophy

by

Samuel Chege Maina
B.Sc.(Hons)(University of Nairobi, Kenya)
M.Sc.(Fin.Math)(Technical University of Kaiserslautern, Germany)

samuel.chege@uts.edu.au

in

School of Finance and Economics
University of Technology, Sydney
PO Box 123 Broadway
NSW 2007, Australia.

June 18, 2011.

Certificate

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirement for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Signed Production Note:
Signature removed prior to publication.

Date 28/06/2011 ....

Acknowledgements

This thesis would not have been possible without the guidance and the help of several individuals who in one way or another extended their invaluable assistance.

First and foremost I would like to express my deepest appreciation to my supervisor, Carl Chiarella, whose support, patience and encouragement throughout the duration of my doctoral studies has been unwavering. Without his guidance and persistent help this thesis would not have been possible. I would also like to express my gratitude to my co-supervisor, Christina Nikitopoulos Sklibosios for her additional supervision, suggestions and invaluable comments. Her thoroughness and meticulous attention to details has come in handy, especially during the write-up phase of the thesis.

I wish to thank my fellow research students and the staff of the School of Finance and Economics at UTS for providing a pleasant environment for the completion of the research work. In particular, I wish to thank Boda Kang, Chih-Ying Hsiao, Min Zheng, Jonathan Ziveyi, Lei Shi, Ke Du and Ji-Won (Stephanie) Ough for the ideas, suggestions and assistance they have provided during the countless discussions we have held. Many thanks also to the head of the School, Tony Hall for his constant support. I am also grateful for the financial assistance I received from Quantitative Finance Research Center (QFRC), Financial Integrity Research Network (FIRN) and the School of Economics and Finance to attend various workshops and conferences.

Finally, I wish to express my appreciation to my wife Christine whose dedication, love, sacrifice and persistent confidence in me, has taken the load off my shoulder and for having single handedly taken care of daughter Sara. My gratitude also goes to our parents for their unconditional support.

Contents

List of Figures

List of Tables

Glossary of Notations

- ATM = At-the-money.
- CIR = Cox-Ingersoll-Ross.
- HJM = Heath-Jarrow-Morton model.
- CDS = Credit Default Swaps.
- HW = Hull-White model.
- ODE = Ordinary differential equation.
- OTM = Out-the-money.
- PDE = Partial differential equation.
- SDE = Stochastic differential equation.
- SIE = Stochastic integral equation.
- $B(t), B_t$: Money market account at time t, bank account at time t.
- $P(t,T,\cdot)$: Price at time t of a default-free zero coupon bond with maturity T.
- $\bar{P}^d(t,T,\cdot)$: Pre-default price at time t of a defaultable zero coupon bond with maturity T.
- $P^d(t,T,\cdot)$: Price at time t of a defaultable zero coupon bond with maturity T.
- $f(t,T,\cdot)$: Instantaneous default-free forward rate of interest prevailing at time t for instantaneous borrowing at T.
- $f^d(t,T,\cdot)$: instantaneous defaultable forward rate of interest prevailing at time t for instantaneous borrowing at T.
- $r(t,\cdot)$: Instantaneous default free short rate of interest at time t.
- $r^d(t,\cdot)$: Instantaneous defaultable short rate of interest at time t.
- $\lambda(t,T,\cdot)$: Instantaneous forward credit spread.
- $c(t,\cdot)$: Instantaneous short term credit spread at time t.

- $N(t)$: Marked point Process at time t.
- $h(t)$: Intensity of a Marked point process at time t under the real world probability measure; $\tilde{h}(t)$: Intensity of a Marked point process at time t under the risk-neutral probability measure.
- $\mathcal{R}(t)$: Fractional recovery process at time t.
- τ_i: Random default time.
- $q(\tau_i)$: Loss rate on the bond's face value at each default time τ_i.
- $V(t)$: Stochastic volatility process at time t.
- $W(t) = \{W^f(t), W^\lambda(t), W^V(t)\}$: Wiener Process at time t under the real world measure;
  $\tilde{W}(t)$: Wiener Process at time t under the risk-neutral measure;
- $X(t)$: Markov chain Process at time t.
- τ_i^x: Jump times of the Markov chain.
- $\mu(\omega; dt, dq)$: Random measure associated with the Marked point process, N.
- $h_{i,j}^X(t)$, $\tilde{h}_{i,j}^X(t)$: Transition intensity at time t of a Markov chain from state i to j.
- $\mathcal{F}^W(t)$, $\mathcal{F}^N(t)$, $\mathcal{F}^X(t)$ Filtrations generated at time t by the Wiener process, Marked point process and Markov chain respectively.
- $\rho_{12} \equiv \rho^{V\lambda}$: Correlation between stochastic volatility and short term credit spread processes; $\rho_{13} \equiv \rho^{Vf}$: Correlation between stochastic volatility and default-free short rate processes; $\rho_{23} \equiv \rho^{\lambda f}$: Correlation between short term credit spread and default-free short rate processes.
- $\phi(t)$: Market price of diffusion risk; $\psi(t)$: Market price of jump risk.
- κ^f, κ^λ, κ^V: Speeds of mean reversion for the risk-free short rate, the short term credit spread and the stochastic volatility processes, respectively.
- $\mathcal{P}(t, r^d, T; T_0, K)$: Price at time t for the put option with maturity T_0, strike K that is knocked out on default of an underlying defaultable bond with a maturity T, under stochastic volatility.
- $\mathcal{C}(t, \bar{P}^d, X(t))$: Price at time t of a call option under regime-switching stochastic volatility.

- $\pi_f(t)$, $\tilde{\pi}_f(t)$, $\bar{\pi}_f(t)$, $\pi_{cpr}(t)$: Price at time t of a credit default swap.
- $\mathcal{C}_{swpt}(t)$, $\tilde{\mathcal{C}}_{swpt}(t)$: Price at time t of a credit default swaption.
- $[X]_t$: Quadratic variation at time t of the process X; $[X,Y]_t$: Quadratic covariation of two processes, X and Y.
- $< \mathbf{a}, \mathbf{b} >=< (a_1, \ldots, a_n), (b_1, \ldots, b_n) >= \sum_{i=1}^n a_i b_i$: Inner product.

Abstract

Empirical evidence strongly suggests that interest rate volatility is stochastic and correlated to changes in interest rates. In addition, the intensity process has been shown to generate heavy-tailed behavior and this has been attributed to stochastic volatility. A good credit risk model should incorporate the correlation between the short rate and credit spread processes as changes in interest rates can directly affect and change the credit spread or indirectly influence the market's perception of default risk which has an impact on credit spreads.

The objective of this thesis is to model credit risk within a Markovian Heath, Jarrow, and Morton [1992] (hereafter HJM) term structure model with stochastic volatility by extending the defaultable framework developed in Schönbucher [1998]. Adapting the HJM framework to include default risk results in a generalised framework that incorporates all the information on the current risk free term structure as well as the credit spread curve. Under some conditions on the specification of the volatility functions, the model admits finite dimensional Markovian realisations and as a result, the default-free yield curve as well as the credit spread curves can be calculated with low computational cost at any given time.

The main contributions of this thesis are:

◇ *Markovian Defaultable HJM Term Structure Models with Unspanned Stochastic Volatility - Chapter 2.* Stochastic volatility is introduced into the Schönbucher [1998] model and we generalise it to allow for a correlation structure between the default-free forward rate, the forward credit spread and stochastic volatility. Under certain level dependent volatility specifications, we derive a Markovian representation of the defaultable short rate in terms of a finite number of state variables which we then express in terms of economic quantities observed in the market, specifically in terms of discrete tenor forward rates. A numerical experiment is then conducted to investigate the distributional properties of the defaultable bond price and bond returns which reveals the existence

of a left tail.

◇ *Credit Derivative Pricing under a Markovian HJM Term Structure Model with (Diffusion Driven) Humped Volatility - Chapter 3.* We verify that under the assumption of a humped volatility specification, the defaultable forward rates admits finite dimensional affine realisations. The default of the underlying reference entity is modelled as a Cox process and we derive exponential affine bond price formulas in the presence of stochastic volatility. We then investigate the pricing of single-name credit default swaps both in the presence and absence of counterparty risk and derive formulas for the valuation of credit default swaptions within the framework. On relaxing the level dependency assumption within the humped volatility specification, we price knocked-out put options on defaultable bonds using the Fourier transform approach.

◇ *Valuation of Bond Options under a Defaultable HJM Class of Models with Regime-Switching Volatility - Chapter 4.* We allow the defaultable forward rate volatility to depend on the current forward rate curve as well as on a modulating continuous time Markov chain making use of the results in Valchev [2004] and Elhouar [2008]. Stochasticity is then introduced to the volatility function by a separable volatility specification which guarantees finite-dimensional Markovian realisations under regime switching. A special case of the short rate class of models, the Hull-White-Extended-Vasicek type of model is obtained in the defaultable setting from which an explicit bond pricing formula is derived. We then apply finite difference methods to price European options under two-state regimes.

We give a summary of all the thesis findings in Chapter 5 where we also present the concluding remarks and directions for future research work.

Chapter 1

Introduction

In this thesis, we model credit risk with the objective of pricing defaultable bonds, credit derivatives and other financial securities that are exposed to credit risk. We develop a general, flexible framework based on the Heath, Jarrow, and Morton [1992](hereafter HJM) theory of the term structure of interest rates which models the evolution of the entire yield curve. In that framework, the instantaneous forward rates which are exogenously specified are used as the building blocks that by construction guarantee the recovery of the currently observed market yield curve. The no-arbitrage drifts of the defaultable forward rates are uniquely specified when the volatilities and correlations are assigned. This yields a model that enables us to capture a number of important stylized facts in credit risk modelling including the significance of correlation between market and default risk and the unspanned stochastic volatility factors that drive interest rate derivatives innovations but do not affect the innovations of interest rates or swap rates.

We adopt the broader definition of credit risk given in Bielecki and Rutkowski [2002] as any risk associated with a credit-linked event, a random event whose occurrence affects the ability of the counterparty to honor its contractual obligations in a financial contract. The possible events include changes in the credit quality of the reference entity or the counterparties (both upgrades and downgrades of their credit ratings), variation of the credit spreads, bankruptcy, restructuring and the default event (obligation default). These are defined with respect to a reference credit and the reference credit assets issued by it.

1.1 Motivation

A major challenge faced by financial institutions including banks, hedge funds, insurance companies, pension funds and brokerage firms is on how to manage or reduce their credit exposures. Different economic and/or regulatory motives dictate the market participants positions at any given time. Credit derivatives allow the market players to hedge credit exposures, transfer credit risk either completely or partially between counterparties, generate leverage, and decompose and separate risks embedded in securities. In addition, they are also used to "synthetically create loan or bond substitutes for entities that have not issued in those markets at chosen maturities" and for expressing a directional or volatility view on an institution.[1]

Credit derivatives offer a higher degree of structural flexibility as compared to the more basic credit instruments such as bank loans and bonds. This arises from their ability to separate credit risk from funding thereby allowing the market players to change their credit risk exposures without the actual buying or selling of the loans or bonds in the primary or secondary markets. There has been an evolution of standardized instruments like Credit Default Swaps that efficiently facilitate the transfer of credit risk between entities and also between different markets for risk. This enhances efficient allocation of credit risk within economies which can be distorted by different capital adequacy requirements for different types of credit investors.

There has been a considerable growth in the credit derivatives market that now exceeds, in notional amounts[2], both the equity derivatives and corporate bond markets. This growth

[1]See Merrill-Lynch [2006a, page 4]
[2]A detailed description of this is given in a survey report by British-Bankers-Association [2006].

has gone hand in hand with the success of quantitative methodologies and technological/computational advances that help practitioners manage the risk. In return, this has spurred further research with the objective of better understanding, modelling and hedging of credit risk in the advent of more complicated synthetic products. It was highlighted in a Merrill-Lynch [2006a] report that, effective from late 2003 the credit derivatives market had evolved from what was predominantly a single-name market to a more complex market comprising in addition, index, correlation and options. This, they noted, arose as a result of the need for yield enhancement, increased leveraging, increased index (CDX and iTraxx) liquidity and continuous product innovation to meet growing investor needs. There exists therefore, an increasing need for credit models that are internally consistent (arbitrage-free), intuitive and that offer easy calibration to market data both for single-name and multi-name products.

The primary motivation behind this thesis is to take advantage of the many appealing properties, benefits and/or advantages of the HJM term structure model as it was applied to credit risk modelling in Schönbucher [2003] and to generalize it to accommodate a number of stylized facts on credit risk. In obtaining finite-dimensional Markovian models, the choices of the volatility functions allow us to introduce stochastic volatility in two ways: via a diffusion process and through a Markov chain.

Introducing stochasticity through a diffusion process allows us to incorporate correlation between stochastic volatility, the interest rates and credit spreads, consistent with stylised facts on interest rate volatility. This generalisation yields closed-form solutions for risky bonds and semi-analytical solutions for contingent claims on the bonds while incorporating the correlation between interest rates and the market perception on the default risk. On making use of the Markov chain to introduce stochastic volatility, we demonstrate that the framework also yields semi-closed form solutions for defaultable bonds. We then we apply finite-difference methods to solve the pricing partial differential equation for the European options on the risky debt.

1.2 Literature Review

Over the last three and half decades, two major arbitrage-free methodologies that model credit risk have evolved; namely, structural models[3] which are more intuitive and reduced-form models[4] that allow for easier calibration to historical data.

1.2.1 Structural Models

Structural models are based on the firm's value and define default as a contingent claim by specifying the default time. The credit event is triggered by the movement of the firm value relative to some random or non-random barrier (also called default threshold) and default time is endogenously specified. The models were initiated by Black and Scholes [1973] and Merton [1974] and offer a link between the credit quality of the firm and its economic/financial condition. The model assumes that default can only occur at the debt's maturity. By applying the Black-Scholes model assumptions, Merton derived a partial differential equation for defaultable bonds within a single rating class. Default time is taken as a predictable stopping time in the model yielding unreasonable (almost zero) short term credit spreads near maturity. This is contrary to the empirical evidence as documented in Jones et al. [1984] which shows that the actual credit spread curves are sometimes flat or even downward-sloping. Zhou [1997, 2001] suggested the introduction of jumps into the model to remedy this shortcoming thereby allowing a firm to suddenly default due to a sudden drop in its value.

Geske [1977] further extended the model to price defaultable coupon bonds by assuming that the equity holders make the coupon payment and thereby own a compound option. He then derived closed form solutions for the coupon bond prices. Cox, Ingersoll, and Ross [1985] (hereafter CIR) applied the Merton model to the valuation of defaultable bonds with

[3]We use this phrase to cover both the Merton class of models and first-passage-time models.

[4]This category includes intensity-based and migration models.

stochastic interest rates in order to identify the variable coupon rate that would eliminate and/or reduce interest rate risk. Shirakawa [1999] extended the Merton model to incorporate stochastic interest rates with dynamics governed by a Vasicek model and derived a closed form solution for the risky bonds. Later extensions of the Merton model including those of Frank and Torous [1989, 1994] who define bankruptcy as an endogenous event driven by factors such as agency and bankruptcy costs. Further extensions have been done in Geske [1979], Jamshidian [1989], Johnson and Stulz [1987], Leland and Toft [1996], Hull and White [1995] and Hilberink and Rogers [2002] among others.

The first-passage models were introduced by Black and Cox [1976], as an extension of the Merton model by incorporating a time-dependent exponential barrier thereby allowing earlier defaults and the derivation of closed form solutions for the price of a defaultable bond. Their approach facilitated the modelling of safety covenants that allow the bondholders to force bankruptcy if some conditions are satisfied. Brennan and Schwartz [1980] applied a constant default barrier in their valuation model for convertible defaultable bonds and solved the resulting pricing partial differential equation numerically. Longstaff and Schwartz [1995a,b] extended the Black-Cox model to incorporate interest rate risk by assuming that the short term interest rates follows the Vasicek model. Default is triggered when the firm's value process hits a constant threshold during the life of the bond. Similar work was done by Briys and de Varenne [1997].

Kim, Ramaswamy, and Sundaresan [1993a,b] considered the case where the bondholders have priority and there are provisions to prohibit the stockholders from selling the firm's assets to pay dividends. Cathcart and El-Jahel [1998] improved on the Longstaff and Schwartz [1995a] model by assuming that the short rate process follows the CIR dynamics and that the default threshold follows a geometric Brownian motion. Shirakawa [1999] investigated the behavior of the credit spreads within the model framework while separating the analysis of the yield spread from the default free interest rate process and evaluated analytically, the arbitrage-free yield spread. However, the assumption on complete information about the asset and the default threshold make default a predictable process and the models therefore still produce credit spreads close to zero for small maturities.

Duffie and Lando [2001] considered the case where investors have incomplete information and can only infer the distribution function for the firm-value process implying that default is not predictable. Giesecke [2004] assumed incomplete information on the default barrier thereby introducing uncertainty in the default threshold and pointed out that the credit yield spread tends to increase with longer maturity, reflecting the increase in uncertainty in the distant future. These constitute what has been referred to in the literature as hybrid models that combine both the structural and reduced-form models.

1.2.2 Intensity Models

In contrast, reduced-form models (also called intensity-based models) assume that an exogenous random variable governs the default process that models the default time but not the severity of loss as in structural models. This process assigns a non-zero default probability over any time interval and default is usually treated as an unpredictable Poisson event. In addition, the recovery rate is assumed to be an exogenously given process in comparison to the structural models where the recovery mechanism is endogenously specified. This approach is attributed to Pye [1974] and Litterman and Iben [1991] and was developed further in later works of Lando [1994], Artzner and Delbaen [1995], Hull and White [1995], Jarrow and Turnbull [1995], Das and Tufano [1996], Duffie and Kan [1996], Jarrow [1996], Duffie [1998], Schönbucher [1998], Lando [1998],Duffie and Singleton [1999], Madan and Unal [1998, 2000], Jarrow and Turnbull [2000], Jarrow and Yu [2001], Bielecki, Jeanblanc, and Rutkowski [2004] and Jamshidian [2004] among others.

Jarrow and Turnbull [1995] assumed a constant, exogenously given default intensity Poisson process and recovery rates which implied that default was equally likely throughout the life of the bond. They derived closed form solutions for defaultable bonds and derivative securities.

This was generalized in Lando [1994, 1998] who considered default intensity driven by a more general Cox processes. Bielecki, Jeanblanc, and Rutkowski [2007] developed a general valuation framework for defaultable basket claims based on Cox processes, from which the implied default correlation was estimated from the prices of the most liquid credit derivatives. The intensity-based approach has also been applied in Duffie et al. [2003] to the valuation of defaultable sovereign debt.

Jarrow, Lando, and Turnbull [1997] extended the work of Jarrow and Turnbull [1995] to allow for credit migration by using constant rating transition intensities although they maintained the independence assumption of the recovery rate to the state variables. This was generalized in Lando [1998] who applied Cox processes to govern the transition intensities thereby introducing time-dependency in the transition probabilities. Das and Tufano [1996] relaxed the independence assumption in Jarrow et al. [1997] by incorporating dependence between the default intensities and the interest rates in addition to allowing for random recovery rates. Das and Tufano [1996] model[5] generated more realistic[6] credit spreads which were linked to more general factors over and above the credit rating class. Kijima and Komoribayashi [1998] modified the risk-neutral default probabilities in Jarrow et al. [1997] to allow for better practical implementation of the model.

Hurd and Kuznetsov [2006, 2007] applied a continuous Markov chain to an independent set of affine processes (stochastic intensities, interest rates and stochastic recoveries) to derive an efficient pricing framework for defaultable securities. In their multi-firm migration framework, the credit migration of each firm is correlated to the market conditions via a stochastic time change variable which governs the migration and default of the firm.

A more exhaustive review and complete list of the literature on the credit risk models is given in Bielecki and Rutkowski [2002], Schönbucher [2003], Duffie and Singleton [2003] and

[5]Their model was structured in a discrete-time HJM framework

[6]This result could be partially attributed to the huge significance of the correlation structure between default intensity and interest rate process in the valuation of options on risky debt. The correlation gives some information about the link between the default free rates and the market's perception of default risk. Evidence of this is given in Longstaff and Schwartz [1995a] and Duffee [1998] among others.

Lipton and Rennie [2011].

1.2.3 Markovian HJM Term Structure Models

The application of the Heath, Jarrow, and Morton [1992] (HJM) model to defaultable term structure modelling falls within the reduced-form class of models. This was first examined in Jarrow and Turnbull [1995] who considered the case where both the underlying and the derivative security are subject to credit risk. Duffie and Singleton [1999] developed a discrete-time reduced-form model that adds a forward spread process to the forward risk-free rate and apply the HJM approach to derive the no-arbitrage drift restriction condition in the defaultable setting. By assuming the recovery of market value, they derived recursive formulas for the contingent claims.

The framework was developed further in Schönbucher [1998] to allow for restructuring of defaultable bonds and multiple recoveries. Various forms of the no-arbitrage drift restriction conditions between the default free and the defaultable term structures are derived from which the term structure of defaultable bond prices is obtained, where the forward credit spread offers the link between the defaultable and default free term structures. Similar results are derived in Pugachevsky [1999] and Maksymiuk and Gatarek [1999] where no requirement is made for the existence of jumps in the forward rates that lead to default in the bond price dynamics but rather default is triggered through exogenously specified point process.

Bielecki and Rutkowski [2000b, 2004] extended the ideas in Schönbucher [1998] to incorporate the probability of migration between rating classes. They derived the valuation formulas for coupon bonds and credit derivatives under various recovery schemes. Eberlein and Özkan [2003] generalized the Bielecki and Rutkowski [2000b] framework using a large and flexible class of Lévy processes to derive an arbitrage-free model of defaultable bonds that incorporates multiple defaults and recoveries in the spirit of Schönbucher [2003].

The resulting short rate prices in the HJM class of models were in general path-dependent where the present level of the short interest rates depend also on the level of interest rates in the past. This is a consequence of the non-Markovian noise terms that occur in the drift of the stochastic differential equation for the short rate. Various attempts to obtain finite-dimensional Markovian models within the HJM framework have been made in the risk-free term structure literature which hinge on particular volatility function specifications. Conditions on level dependent volatility processes that lead to Markovian HJM models under a diffusion forward rate process were obtained in Ritchken and Sankarasubramanian [1995], Bhar and Chiarella [1997] and Inui and Kijima [1998]. Björk and Svensson [2001], Björk and Landèn [2002] and Chiarella and Kwon [2001, 2003] substantially extend these early studies, by considering a rather general level dependent volatility structure, studying the necessary and sufficient conditions for the existence of finite-dimensional-realisations (FDR).

Extensions to jump-diffusion volatility structures have been studied by Björk and Gombani [1999] who examined the necessary and sufficient conditions that guarantee FDR, under a time deterministic jump volatility structure while Chiarella and Nikitopoulos-Sklibosios [2003] considered the necessary conditions on a level dependent jump volatility structure. A defaultable term structure with level dependent volatility and stochastic intensity was studied in CNS. They showed that finite-dimensional-realisations are feasible only for a truncated Markovian system or for constant Poisson volatility functions. Filipovic, Tappe, and Teichmann [2010] establish existence, uniqueness and stability results for mild and weak solutions of stochastic partial differential equations (SPDE's) with path dependent coefficients driven by an infinite dimensional Wiener process and a compensated Poisson random measure. In Berndt, Ritchken, and Sun [2010], Markovian defaultable term structure models with level dependent volatility are considered. Exponential affine representation of riskless and risky bond prices are derived and the model allows for default clustering and contagion.

1.2.4 Stochastic Volatility Models

Stochastic volatility models have been studied extensively for equity markets. They capture some well-known features of the implied volatility surface, such as the volatility smile and skew (slope at-the-money). Past contributions include work by Scott [1987], Hull and White [1987], Stein and Stein [1991] and Heston [1993] who investigated the pricing of European options under a single factor process. However, stochastic models in which volatility is modelled as a one-factor diffusion have been shown to experience difficulties in fitting implied volatility levels across all strikes and maturities. Extensions to the multi-factor case have been investigated by various authors. The extension of Fouque and Lorig [2010] to the Heston Model shows that a multi-scale model can improve calibration to the implied volatility surface produced by the options market.

Research in term structure models with stochastic volatility has mostly revolved around the family of Affine Term Structure Models (ATSM). Longstaff and Schwartz [1992] suggested the existence of randomly changing volatility in interest rates that impacts on derivative prices. Similar results are shown in the contributions by Ball and Torous [1999], Collin-Dufresne and Goldstein [2002] and Trolle and Schwartz [2009].

Hull and White [2004/2005] implemented the Merton model using implied volatilities of options issued by the firm and observed that implied volatility is sufficient to explain and predict credit spreads. In addition, their results show there exists a positive correlation between the implied volatility and credit spreads. Fouque, Sircar, and Solna [2006] investigated the effects of stochastic volatility in the dynamics of the risky debt in the Black and Cox [1976] first passage model of credit risk using the multi-scale model developed in Cotton, Fouque, Papanicolaou, and Sircar [2004]. This was extended in Fouque, Wignall, and Zhou [2008] to allow for a dependency structure in the underlying portfolio of reference single-names for pricing multi-name credit derivatives.

Within the HJM framework, stochastic volatility models were introduced (to the best of our knowledge) in Chiarella and Kwon [2000b] and further advanced by Björk et al. [2004] and Filipovic and Teichmann [2002, 2003] who provided the necessary and sufficient conditions on stochastic volatility for diffusion HJM models to admit FDR, by employing Lie algebra theory. Chiarella, Fanelli, and Musti [2011] modelled the forward credit spread within the HJM framework by assuming a stochastic volatility function linear in the state variables and derived valuation techniques for pricing credit default swaps and swaptions.

1.2.5 Regime Switching in Term Structure Models

Research on regime switching within term structure models dates back to the contributions of Hamilton [1989], Garcia and Perron [1996] and Naik and Lee [1997]. Changes in business cycle conditions and monetary policies may affect real rates and expected inflation, causing interest rates to behave differently in varying time periods. Regime switching models have been used to capture the variations in the stochastic behavior of interest rates over time as they can accommodate regime-dependent mean reversion.

Ang and Bekaert [2002] demonstrated that univariate regime switching models explain adequately the non-linear mean-reversion observed in interest rates. This was extended in Bansal and Zhou [2002] to allow for regime-switching market prices of risk, thereby affecting the entire term structure. Hansen and Poulsen [2000] extended the Vasicek model to incorporate a regime switching long-term level whose change is governed by a Poisson process, thereby allowing for jumps in the drift of the short rate process. They applied simulation techniques for the valuation of bonds and bond option prices.

Landén [2000] developed a pricing model where the drift and diffusion parameters in the short rate process are modulated by a Markov chain whose dynamics follow a jump-diffusion process and derived semi-affine bond price formulas. To introduce a cyclical pattern in the extended-Vasicek Hull-White model, Elliott and Wilson [2007] modelled the mean-reverting level directly as a random process that follows a finite-state, continuous-time Markov chain and deduced the valuation formulas for zero bonds. Elliott and Siu [2009] derived exponential affine bond price formulas under both Markovian regime-switching Hull-white and Cox-Ingersoll-Ross models by allowing the short rate parameters to switch over time according to a continuous-time, finite Markov chain.

Wong and Wong [2007] derived semi closed-form solutions for term structure of defaultable interest rates that incorporates regime-switching using an affine-type model. They also investigated empirically, the impact of the systematic risk of regime switching on the term structure of defaultable bonds in a *two*-state regime within the Cox-Ingersoll-Ross model. Andersson and Vanini [2010] extended the affine Markov chain model in Hurd and Kuznetsov [2007] using a regime switching Markov mixture model that describes both the speed and direction of the migration matrices and derived tractable solutions for credit default swaps.

Not much research work in regime switching has been done within the HJM framework with Valchev [2004] (to the best of our knowledge) being the first to attempt this in the default-free framework. In his work, he extended the class of deterministic volatility HJM models to a stochastic framework via a Markov chain that modulates the exponential function of time-to-maturity volatility. This allows for jump-discontinuities and a broader range of shapes in the term structure of forward rate volatilities. Elhouar [2008] extended the work by Björk and Svensson [2001] to the regime-switching framework and investigated the necessary and sufficient conditions on the volatility function that guarantee finite-dimensional-realisations in Markovian state-space models. Generalisations of the Hull-White and Cox-Ingersoll-Ross models were considered in that work.

1.3 Thesis Structure

The thesis covers *two* broad approaches to the incorporation of stochastic volatility within the Markovian defaultable HJM term structure model. Default time is exogenously specified through a Marked point process and at each default time the value of the defaultable bond changes by a fractional recovery process. The *first* approach is covered in Chapter 2 and Chapter 3 where the stochastic process governing the volatility dynamics is driven by a Wiener process independent of the Wiener processes that drive the forward rate and forward credit spread dynamics. In Chapter 2, the exponentially decaying volatility assumption as a function of maturity is made as compared to the humped volatility specification made in Chapter 3. The *second* approach, covered in Chapter 4, introduces stochasticity to the forward rate volatility using a Markov chain with a finite number of states. Chapter 5 provides a summary of the results and findings, together with potential future research directions.

1.3.1 Markovian Models with Diffusion-Driven Stochastic Volatility

Within the HJM framework, the use of the instantaneous defaultable forward rates as the building blocks of yield curve dynamics coupled with their exogenous specification guarantees that the initial market yield curve can be recovered. It is well known that the no-arbitrage drifts of the forward rates are uniquely specified once the volatilities and the correlations are assigned. Empirical evidence in default-free interest rate markets as given in Collin-Dufresne and Goldstein [2002] and Li and Zhao [2006] supports the existence of an additional source of risk in the volatility of the forward rate that is independent of the risk associated with the term structure. In this case, an additional state variable is required to model the stochastic volatility factor, within a Heston [1993]-type framework.[7]

In this thesis, this notion is extended to the defaultable setting such that both the forward rate and the forward credit spread volatilities depend on an additional state variable that

[7] We note that although option pricing with stochastic volatility has been investigated by Scott [1987], Hull and White [1987] and Stein and Stein [1991] among other authors, Heston [1993] derived a semi-closed form solution involving the evaluation of complex integrals thereby offering convenient computational benefits when calculating the option prices.

models the stochastic volatility factor. This is supported by the observation in D'Souza et al. [2004] who demonstrated that the short rate and intensity processes exhibit stochastic volatility that generates the heavy-tailed behavior observed in the unconditional distribution of their daily movements. We begin Chapter 2 by establishing the correlation structure between the forward rate, forward credit spread and stochastic volatility processes. Evidence of the effects of correlation between stochastic volatility and the short rate on the bond price was investigated in Heston [1993] whereas Jarrow and Turnbull [2000] showed that the correlation between the short rate (forward rate) and the short credit spread (forward credit spread) represents the empirically observed correlation between market risk and default risk. Changes in the default-free short rate compel investors to reassess the probability of default of the defaultable bonds and therefore change the credit spreads.

In addition, adapting the volatility specification in Chiarella and Kwon [2000a] and Björk et al. [2004] to the Schönbucher [1998] framework, we discuss the conditions on the stochastic volatility that would lead to finite dimensional Markovian representations of the defaultable short rate dynamics. We then show that defaultable bond prices across all maturities can be expressed in terms of the default-free short rate, the short term credit spread and a set of Markovian state variables. These state variables are then expressed as a linear combination of fixed tenor forward rates yielding finite dimensional affine realisations in terms of forward rates which allows us to express the defaultable bond price in an exponential affine form in terms of fixed tenor forward rates, which are market observable quantities.

By applying Euler discretisation to the Markovian system, we then discuss the effects of the level of volatility of volatility, the speed of mean reversion and the various correlations on the distribution of defaultable bond prices and returns. The distributional analysis reveals the existence of a long left tail (asymmetry) consistent with the stylistic observation that the upward potential of a bond is limited to the bond's par value whereas the downward risk is unlimited and the investor may lose a large part or the entire investment in the case of bankruptcy.

In Chapter 3, we extend the framework of Chapter 2 by a more generalised volatility specification that allows us to capture a wider range of shapes of the yield curve and in particular, a choice that allows for hump-shaped shocks. It was shown (Collin-Dufresne and Goldstein [2002] and Trolle and Schwartz [2009]) in the default-free setting that the humped volatility improves the model specification, both in terms of likelihood score, analysis of yield errors and caps pricing performance. We then verify that the defaultable forward rates admit finite dimensional affine realisation under the assumption of humped volatility specification and consequently we show that the defaultable bond prices are exponentially affine in the state variables. Expressing the forward rate process as an affine function of the state variables which are jointly Markovian yields faster numerical procedures both for simulation and parameter estimation.

We then demonstrate how the framework can be applied to price credit default swaps and swaptions. We derive some approximating formulas for single-name credit default swap prices and show how this could be extended to include counterparty risk. We then consider a put option that is knocked-out on default of the underlying bond thereby providing price protection. On relaxing the level dependency assumption within the humped volatility specification, we demonstrate how bond options can be priced within the extended framework using the Fourier transform method. The coupled system of differential equations that arise when calculating the cumulative probabilities are solved using numerical integration from which we derive a semi-closed option pricing formula.

1.3.2 Markovian Models with Regime-Switching Stochastic Volatility

In the deterministic volatility HJM models, the volatility curve is fixed and the volatility of a specific forward rate moves along the curve yielding a deterministic motion along a fixed curve. In order to describe the volatility curve effectively, there is need for a process with both deterministic and jump movements. Jump diffusion models are deemed not to be adequate to capture these as they generate jumps too frequently. In Chapter 4, we allow the defaultable forward rate volatility to depend on the current forward rate curve as well as on a modulating continuous time Markov chain. This introduces jump discontinuities as well as other deformations to the term structure of volatilities. We consider an exponentially decaying volatility function in the HJM model driven by a continuous time Markov chain which is independent of the driving Wiener processes. The transition intensity of the Markov chain is assumed to be independent of the jump default intensity.

By extending the results in Valchev [2004] and Elhouar [2008], we discuss the conditions on the defaultable forward rate volatility that would lead to finite dimensional Markovian representations of the defaultable short rate dynamics in the presence of regime-switching. A Markovian two-factor Hull and White [1990]-type model is then derived which allows for better calibration to market data. By construction, the model can be automatically calibrated to the initially observed defaultable and default-free forward curves. On solving the regime-switching bond pricing partial differential equation, we derive a semi-closed form solution for the price of a defaultable bond. This requires solving numerically a coupled system of ordinary differential equations. Using Monte Carlo simulation, we investigate the distributional properties of both the defaultable short rate and bond price dynamics under regime-switching volatility. We observe that increasing the transition intensity and therefore the frequency of regime switching leads to a decrease in the bond prices as investors demand more compensation for the additional source of risk. In addition, increasing the correlation between the market risk and credit risk leads to an increase in the skewness of the bond price distribution in the presence of regime switching.

We finally consider the pricing of a European call option on a defaultable bond with a knock-out provision for the special case of a 2-state regime. By applying finite difference (theta scheme) methods to the coupled option pricing partial differential equations, the option price is approximated on a discrete space-time grid.

Chapter 2

Markovian Defaultable HJM Term Structure Models with Unspanned Stochastic Volatility

This chapter introduces unspanned stochastic volatility into the general defaultable Schönbucher [1998] term structure model. The Wiener processes that determine the uncertainty in the defaultable forward curve are independent to the Wiener processes driving the uncertainty in the stochastic volatility process, a feature that does not naturally arise within the general Heath, Jarrow, and Morton [1992] model (hereafter HJM). Consistent with the stylized facts that interest rate volatility is stochastic and that it is correlated with changes in interest rates, our model is set up in a more generalized framework to incorporate these observations.

As required of a good pricing model in credit risk, the work in this chapter incorporates the correlation between credit spread and short rate processes, as changes in interest rates can directly affect and change the credit spread or indirectly influence the market's perception of default risk, which has an impact on credit spreads. We show that varying these correlations affects the distribution of the defaultable bond prices. Empirical evidence in D'Souza et al.

[2004] demonstrates that short rate and intensity processes exhibit stochastic volatility which generates the heavy-tailed behavior observed in their unconditional distribution of daily movements.

2.1 Introduction

The HJM framework is considered as the most general and flexible setting for the study of interest rate dynamics and the pricing of interest rate derivatives. The only inputs required for the model are the currently observed forward rate curve and the volatility structure of the forward interest rates. The shortcoming of the HJM term structure models is that in the most general setting they are Markovian in the entire yield curve requiring an infinite number of state variables. Since the initial forward rate is completely determined by the market, the only remaining flexibility for obtaining finite dimensional Markovian models within the HJM framework rests in a particular specification of the volatility function. This chapter presents stochastic volatility specifications that will allow the proposed defaultable term structure model to admit finite dimensional Markovian realisations (thereafter FDR), in the spirit of Björk et al. [2004].

Conditions on *level dependent* volatility specifications (that treat the forward rate volatility as deterministic functionals of time to maturity and the short rate and/or fixed tenor forward rates) have been extensively studied within the HJM modelling literature. Some works studying volatility structures for diffusion processes include Ritchken and Sankarasubramanian [1995], Bhar and Chiarella [1997], Björk and Svensson [2001], Björk and Landèn [2002] and Chiarella and Kwon [2001, 2003]. Filipovic and Teichmann [2002, 2003], by employing Lie algebra theory, show that only affine term structure models admit FDR. Extension to volatility structures for jump-diffusion processes have been studied by Björk and Gombani [1999], Chiarella and Nikitopoulos-Sklibosios [2003] and Filipovic et al. [2010].

Several empirical studies support the existence of an additional source of risk in the volatility of the forward rate that is independent of the risk associated with the term structure, see for instance Collin-Dufresne and Goldstein [2002], Li and Zhao [2006] and Trolle and Schwartz [2009]. These empirical findings suggest the suitability of stochastic volatility term structure models, within the Heston [1993]-type stochastic volatility framework,[8] or so-called unspanned stochastic volatility, in which an additional state variable is introduced to model the stochastic volatility factor. Unspanned stochastic volatility specifications within the HJM framework that would lead to a Markovian term structure of interest rates were introduced by Chiarella and Kwon [2000b]. Björk et al. [2004] provide the necessary and sufficient conditions on stochastic volatility for diffusion HJM models to admit FDR.

The modelling of a defaultable term structure using the HJM model was first examined by Jarrow and Turnbull [1995] and Duffie and Singleton [1999]. Schönbucher [1998] proposed a model for the spread of the defaultable interest rates over default-free interest rates that adds a default risk module to an existing model of default free interest rates. Various forms of the no-arbitrage condition between the default free and the defaultable term structures were derived from which the term structure of defaultable bond prices was then obtained. The model developed assumed that a jump in the defaultable forward rate leads to default. In addition, Schönbucher [1998] showed that the forward rate credit spread offers the link between the defaultable and default free term structures. Maksymiuk and Gatarek [1999] obtained the HJM condition for the forward credit spread. They showed that under zero recovery rate and assuming no correlation between defaultable and risk-free rates, the initial spread term structure coincides with the initial term structure of the intensity process.

These results were extended by Pugachevsky [1999] to allow for the case of non-zero correlation. He also derived the relationship between the drift and volatility terms for the spread between forward rates. A defaultable term structure with level dependent volatility

[8] As previously mentioned in Chapter 1, we note that although option pricing with stochastic volatility has been investigated by Scott [1987], Hull and White [1987] and Stein and Stein [1991] among other authors, Heston [1993] derived a semi-closed form solution involving the evaluation of complex integrals thereby offering convenient computational benefits while calculating the option prices.

and stochastic intensity was studied in CNS. They showed that FDR are feasible only for a truncated Markovian system or for constant Poisson volatility functions. A recent paper by Berndt et al. [2010] considers Markovian defaultable term structure models with level dependent volatility and demonstrates the importance of the correlations between interest rates and credit spreads.

The contributions of this chapter are twofold: Firstly, stochastic volatility is introduced within the generalised defaultable term structure model developed by Schönbucher [1998]. The proposed framework models the default time exogenously through a Cox (doubly stochastic Poisson) process and at each default time the value of the defaultable bond is altered by a fractional recovery. The dynamics of the defaultable interest rates are then derived by assuming a diffusion model for the default-free interest rates and the credit spread. The volatilities of both the default-free term structure and the forward credit spread are stochastic as they depend on an non-observable volatility process whose Wiener processes are independent of the Wiener processes driving the default-free term structure and credit spread. By modelling the connection between the default-free and the defaultable term structure, namely the credit spreads, we obtain an arbitrage free model[9] and we are able to accommodate a correlation structure between the credit spread and interest rate as well as the stochastic volatility.

Secondly, we present the necessary conditions on the volatility structure that allows the proposed defaultable term structure model to admit FDR. Precisely in spirit the of Björk et al. [2004], we assume that the default-free forward rate volatility depends on the unobservable volatility process, the current default-free term structure and a quasi exponential time factor. Similarly, the volatility of the credit spread depends on the same hidden Markov volatility process, the current default intensity process and a quasi exponential time factor. Analytical exponential affine bond prices are obtained and the dynamics of the term structure model

[9]When the defaultable term structure and the default-free term structure are modelled independently then a no-arbitrage model can be obtained where negative spreads are possible, see Schönbucher [1998], Section 2.4.

under the risk neutral measure can be described in terms of a finite number of state variables. Then the state space is expressed in terms of fixed tenor forward rates, assigning some economic meaning to the state space. In addition, we conduct a Monte-Carlo simulation experiment to investigate the flexibility of the pricing model and its responsiveness to changes in the underlying correlation structure and volatility.

The structure of the chapter is as follows: In Section 2.2, we introduce stochastic volatility into the defaultable term structure model developed by Schönbucher [1998] and we generalise it to allow for a correlation structure between the default-free forward rate, the forward credit spread and stochastic volatility. In Section 2.3, we assume specific volatility structures and derive a Markovian representation of the defaultable short rate in terms of a finite number of state variables. Furthermore, we express the state variables as a finite dimensional realisation in terms of economic quantities observed in the market, specifically in terms of discrete tenor forward rates. Section 2.4 presents some simulation results on the distributional properties of the defaultable bond price and bond returns. Section 2.5 concludes the chapter. Some technical results are gathered in the Appendix I.

2.2 The Model Setup

We consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ where $\mathbb{P}$ is the real world probability measure and the filtration $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^N$, $t \geq 0$ satisfies the usual conditions. The subfiltration $\mathcal{F}_t^W$ is the $\sigma-$algebra generated by the 3$-$dimensional of standard $\mathbb{P}-$Wiener processes $W(t) = \{W^f(t), W^\lambda(t), W^V(t)\}$,

$$(\mathcal{F}_t^W)_{t\geq 0} = \{\sigma(W(s) : 0 \leq s \leq t)\}_{t\geq 0}, \tag{2.1}$$

and represents the flow of all background information except from default itself which generates the sub-filtration $\mathcal{F}_t^N$.

We denote as $P(t,T,\omega)$ the price at time t of the default-free zero coupon bond with maturity $T \geq t$. We assume a more general modelling setup, where the entire forward rate curve depends on $\omega \in \Omega$ which represents the dependence of the forward rate process on the Wiener paths. This quite general structure will allow us later on to easily introduce the uncertainty associated with stochastic volatility.

Definition 2.1 *1. The instantaneous default-free forward rate of interest prevailing at time t for instantaneous borrowing at T, is defined as* [10]

$$f(t,T,\omega) = -\frac{\partial}{\partial T} \ln P(t,T,\omega), \quad \textit{for all} \quad t \in [0,T]. \tag{2.2}$$

2. The instantaneous default-free short rate is defined as the instantaneously maturing forward rate so that

$$r(t,\omega) = f(t,t,\omega). \tag{2.3}$$

We introduce next the defaultable term structure. We denote as $P^d(t,T,\omega)$ the price at time t of the defaultable zero coupon bond with maturity $T > t$.

Definition 2.2 *1. The instantaneous defaultable forward rate at time t for instantaneous borrowing at T is defined as*

$$f^d(t,T,\omega) = -\frac{\partial}{\partial T} \ln P^d(t,T,\omega), \quad \textit{for all} \quad t \in [0,T]. \tag{2.4}$$

2. The instantaneous defaultable short rate is defined as

$$r^d(t,\omega) = f^d(t,t,\omega). \tag{2.5}$$

[10] Equivalently, on integrating (2.2) with respect to maturity we obtain the following alternative characterisation that defines the bond price in terms of the forward rate; $P^d(t,T,\omega) = \exp\Big(- \int_t^T f^d(t,s,\omega)ds \Big)$.

3. *In addition, the continuously compounded instantaneous forward credit spread is defined as*

$$\lambda(t,T,\omega) = f^d(t,T,\omega) - f(t,T,\omega), \tag{2.6}$$

and the instantaneous short-term credit spread is defined as

$$c(t,\omega) = \lambda(t,t,\omega). \tag{2.7}$$

The default process is modelled via a marked point process, see Jeanblanc et al. [2009]. We let (E,ξ) be a measurable (mark) space. A random measure μ on the space $\mathbb{R}_+ \times E$ is a family of positive measures $\big(\mu(\omega; dt, dq); \omega \in \Omega\big)$ defined on $\mathbb{R}_+ \times E$ such that, for $[0,t] \times A \in \mathcal{B} \otimes \xi$, the map $\omega \longrightarrow \mu(\omega; [0,t], A)$ is $\mathcal{F}^N$−measurable, and $\mu(\omega; \{0\}, E) = 0$. For Borel sets $\mathcal{B}$, note that $E = [0,1]$.

Definition 2.3 1. *A marked point process N is a random sequence (with stochastic jumps) defined by the pair $\{(\tau_i, q_i), i \in \mathbb{N}\}$ with $\tau_i \in \mathbb{R}_+$ and marks $q_i := q(\tau_i) \in E$.*

2. *A random measure μ is associated to the marked point process N by*

$$\mu(\cdot; [0,t], A) = N_A(t),$$

such that

$$\mu(\omega; [0,t], E) = \int_0^t \int_E \mu(\omega; ds, dq) := \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i(\omega) \le t\}} \mathbb{1}_{\{q_i(\omega) \in E\}}.$$

The measure $\mu(\omega; X \times A)$ denotes the number of arrivals during the time set $X \subset \mathbb{R}_+$ that have marks with values in the mark set $A \subset \xi$. If we assume that $\lim_i \tau_i = \infty$, it follows that $\mu(\omega; X \times A) < \infty$ for any bounded interval $X = [0,t] \subset \mathcal{B}(\mathbb{R}_+)$ and that

$\mu(\omega; \mathcal{B}(\mathbb{R}_+) \times A) = \infty$. Therefore, only a finite number of arrivals (defaults) occurs in any bounded time interval although there is an infinite number of defaults overall.

The random measure μ is characterized by the (predictable) compensator random measure ν on the space $\mathbb{R}_+ \times E$, so that for every predictable function $H(\omega; t, q)$, the process defined by

$$M(\omega; t) = \int_0^t \int_E H(\omega; s, q)\mu(\omega; ds, dq) - \int_0^t \int_E H(\omega; s, q)\nu(\omega; ds, dq), \qquad (2.8)$$

is a local martingale. The compensator ν has the form

$$\nu(\omega; dt, dq) = h(\omega; t, dq)dt, \qquad (2.9)$$

where $h(\cdot; t, A)$ is a predictable process.

Remark 2.4 *The most common form of intensity is*

$$h(\omega; t, dq) = h(t)m_t(dq), \qquad (2.10)$$

where $h(t)$ is non-negative $\mathcal{F}_t-$predictable and represents the intensity of a Poisson process while $m_t(dq)$ is the conditional distribution of the marker q on the space (E, ξ). The pair $(h(t), m_t(dq))$ is called the $(\mathbb{P}, \mathcal{F}_t)-$local characteristics of the counting measure $\mu(\omega; dt, dq)$. See Runggaldier [2003].

In practice, a default event does not terminate the debt contract as firms are usually reorganized and the debt is re-floated. The proposed framework will allow for subsequent defaults and hence multiple defaults are possible with the debt restructuring at each default event. The recovery rate $\mathcal{R}(t)$ given default is defined as the extent to which the value of an obligation can be recovered once the obligor has defaulted. This is a measure of the expected fractional recovery in case of default and therefore $\mathcal{R}(t) \in [0, 1]$. In this case, there is an increasing sequence of default times $\{\tau_i\}_{i\in\mathbb{N}}$ and at each τ_i, the defaultable bond's face value is reduced by the loss rate $q(\tau_i) \in [0, 1]$, which can be a random variable.

At maturity T, the defaultable bond subject to multiple defaults has a final payoff

$$\mathcal{R}(T) := \prod_{\tau_i \le T} (1 - q(\tau_i)), \tag{2.11}$$

where $\mathcal{R}(T)$ is the product of the face reductions after all defaults until maturity T. The random loss $q(\tau_i)$ is considered as a random draw at default time τ_i. The fractional recovery process $\mathcal{R}(t)$ can be represented as a Doléans-Dade exponential of the stochastic differential equation, [11]

$$d\mathcal{R}(t) = -\mathcal{R}(t-) \int_E q\mu(dt, dq), \tag{2.12}$$

where $\mu(dt, dq)$ is the random measure associated to the marked point process.

The pre-default price $\bar{P}^d(t, T, \omega)$ at time t of a defaultable zero coupon bond with maturity T, the so-called 'pseudo' bond, is given by

$$\bar{P}^d(t, T, \omega) = \exp\Big(- \int_t^T f^d(t, s, \omega) ds \Big). \tag{2.13}$$

This is the price of the defaultable zero-coupon bond given that it has not defaulted before time t. It then follows that the price of the defaultable bond can be written as

$$P^d(t, T, \omega) = \mathcal{R}(t) \exp\Big(- \int_t^T f^d(t, s, \omega) ds \Big) :\equiv \mathcal{R}(t) \bar{P}^d(t, T, \omega). \tag{2.14}$$

[11] See Appendix A.1 for the proof of this result based on Jacod and Shiryaev [2003, Theorem 4.61, pg.59] and Klebaner [2005, Theorem 8.33 and Section 9.3].

2.2.1 Embedding Stochastic Volatility within the Defaultable HJM framework

Although the volatility processes in the standard HJM framework can be path dependent, they are not considered to be stochastic in the sense of Hull and White [1987], Heston [1993] and Scott [1997]. In our stochastic volatility model for interest rates, the volatility processes is driven by Wiener processes which are independent of the Wiener processes driving the term structure of interest rates. In this way, the stochastic volatility is unspanned.

Chiarella and Kwon [2000b] introduced unspanned stochastic volatility within a class of HJM term structure models in the default free setup and derived bond and bond option prices. Björk et al. [2004] and Filipovic and Teichmann [2002] significantly advanced the study of stochastic volatility for HJM models driven by diffusion processes. We adapt these results to the defaultable Schönbucher [1998] term structure model, where a modenl for the spread between the denfaultable forward rates and default-free forward rates is proposed.

Assumption 2.2.1 *The dynamics of the stochastic volatility process $V = \{V(t), t \in [0, T]\}$ are*

$$dV(t) = \alpha^V(t, V)dt + \sigma^V(t, V)dW^V(t), \tag{2.15}$$

where the drift and diffusion depend only on V.

We further assume that for any function $g(t, T, \omega)$ there exists a function z such that $g(t, T, \omega) = z(t, T, V(t))$. However, for notational convenience we adopt the notation $g(t, T, V) := g(t, T, V(t))$ instead of $z(t, T, V(t))$ from now on.

Assumption 2.2.2 *The instantaneous denfault-free forward rate $f(t, T, V)$ and the instan-*

taneous forward credit spread $\lambda(t,T,V)$ satisfy the stochastic differential equations

$$df(t,T,V) = \alpha^f(t,T,V)dt + \sigma^f(t,T,V)dW^f(t), \tag{2.16}$$

$$d\lambda(t,T,V) = \alpha^\lambda(t,T,V)dt + \sigma^\lambda(t,T,V)dW^\lambda(t), \tag{2.17}$$

respectively, where $W^f(t)$ and $W^\lambda(t)$ are two correlated Wiener processes.

Note that we have consequently assumed that the filtration $\mathcal{F}_t^W$, see (2.1), includes $\mathcal{F}_t^W = \mathcal{F}_t^f \vee \mathcal{F}_t^\lambda \vee \mathcal{F}_t^V$, where

$$\begin{cases} (\mathcal{F}_t^f)_{t\geq 0} = \{\sigma(W_s^f : 0 \leq s \leq t)\}_{t\geq 0}, \\ (\mathcal{F}_t^\lambda)_{t\geq 0} = \{\sigma(W_s^\lambda : 0 \leq s \leq t)\}_{t\geq 0}, \\ (\mathcal{F}_t^V)_{t\geq 0} = \{\sigma(W_s^V : 0 \leq s \leq t)\}_{t\geq 0}. \end{cases} \tag{2.18}$$

The details on the correlation structure are given in Section 2.2.2. By using the equivalent stochastic integral equations imposed by Assumption 2.2.2, the stochastic integral equations for the instantaneous default-free short rate $r(t,V) := f(t,t,V)$ and the instantaneous short-term credit spread $c(t,V) := \lambda(t,t,V)$ are given by

$$r(t,V) = f(0,t,V_0) + \int_0^t \alpha^f(u,t,V)du + \int_0^t \sigma^f(u,t,V)dW^f(u), \tag{2.19}$$

$$c(t,V) = \lambda(0,t,V_0) + \int_0^t \alpha^\lambda(u,t,V)du + \int_0^t \sigma^\lambda(u,t,V)dW^\lambda(u), \tag{2.20}$$

respectively where V_0 is the initial volatility.

By using (2.6) and the dynamics specified in Assumption 2.2.2, the stochastic integral equation for the defaultable forward rate is expressed as

$$f^d(t,T,V) = f^d(0,T,V_0) + \int_0^t \alpha^d(u,T,V)du + \int_0^t \sigma^f(u,T,V)dW^f(u) + \int_0^t \sigma^\lambda(u,T,V)dW^\lambda(u), \tag{2.21}$$

where the initial defaultable forward curve is

$$f^d(0,T,V_0) = f(0,T,V_0) + \lambda(0,T,V_0), \tag{2.22}$$

and the drift coefficient is given by the sum of the individual drift coefficients

$$\alpha^d(t,T,V) = \alpha^f(t,T,V) + \alpha^\lambda(t,T,V). \tag{2.23}$$

In addition, (2.21), for $T = t$, provides the dynamics for the instantaneous defaultable short rate $r^d(t,V) := f^d(t,t,V) = f(t,t,V) + \lambda(t,t,V) = r(t,V) + c(t,V)$ as

$$r^d(t,V) = f^d(0,t,V_0) + \int_0^t \alpha^d(u,t,V)du + \int_0^t \sigma^f(u,t,V)dW^f(u) + \int_0^t \sigma^\lambda(u,t,V)dW^\lambda(u). \tag{2.24}$$

2.2.2 Correlation Structure

Evidence of the effects of correlation between stochastic volatility and the short rate on the bond price were investigated in Heston [1993]. Jarrow and Turnbull [2000] showed that the correlation between the short rate and the credit spread represents the empirically observed correlation between market risk and credit risk. Changes in the default free short rate force investors to reassess the probability of default of defaultable bonds and therefore impact the credit spreads.

We define the correlation matrix between the Wiener processes $W^f(t)$, $W^\lambda(t)$ and $W^V(t)$ by

$$\mathbb{E}\big[(dW^V, dW^\lambda, dW^f)^\top (dW^V, dW^\lambda, dW^f)\big] = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix}, \tag{2.25}$$

where the correlation coefficients ρ_{ij} are given by

$$\rho_{12}dt = \mathbb{E}[dW^V(t) \cdot dW^\lambda(t)], \tag{2.26a}$$

$$\rho_{13}dt = \mathbb{E}[dW^V(t) \cdot dW^f(t)], \tag{2.26b}$$

$$\rho_{23}dt = \mathbb{E}[dW^\lambda(t) \cdot dW^f(t)]. \tag{2.26c}$$

To apply the techniques of the HJM approach, it is convenient to replace the correlated Wiener processes $W^f(t)$, $W^\lambda(t)$ and $W^V(t)$ with uncorrelated processes. We define the uncorrelated Wiener process $W(t) = \big(W_1(t), W_2(t), W_3(t)\big)$ under $\mathbb{P}$ such that

$$\begin{bmatrix} dW^V(t) \\ dW^\lambda(t) \\ dW^f(t) \end{bmatrix} = \begin{bmatrix} \varrho_{11} & \varrho_{12} & \varrho_{13} \\ \varrho_{21} & \varrho_{22} & \varrho_{23} \\ \varrho_{31} & \varrho_{32} & \varrho_{33} \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}. \tag{2.27}$$

Note that the ϱ_{ij}'s are chosen such that the correlation structure of the Wiener processes $W^f(t)$, $W^\lambda(t)$ and $W^V(t)$ is preserved with

$$\sum_{k=1}^{3} \varrho_{ik}\varrho_{jk} = \rho_{ij}, \quad \text{for} \quad i \neq j, \quad j = 1,2,3, \quad \text{and} \quad \sum_{j=1}^{3} \varrho_{ij}^2 = 1, \quad \text{for} \quad i = 1,2,3. \tag{2.28}$$

Then, equations (2.16), (2.17) and (2.15) can be expressed in terms of independent Wiener processes as

$$df(t,T,V) = \alpha^f(t,T,V)dt + \sum_{i=1}^{3} \tilde{\sigma}_i^f(t,T,V)dW_i(t), \tag{2.29a}$$

$$d\lambda(t,T,V) = \alpha^\lambda(t,T,V)dt + \sum_{i=1}^{3} \tilde{\sigma}_i^\lambda(t,T,V)dW_i(t), \tag{2.29b}$$

$$dV(t) = \alpha^V(t,V)dt + \sum_{i=1}^{3} \tilde{\sigma}_i^V(t,V)dW_i(t), \tag{2.29c}$$

where by using transformation (2.27), the volatility functions are defined as

$$\tilde{\sigma}_i^f(t,T,V) = \varrho_{3i}\sigma^f(t,T,V), \quad \tilde{\sigma}_i^\lambda(t,T,V) = \varrho_{2i}\sigma^\lambda(t,T,V), \quad \tilde{\sigma}_i^V(t,V) = \varrho_{1i}\sigma^V(t,V), \tag{2.30}$$

for $i = 1, 2, 3$. Then, (2.21) is expressed as

$$f^d(t,T,V) = f^d(0,T,V_0) + \int_0^t \alpha^d(u,T,V)du + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^d(u,T,V)dW_i(u), \tag{2.31}$$

where, for $i = 1, 2, 3$, the volatility functions are defined as

$$\tilde{\sigma}_i^d(t,T,V) = \tilde{\sigma}_i^f(t,T,V) + \tilde{\sigma}_i^\lambda(t,T,V). \tag{2.32}$$

Setting $T = t$ in (2.31) provides the equation for the defaultable short rate as

$$r^d(t,V) = f^d(0,t,V_0) + \int_0^t \alpha^d(u,t,V)du + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^d(u,t,V)dW_i(u). \tag{2.33}$$

We then have the following result for the price dynamics of the default-free bond and the defaultable bond. We show how multiple defaults and recoveries can be incorporated within the HJM framework when there is no jump in the forward rate dynamics.

Proposition 2.5 *Given the dynamics (2.29a) for the default-free forward rate $f(t,T,V)$, the default-free bond price satisfies the stochastic differential equation*

$$\frac{dP(t,T,V)}{P(t-,T,V)} = \big[r(t,V) + b(t,T,V)\big]dt - \sum_{i=1}^{3}\Big(\int_t^T \tilde{\sigma}_i^f(t,s,V)ds\Big)dW_i(t), \tag{2.34}$$

where

$$b(t,T,V) = -\int_t^T \alpha^f(t,s,V)ds + \frac{1}{2}\sum_{i=1}^{3}\Big(\int_t^T \tilde{\sigma}_i^f(t,s,V)ds\Big)^2. \tag{2.35}$$

Given the dynamics (2.31) for the defaultable forward rate $f^d(t,T,V)$ and the relation (2.14) between defaultable and pseudo bond prices, the defaultable bond price satisfies the stochastic differential equation

$$\frac{dP^d(t,T,V)}{P^d(t-,T,V)} = \left[r^d(t,V) + b^d(t,T,V)\right]dt - \sum_{i=1}^{3}\Big(\int_t^T \tilde{\sigma}_i^d(t,s,V)ds\Big)dW_i(t) - \int_E q\mu(dt,dq), \quad (2.36)$$

where

$$b^d(t,T,V) = -\int_t^T \alpha^d(t,s,V)ds + \frac{1}{2}\sum_{i=1}^{3}\Big(\int_t^T \tilde{\sigma}_i^d(t,s,V)ds\Big)^2. \quad (2.37)$$

Proof: The proof of this result is provided in Appendix A.2. ■

2.2.3 Risk-Neutral Dynamics

The absence of arbitrage opportunities implies that there exists an equivalent probability measure $\tilde{\mathbb{P}}$, namely the risk-neutral measure.[12] For every finite maturity T, there exists a 3-dimensional predictable process $\Phi(t) = \{\phi_1(t), \phi_2(t), \phi_3(t), t \in [0,T]\}$ and a strictly positive measurable function $\psi(t,q)$ satisfying the integrability conditions

$$\int_0^t ||\phi_i(s)||^2 ds < \infty, \quad \text{for} \quad i = 1,2,3, \qquad \int_0^t \int_E |\psi(s,q)h(s,dq)|ds < \infty, \quad (2.38)$$

such that

$$d\tilde{W}_i(t) = dW_i(t) - \phi_i(t)dt, \quad \text{for} \quad i = 1,2,3, \quad (2.39)$$

is a $\tilde{\mathbb{P}}$-Wiener process and the default indicator process $N(t)$ has a $\tilde{\mathbb{P}}$-intensity

$$\tilde{h}(t,dq) = \psi(t,q)h(t,dq). \quad (2.40)$$

[12]This measure is not unique due to market incompleteness which arises from the independent Wiener process that drives the stochastic volatility process.

Proposition 2.6 *Using Girsanov's theorem such that the integrability conditions (2.38) and (2.39) are satisfied, then a risk-neutral measure $\tilde{\mathbb{P}}$ exists, if and only if,*

$$\alpha^f(t,T,V) = -\sum_{i=1}^{3}\tilde{\sigma}_i^f(t,T,V)\Big(\phi_i(t) - \int_t^T \tilde{\sigma}_i^f(t,s,V)ds\Big), \tag{2.41}$$

where $\phi_i(t)$ denotes the market price of interest rate risk associated with the noise process $W_i(t)$. Then the risk-neutral dynamics of the default-free forward rate are

$$df(t,T,V) = \sum_{i=1}^{3}\tilde{\sigma}_i^f(t,T,V)\Big(\int_t^T \tilde{\sigma}_i^f(t,s,V)ds\Big)dt + \sum_{i=1}^{3}\tilde{\sigma}_i^f(t,T,V)d\tilde{W}_i(t), \tag{2.42}$$

and the risk-neutral dynamics of the default-free bond price are

$$\frac{dP(t,T,V)}{P(t,T,V)} = r(t,V)dt - \sum_{i=1}^{3}\Big(\int_t^T \tilde{\sigma}_i^f(t,s,V)ds\Big)d\tilde{W}_i(t), \tag{2.43}$$

Proof: Follows along the lines of Heath et al. [1992] as given in Appendix A.3. ■

Proposition 2.7 *Using Girsanov's theorem such that the integrability condition (2.38) and equations (2.39),(2.40) are satisfied, then a risk-neutral measure $\tilde{\mathbb{P}}$ exists, if and only if,*

$$[r^d(t,V) + b^d(t,T,V)] - \sum_{i=1}^{3}\phi_i(t)\int_t^T \tilde{\sigma}_i^d(t,s,V)ds - \int_E q\,\psi(t,q)h(t,dq) = r(t,V), \tag{2.44}$$

where $\phi_i(t)$ is the market price of interest rate risk associated with the noise process $W_i(t)$ and $\psi(t,q)$ is the market price of default risk.

Proof: Follows similar arguments to those of Björk et al. [1997]. See Appendix A.4. ■

Taking the derivative of (2.44) with respect to T and performing some standard manipulations then yields

$$\alpha^d(t,T,V) = -\sum_{i=1}^{3} \tilde{\sigma}_i^d(t,T,V)\Big(\phi_i(t) - \int_t^T \tilde{\sigma}_i^d(t,s,V)ds\Big), \tag{2.45}$$

which is the corresponding HJM forward rate drift restriction condition for the defaultable bond price. As noted in Schönbucher [2003], the precise knowledge of the nature of the default process N and its compensator M is not necessary in setting up an arbitrage free model for the term structure of defaultable bonds.

Corollary 2.8 *The credit spread drift restriction implied by the proposed model is*

$$\begin{aligned}\alpha^\lambda(t,T,V) = &-\sum_{i=1}^{3} \phi_i(t)\tilde{\sigma}_i^\lambda(t,T,V) + \sum_{i=1}^{3} \tilde{\sigma}_i^\lambda(t,T,V)\int_t^T \tilde{\sigma}_i^\lambda(t,s,V)ds \\ &+ \sum_{i=1}^{3}\Big(\tilde{\sigma}_i^\lambda(t,T,V)\int_t^T \tilde{\sigma}_i^f(t,s,V)ds + \tilde{\sigma}_i^f(t,T,V)\int_t^T \tilde{\sigma}_i^\lambda(t,s,V)ds\Big).\end{aligned} \tag{2.46}$$

Proof: Substitute (2.23) and (2.32) into (2.45) and use condition (2.41). See also Appendix A.5. ■

The drift of the credit spread (Equation (2.46)) is expressed in terms of the volatilities of the default free forward rate and the credit spread. This condition guarantees that the spread cannot become negative because by construction (See Bielecki and Rutkowski [2002, Chapter 13, pages 387 - 390])

$$\bar{P}^d(t,T,V) = P(t,T,V)\exp\Big(-\int_t^T \lambda(t,s,V)ds\Big),$$

from which $\bar{P}^d(t,T,V) < P(t,T,V)$.

Substituting $b^d(t,T,V)$ as given in equation (2.37) into (2.44), as well as using (2.45), it follows that the short term spread is the product of the market price of jump risk, the default intensity and the expected loss quota, that is

$$r^d(t,V) - r(t,V) = \int_E q\,\psi(t,q)h(t,dq). \tag{2.47}$$

Taking into account the fact that the intensity of the default process under the risk-neutral measure is given by (2.40) then

$$r^d(t,V) - r(t,V) = \int_E q\,\tilde{h}(t,dq). \tag{2.48}$$

From (2.7) and (2.48), the short term credit spread, $c(t,V)$ under fractional recovery can then be expressed as

$$c(t,V) \equiv \int_E q\,\tilde{h}(t,dq). \tag{2.49}$$

Formulating the intensity rate as a stochastic process allows rich dynamics for the credit spread process and is flexible enough to capture the empirically observed stochastic credit spreads. As cited in Jarrow and Turnbull [2000], there is considerable empirical evidence that clearly suggests that the credit spread is a function of at least default intensity and the recovery process. Pan and Singleton [2008] further noted that since the default intensity and the recovery process enter symmetrically into pricing under fractional recovery of market value (RMV), they cannot be separately identified using defaultable bond price data alone.

In order to alleviate notational complexity, we define some path dependent quantities $S_j(t,V)$ and $\psi_j(t,V)$ to that regard.

Definition 2.9 *We define the subsidiary state variables*

$$S_1(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,V) \int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu, \tag{2.50a}$$

$$S_2(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,V) \int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvdu, \tag{2.50b}$$

$$S_3(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,V) \int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvdu, \tag{2.50c}$$

$$S_4(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,V) \int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu, \tag{2.50d}$$

$$\psi_1(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,V)d\tilde{W}_i(u), \tag{2.50e}$$

$$\psi_2(t,V) = \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,V)d\tilde{W}_i(u). \tag{2.50f}$$

Lemma 2.10 *Consider the path-dependent state space variables $S_i(t,V)$ and $\psi_i(t,V)$ presented in Definition 2.9. Under $\tilde{\mathbb{P}}$, the defaultable short rate satisfies the stochastic integral equation*

$$r^d(t,V) = f^d(0,t,V_0) + \sum_{i=1}^{4} S_i(t,V) + \sum_{i=1}^{2} \psi_i(t,V), \tag{2.51}$$

the default-free short rate satisfies the stochastic integral equation

$$r(t,V) = f(0,t,V_0) + S_1(t,V) + \psi_1(t,V), \tag{2.52}$$

while the short term spread satisfies the stochastic integral equation

$$c(t,V) = \lambda(0,t,V_0) + \sum_{i=2}^{4} S_i(t,V) + \psi_2(t,V). \tag{2.53}$$

In addition, the state variable $V = \{V(t), t \in 0,T\}$ satisfies the stochastic differential equa-

tion

$$dV(t) = \big[\alpha^V(t,V) + \sum_{i=1}^{3} \phi_i(t)\tilde{\sigma}_i^V(t,V)\big]dt + \sum_{i=1}^{3} \tilde{\sigma}_i^V(t,V)d\tilde{W}_i(t). \tag{2.54}$$

Proof: See Appendix A.6. ■

Corollary 2.11 *The defaultable bond price can be expressed as*

$$P^d(t,T,V) = \tilde{\mathbb{E}}\Big[\exp\Big(-\int_t^T r(s,V)ds\Big)\mathcal{R}(T))\Big|\mathcal{F}_t\Big]. \tag{2.55}$$

Proof: See Appendix A.7. ■

Given the dynamics of the short rate process $r(t,V)$ and the recovery process $\mathcal{R}(t)$, we can use Monte Carlo simulation to calculate the price of the defaultable bond.

The market price of risk $\phi_i(t)$ of the risk factor $W_i(t)$ appears in the drift of the volatility process. Thus our model shares the common feature of the class of Heston [1993] stochastic volatility models, which is that these models do not imply a complete market as they cannot be fully hedged by a portfolio of bonds. Jarrow and Turnbull [2000] suggested that the default intensity process could be assumed to depend on different state variables to reflect a dependency on several macro-economic factors. This requirement could be well captured by a multi-dimensional model of the type in equation (2.53) for the instantaneous spread which is related to the intensity through (2.44).

2.3 The Markovian Term Structure Models with Stochastic Volatility

A key drawback of the HJM approach is the non-Markovian nature of the stochastic differential equation for the short interest rate in its most general form due to the non-Markovian Wiener term that appears in the drift, thereby increasing the computational complexity. More specifically in our modelling framework, the state space variables defined in Definition 2.9 depend on the entire path history of the forward rate and credit spread volatility processes, leading to an infinite dimensional system of stochastic differential equations.

Stochastic volatility specifications within the HJM framework that allow finite dimentional Markovian representations (FDR) have been studied by Chiarella and Kwon [2000b], Björk et al. [2004] and Filipovic and Teichmann [2002]. By employing Lie algebra theory, Björk et al. [2004] examined the necessary and sufficient conditions on stochastic volatility for diffusion default-free HJM models to admit FDR. They demonstrated that a sufficient condition for the existence of FDR is that the volatility function should be the product of a quasi exponential function of the time to maturity and an arbitrary function of the forward rate and the volatility process.[13] We adopt these volatility specifications and consider an application to the defaultable term structure model proposed by Schönbucher [1998].

Thus we consider a class of functional forms for the volatility functions $\sigma^f(t,T,V)$ and $\sigma^\lambda(t,T,V)$, as proposed by Björk et al. [2004], that will allow the non-Markovian representation of $r^d(t,V)$, given by (2.51), to be reduced to a finite dimensional Markovian system of stochastic differential equations.

Assumption 2.3.1 *The volatility functions of the default-free forward interest rate, the*

[13]See Proposition 5.2 of Björk et al. [2004].

forward credit spread and the volatility are of the form

$$\sigma^f(t,T,V) = \sigma_f \sqrt{V(t)r(t,V)}e^{-\kappa_f(T-t)}, \tag{2.56a}$$

$$\sigma^\lambda(t,T,V) = \sigma_\lambda \sqrt{V(t)c(t,V)}e^{-\kappa_\lambda(T-t)}, \tag{2.56b}$$

$$\sigma^V(t,V) = \sigma_V \sqrt{V(t)}, \tag{2.56c}$$

respectively, where $\sigma_f \geq 0$, $\sigma_\lambda \geq 0$, $\sigma_V \geq 0$, κ_f *and* κ_λ *are given constants.*

The volatility specifications (2.56a) and (2.56b) can be considered as an extension of the Ritchken and Sankarasubramanian [1995] volatility structures to stochastic volatility with an application to the defaultable term structure.

Definition 2.12 *Under the volatility specifications of Assumption 2.3.1, we define the additional subsidiary state variables*

$$\eta_1(t,V) = \sum_{i=1}^{3} \varrho_{3i}^2 \int_0^t \sigma_f^2 r(u,V)V(u)e^{-2\kappa_f(t-u)}du, \tag{2.57a}$$

$$\eta_2(t,V) = \sum_{i=1}^{3} \varrho_{2i}^2 \int_0^t \sigma_\lambda^2 c(u,V)V(u)e^{-2\kappa_\lambda(t-u)}du, \tag{2.57b}$$

$$\eta_3(t,V) = \sum_{i=1}^{3} \varrho_{2i}\varrho_{3i} \int_0^t \sigma_f\sigma_\lambda \sqrt{r(u,V)c(u,V)}V(u)e^{-(\kappa_f+\kappa_\lambda)(t-u)}du. \tag{2.57c}$$

Proposition 2.13 *For* $i = 1,2,3$, *the subsidiary state variables* $\eta_i(t,V)$ *of Definition 2.12 satisfy the stochastic differential equations*

$$d\eta_1(t,V) = \Big(\sum_{i=1}^{3} \varrho_{3i}^2\sigma_f^2 r(t,V)V(t) - 2\kappa_f\eta_1(t,V)\Big)dt, \tag{2.58a}$$

$$d\eta_2(t,V) = \Big(\sum_{i=1}^{3} \varrho_{2i}^2\sigma_\lambda^2 c(t,V)V(t) - 2\kappa_\lambda\eta_2(t,V)\Big)dt, \tag{2.58b}$$

$$d\eta_3(t,V) = \Big(\sum_{i=1}^{3} \varrho_{2i}\varrho_{3i}\sigma_f\sigma_\lambda \sqrt{r(t,V)c(t,V)}V(t) - (\kappa_f+\kappa_\lambda)\eta_3(t,V)\Big)dt, \tag{2.58c}$$

and the state variable $S_3(t,V)$*, see (2.50c), satisfies the stochastic differential equation*

$$dS_3(t,V) = \Big[\eta_3(t,V) - \kappa_f S_3(t,V)\Big]dt. \tag{2.59}$$

Proof: Follows from taking the differentials of the state variables defined in Definition 2.12.■

The following proposition shows that the defaultable short rate, the default-free short rate, the short term spread and the defaultable bond price are completely determined by seven state space variables, namely $r(t,V)$, $c(t,V)$, $\eta_i(t,V)$, $(i=1,2,3)$, $S_3(t,V)$ and $V(t)$. Note that an alternative representation that employs ten state space variables is also feasible, see Appendix A.8, though we use the formulation in the set of state space that requires fewer variables and includes model factors such as $r(t,V)$ and $c(t,V)$.

Proposition 2.14 *Under the volatility specification of Assumption 2.3.1 and given the state variable dynamics in Proposition 2.13, the defaultable short rate, default-free short rate, short term credit spread and stochastic volatility processes satisfy the stochastic differential equations*

$$\begin{aligned} dr^d(t,V) &= \big[\theta_d(t,V_0) + \eta_1(t,V) + \eta_2(t,V) + 2\eta_3(t,V) - (\kappa_f - \kappa_\lambda)S_3(t,V) + (\kappa_f - \kappa_\lambda)c(t,V) \\ &\quad - \kappa_f r(t,V)\big]dt + \Big(\sum_{i=1}^{3} \varrho_{3i}\sigma_f\sqrt{r(t,V)V(t)} + \sum_{i=1}^{3}\varrho_{2i}\sigma_\lambda\sqrt{\lambda(t,V)V(t)}\Big)d\tilde{W}_i(t), \end{aligned} \tag{2.60a}$$

$$dr(t,V) = \big[\theta_f(t,V) + \eta_1(t,V) - \kappa_f r(t,V)\big]dt + \sum_{i=1}^{3}\varrho_{3i}\sigma_f\sqrt{r(t,V)V(t)}d\tilde{W}_i(t), \tag{2.60b}$$

$$\begin{aligned} dc(t,V) &= \big[\theta_\lambda(t,V) + \eta_2(t,V) + 2\eta_3(t,V) - (\kappa_f - \kappa_\lambda)S_3(t,V) - \kappa_\lambda c(t,V)\big]dt \\ &\quad + \sum_{i=1}^{3}\varrho_{2i}\sigma_\lambda\sqrt{c(t,V)V(t)}d\tilde{W}_i(t), \end{aligned} \tag{2.60c}$$

$$dV(t) = \big[\alpha^V(t,V) + \sum_{i=1}^{3}\phi_i(t)\sigma_V\sqrt{V(t)}\big]dt + \sum_{i=1}^{3}\tilde{\sigma}_i^V(t,V)d\tilde{W}_i(t). \tag{2.60d}$$

respectively, where the functions in the deterministic drifts are given by

$$\begin{cases} \theta_d(t,V_0) = f_2^d(0,t,V_0) + \kappa_f f(0,t,V_0) + \kappa_\lambda \lambda(0,t,V_0), \\ \theta_f(t,V) = f_2(0,t,V_0) + \kappa_f f(0,t,V_0), \\ \theta_\lambda(t,V) = \lambda_2(0,t,V_0) + \kappa_\lambda \lambda(0,t,V_0). \end{cases} \tag{2.61}$$

Proof: The proof to this proposition is found in Appendix A.8. ■

We show next that the defaultable bond prices across all maturities can be expressed in terms of the default-free short rate, the short rate spread and a set of Markovian state variables.

Proposition 2.15 *Under the Assumption 2.3.1, the price of a $T-$maturity defaultable bond is exponential affine and is given by*

$$P^d(t,T,V) = \frac{\bar{P}^d(0,T,V_0)}{\bar{P}^d(0,t,V_0)} \exp\Big(-\zeta(t,T) - \frac{1}{2}\beta_f^2(t,T)\eta_1(t,V) - \frac{1}{2}\beta_\lambda^2(t,T)\eta_2(t,V) \tag{2.62}$$
$$- \mathfrak{a}(t,T)\eta_3(t,V) - \big[\beta_f(t,T) + \beta_\lambda(t,T)\big] S_3(t,V) - \beta_f(t,T) r(t,V) - \beta_\lambda(t,T) c(t,V)\Big),$$

where

$$\begin{cases} \zeta(t,T) = \ln \mathcal{R}(t) + \beta_f(t,T) f(0,t,V_0) + \beta_\lambda(t,T)\lambda(0,t,V_0)], \\ \mathfrak{a}(t,T) = \frac{1}{\kappa_f}\beta_f(t,T) + \frac{1}{\kappa_\lambda}\beta_\lambda(t,T) + \Big(\frac{1}{\kappa_f} + \frac{1}{\kappa_\lambda}\Big)\Big(\frac{1}{\kappa_f + \kappa_\lambda}\Big)\Big(1 - e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big), \\ \beta_f(t,T) = \int_t^T e^{-\kappa_f(v-t)} dv \quad \textit{and} \quad \beta_\lambda(t,T) = \int_t^T e^{-\kappa_\lambda(v-t)} dv. \end{cases}$$

Proof: See Appendix A.9. ■

Essentially, with this result we have shown that the defaultable bond price takes an exponential affine form in the sense of Duffie and Kan [1996].

2.3.1 Finite Dimensional Realisations in Terms of Defaultable Forward Rates

Some of the Markovian state variables obtained in the proposed Markovian defaultable HJM model do not posses an economic meaning, namely $\eta_i(t,V)$, $(i = 1,2,3)$ and $S_3(t,V)$. Under a default-free term structure setting, Björk and Svensson [2001] and Chiarella and Kwon [2003] have shown that it is possible to express these types of state variables as a linear combination of fixed tenor forward rates, thus obtaining finite dimensional affine realisations in terms of forward rates.

In this section, by adopting their idea, we are able to express the six state variables of the proposed defaultable term structure model, namely $r(t,V)$, $c(t,V)$, $S_3(t,V)$ and $\eta_i(t,V)$, $(i = 1,2,3)$, in terms of defaultable forward rates of six fixed tenors.[14] Consequently, the proposed defaultable term structure is expressed as an exponentially affine term structure in terms of fixed tenor defaultable forward rates. This representation establishes a connection between the defaultable bond price (2.62) and market observable quantities.

Definition 2.16 *We define the deterministic functions*

$$a_1(t,T) = e^{-\kappa_f(T-t)}, \quad a_2(t,T) = e^{-\kappa_\lambda(T-t)}, \quad a_3(t,T) = a_1(t,T)\beta_f(t,T), \quad a_4(t,T) = a_2(t,T)\beta_\lambda(t,T),$$

$$a_5(t,T) = [a_1(t,T)\beta_\lambda(t,T) + a_2(t,T)\beta_f(t,T)], \quad a_6(t,T) = a_1(t,T) + a_2(t,T).$$

Proposition 2.17 *The defaultable forward rate of any maturity can be expressed in terms of six fixed tenor forward rates as*

$$f^d(t,T,V) = D(t,T) + \sum_{j=1}^{6}\sum_{m=1}^{6} a_j(t,T)\hat{a}_{jm} f^d(t,T_m,V), \tag{2.63}$$

[14] Note that state variables $r(t,V)$ and $c(t,V)$ have an economics meaning but are not directly observable. The model could allow to express only the non-observable state variables $\eta_i(t,V)$ for $i = 1,2,3$ and $S_3(t,V)$ in terms of fixed tenor defaultable forward rates, if this is required, see for instance Chiarella and Nikitopoulos-Sklibosios [2003].

where

$$D(t,T) = \widetilde{f}^d(0,t;0,T) + \sum_{j=1}^{6}\sum_{m=1}^{6} a_j(t,T)\hat{a}_{jm}\widetilde{f}^d(0,t;0,T_m),$$

and

$$\widetilde{f}^d(0,t;0,T) = f(0,T,V_0) + \lambda(0,T,V_0) - e^{-\kappa_f(T-t)}f(0,t,V_0) - e^{-\kappa_\lambda(T-t)}\lambda(0,t,V_0), \tag{2.64}$$

and $\hat{a}_{jm}$ *denotes the* jm^{th} *element of the matrix* $A(t)^{-1}$*, which is the inverse of the* 6×6 *square matrix* $A(t)$ *defined as*

$$A(t) = [a_{jm}] \tag{2.65}$$

with $a_{jm} = a_m(t,T_j)$ *as given in Definition 2.16. Assume that* $A(t)$ *is invertable for all* $t \in \{t'; t' = min_i[T_i]\}$*. The state variable* $V = \{V(t), t \in \{t'; t' = min_i[T_i]\}\}$ *satisfies the stochastic differential equation (2.60d).*

Proof: See Appendix A.10. ■

Proposition 2.18 *Given the dynamics for the defaultable forward rate in (2.63), the defaultable bond price can be expressed in an exponential affine form in terms of fixed tenor forward rates as*

$$P^d(t,T,V) = \frac{\bar{P}^d(0,T,V_0)}{\bar{P}^d(0,t,V_0)} \exp\left[-\int_t^T D(t,s)ds - \sum_{m=1}^{6}\sum_{j=1}^{6}\int_t^T a_j(t,s)ds\hat{a}_{jm}f^d(t,T_m,V)\right]. \tag{2.66}$$

Proof: Substitution of (2.63) into the definition (2.14) derives the result. ■

The defaultable bond price expression (2.66) offers an important advantage especially for applications. Market information related to a distinct set of fixed-tenor discrete defaultable forward rates can be embedded into the formula for the defaultable term structure in a very convenient manner due to the tractability of the proposed model.

2.4 Numerical Experiments

In this section, we examine the effect of variations in the parameters of the stochastic volatility and correlations on the distribution of the defaultable bond price and the defaultable bond returns. We first observe that the bond pricing formula in Proposition 2.15 will depend on a particular realisation of the path for the volatility process V. We would simulate the entire system to obtain values of $r(t,V)$, $c(t,V)$, $S_3(t,V)$, $\eta_1(t,V)$, $\eta_2(t,V)$ and $\eta_3(t,V)$ which would then be substituted into the formula (2.62) to obtain the bond price for that particular realisation of V.

2.4.1 Model Inputs

In our numerical investigations we use a typical choice of the system (2.27) so that $\varrho_{12} = \varrho_{13} = \varrho_{23} = 0$. This yields the transformation[15]

$$\begin{bmatrix} dW^V(t) \\ dW^\lambda(t) \\ dW^f(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & \dfrac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \sqrt{\dfrac{1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}}{1-\rho_{12}^2}} \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}. \tag{2.67}$$

In this case, ρ_{12} represents the correlation between the stochastic volatility process and the short term credit spread, ρ_{13} gives the correlation between stochastic volatility and the default free short rate process whereas ρ_{23} represents the correlation between the short term credit spread and short rate process. We have specified the volatility functions for the stochastic

[15] The choice of this specific transformation is necessitated by convenience rather than uniqueness.

volatility process in (2.56c), and note that $\tilde{\sigma}_2^V(t,T,V) = \tilde{\sigma}_3^V(t,T,V) = 0$. By using (2.54), we specify the drift by $\alpha^V(V,t) = \kappa_V\big(\bar{V} - V(t)\big)$, and the market price of risk by $\phi_1(t) = \bar{\phi}\sqrt{V(t)}$ with the scaling factor $\bar{\phi} = 1$[16]. Then the risk neutral dynamics for the volatility process V are

$$dV(t) = \big[\kappa_V\bar{V} - (\kappa_V - \sigma_V)V(t)\big]dt + \sigma_V\sqrt{V(t)}d\tilde{W}_1(t). \tag{2.68}$$

Except for the scenario where we vary the volatility of volatility σ_V, we will use the set of parameters given in Table 2.1 and initial term structures of forward rate and forward credit spread given by

$$f(0,T,V_0) = 0.05 - 0.04\sqrt{V_0}e^{-1.8T} \quad \text{and} \quad \lambda(0,T,V_0) = 0.03 - 0.01\sqrt{V_0}e^{-1.6T}$$

respectively, with the initial volatility chosen to be $V_0 = 0.08$. This implies that the initial short rate and the initial short term credit spread will be $r(0,V_0) = 0.0387$ and $c(0,V_0) = 0.0272$ respectively. The proposed initial term structures provide forward rates between 3.8% and 5% over a period of 20 years and a forward credit spread between 2.7% and 3%. We make

T	σ_f	σ_λ	σ_V	$\bar{V}$	κ_f	κ_λ	κ_V
1.0	0.65	0.45	0.0960	0.0857	0.25	0.3	0.85

Table 2.1: The parameter values used in the simulation experiment.

a simplifying assumption that the initial level of credit spread remains the same irrespective of downgrade on default and subsequent restructuring. Unless otherwise specified, the full correlation structure is given by $\rho_{12} = 0.25$, $\rho_{13} = 0.45$ and $\rho_{23} = -0.30$. [17]

For recovery, we simulate the process $d\mathcal{R}(t) = -\mathcal{R}(t-)q(t)dM(t)$ where $q(\tau_i)$ is the loss rate

[16]This is one possible parameterisation that eventually rescales other parameters in the volatility drift term.

[17]The choice of these parameter values has been motivated by several empirical studies conducted within the default-free HJM framework including Tahani [2004] and Trolle and Schwartz [2009] and which we have adapted to the defaultable framework. We noted that the credit spread levels observed in the market are usually lower than the interest rate levels and therefore their values are significantly lower than the once in the mentioned literature.

at default time τ_i, the compensated process $dM(t) = dN(t) - \tilde{h}(t)dt$ is a martingale and $N(t)$ is a Cox process governing the default dynamics. This is a special case of the general marked point processes in (2.12) as explained in Remark 2.4.[18]

For the simulation experiment, we use an Euler-Maruyama approximation, for $T = 2$ and discretise into 250 subintervals. We generate 200,000 simulated paths for the short rate $r(t, V)$, see (2.60b), and the short rate spread $c(t, V)$, see (2.60c), to obtain the defaultable bond price as in Proposition 2.15 whose distribution we calculate at the point, $t = 1.0$.

2.4.2 Simulation Results

We recall that under recovery of market value (RMV), the recovery ratio is a fraction of the current market value (see Duffie and Singleton [1997], Duffie and Huang [1996] among others). This offers greater computational tractability as compared to other recovery models. In some models, for example, Duffie and Singleton [1999] and Houweling and Vorst [2005], the recovery rate is interwoven with the risk premium, making the distinction between intensity (hazard) and recovery rate difficult and this is the case when using defaultable bond data alone.

Pan and Singleton [2008] have however shown that in the CDS markets, the default intensity and recovery play distinct roles. Given that recovery is a fraction of the face value, the arrival intensity and the recovery parameters can in principle be separately identified using the information contained in the term structure of CDS spreads. In our two-fold analysis, we investigate the distributional properties of a defaultable bond at the point, $t = 1.0$ and then consider the special case of its pre-default (pseudo bond) values under varying parametric specifications.

[18] Although our model allows for multiple defaults and recovery, we assume that the firm's default intensity and recovery rate remain the same even after default and restructuring. A more realistic specification would allow for downgrade in the credit quality thereby increasing the default intensity and reducing the recovery rate in the eventuality of future events. This would require a more general migration model.

Defaultable Bond Analysis

The defaultable bond is assumed to have an average default intensity $\tilde{h}(t) = 0.30$. This risk-neutral default intensity is backed out of bond prices using the formula $\tilde{h} = \frac{s}{1-\mathcal{R}}$, with a yield spread of $s = 1321bps$ as the average yield spread of defaultable bonds over treasuries. This choice falls between $B-$rated bonds risk-neutral default intensity approximated to be 0.0902 and $Caa-$rated bonds whose intensity was estimated to be 0.2130 in Hull et al. [2005]. In addition, we assumed the loss given default to be distributed according to $LGD \sim \mathcal{N}(0.6839, 0.07)$.[19]

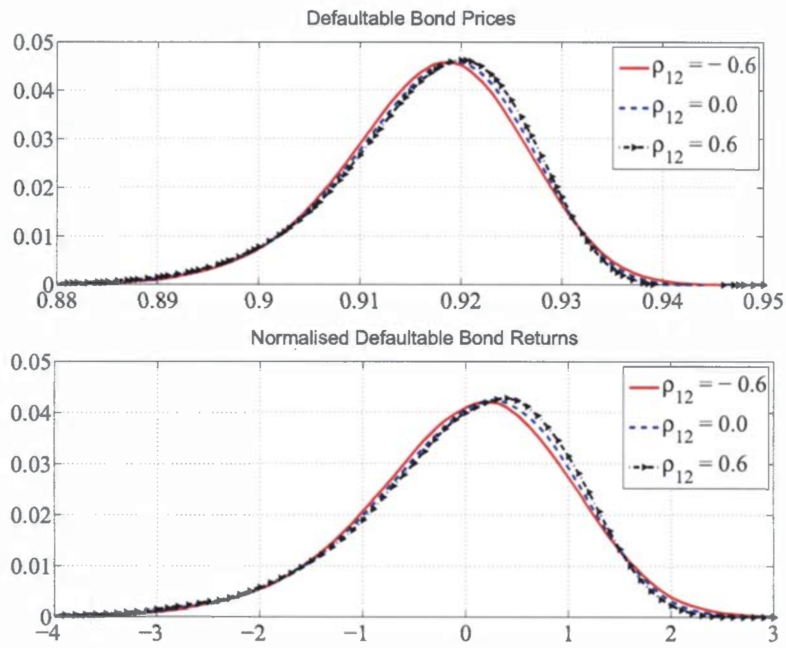


Figure 2.1: Distribution of defaultable bond price and normalised bond returns under varying ρ_{12} while keeping $\rho_{13} = 0.45$ and $\rho_{23} = -0.40$.

Figure 2.1 illustrates the effect of the correlation ρ_{12} between the stochastic volatility process $V(t)$ and the credit spread process $c(t, V)$ on the distribution of defaultable bond price and bond returns. Increasing the correlation ρ_{12} from -0.6 to 0.6 while holding the other correlations at zero tends to increase the (negative) skewness of the two distributions. Table 2.2

[19]This is documented in Moody's [2003] report which gives average recoveries for different rating classes over the time period 1982 – 2003.

gives the values of the changes in skewness and kurtosis with change in the correlation ρ_{12} between short term credit spread and stochastic volatility.

ρ_{12}	-0.60	0.00	0.60
Kurtosis (Price)	4.1178	4.3144	4.6628
Skewness(Price)	-0.8157	-0.9095	-1.0270
Kurtosis (Returns)	4.8161	4.7308	4.6410
Skewness(Returns)	0.1596	0.1708	0.2043

Table 2.2: Effect of correlation ρ_{12} between short term credit spread and stochastic volatility with change in the kurtosis and skewness of defaultable bond price and bond returns.

A similar observation is made in Figure 2.2, where we vary the correlation ρ_{13} between the stochastic volatility process $V(t)$ and the short rate process $r(t, V)$. However increasing the correlation ρ_{13} from -0.6 to 0.6 while holding the other correlations at zero tends to generate a larger variation in the skewness of the bond price and bond returns distribution. This could be attributed to the fact that the short rate has a higher average volatility than the credit spread process. Table 2.3 shows the effect of the correlation ρ_{13} on the defaultable

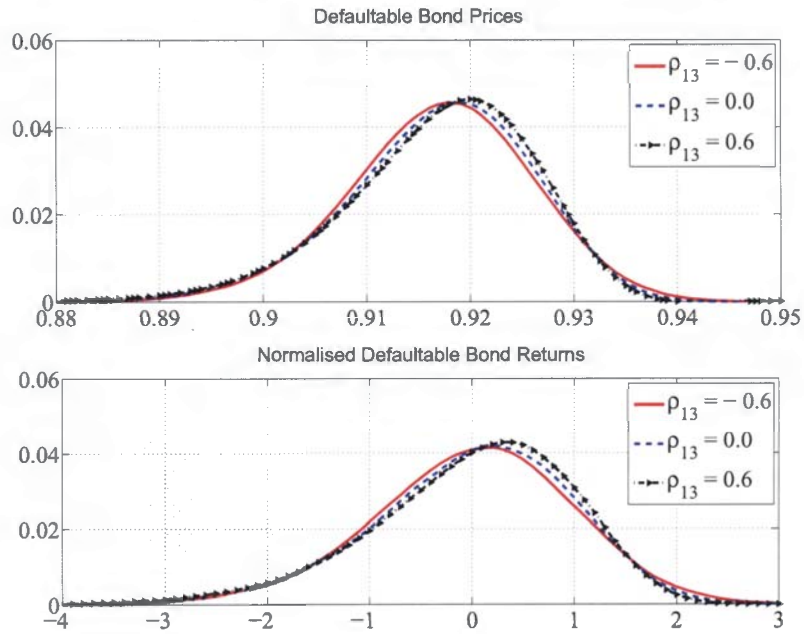


Figure 2.2: Distribution of defaultable bond price and normalised bond returns under varying ρ_{13} while keeping $\rho_{12} = 0.30$ and $\rho_{23} = -0.40$.

bond price and defaultable bond returns distributions.

ρ_{13}	-0.60	0.00	0.60
Kurtosis (Price)	3.2345	3.9043	4.6427
Skewness(Price)	-0.5263	-0.7667	-1.0254
Kurtosis (Returns)	3.7569	4.3455	4.7903
Skewness(Returns)	0.0973	0.1393	0.1997

Table 2.3: Effect of correlation ρ_{13} between short rate and stochastic volatility with change in the kurtosis and skewness of defaultable bond price and bond returns.

Figure 2.3 illustrates the effect of the correlation ρ_{23} between the short-term credit spread $c(t,V)$ and the short rate process $r(t,V)$ on the distribution of defaultable bond prices. Increasing the correlation ρ_{23} tends to decrease both the kurtosis and the (negative) skewness of the distributions. The correlation ρ_{23} between the short-term credit spread $c(t,V)$ and the short rate process $r(t,V)$ conveys information about the covariation between default-free discount rates and the market's perception of default risk. In Longstaff and Schwartz [1995b] and Duffee [1998], it was shown that this relationship is negative for investment-grade, noncallable corporate bonds and strongly negative for lower rated and callable bonds. The magnitude of the effect of this correlation, ρ_{23} is given in Table 2.4.

ρ_{13}	-0.60	0.00	0.60
Kurtosis (Price)	5.6115	4.5180	4.4358
Skewness(Price)	-1.2169	-0.9805	-1.0128
Kurtosis (Returns)	5.9690	4.6780	5.0029
Skewness(Returns)	0.1443	0.1641	0.1933

Table 2.4: Effect of correlation ρ_{23} between the short rate and the short term credit spread on the change in kurtosis and skewness of defaultable bond price and bond returns.

Figure 2.4 illustrates the effect of the volatility of volatility σ^V on the distribution of defaultable bond price and returns, respectively. When $\sigma^V = 0$, the volatility process is deterministic and an increasing σ^V implies that the market has a higher chance of extreme movements. We observe that, increasing volatility of volatility tends to skew the bond price and bond returns distribution to the right and increases the kurtosis of both the bond price and bond returns. Note that, it has been empirically observed that negatively skewed returns

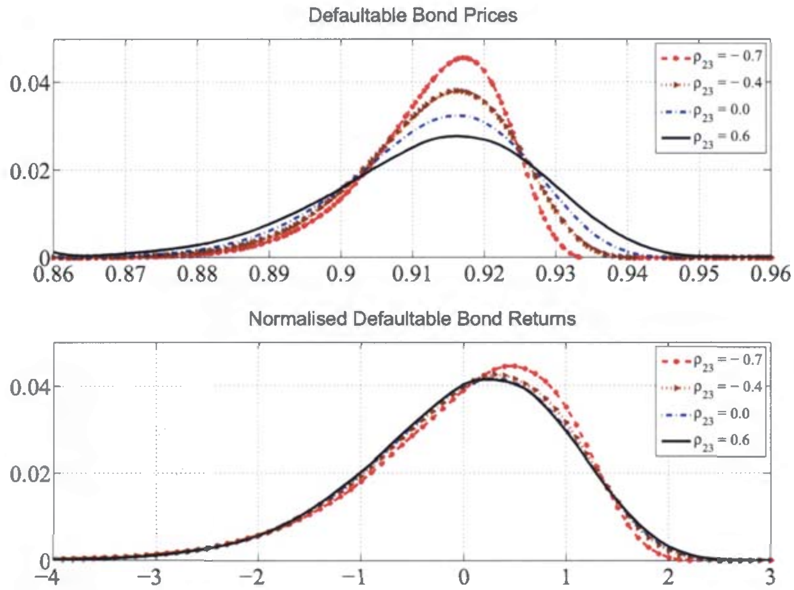


Figure 2.3: Distribution of defaultable bond price and normalised bond returns under varying ρ_{23} while keeping $\rho_{12} = 0.30$ and $\rho_{13} = 0.45$.

(with heavy downside tails) are characteristic of portfolios of defaultable bonds, see D'Souza et al. [2004].

We also investigated the effects of the speed of mean reversion κ_V on the volatility process of the defaultable bond price and returns distribution. From Figure 2.5 we observe that increasing the speed of mean reversion of the volatility process reduces the kurtosis of both the defaultable bond price and returns. Figure 2.6 shows the effect of the default intensity $\tilde{h}(t)$ on distribution of defaultable bond price and bond returns.

We finally tested the deviation from 'normality' of the various price and returns distributions using QQ-plots, the results of which are given in Figure 2.7 under full correlation and the parameter values given in Table 2.1. Of particular mention is Figure 2.7(c) which captures the heavy tail events in the normalised returns of the defaultable bond with the risk-neutral default intensity $\tilde{h}(t) = 0.0507$.

In Figure 2.8 we show the effects of varying the bonds maturity on the bond price and returns

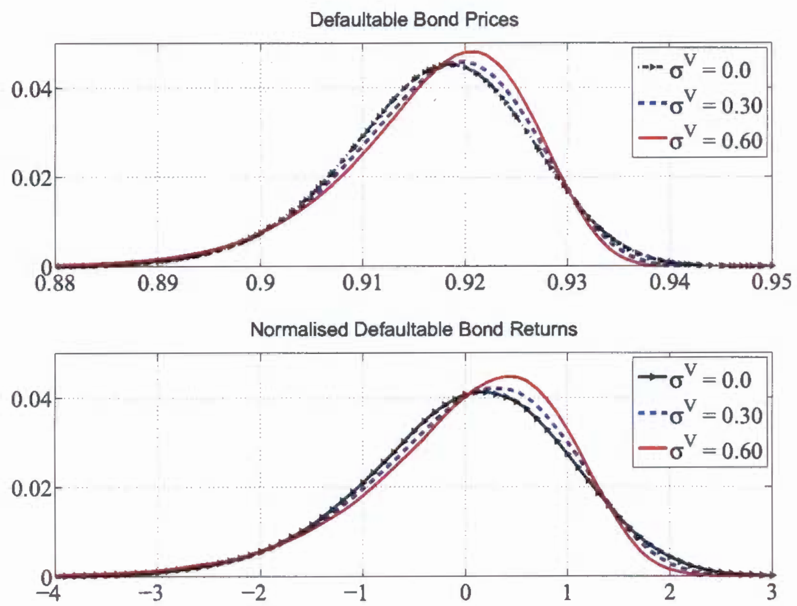


Figure 2.4: Distribution of defaultable short rate and defaultable bond price under varying σ^V while keeping the correlation $\rho_{12} = 0.30$, $\rho_{13} = 0.45$ and $\rho_{23} = -0.40$.

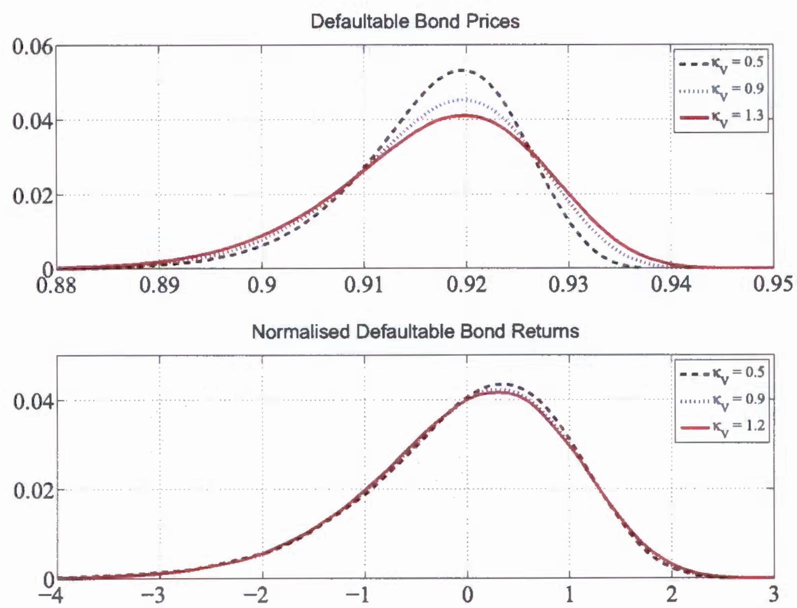


Figure 2.5: Distribution of defaultable bond price and defaultable bond returns under varying κ^V.

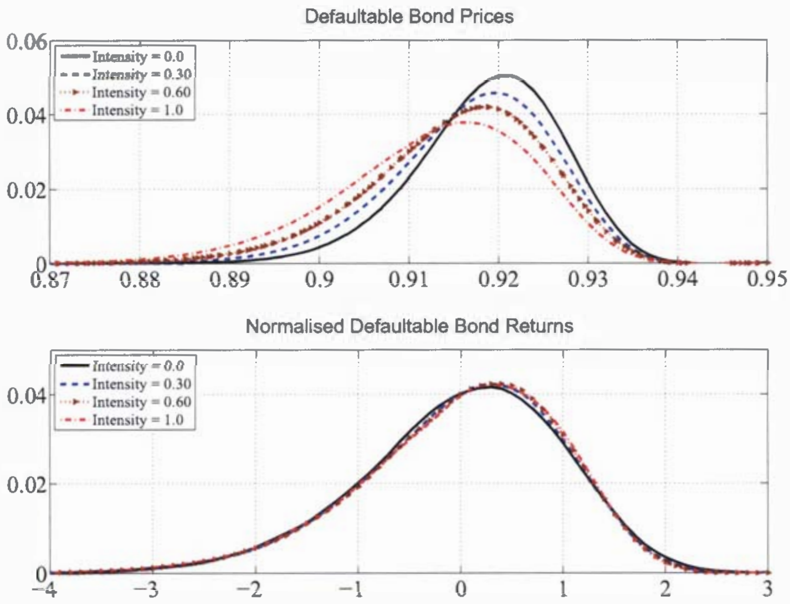


Figure 2.6: Distribution of defaultable bond price and defaultable bond returns under varying default intensity $\tilde{h}(t)$.

distribution. From Figure 2.8(a) and Figure 2.8(b) we observe that increasing the maturity T reduces the negative skewness and lowers the kurtosis of the bond price distributions. However, increasing the maturity was observed to lead to an increment in the kurtosis of the bond returns as shown in Figure 2.8(c) and Figure 2.8(d).

2.4.3 Discussion

Our model results indicate that increasing the correlation between the volatility process $V(t)$ and either the short term credit spread $c(t, V)$ or short rate processes $r(t, V)$ (ρ_{12} or ρ_{13} respectively) increases the negative skewness of the risky and pseudo bond prices. This is attributed to the existence of a long left tail of the distribution. In addition, there is an increase in the positive excess kurtosis due to less frequent large changes, again indicating the presence of long, fat tails.

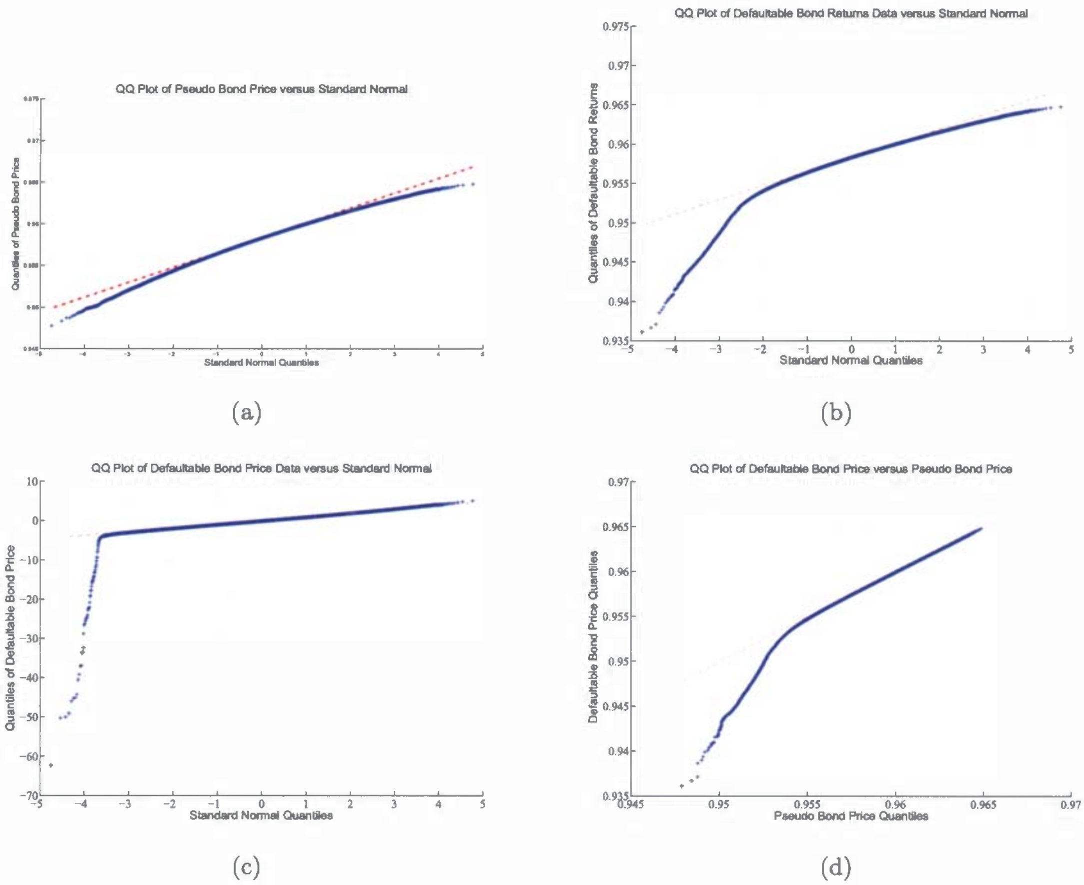
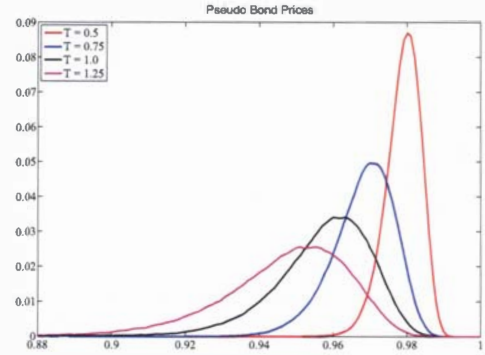
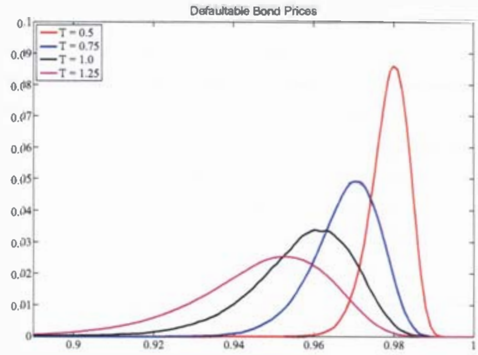


Figure 2.7: A set QQ-Plots of Bond Price quantiles: (a) describes the qq-plot for Pseudo bond price quantiles; (b) describes the qq-plot for defaultable bond price quantiles; (c) describes the qq-plot for defaultable bond returns quantiles; and, (d) describes the qq-plot between the Pseudo-bond price quantiles and defaultable bond price quantiles.

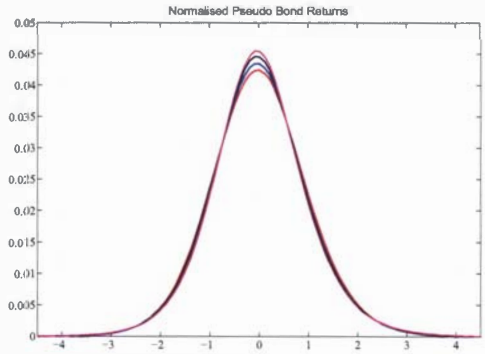
Increasing the correlation ρ_{23} between the credit spread and short rate of the defaultable bond leads to a decrease in the negative skewness and kurtosis of both the defaultable bond price and bond returns. In particular, we observe that there is a reduction in the peakedness of the defaultable bond price distribution. This change is also observed in the distribution of the returns for the defaultable bonds. It was noted in D'Souza et al. [2004] that defaultable bonds have returns that are negatively skewed. This, they observed was due to the fact that the probability of defaultable bonds earning a substantial price appreciation is relatively small but there exists a large probability of earning small profit through interest rates earnings. The distribution, as observed also in our simulated results tends to be skewed around a positive



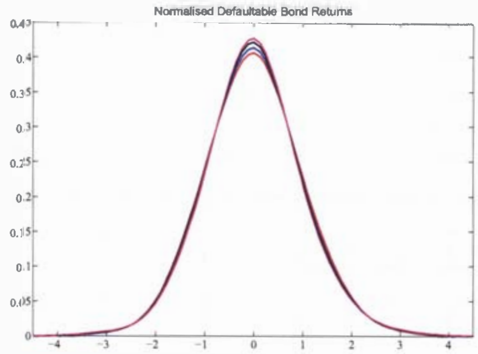
(a)



(b)



(c)



(d)

Figure 2.8: Panel (a) describes pseudo bond price distribution; panel (b) describes the defaultable bond price distribution; panel (c) describes the distribution of the normalised pseudo bond returns while panel (d) describes the distribution of the normalised defaultable bond returns. All these distributions are made under increasing maturity.

value with a small positive tail reflecting the limited upside potential. Adverse movements in credit quality occur with small probability but these can have an adverse negative impact on the value of the asset, generating significant losses. It has been empirically observed that skewed returns with heavy downside tails are characteristic of portfolios of defaultable bonds.

In addition, we observed that increasing the volatility of volatility increases both the kurtosis and negative skewness of the defaultable bond price when the short rate, credit spread and stochastic volatility processes are independent but there is a reduction in the negative skewness under a full correlation structure in addition to an increment in the kurtosis under both cases. We also observe that higher volatility of volatility yields an increase in the kurtosis of the defaultable bond returns under both cases of dependent and independent structures.

2.5 Summary

This chapter develops a class of defaultable HJM term structure models with unspanned stochastic volatility. By modelling the credit spreads, a connection between the default-free and the defaultable forward term structure has been established and a correlation structure between credit spreads, interest rates and stochastic volatility has been incorporated. We specified the default process using a marked point process whose marker models the uncertainty in the recovery rate. We have considered level dependent volatility specifications that reduce the proposed model to admit finite dimensional realisations. In addition, we derived an explicit exponential affine formula for defaultable bond prices in terms of some state variables.

The chapter also attempts to provide a link between the state variables and the market observed quantities, in particular fixed tenor forward rates. This is of significant value when implementing the model and further research into calibration and evaluation of these models. We also expressed the defaultable forward rates in terms of the fixed tenor forward rates and obtained a representation of the defaultable bond price in an exponential affine form, in terms of the fixed tenor forward rates.

Some numerical results have been presented demonstrating how the level of the volatility of volatility, speed of mean reversion of the stochastic volatility process and correlation between the Wiener processes driving the defaultable short rates, short term credit spreads and the stochastic volatility affect the defaultable bond prices and returns. The parameters used in the simulations were chosen from past empirical studies in the default-free HJM framework and adapted to incorporate default risk. Evidence of the existence of fat tails due to the presence of stochastic volatility and defaults in the bond price dynamics was observed in the distributions as depicted by the QQ-plots. In addition, the varying effects of the processes as a result of their varying correlations was observed in the skewness and kurtosis of the distributions. These observations were found to be consistent with the stylised facts on defaultable debt in the presence of stochastic volatility as well as under a correlated system.

Chapter 3

Pricing Defaultable Securities under Humped Volatility

In this chapter, we introduce a framework for pricing credit derivatives within the defaultable Markovian HJM framework under stochastic volatility. We generalise the stochastic exponential decaying volatility specifications developed in Chapter 2 to allow for a term structure model with stochastic humped volatility in addition to depending on the level of the interest rates themselves, in line with the empirical evidence in Chan et al. [1992], Amin and Morton [1995] and Mercurio and Moraleda [2000]. The hump volatility specification allows for sharp curvature changes within the forward rate curves thereby reducing pricing errors in addition to producing various shapes. This framework therefore yields an effective pricing model for options on defaultable bonds as well as for credit default swaps and swaptions.

3.1 Introduction

Over the course of time, the credit derivatives market has evolved from a primarily single-name CDS market into a more complex market that consists of not only the more mainstream single-name CDS (in both high grade and high yield credit) but also the liquid CDS indices (CDX, iTraxx) and the correlation and volatility products. Whereas a single-name CDS is physically settled, CDS indices and tranches are moving towards a standardised cash settlement where following a credit event, the protection seller provides a single cash payment which reflects the extent to which a market valuation of a specific debt obligation of the reference entity has fallen.

Following the events of the recent financial crisis, both the regulators and credit risk models have received a fair share of criticism. Regulatory authorities in the critical financial centres potentially had information on the sub-prime exposure (and hence potential losses) on an institution by institution basis. The entire system of safeguards, consisting of disclosure, regulation and supervision failed and these failures have driven regulators into setting tougher rules and are requiring banks to run stress tests with scenarios that include huge jumps in interest rates. There is also the requirement for 'reverse' stress testing (compulsory in some countries), in which a firm postulates that it has failed and works backwards to determine which vulnerabilities caused the hypothetical collapse.

Berndt, Ritchken, and Sun [2010] noted that the Markovian framework in the Heath, Jarrow, and Morton [1992] model is uniquely suitable for credit stress testing since it is both tractable and general. The state variables can be shown to have some economic interpretation as functions of forward rates of different maturities. The volatilities of the instantaneous interest rates and credit spreads can be chosen arbitrarily and are not restricted to the affine family of Duffie and Kan [1996]. The defaultable HJM framework allows for the risk-free and credit spread curves to be analytically computed at any point in time as a function of some state variables and can be initialized to match the observable term structure of volatilities. The model also allows one to incorporate the correlation between interest rates and credit spreads which has been shown (for example in D'Souza et al. [2004]) to be critical when valuing options on risky debt or when valuing insurance contracts that offer protection against default of a counterparty to an underlying derivatives position. In addition, they showed that Default-free interest rates, credit spreads and stochastic volatility exhibit mean-reversion.

Empirical studies have shown that interest rate volatility is stochastic. Ball and Torous [1999] observed that in contrast to stock returns, interest rate volatility exhibit faster mean reversion behavior and the innovations for interest rate volatility are negligibly correlated to the innovations in interest rates. Collin-Dufresne and Goldstein [2002] and Casassus et al. [2005] postulated unspanned stochastic volatility factors that drive the innovations of interest rate derivatives but do not affect the innovations in the swap rates and therefore the bond prices themselves. They argued this unspanned volatility is the derivatives volatility that cannot be hedged using the yield curve instruments.

Humped volatility improves the model specification, both in terms of likelihood score, analysis of yield errors and cap pricing performance. Reno and Uboldi [2005] argued that a HJM model with humped term-structure volatility could be an alternative to the unspanned volatility model in Collin-Dufresne and Goldstein [2002]. They showed that the R^2 of the regression of observed straddle variations against the straddle price movements from the humped HJM model were close to unity and the model performed well in modelling of straddle prices although their results did not rule out the existence of unspanned volatility. Trolle and Schwartz [2009] showed that a model based on the HJM with a humped term structure and unspanned stochastic volatility matches the implied skews and the dynamic volatilities in the risk-free setup. They estimated the model using a 7-year data set consisting of LIBOR, swap rates, forward swaptions and caps using Quasi maximum likelihood in conjunction with the extended Kalman filter and observed that it gives a good fit.

The chapter makes the following contributions. A generalised defaultable term structure model within the Heath et al. [1992] (hereafter HJM) framework that accommodates unspanned stochastic volatility is presented. More specifically, the proposed model has the following properties. By construction, the model is consistent with the currently observed yield curve and credit spread curve. The model features a default-free term structure that is driven by n factors, a defaultable term structure that is driven by $2n$ factors, and n additional stochastic volatility factors affecting only (interest rate and credit) derivative prices. Additionally, the connection between the two markets (default-free and defaultable) is established through the credit spread, see Schönbucher [1998], which allows us to accommodate a correlation structure between the interest rates, the credit spreads and the stochastic volatility.

However, it is well-known that these models are Markovian in the entire yield curve and credit spread curve thus requiring an infinite number of state variables. Consequently, a quite general volatility specification for the default-free and the defaultable term structure is proposed that leads to finite dimensional realisation of the state space, see for instance Björk et al. [2004], Chiarella and Kwon [2000a] and CNS. The proposed volatility structure allows for level dependency and hump-shaped shocks. In this regard, our model can be considered as an extension of the Berndt et al. [2010] to accommodate unspanned stochastic volatility. In line with the empirical evidence provided by Chan et al. [1992], Amin and Morton [1995] and Mercurio and Moraleda [2000], the volatility structure depends on the level of the short rates and the short-term credit spreads. Under these volatility specifications, the model offers tractability and flexibility as it allows (default-free and defaultable) bond prices to be expressed as exponentially affine functions of state variables which are jointly Markovian. Although the model gives rise to a large (finite) number of state variables, their Markovian structure guarantees that the computational cost remains low.

In addition, pricing of credit derivatives is considered under the proposed model. We derive pricing formulas for single-name credit default swap rates (hereafter CDS rates) and swaptions. Based on approximations proposed by Brigo and Morini [2005], CDS rates are expressed in terms of defaultable bond prices with varying maturities, both in the absence and the presence of counterparty risk. Swaptions have been also priced by using a Black's-type formula. Lastly, the impact of the correlation structure and the stochastic volatility specifications on CDS rates and swaption prices is studied by the means of Monte-Carlo simulations. The simulation results indicate that the correlation between interest rate and credit spread impacts the CDS rate and consequently the swaption prices, similarly to the results derived by Berndt et al. [2010]. This is contrary to the results given in Krekel and Wenzel [2006], who argued that this correlation does not play a significant role in the pricing.

On relaxing the level dependency assumption within the humped volatility specification, we then extend the framework to the option pricing problem. By considering a put option that is 'knocked-out' on default of the underlying bond, we apply Fourier transform methods to derive a semi-closed form solution for the option and obtain the option pricing formula by inverting the semi-closed-form solutions of the characteristic functions derived. The resulting coupled system of differential equations when calculating the exercise probabilities is solved using numerical integration.

The chapter is structured as follows. Section 3.2 presents a multi-dimensional defaultable term structure model with unspanned stochastic volatility. Section 3.3 proposes a hump-shaped level dependent volatility structure for the default-free and the defaultable forward rate that allow this model to admit finite dimensional affine realisations and to produce exponentially affine defaultable bond prices. Section 3.4 considers the pricing of credit default swaps and credit default swaptions. In Section 3.5, the framework is applied to price put options on defaultable bonds where we have assumed that the option is knocked out on the default of the underlying bond. Section 3.6 concludes. The proof of the technical results in this chapter are given in Appendix II.

3.2 A General Defaultable Term Structure Model

We introduce a defaultable term structure model under a general volatility specification that allows a wider range of volatility shocks to the defaultable forward curve. In particular, it represents a term structure in which the volatility is level dependent and contains a hump.

As in Chapter 2, we assume the complete probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ given the full filtration $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^N$ contains default-free information plus explicit monitoring of default up to time t as defined in Section 2.2. In addition, for this chapter, we define default time to be $\tau = \tau_1$.

We consider the model setup of Section 2.2. To allow for various factors in the economy or the market to drive the dynamics of the forward rate and credit spread, we propose the following multi-dimensional model.

Assumption 3.2.1 *The instantaneous default-free forward rate $f(t,T,\mathbb{V})$ and the instantaneous forward credit spread $\lambda(t,T,\mathbb{V})$ satisfy the stochastic integral equations*

$$f(t,T,\mathbb{V}) = f(0,T,\mathbb{V}_0) + \int_0^t \alpha^f(u,T,\mathbb{V})du + \sum_{i=1}^{n}\int_0^t \sigma_i^f(u,T,V_i)dW_i^f(u), \tag{3.1}$$

$$\lambda(t,T,\mathbb{V}) = \lambda(0,T,\mathbb{V}_0) + \int_0^t \alpha^\lambda(u,T,\mathbb{V})du + \sum_{i=1}^{n}\int_0^t \sigma_i^\lambda(u,T,V_i)dW_i^\lambda(u), \tag{3.2}$$

where the stochastic volatility vector process $\mathbb{V} = \{(V_1(t),\ldots,V_n(t)), t \in [0,T]\}$ satisfies the set of the stochastic differential equations

$$dV_i(t) = \alpha^{\mathbb{V}}(t,V_i)dt + \bar{\sigma}_i^{\mathbb{V}}(t,V_i)dW_i^{\mathbb{V}}(t), i = 1,2,...n \tag{3.3}$$

with $\mathbb{V}_0$ being the initial value of the volatility process. As in Section 2.2.1, we assume that the same volatility process V_i is used for both volatility functions $\sigma_i^f(t,T,V_i)$ and $\sigma_i^\lambda(t,T,V_i)$ and that the drift and the diffusion of the volatility process depends only on V_i.

By using definition (2.6) and the dynamics specified in Assumption 3.2.1, the stochastic integral equation for the defaultable forward rate is expressed now as

$$\begin{aligned} f^d(t,T,\mathbb{V}) &= f^d(0,T,\mathbb{V}_0) + \int_0^t (\alpha^f(u,T,\mathbb{V}) + \alpha^\lambda(u,T,\mathbb{V}))du \\ &+ \sum_{i=1}^n \int_0^t \sigma_i^f(u,T,V_i)dW_i^f(u) + \sum_{i=1}^n \int_0^t \sigma_i^\lambda(u,T,V_i)dW_i^\lambda(u), \end{aligned} \tag{3.4}$$

where the initial defaultable forward curve is

$$f^d(0,T,\mathbb{V}_0) = f(0,T,\mathbb{V}_0) + \lambda(0,T,\mathbb{V}_0). \tag{3.5}$$

Furthermore, the specifications of Assumption 3.2.1 imply the following dynamics for the instantaneous default-free short rate $r(t,\mathbb{V}) := f(t,t,\mathbb{V})$ and the instantaneous short-term credit spread $c(t,\mathbb{V}) := \lambda(t,t,\mathbb{V})$

$$r(t,\mathbb{V}) = f(0,t) + \int_0^t \alpha^f(u,t,\mathbb{V})du + \sum_{i=1}^n \int_0^t \sigma_i^f(u,t,V_i)dW_i^f(u), \tag{3.6}$$

$$c(t,\mathbb{V}) = \lambda(0,t) + \int_0^t \alpha^\lambda(u,t,\mathbb{V})du + \sum_{i=1}^n \int_0^t \sigma_i^\lambda(u,t,V_i)dW_i^\lambda(u), \tag{3.7}$$

respectively, where for notional convenience, we have suppressed the explicit dependence on $\mathbb{V}_0$ by setting $f(0,t) = f(0,t,\mathbb{V}_0)$ and $\lambda(0,t) = \lambda(0,t,\mathbb{V}_0)$. For $T = t$, Equation (3.4) provides the dynamics for the instantaneous defaultable short rate $r^d(t,\mathbb{V}) := f^d(t,t,\mathbb{V}) = f(t,t,\mathbb{V}) + \lambda(t,t,\mathbb{V})$ as

$$\begin{aligned} r^d(t,\mathbb{V}) &= f^d(0,t) + \int_0^t (\alpha^f(u,t,\mathbb{V}) + \alpha^\lambda(u,t,\mathbb{V}))du \\ &+ \sum_{i=1}^n \int_0^t \sigma_i^f(u,t,V_i)dW_i^f(u) + \sum_{i=1}^n \int_0^t \sigma_i^\lambda(u,t,V_i)dW_i^\lambda(u). \end{aligned} \tag{3.8}$$

In the subsequent analysis, we consider the correlation structure

$$\mathbb{E}\big[dW_i^x \cdot dW_j^y\big] = \begin{cases} \delta_{ij}\rho_i^{xy}dt & \text{if } x \neq y, \\ \delta_{ij}dt & \text{if } x = y, \end{cases} \tag{3.9}$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$, for $x, y \in \{\mathbb{V}, \lambda, f\}$, $1 \leq i, j \leq n$ and $\rho_i^{xy} \in [-1, 1]$ for all i.

The correlated Wiener processes $W_i^f(t)$, $W_i^\lambda(t)$ and $W_i^{\mathbb{V}}(t)$, for $i = 1, 2, \ldots, n$, can be expressed in terms of independent Wiener processes $W_i(t)$, for $i = 1, 2, \ldots, 3n$. For $i = 1, 2, \ldots, n$, we adopt the following decomposition (for modelling convenience)

$$\begin{aligned} dW_i^f(t) &= z_i^{f_1} dW_i(t), \\ dW_i^\lambda(t) &= z_i^{\lambda_1} dW_i(t) + z_i^{\lambda_2} dW_{n+i}(t), \\ dW_i^{\mathbb{V}}(t) &= z_i^{V_1} dW_i(t) + z_i^{V_2} dW_{n+i}(t) + z_i^{V_3} dW_{2n+i}(t), \end{aligned} \tag{3.10}$$

where the correlation parameters, for $(\rho_i^{\lambda f})^2 \neq 1$, are set as

$$\begin{aligned} &z_i^{f_1} = 1, \\ &z_i^{\lambda_1} = \rho_i^{\lambda f}, \quad z_i^{\lambda_2} = \sqrt{1 - (\rho_i^{\lambda f})^2}, \\ &z_i^{V_1} = \rho_i^{\mathbb{V} f}, \quad z_i^{V_2} = \frac{\rho_i^{\mathbb{V}\lambda} - \rho_i^{\lambda f} \rho_i^{\mathbb{V} f}}{\sqrt{1 - (\rho_i^{\lambda f})^2}}, \quad z_i^{V_3} = \sqrt{\frac{1 - (\rho_i^{\lambda f})^2 - (\rho_i^{\mathbb{V} f})^2 - (\rho_i^{\mathbb{V}\lambda})^2 + 2\rho_i^{\lambda f} \rho_i^{\mathbb{V} f} \rho_i^{\mathbb{V}\lambda}}{1 - (\rho_i^{\lambda f})^2}}. \end{aligned} \tag{3.11}$$

By using the decomposition (3.10), the stochastic integral equations (3.1) and (3.2) as well as the stochastic differential equation (3.3) are expressed in terms of independent Wiener processes as

$$f(t, T, \mathbb{V}) = f(0, T) + \int_0^t \alpha^f(u, T, \mathbb{V}) du + \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^f(u, T, V_i) dW_i(u), \tag{3.12}$$

$$\lambda(t, T, \mathbb{V}) = \lambda(0, T) + \int_0^t \alpha^\lambda(u, T, \mathbb{V}) du + \sum_{i=1}^{2n} \int_0^t \tilde{\sigma}_i^\lambda(u, T, V_i) dW_i(u), \tag{3.13}$$

$$dV_i(t) = \alpha_i^{\mathbb{V}}(t, V_i) dt + \sum_{j=1}^{3} \tilde{\sigma}_{ij}^{\mathbb{V}}(t, V_i) dW_{(j-1)n+i}(t), \tag{3.14}$$

where the volatility functions $\tilde{\sigma}_i^f(t,T,V_i)$, $\tilde{\sigma}_i^\lambda(t,T,V_i)$ and $\tilde{\sigma}_i^{\mathbb{V}}(t,V_i)$ are given by

$$\tilde{\sigma}_i^f(t,T,V_i) = \begin{cases} z_i^{f_1}\sigma_i^f(t,T,V_i), & \text{for } i=1,\dots,n \\ 0, & \text{otherwise;} \end{cases} \tag{3.15}$$

$$\tilde{\sigma}_i^\lambda(t,T,V_i) = \begin{cases} z_i^{\lambda_1}\sigma_i^\lambda(t,T,V_i), & \text{for } i=1,\dots,n\ ; \\ z_{i-n}^{\lambda_2}\sigma_{i-n}^\lambda(t,T,V_{i-n}), & \text{for } i=n+1,\dots,2n; \end{cases} \tag{3.16}$$

while for $j=1,2,3$ and $i=1,\dots,n$,

$$\tilde{\sigma}_{ij}^{\mathbb{V}}(t,V_i) = z_i^{V_j}\sigma_i^{\mathbb{V}}(t,V_i), \quad \text{for} \quad i=1,\dots,n \quad \text{and} \quad j=1,2,3. \tag{3.17}$$

Moreover, when the decomposition (3.10) is applied to (3.4), we have that the defaultable forward rate follows the stochastic integral equation

$$f^d(t,T) = f^d(0,T) + \int_0^t (\alpha^f(u,T) + \alpha^\lambda(u,T))du + \sum_{i=1}^{2n}\int_0^t \tilde{\sigma}_i^d(u,T,V_i)dW_i(u), \tag{3.18}$$

where

$$\tilde{\sigma}_i^d(t,T,V_i) = \tilde{\sigma}_i^f(t,T,V_i) + \tilde{\sigma}_i^\lambda(t,T,V_i). \tag{3.19}$$

The system of equations (3.14), (3.18) and (3.12) embeds stochastic volatility into a traditional defaultable HJM framework. The default-free forward rate is driven by n sources of uncertainty while the (apparently larger) defaultable forward rate is driven by $2n$ sources of uncertainty. The volatility of these forward rate curves, in general, is driven by a total number of $3n$ sources of uncertainty, subject to the correlation structure between the forward rate and their volatilities. Therefore, volatility sensitive instruments, such as interest rate derivatives and credit derivatives are affected by $3n$ factors. Indeed, the proposed framework can be considered as an adaptation of the Heston [1993] stochastic volatility equity model to a defaultable term structure model.

Our extension of the HJM framework to a defaultable setting incorporates also unspanned stochastic volatility. A defaultable term structure model with $2n$ factors is considered, where n factors are associated with the default-free term structure and $3n$ factors associated with their volatilities. Thus the proposed model, subject to the correlation structure allows for n unspanned stochastic volatility factors.

For illustrations purposes, we present the system of the stochastic integral and differential equations (3.12), (3.13) and (3.14) for the special case of $n = 3$.

$$\begin{aligned}
f(t,T,\mathbb{V}) &= f(0,T) + \int_0^t \alpha^f(u,T,\mathbb{V})du + \int_0^t \tilde{\sigma}_1^f(u,T,V_1)dW_1(u) + \int_0^t \tilde{\sigma}_2^f(u,T,V_2)dW_2(u)\\
&\quad + \int_0^t \tilde{\sigma}_3^f(u,T,V_3)dW_3(u),\\
\lambda(t,T,\mathbb{V}) &= \lambda(0,T) + \int_0^t \alpha^\lambda(u,T,\mathbb{V})du + \int_0^t \tilde{\sigma}_1^\lambda(u,T,V_1)dW_1(u) + \int_0^t \tilde{\sigma}_2^\lambda(u,T,V_2)dW_2(u)\\
&\quad + \int_0^t \tilde{\sigma}_3^\lambda(u,T,V_3)dW_3(u) + \int_0^t \tilde{\sigma}_4^\lambda(u,T,V_1)dW_4(u)\\
&\quad + \int_0^t \tilde{\sigma}_5^\lambda(u,T,V_2)dW_5(u) + \int_0^t \tilde{\sigma}_6^\lambda(u,T,V_3)dW_6(u),\\
dV_1(t) &= \alpha_1^{\mathbb{V}}(t,V_1)dt + \tilde{\sigma}_1^{\mathbb{V}}(t,V_1)dW_1(t) + \tilde{\sigma}_4^{\mathbb{V}}(t,V_1)dW_4(t) + \tilde{\sigma}_7^{\mathbb{V}}(t,V_1)dW_7(t),\\
dV_2(t) &= \alpha_2^{\mathbb{V}}(t,V_2)dt + \tilde{\sigma}_2^{\mathbb{V}}(t,V_2)dW_2(t) + \tilde{\sigma}_5^{\mathbb{V}}(t,V_2)dW_5(t) + \tilde{\sigma}_8^{\mathbb{V}}(t,V_2)dW_8(t),\\
dV_3(t) &= \alpha_3^{\mathbb{V}}(t,V_3)dt + \tilde{\sigma}_3^{\mathbb{V}}(t,V_3)dW_3(t) + \tilde{\sigma}_6^{\mathbb{V}}(t,V_3)dW_6(t) + \tilde{\sigma}_9^{\mathbb{V}}(t,V_3)dW_9(t).
\end{aligned} \tag{3.20}$$

Hereafter, we suppress the dependency of the volatility functions $\tilde{\sigma}_i^f$ and $\tilde{\sigma}_i^\lambda$ on V_i for notational simplicity.

The absence of arbitrage implies that there exists an equivalent probability measure $\tilde{\mathbb{P}}$ where for every maturity T there is a $3n$-dimensional process

$$\Phi_n(t) = \{\phi_1(t), \phi_2(t), \ldots, \phi_{3n}(t), t \in [0,T]\},$$

and a strictly positive measurable function $\psi(t,q)$ satisfying the integrability conditions

$$\int_0^t ||\phi_i(s)||^2 ds < \infty, \quad \text{for} \quad i = 1,2,\ldots,3n, \qquad \int_0^t \int_E |\psi(s,q)|h(s,dq)ds < \infty, \tag{3.21}$$

such that

$$d\tilde{W}_i(t) = dW_i(t) - \phi_i(t)dt, \quad \text{for} \quad i = 1, 2, \ldots, 3n, \tag{3.22}$$

is a $\tilde{\mathbb{P}}$-Wiener process and the default indicator process $N(t)$ has a $\tilde{\mathbb{P}}$-intensity

$$\tilde{h}(t, dq) = \psi(t, q)h(t, dq). \tag{3.23}$$

Using Girsanov's theorem (see Heath et al. [1992] and Björk et al. [1997]) and working along the lines of Proposition 2.6 and (2.45), we obtain the multi-dimensional version of the HJM default-free and defaultable forward rate drift restriction, respectively, (see equations (2.41) and (2.45)) as

$$\alpha^f(t,T) = -\sum_{i=1}^{n} \tilde{\sigma}_i^f(t,T)\Big(\phi_i(t) - \int_t^T \tilde{\sigma}_i^f(t,s)ds\Big), \tag{3.24}$$

$$\alpha^d(t,T) =: \alpha^f(t,T) + \alpha^\lambda(t,T) = -\sum_{i=1}^{2n} \tilde{\sigma}_i^d(t,T)\Big(\phi_i(t) - \int_t^T \tilde{\sigma}_i^d(t,s)ds\Big). \tag{3.25}$$

Moreover, by using (3.25) together with (3.24) and (3.19) we derive the credit spread drift restriction which is expressed in terms of the volatilities of the default-free forward rate and the credit spread as

$$\begin{aligned} \alpha^\lambda(t,T) = &-\sum_{i=1}^{2n} \phi_i(t)\tilde{\sigma}_i^\lambda(t,T) + \sum_{i=1}^{2n} \tilde{\sigma}_i^\lambda(t,T)\int_t^T \tilde{\sigma}_i^\lambda(t,s)ds \\ &+\sum_{i=1}^{n}\Big(\tilde{\sigma}_i^\lambda(t,T)\int_t^T \tilde{\sigma}_i^f(t,s)ds + \tilde{\sigma}_i^f(t,T)\int_t^T \tilde{\sigma}_i^\lambda(t,s)ds\Big). \end{aligned} \tag{3.26}$$

By substituting the drift restriction conditions (3.24) and (3.26) into (3.12) and (3.13), respectively, we obtain the dynamics for the forward rate and forward credit spread processes under the risk neutral measure

$$f(t,T) = f(0,T) + \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^f(u,T) \int_u^T \tilde{\sigma}_i^f(u,s) ds du + \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^f(u,T) d\tilde{W}_i(u), \tag{3.27}$$

$$\lambda(t,T) = \lambda(0,T) + \sum_{i=1}^{2n} \int_0^t \tilde{\sigma}_i^\lambda(u,T) \int_u^T \tilde{\sigma}_i^\lambda(u,s) ds du + \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^\lambda(u,T) \int_u^T \tilde{\sigma}_i^f(u,s) ds du$$
$$+ \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^f(u,T) \int_u^T \tilde{\sigma}_i^\lambda(u,s) ds du + \sum_{i=1}^{2n} \int_0^t \tilde{\sigma}_i^\lambda(u,t) d\tilde{W}_i(u). \tag{3.28}$$

Further, by substituting the drift restriction condition (3.25) into (3.18), the defaultable forward rate $f^d(t,T)$ is governed by the system of equations, under the risk-neutral measure,

$$f^d(t,T) = f^d(0,T) + \sum_{i=1}^{2n} \int_0^t \tilde{\sigma}_i^d(u,T) \int_u^T \tilde{\sigma}_i^d(u,s) ds du + \sum_{i=1}^{2n} \int_0^t \tilde{\sigma}_i^d(u,T) d\tilde{W}_i(u), \tag{3.29}$$

where the risk neutral dynamics for the volatility process $\mathbb{V} = \{V_1, V_2, ..., V_n\}$ are expressed as

$$dV_i(t) = \Big[\alpha_i^{\mathbb{V}}(t, V_i) + \sum_{j=1}^{3} \phi_{(j-1)n+i}(t) \tilde{\sigma}_{ij}^{\mathbb{V}}(t, V_i)\Big] dt + \sum_{j=1}^{3} \tilde{\sigma}_{ij}^{\mathbb{V}}(t, V_i) d\tilde{W}_{(j-1)n+i}(t). \tag{3.30}$$

By setting $T = t$ in (3.27) and (3.28), we obtain the risk-neutral dynamics of the short rate and short term spread as

$$r(t) = f(0,t) + \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^f(u,t) \int_u^t \tilde{\sigma}_i^f(u,s) ds du + \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^f(u,t) d\tilde{W}_i(u), \tag{3.31}$$

$$c(t) = \lambda(0,t) + \sum_{i=1}^{2n} \int_0^t \tilde{\sigma}_i^\lambda(u,t) \int_u^t \tilde{\sigma}_i^\lambda(u,s) ds du + \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^\lambda(u,t) \int_u^t \tilde{\sigma}_i^f(u,s) ds du$$
$$+ \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^f(u,t) \int_u^t \tilde{\sigma}_i^\lambda(u,s) ds du + \sum_{i=1}^{2n} \int_0^t \tilde{\sigma}_i^\lambda(u,t) d\tilde{W}_i(u). \tag{3.32}$$

Consequently, the instantaneous defaultable short rate dynamics $r^d(t) = f^d(t,t)$ satisfies the stochastic integral equation

$$r^d(t) = f^d(0,t) + \sum_{i=1}^{2n} \int_0^t \tilde{\sigma}_i^d(u,t) \int_u^t \tilde{\sigma}_i^d(u,s)dsdu + \sum_{i=1}^{2n} \int_0^t \tilde{\sigma}_i^d(u,t)d\tilde{W}_i(u), \tag{3.33}$$

Working along the lines of Proposition 2.5 (but for the multi-dimensional case) and using the drift restriction conditions (3.24) and (3.26), the corresponding dynamics for the defaultable bond price under the risk-neutral measure satisfies the stochastic differential equation

$$\frac{dP^d(t,T)}{P^d(t-,T)} = r(t)dt + \sum_{i=1}^{2n} \tilde{\sigma}_{B,i}^d(t,T)d\tilde{W}_i(t) - d\tilde{M}(\omega,t), \tag{3.34}$$

where the volatility function is given by $\tilde{\sigma}_{B,i}^d(t,T) = -\int_t^T \tilde{\sigma}_i^d(t,s)ds$ and the process $\tilde{M}(\omega,t)$ is defined by

$$d\tilde{M}(\omega,t) = \int_E q\mu(\omega;dt,dq) - \int_E q\tilde{h}(\omega;dt,dq). \tag{3.35}$$

Proposition 3.1 *The defaultable bond price can be expressed as*

$$P^d(t,T) = \mathbb{1}_{\{\tau>t\}}\tilde{\mathbb{E}}\Big[\int_t^T \mathcal{R}(s)\tilde{h}(s,dq)e^{-\int_t^s(r(u,V)+\tilde{h}(u,dq))du}ds\Big|\mathcal{F}_t^W\Big]. \tag{3.36}$$

Proof: Similar to that of Proposition 2.11 but for $2n-$dimensions as given in Appendix A.7. ∎

Under general volatility functions, the defaultable forward rate curve in (3.29) is non-Markovian, thereby leading to computational complexity during derivatives pricing. However in the next section, we propose certain volatility structures that guarantee that the defaultable HJM admits finite dimensional realisations.

3.3 A Specific Volatility Structure

A particular specification of these volatility functions allow us to transform the original non-Markovian structure to Markovian form in line with earlier works in the stochastic volatility yet default-free setting of Chiarella and Kwon [2001] and Björk et al. [2004].

Assumption 3.3.1 *For $1 \leq i \leq n$, the volatility functions are of the form*

$$\sigma_i^f(t,T,V_i) = [a_{0i} + a_{1i}(T-t)]\sqrt{r(t)}\sqrt{V_i(t)}e^{-\kappa_i^f(T-t)}, \tag{3.37}$$

$$\sigma_i^\lambda(t,T,V_i) = [b_{0i} + b_{1i}(T-t)]\sqrt{c(t)}\sqrt{V_i(t)}e^{-\kappa_i^\lambda(T-t)}, \tag{3.38}$$

where κ_i^f, κ_i^λ, a_{0i}, a_{1i}, b_{0i} and b_{1i} are constants.

This class of volatility functions gives rise to a high degree of flexibility in modelling the wide range of shapes of the yield curve by virtue of the polynomial in the deterministic part. These volatility specifications are level dependent and involve unspanned stochastic volatility factors. In addition, the specification allows for hump-shaped shocks that would be essential in matching interest rate derivatives empirically.

Proposition 3.2 *Under the volatility specification for the default-free forward rate and forward credit spread as specified as in Assumption 3.3.1, the default-free forward rate $f(t,T)$ is expressed as*

$$f(t,T) = f(0,T) + \sum_{i=1}^{n} B_{x_{1i}}(T-t)x_{1i}(t) + \sum_{i=1}^{n}\sum_{j=1}^{6} B_{\Phi_{ji}}(T-t)\Phi_{ji}(t), \tag{3.39}$$

and the forward credit spread $\lambda(t,T)$ is expressed as

$$\lambda(t,T) = \lambda(0,T) + \sum_{i=1}^{n}\sum_{j=2}^{3} B_{x_{ji}}(T-t)x_{ji}(t) + \sum_{i=1}^{n}\sum_{j=7}^{20} B_{\Phi_{ji}}(T-t)\Phi_{ji}(t). \tag{3.40}$$

Thus, the defaultable forward rate $f^d(t,T)$ is expressed as

$$f^d(t,T) = f^d(0,T) + \sum_{i=1}^{n}\sum_{j=1}^{3} B_{x_{ji}}(T-t)x_{ji}(t) + \sum_{i=1}^{n}\sum_{j=1}^{20} B_{\Phi_{ji}}(T-t)\Phi_{ji}(t), \qquad (3.41)$$

where

$$\begin{cases} B_{x_{1i}}(T-t) = [a_{0i} + a_{1i}(T-t)]e^{-\kappa_i^f(T-t)}, \\ B_{x_{2i}}(T-t) = z_i^{\lambda_1}[b_{0i} + b_{1i}(T-t)]e^{-\kappa_i^\lambda(T-t)}, \\ B_{x_{3i}}(T-t) = z_i^{\lambda_2}[b_{0i} + b_{1i}(T-t)]e^{-\kappa_i^\lambda(T-t)}, \\ B_{\Phi_{1i}}(T-t) = z_i^{f_1}a_{1i}e^{-\kappa_i^f(T-t)}, \\ B_{\Phi_{2i}}(T-t) = \dfrac{a_{1i}}{\kappa_i^f}\Big(\dfrac{1}{\kappa_i^f} + \dfrac{a_{0i}}{a_{1i}}\Big)[a_{0i} + a_{1i}(T-t)]e^{-\kappa_i^f(T-t)}, \\ B_{\Phi_{3i}}(T-t) = -\Big[\dfrac{a_{1i}a_{1i}}{\kappa_i^f}\Big(\dfrac{1}{\kappa_i^f} + \dfrac{a_{0i}}{a_{1i}}\Big) \\ \qquad\qquad + \dfrac{a_{1i}}{\kappa_i^f}\Big(\dfrac{a_{1i}}{\kappa_i^f} + 2a_{0i}\Big)(T-t) + \dfrac{(a_{1i})^2}{\kappa_i^f}(T-t)^2\Big]e^{-2\kappa_i^f(T-t)}, \\ B_{\Phi_{4i}}(T-t) = \dfrac{(a_{1i})^2}{\kappa_i^f}\Big(\dfrac{1}{\kappa_i^f} + \dfrac{a_{0i}}{a_{1i}}\Big)e^{-\kappa_i^f(T-t)}, \\ B_{\Phi_{5i}}(T-t) = -\dfrac{a_{1i}}{\kappa_i^f}\Big[\dfrac{a_{1i}}{\kappa_i^f} + 2a_{0i} + 2a_{1i}(T-t)\Big]e^{-2\kappa_i^f(T-t)}, \\ B_{\Phi_{6i}}(T-t) = -\dfrac{(a_{1i})^2}{\kappa_i^f}e^{-2\kappa_i^f(T-t)}, \end{cases} \qquad (3.42)$$

$$
\begin{cases}
B_{\Phi_{7i}}(T-t) = z_i^{\lambda_1}\Big[\frac{a_{0i}b_{0i}}{\kappa_i^f} + \frac{a_{1i}b_{0i}}{(\kappa_i^f)^2} + \Big(\frac{a_{0i}b_{1i}}{\kappa_i^f} + \frac{a_{1i}b_{1i}}{(\kappa_i^f)^2}\Big)(T-t)\Big]e^{-\kappa_i^\lambda(T-t)}, \\
B_{\Phi_{8i}}(T-t) = -z_i^{\lambda_1}\Big[\frac{a_{0i}b_{0i}}{\kappa_i^f} + \frac{a_{1i}b_{0i}}{(\kappa_i^f)^2} + \Big(\frac{a_{1i}b_{0i}}{\kappa_i^f} + \frac{a_{0i}b_{1i}}{\kappa_i^f} + \frac{a_{1i}b_{1i}}{(\kappa_i^f)^2}\Big)(T-t) \\
\qquad + \frac{a_{1i}b_{1i}}{\kappa_i^f}(T-t)^2\Big]e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}, \\
B_{\Phi_{9i}}(T-t) = z_i^{\lambda_1}\Big(\frac{a_{0i}b_{0i}}{\kappa_i^f} + \frac{a_{1i}b_{1i}}{(\kappa_i^f)^2}\Big)e^{-\kappa_i^\lambda(T-t)}, \\
B_{\Phi_{10i}}(T-t) = -z_i^{\lambda_1}\Big[\frac{a_{1i}b_{0i}}{\kappa_i^f} + \frac{a_{0i}b_{1i}}{\kappa_i^f} + \frac{a_{1i}b_{1i}}{(\kappa_i^f)^2} - 2\frac{a_{1i}b_{1i}}{\kappa_i^f}(T-t)\Big]e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}, \\
B_{\Phi_{11i}}(T-t) = -z_i^{\lambda_1}\frac{a_{1i}b_{1i}}{\kappa_i^f}e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}, \\
B_{\Phi_{12i}}(T-t) = z_i^{\lambda_1}\Big[\frac{a_{0i}b_{0i}}{\kappa_i^\lambda} + \frac{a_{0i}b_{1i}}{(\kappa_i^\lambda)^2} + \Big(\frac{a_{1i}b_{0i}}{\kappa_i^\lambda} + \frac{a_{1i}b_{1i}}{(\kappa_i^\lambda)^2}\Big)(T-t)\Big]e^{-\kappa_i^f(T-t)}, \\
B_{\Phi_{13i}}(T-t) = z_i^{\lambda_1}\Big(\frac{a_{0i}b_{0i}}{\kappa_i^\lambda} + \frac{a_{1i}b_{1i}}{(\kappa_i^\lambda)^2}\Big)e^{-\kappa_i^f(T-t)},
\end{cases}
\tag{3.43}
$$

$$
\begin{cases}
B_{\Phi_{14i}}(T-t) = \frac{b_{1i}}{\kappa_i^\lambda}\Big(\frac{1}{\kappa_i^\lambda} + \frac{b_{0i}}{b_{1i}}\Big)[b_{0i} + b_{1i}(T-t)]e^{-\kappa_i^\lambda(T-t)}, \\
B_{\Phi_{15i}}(T-t) = \Big[\frac{b_{1i}b_{1i}}{\kappa_i^\lambda}\Big(\frac{1}{\kappa_i^\lambda} + \frac{b_{0i}}{b_{1i}}\Big) + \frac{b_{1i}}{\kappa_i^\lambda}\Big(\frac{b_{1i}}{\kappa_i^\lambda} + 2b_{0i}\Big)(T-t) + \frac{(b_{1i})^2}{\kappa_i^\lambda}(T-t)^2\Big]e^{-2\kappa_i^\lambda(T-t)}, \\
B_{\Phi_{16i}}(T-t) = \frac{(b_{1i})^2}{\kappa_i^\lambda}\Big(\frac{1}{\kappa_i^\lambda} + \frac{b_{0i}}{b_{1i}}\Big)e^{-\kappa_i^\lambda(T-t)}, \\
B_{\Phi_{17i}}(T-t) = -\frac{b_{1i}}{\kappa_i^\lambda}\Big[\frac{b_{1i}}{\kappa_i^\lambda} + 2b_{0i} + 2b_{1i}(T-t)\Big]e^{-2\kappa_i^\lambda(T-t)}, \\
B_{\Phi_{18i}}(T-t) = -\frac{(b_{1i})^2}{\kappa_i^\lambda}e^{-2\kappa_i^\lambda(T-t)}, \\
B_{\Phi_{19i}}(T-t) = z_i^{\lambda_1}b_{1i}e^{-\kappa_i^\lambda(T-t)}, \\
B_{\Phi_{20i}}(T-t) = z_i^{\lambda_2}b_{1i}e^{-\kappa_i^\lambda(T-t)},
\end{cases}
\tag{3.44}
$$

and the state variables $x_{ji}(t)$ and $\Phi_{ji}(t)$ satisfy the stochastic differential equations given in Corollary 3.3.

Proof: See Appendix B.1 for the technical details. ■

Corollary 3.3 *The state variables $x_{ji}(t)$ and $\Phi_{ji}(t)$ satisfy the stochastic differential equa-*

tions

$$\left\{\begin{aligned}
&dx_{1i}(t) = -\kappa_i^f x_{1i}(t)dt + \sqrt{r(t)V_i(t)}d\tilde{W}_i(t),\\
&dx_{2i}(t) = -\kappa_i^\lambda x_{2i}(t)dt + \sqrt{c(t)V_i(t)}d\tilde{W}_i(t),\\
&dx_{3i}(t) = -\kappa_i^\lambda x_{3i}(t)dt + \sqrt{c(t)V_i(t)}d\tilde{W}_{n+i}(t),\\
&d\Phi_{1i}(t) = \left[x_{1i}(t) - \kappa_i^f \Phi_{1i}(t)\right]dt, \quad d\Phi_{2i}(t) = \left[r(t)V_i(t) - \kappa_i^f \Phi_{2i}(t)\right]dt,\\
&d\Phi_{3i}(t) = \left[r(t)V_i(t) - 2\kappa_i^f \Phi_{3i}(t)\right]dt, \quad d\Phi_{4i}(t) = \left[\Phi_{2i}(t) - \kappa_i^f \Phi_{4i}(t)\right]dt,\\
&d\Phi_{5i}(t) = \left[\Phi_{3i}(t) - 2\kappa_i^f \Phi_{5i}(t)\right]dt, \quad d\Phi_{6i}(t) = \left[2\Phi_{5i}(t) - 2\kappa_i^f \Phi_{6i}(t)\right]dt,\\
&d\Phi_{7i}(t) = \left[V_i(t)\sqrt{r(t)c(t)} - \kappa_i^\lambda \Phi_{7i}(t)\right]dt, \quad d\Phi_{8i}(t) = \left[V_i(t)\sqrt{r(t)c(t)} - (\kappa_i^f + \kappa_i^\lambda)\Phi_{8i}(t)\right]dt,\\
&d\Phi_{9i}(t) = \left[\Phi_{9i}(t) - \kappa_i^\lambda \Phi_{9i}(t)\right]dt, \quad d\Phi_{10i}(t) = \left[\Phi_{10i}(t) - (\kappa_i^f + \kappa_i^\lambda)\Phi_{10i}(t)\right]dt,\\
&d\Phi_{11i}(t) = \left[2\Phi_{12i}(t) - (\kappa_i^f + \kappa_i^\lambda)\Phi_{11i}(t)\right]dt, \quad d\Phi_{12i}(t) = \left[V_i(t)\sqrt{r(t)c(t)} - \kappa_i^f \Phi_{12i}(t)\right]dt,\\
&d\Phi_{13i}(t) = \left[\Phi_{14i}(t) - \kappa_i^f \Phi_{13i}(t)\right]dt, \quad d\Phi_{14i}(t) = \left[c(t)V_i(t) - \kappa_i^\lambda \Phi_{14i}(t)\right]dt,\\
&d\Phi_{15i}(t) = \left[c(t)V_i(t) - 2\kappa_i^\lambda \Phi_{15i}(t)\right]dt, \quad d\Phi_{16i}(t) = \left[\Phi_{16i}(t) - \kappa_i^\lambda \Phi_{16i}(t)\right]dt,\\
&d\Phi_{17i}(t) = \left[\Phi_{15i}(t) - 2\kappa_i^\lambda \Phi_{17i}(t)\right]dt, \quad d\Phi_{18i}(t) = \left[2\Phi_{17i}(t) - 2\kappa_i^\lambda \Phi_{18i}(t)\right]dt,\\
&d\Phi_{19i}(t) = \left[x_{2i}(t) - \kappa_i^\lambda \Phi_{19i}(t)\right]dt, \quad d\Phi_{20i}(t) = \left[x_{3i}(t) - \kappa_i^\lambda \Phi_{20i}(t)\right]dt,
\end{aligned}\right. \tag{3.45}$$

subject to the initial conditions $x_{ji}(0) = \Phi_{ji}(0) = 0$ *for* $i = 1, ..., n$ *and* $j = 1, ..., 20$.

Proof: Take the stochastic differential of (B.1.4), (B.1.10) and (B.1.12) in Appendix B.1 to obtain the stochastic differential equations.

Corollary 3.4 *The short rate and the short term credit spread processes can be expressed as*

$$\begin{aligned}
r(t) &= f(0,t) + \sum_{i=1}^{n} \alpha_{1i} x_{1i}(t) + \sum_{i=1}^{n}\sum_{j=1}^{6} \beta_{ji}\Phi_{ji}(t),\\
c(t,T) &= \lambda(0,T) + \sum_{i=1}^{n}\sum_{j=2}^{3} \alpha_{ji} x_{ji}(t) + \sum_{i=1}^{n}\sum_{j=7}^{20} \beta_{ji}\Phi_{ji}(t),
\end{aligned} \tag{3.46}$$

where $\alpha_{ji} = B_{x_{ji}}(0)$ *and* $\beta_{ji} = B_{\Phi_{ji}}(0)$.

Proof: By using (3.39) and $r(t) = f(t,t)$, as well as (3.40) and $c(t) = \lambda(t,t)$. ■

For simplicity, we will assume that on specifying the market price of risk $\phi_{(j-1)n+i}(t)$, the drift function given in (3.30) can be simplified to the general form $\kappa_i^{\mathbb{V}}(\bar{V}_i - V_i)$ and (3.30) could be written as

$$dV_i(t) = \kappa_i^{\mathbb{V}}(\bar{V}_i - V_i)dt + \sum_{j=1}^{3} \bar{\sigma}_{ij}^{\mathbb{V}} \sqrt{V_i(t)} d\tilde{W}_{(j-1)n+i}(t). \tag{3.47}$$

Figure 3.1 illustrates a possible evolution of the defaultable forward rate curve surface (3.41) for $n = 1$. The surface is generated over a maturity $T = 2$ by assuming that the initial term structures of the forward rate and forward credit spread are given by

$$f(0,T) = 0.05 - 0.03\sqrt{V(0)}e^{-0.18T} \quad \text{and} \quad \lambda(0,T) = 0.03 - 0.01\sqrt{V(0)}e^{-0.16T},$$

respectively with the initial volatility chosen to be $V(0) = 0.08$. In addition, Table 3.1 specifies the parameter values used in the illustration. [20]

a_{01}	a_{11}	b_{01}	b_{11}	$\bar{V}$	$\tilde{\sigma}^{\mathbb{V}}$	κ_V	κ_f	κ_λ	$\rho^{\mathbb{V}\lambda}$	$\rho^{\mathbb{V}f}$	$\rho^{f\lambda}$
0.0045	0.0131	0.0025	0.011	0.0857	0.096	0.85	0.3341	0.25	0.2720	0.4615	-0.40

Table 3.1: The parameter values used in simulating forward rate and price surfaces, where $\tilde{\sigma}^{\mathbb{V}} = \{\bar{\sigma}_{11}^{\mathbb{V}}, \bar{\sigma}_{12}^{\mathbb{V}}, \bar{\sigma}_{13}^{\mathbb{V}}\}$.

As evident from Figure 3.1, the proposed hump-shaped level dependent stochastic volatility model can generate a variety of shapes for the defaultable forward rate curve.

Proposition 3.5 *If the default-free forward rate dynamics satisfies the dynamics in (3.39),*

[20] The model is sensitive to the choice of the parameters a_{01}, a_{11}, b_{01}, b_{11}, κ_f and κ_λ. A parameter estimation exercise would add a lot of value to the simulation results, more so for the forward credit spread dynamics where this (to the best of our knowledge) has not been attempted within the HJM framework with stochastic hump shaped volatilities. For the default-free parameters, we experimented with the parameters estimated in Trolle and Schwartz [2009].

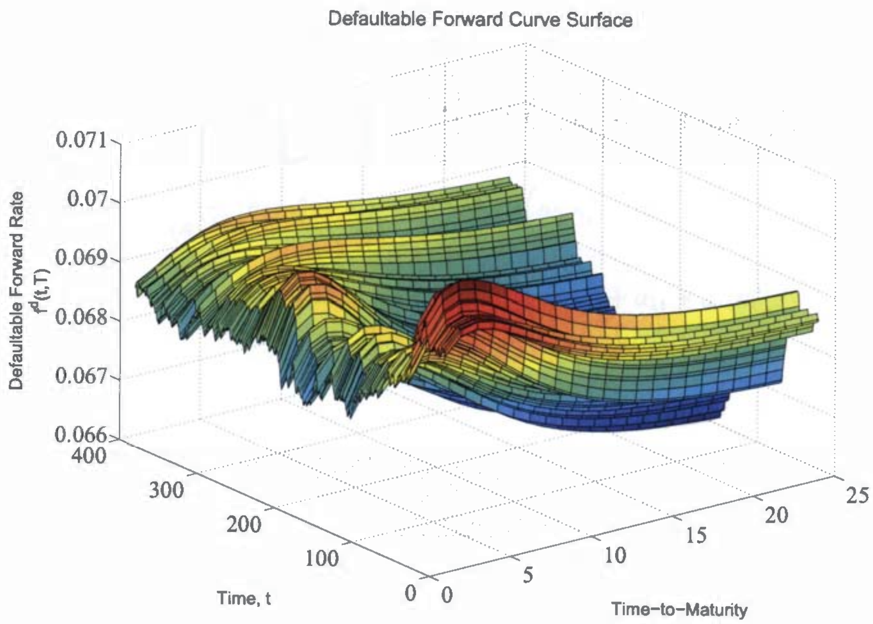


Figure 3.1: A sample evolution of the defaultable forward curve surface simulated using (3.41) for $T = 2$ years.

the price of a default-free bond is expressed in the exponential affine form and of the form

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\Big(- \sum_{i=1}^{n} D_{x_{1i}}(T-t)x_{1i}(t) - \sum_{i=1}^{n}\sum_{j=1}^{6} D_{\Phi_{ji}}(T-t)\Phi_{ji}(t)\Big). \quad (3.48)$$

In addition, assuming that the defaultable forward rate process satisfies the dynamics given in Proposition 3.2, then the defaultable bond price $P^d(t,T) = \mathcal{R}(t)\bar{P}^d(t,T)$ is exponential affine and has the representation

$$\bar{P}^d(t,T) = \frac{\bar{P}^d(0,T)}{\bar{P}^d(0,t)} \exp\Big(- \sum_{i=1}^{n}\sum_{j=1}^{3} D_{x_{ji}}(T-t)x_{ji}(t) - \sum_{i=1}^{n}\sum_{j=1}^{20} D_{\Phi_{ji}}(T-t)\Phi_{ji}(t)\Big), \quad (3.49)$$

where $x_{ji}(t)$ and $\Phi_{ji}(t)$ are specified in Corollary 3.3 and the deterministic functions $D_{x_{ji}}(T-$

t) and $D_{\Phi_{ji}}(T-t)$ *are given by*

$$\begin{cases} D_{x_{1i}}(T-t) = \dfrac{z_i^{f_1}}{(\kappa_i^f)^2}\Big[a_{0i}\kappa_i^f + a_{1i} - e^{-\kappa_i^f(T-t)}\Big(a_{0i}\kappa_i^f + a_{1i} + a_{1i}\kappa_i^f(T-t)\Big)\Big], \\ D_{x_{2i}}(T-t) = \dfrac{z_i^{\lambda_1}}{(\kappa_i^\lambda)^2}\Big[a_{0i}\kappa_i^\lambda + a_{1i} - e^{-\kappa_i^\lambda(T-t)}\Big(a_{0i}\kappa_i^\lambda + a_{1i} + a_{1i}\kappa_i^\lambda(T-t)\Big)\Big], \\ D_{x_{3i}}(T-t) = \dfrac{z_i^{\lambda_2}}{(\kappa_i^\lambda)^2}\Big[a_{0i}\kappa_i^\lambda + a_{1i} - e^{-\kappa_i^\lambda(T-t)}\Big(a_{0i}\kappa_i^\lambda + a_{1i} + a_{1i}\kappa_i^\lambda(T-t)\Big)\Big], \end{cases} \tag{3.50}$$

$$\begin{cases} D_{\Phi_{1i}}(T-t) = \dfrac{z_i^{f_1}a_{1i}}{\kappa_i^f}\big(1 - e^{-\kappa_i^f(T-t)}\big), \\ D_{\Phi_{2i}}(T-t) = \Big(\dfrac{a_{1i}}{\kappa_i^f}\Big)^2\Big(\dfrac{1}{\kappa_i^f} + \dfrac{a_{0i}}{a_{1i}}\Big)\Big[\Big(\dfrac{1}{\kappa_i^f} + \dfrac{a_{0i}}{a_{1i}}\Big)\Big(e^{-\kappa_i^f(T-t)} - 1\Big) + (T-t)e^{-\kappa_i^f(T-t)}\Big], \\ D_{\Phi_{3i}}(T-t) = -\Big(\dfrac{a_{1i}}{(\kappa_i^f)^2}\Big)\Big[\Big(\dfrac{a_{1i}}{2(\kappa_i^f)^2} + \dfrac{a_{0i}}{\kappa_i^f} + \dfrac{(a_{0i})^2}{2a_{1i}}\Big)\big(e^{-2\kappa_i^f(T-t)} - 1\big) \\ \qquad + \Big(\dfrac{a_{1i}}{\kappa_i^f} + a_{0i}\Big)(T-t)e^{-2\kappa_i^f(T-t)} + \dfrac{a_{1i}}{2}(T-t)^2 e^{-2\kappa_i^f(T-t)}\Big], \\ D_{\Phi_{4i}}(T-t) = \Big(\dfrac{a_{1i}}{\kappa_i^f}\Big)^2\Big(\dfrac{1}{\kappa_i^f} + \dfrac{a_{0i}}{a_{1i}}\Big)\big(e^{-\kappa_i^f(T-t)} - 1\big), \\ D_{\Phi_{5i}}(T-t) = -\Big(\dfrac{a_{1i}}{(\kappa_i^f)^2}\Big)\Big[\Big(\dfrac{a_{1i}}{\kappa_i^f} + a_{0i}\Big)\big(e^{-2\kappa_i^f(T-t)} - 1\big) + a_{1i}(T-t)e^{-2\kappa_i^f(T-t)}\Big], \\ D_{\Phi_{6i}}(T-t) = -\dfrac{1}{2}\Big(\dfrac{a_{1i}}{\kappa_i^f}\Big)^2\big(e^{-2\kappa_i^f(T-t)} - 1\big), \end{cases} \tag{3.51}$$

$$
\begin{cases}
D_{\Phi_{7i}}(T-t) = \frac{z_i^{\lambda_1}}{\kappa_i^f \kappa_i^\lambda}\Big[\Big(a_{0i} + \frac{a_{1i}}{k_i^f}\Big) b_{1i}(T-t)e^{-\kappa_i^\lambda (T-t)}\Big], \\
D_{\Phi_{8i}}(T-t) = -\frac{z_i^{\lambda_1}}{\kappa_i^f + \kappa_i^\lambda}\Big\{\Big(\frac{a_{0i}b_{0i}}{\kappa_i^f} \\
\qquad + \frac{a_{1i}b_{0i}}{(\kappa_i^f)^2}\Big)\Big(e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)} - 1\Big) + \frac{1}{\kappa_i^f + \kappa_i^\lambda}\Big(\frac{a_{1i}b_{0i}}{\kappa_i^f} + \frac{a_{0i}b_{1i}}{\kappa_i^f} + \frac{a_{1i}b_{1i}}{(\kappa_i^f)^2}\Big) \\
\qquad \Big[1 - e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)} - (\kappa_i^f + \kappa_i^\lambda)(T-t)e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\Big] + \frac{1}{\kappa_i^f + \kappa_i^\lambda}\Big(\frac{a_{1i}b_{1i}}{\kappa_i^f}\Big) \\
\qquad \Big[2 - e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\Big(2 - (\kappa_i^f + \kappa_i^\lambda)(T-t)(2 - (\kappa_i^f + \kappa_i^\lambda)(T-t))\Big)\Big]\Big\}, \\
D_{\Phi_{9i}}(T-t) = \frac{z_i^{\lambda_1}}{\kappa_i^\lambda}\Big(\frac{a_{0i}b_{0i}}{\kappa_i^f} + \frac{a_{1i}b_{1i}}{(\kappa_i^f)^2}\Big)\big(1 - e^{-\kappa_i^\lambda (T-t)}\big), \\
D_{\Phi_{10i}}(T-t) = -\frac{z_i^{\lambda_1}}{\kappa_i^f(\kappa_i^f + \kappa_i^\lambda)}\Big\{\Big((a_{0i}b_{1i} + \frac{a_{1i}b_{1i}}{\kappa_i^f}\Big)\Big(b_{0i} - 2\frac{b_{1i}}{\kappa_i^f + \kappa_i^\lambda}\Big)\Big(1 - e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\Big) \\
\qquad + 2b_{1i}(T-t)e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\Big\}, \\
D_{\Phi_{11i}}(T-t) = -\frac{z_i^{\lambda_1}}{(\kappa_i^f + \kappa_i^\lambda)}\frac{a_{1i}b_{1i}}{\kappa_i^f}\big(1 - e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\big), \\
D_{\Phi_{12i}}(T-t) = \frac{z_i^{\lambda_1}}{\kappa_i^f \kappa_i^\lambda}\Big[\Big(b_{0i} + \frac{b_{1i}}{k_i^f}\Big)a_{1i}(T-t)e^{-\kappa_i^f (T-t)}\Big], \\
D_{\Phi_{13i}}(T-t) = \frac{z_i^{\lambda_1}}{\kappa_i^f}\Big(\frac{a_{0i}b_{0i}}{\kappa_i^\lambda} + \frac{a_{1i}b_{1i}}{(\kappa_i^\lambda)^2}\Big)\big(1 - e^{-\kappa_i^f (T-t)}\big),
\end{cases}
\tag{3.52}
$$

and finally we have

$$
\begin{cases}
D_{\Phi_{14i}}(T-t) = \Big(\frac{b_{1i}}{\kappa_i^\lambda}\Big)^2\Big(\frac{1}{\kappa_i^\lambda} + \frac{b_{0i}}{b_{1i}}\Big)\Big[\Big(\frac{1}{\kappa_i^\lambda} + \frac{b_{0i}}{b_{1i}}\Big)\Big(e^{-\kappa_i^\lambda (T-t)} - 1\Big) + (T-t)e^{-\kappa_i^\lambda (T-t)}\Big], \\
D_{\Phi_{15i}}(T-t) = -\Big(\frac{b_{1i}}{(\kappa_i^\lambda)^2}\Big)\Big[\Big(\frac{b_{1i}}{2(\kappa_i^\lambda)^2} + \frac{b_{0i}}{\kappa_i^\lambda} + \frac{(b_{0i})^2}{2b_{1i}}\Big)\big(e^{-2\kappa_i^\lambda (T-t)} - 1\big) \\
\qquad + \Big(\frac{b_{1i}}{\kappa_i^\lambda} + b_{0i}\Big)(T-t)e^{-2\kappa_i^\lambda (T-t)} + \frac{b_{1i}}{2}(T-t)^2 e^{-2\kappa_i^\lambda (T-t)}\Big], \\
D_{\Phi_{16i}}(T-t) = \Big(\frac{b_{1i}}{\kappa_i^\lambda}\Big)^2\Big(\frac{1}{\kappa_i^\lambda} + \frac{b_{0i}}{b_{1i}}\Big)\big(e^{-\kappa_i^\lambda (T-t)} - 1\big), \\
D_{\Phi_{17i}}(T-t) = -\Big(\frac{b_{1i}}{(\kappa_i^\lambda)^2}\Big)\Big[\Big(\frac{b_{1i}}{\kappa_i^\lambda} + b_{0i}\Big)\big(e^{-2\kappa_i^\lambda (T-t)} - 1\big) + b_{1i}(T-t)e^{-2\kappa_i^\lambda (T-t)}\Big], \\
D_{\Phi_{18i}}(T-t) = -\frac{1}{2}\Big(\frac{b_{1i}}{\kappa_i^\lambda}\Big)^2\big(e^{-2\kappa_i^\lambda (T-t)} - 1\big), \quad D_{\Phi_{19i}}(T-t) = \frac{z_i^{\lambda_1} b_{1i}}{\kappa_i^\lambda}\big(1 - e^{-\kappa_i^\lambda (T-t)}\big), \\
D_{\Phi_{20i}}(T-t) = \frac{z_i^{\lambda_2} b_{1i}}{\kappa_i^\lambda}\big(1 - e^{-\kappa_i^\lambda (T-t)}\big).
\end{cases}
\tag{3.53}
$$

Proof: See Appendix B.2. ■

Figure 3.2 illustrates the price surface for defaultable bonds with maturity $T = 10$ years using (3.49).

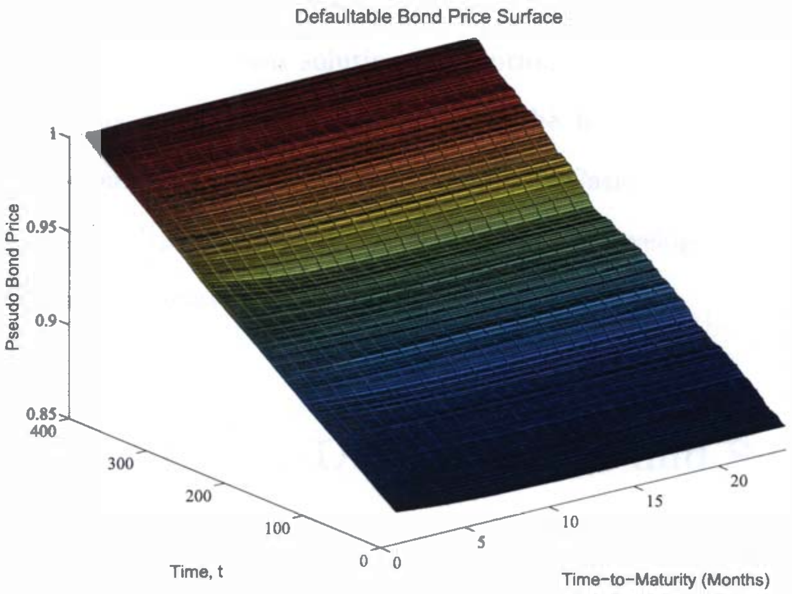


Figure 3.2: Price surface for the pseudo bond for a possible evolution of the state variables $x_{ji}(t)$ and $\Phi_{ji}(t)$ for $T = 2$ years.

The default-free bond price can be expressed in terms of $8n = 7n + n$ state variables while the defaultable bond price can be expressed in terms of $24n = 23n + n$ state variables, where the n state variables are associated with n stochastic volatility variables. Both the default-free and defaultable bond prices are exponential affine functions of these state variables and the $7n$ state variables driving the default-free prices constitute a common set for both default-free and defaultable bond prices. Even though the dimensions of the state space are relatively large, the driving sources of uncertainty of the entire state space are only $3n$. Thus assessing the model's flexibility and suitability for estimation or calibration applications, this Markovian defaultable model can be computationally moderate. Note that the model can be easily adjusted to accommodate exponentially decaying volatility functional forms for the forward rates and credit spreads resulting in a Markovian system with lower dimension state space as shown in Chapter 2.

The proposed defaultable term structure model will be used next to explicitly price credit derivatives, for instance, credit default swaps and swaptions. In addition, we demonstrate in Section 3.5 that by excluding the level dependence on the forward rate and credit spread volatility specifications, semi-closed solutions for options of defaultable bonds can be derived. Pricing options on defaultable bonds under the broad volatility specifications in Assumption 3.3.1 is perfomed only numerically via Monte Carlo simulations, see for instance Chiarella et al. [2011], where a defaultable term structure is developed with level dependent volatilities yet without stochastic volatility.

3.4 Pricing of Credit Default Swaps and Swaptions

In this section, we derive pricing formulas for single-name credit default swaps (CDS) and default swaptions. For simplicity, the main focus is on the case where there is no counterparty risk although we also show that this could also be incorporated into the pricing model.

3.4.1 CDS with no Counterparty Risk

A CDS contract involves three parties, the insurance buyer or insured party, the insurance seller or insurer and the reference obligor. A CDS contract with maturity T allows the insured party to receive protection, up to time T, from the insurer against default of the reference obligor. The insured party pays a regular fee π (*premium*) to the insurer in return of a protection payment upon default. In the absence of counterparty risk, default time $\tau < T$ represents the time that the reference obligor fails to make the required payments on the structured reference bonds. When $t < \tau \leq T$, then the insurer has to make a protection payment $(1 - \mathcal{R})$ at default time τ, where $\mathcal{R}$ is the recovery rate prevailing at the default time τ, which we assume to be given.[21] Then the discounted payoff of the protection leg,

[21] We note that although we use $\mathcal{R}$ for the recovery rate of the underlying risky asset, this is not the recovery rate as used in Section 3.3.

under the physical settlement assumption,[22] at $t < \tau$ is

$$W_{prt}(t) = e^{-\int_t^\tau r(s)ds}(1-\mathcal{R})\mathbb{1}_{\{t<\tau\leq T\}}. \tag{3.54}$$

The insured party pays the premium π at times $t_i, i = 1, 2, ..., N$ until either the contract maturity $t_N = T$, if no default occurs, or until default, if $\tau \leq T$. By denoting as $\delta_i = t_{i-1} - t_i$, then the value at time $t < t_1$ of the premium leg, including the accrual payment for the fraction of time in which default occurs, is given by

$$W_{prm}(t) = \pi \sum_{i=1}^{N} \delta_i e^{-\int_t^{t_i} r(s)ds}\mathbb{1}_{\{\tau>t_i\}} + \pi(\tau - t_{\tau-1})e^{-\int_t^\tau r(s)ds}\mathbb{1}_{\{t<\tau<T\}}, \tag{3.55}$$

where t_1 is the first premium payment date and $t_{\tau-1}$ is the last premium payment date t_i before default time τ.

For notational simplicity, we assume that the default intensity (under the risk-neutral measure) is $\tilde{h}(t, dq) = \tilde{h}(t)$. Under no-arbitrage pricing, the value of the CDS can be expressed under the risk-neutral probability measure as

$$CDS_\pi(t) = \tilde{\mathbb{E}}\Big[W_{prt}(t) - W_{prm}(t)\Big|\mathcal{F}_t\Big]. \tag{3.56}$$

The fair premium rate $\tilde{\pi}(t)$, the so called CDS spread, is the rate that will make the value of the CDS equal to zero.

Proposition 3.6 *When the contract is settled (that is, the protection payment is made)*

[22] Default can be settled *physically* where A and B exchange one of the specified reference bonds at its par value or alternatively as a *cash-settlement* as is the common market practice. In this case, several independent dealers are asked to provide quotes on the defaulted bond, and party B pays party A the difference between the average quoted value and the par bond value. In a CDS, the protection buyer is effectively long on a delivery option which gives the buyer the right to deliver the 'cheapest-to-deliver' asset to the protection seller.

immediately on default of the reference obligor, the CDS spread is given by

$$\tilde{\pi}(t) = \frac{\mathbb{1}_{\{\tau>t\}}(1-\mathcal{R})\int_t^T \tilde{\mathbb{E}}\left[\tilde{h}(u)e^{-\int_t^u (r(s)+\tilde{h}(s))ds}\middle|\mathcal{F}_t^W\right]du}{\mathbb{1}_{\{\tau>t\}}\sum_{i=1}^N \delta_i \bar{P}^d(t,t_i) + \tilde{\mathbb{E}}\left[(\tau - t_{\tau-1})e^{-\int_t^\tau r(s)ds}\mathbb{1}_{\{t<\tau<T\}}\middle|\mathcal{F}_t\right]}. \tag{3.57}$$

Proof: See Appendix B.3.3. ■

Instead of allowing the protection payment to be made on default time τ, the protection payment could be deferred to the first premium payment date t_i following default time $(t_i > \tau)$. This gives rise to the postponed running CDS whose main advantage is that the absence of accrued-interest term in $(\tau - t_{\tau-1})$ ensures that all payments occur at the canonical grid of the t_i's. We then have the following result.

Corollary 3.7 *By assuming that the protection payment is postponed to the first premium payment date t_i following default time, the CDS spread can be approximated by*

$$\tilde{\pi}(t) \approx \frac{(1-\mathcal{R})\sum_{i=1}^N \left[\bar{P}^d(t,t_{i-1}) - \bar{P}^d(t,t_i)\right]}{\sum_{i=1}^N (t_i - t_{i-1})\bar{P}^d(t,t_i)}. \tag{3.58}$$

Proof: Work along the lines of Brigo and Morini [2005]. See also Appendix B.3.5. ■

Numerical Study - CDS

The shape of the credit curves is influenced by the demand and supply for credit protection in the CDS market and reflects the credit quality of the reference entities. Unless otherwise stated, we assume the parameter values given in Table 3.2 with $N = 400$ and a maturity $T = 2$ of the underlying defaultable bond.

a_{01}	a_{11}	b_{01}	b_{11}	$\bar{V}$	$\tilde{\sigma}^{\mathbf{V}}$	κ_V	κ_f	κ_λ	$\rho^{\mathbf{V}\lambda}$	$\rho^{\mathbf{V}f}$	$\rho^{f\lambda}$
0.158	0.0139	0.021	0.0139	0.7542	0.6	2.1476	0.8	0.95	0.2720	0.4615	-0.40

Table 3.2: The parameter values used in simulating CDS and CDS option.

In Table 3.3, we give the simulated results of the CDS spread for a contract on a defaultable zero coupon bond obtained from (3.58), given a recovery rate of 40%,[23] under varying correlation $\rho^{f\lambda} \in \{-0.8, 0.8\}$ between short term credit spread and the short interest rates. Although Krekel and Wenzel [2006] argued that the effect of this correlation is not very significant and need not be considered when calculating CDS spread, our results suggests the contrary.

$\rho^{f\lambda}$	- 0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
$\bar{\pi}(t), (Bps)$	412	391	381	379	384	393	408	430	462

Table 3.3: Numerical results on the CDS spread under varying correlation $\rho^{f\lambda}$.

We observe that varying the correlation $\rho^{f\lambda}$ from negative to positive leads to an increase in the CDS spread, after an initial decline. However, increasing the correlations $\rho^{\mathbb{V}\lambda}$ and $\rho^{\mathbb{V}f}$ leads to a decrease in the CDS spread. Table 3.4 illustrates the effect of varying $\rho^{\mathbb{V}f}$ while holding $\rho^{f\lambda} = -0.4$ and $\rho^{\mathbb{V}\lambda} = 0.2720$.

$\rho^{f\mathbb{V}}$	- 0.8	-0.6	-0.3	0	0.3	0.6	0.8
$\bar{\pi}(t), (Bps)$	414	407	398	391	384	379	378

Table 3.4: Numerical results on the CDS spread under varying correlation $\rho^{f\mathbb{V}}$.

We also investigated the effect of the volatility of volatility $\bar{\sigma}^{\mathbb{V}}_{1j}$, $(j = 1, 2, 3)$ on the CDS spread. For simplicity, we assumed that $\bar{\sigma}^{\mathbb{V}}_{11} = \bar{\sigma}^{\mathbb{V}}_{12} = \bar{\sigma}^{\mathbb{V}}_{13}$ while the correlation parameters are given by $\rho^{\mathbb{V}f} = 0.4615$, $\rho^{f\lambda} = -0.4$ and $\rho^{\mathbb{V}\lambda} = 0.2720$ and the stochastic volatility process parameters are given by $\bar{V} = 0.0857$ and $\kappa_V = 0.45$. From Table 3.5, we observe that an increase in the volatility of volatility initially leads to a a decrease in the CDS spread but after a certain threshold is exceeded, further increase in the volatility of volatility leads to an upswing in the CDS spread.

The effect of increasing maturity on the the CDS spread is shown in Table 3.6. The resulting shape from CDS spread for varying maturities is typical for volatile market conditions where

[23]We adopt recovery assumptions as proposed in Pan and Singleton [2008].

$\bar{\sigma}^{\mathbb{V}}_{1j}$, $(j = 1, 2, 3)$	0.0	0.1	0.3	0.5	0.7	1.0
$\tilde{\pi}(t), (Bps)$	399	395	389	383	379	374

Table 3.5: Numerical results of CDS spread under varying volatility of volatility, $\bar{\sigma}^{\mathbb{V}}_{1j}$, $(j = 1, 2, 3)$.

the higher cost of short-term protection leads to the inverted CDS curve. This implies that a firm faces a greater chance of defaulting within a short term period than in the long term. In addition, fears of a sharp rise in the rate of high-yield corporate defaults could prompt investors to seek more short-dated credit protection in a bid to reduce risk.[24] For

Maturity, T	0.25	0.50	1	2	3	5	7	10
$\tilde{\pi}(t), (Bps)$	659	540	427	381	427	443	478	509

Table 3.6: Numerical results on the CDS spread with increasing maturity.

a certain recovery assumption, say 40%, [25] the fair CDS rate increase can be thought to be a consequence of the decreasing survival probability leading to widening spreads. Typically, the slope of the CDS spread curve is flatter for higher premium levels and steeper for lower premium levels. Any changes in the shape and perceptions of the fair premium for credit default swap protection are reflected in the spreads observed in the market. A curve of the survival probability for a reference entity can be inferred from the CDS curve and can be seen to be a decreasing function to maturity. As expected, we observed that higher recovery rates implied by different ratings classes gave rise to lower the CDS spread as shown in Table 3.7.

When using CDS spreads as a default risk indicator[26], it is important to note that spreads

[24]This result shows the robustness of the model in its ability capture various shapes on CDS spread curves. Under nonvolatile market conditions, the cost of protection over a longer term is usually higher as it is difficult to predict cash flows and future events that affect the profitability of a firm over a longer period. This would give rise to an increasing credit default swap curve.

[25]As highlighted in Pan and Singleton [2008], the CDS price under the recovery of market value (RMV) framework is given as a product of the loss given default $L \equiv LGD$ and the default intensity $\tilde{h}(t)$ in the sense that $CDS^{RMV}(t) = g(\tilde{h}(t)L)$ for some function g. This implies that the default intensity and the loss given default cannot be identified separately using defaultable bond data alone. Under the recovery of face value (RFV) framework, these two play distinct roles and the CDS pricing relationship is of the form $CDS^{RFV}(t) = Lf(\tilde{h}(t))$. This has the immediate consequence that under RFV, the explicit dependence of $CDS(t)$ on LGD implies that the ratio of two CDS spreads on contracts of different maturities does not depend of LGD but contains information about $\tilde{h}(t)$. We adopt this model of recovery in our formulation.

[26]Grossman and Hansen [2010] show that to estimate the default risk implied by CDS spreads at a given

Recovery rate, $\mathcal{R}$	0.10	0.40	0.70	0.90
CDS spread, $\tilde{\pi}(t)$, (Bps)	572	381	191	64

Table 3.7: Effects of recovery rate on CDS spread for bonds with 2−year maturities.

can be driven by several factors that may not be directly related to the reference entity's fundamental credit worthiness which include leverage interest in CDS trading, counterparty risk and risk-aversion of market participants.

In practice, questions arise on the ability of the protection seller to fulfil its obligation to make compensation payment at the end of the settlement period, given that its credit quality may have deteriorated due to contagion effects that may arise from the default of reference bond. Whereas counterparties tend to be of high credit quality, it has been observed that their credit quality can deteriorate, sometimes almost in parallel with the firms for which they provide credit protection with some protection sellers actually defaulting on their obligations from credit derivatives contracts.

3.4.2 CDS with Counterparty Risk

To determine the fair CDS rate in the presence of counterparty risks, the inter-dependent default risk structures between these parties should be considered simultaneously.[27] It was shown in Jarrow and Yu [2001] that a CDS may be significantly overpriced if the default correlation between protection seller and reference entity is ignored. Hull and White [2001] argue that if the default correlation is positive, then the default of the counterparty will result in a positive replacement cost for the protection buyer.

point, the average spread for the entity is calculated and then converted to a Probability of Default (PD) value using the formula: PD(1 year) = CDS spread (annualized)/Loss Severity$(1-\mathcal{R})$. It has however been noted that while using annualised spreads to imply annual PD, if the market perceives an entity's default to be definite then the resulting PD could exceed 100%. The approach is however tractable, intuitive and can be directly related to the credit performance of the underlying credit reference entity.

[27]If the insurance seller defaults before the reference obligor the insured party is left without any protection and they would be forced to go into the marketplace and purchase protection on the reference obligor from another insurance seller at an additional cost (at the current market spread level). Conversely, the insured party themselves may also default before the reference obligor. In this case the insurance seller's obligations to make a contingent payment cease, but on the other hand the positive impact of the fee payments is also lost.

Following Chen and Filipovic [2007], we let τ_1, τ_2, τ_3 be the default times of the reference obligor, the insured party and the insurance seller, respectively. The insured party pays the premium π at times t_i only given the events that happened in the preceding periods. At time $t_i < \tau_1 \wedge \tau_2 \wedge \tau_3$, the insured party pays to the seller the fixed rate π if no default has taken place. If the reference obligor has defaulted in the period $(t_{i-1}, t_i]$, that is $t_{i-1} < \tau_1 \leq t_i$ and the insured party has not defaulted by time t_{i-1}, $\tau_2 > t_{i-1}$ and the insurance seller has not yet defaulted by time t_i with $\tau_3 > t_i$, then the seller pays $(1 - \mathcal{R})$ and the contract terminates. Otherwise if either the insured party or the insurance seller defaults before then, there is no payment and the contract terminates. The protection payment is therefore made only on the occurrence of event τ_1 and zero otherwise.

Assuming a postponed running CDS, the discounted payoff at time $t < t_1$ of the premium leg is given by

$$W^{cpr}_{prm}(t) = \pi \sum_{i=1}^{N} \delta_i e^{-\int_t^{t_i} r(s)ds} \mathbb{1}_{\{\tau_1 \wedge \tau_2 \wedge \tau_3 > t_i\}}, \tag{3.59}$$

and similarly, we can express the discounted payoff of the protection leg as

$$W^{cpr}_{prt}(t) = (1 - \mathcal{R}) \sum_{i=1}^{N} e^{-\int_t^{t_i} r(s)ds} \mathbb{1}_{\{t_{i-1} < \tau_1 \leq t_i\}} \mathbb{1}_{\{\tau_2 > t_{i-1}\}} \mathbb{1}_{\{\tau_3 > t_i\}}. \tag{3.60}$$

The fair CDS spread $\pi_{cpr}(t)$, in the presence of counterparty risk, at time $t < t_1$ is the fixed rate which guarantees that the value of the CDS is zero, namely,

$$CDS_{cpr}(t) = \tilde{\mathbb{E}}\left[W^{cpr}_{prt}(t) - W^{cpr}_{prm}(t) \middle| \mathcal{F}_t\right] = 0, \tag{3.61}$$

and is expressed as

$$\tilde{\pi}_{cpr}(t) = \frac{(1 - \mathcal{R}) \sum_{i=1}^{N} \tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds} \left(\mathbb{1}_{\{\tau_1 > t_{i-1}\}} - \mathbb{1}_{\{\tau_1 > t_i\}}\right) \mathbb{1}_{\{\tau_2 > t_{i-1}\}} \mathbb{1}_{\{\tau_3 > t_i\}} \middle| \mathcal{F}_t\right]}{\delta \sum_{i=1}^{N} \tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds} \mathbb{1}_{\{\tau_1 \wedge \tau_2 \wedge \tau_3 > t_i\}} \middle| \mathcal{F}_t\right]}. \tag{3.62}$$

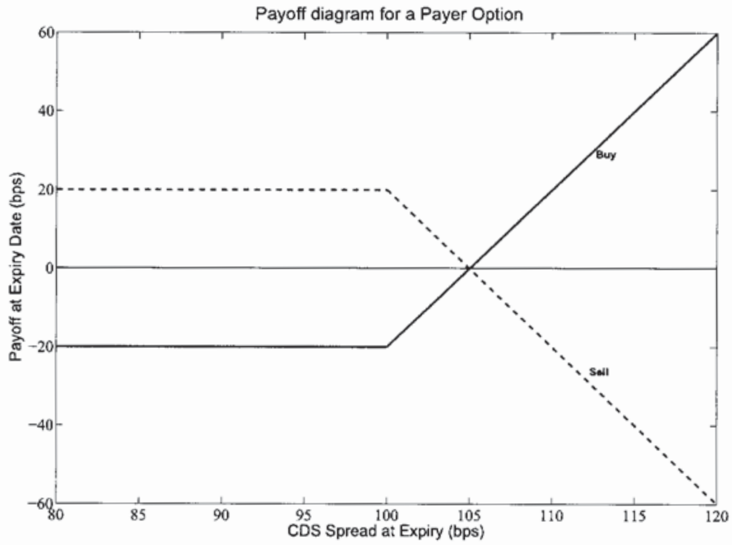
The CDS spread in (3.62) can be approximated by a ratio of pseudo bonds of various maturities, similarly to expression (3.58). It was however noted in Schönbucher [2003] that the default correlation levels that can be reached through this approach are typically too low when compared with empirical default correlations in the addition to the level of complexity involved in deriving and analysing the resulting dependency structure. In Schönbucher and Schubert [2001], the authors propose an extension of the intensity-based approach to incorporate default correlations using copulas which have been shown to generate realistic time-distribution of the default times. Since the dependency structure is completely described by the copula function, there is liberty in the specification of the copula used in the model which allows for reproduction of various dependency structures between the default times.

3.4.3 Credit Default Swaptions

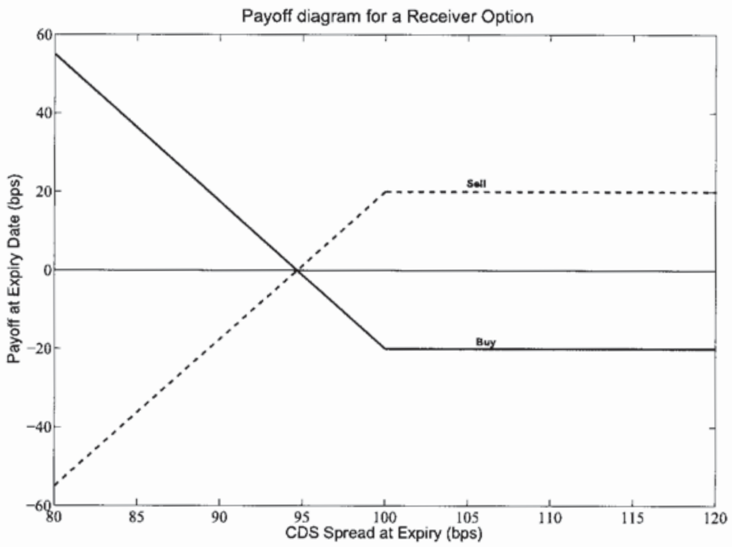
Credit default swaptions allow investors to hedge risk or to express a *directional* view on credit spreads. A payer option gives the right to buy credit protection at a pre-specified level (the strike) on a future date. This can be considered as put option on credit or as a call option on credit spread. The investor will profit if credit default spreads widen sufficiently to recover the premium paid for the option. It has been noted (see Merrill-Lynch [2006b]) that buying a payer option is an expensive way to short credit but is often more appropriate for an investor with a bearish outlook, who also believes that there is a significant probability that he/she will be wrong and that spreads may tighten.

Alternatively, an investor may buy a payer option for hedging against the risk that spreads will widen significantly. In this case, he/she buys a deep out-of-the-money payer option to insure against a worst case scenario. The various payoff diagram at the maturity date for the different strategies on the payer option are given in Figure 3.3(a).

A receiver option gives the right to sell credit default protection at a pre-specified level



(a)



(b)

Figure 3.3: Panel (a) illustrates the payoff diagram for a payer option (buy or sell the right to buy protection) while Panel (b) gives the payoff for a receiver option (buy or sell the right to sell protection). The figures are adapted from Merrill-Lynch [2006b, page 177].

(strike) on a future date. In this case, the investor profits if spreads tighten through the strike by enough to recuperate the option premium but if the spreads widen, the option expires worthless and the investor loses the premium paid. It can also be used to hedge against downside loss should the spreads tighten for an investor who already owns credit default protection. A receiver option is therefore a call option on credit since the buyer gains when credit quality improves but it may also be considered as a put option on spreads, because spreads tighten when credit quality improves. Figure 3.3(b) illustrates the payoff diagrams for the various strategies on the receiver option at the expiry date.

From Figure 3.3, we observe that selling a receiver option has a different payoff than buying a payer option, even though both express bearish views. Whereas the buyer of a payer option pays money upfront (negative carry) and profits if the spreads widen sufficiently, the seller of a receiver option receives money upfront (positive carry) and profits as long as spreads do not tighten.

Most contracts contain knockout provisions. A receiver option becomes worthless following a credit event. The buyer of a receiver contract does not exercise, because he would sell credit default protection and immediately owe par minus recovery which would result in an overall loss. Instead, the buyer of a receiver option loses the premium paid to the seller. A payer option becomes worthless following a credit event only if there is a knockout provision specifying that the option contract automatically terminates following a credit event, with the buyer losing the premium paid to the seller. If there is not a knockout provision, the buyer of a payer option exercises. The buyer receives par, delivers a physical bond (or cash settlement at the recovery rate) and earns $(1-\mathcal{R})$ times the notional amount of the contract, less the premium paid. The seller keeps the premium but loses $(1-\mathcal{R})$ times the notional.[28]

[28] In case there is no knockout provision in the contract, the value of the contingent payment due on the occurrence of the credit event must be considered. However, this is only relevant for a payer swaption since if there is severe deterioration in the credit quality of the underlying reference credit (widening spreads), the payer could be faced with a default of the underlying prior to the options expiry. In that case, the payer receives a payment in the event of default of the underlying prior to the expiry of the option. The contingent payment does not apply to a receiver since as the owner of the call option, the receiver has value when spreads contract but does not have any on worsening credit quality.

In practice, the most liquid options are 3-month and 6-month options which are commonly bought or sold on the 5 year CDS (from the transaction date). Options with expiry dates that match standard maturities on the CDS index or single-name CDS have better liquidity.

Credit default swaptions (hereafter CDS options) are options written on CDS contracts. More precisely, a *plain-vanilla* CDS option with maturity T_m is a European option on a *forward* credit default swap (hereafter forward CDS). The underlying forward CDS is a CDS contract issued at time s with a start date T_m and maturity T, with $s < T_m < T$, see Bielecki et al. [2007] for a formal definition. This contract gives default protection over the future interval $[T_m, T]$ but if the reference obligor defaults before the start date, that is $\tau < T_m$, the contract is terminated and no payments are made. The value of the forward CDS at time $t \in [s, T_m]$ under the risk-neutral measure is given by

$$CDS_f(t, \pi_f) = \tilde{\mathbb{E}}\Big[W^f_{prt}(t) - \pi_f \bar{W}^f_{prm}(t) \Big| \mathcal{F}_t\Big], \tag{3.63}$$

where, $W^f_{prt}(t)$ is the value of the CDS protection leg and $\pi_f \bar{W}^f_{prm}(t) = W^f_{prm}(t)$ is the value of the CDS premium leg. In addition, we define the forward CDS spread, $\tilde{\pi}_f(t, T_m, T)$, as the variable which will makes the value of the forward CDS contract to be zero at time t.

We consider a payer CDS option with a strike rate K and maturity T_m on a forward CDS maturing at T and with tenor payment dates $\mathrm{t}_1 = T_m + \delta$, $\mathrm{t}_2 = T_m + 2\delta$, $\ldots$, $\mathrm{t}_N = T_m + N\delta$, with $\delta = (T - T_m)/N$. Upon the option's exercise, which will occur if the reference obligor does not default before T_m, this is $\tau > T_m$, the strike spread K is the fixed rate to be paid (instead of the CDS spread $\pi_f(T_m)$) on the tenor payment dates $\mathrm{t}_i, i = 1, 2, \ldots, N$, in exchange of the CDS default protection. If the reference obligor defaults before T_m, the contract will terminate with no payments exchange. Therefore the payoff $V(T_m)$ of the payer

CDS option[29] at the option maturity T_m is given by

$$V(T_m) = \mathbb{1}_{\{\tau > T_m\}} \Big(CDS_f(T_m, K) - CDS_f(T_m, \tilde{\pi}_f(T_m)) \Big)^+, \tag{3.64}$$

where, by definition, $CDS_f(T_m, \tilde{\pi}_f(T_m)) = 0$. As the option will be only exercised if $\tilde{\pi}_f(T_m) > K$, by using (3.63), the payoff function can equivalently be written as

$$\begin{aligned} V(T_m) &= \mathbb{1}_{\{\tau > T_m\}} \Big(\tilde{\mathbb{E}}\Big[W^f_{prt}(T_m) \Big| \mathcal{F}_t \Big] - K \tilde{\mathbb{E}}\Big[\bar{W}^f_{prm}(T_m) \Big| \mathcal{F}_t \Big] \Big)^+ \\ &= \mathbb{1}_{\{\tau > T_m\}} \mathbb{1}_{\{\tilde{\pi}_f(T_m) > K\}} \tilde{\mathbb{E}}\Big[W^f_{prt}(T_m) \Big| \mathcal{F}_t \Big] - K \mathbb{1}_{\{\tau > T_m\}} \mathbb{1}_{\{\tilde{\pi}_f(T_m) > K\}} \tilde{\mathbb{E}}\Big[\bar{W}^f_{prm}(T_m) \Big| \mathcal{F}_t \Big]. \end{aligned} \tag{3.65}$$

Alternatively, by substituting the values $CDS_f(T_m, K)$ and $CDS_f(T_m, \tilde{\pi}_f(T_m))$ of the forward CDS contracts from (3.63) into the payoff function (3.64), we obtain an expression for the payoff of the payer CDS option in terms of spreads as

$$V(T_m) = \mathbb{1}_{\{\tau > T_m\}} \tilde{\mathbb{E}}\Big[\bar{W}^f_{prm}(T_m) \Big| \mathcal{F}^W_t \Big] \big(\tilde{\pi}_f(t, T_m) - K \big)^+. \tag{3.66}$$

As noted in Bielecki et al. [2008], we observe from (3.64) and (3.66) that a call option with zero strike on the value of the forward CDS with spread K is equivalent to a call option on the forward CDS rate K. The value of the CDS option $\mathcal{C}_{swpt}(t)$ at any time $t \in [s, T_m]$ can then be expressed under the risk neutral measure as a discounted payoff, namely,

$$\mathcal{C}_{swpt}(t) = \mathbb{1}_{\{\tau > t\}} \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_m} r(s)ds} V(T_m) | \mathcal{F}_t \Big]. \tag{3.67}$$

When the payoff function is given by (3.65), the expectations in (3.67) can be calculated using Monte-Carlo simulations to give the default swaption price. Applying the techniques used in Section 3.4.1 to the expectations in (3.65) gives the following result.

[29] From the Banc of America Securities, Guide to Credit Default Swaptions, we quote the following: "Credit default swaptions use the lingo *payer* and *receiver*, instead of *put* and *call*: a payer option is both a put option on credit quality - a bet that credit will deteriorate - and a call option on spreads - a bet that spreads will widen".

Proposition 3.8 *The price at time t of a credit default swaption on a forward CDS with a strike rate K and maturity date T_m can be approximated* [30] *by*

$$\begin{aligned}\mathcal{C}_{swpt}(t) \approx \mathbb{1}_{\{\tau>t\}} LGD \sum_{i=1}^{N} \bar{P}^d(t,t_{i-1}) Pr_t^{t_{i-1}}(u(T_m) > \ln K) \\ -\mathbb{1}_{\{\tau>t\}} \sum_{i=1}^{N} \bar{P}^d(t,t_i)\big(\delta K + LGD\big) Pr_t^{t_i}(u(T_m) > \ln K),\end{aligned} \tag{3.68}$$

where $Pr_t^S(u(T_m) > \ln K)$ is the conditional probability of the event $\{u(T_m) > \ln K\}$ based on the $S-$forward measure $\mathcal{Q}^S$ induced on $\tilde{\mathbb{P}}$ by the price of the zero recovery, zero-coupon bond issued at time t and $LGD = (1-\mathcal{R})$.

Proof: See Appendix B.3.6.

By using the expression (3.66) for the payoff function, the expectation in (3.67) can be reduced to a Black's formula, as proposed by Rutkowski and Armstrong [2009]. By an appropriate choice of the numeraire that depends on the value of the premium leg and the survival process of τ $Pr(\tau > t \big| F_t^W)$, Rutkowski and Armstrong [2009] define an equivalent probability measure $\bar{\mathcal{Q}}$, and show that the price of the CDS option can be expressed as

$$\tilde{\mathcal{C}}_{swpt}(t) = \mathbb{1}_{\{\tau>t\}} \tilde{A}(t) \bar{\mathbb{E}}\Big[\big(\tilde{\pi}_f(t,T_m) - K\big)\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}\Big|\mathcal{F}_t\Big], \tag{3.69}$$

where $\bar{\mathbb{E}}$ is the expectation under the $\bar{\mathcal{Q}}$ measure and

$$\tilde{A}(t) = Pr(\tau > t \big| F_t^W) \tilde{\mathbb{E}}\Big[\bar{W}^f_{prm}(T_m)\Big|\mathcal{F}_t\Big]. \tag{3.70}$$

In addition, under this new measure, the forward CDS spread, $\tilde{\pi}_f(t,T_m)$, is an $(\mathcal{F},\bar{\mathcal{Q}})-$martingale and its dynamics follow the driftless stochastic differential equation

$$d\tilde{\pi}_f(t,T_m) = \sigma_f^V(t)\tilde{\pi}_f(t,T_m)d\bar{W}(t), \tag{3.71}$$

[30] We use the approximation $e^{-\int_t^{t_i} r(s)ds} \approx e^{-\int_t^{t_{i-1}} r(s)ds}$ made in (B.3.8) whose error is of a very small order for small values of δ.

where $\bar{W}$ is a $\bar{\mathcal{Q}}$–Brownian motion. If we assume that the volatility σ_f^V for different tenor dates is a constant, the value of a credit default swaption with strike K and maturity T_m is given by the Black's formula[31]

$$\tilde{\mathcal{C}}_{swpt}(t) = \mathbb{1}_{\{\tau>t\}}\tilde{\mathbb{E}}\big[\bar{W}^f_{prm}(T_m)\big|\mathcal{F}_t\big]\big(\tilde{\pi}_f(t,T_m)N(d_1) - KN(d_2)\big), \tag{3.72}$$

where

$$d_1 = \frac{\ln\big(\frac{\tilde{\pi}_f(t,T_m)}{K}\big) + \frac{(\sigma_f^V)^2}{2}(T_m - t)}{\sigma_f^V\sqrt{T_m - t}} \quad \text{and} \quad d_2 = d_1 - \sigma_f^V\sqrt{T_m - t}, \tag{3.73}$$

and σ_f^V is the only parameter to be inferred from market data. Although the model is not easily calibrated to quoted data if the market is illiquid, it provides a platform where prices of different options can be translated into implied volatilities thereby giving more information on the market. In addition as noted in Brigo and Morini [2005], the computation of the implied volatilities allows us to assess the implications of different models on the classic strike volatility curve (smile or skew).

Numerical Study - CDS Options

To illustrate the model, we compute the swaption prices based on (3.72) for varying parameter values. To calculate $\tilde{\mathbb{E}}\big[\bar{W}_{prm}(t)\big|\mathcal{F}_t\big]$ in (3.72), we use the approximations made in Corollary 3.7. At time $t = 0$, we calculate the price of a swaption with maturity $T_m = 0.5$ issued on a credit default swap that has a defaultable bond (based on the framework developed in Section 3.3) with a maturity of $T = 2$ years as its underlying and the default protection is required for the period $[0.5, 2.5]$. In addition, we use the following parameters for the simulation: recovery rate $\mathcal{R} = 40\%$, volatility of the forward CDS rate $\sigma_f^V = 0.4$ [32]

[31] It was remarked in Brigo and Morini [2005] that this distributional assumption is inspired by standard models used to model equity and interest rate markets. Jabbour et al. [2008] however rejected this hypothesis by showing that the log forward CDS spreads exhibit large positive skewness and excess kurtosis.

[32] It was noted in Schönbucher [2004] that the value of the volatility $\bar{\sigma}(t) = 0.4$ is of an acceptable level for simulation purposes.

with $N = 400$.

Figure 3.4 gives the credit default swaption price under varying strike rates and correlation $\rho^{f\lambda}$ between short rate and short term credit spread[33]. We observe that increasing K leads to

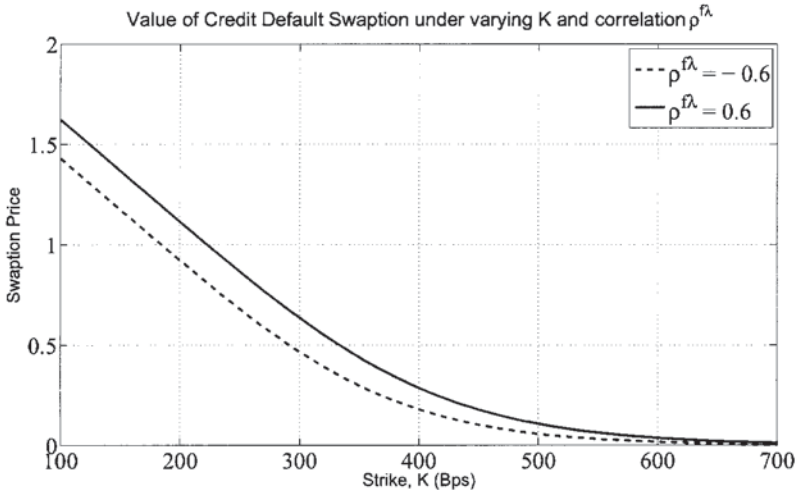


Figure 3.4: Credit swaption prices given by equation (3.72) under varying strikes and correlation $\rho^{f\lambda}$ between interest rate $r(t)$ and short term credit spread $c(t)$.

a decrease in the credit default swaption price which is in line with the market behavior that deep in-the-money options trade higher than at-the-money and out-of-the-money options. For deep in-the-money options, negative correlation $\rho^{f\lambda}$ is seen to produce lower swaption prices.

However, from Figure 3.5 we observe that negative correlation $\rho^{f\mathbb{V}}$ produces higher swaption prices for deep in-the-money options as compared to positive correlation although the overall swaption prices are lower when compared to the values produced in Figure 3.4.

In Figure 3.6, the value of an at-the-money payer swaption is shown as a function of its time-to-maturity. Similar simulation parameters as in Figure 3.4 were used in addition to varying the maturity of the defaultable bond for which default protection is required after 0.5 years using the forward CDS. As expected, we observe that when protection is sought for

[33]We recall that in this case, the short term credit spread coincides with the default intensity given that we are considering the pre-default bond price.

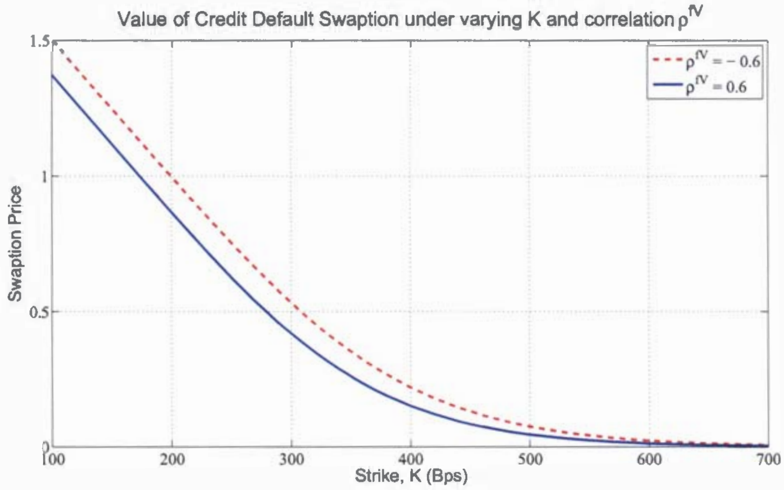


Figure 3.5: Credit swaption prices given by equation (3.72) under varying strikes and correlation $\rho^{f\mathbf{V}}$ between interest rate $r(t)$ and stochastic volatility $V(t)$.

bonds with shorter maturities, we have lower swaption prices and this is lower, the further we are from the maturity of the option. The recovery rate $\mathcal{R}$ enters implicitly into (3.72)

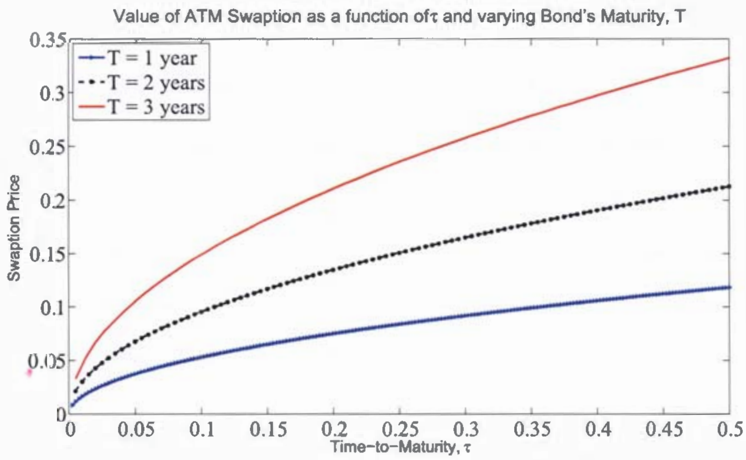


Figure 3.6: The value of an ATM swaption as a function of time to maturity of the option under varying maturity of the defaultable bond for which CDS protection is required.

through the current forward CDS rate $\bar{\pi}_f(t)$. Figure 3.7 investigates the time sensitivity of the swaption and we observe that the credit swaption becomes more sensitive to changes in the recovery assumption the longer the time-to-maturity. A similar observation was made in the model by Hull and White [2003][34].

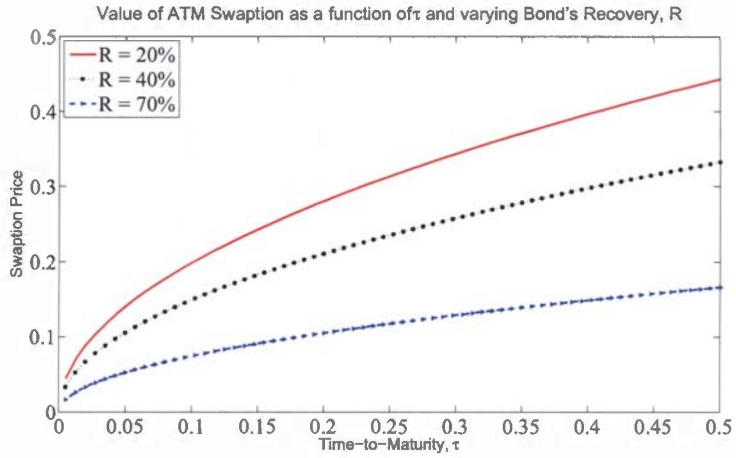


Figure 3.7: The value of an ATM swaption for varying time to maturity of the option and different recovery values for the defaultable bond.

In Figure 3.8, the value of the CDS swaption price is given as a function of the volatility of volatility, σ_{1j}^V, (j=1,2,3). We observe that increasing the volatility of volatility leads to a decrease in the swaption price, with the effect being more for deep in the money options.

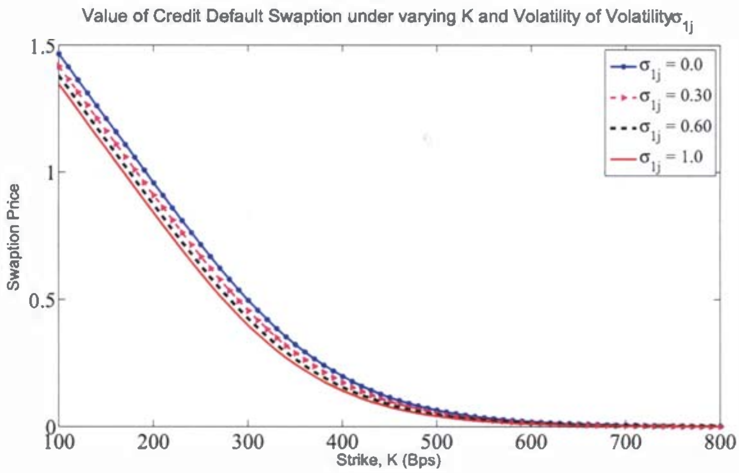
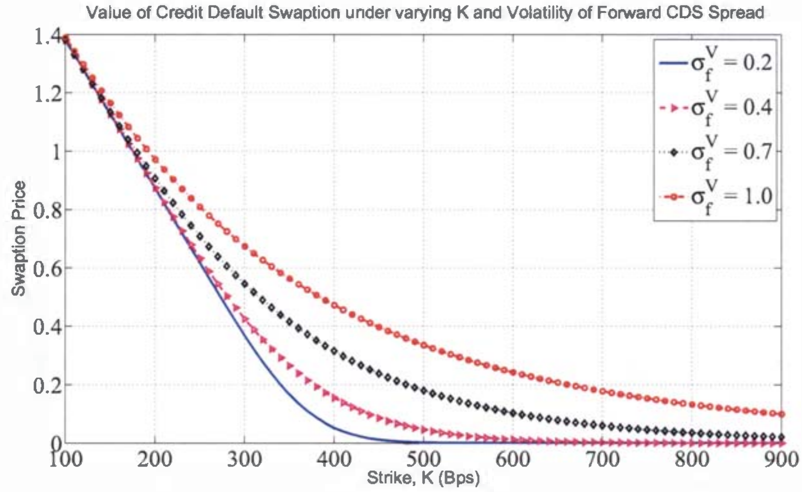


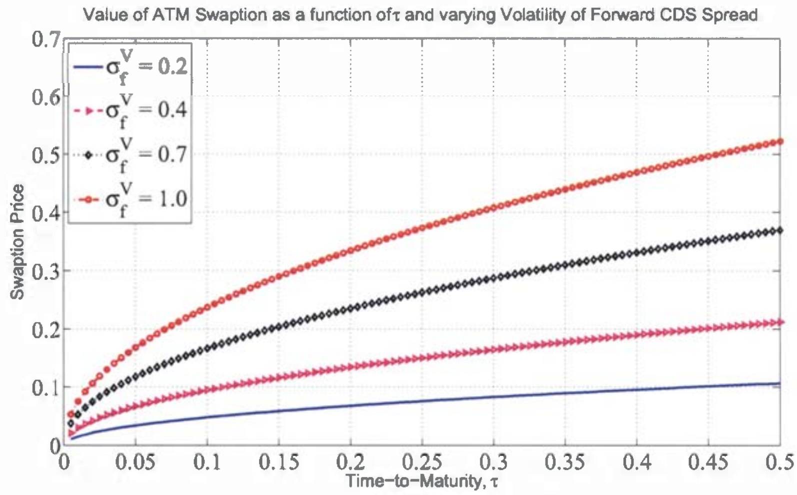
Figure 3.8: Credit swaption prices under varying strikes and volatility of volatility σ_{1j}^V, (j=1,2,3).

[34]Their results also showed that this increasing percentage impact does not become very large.

Finally, in Figure 3.9 we investigate the effect of varying the volatility of the forward CDS spread on the swaption price where we observe that increasing σ_f^V leads to a decrease in the swaption price.



(a)



(b)

Figure 3.9: Panel (a) gives the value of a credit default swaption for varying strikes and different volatilities of the forward CDS spread while panel (b) gives the value of an ATM swaption for varying time-to-maturity of the option and different volatilities of the forward CDS spread.

3.5 Pricing Put Options on Defaultable Bonds

In this section, we focus on the evaluation of options on a defaultable bond. Such an evaluation could include cases where there is no default risk on the option as well as where the option is also subject to default risk. In the latter case, a simplifying assumption would be that the default of the writer of the vulnerable option and the default of the issuer of the defaultable underlying asset are independent events. If the writer's default occurs before the option's maturity, then the buyer would suffer fractional loss of the option's market value and the writer would pay back the residual value. In this case, the vulnerable option can be valued in a similar manner as a defaultable bond. Our study however will exclude vulnerable options with a risky asset or risk free asset as the underlying.

A European put option with maturity T_0 and exercise price K on a defaultable zero coupon bond with maturity T and time t price $P^d(t,T)$ has a payoff $\left(K - P^d(T_0,T)\right)^+$, given that $T_0 < T$ and if the option is knocked out on default, then its payoff given by

$$\mathbb{1}_{\{\tau > T_0\}}\left(K - P^d(T_0,T)\right)^+,$$

where τ is the default time of the underlying defaultable bond. The option protects the buyer against risks that arise from interest rates, credit spreads and default and that could cause the price of the defaultable bond to fall below the strike level K.

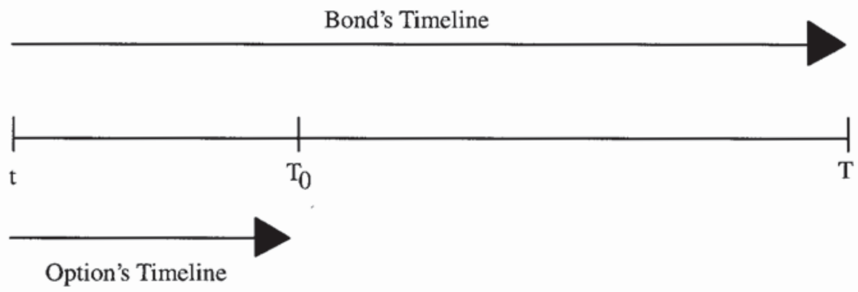


Figure 3.10: The Timeline for an option on defaultable bond.

If we assume that there is no default risk with the issuer of the option, then the price of the put option is given by [35]

$$\begin{cases} \mathcal{P}(t, r^d, T; T_0, K) = \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r(u)du}\Big(K - \bar{P}^d(T_0, T)\Big)^+\Big|\mathcal{F}_t\Big], \\ \qquad = \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r^d(u)du}\Big(K - \bar{P}^d(T_0, T)\Big)^+\Big|\mathcal{F}_t\Big] + KP(t, T_0) - P^d(t, T) \\ \qquad\quad -\tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r^d(u)du}\Big(K - \bar{P}^d(T_0, T)\Big)\Big|\mathcal{F}_t\Big], \end{cases} \tag{3.74}$$

where $\bar{P}^d(t, T)$ is the pre-default value of the defaultable bond. In this case, the post-default price behavior will also come into consideration. The bond could continue being traded with a market price dropped to the recovery multiplied by the pre-default market value of the debt. Alternatively, the proceeds from recovery can be reinvested at the default-free rate and rolled over until expiry. If the credit option survives a credit event, default risk and spread risk are transferred between the counterparties but if it is knocked out on default, only the spread risk is transferred.

A put option on a defaultable bond can be decomposed into default protection and price protection components (See Schönbucher [2000] and Appendix B.4). If the option is knocked out at default, then it has only the value of the price protection component. If the put is knocked out at default (zero recovery assumption), then the put price at time $t \leq T_0$ is given by

$$\mathcal{P}(t, r^d, T; T_0, K) = \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r^d(u)du}\Big(K - \bar{P}^d(T_0, T)\Big)^+\Big|\mathcal{F}_t\Big]. \tag{3.75}$$

We can apply Monte Carlo methods to easily compute the expectation in (3.75) over paths of stochastic process $r^d(t)$ since the pseudo bond price $\bar{P}^d(T_0, T)$ is given in closed form by (3.49) thereby saving on the computational time.

However, the presence of level dependence in the volatility specification of Assumption 3.3.1 makes it hard, if not impossible, to obtain a closed form solution for the option prices. In this last section, we make the following simplifying assumption on the structure of the humped

[35] The proof follows from Schönbucher [2000, lemma 3.5], which we give in Appendix B.4 for completeness.

volatilities.

Assumption 3.5.1 *For $1 \le i \le n$, the volatility functions are of the form*

$$\sigma_i^f(t,T,V_i) = [a_{0i} + a_{1i}(T-t)]\sqrt{V_i(t)}e^{-\kappa_i^f(T-t)}, \tag{3.76}$$

$$\sigma_i^\lambda(t,T,V_i) = [b_{0i} + b_{1i}(T-t)]\sqrt{V_i(t)}e^{-\kappa_i^\lambda(T-t)}, \tag{3.77}$$

where κ_i^f, κ_i^λ, a_{0i}, a_{1i}, b_{0i} and b_{1i} are constants.

Proposition 3.9 *The defaultable bond price dynamics satisfy the stochastic differential equation*

$$\frac{dP^d(t,T)}{P^d(t-,T)} = r(t)dt - \sum_{i=1}^{2n} d_i(t,T)\sqrt{V_i(t)}d\tilde{W}_i(t) - d\tilde{M}(\omega;t), \tag{3.78}$$

where for $i = 1, \dots, n$ we define the deterministic functions

$$d_i(t,T) = \begin{cases} \bar{\beta}_{1i}(t,T) + \bar{\beta}_{2i}(t,T), & \text{for } i = 1,\dots,n; \\ \bar{\beta}_{3i}(t,T), & \text{for } i = n+1,\dots,2n; \end{cases} \tag{3.79}$$

given that

$$\begin{cases} \bar{\beta}_{1i}(t,T) = \dfrac{a_{0i}}{\kappa_i^f}\left(1-e^{-\kappa_i^f(T-t)}\right) + \dfrac{a_{1i}}{\kappa_i^f}\left[\dfrac{1}{\kappa_i^f}\left(1-e^{-\kappa_i^f(T-t)}\right) - (T-t)e^{-\kappa_i^f(T-t)}\right], \\ \bar{\beta}_{2i}(t,T) = z_i^{\lambda_1}\left(\dfrac{b_{0i}}{\kappa_i^\lambda}\left(1-e^{-\kappa_i^\lambda(T-t)}\right) + \dfrac{b_{1i}}{\kappa_i^\lambda}\left[\dfrac{1}{\kappa_i^\lambda}\left(1-e^{-\kappa_i^\lambda(T-t)}\right) - (T-t)e^{-\kappa_i^\lambda(T-t)}\right]\right), \\ \bar{\beta}_{3i}(t,T) = z_i^{\lambda_2}\left(\dfrac{b_{0i}}{\kappa_i^\lambda}\left(1-e^{-\kappa_i^\lambda(T-t)}\right) + \dfrac{b_{1i}}{\kappa_i^\lambda}\left[\dfrac{1}{\kappa_i^\lambda}\left(1-e^{-\kappa_i^\lambda(T-t)}\right) - (T-t)e^{-\kappa_i^\lambda(T-t)}\right]\right). \end{cases} \tag{3.80}$$

In addition, the pseudo bond price dynamics follow the stochastic differential equation

$$\frac{d\bar{P}^d(t,T)}{\bar{P}^d(t,T)} = r(t)dt - \sum_{i=1}^{2n} d_i(t,T)\sqrt{V_i(t)}d\tilde{W}_i(t). \tag{3.81}$$

Proof: See Appendix B.5 for the details. ■

The first term on the right hand side of (3.78) represents the risk-neutral drift of the defaultable bond price. The Wiener terms $\tilde{W}_i(t)$ introduce the interest rate uncertainty into the price dynamics through the bond price volatility. The last term with the jump martingale models the credit risk associated with the jump time and jump size. At the jump time τ_i for $i \in \mathbb{N}$, $\tilde{M}$ jumps to one and the defaultable bond jumps down by the fraction q.

We define the log price process for the pseudo bond with maturity T_0 by $\bar{X}^d(t,T_0) = \log\bar{P}^d(t,T_0)$ and from Itô's lemma, we can write the dynamics for the log bond price $\bar{X}^d(t,T_0)$ as

$$d\bar{X}^d(t,T_0) = \Big(r(t) - \frac{1}{2}\sum_{i=1}^{2n} d_i^2(t,T_0)V_i(t)\Big)dt - \sum_{i=1}^{2n} d_i(t,T_0)\sqrt{V_i(t)}d\tilde{W}_i(t), \tag{3.82}$$

and the pseudo bond price is given by

$$\bar{P}^d(t,T_0) = \exp\Big(\big[r(t) - \frac{1}{2}\sum_{i=1}^{2n} d_i^2(t,T_0)V_i(t)\big]dt - \sum_{i=1}^{2n} d_i(t,T_0)\sqrt{V_i(t)}d\tilde{W}_i(t)\Big). \tag{3.83}$$

From Girsanov's theorem, we introduce a Radon-Nikodym derivative $\leftthreetimes(T_0)$, such that its stochastic exponential is a uniformly integrable martingale. Application of Itô's lemma to (3.83) leads to

$$\frac{d\leftthreetimes(T_0)}{\leftthreetimes(T_0)} = -\sum_{i=1}^{2n} d_i(t,T_0)\sqrt{V_i(t)}d\tilde{W}_i(t),$$

and if the Novikov's boundedness condition $\tilde{\mathbb{E}}\Big[-\exp\sum_{i=1}^{2n}\Big(\int_t^{T_0} d_i^2(s,T_0)V_i(s)ds\big|\mathcal{F}_t\Big)\Big] < \infty$ is satisfied, then

$$\frac{d\tilde{\mathbb{P}}^{T_0}}{d\tilde{\mathbb{P}}}\Big|_{\mathcal{F}_t} = \leftthreetimes(T_0),$$

defines a forward probability measure $\tilde{\mathbb{P}}^{T_0}$ equivalent to $\tilde{\mathbb{P}}$. Furthermore, the process $\tilde{W}_i(t)$ is expressed as

$$d\tilde{W}_i(t) = d\tilde{W}_i^{T_0}(t) - d_i(t,T_0)\sqrt{V_i(t)}dt, \tag{3.84}$$

where $d\tilde{W}_i^{T_0}(t)$ is a $\tilde{\mathbb{P}}^{T_0}$–Brownian motion. The log price dynamics for the zero recovery bond with maturity T under the $\tilde{\mathbb{P}}^{T_0}$–forward measure can written as

$$\begin{aligned} d\bar{X}^d(t,T) = &\Big(r(t) - \frac{1}{2}\sum_{i=1}^{2n} d_i^2(t,T)V_i(t) + \sum_{i=1}^{2n} d_i(t,T_0)d_i(t,T)V_i(t)\Big)dt \\ &- \sum_{i=1}^{2n} d_i(t,T)\sqrt{V_i(t)}d\tilde{W}_i^{T_0}(t). \end{aligned} \tag{3.85}$$

We define a random variable $\hat{X}(t,T,T_0) = \log\Big(\dfrac{\bar{P}^d(t,T)}{\bar{P}^d(t,T_0)}\Big)$. Then, from (3.85)

$$\begin{aligned} d\hat{X}(t,T,T_0) &= d\bar{X}^d(t,T) - d\bar{X}^d(t,T_0). \\ &= -\frac{1}{2}\sum_{i=1}^{2n}\Big(d_i(t,T) - d_i(t,T_0)\Big)^2 V_i(t)dt - \sum_{i=1}^{2n}\Big(d_i(t,T) - d_i(t,T_0)\Big)\sqrt{V_i(t)}d\tilde{W}_i^{T_0}(t). \end{aligned} \tag{3.86}$$

Assumption 3.5.2 *For $1 \le i \le n$ and $j = 1, 2, 3$, the volatility function $\tilde{\sigma}_{ij}^{\mathbb{V}}(t, V_i)$ is of the form*

$$\tilde{\sigma}_{ij}^{\mathbb{V}}(t, V_i) = \bar{\sigma}_{ij}^{\mathbb{V}}\sqrt{V_i(t)}, \tag{3.87}$$

where $\bar{\sigma}_{ij}^{\mathbb{V}}$ is a constant.

Using (3.84), the stochastic volatility dynamics in (3.30) under the forward measure can be written as

$$dV_i(t) = \bar{\alpha}_i^V(t, V_i)dt + \sum_{j=1}^{3}\bar{\sigma}_{ij}^{\mathbb{V}}\sqrt{V_i(t)}d\tilde{W}_{(j-1)n+i}^{T_0}(t), \tag{3.88}$$

where the drift term is given by

$$\bar{\alpha}_i^V(t, V_i) = \alpha_i^V(t, V_i) + \sum_{j=1}^{3}\big(\phi_{(j-1)n+i}(t) - d_i(t,T_0)\big)\bar{\sigma}_{ij}^{\mathbb{V}}\sqrt{V_i(t)}. \tag{3.89}$$

For simplicity, we will assume that on specifying the market price of risk $\phi_{(j-1)n+i}(t)$, the drift function given in (3.89) coupled with Assumption 3.5.2 can be simplified to the general form $\kappa_i^{\mathbb{V}}\big(\bar{V}_i - V_i\big)$ and (3.88) could be written as

$$dV_i(t) = \kappa_i^{\mathbb{V}}\big(\bar{V}_i - V_i\big)dt + \sum_{j=1}^{3} \bar{\sigma}_{ij}^{\mathbb{V}} \sqrt{V_i(t)} d\tilde{W}_{(j-1)n+i}^{T_0}(t). \tag{3.90}$$

3.5.1 Pricing Methodology for a Knocked-Out Put Option

We observe from (3.74) and (3.83) that both the discount factor and the payoff functions depend on the same stochastic process $r(t)$. We therefore cannot evaluate the expectations separately and then multiply them afterwards as would be the case with deterministic discount rates. There is a need to derive the solution under joint stochastic dynamics. Given that the discount factor has an exponential affine representation, we can use a generalised characteristic function. It is well known that expressing a probability via its characteristic function is equivalent to expressing the probability via density functions and if the stochastic process consists only of one variable, then the characteristic function is just the Fourier transform of the particular transition density function.

From Bakshi and Madan [2000], the characteristic function can be interpreted as a hypothetical contingent claim. When the *state-price density*, given by $\vartheta(v)\exp\big(-\int_t^{T_0} r^d(u)du\big)$ is known and tractable, the option valuation problem is significantly simplified because although the state-price density may at times be complicated, its characteristic function remains uncomplicated and a closed-form formulation of the latter is all that is needed for derivative-security valuation. Whereas probability functions (on solving the respective partial differential equations) and consequently derivative prices can be hard to obtain in some instances due to discontinuous terminal conditions, the solution for the particular general characteristic function can be recovered since their terminal conditions are smooth, infinitely differentiable with 'finite algebraic moments of all orders' and therefore mathematically tractable. This arises, amongst many reasons, due to the one-to-one correspondence between the characteristic function and its distribution, guaranteeing a unique form of the option pricing formula.

We recall from (3.75) that the price of a put option at time $t \leq T_0$

$$\mathcal{P}(t, r^d, T; T_0, K) = \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r^d(u)du}\Big(K - \bar{P}^d(T_0, T)\Big)^+ \Big| \mathcal{F}_t\Big], \tag{3.91}$$

and this could be expressed as

$$\mathcal{P}(t, r^d, T; T_0, K) = \int_\chi e^{-\int_t^{T_0} r^d(u)du}\big(K - \bar{P}^d(T_0, T)\big)\vartheta(v)dv, \tag{3.92}$$

where $\vartheta(v)$[36] is the risk-neutral joint density function of the future uncertainty $v \equiv \Big(\int_t^{T_0} r^d(u)du, \bar{P}^d(T_0, T)\Big)$ and $\chi = \{\bar{P}^d(t, T) < K\}$ is the exercise region of the put option.

A put option is in the money at maturity T_0 if $\bar{P}^d(T_0, T) < K$ and the pricing formula in (3.91) can be written as

$$\begin{aligned}\mathcal{P}(t, r^d, T; T_0, K) &= \tilde{\mathbb{E}}_t\Big[e^{-\int_t^{T_0} r^d(u)du}\Big(K - \bar{P}^d(T_0, T)\Big)\mathbb{1}_{\{\bar{P}^d(T_0,T)<K\}}\Big] \\ &= K\tilde{\mathbb{E}}_t\Big[e^{-\int_t^{T_0} r^d(u)du}\mathbb{1}_{\{\bar{X}^d(T_0,T)<\xi\}}\Big] - \tilde{\mathbb{E}}_t\Big[e^{-\int_t^{T_0} r^d(u)du+\bar{X}^d(T_0,T)}\mathbb{1}_{\{\bar{X}^d(T_0,T)<\xi\}}\Big],\end{aligned} \tag{3.93}$$

where in addition, we have defined the log strike price $\xi = \log(K)$. However, these expectations are not Arrow-Debreu securities in the sense of:

Definition 3.10 *An Arrow-Debreu security is a contingent claim that pays one unit of money at the maturity date T, if and only if, a specified state $\mathcal{A}$ occurs. In this case, the value of an Arrow-Debreu security under a probability measure $\hat{\mathbb{Q}}$ is given by*

$$AD(x_t, t, T_0) = E^{\hat{\mathbb{Q}}}\big(\mathbb{1}_{\mathcal{A}}\big),$$

where x_t is the vector of the underlying processes.

[36]See Bouziane [2008] and Bakshi and Madan [2000].

Then starting from the risk-neutral bond price dynamics in (3.83) and the generalised put price formula (3.93), we express the price of an option on a defaultable zero coupon bond as follows.

Proposition 3.11 *The price of a European put option with maturity T_0 and strike K on a defaultable bond with maturity T that is knocked-out on default is given by*

$$\mathcal{P}(t, r^d, T; T_0, K) = K\bar{P}^d(t, T_0)\big[1 - \Pi_1(\xi)\big] - G(t, T_0, T)\big[1 - \Pi_2(\xi)\big], \qquad (3.94)$$

where $\Pi_1(t, T_0)$ and $\Pi_2(t, T_0)$ are exercise probabilities, $\bar{P}^d(t, T_0)$ is the price of the defaultable zero coupon bond and

$$G(t, T_0, T) = \tilde{\mathbb{E}}_t\Big[e^{-\int_t^{T_0} r^d(u)du + \bar{X}^d(T_0,T)}\Big]$$

is the scaled forward price, representing the time t price of a commitment to deliver at time T_0 the quantity $\bar{X}^d(T_0, T)$. In addition, the risk-neutral probabilities are given by

$$\Pi_j(\xi) = \frac{1}{2} + \frac{1}{\pi}\int_{0^+}^{\infty} Re\Bigg[\frac{e^{-i\phi\xi} f_j(t, T_0; \phi)}{i\phi}\Bigg]d\phi, \quad j = 1, 2 \qquad (3.95)$$

where $f_j(t, T_0\phi)$ are the characteristic functions for each security defined by

$$f_1(t, T_0; \phi) = \bar{P}^d(t, T_0)^{-1}\tilde{\mathbb{E}}_t\big[e^{-\int_t^{T_0} r^d(u)du + i\phi\bar{X}^d(T_0,T)}\big], \qquad (3.96a)$$

$$f_2(t, T_0; \phi) = G(t, T_0, T)^{-1}\tilde{\mathbb{E}}_t\big[e^{-\int_t^{T_0} r^d(u)du + (1+i\phi)\bar{X}^d(T_0,T)}\big]. \qquad (3.96b)$$

Proof: See Appendix B.6. ■

To calculate these probabilities, it remains to solve for the associated characteristic functions and to this end we introduce a transform

$$\Psi(z) = \tilde{\mathbb{E}}_t\big[e^{-\int_t^{T_0} r^d(u)du + z\bar{X}^d(T_0,T)}\big].$$

From (3.96a) and (3.96b) we note that the characteristic functions can be expressed as

$$f_1(t,T_0;\phi) = \frac{\Psi(i\phi)}{\bar{P}^d(t,T_0)} \quad \text{and} \quad f_2(t,T_0;\phi) = \frac{\Psi(1+i\phi)}{G(t,T_0,T)}. \tag{3.97}$$

In particular, we observe that we can write this transform in a more general form $\Psi(a+i\phi)$, for $a = \{0,1\}$. Following the approach used in Repplinger [2008], by changing to the T_0−forward measure we can write the transform

$$\Psi(z) = \bar{P}^d(t,T_0)\tilde{\mathbb{E}}_t\Big[\frac{e^{-\int_t^{T_0} r^d(u)du}}{\bar{P}^d(t,T_0)} e^{z\bar{X}^d(T_0,T)}\Big] := \bar{P}^d(t,T_0)\Upsilon_t(z), \tag{3.98}$$

where the function $\Upsilon_t(z)$ is defined by

$$\Upsilon_t(z) = \tilde{\mathbb{E}}_t^{T_0}\big[e^{z\bar{X}^d(T_0,T)}\big]. \tag{3.99}$$

The cumulative probabilities in (3.95) of Proposition 3.11 can then be written as

$$\begin{aligned} \Pi_1(\xi) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{\Upsilon_t(i\phi)e^{-i\phi\xi}}{i\phi}\Big]d\phi, \\ \Pi_2(\xi) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{\bar{P}^d(t,T_0)}{G(t,T_0,T)}\frac{\Upsilon_t(1+i\phi)e^{-i\phi\xi}}{i\phi}\Big]d\phi. \end{aligned} \tag{3.100}$$

It has been shown (see Keith and Stuart [1994]) that the integrals in (3.100) are well defined and convergent over the interval $\phi \in [0,\infty)$.

Proposition 3.12 *The expectation for $\Upsilon_t(z)$ in (3.99) takes an exponential affine form*

$$\Upsilon_t(z) = \exp\Big(z\hat{X}(t) + A(t,z) + \sum_{i=1}^{2n} C_i(t,z)V_i(t)\Big), \tag{3.101}$$

where the deterministic functions $A(t,z)$ and $C_i(t,z)$, $i = 1,2,...,2n$ solve the system of *ordinary differential equations*

$$\begin{cases} \dfrac{dC_i(t,z)}{dt} = -\dfrac{1}{2}\displaystyle\sum_{j=1}^{3}(\bar{\sigma}_{ij}^{\mathbb{V}})^2 C_i^2(t,z) + \Big(\kappa_i^{\mathbb{V}} + z\sum_{j=1}^{3}(d_i(t,T) - d_i(t,T_0))\bar{\sigma}_{ij}^{\mathbb{V}}\Big)C_i(t,z) \\ \qquad\qquad + \dfrac{1}{2}\big(z - z\bar{z}\big)\big(d_i(t,T) - d_i(t,T_0)\big)^2, \\ \dfrac{dA(t,z)}{dt} = -\displaystyle\sum_{i=1}^{2n} C_i(t,z)\kappa_i^{\mathbb{V}}\bar{V}_i, \end{cases} \tag{3.102}$$

subject to the boundary conditions $\bar{P}^d(T_0, T_0) = 1$ *and* $A(T_0, z) = C_i(T_0, z) = 0$.

Proof: See Appendix B.7.

We observe that the first ordinary differential equation in (3.102) (or equivalently (B.7.6)) are of the Riccati type with complex coefficients. It was noted in Tahani [2004] that although closed-form solution for such a system can be derived, it usually involves complex algebra with Whittaker and hypergeometric functions and a numerical approach was shown to provide far more efficient and accurate solutions at a lower computational cost. The characteristic functions can then be inverted to give the desired cumulative probabilities $\Pi_j(\xi, \phi)$ since the integrand in (3.95) is smooth and decays rapidly. Repplinger [2008, Section 7.2.2] derived an explicit solution for a system of ordinary differential equations similar to (3.102) in the default-free framework by making use of degenerate hypergeometric-functions.

3.6 Summary

In this chapter, we have developed a Markovian HJM model for the defaultable term structure where the volatility function is dependent on the time to maturity, default-free short rate, short term credit spread and volatility. This class of volatility functions allow for a high degree of flexibility in modelling the wide range of shapes of the yield curve in addition to allowing for dependence on the driving stochastic variables. Making the short rate process Markovian has great computational advantage since any Markov process can always be mapped into a recombining lattice whose number of nodes grows linearly with the number of time steps. The dependence of the defaultable forward rate volatility function on the path enters through its dependence on the short rate and the credit spread processes.

We obtain a generalisation that expresses the defaultable forward rate curve as an affine function of a set of state variables which summarises the history of the forward rate curve evolution. This an extension of the work in Chiarella and Kwon [2003] to the defaultable HJM framework where we adopt the humped volatility specification. We then derived a closed form formula for the defaultable zero coupon bond prices, expressing them as an exponential affine function of state variables. We then demonstrated how the framework can be applied to price credit default swaps and swaptions and derived some approximating formulas for single-name CDS prices and show how this could extended to include counterparty risk. From the conducted numerical simulations, we observed that the model captures the stylistic features of the credit default swaps and swaptions with respect to the assumptions on the recovery of the defaultable bond as well as the time to maturity of the option.

The model is finally applied to price options on a defaultable asset, in this case, a defaultable bond with a knock-out provision. We made a simplifying assumption by relaxing level dependence in the structure of the volatility functions to facilitate the computation of a closed-form solution. We solve the coupled system of differential equation that arise when calculating the cumulative probabilities using numerical integration (Fourier transform methods) from which we derive a semi-closed option pricing formula.

Chapter 4

Defaultable HJM Class of Models with Regime-Switching Volatility

In this chapter, we present a model for pricing defaultable bonds within the regime-switching HJM class of models under fractional recovery. We follow two approaches to incorporate regime-switching within our model. In the first case, stochasticity is introduced to the volatility function by a modulating Markov chain via the separable volatility specification. Some special cases of short rate models are then obtained, from which explicit bond price formulas are derived. We then look at the general case where the volatility function of the stochastic volatility process is modulated by an underlying Markov chain.

4.1 Introduction

Due to the close relationship between income growth (which fluctuates with the business cycle) and the demand and supply for money, the time series of interest rates exhibits cyclic patterns. Early empirical evidence as documented in Hamilton [1989] suggests that economies experience recurrent shifts between distinct regimes of the business cycles whose expansion and recession have regime-switching effects on nominal interest rates as well as changes in monetary policy and exchange rate regime. The distinct regimes allow for the underlying processes to follow different dynamics while in different states of the world; for example 'good' and 'bad' economic environments. Lam and Li [1998] combined a first order Markov process and a stochastic volatility model and found that the volatility of the S&P 500 can be well captured by a two-state regime-switching, stochastic volatility model whereas Sola and Driffill [1994] and Garcia and Perron [1996] investigated regime shifts in real interest rates. As noted in Wu and Zeng [2005], it is most likely that Markov regime shifts represent a systematic risk that should be priced in term structure models. This implies that the bond risk premium consists of risk arising from both the diffusion risk and regime-switching risk where the latter stems from systematic risk of periodic shifts on interest rates and/or bond prices as a result of changing regimes.

Research to investigate the impact of switching regimes on the yield curve has involved incorporating hidden Markov chains into the stochastic processes of the state variables and pricing kernels. Empirical evidence indicates that the yield curve shows varying properties across regimes and therefore changing regimes affect bond returns. Landén [2000] considered a diffusion type model for the short rate with both the diffusion and drift parameters being modulated by an underlying Markov process and derived a closed form solution for bond prices under the risk-neutral measure. Kalimipalli and Susmel [2004] introduced regime-switching in a 2-factor stochastic volatility model and modelled the volatility of short-term interest rates as a stochastic volatility process whose mean is subject to shifts in regime. Their in-sample results favor regime-switching stochastic volatility (RSV) model as compared to a single-state stochastic volatility model or a GARCH family of models. However, their out-of-sample results were mixed and provided weak support for regime-switching stochastic volatility models.

Wu and Zeng [2005] obtained closed form solution of the term structure of interest rates under an affine-type (CIR) model and showed that with regime-switching risk, the model captures the empirical features in the term structure of interest rates. This was further extended in Wu and Zeng [2006] who proposed a regime-dependent jump-diffusion model of the term structure of interest rates to capture the effects of discrete jumps in the interest rates coupled with the shifts in policy regime that induce systematic risk. Elliott and Wilson [2007, Chapter 2] model the short rate as a random process where they assume the mean-reverting level follows a finite-state, continuous-time Markov chain that switches to different levels producing a cyclical pattern in the short rate. The randomness of the Markov chain prevents the business cycle lengths and intensities from being predictable.

There is less literature that focusses on the dynamics of the term structure of volatility of the forward rates as a function of maturity. In the deterministic volatility Heath et al. [1992] (HJM) models, the volatility curve is fixed and the volatility of a specific forward rate moves along the curve. Thus, there is a deterministic motion along a fixed curve. However, to be able to describe the volatility curve effectively, there is need for a process with both deterministic and jump movements. Jump diffusion models are not adequate to capture these as they generate jumps too frequently.

A class of piecewise-deterministic Markov processes was introduced in Davis [1984] which allows deterministic motion and random jumps. This class, which includes as special cases, various non-diffusion models provides a broad modelling framework for problems of this nature and is closely related to a class of stochastic jump processes for which stochastic calculus tools are readily available. Valchev [2004] introduced a continuous-time Markov chain parameterizations of volatility within the HJM model. This specification allows for jump discontinuities as well as other deformations of the term structure of volatilities and provides an extension of the class of deterministic volatility HJM models to a wider class of models with piecewise-deterministic volatility.

Elhouar [2008] investigated the HJM models with regime-switching stochastic volatility and established the necessary and sufficient conditions on the volatility that guarantee finite dimensional realizations. The forward rate volatility is allowed to depend on the current forward rate curve as well as on a Markov chain with finite number of states. Valchev and Elliott [2004] provided regime-switching stochastic volatility extensions of the LIBOR market model where the instantaneous forward LIBOR volatility is modulated by a continuous time homogeneous Markov chain. Jeanblanc and Valchev [2004] developed a pricing model for defaultable bonds by assuming that the default intensity is driven by a Markov chain which accounts for both default and liquidity risks.

There are three main contributions of this chapter: Firstly, we establish the conditions on the defaultable forward rate volatility that would lead to finite dimensional Markovian realisations of the defaultable short rate dynamics in the presence of regime-switching. By considering a choice of exponentially decaying volatility functions modulated by a continuous time Markov chain which is independent of the driving Wiener processes, we derive a two-factor Hull-White-Extended-Vasicek type of model. In addition to offering better calibration to market data, the model can be automatically calibrated to the currently observed yield curve due to the inherent advantages of the underlying HJM framework.

Secondly, by expressing the defaultable bond price in an exponentially affine form we solve the regime-switching bond pricing partial differential equation and derive a semi-closed form solution for the price of the bond. This is achieved by numerically solving a coupled system of ordinary differential equations. Using Monte Carlo simulation, we investigated the distributional properties of both the defaultable short rate and bond price dynamics under regime-switching volatility. Thirdly, we consider option pricing within this framework. In particular, we price a European call option on a defaultable bond with a knock-out provision for the special case of 2-states regimes. By applying finite difference (theta scheme) methods to the coupled option pricing partial differential equations, we approximate the option price on a discretely space-time grid.

The structure of this chapter is as follows: In Section 4.2, we give a brief review of the key mathematical tools in the theory of Markov chains that we will use in this chapter. Section 4.2.2 introduces the defaultable HJM framework with regime-switching. This can be seen as a *variation of the framework developed in Chapter 2 to allow for regime-switching* stochastic volatility. In Section 4.3, we develop a Markovian HJM framework in the presence of regime-switching and derive a semi-explicit bond price formulation under the special case of two-state regimes. In Section 4.4 we investigate the option pricing problem in the presence of regime-switching stochastic volatility for the special case of two-state regimes and Section 4.5 concludes the chapter. We then provide the proof of some of the technical results in Appendix III.

4.2 The Model Setup

We consider an arbitrage-free bond market modelled on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ where $\mathbb{P}$ is the real world probability measure. The augmented and right continuous filtration is given by $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^N \vee \mathcal{F}_t^X$ where the filtrations $\mathcal{F}_t^W$ and $\mathcal{F}_t^N$ are generated by the observations of the Wiener process and counting process respectively and are as defined in Chapter 2 whereas $\mathcal{F}_t^X$ is the filtration generated by the Markov chain to be introduced in Section 4.2.1 satisfying the usual conditions.

4.2.1 Markov Chain Framework

Following Elliott et al. [1994], we let $X(t)$, $t \geq 0$ be a finite state Markov chain with state space $\mathcal{S} = \{s_1, s_2, ..., s_N\}$ defined on the above probability space. The s_i's are points in $\mathbb{R}^N$ and can be used to model factors of the economy which for simplicity can be identified with unit vectors $\{e_1, e_2, ..., e_N\}$ with $e_i = (0, ..., 0, 1, 0, ..., 0)^\top \in \mathbb{R}^N$. At any given time t, the state $X(t)$ of the Markov chain is one of the unit vectors, $e_1, e_2, ..., e_N$. For any real valued function of $X(t)$, say $g(X(t))$, with $g_i = g(e_i)$ and $g = (g_1, g_2, ..., g_N)^\top$, we have that $g(X(t)) = \langle g, X(t)\rangle$ where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^N$.

The unconditional distribution of $X(t)$ is the vector $\mathbb{E}[X(t)] = p_t = (p_t^1, p_t^2, ..., p_t^N)$, where

$$p_t^i = P(X(t) = e_i) = \mathbb{E}\big[\langle e_i, X(t)\rangle\big], \quad \text{for} \quad 1 \leq i \leq N.$$

Likewise, the $K-$dimensional row vector $p_0 = [p_0(i)]_{1\leq i\leq N} = [P\{X(0) = i\}]_{1\leq i\leq N}$ denotes the initial probability distribution for the Markov chain $X(t)$ under measure $\mathbb{P}$.

Definition 4.1 *A two-parameter family $\mathbf{P}(s,t)$, $s,t \in \mathbb{R}_+, s \leq t$ of stochastic matrices is called the family of transition probability matrices for the $\mathcal{F}_t^X-$Markov Chain X if, for every $s,t \in \mathbb{R}_+, s \leq t$*

$$P(X_t = j | X_s = i) = p_{ij}(s,t), \quad \forall i,j \in \mathcal{S},$$

and in particular, the equality $\mathbf{P}(s,s) = I$ is satisfied for every $s \in \mathbb{R}_+$. In addition, $\mathbf{P}(s,t)$ satisfies the Chapman-Kolmogorov equation

$$\mathbf{P}(s,t) = \mathbf{P}(s,u)\mathbf{P}(u,t), \quad \forall 0 \leq s \leq u \leq t.$$

Let $\mathbf{P}(s, t + \triangle t)$ be right continuous at $\triangle t = 0$. It has been shown[37] that right continuity of the family implies its right-hand differentiability at $\triangle t = 0$. More specifically, the finite limit

$$h_{ij}^X(t) = lim_{\triangle t\downarrow 0}\frac{p_{ij}(t,t+\triangle t) - \delta_{ij}}{\triangle t},$$

exists where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Given that $h_{ij}^X(t) \geq 0$ for arbitrary $i \neq j$,

$$h_{ii}^X(t) = lim_{\triangle t\downarrow 0}\frac{p_{ii}(t,t+\triangle t) - 1}{\triangle t} = -lim_{\triangle t\downarrow 0}\frac{\sum_{i\neq j,j=1}^N p_{ij}(t,t+\triangle t)}{\triangle t} = -\sum_{i\neq j,j=1}^N h_{ij}^X(t). \quad (4.1)$$

[37]For instance, Theorem 8.1.2 in Rolski et al. [1998]

The variable $h_{ij}^X(t)$ is called the transition intensity from state i to j of the Markov chain and the function $H(t) = \{h_{ij}^X(t)\}_{1\leq i,j\leq N}$ denotes the infinitesimal generator matrix (also called the intensity matrix) associated with the time homogenous Markov chain. For any two arbitrary states $i, j \in \mathcal{S}$, the Chapman-Kolmogorov equation yields

$$p_{ij}(s, t+\triangle t) = \sum_{k=1}^{N} p_{ik}(s,t)p_{kj}(t, t+\triangle t), \quad 0 \leq s \leq t \leq t+\triangle t.$$

It then follows that,

$$lim_{\triangle t\downarrow 0}\frac{p_{ij}(s,t+\triangle t) - p_{ij}(s,t)}{\triangle t} = \sum_{k=1}^{N} p_{ik}(s,t)h_{kj}^X(t),$$

which is the forward Kolmogorov equation

$$\frac{d\mathbf{P}(s,t)}{dt} = \mathbf{P}(s,t)H(t), \quad \mathbf{P}(s,s) = I. \tag{4.2}$$

It can be shown that $\mathbf{P}(s,t)$ also satisfies the Kolmogorov backward equation

$$\frac{d\mathbf{P}(s,t)}{dt} = -H(s)\mathbf{P}(s,t), \quad \mathbf{P}(t,t) = I. \tag{4.3}$$

For the time homogenous Markov chain, we have the following:

Definition 4.2 *A one-parameter family* $\mathbf{P}(t)$, $t \in \mathbb{R}_+$ *of stochastic matrices is called the family transition probability matrices for the* $\mathcal{F}_t^X-$*Markov chain* X *if, for every* $s, t \in \mathbb{R}_+$,

$$P(X_{s+t} = j|X_s = i) = p_{ij}(t), \quad \forall i, j \in \mathcal{S}.$$

The transition intensity process can then be shown to satisfy

$$h_{ij}^X = lim_{t\downarrow 0}\frac{p_{ij}(t) - p_{ij}(0)}{t} = lim_{t\downarrow 0}\frac{p_{ij}(t) - \delta_{ij}}{t}, \text{with} \quad h_{ii}^X = -\sum_{i\neq j} h_{ij}^X.$$

In this case, the Kolmogorov forward and backward equations are given by

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{P}(t)H, \quad \mathbf{P}(0) = I, \tag{4.4}$$

$$\frac{d\mathbf{P}(t)}{dt} = H\mathbf{P}(t), \quad \mathbf{P}(0) = I, \tag{4.5}$$

respectively and have the same unique solution

$$\mathbf{P}(t) = e^{tH} := I + \sum_{n\geq 1} \frac{(tH)^n}{n!}, \quad \forall t \in \mathbb{R}_+. \tag{4.6}$$

The generator matrix $H = \sum_{i,j=1}^{N} h_{i,j}^X$ uniquely determines all the relevant probabilistic properties of the time homogenous Markov chain.

A well known result (see Elliott et al. [1994]) is that $X(t)$ admits a semi-martingale representation of the form

$$X(t) = X(0) + \int_0^t HX(s)ds + M^X(t), \tag{4.7}$$

where $M^X(t)$ is an $\mathcal{F}_t^X$–martingale such that $\mathbb{E}[M^X(t)|\mathcal{F}_s^X] = M^X(s)$. A function of the Markov chain $X(t) \in \mathcal{S}$ can be represented by a vector $g(t) = (g_1(t), ..., g_N(t))$ so that

$$g(t, X(t)) = g(t)^\top \cdot X(t) = \langle g(t), X(t) \rangle.$$

4.2.2 Defaultable HJM Model with Markov Chain Volatility

Similar to Definition 2.3, the default time τ_i corresponds to the *i-th* jump (the *i-th* default) of a marked point process $N(t)$ which is characterized by a general intensity process $h^N(t, dq) := h^N(t)$. This intensity is independent of previous defaults and the default time is a totally inaccessible stopping time. The default intensity is also assumed to be independent of the transition intensity $h^X(t)$ of the Markov chain $X(t)$ and as in Chapter 2, we assume that on default, a firm is reorganized and the debt re-floated allowing for multiple defaults and recoveries where the recovery rate $\mathcal{R}(t)$ is a measure of the expected fractional recovery on default. At maturity, the defaultable bond has a final payoff

$$\mathcal{R}(T) := \prod_{\tau_i \leq T} (1 - q(\tau_i)), \quad \mathcal{R}(t) \in [0, 1], \tag{4.8}$$

where $\mathcal{R}(T)$ is the product of the face reductions $q(\tau_i)$ after all defaults until maturity T.

We denote as $P(t, T, X(t))$ the price at time t of the default-free zero coupon bond with maturity $T \geq t$ and $P^d(t, T, X(t))$ the price at time t of the defaultable zero coupon bond with maturity $T > t$. The following definition on the default-free forward rate, defaultable forward rate and forward credit spread is an analogue of Definition 2.1 and Definition 2.2 modified to incorporate regime switching.

Definition 4.3 *1. The instantaneous default-free forward rate of interest prevailing at time t for instantaneous borrowing at T, is defined as*

$$f(t, T, X(t)) = -\frac{\partial}{\partial T} \ln P(t, T, X(t)), \quad \textit{for all} \quad t \in [0, T]. \tag{4.9}$$

2. The instantaneous defaultable forward rate of interest prevailing at time t with maturity $T > t$ is defined by

$$f^d(t, T, X(t)) = -\frac{\partial}{\partial T} \ln P^d(t, T, X(t)), \quad \textit{for all} \quad t \in [0, T]. \tag{4.10}$$

3. In addition, we define the continuously compounded instantaneous forward credit spread as

$$\lambda(t, T, X(t)) = f^d(t, T, X(t)) - f(t, T, X(t)). \tag{4.11}$$

4. The instantaneous default-free short rate is defined as $r(t, X(t)) = f(t, t, X(t))$, the in-

stantaneous defaultable short-rate is defined by $r^d(t, X(t)) = f^d(t, t, X(t))$ *and following (4.11), the short-term credit spread is defined by* $c(t, X(t)) = \lambda(t, t, X(t))$.

We adopt the approach used in Schönbucher [1998] who showed that a model of the spread for the defaultable forward rates over the default free forward rates may be used to add a default-risk module to an existing default-free model of forward rates.

The pre-default price $\bar{P}^d(t, T, X(t))$ at time t of a defaultable zero-coupon bond with maturity T, a so-called 'pseudo' bond, is the price of the defaultable zero-coupon bond given that it has not defaulted before time t and is given by

$$\bar{P}^d(t, T, X(t)) = \exp\Big(- \int_t^T f^d(t, v, X(t)) dv \Big). \tag{4.12}$$

The price of the defaultable bond can then be written as

$$P^d(t, T, X(t)) = \mathcal{R}(t) \bar{P}^d(t, T, X(t)). \tag{4.13}$$

We assume that $f(t, T, X(t))$ and $\lambda(t, T, X(t))$ are the unique strong solutions to the stochastic integral equations

$$f(t, T, X(t)) = f(0, T) + \int_0^t \alpha^f(u, T, X(u)) du + \int_0^t \sigma^f(u, T, X(u)) dW^f(u), \tag{4.14a}$$

$$\lambda(t, T, X(t)) = \lambda(0, T) + \int_0^t \alpha^\lambda(u, T, X(u)) du + \int_0^t \sigma^\lambda(u, T, X(u)) dW^\lambda(u), \tag{4.14b}$$

respectively, where we have dropped the dependence on X_0 in the initial curves $f(0, T)$ and $\lambda(0, T)$ for notational simplicity, with the driving Wiener processes $W^f(t)$ and $W^\lambda(t)$ being correlated under the real world probability measure $\mathbb{P}$. Setting $T = t$, equations (4.14a) and

(4.14b) yield the short rate and short term credit spread dynamics

$$r(t, X(t)) = f(0,t) + \int_0^t \alpha^f(u,t,X(u))du + \int_0^t \sigma^f(u,t,X(u))dW^f(u), \tag{4.15a}$$

$$c(t, X(t)) = \lambda(0,t) + \int_0^t \alpha^\lambda(u,t,X(u))du + \int_0^t \sigma^\lambda(u,t,X(u))dW^\lambda(u), \tag{4.15b}$$

respectively.

To apply the techniques of Heath et al. [1992], it is convenient to replace the correlated Wiener processes $W^f(t)$ and $W^\lambda(t)$ with uncorrelated processes. We define uncorrelated Wiener processes $W_i(t)$, $i = 1, 2$ such that

$$\begin{bmatrix} dW^f(t) \\ dW^\lambda(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}. \tag{4.16}$$

where the $(a_{ij})'s$, $i, j = 1, 2$ are chosen so as to preserve the correlation structure of the Wiener processes $W^f(t)$ and $W^\lambda(t)$. A possible characterisation that yields a system with 2 noise processes driving the defaultable dynamics and 1 noise process for the default-free dynamics is made by

$$a_{11} = 1, \quad a_{12} = 0, \quad a_{21} = \rho, \quad a_{22} = \sqrt{1-\rho^2},$$

from which it follows that

$$dW^f(t) = dW_1(t), \tag{4.17a}$$

$$dW^\lambda(t) = \rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t). \tag{4.17b}$$

The forward rate and forward credit spread stochastic integral equations (4.14a) and (4.14b) can be expressed as

$$f(t,T,X(t)) = f(0,T) + \int_0^t \alpha^f(u,T,X(u))du + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^f(u,T,X(u))dW_i(u), \tag{4.18a}$$

$$\lambda(t,T,X(t)) = \lambda(0,T) + \int_0^t \alpha^\lambda(u,T,X(u))du + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^\lambda(u,T,X(u))dW_i(u), \tag{4.18b}$$

where

$$\tilde{\sigma}_1^f(t,T,X(t)) = \sigma^f(t,T,X(t)), \quad \tilde{\sigma}_2^f(t,T,X(t)) = 0, \tag{4.19}$$

$$\tilde{\sigma}_1^\lambda(t,T,X(t)) = \rho\sigma^\lambda(t,T,X(t)) \quad \text{and} \quad \tilde{\sigma}_2^\lambda(t,T,X(t)) = \sqrt{1-\rho^2}\sigma^\lambda(t,T,X(t)). \tag{4.20}$$

Using equations (4.11), (4.18a), (4.18b) and the transformation in (4.16), the defaultable forward rate follows stochastic integral equation

$$f^d(t,T,X(t)) = f^d(0,T) + \int_0^t \alpha^d(u,T,X(u))du + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^d(u,T,X(u))dW_i(u), \tag{4.21}$$

where the drift and the volatility functions are given by

$$\alpha^d(t,T,X(t)) = \alpha^f(t,T,X(t)) + \alpha^\lambda(t,T,X(t)),$$
$$\tilde{\sigma}_i^d(t,T,X(t)) = \tilde{\sigma}_i^f(t,T,X(t)) + \tilde{\sigma}_i^\lambda(t,T,X(t)),$$

respectively, for $i = 1, 2$. For $T = t$, the defaultable short rate dynamics satisfy the stochastic integral equation

$$r^d(t,X(t)) = f^d(0,t) + \int_0^t \alpha^d(u,t,X(u))du + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^d(u,t,X(u))dW_i(u). \tag{4.22}$$

Substituting (4.21) into (4.10) then applying the stochastic Fubini theorem and Itô's lemma, we observe that the defaultable bond price satisfies the stochastic differential equation[38]

$$\frac{dP^d(t,T,X(t))}{P^d(t-,T,X(t-))} = \mu^d(t,T,X(t))dt + \sum_{i=1}^{2} \tilde{\sigma}^d_{B,i}(t,T,X(t))dW_i(t) - \int_E q\mu(dt,dq), \quad (4.23)$$

where

$$\begin{cases} \mu^d(t,T,X(t)) = r^d(t,X(t)) + b^d(t,T,X(t)), \\ b^d(t,T,X(t)) = -\alpha^d_B(t,T,X(t)) + \frac{1}{2}\sum_{i=1}^{2}\left(\tilde{\sigma}^d_{B,i}(t,T,X(t))\right)^2, \\ \alpha^d_B(t,T,X(t)) = \int_t^T \alpha^d(t,v,X(t))dv, \\ \tilde{\sigma}^d_{B,i}(t,T,X(t)) = -\int_t^T \tilde{\sigma}^d_i(t,v,X(t))dv. \end{cases} \quad (4.24)$$

As in Chapter 2, we note that the absence of arbitrage opportunities implies that there exists an equivalent probability measure $\tilde{\mathbb{P}}$, namely the risk-neutral measure. For every finite maturity T, there exists a 3-dimensional predictable process $\Phi(t) = \{\phi_1(t), \phi_2(t), t \in [0,T]\}$ and a strictly positive measurable function $\psi(t,q)$ satisfying the integrability conditions

$$\int_0^t ||\phi_i(u)||^2 du < \infty, \quad \text{for} \quad i = 1, 2, \qquad \int_0^t \int_E |\psi(u,q)h(u,dq)|du < \infty, \quad (4.25)$$

such that

$$d\tilde{W}_i(t) = dW_i(t) - \phi_i(t)dt, \quad \text{for} \quad i = 1, 2 \quad (4.26)$$

is a $\tilde{\mathbb{P}}$-Wiener process and the default indicator process $N(t)$ has a $\tilde{\mathbb{P}}$-intensity

$$\tilde{h}(t,dq) = \psi(t,q)h(t,dq). \quad (4.27)$$

This market price of risk incorporates the market prices of interest rate and credit spread

[38] A detailed formulation of this result follows the approach given Appendix A.2 although in the Chapter 2 we had dependence on the stochastic volatility process.

risks. It should be noted that the regime-switching risk will feature in the stochastic dynamics for the Markov chain through the modified compensator process $\tilde{H}$ under the risk-neutral measure. In this case, the corresponding HJM defaultable forward rate drift restriction condition is given by

$$\alpha^d(t,T,X(t)) = -\sum_{i=1}^{2}\tilde{\sigma}_i^d(t,T,X(t))\Big(\phi_i(t) - \int_t^T \tilde{\sigma}_i^d(t,s,X(t))ds\Big). \tag{4.28}$$

Following Heath et al. [1992] and Schönbucher [1998] who derive the no-arbitrage, drift-restriction conditions for the default-free and defaultable forward rate respectively, we observe that under the risk-neutral measure $\tilde{\mathbb{P}}$ the defaultable forward rate follows the stochastic integral equation[39]

$$\begin{aligned} f^d(t,T,X(t)) &= f^d(0,T) + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^f(u,T,X(u))\int_u^T \tilde{\sigma}_i^f(u,v,X(u))dvdu \\ &+ \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^\lambda(u,T,X(u))\int_u^T \tilde{\sigma}_i^\lambda(u,v,X(u))dvdu + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^\lambda(u,T,X(u))\int_u^T \tilde{\sigma}_i^f(u,v,X(u))dvdu \\ &+ \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^f(u,T,X(u))\int_u^T \tilde{\sigma}_i^\lambda(u,v,X(u))dvdu \\ &+ \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^f(u,T,X(u))d\tilde{W}_i(u) + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^\lambda(u,T,X(u))d\tilde{W}_i(u). \end{aligned} \tag{4.29}$$

Similarly, using the no-arbitrage drift restriction condition derived in Pugachevsky [1999], the forward credit spread satisfies the stochastic integral equation

$$\begin{aligned} \lambda(t,T,X(t)) &= \lambda(0,T) + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^\lambda(u,T,X(u))\int_u^T \tilde{\sigma}_i^\lambda(u,v,X(u))dvdu \\ &+ \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^f(u,T,X(u))\int_u^T \tilde{\sigma}_i^\lambda(u,v,X(u))dvdu + \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^\lambda(u,T,X(u))\int_u^T \tilde{\sigma}_i^f(u,v,X(u))dvdu \\ &+ \sum_{i=1}^{2}\int_0^t \tilde{\sigma}_i^\lambda(u,T,X(u))d\tilde{W}_i(u). \end{aligned} \tag{4.30}$$

By setting $T = t$ in (4.29), we find that the instantaneous defaultable short interest rate

[39] This mirrors the approach that we employed in Section 2.2.3.

dynamics follow the stochastic integral equation

$$\begin{aligned}
r^d(t, X(t)) &= f^d(0,t) + \sum_{i=1}^{2} \int_0^t \tilde{\sigma}_i^f(u,t,X(u)) \int_u^t \tilde{\sigma}_i^f(u,v,X(u))dvdu \\
&+ \sum_{i=1}^{2} \int_0^t \tilde{\sigma}_i^\lambda(u,t,X(u)) \int_u^t \tilde{\sigma}_i^\lambda(u,v,X(u))dvdu \\
&+ \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,X(u)) \int_u^t \tilde{\sigma}_i^f(u,v,X(u))dvdu + \sum_{i=1}^{2} \int_0^t \tilde{\sigma}_i^f(u,t,X(u)) \int_u^t \tilde{\sigma}_i^\lambda(u,v,X(u))dvdu \\
&+ \sum_{i=1}^{2} \int_0^t \tilde{\sigma}_i^f(u,t,X(u))d\tilde{W}_i(u) + \sum_{i=1}^{2} \int_0^t \tilde{\sigma}_i^\lambda(u,t,X(u))d\tilde{W}_i(u),
\end{aligned} \tag{4.31}$$

whereas the short term credit spread satisfies

$$\begin{aligned}
c(t, X(t)) &= \lambda(0,t) + \sum_{i=1}^{2} \int_0^t \tilde{\sigma}_i^\lambda(u,t,X(u)) \int_u^t \tilde{\sigma}_i^\lambda(u,v,X(u))dvdu \\
&+ \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^\lambda(u,t,X(u)) \int_u^t \tilde{\sigma}_i^f(u,v,X(u))dvdu + \sum_{i=1}^{2} \int_0^t \tilde{\sigma}_i^f(u,t,X(u)) \int_u^t \tilde{\sigma}_i^\lambda(u,v,X(u))dvdu \\
&+ \sum_{i=1}^{2} \int_0^t \tilde{\sigma}_i^\lambda(u,t,X(u))d\tilde{W}_i(u).
\end{aligned} \tag{4.32}$$

The Markov chain $X(t)$ in (4.7) represents N distinct regimes taking on values $1, 2, ..., N$. We can write its dynamics in differential form under the risk neutral measure as

$$dX(t) = \tilde{H}X(t)dt + d\tilde{M}^X(t), \tag{4.33}$$

where $\tilde{M}^X(t)$ is a martingale under the risk-neutral measure $\tilde{\mathbb{P}}$. By making the assumption that the market price of regime-switching risk is zero, the transition intensity remains the same under both measures since the generator is given by $\tilde{H} = \sum_{i,j=1}^{N} h_{i,j}^X$. As in (2.55), the defaultable bond price in this case can be shown to satisfy the expectation

$$\begin{aligned}
P^d(t,T,X(t)) &= \tilde{\mathbb{E}}\Big[\exp\Big(- \int_t^T r(u,X(u))du\Big)\mathcal{R}(T)\Big|\mathcal{F}_t\Big] \\
&= \mathbb{1}_{\{\tau > t\}}\tilde{\mathbb{E}}\Big[\int_t^T \mathcal{R}(u)\tilde{h}^N(u,dq)e^{-\int_t^u (r(s,X(s))+\tilde{h}^N(s,dq))ds}du\Big|\mathcal{F}_t^J\Big],
\end{aligned} \tag{4.34}$$

where the filtration $\mathcal{F}_t^J = \mathcal{F}_t^W \vee \mathcal{F}_t^X$ contains all other information except information on default. Monte Carlo simulation can be used to price the defaultable bond using (4.34).

4.3 Hull-White-Extended-Vasicek Model with Regime-Switching

In this section, we extend the results in Valchev [2004] to the defaultable HJM framework. We assume that the volatility functions $\sigma^f(t,T,X(t))$ and $\sigma^\lambda(t,T,X(t))$ are modulated by the same Markov chain $X(t)$ which could be a vector incorporating different states. For notational simplicity, we suppress the dependency of the defaultable forward rate, defaultable short rate, forward credit spread and short term credit spread on the Markov chain $X(t)$.

4.3.1 Model Formulation

Let $\tau_1^x, ..., \tau_k^x, ...$ denote the jump times of the Markov chain and $\mathcal{J}_t$ the total number of jumps by time t. A general volatility function $\sigma(t,T,\cdot)$ takes any of the N possible values corresponding to the states of the Markov chain. Given that $e_i = \{0,..,0,1,0,...,0\}$, we observe that $\sigma(t,T,X(t)) = (\sigma(t,T,e_1),....,\sigma(t,T,e_N))^\top$ and the volatility functions $\sigma^f(t,T,X(t))$ and $\sigma^\lambda(t,T,X(t))$ can be represented as

$$\sigma^f(t,T,X(t)) = \langle\bar{\sigma}^f(t,T),X(t)\rangle \quad \text{and} \quad \sigma^\lambda(t,T,X(t)) = \langle\bar{\sigma}^\lambda(t,T),X(t)\rangle.$$

Assumption 4.3.1 *We shall assume the particular functional forms*

$$\bar{\sigma}^f(t,T) = \left(\sigma^f(e_1)e^{-\kappa_f(e_1)(T-t)},.....,\sigma^f(e_N)e^{-\kappa_f(e_N)(T-t)}\right),$$
$$\bar{\sigma}^\lambda(t,T) = \left(\sigma^\lambda(e_1)e^{-\kappa_\lambda(e_1)(T-t)},.....,\sigma^\lambda(e_N)e^{-\kappa_\lambda(e_N)(T-t)}\right).$$

Between any two jump times $t \in [\tau_k^x, \tau_{k+1}^x[$, given that the coefficients $\sigma^f(X_{\tau_k^x})$, $\sigma^\lambda(X_{\tau_k^x})$,

$\kappa_f(X_{\tau_k^x})$ and $\kappa_\lambda(X_{\tau_k^x})$ are piecewise constant, the volatility functions are piecewise-deterministic and of the form

$$\sigma^f(t,T,X(t)) = \sigma^f(X_{\tau_k^x})e^{-\kappa_f(X_{\tau_k^x})(T-t)}, \tag{4.35a}$$

$$\sigma^\lambda(t,T,X(t)) = \sigma^\lambda(X_{\tau_k^x})e^{-\kappa_\lambda(X_{\tau_k^x})(T-t)}. \tag{4.35b}$$

By substituting (4.35a) and (4.35b) into (4.29), the defaultable forward rate can be written as

$$\begin{aligned}
f^d(t,T) &= f^d(\tau_k^x,T) + \sigma^f(X_{\tau_k^x})^2\int_{\tau_k^x}^t e^{-\kappa_f(X_{\tau_k^x})(T-s)}\int_s^T e^{-\kappa_f(X_{\tau_k^x})(u-s)}duds\\
&+\sigma^\lambda(X_{\tau_k^x})^2\int_{\tau_k^x}^t e^{-\kappa_\lambda(X_{\tau_k^x})(T-s)}\int_s^T e^{-\kappa_\lambda(X_{\tau_k^x})(u-s)}duds\\
&+\sqrt{1-\rho^2}\sigma^f(X_{\tau_k^x})\sigma^\lambda(X_{\tau_k^x})\Big[\int_{\tau_k^x}^t e^{-\kappa_f(X_{\tau_k^x})(T-s)}\int_s^T e^{-\kappa_\lambda(X_{\tau_k^x})(u-s)}duds\\
&+\int_{\tau_k^x}^t e^{-\kappa_\lambda(X_{\tau_k^x})(T-s)}\int_s^T e^{-\kappa_f(X_{\tau_k^x})(u-s)}duds\Big]\\
&+\int_{\tau_k^x}^t\Big(\sigma^f(X_{\tau_k^x})e^{-\kappa_f(X_{\tau_k^x})(T-s)}+\rho\sigma^\lambda(X_{\tau_k^x})e^{-\kappa_\lambda(X_{\tau_k^x})(T-s)}\Big)d\tilde{W}_1(s)\\
&+\sqrt{1-\rho^2}\int_{\tau_k^x}^t\sigma^\lambda(X_{\tau_k^x})e^{-\kappa_\lambda(X_{\tau_k^x})(T-s)}d\tilde{W}_2(s),
\end{aligned}\tag{4.36}$$

from which the defaultable short rate process dynamics are given by

$$\begin{aligned}
r^d(t) &= f^d(\tau_k^x,t) + \sigma^f(X_{\tau_k^x})^2\int_{\tau_k^x}^t e^{-\kappa_f(X_{\tau_k^x})(t-s)}\int_s^t e^{-\kappa_f(X_{\tau_k^x})(u-s)}duds\\
&+\sigma^\lambda(X_{\tau_k^x})^2\int_{\tau_k^x}^t e^{-\kappa_\lambda(X_{\tau_k^x})(t-s)}\int_s^t e^{-\kappa_\lambda(X_{\tau_k^x})(u-s)}duds\\
&+\sqrt{1-\rho^2}\sigma^f(X_{\tau_k^x})\sigma^\lambda(X_{\tau_k^x})\Big[\int_{\tau_k^x}^t e^{-\kappa_f(X_{\tau_k^x})(t-s)}\int_s^t e^{-\kappa_\lambda(X_{\tau_k^x})(u-s)}duds\\
&+\int_{\tau_k^x}^t e^{-\kappa_\lambda(X_{\tau_k^x})(t-s)}\int_s^t e^{-\kappa_f(X_{\tau_k^x})(u-s)}duds\Big]\\
&+\int_{\tau_k^x}^t\Big(\sigma^f(X_{\tau_k^x})e^{-\kappa_f(X_{\tau_k^x})(t-s)}+\rho\sigma^\lambda(X_{\tau_k^x})\int_{\tau_k^x}^t e^{-\kappa_\lambda(X_{\tau_k^x})(t-s)}\Big)d\tilde{W}_1(s)\\
&+\sqrt{1-\rho^2}\int_{\tau_k^x}^t\sigma^\lambda(X_{\tau_k^x})e^{-\kappa_\lambda(X_{\tau_k^x})(t-s)}d\tilde{W}_2(s).
\end{aligned}\tag{4.37}$$

Figure 4.1 illustrates the evolution of the defaultable forward rate curve together with its modulating Markov chain over 500 time-steps for an economy with high and low volatility regimes given by the parameters $\sigma_0^f = [0.02, 0.04]$, $\sigma_0^\lambda = [0.01, 0.02]$, $\kappa_f = [0.6, 0.8]$ and $\kappa_\lambda = [0.7, 0.9]$. In addition, we assume a constant initial forward curve $f^d(0,T) = 0.07$ while the transition probability matrix is given by

$$\mathbf{P} = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{bmatrix}, \tag{4.38}$$

with an initial distribution

$$\mathbf{p}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}. \tag{4.39}$$

Such a specification of transition probability matrix guarantees that $h_{1,2}^X = h_{2,1}^X$ which implies the same transition intensities for a given state. In addition, we choose the values in (4.38) to ensure that the frequency of regime switching remains low.

From Figure 4.1, we observe that when the regime switches from state 2 to 1, there is a change in the simulated path of defaultable forward rates from high volatility to low volatility. A key difference from a jump process is that the process does not jump to the new state at once but drifts towards it under the new parameters. Since the long term average varies between regimes, this explains the changes in the drift of the forward rate process leading to rapid shifts in the term structure after a switch although the process remains continuous. In addition, the term structures generated over time can cross each other and therefore the traditional static duration measures would perform badly as noted in Hansen and Poulsen [2000].

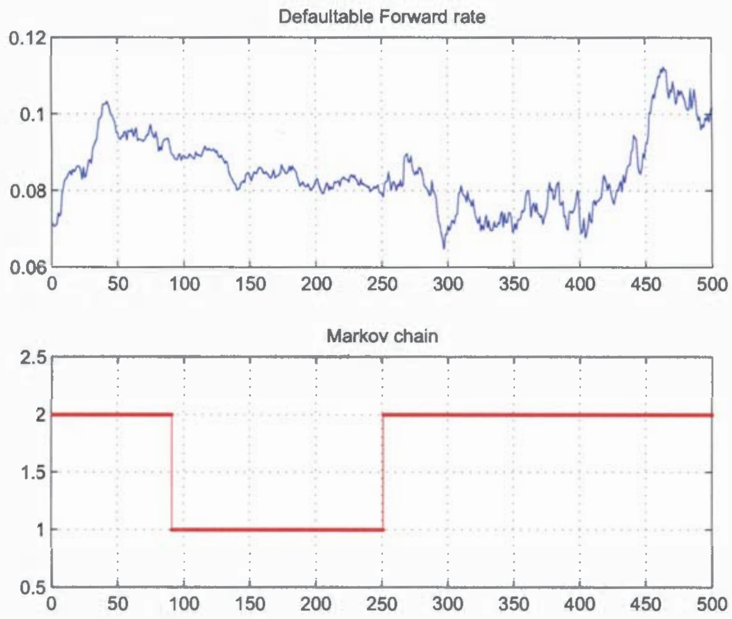


Figure 4.1: Defaultable forward rate dynamics and the modulating Markov chain.

Taking differentials in (4.37) yields the stochastic differential equation

$$
\begin{aligned}
dr^d(t) = \Big[\frac{\partial \mu^d(t,x)}{\partial t} &- \int_{\tau_k}^{t}\Big(\sigma^f(X_{\tau_k})\kappa_f(X_{\tau_k})e^{-\kappa_f(X_{\tau_k})(t-u)} - \rho\sigma^\lambda(X_{\tau_k})\kappa_\lambda(X_{\tau_k})e^{-\kappa_\lambda(X_{\tau_k})(t-u)}\Big)d\tilde{W}_1(u) \\
&- \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k})\kappa_\lambda(X_{\tau_k})\int_{\tau_k}^{t} e^{-\kappa_\lambda(X_{\tau_k})(t-u)}d\tilde{W}_2(u)\Big]dt + \big(\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big)d\tilde{W}_1(t) \\
&+ \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k})d\tilde{W}_2(t),
\end{aligned}
\tag{4.40}
$$

where the $\mu^d(t,x)$ term in the drift is given by

$$
\begin{aligned}
\mu^d(t,x) = f^d(\tau_k,t) &+ \frac{\sigma^f(X_{\tau_k})^2}{2\kappa_f(X_{\tau_k})^2}\big(1-e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\big)^2 + \frac{\sigma^\lambda(X_{\tau_k})^2}{2\kappa_\lambda(X_{\tau_k})^2}\big(1-e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\big)^2 \\
&- \frac{\sigma^f(X_{\tau_k})\sigma^\lambda(X_{\tau_k})}{\kappa_f(X_{\tau_k})\kappa_\lambda(X_{\tau_k})}\sqrt{1-\rho^2}\Big[3 - e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\Big(1+\frac{\kappa_f(X_{\tau_k})}{\kappa_\lambda(X_{\tau_k})+\kappa_f(X_{\tau_k})}e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\Big) \\
&- e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\Big(1+\frac{\kappa_\lambda(X_{\tau_k})}{\kappa_\lambda(X_{\tau_k})+\kappa_f(X_{\tau_k})}e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\Big)\Big].
\end{aligned}
$$

Proposition 4.4 *Given that the volatility functions satisfy Assumption 4.3.1, the defaultable short rate dynamics under the risk-neutral measure $\tilde{\mathbb{P}}$ follow a two-factor Hull and White*

[1990] type of model

$$dr^d(t) = \kappa_\lambda(X_{\tau_k^x})\big[m_d(t,x) - m(t,x)r(t) - r^d(t)\big]dt$$
$$+ \big[\sigma^f(X_{\tau_k^x}) + \rho\sigma^\lambda(X_{\tau_k^x})\big]d\tilde{W}_1(t) + \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k^x})d\tilde{W}_2(t), \tag{4.41a}$$
$$dr(t) = \kappa_f(X_{\tau_k^x})[m_f(t,x) - r(t)]dt + \sigma^f(X_{\tau_k^x})d\tilde{W}_1(t), \quad \textit{for} \quad t \in [\tau_k^x, \tau_{k+1}^x[, \tag{4.41b}$$

where the coefficients in the drift are defined by

$$m_d(t,x) = \frac{\theta(t,x)}{\kappa_\lambda(X_{\tau_k^x})}, \quad m_f(t,x) = \frac{\theta_f(t,x)}{\kappa_f(X_{\tau_k^x})} \quad \textit{and} \quad m(t,x) = \frac{\kappa_f(X_{\tau_k^x}) + \kappa_\lambda(X_{\tau_k^x})}{\kappa_\lambda(X_{\tau_k^x})},$$

where

$$\begin{cases} \theta(t,x) = \dfrac{\partial \mu^d(t,x)}{\partial t} + \kappa_f(X_{\tau_k^x})\mu_f(t,x) + \kappa_\lambda(X_{\tau_k^x})\mu_\lambda(t,x), \\ \theta_f(t,x) = f_2(\tau_k^x, t) + \dfrac{\sigma^f(X_{\tau_k^x})^2}{\kappa_f(X_{\tau_k^x})}\big(1 - e^{-\kappa_f(X_{\tau_k^x})(t-\tau_k^x)}\big) + \kappa_f(X_{\tau_k^x})\mu_f(t,x), \\ \mu^d(t,x) = \mu^f(t,x) + \mu^\lambda(t,x). \end{cases} \tag{4.42}$$

Proof: The proof is given in Appendix C.1. ∎

The coefficients $\kappa_\lambda(X_{\tau_k^x})$, $m_d(t,x)$ and $m(t,x)$ are functions of the modulating Markov chain $X(t)$ which are switching between different values with the jumps of the Markov chain. We observe that due to path dependence on the Wiener processes $\tilde{W}_i(t)$, $(i = 1, 2)$ in (4.40), the system of stochastic differential equations (4.41a) - (4.41b) is non-Markovian. We can write the stochastic integral equation for the default-free and defaultable short rate as

$$r(t) = f(\tau_k^x, t) + S_1(t,x) + \varphi_1(t,x), \tag{4.43a}$$
$$r^d(t) = f^d(\tau_k^x, t) + \sum_{i=1}^{4} S_i(t,x) + \sum_{i=1}^{3} \varphi_i(t,x), \tag{4.43b}$$

respectively, where we define the state variables

$$
\begin{aligned}
S_1(t,x) &= \int_{\tau_k^x}^{t} \sigma^f(X_{\tau_k^x}) e^{-\kappa_f(X_{\tau_k^x})(t-s)} \int_s^t \sigma^f(X_{\tau_k^x}) e^{-\kappa_f(X_{\tau_k^x})(u-s)} du ds,\\
S_2(t,x) &= \int_{\tau_k^x}^{t} \sigma^\lambda(X_{\tau_k^x}) e^{-\kappa_\lambda(X_{\tau_k^x})(t-s)} \int_s^t \sigma^\lambda(X_{\tau_k^x}) e^{-\kappa_\lambda(X_{\tau_k^x})(u-s)} du ds,\\
S_3(t,x) &= \sqrt{1-\rho^2}\int_{\tau_k^x}^{t} \sigma^f(X_{\tau_k^x}) e^{-\kappa_f(X_{\tau_k^x})(t-s)} \int_s^t \sigma^\lambda(X_{\tau_k^x}) e^{-\kappa_\lambda(X_{\tau_k^x})(u-s)} du ds,\\
S_4(t,x) &= \sqrt{1-\rho^2}\int_{\tau_k^x}^{t} \sigma^\lambda(X_{\tau_k^x}) e^{-\kappa_\lambda(X_{\tau_k^x})(t-s)} \int_s^t \sigma^f(X_{\tau_k^x}) e^{-\kappa_f(X_{\tau_k^x})(u-s)} du ds, \qquad (4.44)\\
\varphi_1(t,x) &= \int_{\tau_k^x}^{t} \sigma^f(X_{\tau_k^x}) e^{-\kappa_f(X_{\tau_k^x})(t-s)} d\tilde{W}_1(s),\\
\varphi_2(t,x) &= \rho\int_{\tau_k^x}^{t} \sigma^\lambda(X_{\tau_k^x}) e^{-\kappa_\lambda(X_{\tau_k^x})(t-s)} d\tilde{W}_1(s),\\
\varphi_3(t,x) &= \sqrt{1-\rho^2}\int_{\tau_k^x}^{t} \sigma^\lambda(X_{\tau_k^x}) e^{-\kappa_\lambda(X_{\tau_k^x})(t-s)} d\tilde{W}_2(s).
\end{aligned}
$$

The stochastic differential (4.40) can then be written as[40]

$$
\begin{aligned}
dr^d(t) = \Big[&f_2^d(\tau_k^x,t) + \frac{\partial}{\partial t}\sum_{i=1}^{4} S_i(t,x) - \kappa_f(X_{\tau_k^x})\varphi_1(t,x) - \kappa_\lambda(X_{\tau_k^x})\varphi_2(t,x)\\
&- \kappa_\lambda(X_{\tau_k^x})\Big(r^d(t) - f^d(\tau_k^x,t) - \varphi_1(t,x) - \varphi_2(t,x) - \sum_{i=1}^{4} S_i(t,x)\Big)\Big]dt\\
&+ \left[\sigma^f(X_{\tau_k^x}) + \rho\sigma^\lambda(X_{\tau_k^x})\right] d\tilde{W}_1(t) + \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k^x}) d\tilde{W}_2(t).
\end{aligned}
$$

The following proposition is a parallel one to Proposition 4.4.

Proposition 4.5 *Using (4.43a), we can write the Markovian dynamics of the defaultable*

[40] Although both $\varphi_1(t,x)$ and $\varphi_2(t,x)$ are driven by the same Wiener process $\tilde{W}_1(t)$, we identify them separately since $\varphi_1(t,x)$ arises from the default-free model whereas $\varphi_2(t,x)$ is from the credit spread dynamics.

short rate as the two-factor Hull and White [1990] type model

$$dr^d(t) = \kappa_\lambda(X_{\tau_k^x})\Big[\bar{\Theta}_d(t,x) - \bar{\kappa}_d(X_{\tau_k^x})r(t) - r^d(t)\Big]dt + \big(\sigma^f(X_{\tau_k^x}) + \rho\sigma^\lambda(X_{\tau_k^x})\big)d\tilde{W}_1(t)$$
$$+ \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k^x})d\tilde{W}_2(t), \tag{4.45a}$$
$$dr(t) = \kappa_f(X_{\tau_k^x})\Big[\bar{\Theta}_f(t,x) - r(t)\Big]dt + \sigma^f(X_{\tau_k^x})d\tilde{W}_1(t), \tag{4.45b}$$

and the coefficients in the drift of the stochastic differential equations are given by

$$\begin{cases} \bar{\Theta}_d(t,x) = \dfrac{\Theta_d(t,x) + \kappa_d(X_{\tau_k^x})\big(f(\tau_k^x,t) + S_1(t,x)\big)}{\kappa_\lambda(X_{\tau_k^x})}, \\ \Theta_d(t,x) = f_2^d(\tau_k^x,t) + \kappa_\lambda(X_{\tau_k^x})f^d(\tau_k^x,t) + \dfrac{\partial}{\partial t}\displaystyle\sum_{i=1}^4 S_i(t,x) + \kappa_\lambda(X_{\tau_k^x})\sum_{i=1}^4 S_i(t,x), \\ \kappa_d(X_{\tau_k^x}) = \kappa_f(X_{\tau_k^x}) - \kappa_\lambda(X_{\tau_k^x}), \quad \bar{\kappa}_d(X_{\tau_k^x}) = \dfrac{\kappa_d(X_{\tau_k^x})}{\kappa_\lambda(X_{\tau_k^x})}, \quad \bar{\Theta}_f(t,x) = \dfrac{\Theta_f(t,x)}{\kappa_f(X_{\tau_k^x})}, \\ \Theta_f(t,x) = f_2(\tau_k^x,t) + \kappa_f(X_{\tau_k^x})f(\tau_k^x,t) + \kappa_f(X_{\tau_k^x})S_1(t,x) + \dfrac{\partial}{\partial t}S_1(t,x). \end{cases} \tag{4.46}$$

Proof: Proof given in Appendix C.2. ■

We observe that the functions $\bar{\Theta}_d(t,x)$ and $\bar{\Theta}_f(t,x)$ in the stochastic differential equations (4.45a) and (4.45b) are automatically calibrated to the initially observed forward curves $f^d(\tau_k^x,t)$ and $f(\tau_k^x,t)$ respectively.

Figure 4.2 illustrates the evolution of the defaultable short rate curve together with its modulating Markov chain over 500 time-steps for an economy with high and low volatility regimes. The levels of the parameters σ_0^f and σ_0^λ under different regimes is given by $\sigma_0^f = [0.03, 0.04]$, $\sigma_0^\lambda = [0.01, 0.02]$ whereas the speed of mean reversion are assumed to remain constant speed irrespective of the regime, $\kappa_f = 0.6$ and $\kappa_\lambda = 0.4$. In addition, we are assuming that the correlation between market risk and credit risk, $\rho = 0.5$. The effect of the regime-switching can be observed in the short rate curve where we compare the difference in the regime-switching defaultable short rate curve with the case where the underlying Markov chain does not switch given that both start from the same volatility regime.

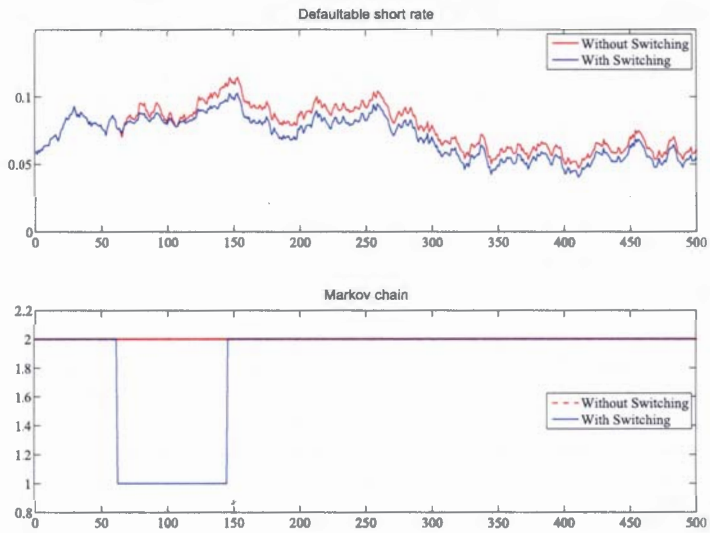


Figure 4.2: Comparison between regime-switching defaultable short rate and a non-switching term structure.

The two-factor Hull-White type model that we have derived allows for better calibration to market data. This forms part of the key difference of our model as compared to the models developed in Landén [2000], Wong and Wong [2007] and Elliott and Siu [2009]. In addition, our model also benefits from the inherent advantages of the underlying HJM framework in that, it is automatically calibrated to the currently observed yield curve. The model is also complete in the sense that, it does not involve the market price of risk directly (except the transition risk which is captured within the modified generator matrix) and therefore no assumption to that effect need to be made.

4.3.2 Defaultable Bond Pricing

In Landén [2000] and Hansen and Poulsen [2000], the problem of risk-free bond pricing with hidden Markov models was undertaken. They considered a Vasicek [1977] short-rate type model where the interest rate process is mean reverting towards a shifting long-term average with a constant speed of mean reversion κ and with a constant volatility function.

Hansen and Poulsen [2000] developed probabilistic algorithms to compute the bond prices whereas Landén [2000] used a semi-affine term structure and derived closed form solutions by making use of Whittaker functions. Naik and Lee [1997] studied a generalization of the model where in addition, the volatility also depends on an underlying Markov chain although they did not obtain closed form solutions for the bond price.

We seek to derive a 'semi-closed form' solution for the bond price within the two factor Hull and White [1990] model framework that we have developed. The speed of mean reversion, the long term average and the volatility are all functions of the underlying Markov chain. It was shown in Elhouar [2008] that the extended Vasicek Hull-White model admits finite dimensional realisations with the choice of parameter specification that we have made. The short rate process is mean reverting towards a shifting long term level at a shifting speed of mean reversion with a shifting volatility function. The process however remains continuous although the model experiences rapid shifts in its term structure.

We recall that the defaultable short rate between default times is determined by the following system of stochastic differential equations

$$\begin{aligned} dr^d(t) &= \kappa_\lambda(X_{\tau_k^x})\big[\bar{\Theta}_d(t,x) - \bar{\kappa}_d(X_{\tau_k^x})r(t) - r^d(t)\big]dt + \big(\sigma^f(X_{\tau_k^x}) + \rho\sigma^\lambda(X_{\tau_k^x})\big)d\tilde{W}_1(t) \\ &\quad + \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k^x})d\tilde{W}_2(t), \qquad (4.47a) \\ dr(t) &= \kappa_f(X_{\tau_k^x})\big[\bar{\Theta}_f(t,x) - r(t)\big]dt + \sigma^f(X_{\tau_k^x})d\tilde{W}_1(t). \qquad (4.47b) \end{aligned}$$

Since the defaultable short rate process (4.47a) is modelled directly under the risk-neutral measure, the bond price formula

$$P^d(t,T,X(t)) = \mathcal{R}(t)\bar{P}^d(t,T,X(t)), \qquad (4.48)$$

provides a convenient way to get the prices. However, the estimation of the model parameters could provide a source of difficulty as one can only estimate them under the real world probability. Elliott and Siu [2009] noted that given bond price data one might recover the risk neutral parameters of the short rate model implied by the data. The approach is however much easier to implement in practice if an analytically tractable formula is available for evaluating the risk neutral valuation formula.

We observe that under the risk-neutral measure $\tilde{\mathbb{P}}$, the price at time t of a pseudo-bond with maturity T, denoted by $\bar{P}^d(t, X_t, T)$, is a function of the two underlying factors $r^d(t)$ and $r(t)$.

Definition 4.6 *A defaultable short rate model is said to exhibit exponential affine term structure if the pseudo-bond prices can be written as*

$$\bar{P}^d(t, X_t, T) = \exp\Big(A(t, x, T) - B(t, x, T)r - C(t, x, T)r^d\Big), \tag{4.49}$$

where A, B and C are deterministic functions having terminal values

$$A(T, x, T) = B(T, x, T) = C(T, x, T) = 0.$$

Applying Itô's differentiation rule to $\bar{P}^d(t, X_t, T)$, we obtain

$$\begin{aligned} d\bar{P}^d =& \Big[\frac{\partial \bar{P}^d}{\partial t} + \mu_r(x,t)\frac{\partial \bar{P}^d}{\partial r} + \mu_{r^d}(x,t)\frac{\partial \bar{P}^d}{\partial r^d} + \frac{1}{2}\sigma^f(X_{\tau_k^x})^2\frac{\partial^2 \bar{P}^d}{\partial r^2} \\ &+ \sigma^f(X_{\tau_k^x})\Big(\sigma^f(X_{\tau_k^x}) + \rho\sigma^\lambda(X_{\tau_k^x})\Big)\frac{\partial^2 \bar{P}^d}{\partial r \partial r^d} \\ &+ \frac{1}{2}\Big(\sigma^f(X_{\tau_k^x})^2 + 2\rho\sigma^f(X_{\tau_k^x})\sigma^\lambda(X_{\tau_k^x}) + \sigma^\lambda(X_{\tau_k^x})^2\Big)\frac{\partial^2 \bar{P}^d}{\partial (r^d)^2}\Big]dt \\ &+ \Big[\sigma^f(X_{\tau_k^x})\frac{\partial \bar{P}^d}{\partial r} + \Big(\sigma^f(X_{\tau_k^x}) + \rho\sigma^\lambda(X_{\tau_k^x})\Big)\frac{\partial \bar{P}^d}{\partial r^d}\Big]d\tilde{W}_1(t) + \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k^x})\frac{\partial \bar{P}^d}{\partial r^d}d\tilde{W}_2(t) \\ &+ < \bar{P}^d, dX(t) >, \end{aligned} \tag{4.50}$$

with time dependent coefficients

$$\mu_r(x,t) = \kappa_f(X_{\tau_k^x})\big[\bar{\Theta}_f(t,x) - r(t)\big],$$
$$\mu_{r^d}(x,t) = \kappa_\lambda(X_{\tau_k^x})\big[\bar{\Theta}_d(t,x) - \bar{\kappa}_d(X_{\tau_k^x})r(t) - r^d(t)\big].$$

Since $\bar{P}^d(t, X_t, T)$ is a martingale under the risk neutral measure, the bounded variation terms which are not martingales in (4.50) above must sum to zero. We then have the partial differential equation

$$\frac{\partial \bar{P}^d}{\partial t} + \mu_r \frac{\partial \bar{P}^d}{\partial r} + \mu_{r^d}\frac{\partial \bar{P}^d}{\partial r^d} + \frac{1}{2}\sigma^f(X_{\tau_k^x})^2 \frac{\partial^2 \bar{P}^d}{\partial r^2} + \sigma^f(X_{\tau_k^x})\Big(\sigma^f(X_{\tau_k^x}) + \rho\sigma^\lambda(X_{\tau_k^x})\Big)\frac{\partial^2 \bar{P}^d}{\partial r \partial r^d}$$
$$+ \frac{1}{2}\Big(\sigma^f(X_{\tau_k^x})^2 + 2\rho\sigma^f(X_{\tau_k^x})\sigma^\lambda(X_{\tau_k^x}) + \sigma^\lambda(X_{\tau_k^x})^2\Big)\frac{\partial^2 \bar{P}^d}{\partial (r^d)^2} + < \bar{P}^d, \tilde{H}X(t) >= 0, \tag{4.51}$$

subject to the terminal condition that $\bar{P}^d(T, X_T, T) = 1$ where

$$< \bar{P}, \tilde{H}X(t) > = \sum_{i,j=1}^{N} e^{A_j - Br - Cr^d}\tilde{h}_{ij}^X. \tag{4.52}$$

Given that the partial differential equation (4.51) is satisfied, then from the Feynman-Kac formula, the arbitrage free price at time t of a pseudo-bond $\bar{P}^d(t, X_t, T)$ with maturity time T satisfies the regime-switching (partial differential) term structure equation

$$\frac{\partial \bar{P}^d}{\partial t} + \mathcal{K}\bar{P}^d - r^d\bar{P}^d = 0, \tag{4.53}$$

subject to terminal condition $\bar{P}^d(T, X_T, T) = 1$, where

$$\mathcal{K}\bar{P}^d = \mu_r \frac{\partial \bar{P}^d}{\partial r} + \mu_{r^d}\frac{\partial \bar{P}^d}{\partial r^d} + \frac{1}{2}\sigma^f(X_{\tau_k^x})^2 \frac{\partial^2 \bar{P}^d}{\partial r^2} + \sigma^f(X_{\tau_k^x})\Big(\sigma^f(X_{\tau_k^x}) + \rho\sigma^\lambda(X_{\tau_k^x})\Big)\frac{\partial^2 \bar{P}^d}{\partial r \partial r^d}$$
$$+ \frac{1}{2}\Big(\sigma^f(X_{\tau_k^x})^2 + 2\rho\sigma^f(X_{\tau_k^x})\sigma^\lambda(X_{\tau_k^x}) + \sigma^\lambda(X_{\tau_k^x})^2\Big)\frac{\partial^2 \bar{P}^d}{\partial (r^d)^2} + < \bar{P}^d, \tilde{H}X(t) > .$$

In addition, from (4.49) it follows that

$$\frac{\partial \bar{P}^d}{\partial t} = \left[\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r - \frac{\partial C}{\partial t} r^d\right] \bar{P}^d, \quad \frac{\partial \bar{P}^d}{\partial r} = -B\bar{P}^d, \quad \frac{\partial^2 \bar{P}^d}{\partial (r)^2} = B^2 \bar{P}^d,$$

$$\frac{\partial \bar{P}^d}{\partial r^d} = -C\bar{P}^d, \quad \frac{\partial^2 \bar{P}^d}{\partial (r^d)^2} = C^2 \bar{P}^d, \quad \frac{\partial^2 \bar{P}^d}{\partial r \partial r^d} = BC\bar{P}^d.$$

Proposition 4.7 *If $i = 1, 2, ..., N$ represents the states of the Markov chain, we have the set of three ordinary differential equations for C, B and A,*

$$\frac{\partial C_i}{\partial t} = \kappa_\lambda^i C_i - 1, \tag{4.54a}$$

$$\frac{\partial B_i}{\partial t} = \kappa_f^i B_i + (\kappa_f^i - \kappa_\lambda^i) C_i, \tag{4.54b}$$

$$\begin{aligned} \frac{\partial A_i}{\partial t} &= \kappa_f^i \bar{\Theta}_f(t,i) B_i + \kappa_\lambda^i \bar{\Theta}_d(t,i) C_i - \frac{1}{2}\Big[(\sigma_i^f)^2 B_i^2 + 2\sigma_i^f\big(\sigma_i^f + \rho\sigma_i^\lambda\big) C_i B_i \\ &\quad + \big((\sigma_i^f)^2 + 2\rho\sigma_i^f\sigma_i^\lambda + (\sigma_i^\lambda)^2\big) C_i^2\Big] - \sum_{j=1}^N \tilde{h}_{i,j}^X e^{A_j - A_i}, \end{aligned} \tag{4.54c}$$

where we have defined $A_i(t,T) := A(t,i,T)$, $B_i(t,T) := B(t,i,T)$ and $C_i(t,T) := C(t,i,T)$.

Proof: See Appendix C.3. ■

To obtain an explicit affine bond price formula, it only remains to solve the set of ordinary differential equations in Proposition 4.7 for the coefficients $A(t,x,T)$, $B(t,x,T)$ and $C(t,x,T)$. We express the coefficients as $A_i(t,T)$, $B_i(t,T)$ and $C_i(t,T)$ to reflect the possible regimes that the Markov chain visits.

Consider the special case where $N = 2$ such that the Markov chain switches between the two states $i = 1, 2$ representing two different regimes. For the ordinary differential equations in (4.54a), it follows that the solution is given by

$$C_i(t,T) = \frac{1}{\kappa_\lambda^i}\big(1 - e^{-\kappa_\lambda^i (T-t)}\big). \tag{4.55}$$

We observe that (4.54b) can be written as

$$\frac{\partial B_i}{\partial t} - \kappa_f^i B_i - (\kappa_f^i - \kappa_\lambda^i)C_i = 0 \tag{4.56}$$

The integrating factor in this case is given by $e^{-\kappa_f^i t}$ which on multiplying with equation (4.56) and solving yields

$$B_i(t,T) = \frac{\kappa_f^i - \kappa_\lambda^i}{\kappa_f^i \kappa_\lambda^i}\Big(e^{-\kappa_f^i(T-t)} - 1\Big) - \frac{1}{\kappa_\lambda^i}\Big(e^{-\kappa_f^i(T-t)} - e^{-\kappa_\lambda^i(T-t)}\Big). \tag{4.57}$$

We can write explicitly the ordinary differential equations for A in (4.54c) as

$$\begin{aligned}\frac{\partial A_1}{\partial t} &= \kappa_f^1 \bar{\Theta}_f(t,1)B_1 + \kappa_\lambda^1 \bar{\Theta}_d(t,1)C_1 - \frac{1}{2}\Big[(\sigma_1^f)^2 B_1^2 + 2\sigma_1^f\big(\sigma_1^f + \rho\sigma_1^\lambda\big)C_1 B_1 \\ &\quad + \big((\sigma_1^f)^2 + 2\rho\sigma_1^f\sigma_1^\lambda + (\sigma_1^\lambda)^2\big)C_1^2\Big] - \tilde{h}_{11}^X - \tilde{h}_{12}^X e^{A_2 - A_1}, \qquad (4.58a)\\ \frac{\partial A_2}{\partial t} &= \kappa_f^2 \bar{\Theta}_f(t,2)B_2 + \kappa_\lambda^2 \bar{\Theta}_d(t,2)C_2 - \frac{1}{2}\Big[(\sigma_2^f)^2 B_2^2 + 2\sigma_2^f\big(\sigma_2^f + \rho\sigma_2^\lambda\big)C_2 B_2 \\ &\quad + \big((\sigma_2^f)^2 + 2\rho\sigma_2^f\sigma_2^\lambda + (\sigma_2^\lambda)^2\big)C_2^2\Big] - \tilde{h}_{21}^X e^{A_1 - A_2} - \tilde{h}_{22}^X, \qquad (4.58b)\end{aligned}$$

given that the transition intensities satisfy the condition

$$\sum_{j=1}^{2} \tilde{h}_{i,j}^X = 0, \quad i = 1,2. \tag{4.59}$$

We observe that $\tilde{h}_{1,1}^X = -\tilde{h}_{1,2}^X$ whereas $\tilde{h}_{2,2}^X = -\tilde{h}_{2,1}^X$. It then follows that the system of ordinary differential equations above can be written as

$$\begin{aligned}\frac{\partial A_1}{\partial t} - \jmath_1(t) + h_{1,2}^X e^{A_2 - A_1} - \tilde{h}_{1,2}^X = 0,\\ \frac{\partial A_2}{\partial t} - \jmath_2(t) + h_{2,1}^X e^{A_1 - A_2} - \tilde{h}_{2,1}^X = 0,\end{aligned} \tag{4.60}$$

where we have defined the function

$$\jmath_i(t) = \kappa_f^i \bar{\Theta}_f(t,i)B_i + \kappa_\lambda^i \bar{\Theta}_d(t,i)C_i \\ - \frac{1}{2}\Big[(\sigma_i^f)^2 B_i^2 + 2\sigma_i^f(\sigma_i^f + \rho\sigma_i^\lambda)C_i B_i + \big((\sigma_i^f)^2 + 2\rho\sigma_i^f\sigma_i^\lambda + (\sigma_i^\lambda)^2\big)C_i^2\Big].$$

The coupled system of ordinary differential equations (4.60) is similar to the one obtained in Landén [2000] and Elliott and Siu [2009].

We now briefly investigate the effect that correlation between credit risk and market risk has on the normalised distribution of both the defaultable bond price and defaultable short rate under regime switching and compare with the case when the model parameters do not undergo regime-switching. We then investigate the effect of the transition intensities on the distributional properties of both the short rate and bond price.

In the following simulation examples, we have assumed the same transition intensities for a state $\tilde{h}^X_{1,2} = \tilde{h}^X_{2,1}$. In addition, we take the speeds of mean reversion to be constant for both regimes with $\kappa_f = 0.6$, $\kappa_\lambda = 0.4$. The volatility parameters are given by $\sigma_f = [0.02, 0.03]$, $\sigma_\lambda = [0.01, 0.02]$, $\rho = 0.5$ and a constant initial forward curve $f^d(\tau_k^x, t) = 0.07$ over a 2-year maturity horizon. To reduce the frequency and number of state transitions, we assume a probability matrix and initial distribution given by

$$\mathbf{P} = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \tag{4.61}$$

respectively.

Figure 4.3 shows the evolution of the default free short rate, defaultable short rate and pseudo bond price with an underlying Markov chain where the effect of the changing volatility regimes on the dynamics of the short rate and bond price evolution are observed.

From Table 4.1, we observe that in the presence of regime-switching, increasing the correla-

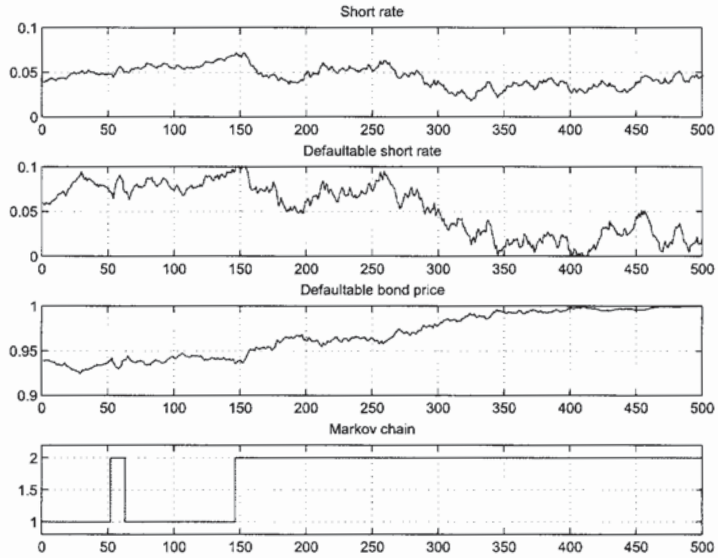


Figure 4.3: Evolution of default free short rate, defaultable short rate and defaultable bond price under regime-switching dynamics.

tion between credit spread and short rate increases the positive skewness in the normalised bond price distribution. However, as Table 4.2 shows, in the absence of regime-switching, an increase in correlation causes a bigger change in the skewness of the normalised bond price distribution.

ρ	Kurtosis(Rate)	Skewness(Rate)	Kurtosis(Price)	Skewness(Price)
-0.5	3.3249	0.0158	3.3685	0.0292
0.5	3.3276	-0.0107	3.3557	0.0980

Table 4.1: An analysis of the effect of increasing the correlation between short term credit spread and short rate on the kurtosis and skewness of the normalised distributions in the *presence* of regime-switching.

We also investigated how frequent regime switching affects the distributions. As expected, this causes a reduction in the bond prices due to the additional volatility risk and therefore increasing the transition intensities implies a decrease in the bond prices as investors would require more compensation for the higher risk. From Figure 4.4 and Table 4.3, we observe that increasing the transition intensity causes a reduction in both the kurtosis and skewness

ρ	Kurtosis(Rate)	Skewness(Rate)	Kurtosis(Price)	Skewness(Price)
-0.5	4.2145	0.0960	4.2778	- 0.0723
0.5	4.3424	0.0666	4.3727	0.0361

Table 4.2: Effect of increasing the correlation between credit spread and short rate on the kurtosis and skewness of the normalised distributions in the *absence* of regime switching.

of the normalised bond price distribution.

$h^X_{12} = h^X_{21}$	Kurtosis(Rate)	Skewness(Rate)	Kurtosis(Price)	Skewness(Price)
0.0	4.3424	0.0666	4.3727	0.0361
0.0051	3.3276	-0.0107	3.3557	0.0980
0.4024	3.0319	0.0400	3.0239	0.0370

Table 4.3: Effect of increasing the transition intensity, $h^X_{i,j}$ on the kurtosis and skewness of the distributions of the defaultable short rate and pseudo bond price.

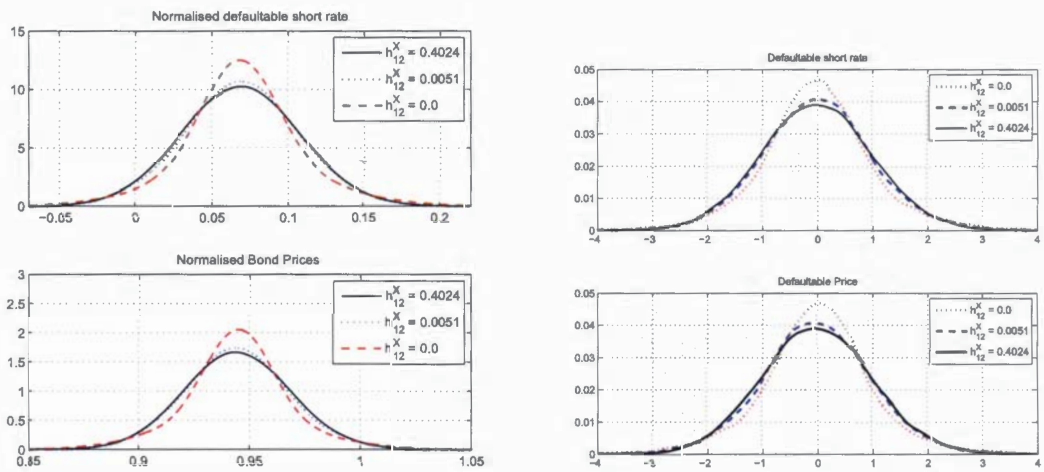


Figure 4.4: Effect of transition intensity on the kurtosis and skewness of the normalised defaultable short rate and normalised defaultable bond price distributions.

4.4 Option Pricing under Regime-Switching

In this section, we seek to derive the valuation framework for a European call option[41] with maturity T_0 and strike K written on a defaultable zero-coupon bond with maturity T given

[41] We also assume in this case that this option is not subject to counterparty risk.

that $0 \leq t \leq T_0 < T$. Let $\mathcal{C}(t, \bar{P}^d, X(t), T_0)$ denote the price at time t of a call option, which for notational simplicity we will write as $\mathcal{C}(t, \bar{P}^d, X(t))$. As shown in Section 3.5[42], the price at time t of the call option knocked out on default of the underlying is given by

$$\mathcal{C}(t, \bar{P}^d, X(t)) = \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r^d(s)ds}\big(\bar{P}^d(T_0, T) - K\big)^+\Big|\mathcal{F}_t\Big],$$

where we use $\bar{P}^d(T_0, T) = \bar{P}^d(T_0, X(T_0), T)$ to denote the defaultable zero coupon bond price for notational simplicity. The option can also be interpreted as a defaultable investment with zero recovery as the buyer of the option receives nothing on default of the underlying asset.

The pseudo bond price dynamics for $t \in [\tau_k^x, \tau_{k+1}^x[$ satisfy the stochastic differential equation

$$d\bar{P}^d = \mu^P(x, t)dt + \sum_{i=1}^{2}\sigma_i^P(x, t)d\tilde{W}_i(t) + < \bar{P}^d, dX(t) >, \tag{4.62}$$

where the drift and volatility functions are given by

$$\begin{aligned}
\mu^P(x, t) &= \frac{\partial \bar{P}^d}{\partial t} + \mu_r(x, t)\frac{\partial \bar{P}^d}{\partial r} + \mu_{r^d}(x, t)\frac{\partial \bar{P}^d}{\partial r^d} + \frac{1}{2}\sigma^f(X_{\tau_k^x})^2\frac{\partial^2 \bar{P}^d}{\partial r^2} \\
&\quad + \sigma^f(X_{\tau_k^x})\Big(\sigma^f(X_{\tau_k^x}) + \rho\sigma^\lambda(X_{\tau_k^x})\Big)\frac{\partial^2 \bar{P}^d}{\partial r \partial r^d} \\
&\quad + \frac{1}{2}\Big(\sigma^f(X_{\tau_k^x})^2 + 2\rho\sigma^f(X_{\tau_k^x})\sigma^\lambda(X_{\tau_k^x}) + \sigma^\lambda(X_{\tau_k^x})^2\Big)\frac{\partial^2 \bar{P}^d}{\partial (r^d)^2}, \\
\sigma_1^P(x, t) &= \sigma^f(X_{\tau_k^x})\frac{\partial \bar{P}^d}{\partial r} + \Big(\sigma^f(X_{\tau_k^x}) + \rho\sigma^\lambda(X_{\tau_k^x})\Big)\frac{\partial \bar{P}^d}{\partial r^d}, \\
\sigma_2^P(x, t) &= \sqrt{1 - \rho^2}\sigma^\lambda(X_{\tau_k^x})\frac{\partial \bar{P}^d}{\partial r^d},
\end{aligned}$$

and when $< \bar{P}^d, dX(t) >$ is defined in (4.33) and (4.52). Given that the pseudo bond price satisfies an exponential affine form

$$\bar{P}^d(t, X(t), T) = \exp\Big(A(t, x, T) - B(t, x, T)r - C(t, x, T)r^d\Big), \tag{4.63}$$

[42]However, we note that the option considered in Chapter 3, (3.75) was within the stochastic volatility framework.

we can evaluate the partial differentials $\frac{\partial \bar{P}^d}{\partial t}$, $\frac{\partial \bar{P}^d}{\partial r}$, $\frac{\partial \bar{P}^d}{\partial r^d}$, $\frac{\partial^2 \bar{P}^d}{\partial r^2}$, $\frac{\partial^2 \bar{P}^d}{\partial (r^d)^2}$ and $\frac{\partial^2 \bar{P}^d}{\partial r \partial r^d}$ explicitly.

Let the quantity $\mathcal{V}(t, \bar{P}^d, X(t))$, the discounted option price be defined by

$$\begin{aligned}\mathcal{V}(t, \bar{P}^d, X(t)) &= e^{-\int_0^t r^d(s)ds}\mathcal{C}(t, \bar{P}^d, X(t)),\\ &= \tilde{\mathbb{E}}\Big[e^{-\int_0^{T_0} r^d(s)ds}\big(\bar{P}^d(T_0, T) - K\big)^+\Big|\mathcal{F}_t\Big].\end{aligned} \tag{4.64}$$

Given that

$$\mathbf{V}(t, \bar{P}^d) = \Big(\mathcal{V}(t, \bar{P}^d, e_1), \mathcal{V}(t, \bar{P}^d, e_2), ..., \mathcal{V}(t, \bar{P}^d, e_N)\Big),$$

we can then express $\mathcal{V}\big(t, \bar{P}^d, X(t)\big) = \big\langle \mathcal{V}(t, \bar{P}^d), X(t)\big\rangle$. Applying Itô's rule to V we have

$$\mathcal{V}(t, \bar{P}^d, X(t)) = \mathcal{V}(0, \bar{P}^d, X(t)) + \int_0^t \frac{\partial \mathcal{V}}{\partial s} ds + \int_0^t \frac{\partial \mathcal{V}}{\partial \bar{P}^d} d\bar{P}^d + \frac{1}{2}\int_0^t \frac{\partial^2 \mathcal{V}}{\partial (\bar{P}^d)^2} d\big(\bar{P}^d\big)^2 + \big\langle \mathcal{V}, dX(t)\big\rangle, \tag{4.65}$$

whereas the Markov chain dynamics follow $dX(t) = \tilde{H}X(t)dt + d\tilde{M}^X(t)$.

By definition of a martingale, all time integrals sum up to zero from which we have that

$$\frac{\partial \mathcal{V}}{\partial t} + \mu^P(x,t)\frac{\partial \mathcal{V}}{\partial \bar{P}^d} + \frac{1}{2}\big([(\sigma_1^P(x,t))^2 + (\sigma_2^P(x,t))^2\big]\frac{\partial^2 \mathcal{V}}{\partial (\bar{P}^d)^2} + \big\langle \mathcal{V}, \tilde{H}X(t)\big\rangle = 0. \tag{4.66}$$

Given that $\mathcal{V} = e^{-\int_0^t r^d(s)ds}\mathcal{C}$, the partial differential equation in (4.66) can be written as

$$e^{-\int_0^t r^d(s)ds}\Big(\frac{\partial \mathcal{C}}{\partial t} + \mu^P(x,t)\frac{\partial \mathcal{C}}{\partial \bar{P}^d} + \frac{1}{2}\big[(\sigma_1^P(x,t))^2 + (\sigma_2^P(x,t))^2\big]\frac{\partial^2 \mathcal{C}}{\partial (\bar{P}^d)^2} + \big\langle \mathcal{C}, \tilde{H}X(t)\big\rangle - r^d(t)\mathcal{C}\Big) = 0, \tag{4.67}$$

subject to the terminal conditions $\mathcal{C}(T_0, \bar{P}^d, e_i) = \big(\bar{P}^d(T_0, T) - K\big)^+$.

4.4.1 Finite Difference Methods: Theta Scheme

To solve the pricing partial differential equation in (4.68), we adopt the θ-method technique as used in Khaliq and Liu [2009] where they extended the penalty method for pricing American options. A notable feature of our model is that to model the underlying asset, the defaultable bond, we will also need to solve a coupled system of ODE's in addition to the coupled PDE arising in the option pricing problem. If the governing Markov chain occupies only two states $i = \{1, 2\}$ such that we can write $\mathbf{C} = \big(\mathcal{C}_1, \mathcal{C}_2\big)$ with $\mathcal{C}_i = \mathcal{C}(t, \bar{P}^d, e_i)$, then $\mathbf{C}$ satisfies the system of partial differential equations

$$\frac{\partial \mathcal{C}_i}{\partial t} + \mu_i^P \frac{\partial \mathcal{C}_i}{\partial \bar{P}^d} + \frac{1}{2}\big[(\sigma_{1,i}^P)^2 + (\sigma_{2,i}^P)^2\big]\frac{\partial^2 \mathcal{C}_i}{\partial (\bar{P}^d)^2} + \big\langle \mathbf{C}, He_i \big\rangle - r_i^d \mathcal{C}_i = 0, \tag{4.68}$$

or equivalently

$$\frac{\partial \mathcal{C}_i}{\partial t} + \mu_i^P \frac{\partial \mathcal{C}_i}{\partial \bar{P}^d} + \frac{1}{2}\big[(\sigma_{1,i}^P)^2 + (\sigma_{2,i}^P)^2\big]\frac{\partial^2 \mathcal{C}_i}{\partial (\bar{P}^d)^2} + \sum_{j\neq i} \tilde{h}_{ij}^X \big(\mathcal{C}_j - \mathcal{C}_i\big) - r_i^d \mathcal{C}_i = 0, \tag{4.69}$$

subject to the terminal conditions

$$\mathcal{C}_i(T_0, \bar{P}^d) = \big(\bar{P}^d(T_0, T) - K\big)^+,$$

for $i = 1, 2$. The option pricing problem then reduces to a problem of solving a system of coupled partial differential equations[43]

$$\begin{cases} \dfrac{\partial \mathcal{C}_1}{\partial t} + \mu_1^P \dfrac{\partial \mathcal{C}_1}{\partial \bar{P}^d} + \dfrac{1}{2}\dfrac{\partial^2 \mathcal{C}_1}{(\partial \bar{P}^d)^2}\big((\sigma_{1,1}^P)^2 + (\sigma_{2,1}^P)^2\big) + (\mathcal{C}_1 - \mathcal{C}_2)\tilde{h}_{11}^X - r_1^d \mathcal{C}_1 = 0, \\ \dfrac{\partial \mathcal{C}_2}{\partial t} + \mu_2^P \dfrac{\partial \mathcal{C}_2}{\partial \bar{P}^d} + \dfrac{1}{2}\dfrac{\partial^2 \mathcal{C}_2}{\partial (\bar{P}^d)^2}\big((\sigma_{1,2}^P)^2 + (\sigma_{2,2}^P)^2\big) + (\mathcal{C}_2 - \mathcal{C}_1)\tilde{h}_{22}^X - r_2^d \mathcal{C}_2 = 0, \\ \mathcal{C}_i(T_0, \bar{P}^d) = \big(\bar{P}^d(T_0, T) - K\big)^+. \end{cases} \tag{4.70}$$

[43]We observe that $\sum_{j\neq i} \tilde{h}_{ij}^X(\mathcal{C}_j - \mathcal{C}_i) = \sum_{j\neq i} \tilde{h}_{ij}^X \mathcal{C}_j + \tilde{h}_{ii}^X \mathcal{C}_i = \sum_j \tilde{h}_{ij}^X \mathcal{C}_j$ from which we get the coupled system of partial differential equations in (4.70).

It would be extremely difficult to obtain a closed-form solution to (4.70) and therefore we resort to efficient numerical methods for the solution of the coupled partial differential equations.

Let $\bar{P}^d_{max}$ be the maximum possible bond price chosen as the upper bound (we could take this to be $\bar{P}^d_{max} = 1$ if $T_0 = T$). Equation (4.69) can be written as

$$\begin{cases} \dfrac{\partial \mathcal{C}_i}{\partial t} + \mu_i^P \dfrac{\partial \mathcal{C}_i}{\partial \bar{P}^d} + \dfrac{1}{2}\big[(\sigma_{1,i}^P)^2 + (\sigma_{2,i}^P)^2\big] \dfrac{\partial^2 \mathcal{C}_i}{\partial (\bar{P}^d)^2} - \big(r_i^d - \tilde{h}_{ii}^X\big)\mathcal{C}_i + \displaystyle\sum_{j \neq i} \tilde{h}_{ij}^X \mathcal{C}_j = 0, \\ \mathcal{C}_i(\bar{P}^d, T_0) = \big(\bar{P}^d(T_0, T) - K\big)^+, \\ \mathcal{C}_i(\bar{P}^d_{min}, t) = 0, \\ \mathcal{C}_i(\bar{P}^d_{max}, t) = e^{-r^d(T_0 - t)}\big(\bar{P}^d_{max} - K\big), \end{cases} \tag{4.71}$$

where $\mathcal{C}_i(\bar{P}^d, t)$ is the solution of (4.71). The quantity $\big(\bar{P}^d_{max} - K\big)$ is always positive since the strike $K < \bar{P}^d_{max}$.

If $\bar{P}^d = \bar{P}^d_{min}$, the option is worthless and therefore the option's lower bound is $\mathcal{C}_i(\bar{P}^d_{min}, T_0) = 0$. To discretize the domain $(\bar{P}^d, t) \in [\bar{P}^d_{min}, \bar{P}^d_{max}] \times [0, T_0]$, for positive integers M and N we let $\Delta \bar{P} = \dfrac{\bar{P}^d_{max}}{M+1}$ and $\Delta t = \dfrac{T_0}{N+1}$ be the grid sizes for $\bar{P}^d$ and t respectively. In addition, for notational simplicity we define $\bar{\sigma}_i^2 = (\sigma_{1,i}^P)^2 + (\sigma_{2,i}^P)^2$.

For $0 \leq m \leq M+1$, $0 \leq n \leq N+1$ and states $i = \{1, 2\}$, we let $\mathcal{C}_i^{m,n} = \mathcal{C}_i(m\Delta \bar{P}^d, n\Delta t)$. The grid coordinates (m, n) allow us to compute the solution at discrete points. This point corresponds to time $n\Delta t$, for $n = 0, 1, 2, ..., N+1$ and bond price $\Delta \bar{P}^d$ for $m = 0, 1, 2, ..., M+1$.

Then using discretization and $\theta-$methods (see Topper [2005]) we have the system of equations

$$\frac{\mathcal{C}_i^{m,n+1}-\mathcal{C}_i^{m,n}}{\Delta t}+\frac{1}{2}\bar{\sigma}_i^2\left[\theta\frac{\delta_p^2\mathcal{C}_i^{m,n+1}}{(\Delta\bar{P})^2}+(1-\theta)\frac{\delta_p^2\mathcal{C}_i^{m,n}}{(\Delta\bar{P})^2}\right]+\mu_i^P\left[\theta\frac{\delta_p\mathcal{C}_i^{m,n+1}}{(\Delta\bar{P})}+(1-\theta)\frac{\delta_p\mathcal{C}_i^{m,n}}{(\Delta\bar{P})}\right]$$
$$-\left(r_i^d-\tilde{h}_{ii}^X\right)\left[\theta\mathcal{C}_i^{m,n+1}+(1-\theta)\mathcal{C}_i^{m,n}\right]+\sum_{j\neq i}\tilde{h}_{ij}^X\left[\theta\mathcal{C}_j^{m,n+1}+(1-\theta)\mathcal{C}_j^{m,n}\right]=0, \tag{4.72}$$

where

$$\begin{cases}\delta_p^2\mathcal{C}_i^{m,n}=\mathcal{C}_i^{m+1,n}-2\mathcal{C}_i^{m,n}+\mathcal{C}_i^{m-1,n},\\ \delta_p\mathcal{C}_i^{m,n}=\mathcal{C}_i^{m+1,n}-\mathcal{C}_i^{m,n},\quad\text{for}\quad 0\leq\theta\leq 1.\end{cases} \tag{4.73}$$

On simplification, (4.72) can be written as

$$[1+(1-\theta)W_i^m]\mathcal{C}_i^{m,n}=\theta L_i^m\mathcal{C}_i^{m-1,n+1}+[1-\theta W_i^m]\mathcal{C}_i^{m,n+1}+\theta F_i^m\mathcal{C}_i^{m+1,n+1}+(1-\theta)L_i^m\mathcal{C}_i^{m-1,n}$$
$$+(1-\theta)F_i^m\mathcal{C}_i^{m+1,n}+\Delta t\sum_{j\neq i}\tilde{h}_{ij}^X[\theta\mathcal{C}_j^{m,n+1}+(1-\theta)\mathcal{C}_j^{m,n}], \tag{4.74}$$

where

$$L_i^m=\alpha\bar{\sigma}_i^2,\quad W_i^m=2\alpha\bar{\sigma}_i^2+\beta\mu_i^P+\left(r_i^d-\tilde{h}_{ii}^X\right)\Delta t\quad\text{and}\quad F_i^m=\alpha\bar{\sigma}_i^2+\beta\mu_i^P,$$

with $\alpha=\dfrac{\Delta t}{2(\Delta\bar{P})^2}$ and $\beta=\dfrac{\Delta t}{\Delta\bar{P}}$. Equation (4.74) can further be simplified as

$$\theta_1L_i^m\mathcal{C}_i^{m-1,n}+(1-\theta_1W_i^m)\mathcal{C}_i^{m,n}+\theta_1F_i^m\mathcal{C}_i^{m+1,n}+\theta_1\Delta t\sum_{j\neq i}\tilde{h}_{ij}^X\mathcal{C}_j^{m,n}$$
$$=\theta L_i^m\mathcal{C}_i^{m-1,n+1}+(1-\theta W_i^m)\mathcal{C}_i^{m,n+1}+\theta F_i^m\mathcal{C}_i^{m+1,n+1}+\theta\Delta t\sum_{j\neq i}\tilde{h}_{ij}^X\mathcal{C}_j^{m,n+1}, \tag{4.75}$$

where we have defined $\theta_1=(\theta-1)$.

For each regime $i=1,2$, we define the vector $\mathbf{C}_i^n=[\mathcal{C}_i^{1,n},\mathcal{C}_i^{2,n},....,\mathcal{C}_i^{M,n}]^\top$ such that (4.75) can be written in vector form as

$$\mathbf{A}_i\mathbf{C}_i^n + \theta_1\Delta t\sum_{j\neq i}\tilde{h}_{ij}^X\mathbf{C}_j^n = \mathbf{B}_i\mathbf{C}_i^{n+1} + \theta\Delta t\sum_{j\neq i}\tilde{h}_{ij}^X\mathbf{C}_j^{n+1} + \mathbf{g}_i^{n+1}, \tag{4.76}$$

where the tridiagonal square matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ which arise from central difference approximations to the spatial derivatives are given by

$$\mathbf{A}_i = \begin{bmatrix} 1-\theta_1 W_i^1 & \theta_1 F_i^1 & 0 & \cdots & & & 0 \\ \theta_1 L_i^2 & 1-\theta_1 W_i^2 & \theta_1 F_i^2 & 0 & \cdots & & \vdots \\ 0 & \theta_1 L_i^3 & 1-\theta_1 W_i^3 & \theta_1 F_i^3 & 0 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & 0 \\ & & & & \theta_1 L_i^{M-1} & 1-\theta_1 W_i^{M-1} & \theta_1 F_i^{M-1} \\ 0 & 0 & \cdots & & 0 & \theta_1 L_i^M & 1-\theta_1 W_i^M \end{bmatrix}, \tag{4.77}$$

and

$$\mathbf{B}_i = \begin{bmatrix} 1-\theta W_i^1 & \theta F_i^1 & 0 & \cdots & & & 0 \\ \theta L_i^2 & 1-\theta W_i^2 & \theta F_i^2 & 0 & \cdots & & \vdots \\ 0 & \theta L_i^3 & 1-\theta W_i^3 & \theta F_i^3 & 0 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & & \theta L_i^{M-1} & 1-\theta W_i^{M-1} & \theta F_i^{M-1} \\ 0 & 0 & \cdots & & 0 & \theta L_i^M & 1-\theta W_i^M \end{bmatrix}, \tag{4.78}$$

respectively with

$$\mathbf{g}_i^{n+1} = \begin{bmatrix} 0 \\ \vdots \\ F_i^M e^{-r^d(T_0-t)}(\bar{P}_{max}^d - K) \end{bmatrix}. \tag{4.79}$$

For the special case of 2−state regimes, we let

$$\mathbf{C}^n = \begin{bmatrix} \mathbf{C}_1^n \\ \mathbf{C}_2^n \end{bmatrix}, \tag{4.80}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \theta_1 \tilde{h}_{12}^X \Delta t \mathbf{I} \\ \theta_1 \tilde{h}_{21}^X \Delta t \mathbf{I} & \mathbf{A}_2 \end{bmatrix}, \tag{4.81}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \theta_1 \tilde{h}_{12}^X \Delta t \mathbf{I} \\ \theta_1 \tilde{h}_{21}^X \Delta t \mathbf{I} & \mathbf{B}_2 \end{bmatrix}, \tag{4.82}$$

where $\mathbf{I}$ denotes the 2×2 unit matrix and

$$\mathbf{g}^{n+1} = \begin{bmatrix} \mathbf{g}_1^{n+1} \\ \mathbf{g}_2^{n+1} \end{bmatrix}. \tag{4.83}$$

Then (4.76) can be written as the system

$$\mathbf{A}\mathbf{C}^n = \mathbf{B}\mathbf{C}^{n+1} + \mathbf{g}^{n+1}, \tag{4.84}$$

with $n = N+1, N, .., 1, 0$, and terminal condition $\mathbf{C}^{N+1}$ determined by the option's payoff $(\bar{P}^d - K)^+$. The scheme therefore requires solving an $M \times 2$ linear system of equations at each step.

If for $j \neq i$ we replace $\mathcal{C}_j^{m,n}$ with $\mathcal{C}_j^{m,n+1}$, then (4.75) can be reduced to a similar form

$$\begin{aligned} &\theta_1 L_i^m \mathcal{C}_i^{m-1,n} + (1 - \theta_1 W_i^m)\mathcal{C}_i^{m,n} + \theta_1 F_i^m \mathcal{C}_i^{m+1,n} \\ &\quad = \theta L_i^m \mathcal{C}_i^{m-1,n+1} + (1 - \theta W_i^m)\mathcal{C}_i^{m,n+1} + \theta F_i^m \mathcal{C}_i^{m+1,n+1} + \theta \Delta t \sum_{j \neq i} \tilde{h}_{ij}^X \mathcal{C}_j^{m,n+1}, \end{aligned} \tag{4.85}$$

or equivalently;

$$\mathbf{A}_i\mathbf{C}_i^n = \mathbf{B}_i\mathbf{C}_i^{n+1} + \Delta t\sum_{j\neq i}\tilde{h}_{ij}^X\mathbf{C}_j^{n+1} + \mathbf{g}_i^{n+1}, \quad i = 1, 2, \tag{4.86}$$

with $\mathbf{C}_i^n = [\mathcal{C}_i^{1,n}, \mathcal{C}_i^{2,n}, ..., \mathcal{C}_i^{M,n}]^\top$. The equations in (4.86) can be solved separately yielding a variant of the system in (4.84), namely

$$\bar{\mathbf{A}}\mathbf{C}^n = \mathbf{B}\mathbf{C}^{n+1} + \mathbf{g}^{n+1}, \tag{4.87}$$

where $\bar{\mathbf{A}}$ is given by

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}. \tag{4.88}$$

It is well known that the $\theta-$methods include the forward Euler (fully implicit) method with $\theta = 1$, the backward Euler (fully explicit) method for $\theta = 0$ and the Crank-Nicholson method for $\theta = \frac{1}{2}$ as special cases. In addition, explicit finite difference schemes are conditionally stable[44] whereas the implicit and Crank-Nicholson methods are both unconditionally stable. The convergence of a particular scheme is known to depend fully on its stability and this is more or less the main benefit offered by the latter schemes over the former as opposed to their accuracy.

4.5 Summary

In this chapter, we have extended work by Valchev [2004] to the defaultable HJM term structure model. By choosing an exponentially decaying volatility function whose parameters are modulated by the Markov chain, we derived a defaultable two-factor Hull-White-Extended-

[44] A scheme is said to be stable if the eigenvalues of A^{-1} are less than 1 in absolute value.

Vasicek type model. By construction, this model can be automatically calibrated to the currently observed defaultable and default-free forward curves. The parameters in both the drift and volatility functions are modulated by a continuous time Markov chain which implies that the speed of mean reversion, long-term average and volatility vary between the different regimes. The default-free and defaultable short rate processes are mean reverting towards a shifting long term average level at a shifting speed of mean reversion with a shifting volatility function.

We then express the defaultable bond price in an exponential affine formula, linear in both the default-free and defaultable short rate processes for the special case of two regimes, although the results still hold for more regimes. This is achieved by solving the regime-switching bond pricing partial differential equation which further requires solving numerically, a coupled system of ordinary differential equations. Some numerical results to illustrate the response of the bond price density to changes in the correlation between the credit spreads (which gives the market perception of the default risk) and interest rate and also due to the increase in the frequency of regime changes were also presented. In particular, we observed that increasing the correlation between the market risk and credit risk increases the skewness of the bond price distribution in the presence of regime switching. In addition, we observed that increasing the transition intensity and therefore the frequency of regime switching leads to a decrease in the kurtosis of the defaultable bond price distribution as investors demand more compensation for the additional source of risk.

We then investigated the option pricing problem on the defaultable bond in the presence of regime switching using finite difference methods. Our method follows the approach used by Khaliq and Liu [2009] where they applied the theta-adaptive methods to pricing American options in the presence of regime switching. This yields a system of equations which when solved subject to the boundary conditions gives the price of a call option on a defaultable bond under the fully implicit, fully explicit and the Crank-Nicolson schemes.

Chapter 5

Conclusion and Further Directions for Research

5.1 Summary of Findings

As a result of the diverging pros and cons of the two main classes of modern credit derivatives models, namely the structural and reduced form models, there is no standard model for credit and the choice is dependent on what the model is to be used for. Within the reduced form class of models (wherein this thesis falls), the likelihood of default and/or downgrade is modelled directly. In some cases, the 'forward curve' of default probabilities which may be used to price instruments of varying maturities can also be modelled. These probabilities play the central role in the distribution of credit losses and to estimate them, models of investor uncertainty and evolution of the available information over time are needed as well as the definition of the default event.

To price credit sensitive instruments, in addition to a model on default probabilities, one needs a model for the default-free interest rates, a model of recovery given default and a model of the premium (spread) that investors require as compensation for bearing systematic credit risk. Empirical evidence indicates that the time series of the default free rates and credit spreads exhibit mean reversion and fat tails and D'Souza et al. [2004] attributed the latter to the existence of stochastic volatility dynamics. Default-free interest rates as state variables affect the credit spread of the defaultable bonds as changes in interest rates compel investors to re-evaluate their assessment of the default probabilities of all risky bonds.

To model credit risk, we have adopted the Markovian defaultable HJM framework where a model for spread of the defaultable interest rates over default-free interest rates adds a default risk module to an existing model of the default free interest rates. By extending the HJM framework to include default risk, we obtain a generalised framework that fully incorporates all the information on the current risk free term structure as well as the credit spread curve. By imposing restrictions on the forward rate and forward credit spread volatility, the defaultable HJM admits finite dimensional realisations making the class of models computationally tractable. The default process has been specified using a marked point process thereby using the mark of the point process to model the uncertainty in the recovery rate.

We have explored two broad ways of incorporating stochastic volatility within the defaultable HJM term structure class of models. The first approach was covered in Chapter 2 and Chapter 3 where the stochastic process governing the volatility dynamics was driven by a Wiener process independent of the Wiener processes driving the forward rate and forward credit spread dynamics. In Chapter 2, we made the assumption that the volatility function is exponentially decaying as a function of maturity and is level dependent (on the short rate or credit spread and stochastic volatility) and obtained exponential affine defaultable bond prices. In Chapter 3, we assumed the hump volatility specification and verified that the defaultable forward rates admits finite dimensional affine realisation and showed that the defaultable bond prices are exponentially affine in some state variables. We then applied the framework developed to the pricing of credit derivatives and in particular to pricing credit default swaps, credit default swaptions and bond options, the latter under some further simplifying assumptions on the hump volatility specifications.

In the second approach covered in Chapter 4, we allowed for stochasticity to be introduced to the defaultable forward rate volatility using a Markov chain with a finite number of states. We then discussed the conditions on the defaultable forward rate volatility that would lead to finite dimensional Markovian representations of the defaultable short rate dynamics in the presence of regime-switching. We then applied the framework developed pricing defaultable securities with special focus on pricing defaultable bonds and call options on the defaultable bonds.

5.1.1 Markovian Defaultable HJM Class of Models with Unspanned Stochastic Volatility

In Chapter 2, we established a defaultable HJM framework based on Schönbucher [1998] that incorporates a correlation structure between the forward rate, forward credit spread and stochastic volatility processes. This was motivated by the evidence in Heston [1993] where the effects of the correlation between stochastic volatility and short rate on the bond price were investigated. In addition, Jarrow and Turnbull [2000] showed that the correlation between the short rate (forward rate) and the short credit spread (forward credit spread) represents the empirically observed correlation between market risk and default risk.

By adapting the volatility specifications of the type developed by Chiarella and Kwon [2000b], Björk et al. [2004] and Filipovic and Teichmann [2002] to the Schönbucher [1998] model, we established the necessary and sufficient conditions on the volatility structure that allow the defaultable term structure model with stochastic volatility to admit finite-dimensional-realisations in the defaultable short rate dynamics. The volatility functions were a product of a quasi exponential function of the time-to-maturity and an arbitrary function of the forward rate and the volatility process. We also showed that the defaultable bond prices across all maturities can be expressed in terms of the default-free short rate, the short term credit spread and a set of Markovian state variables. We expressed these Markovian state variables in terms of defaultable forward rates of a number of fixed tenors thereby establishing a connection between the defaultable bond price and market observable quantities. In addition, we derived an expression for the defaultable forward rate of any maturity in terms of the fixed tenor forward rates.

By performing simulations in Chapter 2, we investigated the distributional properties of the closed form solution for the defaultable bond price. In particular, we focused on the effects of the correlations in our model and stochastic volatility on the distribution of defaultable bond prices and returns. It was shown that increasing the correlation between the stochastic volatility process and the credit spread or short rate processes from negative to positive increases the negative skewness of the defaultable bond price distribution (conditional on the stochastic volatility process). In addition, increasing both correlations yields higher (excess) kurtosis on the distribution of the normalised defaultable bond returns. We also observed that increasing the correlation between the short-term credit spread and the short rate process leads to a decrease in both the kurtosis and the (negative) skewness of both the defaultable bond price and normalised returns distributions. This correlation conveys information about the co-variation between default-free discount rates and the market's perception of default risk.

The simulation results also indicated that the presence of stochastic volatility in the model affects the skewness and the kurtosis of the defaultable bond price distribution. In particular, when the model state variables (short rate, short term credit spread and stochastic volatility) are uncorrelated, increasing the volatility of volatility, and so the stochastic volatility, leads to an increase in the negative skewness in the defaultable bond price distribution. However, when the state variables are correlated and noting in particular that following the observation made in Longstaff and Schwartz [1995b] and Duffee [1998], that the correlation between the short-term credit spread and the short rate processes is negative for investment-grade noncallable corporate bonds, increasing the stochastic volatility increases the positive skewness in the defaultable bond price distribution. In addition, we observed that this leads to an increase in the excess kurtosis of the normalised bond returns.

By building on the framework developed in Chapter 2, we developed in Chapter 3 a generalised volatility specification that allows for hump-shaped shocks, state dependency and unspanned volatility. This class of volatility functions gives rise to a higher degree of flexibility in modelling the wide range of shapes of the yield curve by virtue of the polynomial in the deterministic part. The hump-shaped shocks are essential in matching interest rate derivatives empirically. We then expressed the defaultable forward rate process as an affine function of some state variables which are jointly Markovian from which we showed that the defaultable bond price is exponential affine in the state variables.

This extended framework was then applied to price credit derivatives. We first derived the approximating pricing formulas for single-name credit default swaps and swaptions within the model. The simulation results showed that the valuation formulas derived from our class of models capture the stylised empirical facts on credit default swaps and swaption prices. A notable observation is that the correlation between short term interest rate and short term credit spread has an impact on the fair credit default swap rate and consequently the swaption prices. By applying Fourier transform methods, we derived a semi-closed form solution to the pricing problem of a put option with a knock-out provision on default of the underlying defaultable bond. We achieved this by making a simplifying assumption on the nature of the hump volatility specification wherein we relaxed the requirement of level dependency. We then solved the resulting coupled system of differential equations that arises when calculating the exercise probabilities using numerical integration to obtain a semi-closed form solution.

5.1.2 Markovian Defaultable HJM Class of Models with Regime-Switching Stochastic Volatility

In Chapter 4, we presented a defaultable HJM framework that introduced stochastic volatility into the defaultable forward rate volatility functions using a continuous time Markov chain with a finite number of states. Following Valchev [2004] and Elhouar [2008], we provided the necessary and sufficient conditions on the volatility function that guarantees finite dimensional Markovian realisations in the presence of regime-switching. We observed that we can reduce the Markovian dynamics of the defaultable short rate to a two-factor Hull and White [1990] type model that allows for better calibration to market data.

By solving the regime-switching bond pricing partial differential equation (which required solving a coupled system of ordinary differential equations numerically), we derived an exponential affine defaultable bond price formula with linearity in both the default-free and defaultable short rate processes for the special case of two regimes, although the results would still hold for more regimes. Chapter 4 also presented some numerical results to illustrate the response of the defaultable bond price distribution to changes in the correlation between the short term credit spread (which gives the market perception of the default risk) and the interest rate. We demonstrated that as in Chapter 2, increasing this correlation implied an increase in the negative skewness of the defaultable bond price distribution in the presence of regime switching stochastic volatility. In addition, it was noted that an increase in the transition intensity and therefore the frequency of regime switching led to a decrease in the kurtosis of the defaultable bond price distribution, a result that can be attributed to investors demanding more compensation for the additional sources of risk.

In conclusion, Chapter 4 also presented results on the option pricing problem in the presence of regime switching volatility. To price a European call option on a defaultable bond with a knock-out provision (for the special case of 2-states regimes), we applied finite difference (theta scheme) methods to the resulting, coupled option pricing partial differential equations and approximated the option price on a discrete space-time grid.

5.2 Directions for Future Research

The results presented in this thesis lead to several possible avenues for future research. The stochastic volatility models developed in Chapter 2 and Chapter 3 are similar to their counterpart in Heston [1993] and Hull and White [1987]. These are known to be incomplete as they involve the market price of risk that arises from the independent Wiener process driving the stochastic volatility process. It would be of interest to investigate how the complete Markovian stochastic volatility model introduced in Hobson and Rogers [1998] and Chiarella and Kwon [2000a] can be extended to the defaultable HJM setting.

Throughout the thesis, we have consistently worked with the assumption of fractional recovery on default on the risky entity. We noted in Chapter 2 that this framework is not broad enough to capture a bond's downgrades or upgrades over time. The pricing model within a multiple defaults and recovery framework is therefore weakened because following default and subsequent repackaging and recovery, a bond issue would ideally receive a downgrade in its credit ratings. Another possible extension would be to consider working within the affine Markov chain (AMC) model for the multifirm credit migration framework proposed in Hurd and Kuznetsov [2007], which was shown to extend easily to the multiple firm framework in addition to allowing for up/downgrading of the risky names. The AMC model could also be a good candidate for modelling default correlation of any two names since as we noted in Chapter 3 and also by Schönbucher [2003, Chapter 10], the default correlation levels that could be achieved within our framework are typically too low when compared with empirical default correlations in addition to the level of complexity involved in deriving and analysing the resulting dependency structure. In addition, this extension could also incorporate jumps within defaultable term structure and thereby allow the for defaultable rates to jump in the event of default.

The option pricing models developed in Chapter 3 and Chapter 4 could also be applied to complex payoff functions, in addition to investigating pricing within the American options context. The latter would constitute an interesting formulation problem especially in the framework of Chapter 4 given the existence of default uncertainty and regime switching risk as well as the early exercise complication. Throughout the thesis, with the exception of Section 3.4.2, we have assumed the absence of counterparty risk. The rapid growth in the OTC derivatives market has pushed to the fore the problem of counterparty credit risk (CCR). Setting limits against future credit exposures and verifying potential trades against these limits has the downside of rejecting trading opportunities with large exposures that exceed the set limits. Credit value adjustment (CVA), being the market value of CCR allows financial institutions to dynamically price CCR directly into new trades. Research on this pertinent topic remains an ongoing concern. In addition, given that credit options are predominantly over-the-counter financial contracts where there is no guarantee from a third party such as a clearing house, another possible topic of study would involve investigating option pricing in the case of vulnerable options within the framework developed in Chapter 3. Little work has been done in this area, and to the best of our knowledge, none within the defaultable HJM framework.

In the simulation experiments that we conducted in the thesis, we relied on parameters values estimated within the default-free HJM framework by several authors and used them to 'approximate' what would be 'reasonable' parameter values in the defaultable setting. An empirical study conducted within the our framework would go a long way in filling the gap that exists in this area and yield parameter values that could further assist in deepening research within the defaultable HJM framework. A prominent feature of the stochastic volatility model is that the likelihood function is expressed by a high dimensional integral which cannot be analytically solved as a result of the latent volatility process and one has to revert to numerically integrating out the latent volatility process using importance sampling techniques followed by numerical maximisation of the approximate likelihood function as in

Sandmann and Koopman [1998] and Durham [2006].

The model we have developed in this thesis offers a general yet tractable framework for analysing and measuring the extent to which volatility can be spanned in credit risk market. To our knowledge there is no empirical study on this very important feature of the credit risk volatility which plays a vital role in credit derivative hedging and pricing. Trolle and Schwartz [2009] studied this within a default-free setting.

Given that the CDS spread can be driven by several factors that may not be directly related to the reference entity's fundamental credit worthiness including leverage interest in CDS trading, counterparty risk and risk-aversion of market participants, another possible research topic would be to investigate the extent to which the volatility of CDS spreads is an indicator of default risk. During the recent financial crisis, it was noted (see Grossman and Hansen [2010, Fitch Ratings Research on Credit spreads and Default risk]) that although widening CDS spreads normally imply deterioration in the credit quality followed by default, there exists what has come to be referred to as 'false positives' when the spreads widen but there are no subsequent defaults. In this case, the spreads overstate the subsequent realised default experience of corporate issuers, imposing significant costs on market participants who rely on them as default risk indicators.

Appendix I

A.1 Doléans-Dade Exponential Formula

In this appendix, we provide this key result and sketch the proof. For a more detailed proof, the reader is referred to Jacod and Shiryaev [2003, Theorem 4.61, pg.59] and Klebaner [2005, Theorem 8.33 and Section 9.3].

Theorem A.1 *Let the process X be a semi-martingale. Then the stochastic equation*

$$\mathcal{R}(t) = 1 + \int_0^t \mathcal{R}(s-)dX(s), \tag{A.1.1}$$

has a unique solution, the stochastic exponential of X, that is given by

$$\mathcal{E}(X)(t) := \mathcal{R}(t) = e^{X(t)-X(0)-\frac{1}{2}[X,X]^c(t)} \prod_{s\leq t} \big(1 + \Delta X(s)\big)e^{-\Delta X(s)}. \tag{A.1.2}$$

We note that (A.1.1) can equivalently be written as $d\mathcal{R}(t) = \mathcal{R}(t-)dX(t)$, $\mathcal{R}(0) = 1$ and in this thesis, we have specified the process X by

$$dX(t) = -\int_0^1 q\mu(dt, dq), \tag{A.1.3}$$

or alternatively,

$$X(t) = X(0) - \int_0^t \int_0^1 q\mu(ds, dq). \tag{A.1.4}$$

Given that X is a semi-martingale, the infinite product in (A.1.2) is almost surely absolutely convergent, since $\sum_{s\le t} \big(\Delta X(s)\big)^2 \le [X,X]_t < \infty$. The process $\mathcal{E}(X)$ in (A.1.2) is called the Doléans-Dade exponential of X. When X is continuous, we get an exponential martingale $\mathcal{E}(X)(t) = e^{X(t)-X(0)-\frac{1}{2}[X,X]^c(t)}$. If X is a process with local finite variation, then $[X^c, X^c]_t = 0$ and its continuous part can be shown to satisfy $dX^c(t) = dX(t) - \Delta X(s)$. In addition, given that $X^c(t) = X(t) - X(0) - \sum_{s\le t} \Delta X(s)$, the stochastic exponential in (A.1.2) can be written as

$$\mathcal{R}(t) = e^{X^c(t)} \prod_{s\le t} \big(1 + \Delta X(s)\big). \tag{A.1.5}$$

If the continuous part $X^c \equiv 0$, the formula (A.1.13) reduces to an identity and from (A.1.4), we have

$$\mathcal{R}(t) = \prod_{\tau_i \le T} \big(1 - q_i\big). \tag{A.1.6}$$

Proof: To check if $\mathcal{R}(t)$ is a solution, we let $Z(t) = X(t) - X(0) - \dfrac{1}{2}[X,X]^c(t)$ and $Y(t) = \prod_{s\le t} \big(1 + \Delta X(s)\big)e^{-\Delta X(s)}$ such that $\mathcal{R}(t) = e^{Z(t)}Y(t)$.

By applying Itô's lemma to $\mathcal{R}(t)$ we have that

$$\begin{aligned}\mathcal{R}(t) &= 1 + \mathcal{R}(t-) \cdot Z(t) + e^{Z(t-)} \cdot Y(t) + \frac{1}{2}\mathcal{R}(t-) \cdot [Z^c, Z^c]_t \\ &\quad + \sum_{s\le t} \big[\Delta\mathcal{R}(s) - \mathcal{R}(s-)\Delta Z(s) - e^{Z(s-)}\Delta Y(s)\big].\end{aligned} \tag{A.1.7}$$

We know from the definition of Z that $Z^c = X^c$ from which it follows that

$$Z(t) + \frac{1}{2}[Z^c, Z^c]_t = X(t) - X(0). \tag{A.1.8}$$

Similarly, we know that

$$e^{Z(t-)}Y(t) = \sum_{s \leq t} e^{Z(s-)} \Delta Y(s), \tag{A.1.9}$$

since the process Y is of pure jump type. Moreover,

$$\Delta \mathcal{R}(s) = e^{Z(s-)+\Delta Z(s)} Y(s-)\big(1 + \Delta X(s)\big)e^{-\Delta X(s)} - e^{Z(s-)}Y(s) = \mathcal{R}(s-)\Delta Z(s), \tag{A.1.10}$$

since $\Delta X(s) = \Delta Z(s)$. Substituting (A.1.8), (A.1.9) and (A.1.10) into (A.1.7), it follows that

$$\mathcal{R}(t) = 1 + \mathcal{R}(t-)\big(X(t) - X(0)\big) = 1 + \mathcal{R}(t-) \cdot X(t). \tag{A.1.11}$$

To prove uniqueness, we let $\mathcal{R}$ be an arbitrary solution and $Y = \mathcal{R}e^{-Z}$ where we suppress dependence on time for notational simplicity. On applying Itô's lemma, we have that

$$\begin{aligned} Y - 1 &= e^{-Z_-} \cdot \mathcal{R} - Y_- \cdot Z + \frac{1}{2} Y_- \cdot [X, X]^c - e^{-Z_-} \cdot [X, \mathcal{R}]^c + \sum \{\Delta Y + Y_- \Delta Z - e^{-Z_-} \Delta \mathcal{R}\} \\ &= Y_- \cdot X - Y_- \cdot X + \frac{1}{2} Y_- \cdot [X, X]^c + \frac{1}{2} Y_- \cdot [X, X]^c - Y_- \cdot [X, X]^c + \sum \{\Delta Y + Y_- \Delta X - Y_- \Delta X\} \\ &= \sum \Delta Y. \end{aligned} \tag{A.1.12}$$

This shows that Y is a purely discontinuous process of locally finite variation. In addition, we have that

$$\begin{aligned} \Delta Y &= \mathcal{R}e^{-Z} - e^{-Z_-}\mathcal{R}_- = (\mathcal{R}_- + \Delta \mathcal{R})e^{-Z_- - \Delta Z} - e^{-Z_-}\mathcal{R}_- \\ &= \mathcal{R}_- \{(1 + \Delta X)e^{-X} - 1\}, \end{aligned} \tag{A.1.13}$$

which implies that

$$Y = 1 + Y_{-} \cdot A, \quad \text{where} \quad A = \sum \{(1 + \Delta X)e^{-X} - 1\}. \tag{A.1.14}$$

In addition, it was shown in Kallenberg [1997, Theorem 23.8, pg.442] that the homogenous equation $Y = Y_{-} \cdot A$ has a unique solution, $Y = 0$. Hence the proof follows. ♦

A.2 Proof of Proposition 2.5 on Bond Price Dynamics with Stochastic Volatility

The proof of (2.34) in the first part of the proposition mirrors the derivation in the original Heath et al. [1992] paper as the stochastic volatility does not enter directly into the formulation. An idea of the proof is given in Appendix A.3 and we therefore omit it and proceed to derive the result for the second part, which follows the idea in Schönbucher [1998], Theorem 2, equation 44.

From the definition of the defaultable bond in (2.14) we recall that

$$P^d(t,T,V) = \mathcal{R}(t)\bar{P}^d(t,T,V), \tag{A.2.1}$$

where $\bar{P}^d(t,T,V)$, the pseudo-bond is as defined in (2.13) and $\mathcal{R}(t)$ is the remainder of all fractional default losses at time t. We observe that the pre-default value of the bond is given by

$$P^d(t-,T,V) = \mathcal{R}(t-)\bar{P}^d(t,T,V), \tag{A.2.2}$$

where $\bar{P}^d(t,T,V)$ is a continuous function as per the definition of the pseudo-bond.

Applying Itô's lemma to (A.2.2) yields

$$dP^d(t,T,V) = \bar{P}^d(t,T,V)d\mathcal{R}(t) + \mathcal{R}(t-)d\bar{P}^d(t,T,V) + d\mathcal{R}(t)d\bar{P}^d(t,T,V),$$

which is equivalent to

$$\begin{aligned} dP^d(t,T,V) &= \mathcal{R}(t-)d\bar{P}^d(t,T,V) + d\mathcal{R}(t)\big(\bar{P}^d(t,T,V) + d\bar{P}^d(t,T,V)\big), \\ &= \mathcal{R}(t-)d\bar{P}^d(t,T,V) + \bar{P}^d(t,T,V)d\mathcal{R}(t). \end{aligned}$$

The last equation can be written as

$$dP^d(t,T,V) = \mathcal{R}(t-)\bar{P}^d(t-,T,V)\frac{d\bar{P}^d(t,T,V)}{\bar{P}^d(t-,T,V)} + \bar{P}^d(t,T,V)\mathcal{R}(t-)\frac{d\mathcal{R}(t)}{\mathcal{R}(t-)},$$

and given that $\bar{P}^d(t-,T,V) = \bar{P}^d(t,T,V)$ this reduces to

$$dP^d(t,T,V) = P^d(t-,T,V)\underbrace{\frac{d\bar{P}^d(t,T,V)}{\bar{P}^d(t,T,V)}}_{A} + P^d(t-,T,V)\underbrace{\frac{d\mathcal{R}(t)}{\mathcal{R}(t-)}}_{B}. \tag{A.2.3}$$

It remains to derive the stochastic differential equation for the pseudo-bond $\bar{P}^d(t,T,V)$.

On substituting (2.31) into (2.13) we have

$$\bar{P}^d(t,T,V) = \exp\bigg(- \int_t^T f^d(0,s,V_0)ds - \int_t^T \int_0^t \alpha^d(u,s,V)duds - \sum_{i=1}^3 \int_t^T \int_0^t \tilde{\sigma}_i^d(u,s,V)dW_i(u)ds\bigg). \tag{A.2.4}$$

In addition, following (2.31) we observe that

$$\int_t^T f^d(t,s,V)ds = \int_t^T f^d(0,s,V_0)ds + \int_t^T \int_0^t \alpha^d(u,s,V)duds + \sum_{i=1}^3 \int_t^T \int_0^t \tilde{\sigma}_i^d(u,s,V)dW_i(u)ds. \tag{A.2.5}$$

On using stochastic Fubini's theorem in the term in brackets on the RHS of (A.2.4), we find that

$$\int_t^T f^d(t,s,V)ds = \int_t^T f^d(0,s,V_0)ds + \int_0^t \int_t^T \alpha^d(u,s,V)dsdu + \sum_{i=1}^3 \int_0^t \int_t^T \tilde{\sigma}_i^d(u,s,V)dsdW_i(u). \tag{A.2.6}$$

This can be written as

$$\begin{aligned}\int_t^T f^d(t,s,V)ds &= \int_0^T f^d(0,s,V_0)ds + \int_0^t \int_u^T \alpha^d(u,s,V)dsdu + \sum_{i=1}^3 \int_0^t \int_u^T \tilde{\sigma}_i^d(u,s,V)dsdW_i(u) \\ &- \int_0^t f^d(0,s,V_0)ds - \int_0^t \int_u^t \alpha^d(u,s,V)dsdu - \sum_{i=1}^3 \int_0^t \int_u^t \tilde{\sigma}_i^d(u,s,V)dsdW_i(u).\end{aligned} \tag{A.2.7}$$

We recall from (2.33) that the defaultable short rate follows the stochastic integral equation

$$r^d(t,V) = f^d(0,t,V_0) + \int_0^t \alpha^d(u,t,V)du + \sum_{i=1}^3 \int_0^t \tilde{\sigma}_i^d(u,t,V)dW_i(u), \tag{A.2.8}$$

which on integrating from 0 to t yields

$$\int_0^t r^d(s,V)ds = \int_0^t f^d(0,s,V_0)ds + \int_0^t \int_0^s \alpha^d(u,t,V)duds + \sum_{i=1}^3 \int_0^t \int_0^s \tilde{\sigma}_i^d(u,t,V)dW_i(u)ds. \tag{A.2.9}$$

Following Fubini's theorem and change of the limits of integration, (A.2.9) can be expressed as

$$\int_0^t r^d(s,V)ds = \int_0^t f^d(0,s,V_0)ds + \int_0^t \int_u^t \alpha^d(u,t,V)dsdu + \int_0^t \int_u^t \tilde{\sigma}_i^d(u,t,V)dsdW_i(u). \tag{A.2.10}$$

On substituting (A.2.10) into (A.2.7), we have the expression

$$\int_t^T f^d(t,s,V)ds = -\ln P^d(0,T,V_0) - \int_0^t r^d(s,V)ds + \int_0^t \int_u^T \alpha^d(u,s,V)dsdu + \sum_{i=1}^{3} \int_0^t \int_u^T \tilde{\sigma}_i^d(u,s,V)dsdW_i(u), \quad \text{(A.2.11)}$$

where we have used the definition of the defaultable bond as a function of the pseudo-bond to show that

$$P^d(0,T,V_0) = \mathcal{R}(0)\bar{P}^d(0,T,V_0) = \bar{P}^d(0,T,V_0).$$

We define the log pseudo bond price $\bar{B}^d(t,T,V) = \ln \bar{P}^d(t,T,V)$. Then, using equations (A.2.11) and (2.13) the log pseudo bond price can then be expressed as the stochastic integral equation

$$\bar{B}^d(t,T,V) = \int_0^t r^d(s,V)ds + \ln P^d(0,T,V_0) - \int_0^t \int_u^T \alpha^d(u,s,V)dsdu - \sum_{i=1}^{3} \int_0^t \int_u^T \tilde{\sigma}_i^d(u,s,V)dsdW_i(u). \quad \text{(A.2.12)}$$

Equivalently, this can be expressed as the stochastic differential equation

$$d\bar{B}^d(t,T,V) = [r^d(t,V) - \alpha_B^d(t,T,V)]dt + \sum_{i=1}^{3} \tilde{\sigma}_{B,i}^d(t,T,V)dW_i(t), \quad \text{(A.2.13)}$$

where

$$\alpha_B^d(u,T,V) = \int_u^T \alpha^d(u,s,V)ds, \quad \text{and} \quad \tilde{\sigma}_{B,i}^d(u,T,V) = -\int_u^T \tilde{\sigma}_i^d(u,s,V)ds. \quad \text{(A.2.14a)}$$

It can then be seen that the pseudo-bond dynamics satisfy the stochastic differential equation

$$d\bar{P}^d(t,T,V) = \bar{P}^d(t,T,V)\big(r^d(t,V) + b^d(t,T,V)\big)dt + \bar{P}^d(t,T,V)\sum_{i=1}^{3}\tilde{\sigma}^d_{B,i}(t,T,V)dW_i(t), \tag{A.2.15}$$

where the coefficients in the drift and diffusion are given by

$$b^d(t,T,V) = -\alpha^d_B(t,T,V) + \frac{1}{2}\sum_{i=1}^{3}\big(\tilde{\sigma}^d_{B,i}(t,T,V)\big)^2, \tag{A.2.16}$$

$$\tilde{\sigma}^d_{B,i}(t,T,V) = -\int_t^T \tilde{\sigma}^d_i(t,s,V)ds. \tag{A.2.17}$$

Substituting (A.2.15) into (A.2.3) then yields the stochastic differential equation for the defaultable bond

$$\frac{dP^d(t,T,V)}{P^d(t-,T,V)} = \big(r^d(t,V) + b^d(t,T,V)\big)dt + \sum_{i=1}^{3}\tilde{\sigma}^d_{B,i}(t,T,V)dW_i(t) - \int_E q\mu(dt,dq). \tag{A.2.18}$$

Equivalently, we note that $dM(\omega,t) = \int_E q\mu(dt,dq) - \int_E q\nu(dt,dq)$ such that $M(\omega,t)$ is a local martingale. The defaultable price dynamics can alternatively be written as

$$\frac{dP^d(t,T,V)}{P^d(t-,T,V)} = \big(r^d(t,V) + b^d(t,T,V) - \int_E qh(dt,dq)\big)dt + \sum_{i=1}^{3}\tilde{\sigma}^d_{B,i}(t,T,V)dW_i(t) - dM(\omega,t), \tag{A.2.19}$$

where we have used (2.9), with $\nu(\omega;dt,dq) = h(\omega;t,dq)dt$. Hence the proof follows. ♦

A.3 Proof of Proposition 2.6 on Bond Pricing under Risk-Neutral Dynamics

We only provide the idea of the proof to this key result. For the complete proof, the reader is referred to the original paper by Heath, Jarrow, and Morton [1992].

Using (2.2) and the stochastic integral equation equivalent to (2.29a), then by applying Ito's lemma and simplifying we find the corresponding stochastic differential equation for the bond price to be

$$\frac{dP(t,T,V)}{P(t,T,V)} = \Big(r(t,V) - \alpha^f_B(t,T,V) + \frac{1}{2}\sum_{i=1}^{3}\big(\tilde{\sigma}^f_{B,i}(t,T,V)\big)^2\Big)dt + \sum_{i=1}^{3}\tilde{\sigma}^f_{B,i}(t,T,V)dW_i(t), \tag{A.3.1}$$

where the coefficients are given by

$$\alpha^f_B(t,T,V) = \int_t^T \alpha^f(t,s,V)ds, \tag{A.3.2}$$

$$\tilde{\sigma}^f_{B,i}(t,T,V) = -\int_t^T \tilde{\sigma}^f_i(t,s,V)ds. \tag{A.3.3}$$

Following the hedging argument, in order that there not exist riskless arbitrage opportunities between bonds of different maturities then the instantaneous excess bond return, risk adjusted by its maturity must equal the market price of interest rate risk. It then follows that

$$\sum_{i=1}^{3}\phi_i\big(\tilde{\sigma}^f_{B,i}(t,T,V) - \alpha^f_B(t,T,V) + \frac{1}{2}\sum_{i=1}^{3}\big(\tilde{\sigma}^f_{B,i}(t,T,V) = 0, \tag{A.3.4}$$

which on integrating with respect to maturity and re-arranging yields

$$\alpha^f(t,T,V) = -\sum_{i=1}^{3}\tilde{\sigma}^f_i(t,T,V)\Big(\phi_i(t) - \int_t^T \tilde{\sigma}^f_i(t,s,V)ds\Big), \tag{A.3.5}$$

the forward rate drift restriction which as was shown by HJM is a necessary and sufficient condition for the absence of riskless arbitrage opportunities.

We observe that under the risk neutral measure, the forward rate dynamics follow the stochastic integral equation

$$f(t,T,V) = f(0,T,V) + \int_0^t \alpha^f(u,T,V)du + \sum_{i=1}^{3}\int_0^t \phi_i(u)\tilde{\sigma}_i^f(u,T,V)du + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,T,V)d\tilde{W}_i(u), \tag{A.3.6}$$

and on substituting (A.3.5) into (A.3.6) yields

$$f(t,T,V) = f(0,T,V) + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,T,V)\int_u^T \tilde{\sigma}_i^f(u,s,V)duds + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,T,V)d\tilde{W}_i(u). \tag{A.3.7}$$

Similarly, using the drift restriction condition (A.3.1) in (A.3.1) and simplifying gives

$$dP(t,T,V) = P(t,T,V)r(t,V)dt - P(t,T,V)\sum_{i=1}^{3}\Big(\int_t^T \tilde{\sigma}_i^f(t,s,V)ds\Big)d\tilde{W}_i(t). \tag{A.3.8}$$

Hence the proof to Proposition 2.6 follows. ♦

A.4 Proof of Proposition 2.7 on the Existence of the Risk-Neutral Measure

From the version of the Girsanov's Theorem presented in Björk et al. [1997] (Theorem 3.12) which there is applied to the default-free framework, the defaultable bond price dynamics in equation (2.36) under the risk-neutral measure can be written as

$$\frac{dP^d(t,T,V)}{P^d(t-,T,V)} = \Big(r^d(t,V) + b^d(t,T,V)\Big)dt + \sum_{i=1}^{3} \tilde{\sigma}^d_{B,i}(t,T,V)\big[d\tilde{W}_i(t) + \phi_i(t)dt\big]$$
$$- \Big(\int_E q\mu(dt,dq) - \int_E q\tilde{h}(dt,dq)dt\Big) - \int_E q\tilde{h}(dt,dq)dt. \qquad \text{(A.4.1)}$$

This can be written as

$$\frac{dP^d(t,T,V)}{P^d(t-,T,V)} = \Big(r^d(t,V) + b^d(t,T,V) + \sum_{i=1}^{3} \phi_i(t)\tilde{\sigma}^d_{B,i}(t,T,V) - \int_E q\tilde{h}(dt,dq)\Big)dt$$
$$+ \sum_{i=1}^{3} \tilde{\sigma}^d_{B,i}(t,T,V)d\tilde{W}_i(t) - d\tilde{M}(\omega,t), \qquad \text{(A.4.2)}$$

where $d\tilde{M}(\omega,t)$ is a local martingale under the risk-neutral measure.

From the fundamental theorem of asset pricing we know that a measure $\tilde{\mathbb{P}}$ is a risk-neutral measure if and only if the discounted bond price is a martingale such that the bond dynamics are of the form

$$dP^d(t,T,V) = P^d(t,T,V)r(t,V) + d\tilde{V}(t), \qquad \text{(A.4.3)}$$

where $\tilde{V}$ is a $\tilde{\mathbb{P}}-$local martingale. Then, comparing the drifts of equations (A.4.2) and (A.4.3) we observe that we require

$$r^d(t,V) + b^d(t,T,V) + \sum_{i=1}^{3} \phi_i(t)\tilde{\sigma}^d_{B,i}(t,T,V) - \int_E q\tilde{h}(dt,dq) = r(t,V),$$

inorder that the discounted defaultable bond price is a martingale. Hence the proof follows.

♦

A.5 Credit Spread Drift Restriction Condition.

By using Equation (2.32), the drift restriction condition (2.45) can be expanded to

$$\begin{aligned}\alpha^d(t,T,V) &= -\sum_{i=1}^{3}\phi_i(t)\tilde{\sigma}_i^f(t,T,V) - \sum_{i=1}^{3}\phi_i(t)\tilde{\sigma}_i^\lambda(t,T,V)\\ &+\sum_{i=1}^{3}\tilde{\sigma}_i^f(t,T,V)\int_t^T\tilde{\sigma}_i^f(t,s,V)ds + \sum_{i=1}^{3}\tilde{\sigma}_i^\lambda(t,T,V)\int_t^T\tilde{\sigma}_i^\lambda(t,s,V)ds\\ &+\sum_{i=1}^{3}\tilde{\sigma}_i^\lambda(t,T,V)\int_t^T\tilde{\sigma}_i^f(t,s,V)ds + \sum_{i=1}^{3}\tilde{\sigma}_i^f(t,T,V)\int_t^T\tilde{\sigma}_i^\lambda(t,s,V)ds. \quad \text{(A.5.1)}\end{aligned}$$

Substitute (2.23) and (2.32) into equation (2.45) and use the condition in (A.5.1) to obtain Equation (2.46). ♦

A.6 Proof of Lemma 2.10

By using equations (2.31), (2.39) and the drift restriction condition (2.46), the defaultable forward rate risk-neutral dynamics becomes

$$\begin{aligned}f^d(t,T,V) &= f^d(0,T,V_0) + \sum_{i=1}^{3}\int_0^t\tilde{\sigma}_i^f(u,T,V)\int_u^T\tilde{\sigma}_i^f(u,s,V)dsdu\\ &+\sum_{i=1}^{3}\int_0^t\tilde{\sigma}_i^\lambda(u,T,V)\int_u^T\tilde{\sigma}_i^\lambda(u,s,V)dsdu + \sum_{i=1}^{3}\int_0^t\tilde{\sigma}_i^\lambda(u,T,V)\int_u^T\tilde{\sigma}_i^f(u,s,V)dsdu\\ &+\sum_{i=1}^{3}\int_0^t\tilde{\sigma}_i^f(u,T,V)\int_u^T\tilde{\sigma}_i^\lambda(u,s,V)dsdu\\ &+\sum_{i=1}^{3}\int_0^t\tilde{\sigma}_i^f(u,T,V)d\tilde{W}_i(u) + \sum_{i=1}^{3}\int_0^t\tilde{\sigma}_i^\lambda(u,T,V)d\tilde{W}_i(u). \quad \text{(A.6.1)}\end{aligned}$$

On setting $T = t$ in (A.6.1) it follows that the instantaneous defaultable short rate dynamics $r^d(t, V)$ under the risk neutral measure are given by the stochastic integral equation

$$\begin{aligned} r^d(t,V) &= f^d(0,t,V_0) + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,s,V)dsdu + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,s,V)dsdu \\ &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,s,V)dsdu + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,s,V)dsdu \\ &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,t,V)d\tilde{W}_i(u) + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)d\tilde{W}_i(u). \end{aligned} \tag{A.6.2}$$

By using the state variables of Definition 2.9 this can be written as

$$r^d(t,V) = f^d(0,t,V_0) + \sum_{j=1}^{4} S_j(t,V) + \sum_{j=1}^{2} \psi_j(t,V), \tag{A.6.3}$$

whose differential form is given by

$$dr^d(t,V) = \left[f_2^d(0,t,V_0) + \sum_{j=1}^{4}\frac{\partial}{\partial t}S_j(t,V)\right]dt + \sum_{j=1}^{2} d\psi_j(t,V). \tag{A.6.4}$$

Similarly, from condition (2.46) the forward credit spread dynamics $\lambda(t,T,V)$ in Equation (2.29b) can be written as

$$\begin{aligned} \lambda(t,T,V) &= \lambda(0,T,V_0) + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,T,V)\int_u^T \tilde{\sigma}_i^\lambda(u,s,V)dsdu \\ &+ \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,T,V)\int_u^T \tilde{\sigma}_i^\lambda(u,s,V)dsdu + \sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,T,V)\int_u^T \tilde{\sigma}_i^f(u,s,V)dsdu \\ &+ \sum_{i=1}^{3} \tilde{\sigma}_i^\lambda(t,t,V)d\tilde{W}_i(t). \end{aligned} \tag{A.6.5}$$

Setting $T = t$ in (A.6.5) and using (2.7), then Equation (2.53) is derived.

Furthermore, the default free instantaneous short rate in equation (2.19) follows the stochastic integral equation

$$r(t,V) = f(0,t,V_0) + \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,V) \int_u^t \tilde{\sigma}_i^f(u,s,V)dsdu + \sum_{i=1}^{3} \int_0^t \tilde{\sigma}_i^f(u,t,V)d\tilde{W}_i(u). \tag{A.6.6}$$

Using the quantities defined in (2.50a) and (2.50e), Equation (A.6.6) can be expressed as (2.52). Furthermore, recalling (2.29c) the stochastic volatility process $V(t)$, will follow the stochastic differential (2.54) under the risk-neutral measure $\tilde{\mathbb{P}}$. ♦

A.7 Proof of Corollary 2.11

By using Condition 2.44, Equation (2.36) becomes

$$\frac{dP^d(t,T,V)}{P^d(t-,T,V)} = r(t,V)dt + \sum_{i=1}^{3} \tilde{\sigma}_{B,i}^d(t,T,V)\big(dW_i(t) - \phi_i(t)dt\big) - \left(\int_E q\mu(dt,dq) - \int_E \psi(t,q)h(t,dq)dt\right), \tag{A.7.1}$$

and by making use of (2.39) and (2.40), (A.7.1) can be written as

$$\frac{dP^d(t,T,V)}{P^d(t-,T,V)} = r(t,V)dt + \sum_{i=1}^{3} \tilde{\sigma}_{B,i}^d(t,T,V)d\tilde{W}_i(t) - dM(\omega;t). \tag{A.7.2}$$

These are the dynamics of the defaultable bond price under the risk-neutral measure in which the process

$$\begin{aligned}\tilde{M}(\omega;t) &= \int_0^t \int_E q\,\mu(\omega;ds,dq) - \int_0^t \int_E q\,\psi(s,q)h(s,dq)ds \\ &= \int_0^t \int_E q\,\mu(\omega;ds,dq) - \int_0^t \int_E q\,\tilde{h}(s,dq)ds,\end{aligned}$$

is a local martingale.

We define the relative defaultable bond price by

$$Z^d(t,T,V) = \frac{P^d(t,T,V)}{B(t,V)},$$

where $B(t,V) = \exp\Big(\int_0^t r(s,V)ds\Big)$ is the accumulated money market account. Applying Itô's quotient rule, the stochastic differential equation for $Z^d(t,T,V)$ is

$$\frac{dZ^d(t,T,V)}{Z^d(t-,T,V)} = \sum_{i=1}^{3} \tilde{\sigma}^d_{B,i}(t,T,V)d\tilde{W}_i(t) - \int_E q\,\tilde{h}(t,dq)dt. \tag{A.7.3}$$

If we let $\tilde{\mathbb{E}}$ denote mathematical expectation with respect to the risk neutral probability measure, it then follows that

$$\tilde{\mathbb{E}}\big[dZ^d(t,T,V)\big|\mathcal{F}_t\big] = 0.$$

This implies that

$$\tilde{\mathbb{E}}\big[Z^d(T,T,V)\big|\mathcal{F}_t\big] = Z^d(t,T,V),$$

and given that $P^d(T,T,V) = \bar{P}^d(T,T,V)\mathcal{R}(T)$, the defaultable bond price satisfies

$$P^d(t,T,V) = \tilde{\mathbb{E}}\Big[\exp\Big(-\int_t^T r(s,V)ds\Big)\mathcal{R}(T)\Big|\mathcal{F}_t\Big]. \tag{A.7.4}$$

We observe that if there is no default prior to maturity the payout is 1 whereas $\mathcal{R}(\tau_i)$ is the actual payment if there is default before T. Equation (A.7.4) can then be written as

$$P^d(t,T,V) = \tilde{\mathbb{E}}\Big[\exp\Big(-\int_t^{\tau} r(s,V)ds\Big)\mathcal{R}(\tau_i)\Big|\mathcal{F}_t\Big]. \tag{A.7.5}$$

We note that the quantity $\exp\big(-\int_t^T r(s,V)ds\big)$ is the stochastic discount factor under the measure $\tilde{\mathbb{P}}$ used to discount back to time t the \$1 payoff to be received at time T. ♦

A.8 Proof of Proposition 2.14

Using the stochastic integral equation dynamics (2.51) of the defaultable short rate and the volatility specifications of Assumption 2.3.1 and taking differentials, then $r^d(t, V)$ is found to satisfy the stochastic differential equation

$$dr^d(t,V) = \big[\frac{\partial}{\partial t} f^d(0,t,V_0) + \sum_{j=1}^{4} \frac{\partial}{\partial t} S_j(t,V) - \kappa_f \psi_1(t,V) - \kappa_\lambda \psi_2(t,V)\big] dt \\ + \sum_{i=1}^{3} \Big(\varrho_{3i}\sigma_f \sqrt{r(t,V)V(t)} + \varrho_{2i}\sigma_\lambda \sqrt{\lambda(t,V)V(t)} \Big) d\tilde{W}_i(t). \tag{A.8.1}$$

Furthermore, by using the additional state variables $\eta_i, i = 1, 2, 3$, as given in Definition 2.12, Equation (A.8.1) yields

$$dr^d(t,V) = \big[f_2^d(0,t,V_0) + \eta_1(t,V) - \kappa_f S_1(t,V) + \eta_2(t,V) - \kappa_\lambda S_2(t,V) \\ + 2\eta_3(t,V) - \kappa_f S_3(t,V) - \kappa_\lambda S_4(t,V) - \kappa_f \psi_1(t,V) - \kappa_\lambda \psi_2(t,V)\big] dt \\ + \sum_{i=1}^{3} \Big(\varrho_{3i}\sigma_f \sqrt{r(t,V)V(t)} + \varrho_{2i}\sigma_\lambda \sqrt{\lambda(t,V)V(t)} \Big) d\tilde{W}_i(t). \tag{A.8.2}$$

From the short-term credit spread dynamics (2.53), the variable $\psi_2(t, V)$ can be expressed as

$$\psi_2(t,V) = c(t,V) - \lambda(0,t,V_0) - \sum_{j=2}^{4} S_j(t,V). \tag{A.8.3}$$

By rewriting the default free dynamics (2.52) of the forward rate, we have that

$$\psi_1(t,V) = r(t,V) - f(0,t,V_0) - S_1(t,V). \tag{A.8.4}$$

Substituting equations (A.8.3) and (A.8.4) into (A.8.2), we obtain

$$\begin{aligned} dr^d(t,V) &= \big[f_2^d(0,t,V_0) + \kappa_f f(0,t,V_0) + \kappa_\lambda \lambda(0,t,V_0) + \eta_1(t,V) + \eta_2(t,V) + 2\eta_3(t,V) \\ &\quad - (\kappa_f - \kappa_\lambda) S_3(t,V) - \kappa_f r(t,V) - \kappa_\lambda c(t,V)\big] dt \\ &\quad + \Big(\sum_{i=1}^{3} \varrho_{3i}\sigma_f \sqrt{r(t,V)V(t)} + \sum_{i=1}^{3} \varrho_{2i}\sigma_\lambda \sqrt{\lambda(t,V)V(t)}\Big) d\tilde{W}_i(t). \end{aligned} \tag{A.8.5}$$

This can be rearranged to yield the result in Proposition 2.14. Similarly by using the state variables $\eta_i, i = 1, 2, 3$, as given in Definition 2.12, the results for the default-free short rate and the short-term credit spread can be obtained. ♦

A.9 Proof of Proposition 2.15

By substituting the drift condition (2.45) into the dynamics (2.31), the stochastic integral equation for the defaultable forward rate under the risk-neutral measure may be expressed as

$$f^d(t,T,V) = f^d(0,T,V_0) + \sum_{i=1}^{3} \Big[\int_0^t \tilde{\sigma}_i^{d*}(u,T,V)du + \int_0^t \tilde{\sigma}_i^d(u,T,V)d\tilde{W}_i(u)\Big], \tag{A.9.1}$$

where

$$\tilde{\sigma}_i^{d*}(t,T,V) = \tilde{\sigma}_i^d(t,T,V) \int_t^T \tilde{\sigma}_i^d(t,s,V)ds. \tag{A.9.2}$$

Then, from equation (2.13), the 'pseudo' bond is expressed as

$$\begin{aligned} \bar{P}^d(t,T,V) = \exp\Big[-\sum_{i=1}^{3} \Big(\int_t^T f^d(0,s,V_0)ds + \int_t^T \int_0^t \tilde{\sigma}_i^{d*}(u,s,V)duds \\ + \int_t^T \int_0^t \tilde{\sigma}_i^d(u,s,V)d\tilde{W}_i(u)ds \Big)\Big]. \end{aligned} \tag{A.9.3}$$

We define a new variable I such that

$$I = \int_t^T \int_0^t \tilde{\sigma}_i^{d*}(u,s,V)duds + \int_t^T \int_0^t \tilde{\sigma}_i^d(u,s,V)d\tilde{W}_i(u)ds. \qquad \text{(A.9.4)}$$

By applying Fubini's theorem, this can be rewritten as

$$I = \int_0^t \int_t^T \tilde{\sigma}_i^{d*}(u,s,V)dsdu + \int_0^t \int_t^T \tilde{\sigma}_i^d(u,s,V)dsd\tilde{W}_i(u), \qquad \text{(A.9.5)}$$

so that, $I = I_1 + I_2$. We note that for $u < t < s$,

$$\int_t^T \tilde{\sigma}_i^{d*}(u,s,V)ds = \int_t^T \tilde{\sigma}_i^d(u,s,V)\int_u^t \tilde{\sigma}_i^d(u,v,V)dvds + \int_t^T \tilde{\sigma}_i^d(u,s,V)\int_t^s \tilde{\sigma}_i^d(u,v,V)dvds,$$

and by using (2.32) we can expand further to obtain

$$\begin{aligned}\int_t^T \tilde{\sigma}_i^{d*}(u,s,V)ds &= \int_t^T \tilde{\sigma}_i^f(u,s,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvds + \int_t^T \tilde{\sigma}_i^f(u,s,V)\int_t^s \tilde{\sigma}_i^f(u,v,V)dvds\\ &+ \int_t^T \tilde{\sigma}_i^\lambda(u,s,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvds + \int_t^T \tilde{\sigma}_i^\lambda(u,s,V)\int_t^s \tilde{\sigma}_i^\lambda(u,v,V)dvds \qquad \text{(A.9.6)}\\ &+ \int_t^T \tilde{\sigma}_i^f(u,s,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvds + \int_t^T \tilde{\sigma}_i^f(u,s,V)\int_t^s \tilde{\sigma}_i^\lambda(u,v,V)dvds\\ &+ \int_t^T \tilde{\sigma}_i^\lambda(u,s,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvds + \int_t^T \tilde{\sigma}_i^\lambda(u,s,V)\int_t^s \tilde{\sigma}_i^f(u,v,V)dvds.\end{aligned}$$

From (2.30) and by using the volatility specifications of Assumption 2.3.1, we obtain

$$\begin{aligned}\int_t^T \tilde{\sigma}_i^d(u,s,V)ds &= \int_t^T \tilde{\sigma}_i^f(u,s,V)ds + \int_t^T \tilde{\sigma}_i^\lambda(u,s,V)ds\\ &= \beta_f(t,T)\tilde{\sigma}_i^f(u,t,V) + \beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,V), \qquad \text{(A.9.7)}\end{aligned}$$

where the deterministic functions $\beta_f(t,T)$ and $\beta_\lambda(t,T)$ defined by

$$\begin{cases} \beta_f(t,T) = \int_t^T e^{-\kappa_f(v-t)}dv, \\ \beta_\lambda(t,T) = \int_t^T e^{-\kappa_\lambda(v-t)}dv. \end{cases} \qquad \text{(A.9.8)}$$

Additionally, note that[45]

$$\begin{aligned}
\int_t^T \tilde{\sigma}_i^f(u,s,V)\int_t^s \tilde{\sigma}_i^f(u,v,V)dvds &= \frac{1}{2}[\beta_f(t,T)\tilde{\sigma}_i^f(u,t,V)]^2,\\
\int_t^T \tilde{\sigma}_i^\lambda(u,s,V)\int_t^s \tilde{\sigma}_i^\lambda(u,v,V)dvds &= \frac{1}{2}[\beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,V)]^2,\\
\int_t^T \tilde{\sigma}_i^f(u,s,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvds &= \Big(\frac{\beta_f(t,T)}{\kappa_\lambda}+\frac{1-e^{-(\kappa_f+\kappa_\lambda)(T-t)}}{\kappa_\lambda(\kappa_f+\kappa_\lambda)}\Big)\tilde{\sigma}_i^\lambda(u,t,V)\tilde{\sigma}_i^f(u,t,V),\\
\int_t^T \tilde{\sigma}_i^\lambda(u,s,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvds &= \Big(\frac{\beta_\lambda(t,T)}{\kappa_f}+\frac{1-e^{-(\kappa_\lambda+\kappa_f)(T-t)}}{\kappa_f(\kappa_f+\kappa_\lambda)}\Big)\tilde{\sigma}_i^\lambda(u,t,V)\tilde{\sigma}_i^f(u,t,V).
\end{aligned} \tag{A.9.9}$$

Thus from (A.9.7) and (A.9.9), the equation (A.9.6) becomes

$$\begin{aligned}
\int_t^T \tilde{\sigma}_i^{d*}(u,s,V)ds &= \beta_f(t,T)\tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dv + \frac{1}{2}[\beta_f(t,T)\tilde{\sigma}_i^f(u,t,V)]^2\\
&+ \beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dv + \frac{1}{2}[\beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,V)]^2\\
&+ \beta_f(t,T)\tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dv + \beta_\lambda(t,T)\tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dv\\
&+ \Big[\frac{\beta_f(t,T)}{\kappa_\lambda}+\frac{\beta_\lambda(t,T)}{\kappa_f}+\Big(\frac{1}{\kappa_f}+\frac{1}{\kappa_\lambda}\Big)\Big(\frac{1-e^{-(\kappa_f+\kappa_\lambda)(T-t)}}{\kappa_f+\kappa_\lambda}\Big)\Big]\tilde{\sigma}_i^f(u,t,V)\tilde{\sigma}_i^\lambda(u,t,V).
\end{aligned} \tag{A.9.10}$$

[45]Note that

$$\int_t^T e^{-\kappa_f(s-t)}\int_t^s e^{-\kappa_\lambda(v-t)}dvds = \frac{1}{\kappa_\lambda}\beta_f(t,T) + \frac{1}{\kappa_\lambda(\kappa_f+\kappa_\lambda)}\Big(1-e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big),$$

and

$$\int_t^T e^{-\kappa_\lambda(s-t)}\int_t^s e^{-\kappa_f(v-t)}dvds = \frac{1}{\kappa_f}\beta_\lambda(t,T) + \frac{1}{\kappa_f(\kappa_f+\kappa_\lambda)}\Big(1-e^{-(\kappa_\lambda+\kappa_f)(T-t)}\Big).$$

By substituting (A.9.7) and (A.9.10) into (A.9.5), it follows that

$$
\begin{aligned}
I = &\boxed{\beta_f(t,T)\Big(\int_0^t \tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu + \int_0^t \tilde{\sigma}_i^f(u,t,V)d\tilde{W}_i(u)\Big)} \\
&\boxed{\begin{aligned}&+\beta_\lambda(t,T)\Big(\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvdu + \int_0^t \tilde{\sigma}_i^f(u,t,V)\int_u^t \tilde{\sigma}_i^\lambda(u,v,V)dvdu \\
&\quad + \int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu + \int_0^t \tilde{\sigma}_i^\lambda(u,t,V)d\tilde{W}_i(u)\Big)\end{aligned}} \\
&- \beta_\lambda(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu + \beta_f(t,T)\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu \\
&+ \Big[\frac{1}{\kappa_\lambda}\beta_f(t,T) + \frac{1}{\kappa_f}\beta_\lambda(t,T) + \Big(\frac{1}{\kappa_f}+\frac{1}{\kappa_\lambda}\Big)\Big(\frac{1}{\kappa_f+\kappa_\lambda}\Big)\Big(1-e^{-(\kappa_f+\kappa_\lambda)(T-t)}\Big)\Big] \\
&\int_0^t \tilde{\sigma}_i^f(u,t,V)\tilde{\sigma}_i^\lambda(u,t,V)du + \frac{1}{2}\Big[\beta_f^2(t,T)\int_0^t \tilde{\sigma}_i^{2f}(u,t,V)du + \beta_\lambda^2(t,T)\int_0^t \tilde{\sigma}_i^{2\lambda}(u,t,V)du\Big],
\end{aligned}
\tag{A.9.11}
$$

which by employing Definition 2.9 and (2.52) to the expression in the first box and (2.53) to the expression in the second box of (A.9.11) allows it to be written as

$$
\begin{aligned}
I = &\beta_f(t,T)[r(t,V) - f(0,t,V_0)] + \beta_\lambda(t,T)[c(t,V) - \lambda(0,t,V_0)] \\
&+ [\beta_f(t,T) - \beta_\lambda(t,T)]\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu + \mathfrak{a}(t,T)\int_0^t \tilde{\sigma}_i^f(u,t,V)\tilde{\sigma}_i^\lambda(u,t,V)du \\
&+ \frac{1}{2}\Big[\beta_f^2(t,T)\int_0^t \tilde{\sigma}_i^{2f}(u,t,V)du + \beta_\lambda^2(t,T)\int_0^t \tilde{\sigma}_i^{2\lambda}(u,t,V)du\Big],
\end{aligned}
\tag{A.9.12}
$$

where

$$
\mathfrak{a}(t,T) = \frac{\beta_f(t,T)}{\kappa_\lambda} + \frac{\beta_\lambda(t,T)}{\kappa_f} + \Big(\frac{1}{\kappa_f}+\frac{1}{\kappa_\lambda}\Big)\frac{1-e^{-(\kappa_f+\kappa_\lambda)(T-t)}}{\kappa_f+\kappa_\lambda}.
$$

We can then write equation (A.9.3)

$$\bar{P}^d(t,T,V) = \frac{\bar{P}^d(0,T,V_0)}{\bar{P}^d(0,t,V_0)}\exp\Big[-\beta_f(t,T)[r(t,V)-f(0,t,V_0)] - \beta_\lambda(t,T)[c(t,V)-\lambda(0,t,V_0)]$$
$$-[\beta_f(t,T)-\beta_\lambda(t,T)]\sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^\lambda(u,t,V)\int_u^t \tilde{\sigma}_i^f(u,v,V)dvdu$$
$$-\mathfrak{a}(t,T)\sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^f(u,t,V)\tilde{\sigma}_i^\lambda(u,t,V)du$$
$$-\frac{1}{2}\Big[\beta_f^2(t,T)\sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{2f}(u,t,V)du + \beta_\lambda^2(t,T)\sum_{i=1}^{3}\int_0^t \tilde{\sigma}_i^{2\lambda}(u,t,V)du\Big]\Big]. \qquad \text{(A.9.13)}$$

Given (2.57a), (2.57b), (2.57c) and (2.50c) in Proposition 2.14, the equation for the pseudo bond above reduces to

$$\bar{P}^d(t,T,V) = \frac{\bar{P}^d(0,T,V_0)}{\bar{P}^d(0,t,V_0)}\exp\Big(-\frac{1}{2}\beta_f^2(t,T)\eta_1(t,V) - \frac{1}{2}\beta_\lambda^2(t,T)\eta_2(t,V) - \mathfrak{a}(t,T)\eta_3(t,V)$$
$$-\big[\beta_f(t,T)-\beta_\lambda(t,T)\big]S_3(t,V) - \beta_f(t,T)[r(t,V)-f(0,t,V_0)] - \beta_\lambda(t,T)[c(t,V)-\lambda(0,t,V_0)]\Big). \qquad \text{(A.9.14)}$$

By using the definition of the defaultable bond in terms of the pseudo bond given in Equation (2.14), the expression for the defaultable bond yields equation (2.62). ♦

A.10 Proof of Proposition 2.17

From the definition given in (2.14) we have that

$$f^d(t,T,V) = -\frac{\partial}{\partial T}\ln P^d(t,T,V), \quad \text{for all} \quad t\in[0,T]. \qquad \text{(A.10.1)}$$

On recalling (2.62) and integrating it with respect to T then taking the exponent, we see that (A.10.1) can be written as

$$P^d(t,T) = \exp\left[-\int_t^T f^d(0,s,V_0)ds - \zeta(t,T) - \frac{1}{2}\beta_f^2(t,T)\eta_1(t,V) - \frac{1}{2}\beta_\lambda^2(t,T)\eta_2(t,V)\right. \quad \text{(A.10.2)}$$
$$\left. -\mathfrak{a}(t,T)\eta_3(t,V) - \left[\beta_f(t,T)+\beta_\lambda(t,T)\right]S_3(t,V) - \beta_f(t,T)r(t,V) - \beta_\lambda(t,T)c(t,V)\right].$$

Since we have that (set $f^d(0,T,V_0) = f^d(0,T)$), then (A.10.2) becomes

$$f^d(t,T,V) - f^d(0,T) = -e^{-\kappa_f(T-t)}f(0,t) - e^{-\kappa_\lambda(T-t)} + e^{-\kappa_f(T-t)}\beta_f(t,T)\eta_1(t,V)\lambda(0,t)$$
$$+e^{-\kappa_\lambda(T-t)}\beta_\lambda(t,T)\eta_2(t,V) + [e^{-\kappa_f(T-t)}\beta_\lambda(t,T) + e^{-\kappa_\lambda(T-t)}\beta_f(t,T)]\eta_3(t,V)$$
$$+[e^{-\kappa_f(T-t)} + e^{-\kappa_\lambda(T-t)}]S_3(t,V) + e^{-\kappa_f(T-t)}r(t,V) + e^{-\kappa_\lambda(T-t)}c(t,V).$$

(A.10.3)

By using the deterministic functions $a_i(t,T)$ of Definition 2.16 and the term (2.64) that includes the information of the initial term structure of the forward rates and credit spread, equation (A.10.3) can be expressed as

$$f^d(t,T,V) - \widetilde{f}^d(0,t;0,T) = \mathcal{A}\mathcal{X}^\top \quad \text{(A.10.4)}$$

with the matrices $\mathcal{A} = [a_i(t,T)], i = 1,2,\ldots,6$, and $\mathcal{X} = [r(t,V)c(t,V)\eta_1(t,V)\eta_2(t,V)\eta_3(t,V)S_3(t,V)]$. Since $a_i(t,T)$ are deterministic functions, the value of $f^d(t,T,V)$ can be expressed as a linear combination of the six state variables $r(t,V)$, $c(t,V)$, $S_3(t,V)$ and $\eta_i(t,V)$ for $i = 1,2,3$. Then equation (A.10.4) can be used to express the state variables as a linear combination of a finite set of six forward rates.

By working under the parameterisation of the maturity variable introduced in Brace, Gatarek, and Musiela [1997], we let $T = t + \tau$ and denote $\Delta f_\tau^d(t,V) := f^d(t,t+\tau,V) - \widetilde{f}^d(0,t;0,t+\tau)$. Then by fixing six tenors $0 \leq \tau_1 \leq \ldots \leq \tau_6$ and setting $\tau = \tau_i$, for

$i = 1, 2, \ldots, 6$, Equation (A.10.4) yields the system

$$\mathcal{X}^\top = \begin{bmatrix} \Delta f^d_{\tau_1}(t,V) \\ \Delta f^d_{\tau_2}(t,V) \\ \vdots \\ \Delta f^d_{\tau_6}(t,V) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{16} \\ a_{21} & a_{22} & \ldots & a_{26} \\ \vdots & \vdots & \ddots & \\ a_{61} & a_{62} & \ldots & a_{66} \end{bmatrix}}_{A(t)} \begin{bmatrix} r(t,V) \\ c(t,V) \\ \eta_1(t,V) \\ \eta_2(t,V) \\ \eta_3(t,V) \\ S_3(t,V) \end{bmatrix}, \tag{A.10.5}$$

where $a_{jm} = a_m(t, T_j)$, see Definition 2.16. Assume that the determinant of the matrix $A(t)$ exists and is not equal to zero i.e, $detA(t) \neq 0$, then the system of equations (A.10.5) is invertible. If the matrix $A(t)$ is invertible, then the corresponding HJM model admits an affine Markovian realization in terms of the forward rates $f^d(t, t+\tau_i, V)$ for $i = 1, 2, ..., 6$. We can then write the state variables as linear combinations of forward rates $f^d_{\tau_1}(t,V)$, ..., $f^d_{\tau_6}(t,V)$ in the form,

$$\begin{bmatrix} r(t,V) \\ c(t,V) \\ \eta_1(t,V) \\ \eta_2(t,V) \\ \eta_3(t,V) \\ S_3(t,V) \end{bmatrix} = A(t)^{-1} \begin{bmatrix} \Delta f^d_{\tau_1}(t,V) \\ \Delta f^d_{\tau_2}(t,V) \\ \vdots \\ \Delta f^d_{\tau_6}(t,V) \end{bmatrix}. \tag{A.10.6}$$

By substituting the expressions (A.10.6) for the state variables into the equation (A.10.4) and collecting like terms the equation (2.63) is obtained. ♦

Appendix II

B.1 Proof of Proposition 3.2 on the Defaultable Forward Rate

We recall from (3.27) that under the risk-neutral measure, the dynamics of the default-free forward rate can be written as

$$f(t,T) = f(0,T) + \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^f(u,T,V_i) \int_u^T \tilde{\sigma}_{i1}^f(u,s,V_i) ds du + \sum_{i=1}^{n} \int_0^t \tilde{\sigma}_i^f(u,T,V_i) d\tilde{W}_i(u). \tag{B.1.1}$$

Using the volatility specifications (3.76), the stochastic integral equation (B.1.1) becomes

$$\begin{aligned} f(t,T) = f(0,T) &+ \sum_{i=1}^{n} \int_0^t r(u)V_i(u)[a_{0i} + a_{1i}(T-u)]e^{-\kappa_i^f(T-u)} \int_u^T [a_{0i} + a_{1i}(s-u)]e^{-\kappa_i^f(s-u)} ds du \\ &+ \sum_{i=1}^{n} \int_0^t \sqrt{r(u)V_i(u)}[a_{0i} + a_{1i}(T-u)]e^{-\kappa_i^f(T-u)} d\tilde{W}_i(u). \end{aligned} \tag{B.1.2}$$

By using the property $T-u=(T-t)+(t-u)$, and perform standard algebraic manipulations, equation (B.1.2) becomes

$$
\begin{aligned}
f(t,T) &= f(0,T) + \sum_{i=1}^{n} \frac{a_{1i}}{\kappa_i^f}\Big(\frac{1}{\kappa_i^f} + \frac{a_{0i}}{a_{1i}}\Big)[a_{0i} + a_{1i}(T-t)]e^{-\kappa_i^f(T-t)}\Phi_{2i}(t) \\
&+ \sum_{i=1}^{n}\Big[\frac{a_{1i}a_{1i}}{\kappa_i^f}\Big(\frac{1}{\kappa_i^f} + \frac{a_{0i}}{a_{1i}}\Big) + \frac{a_{1i}}{\kappa_i^f}\Big(\frac{a_{1i}}{\kappa_i^f} + 2a_{0i}\Big)(T-t) + \frac{a_{1i}}{\kappa_i^f}(T-t)^2\Big]e^{-\kappa_i^f(T-t)}\Phi_{3i}(t) \\
&+ \sum_{i=1}^{n}\frac{(a_{1i})^2}{\kappa_i^f}\Big(\frac{1}{\kappa_i^f} + \frac{a_{0i}}{a_{1i}}\Big)e^{-\kappa_i^f(T-t)}\Phi_{4i}(t) \\
&- \sum_{i=1}^{n}\frac{a_{1i}}{\kappa_i^f}\Big[\frac{a_{1i}}{\kappa_i^f} + 2a_{0i} + 2a_{1i}(T-t)\Big]e^{-2\kappa_i^f(T-t)}\Phi_{5i}(t) - \sum_{i=1}^{n}\frac{(a_{1i})^2}{\kappa_i^f}e^{-2\kappa_i^f(T-t)}\Phi_{6i}(t) \\
&+ \sum_{i=1}^{n} z_i^{f_1}[a_{0i} + a_{1i}(T-t)]e^{-\kappa_i^f(T-t)}x_{1i}(t) + \sum_{i=1}^{n} z_i^{f_1}a_{1i}e^{-\kappa_i^f(T-t)}\Phi_{1i}(t),
\end{aligned}
\tag{B.1.3}
$$

where

$$
\begin{cases}
x_{1i}(t) = \int_0^t \sqrt{r(u)V_i(u)}e^{-\kappa_i^f(t-u)}d\tilde{W}_i(u), \\
\Phi_{1i}(t) = \int_0^t \sqrt{r(u)V_i(u)}(t-u)e^{-\kappa_i^f(t-u)}d\tilde{W}_i(u), \\
\Phi_{2i}(t) = \int_0^t r(u)V_i(u)e^{-\kappa_i^f(t-u)}du, \\
\Phi_{3i}(t) = \int_0^t r(u)V_i(u)e^{-2\kappa_i^f(t-u)}du, \\
\Phi_{4i}(t) = \int_0^t r(u)V_i(u)(t-u)e^{-\kappa_i^f(t-u)}du, \\
\Phi_{5i}(t) = \int_0^t r(u)V_i(u)(t-u)e^{-2\kappa_i^f(t-u)}du, \\
\Phi_{6i}(t) = \int_0^t r(u)V_i(u)(t-u)^2e^{-2\kappa_i^f(t-u)}du,
\end{cases}
\tag{B.1.4}
$$

Thus, the forward rate process is Markovian and more specifically is affine in the state space variables since

$$
f(t,T) = f(0,T) + \sum_{i=1}^{n} B_{x_{1i}}(T-t)x_{1i}(t) + \sum_{i=1}^{n}\sum_{j=1}^{6} B_{\Phi_{ji}}(T-t)\Phi_{ji}(t), \tag{B.1.5}
$$

where $B_{x_{1i}}(T-t)$ and $B_{\Phi_{ji}}(T-t)$ are given in (3.42).

Similarly, we recall from (3.28) that the forward credit spread dynamics under the risk-neutral measure can be written as

$$\lambda(t,T) = \sum_{i=1}^{2n}\int_0^t \tilde{\sigma}_i^\lambda(u,T)\int_u^T \tilde{\sigma}_i^\lambda(u,s)dsdu + \sum_{i=1}^{2n}\int_0^t \Big(\tilde{\sigma}_i^\lambda(u,T)\int_u^T \tilde{\sigma}_i^f(u,s)ds + \tilde{\sigma}_i^f(u,T)\int_u^T \tilde{\sigma}_i^\lambda(u,s)ds\Big)du$$
$$+ \sum_{i=1}^{2n}\int_0^t \tilde{\sigma}_i^\lambda(u,T)d\tilde{W}_i(u). \tag{B.1.6}$$

Then by using the volatility specifications of Assumption 3.3.1, the drift components in the forward credit spread representation (B.1.6) are given by (recall that $z_i^{f_1} = 1$ so that $z_i^{\lambda_1} z_i^{f_1} = z_i^{\lambda_1}$)

$$\sum_{i=1}^{2n}\int_0^t \tilde{\sigma}_i^\lambda(u,T)\int_u^T \tilde{\sigma}_i^f(u,s)dsdu$$
$$= \sum_{i=1}^{n} z_i^{\lambda_1}\Big\{\Big[\frac{a_{0i}b_{0i}}{\kappa_i^f} + \frac{a_{1i}b_{0i}}{(\kappa_i^f)^2} + \Big(\frac{a_{0i}b_{1i}}{\kappa_i^f} + \frac{a_{1i}b_{1i}}{(\kappa_i^f)^2}\Big)(T-t)\Big]e^{-\kappa_i^\lambda(T-t)}\Phi_{7i}(t)$$
$$- \Big[\frac{a_{0i}b_{0i}}{\kappa_i^f} + \frac{a_{1i}b_{0i}}{(\kappa_i^f)^2} + \Big(\frac{a_{1i}b_{0i}}{\kappa_i^f} + \frac{a_{0i}b_{1i}}{\kappa_i^f} + \frac{a_{1i}b_{1i}}{(\kappa_i^f)^2}\Big)(T-t) + \frac{a_{1i}b_{1i}}{\kappa_i^f}(T-t)^2\Big]e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\Phi_{8i}(t)$$
$$+ \Big(\frac{a_{0i}b_{0i}}{\kappa_i^f} + \frac{a_{1i}b_{1i}}{(\kappa_i^f)^2}\Big)e^{-\kappa_i^\lambda(T-t)}\Phi_{9i}(t)$$
$$- \Big[\frac{a_{1i}b_{0i}}{\kappa_i^f} + \frac{a_{0i}b_{1i}}{\kappa_i^f} + \frac{a_{1i}b_{1i}}{(\kappa_i^f)^2} - 2\frac{a_{1i}b_{1i}}{\kappa_i^f}(T-t)\Big]e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\Phi_{10i}(t)$$
$$- \frac{a_{1i}b_{1i}}{\kappa_i^f}e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\Phi_{11i}(t)\Big)\Big\}, \tag{B.1.7}$$

and

$$\sum_{i=1}^{2n}\int_0^t \tilde{\sigma}_i^f(u,T)\int_u^T \tilde{\sigma}_i^\lambda(u,s)dsdu = \sum_{i=1}^{n} z_i^{\lambda_1}\Big\{$$
$$- \Big[\frac{a_{0i}b_{0i}}{\kappa_i^\lambda} + \frac{a_{0i}b_{1i}}{(\kappa_i^\lambda)^2} + \Big(\frac{a_{0i}b_{1i}}{\kappa_i^\lambda} + \frac{a_{1i}b_{0i}}{\kappa_i^\lambda} + \frac{a_{1i}b_{1i}}{(\kappa_i^\lambda)^2}\Big)(T-t) + \frac{a_{1i}b_{1i}}{\kappa_i^\lambda}(T-t)^2\Big]e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\Phi_{8i}(t)$$
$$- \Big[\frac{b_{1i}a_{0i}}{\kappa_i^\lambda} + \frac{b_{0i}a_{1i}}{\kappa_i^\lambda} + \frac{a_{1i}b_{1i}}{(\kappa_i^\lambda)^2} - 2\frac{a_{1i}b_{1i}}{\kappa_i^\lambda}(T-t)\Big]e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\Phi_{10i}(t)$$
$$- \frac{a_{1i}b_{1i}}{\kappa_i^\lambda}e^{-(\kappa_i^f+\kappa_i^\lambda)(T-t)}\Phi_{11i}(t) + \Big[\frac{a_{0i}b_{0i}}{\kappa_i^\lambda} + \frac{a_{0i}b_{1i}}{(\kappa_i^\lambda)^2} + \Big(\frac{a_{1i}b_{0i}}{\kappa_i^\lambda} + \frac{a_{1i}b_{1i}}{(\kappa_i^\lambda)^2}\Big)(T-t)\Big]e^{-\kappa_i^f(T-t)}\Phi_{12i}(t)$$
$$+ \Big(\frac{a_{0i}b_{0i}}{\kappa_i^\lambda} + \frac{a_{1i}b_{1i}}{(\kappa_i^\lambda)^2}\Big)e^{-\kappa_i^f(T-t)}\Phi_{13i}(t)\Big\}, \tag{B.1.8}$$

and finally

$$
\begin{aligned}
\sum_{i=1}^{2n}\int_0^t \tilde{\sigma}_i^{\lambda}(u,T)\int_u^T \tilde{\sigma}_i^{\lambda}(u,s)dsdu &= \sum_{i=1}^{n}\Big\{\frac{b_{1i}}{\kappa_i^{\lambda}}\Big(\frac{1}{\kappa_i^{\lambda}}+\frac{b_{0i}}{b_{1i}}\Big)[b_{0i}+b_{1i}(T-t)]e^{-\kappa_i^{\lambda}(T-t)}\Phi_{14i}(t) \\
&+\Big[\frac{b_{1i}b_{1i}}{\kappa_i^{\lambda}}\Big(\frac{1}{\kappa_i^{\lambda}}+\frac{b_{0i}}{b_{1i}}\Big)+\frac{b_{1i}}{\kappa_i^{\lambda}}\Big(\frac{b_{1i}}{\kappa_i^{\lambda}}+2b_{0i}\Big)(T-t)+\frac{b_{1i}}{\kappa_i^{\lambda}}(T-t)^2\Big]e^{-\kappa_i^{\lambda}(T-t)}\Phi_{15i}(t) \\
&+\frac{(b_{1i})^2}{\kappa_i^{\lambda}}\Big(\frac{1}{\kappa_i^{\lambda}}+\frac{b_{0i}}{b_{1i}}\Big)e^{-\kappa_i^{\lambda}(T-t)}\Phi_{16i}(t) \\
&-\frac{b_{1i}}{\kappa_i^{\lambda}}\Big[\frac{b_{1i}}{\kappa_i^{\lambda}}+2b_{0i}+2b_{1i}(T-t)\Big]e^{-2\kappa_i^{\lambda}(T-t)}\Phi_{17i}(t)-\frac{(b_{1i})^2}{\kappa_i^{\lambda}}e^{-2\kappa_i^{\lambda}(T-t)}\Phi_{18i}(t)\Big\},
\end{aligned}
\tag{B.1.9}
$$

where we introduce the additional state variables

$$
\begin{cases}
\Phi_{7i}(t)=\int_0^t \eta_i(u)e^{-\kappa_i^{\lambda}(t-u)}du, \quad \Phi_{8i}(t)=\int_0^t \eta_i(u)e^{-(\kappa_i^f+\kappa_i^{\lambda})(t-u)}du, \\
\Phi_{9i}(t)=\int_0^t \eta_i(u)(t-u)e^{-\kappa_i^{\lambda}(t-u)}du, \quad \Phi_{10i}(t)=\int_0^t \eta_i(u)(t-u)e^{-(\kappa_i^f+\kappa_i^{\lambda})(t-u)}du, \\
\Phi_{11i}(t)=\int_0^t \eta_i(u)(t-u)^2e^{-(\kappa_i^f+\kappa_i^{\lambda})(t-u)}du, \quad \Phi_{12i}(t)=\int_0^t \eta_i(u)e^{-\kappa_i^f(t-u)}du, \\
\Phi_{13i}(t)=\int_0^t \eta_i(u)(t-u)e^{-\kappa_i^f(t-u)}du, \quad \Phi_{14i}(t)=\int_0^t c(u)V_i(u)e^{-\kappa_i^{\lambda}(t-u)}du, \\
\Phi_{15i}(t)=\int_0^t c(u)V_i(u)e^{-2\kappa_i^{\lambda}(t-u)}du, \quad \Phi_{16i}(t)=\int_0^t c(u)V_i(u)(t-u)e^{-\kappa_i^{\lambda}(t-u)}du, \\
\Phi_{17i}(t)=\int_0^t c(u)V_i(u)(t-u)e^{-2\kappa_i^{\lambda}(t-u)}du, \quad \Phi_{18i}(t)=\int_0^t c(u)V_i(u)(t-u)^2e^{-2\kappa_i^{\lambda}(t-u)}du,
\end{cases}
\tag{B.1.10}
$$

where $\eta_i(t)=V_i(t)\sqrt{r(t)c(t)}$. We note in addition that,,

$$
\begin{aligned}
\sum_{i=1}^{2n}\int_0^t \tilde{\sigma}_i^{\lambda}(u,T)d\tilde{W}_i(u) &= \sum_{i=1}^{n} z_i^{\lambda_1}[b_{0i}+b_{1i}(T-t)]e^{-\kappa_i^{\lambda}(T-t)}x_{2i}(t) \\
&+\sum_{i=1}^{n} z_i^{\lambda_2}[b_{0i}+b_{1i}(T-t)]e^{-\kappa_i^{\lambda}(T-t)}x_{3i}(t)+\sum_{i=1}^{n} z_i^{\lambda_1}b_{1i}e^{-\kappa_i^{\lambda}(T-t)}\Phi_{19i}(t) \\
&+\sum_{i=1}^{n} z_i^{\lambda_2}b_{1i}e^{-\kappa_i^{\lambda}(T-t)}\Phi_{20i}(t),
\end{aligned}
\tag{B.1.11}
$$

where we introduce the state variables

$$\begin{cases} x_{2i}(t) = \int_0^t \sqrt{c(u)V_i(u)}e^{-\kappa_i^\lambda(t-u)}d\tilde{W}_i(u), \quad x_{3i}(t) = \int_0^t \sqrt{c(u)V_i(u)}e^{-\kappa_i^\lambda(t-u)}d\tilde{W}_{n+i}(u), \\ \Phi_{19i}(t) = \int_0^t c(u)V_i(u)(t-u)e^{-\kappa_i^\lambda(t-u)}d\tilde{W}_i(u), \quad \Phi_{20i}(t) = \int_0^t c(u)V_i(u)(t-u)e^{-\kappa_i^\lambda(t-u)}d\tilde{W}_{n+i}(u). \end{cases} \tag{B.1.12}$$

Consequently, the forward credit spread satisfies

$$\lambda(t,T) = \lambda(0,T) + \sum_{i=1}^{n}\sum_{j=2}^{3} B_{x_{ji}}(T-t)x_{ji}(t) + \sum_{i=1}^{n}\sum_{j=7}^{20} B_{\Phi_{ji}}(T-t)\Phi_{ji}(t), \tag{B.1.13}$$

where the coefficients $B_{x_{ji}}(T-t)$ and $B_{\Phi_{ji}}(T-t)$ are specified in (3.43) and (3.44). It follows from the definition $f^d(t,T) = f(t,T) + \lambda(t,T)$ and (B.1.5) and (B.1.13) that the defaultable forward rate admits finite dimensional affine realisation

$$f^d(t,T) = f^d(0,T) + \sum_{i=1}^{n}\sum_{j=1}^{3} B_{x_{ji}}(T-t)x_{ji}(t) + \sum_{i=1}^{n}\sum_{j=1}^{20} B_{\Phi_{ji}}(T-t)\Phi_{ji}(t). \tag{B.1.14}$$

Hence the proof of Proposition 3.2 is established. ♦

B.2 Proof of Proposition 3.5 for the Exponential Affine Bond Price formula

Straightforward application of (B.1.14) for the affine Markovian forward rate into the definition of the defaultable bond price formula, $\bar{P}^d(t,T) = \exp\Big(-\int_t^T f^d(t,s)ds\Big)$ yields

$$\bar{P}^d(t,T) = \frac{\bar{P}^d(0,T)}{\bar{P}^d(0,t)}\exp\Big(-\sum_{i=1}^{n}\sum_{j=1}^{3} x_{ji}(t)\int_t^T B_{x_{ji}}(s-t)ds - \sum_{i=1}^{n}\sum_{j=1}^{20}\Phi_{ji}(t)\int_t^T B_{\Phi_{ji}}(s-t)ds\Big). \tag{B.2.1}$$

We proceed to integrate the deterministic functions $B_{x_{ji}}$ and $B_{\Phi_{ji}}$ (See Proposition 3.2) in the exponent with respect to maturity. Substituting equations (3.50) - (3.53) into the general exponential defaultable bond price expression (B.2.1) yields (3.49) in Proposition 3.5.

Hence the proof of Proposition 3.5 is established. ♦

B.3 Some Important Results for Section 3.4

B.3.1 Pseudo-Bond Price Formula

Given a risk-neutral measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ and a $\mathcal{F}_t^W$−measurable short rate $r(t)$, we assume that there exists the non-negative process $\tilde{h}(t)$ defined in Remark 2.4 and Equation (2.40) such that

$$\int_0^t (r(s) + \tilde{h}(s))ds < \infty, \quad \text{for all} \quad t \in \mathbb{R}_+.$$

The conditional survival probability satisfies $\tilde{\mathbb{P}}\big(\tau > t \big| \mathcal{F}_t^W\big) = \tilde{\mathbb{E}}\big(e^{-\int_0^t \tilde{h}(s)ds} \big| \mathcal{F}_t^W\big), \quad t \geq 0$. As noted in Jamshidian [2004], "the conditional survival probability may then be interpreted as an agent's best probabilistic estimate as to whether the firm has survived up to time t or not, which agent observes everything in $\mathcal{F}_t^W$, but is somehow denied the exact information in $\mathcal{F}_t^N$ as to whether or not default has actually occurred by time t." It is also well known that the conditional default probability is uniformly distributed on $(0, 1)$ and independent of $\mathcal{F}_t^W$.

Then, the arbitrage price satisfies

$$\begin{aligned}
\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\mathbb{1}_{\{\tau>t_i\}}\middle|\mathcal{F}_t\right] &= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \tilde{h}(s)ds}\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\mathbb{1}_{\{\tau>t_i\}}\middle|\mathcal{F}_t^W\right] \\
&= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \tilde{h}(s)ds}\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\underbrace{\tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau>t_i\}}\middle|\mathcal{F}_{t_i}^W\right]}_{e^{-\int_0^{t_i}\tilde{h}(s)ds}}\middle|\mathcal{F}_t^W\right], \qquad \text{(B.3.1)} \\
&= \mathbb{1}_{\{\tau>t\}}\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i}(r(s)+\tilde{h}(s))ds}\middle|\mathcal{F}_t^W\right].
\end{aligned}$$

In this case, pricing a defaultable bond reduces to the case of pricing a non-defaultable (pseudo) bond with an 'adjusted' short rate $r(t)+\tilde{h}(t)=r^d(t)$. Equation (B.3.1) can then be written as

$$\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\mathbb{1}_{\{\tau>t_i\}}\middle|\mathcal{F}_t\right] = \mathbb{1}_{\{\tau>t\}}\bar{P}^d(t,t_i). \qquad \text{(B.3.2)}$$

♦

B.3.2 Simplifying Relations for Standard Running CDS

The *fair price* of a running CDS with whose compensation is made immediately on default is given by

$$\tilde{\pi}(t) = \frac{(1-\mathcal{R})\tilde{\mathbb{E}}\left[e^{-\int_t^{\tau} r(s)ds}\mathbb{1}_{\{t<\tau\le T\}}\middle|\mathcal{F}_t\right]}{\delta\sum_{i=1}^{N}\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\mathbb{1}_{\{\tau>t_i\}}\middle|\mathcal{F}_t\right]+\tilde{\mathbb{E}}\left[(\tau-t_{\tau-1})e^{-\int_t^{\tau} r(s)ds}\mathbb{1}_{\{t<\tau<T\}}\middle|\mathcal{F}_t\right]}. \qquad \text{(B.3.3)}$$

In addition, we note that $\mathbb{1}_{\{\tau>T\}}=\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau>T\}}$ and $\mathbb{1}_{\{t<\tau\le T\}}=\mathbb{1}_{\{\tau>t\}}-\mathbb{1}_{\{\tau>T\}}$. Following Filipovic [2009], we remark that every $\mathcal{F}_\infty^W$−measurable random variable satisfies $\mathbb{E}[X|\mathcal{F}_t]=$

$\mathbb{E}[X|\mathcal{F}_t^W]$ with $\tilde{\mathbb{P}}(\tau > t|\mathcal{F}_\infty^W) = \tilde{\mathbb{P}}(\tau > t|\mathcal{F}_t^W)$, $t \geq 0$. Then,

$$\begin{aligned}\tilde{\mathbb{P}}\big(t < \tau \leq u \big| \mathcal{F}_\infty^W \vee \mathcal{F}_t^N\big) &= \mathbb{1}_{\{\tau > t\}} e^{\int_0^t \tilde{h}(s)ds} \tilde{\mathbb{E}}\big[\mathbb{1}_{\{t<\tau\leq u\}} \big| \mathcal{F}_\infty^W\big] \\ &= \mathbb{1}_{\{\tau > t\}} e^{\int_0^t \tilde{h}(s)ds} \Big(e^{-\int_t^u \tilde{h}(s)ds} - e^{-\int_0^u \tilde{h}(s)ds}\Big) \\ &= \mathbb{1}_{\{\tau > t\}} \big(1 - e^{-\int_t^u \tilde{h}(s)ds}\big).\end{aligned} \tag{B.3.4}$$

This is the $\mathcal{F}_\infty^W \vee \mathcal{F}_t^N$ conditional distribution of $\tau|\tau > t$ and on differentiating (B.3.4) with respect to u yields

$$\mathbb{1}_{\{\tau > t\}} \tilde{h}(u) e^{-\int_t^u \tilde{h}(s)ds} \mathbb{1}_{\{\tau \leq u\}}.$$

It then follows that we can now simplify the expectation in the numerator

$$\begin{aligned}\tilde{\mathbb{E}}\Big[e^{-\int_t^\tau r(s)ds} \mathbb{1}_{\{t<\tau\leq T\}} \Big| \mathcal{F}_t\Big] &= \tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\Big[e^{-\int_t^\tau r(s)ds} \mathbb{1}_{\{t<\tau\leq T\}} \Big| \mathcal{F}_\infty^W \vee \mathcal{F}_t^N\Big] \Big| \mathcal{F}_t\Big] \\ &= \mathbb{1}_{\{\tau > t\}} \tilde{\mathbb{E}}\Big[\int_t^T \tilde{h}(u) e^{-\int_t^u r(s)ds} e^{-\int_t^u \tilde{h}(s)ds} du \Big| \mathcal{F}_t\Big] \\ &= \mathbb{1}_{\{\tau > t\}} \int_t^T \tilde{\mathbb{E}}\Big[\tilde{h}(u) e^{-\int_t^u (r(s)+\tilde{h}(s))ds} \Big| \mathcal{F}_t^W\Big] du.\end{aligned} \tag{B.3.5}$$

♦

B.3.3 Proof of Proposition 3.6 for Standard CDS

Given that the fair credit swap spread is the value of π that makes the value of the swap contract to be zero, from (3.55) and (3.54) it follows that

$$\tilde{\pi}(t) = \frac{(1-\mathcal{R})\tilde{\mathbb{E}}\Big[e^{-\int_t^\tau r(s)ds} \mathbb{1}_{\{t<\tau\leq T\}} \Big| \mathcal{F}_t\Big]}{\delta \sum_{i=1}^N \tilde{\mathbb{E}}\Big[e^{-\int_t^{t_i} r(s)ds} \mathbb{1}_{\{\tau > t_i\}} \Big| \mathcal{F}_t\Big] + \tilde{\mathbb{E}}\Big[(\tau - t_{\tau-1}) e^{-\int_t^\tau r(s)ds} \mathbb{1}_{\{t<\tau<T\}} \Big| \mathcal{F}_t\Big]}. \tag{B.3.6}$$

Substituting (B.3.2) in Appendix B.3.1 and (B.3.5) in Appendix B.3.2 into (B.3.6) we then have that

$$\sum_{i=1}^N \tilde{\mathbb{E}}\Big[e^{-\int_t^{t_i} r(s)ds} \mathbb{1}_{\{\tau > t_i\}} \Big| \mathcal{F}_t\Big] = \mathbb{1}_{\{\tau > t\}} \sum_{i=1}^N \bar{P}^d(t, t_i),$$

and

$$\tilde{\mathbb{E}}\left[e^{-\int_t^T r(s)ds}\mathbb{1}_{\{t<\tau\le T\}}\Big|\mathcal{F}_t\right] = \mathbb{1}_{\{\tau>t\}}\int_t^T \tilde{\mathbb{E}}\left[\tilde{h}(u)e^{-\int_t^u(r(s)+\tilde{h}(s))ds}\Big|\mathcal{F}_t^W\right]du.$$

from which (3.57) follows. Hence the proof is established. ♦

B.3.4 Simplifying Relations for Postponed Running CDS

We recall that the *fair postponed CDS price* is given by

$$\tilde{\pi}(t) = \frac{(1-\mathcal{R})\sum_{i=1}^N \tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\mathbb{1}_{\{t_{i-1}<\tau\le t_i\}}\Big|\mathcal{F}_t\right]}{\delta(t)\sum_{i=1}^N \tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\mathbb{1}_{\{\tau>t_i\}}\Big|\mathcal{F}_t\right]}, \tag{B.3.7}$$

The conditional expectation in the numerator can then be expressed as

$$\begin{aligned}
\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\mathbb{1}_{\{t_{i-1}<\tau<t_i\}}\Big|\mathcal{F}_t\right] &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\left(\mathbb{1}_{\{\tau<t_{i-1}\}}-\mathbb{1}_{\{\tau<t_i\}}\right)\Big|\mathcal{F}_{t_i}^W\vee\mathcal{F}_t^N\right]\Big|\mathcal{F}_t\right]\\
&= \mathbb{1}_{\tau>t}\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\left(e^{-\int_t^{t_{i-1}}\tilde{h}(s)ds}-e^{-\int_t^{t_i}\tilde{h}(s)ds}\right)\Big|\mathcal{F}_t^W\right]\\
&= \mathbb{1}_{\tau>t}\left\{\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}e^{-\int_t^{t_{i-1}}\tilde{h}(s)ds}\Big|\mathcal{F}_t^W\right]-\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}e^{-\int_t^{t_i}\tilde{h}(s)ds}\Big|\mathcal{F}_t^W\right]\right\}\\
&= \mathbb{1}_{\tau>t}\left\{\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}e^{-\int_t^{t_{i-1}}\tilde{h}(s)ds}\Big|\mathcal{F}_t^W\right]-\bar{P}^d(t,t_i)\right\}.
\end{aligned}$$

Following Brigo and Morini [2005], by using the approximation $e^{-\int_t^{t_i} r(s)ds}\approx e^{-\int_t^{t_{i-1}} r(s)ds}$, then we can write

$$\begin{cases}
\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}e^{-\int_t^{t_{i-1}}\tilde{h}(s)ds}\Big|\mathcal{F}_t^W\right] = \bar{P}^d(t,t_{i-1}),\\
\tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds}\mathbb{1}_{\{t_{i-1}<\tau<t_i\}}\Big|\mathcal{F}_t\right] = \bar{P}^d(t,t_{i-1})-\bar{P}^d(t,t_i).
\end{cases} \tag{B.3.8}$$

♦

B.3.5 Proof of Proposition 3.7 for Postponed Running CDS

In this case of postponed CDS, the value of premium leg will not have the accrual component and can be written as

$$\tilde{W}_{prm}(t) = \pi\delta \sum_{i=1}^{N} e^{-\int_t^{t_i} r(s)ds} \mathbb{1}_{\{\tau > t_i\}}. \tag{B.3.9}$$

The value the protection leg in this case is given by

$$\tilde{W}_{prt}(t) = (1-\mathcal{R}) \sum_{i=1}^{N} e^{-\int_t^{t_i} r(s)ds} \mathbb{1}_{\{t_{i-1} < \tau \le t_i\}}. \tag{B.3.10}$$

It then follows that the pre-default fair forward credit default swap spread of a postponed CDS at time t can be expressed as

$$\tilde{\pi}(t) = \frac{(1-\mathcal{R}) \sum_{i=1}^{N} \tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds} \mathbb{1}_{\{t_{i-1} < \tau \le t_i\}} \middle| \mathcal{F}_t \right]}{\delta \sum_{i=1}^{N} \tilde{\mathbb{E}}\left[e^{-\int_t^{t_i} r(s)ds} \mathbb{1}_{\{\tau > t_i\}} \middle| \mathcal{F}_t \right]}. \tag{B.3.11}$$

Using some results from Appendix B.3.1 and the approximation $e^{-\int_t^{t_i} r(s)ds} \approx e^{-\int_t^{t_{i-1}} r(s)ds}$ in Appendix B.3.4, (B.3.11) reduces to

$$\tilde{\pi}(t) \approx \frac{(1-\mathcal{R}) \sum_{i=1}^{N} \left[\bar{P}^d(t, t_{i-1}) - \bar{P}^d(t, t_i) \right]}{\delta \sum_{i=1}^{N} \bar{P}^d(t, t_i)}. \tag{B.3.12}$$

Hence the proof follows. ♦

B.3.6 Proof of Proposition 3.8 on Swaption Price

By observing from equations (B.3.9) and (B.3.10) that we could write the values of the premium leg and protection leg as

$$\begin{aligned}\bar{W}_{prm}(t_0) &= \delta \sum_{i=1}^{N} e^{-\int_{T_m}^{t_i} r(s)ds} \mathbb{1}_{\{\tau > t_i\}},\\ \tilde{W}_{prt}(T_m) &= (1-\mathcal{R}) \sum_{i=1}^{N} e^{-\int_{T_m}^{t_i} r(s)ds} \mathbb{1}_{\{t_{i-1} < \tau \le t_i\}},\end{aligned} \tag{B.3.13}$$

respectively. Hence, the price of the swaption in (3.65) can be written as

$$\begin{aligned}\mathcal{C}_{swpt}(t) = (1-\mathcal{R})\tilde{\mathbb{E}}\Big[e^{-\int_t^{T_m} r(s)ds}\mathbb{1}_{\{\tau>T_m\}}\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}\sum_{i=1}^{N}\underbrace{\tilde{\mathbb{E}}\Big[e^{-\int_{T_m}^{t_i} r(s)ds}\mathbb{1}_{\{t_{i-1}<\tau\le t_i\}}\Big|\mathcal{F}_{T_m}\Big]}_{use\ equation (B.3.8)}\Big|\mathcal{F}_t\Big]\\ -K\delta\tilde{\mathbb{E}}\Big[e^{-\int_t^{T_m} r(s)ds}\mathbb{1}_{\{\tau>T_m\}}\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}\sum_{i=1}^{N}\underbrace{\tilde{\mathbb{E}}\Big[e^{-\int_{T_m}^{t_i} r(s)ds}\mathbb{1}_{\{\tau>t_i\}}\Big|\mathcal{F}_{T_m}\Big]}_{use\ equation (B.3.1)}\Big|\mathcal{F}_t\Big].\end{aligned} \tag{B.3.14}$$

Equation (B.3.14) can be approximated by [46]

$$\begin{aligned}\mathcal{C}_{swpt}(t) \approx (1-\mathcal{R})\sum_{i=1}^{N}\tilde{\mathbb{E}}\Big[e^{-\int_t^{T_m} r(s)ds}\Big(\bar{P}^d(T_m,t_{i-1})-\bar{P}^d(T_m,t_i)\Big)\mathbb{1}_{\{\tau>T_m\}}\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}\Big|\mathcal{F}_t\Big]\\ -K\delta\sum_{i=1}^{N}\tilde{\mathbb{E}}\Big[e^{-\int_t^{T_m} r(s)ds}\bar{P}^d(T_m,t_i)\mathbb{1}_{\{\tau>T_m\}}\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}\Big|\mathcal{F}_t\Big].\end{aligned} \tag{B.3.15}$$

Following the argument used Appendix B.3.1, the default swaption pricing formula can then

[46] We use the approximation $e^{-\int_t^{t_i} r(s)ds} \approx e^{-\int_t^{t_{i-1}} r(s)ds}$ made in (B.3.8) in Appendix B.3.4.

be reduced to

$$\mathcal{C}_{swpt}(t) \approx (1-\mathcal{R})\mathbb{1}_{\{\tau>t\}} \sum_{i=1}^{N} \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\big(\bar{P}^d(T_m,t_{i-1}) - \bar{P}^d(T_m,t_i)\big)\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}\Big|\mathcal{F}_t^W\Big]$$
$$-K\delta\mathbb{1}_{\{\tau>t\}} \sum_{i=1}^{N} \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\bar{P}^d(T_m,t_i)\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}\Big|\mathcal{F}_t^W\Big]. \tag{B.3.16}$$

In addition, the second expectation in (B.3.16) can be written as

$$\begin{aligned}&\tilde{\mathbb{E}}\big[e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\bar{P}^d(T_m,t_i)\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}\Big|\mathcal{F}_t\big]\\ &= \tilde{\mathbb{E}}\big[e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\bar{P}^d(T_m,t_i)\Big|\mathcal{F}_t\big]\tilde{\mathbb{E}}\left[\frac{e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\bar{P}^d(T_m,t_i)\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}}{\tilde{\mathbb{E}}\big[e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\bar{P}^d(T_m,t_i)\Big|\mathcal{F}_t\big]}\Bigg|\mathcal{F}_t\right],\end{aligned} \tag{B.3.17}$$

and from the tower property of conditional expectation, it can be shown that

$$\tilde{\mathbb{E}}\big[e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\bar{P}^d(T_m,t_i)\Big|\mathcal{F}_t\big] = \bar{P}^d(t,t_i). \tag{B.3.18}$$

By defining the Radon-Nikodymn derivative

$$\frac{d\mathcal{Q}^{t_i}}{d\tilde{\mathbb{P}}} = \frac{e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\bar{P}^d(T_m,t_i)}{\bar{P}^d(t,t_i)}, \tag{B.3.19}$$

equation (B.3.17) can be then be reduced to

$$\tilde{\mathbb{E}}\big[e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\bar{P}^d(T_m,t_i)\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}\big|\mathcal{F}_t\big] = \bar{P}^d(t,t_i)\mathbb{E}_t^{\mathcal{Q}^{t_i}}\big[\mathbb{1}_{\{u(T_m)>\ln K\}}\big|\mathcal{F}_t\big], \tag{B.3.20}$$

where the expectation of the right hand side is taken under the $\mathcal{Q}^{t_i}$–forward measure with $u(T_m) = \ln\big(\tilde{\pi}_f(T_m)\big)$.

Similarly, we define another Radon-Nikodymn derivative

$$\frac{d\mathcal{Q}^{t_{i-1}}}{d\tilde{\mathbb{P}}} = \frac{e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\bar{P}^d(T_m, t_{i-1})}{\bar{P}^d(t, t_{i-1})}, \tag{B.3.21}$$

which together with the argument used in (B.3.18) shows that

$$\begin{aligned} &\tilde{\mathbb{E}}\big[e^{-\int_t^{T_m}[r(s)+\tilde{h}(s)]ds}\big(\bar{P}^d(T_m, t_{i-1}) - \bar{P}^d(T_m, t_i)\big)\mathbb{1}_{\{\tilde{\pi}_f(T_m)>K\}}\big|\mathcal{F}_t\big] \\ &\quad = \bar{P}^d(t, t_{i-1})\mathbb{E}^{\mathcal{Q}^{t_{i-1}}}\big[\mathbb{1}_{\{u(T_m)>\ln K\}}\big|\mathcal{F}_t\big] - \bar{P}^d(t, t_i)\mathbb{E}^{\mathcal{Q}^{t_i}}\big[\mathbb{1}_{\{u(T_m)>\ln K\}}\big|\mathcal{F}_t\big]. \end{aligned} \tag{B.3.22}$$

The credit default swaption formula in (B.3.16) can then be written as

$$\begin{aligned} \mathcal{C}_{swpt}(t) \approx\ & \mathbb{1}_{\{\tau>t\}}LGD\sum_{i=1}^{N}\bar{P}^d(t, t_{i-1})\mathbb{E}^{\mathcal{Q}^{t_{i-1}}}\big[\mathbb{1}_{\{u(T_m)>\ln K\}}\big|\mathcal{F}_t\big] \\ & -\mathbb{1}_{\{\tau>t\}}\sum_{i=1}^{N}\big(\delta K + LGD\big)\bar{P}^d(t, t_i)\mathbb{E}^{\mathcal{Q}^{t_i}}\big[\mathbb{1}_{\{u(T_m)>\ln K\}}\big|\mathcal{F}_t\big], \end{aligned} \tag{B.3.23}$$

or equivalently,

$$\begin{aligned} \mathcal{C}_{swpt}(t) \approx\ & \mathbb{1}_{\{\tau>t\}}LGD\sum_{i=1}^{N}\bar{P}^d(t, t_{i-1})Pr_t^{t_{i-1}}(u(T_m) > \ln K) \\ & -\mathbb{1}_{\{\tau>t\}}\sum_{i=1}^{N}\bar{P}^d(t, t_i)\big(\delta K + LGD\big)Pr_t^{t_i}(u(T_m) > \ln K). \end{aligned} \tag{B.3.24}$$

$Pr_t^S(u(T_m) > \ln K)$ is the conditional probability of the event $\{u(T_m) > \ln K\}$ based on the $S-$forward measure $\mathcal{Q}^S$ induced on $\tilde{\mathbb{P}}$ by the price of the zero recovery, zero-coupon bond issued at time t and $LGD = (1 - \mathcal{R})$. ♦

B.4 Proof of Result (3.74) on the Price of a Put Option on a Defaultable Bond

The proof of this result follows from Schönbucher [2000], lemma 3.5. Let $\gamma(t,T_0) = \exp\big(-\int_t^{T_0} r(u)du\big)$ be the discount factor using the default free rate over the period $[t,T_0]$. At any time $t < T_0$, the price of a put option on a defaultable bond can be expressed as the sum of the payoffs with and without defaults

$$\mathcal{P}(t,r^d,T;T_0,K) = \tilde{\mathbb{E}}\Big[\gamma(t,T_0)\big(K - P^d(T_0,T)\big)^+\Big|\mathcal{F}_t\Big] \tag{B.4.1}$$

$$= \tilde{\mathbb{E}}\Big[\mathbb{1}_{\{\tau>T_0\}}\gamma(t,T_0)\big(K - \bar{P}^d(T_0,T)\big)^+\Big|\mathcal{F}_t\Big] + \tilde{\mathbb{E}}\Big[\mathbb{1}_{\{\tau\le T_0\}}\gamma(t,T_0)\big(K - R(T_0)\bar{P}^d(T_0,T)\big)\Big|\mathcal{F}_t\Big]. \tag{B.4.2}$$

Substituting $\mathbb{1}_{\{\tau\le T_0\}} = 1 - \mathbb{1}_{\{\tau>T_0\}}$ into (B.4.2), we then have that

$$\mathcal{P}(t,r^d,T;T_0,K) = \tilde{\mathbb{E}}\Big[\mathbb{1}_{\{\tau>T_0\}}\gamma(t,T_0)\big(K - \bar{P}^d(T_0,T)\big)^+\Big|\mathcal{F}_t\Big]$$
$$+\tilde{\mathbb{E}}\Big[\gamma(t,T_0)\big(K - \mathcal{R}(T_0)\bar{P}^d(T_0,T)\big)\Big|\mathcal{F}_t\Big] - \tilde{\mathbb{E}}\Big[\mathbb{1}_{\{\tau>T_0\}}\gamma(t,T_0)\big(K - \mathcal{R}(T_0)\bar{P}^d(T_0,T)\big)\Big|\mathcal{F}_t\Big]. \tag{B.4.3}$$

Given that there are no defaults until T_0, then $\mathcal{R}(T_0) = 1$ thus $\mathbb{1}_{\{\tau>T_0\}}\mathcal{R}(T_0) = \mathbb{1}_{\{\tau>T_0\}}$ and (B.4.3) reduces to

$$\mathcal{P}(t,r^d,T;T_0,K) = \tilde{\mathbb{E}}\Big[\mathbb{1}_{\tau>T_0}\gamma(t,T_0)\big(K - \bar{P}^d(T_0,T)\big)^+\Big|\mathcal{F}_t\Big] + KP(t,T_0) - P^d(t,T)$$
$$- \tilde{\mathbb{E}}\Big[\mathbb{1}_{\{\tau>T_0\}}\gamma(t,T_0)\big(K - \bar{P}^d(T_0,T)\big)\Big|\mathcal{F}_t\Big]. \tag{B.4.4}$$

By noting that the counting process is $\mathcal{F}_t$−measurable the above equation can then be written as

$$\mathcal{P}(t,r^d,T;T_0,K) = \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r^d(u)du}\big(K - \bar{P}^d(T_0,T)\big)^+\Big|\mathcal{F}_t\Big] + KP(t,T_0) - P^d(t,T)$$
$$- \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r^d(u)du}\big(K - \bar{P}^d(T_0,T)\big)\Big|\mathcal{F}_t\Big]. \tag{B.4.5}$$

Hence the proof follows. ♦

B.5 Proof of Proposition 3.9.

We recall from (3.34) that the defaultable bond price under the risk-neutral measure satisfies the stochastic differential equation

$$\frac{dP^d(t,T)}{P^d(t-,T)} = r(t)dt + \sum_{i=1}^{2n} \tilde{\sigma}^d_{B,i}(t,T)d\tilde{W}_i(t) - d\tilde{M}(\omega,t), \tag{B.5.1}$$

where the volatility function is given by

$$\tilde{\sigma}^d_{B,i}(t,T) = -\int_t^T \tilde{\sigma}^d_i(t,s)ds = -\Big(\int_t^T \tilde{\sigma}^d_i(t,s)ds + -\int_t^T \tilde{\sigma}^\lambda_i(t,s)ds\Big), \tag{B.5.2}$$

and the process $\tilde{M}(\omega,t)$ is a martingale.

From Assumption 3.5.1, it follows that

$$\begin{cases} \int_t^T \tilde{\sigma}^f_i(t,s)ds = \sqrt{V_i(t)}\int_t^T [a_{0i} + a_{1i}(s-t)]e^{-\kappa^f_i(s-t)}ds, & \text{for } i = 1,\dots,n, \\ \int_t^T \tilde{\sigma}^\lambda_i(t,s)ds = z_i^{\lambda_1}\sqrt{V_i(t)}\int_t^T [b_{0i} + b_{1i}(s-t)]e^{-\kappa^\lambda_i(s-t)}ds, & \text{for } i = 1,\dots,n, \\ \int_t^T \tilde{\sigma}^\lambda_i(t,s)ds = z_i^{\lambda_2}\sqrt{V_i(t)}\int_t^T [b_{0i} + b_{1i}(s-t)]e^{-\kappa^\lambda_i(s-t)}ds, & \text{for } i = n+1,\dots,2n \end{cases} \tag{B.5.3}$$

which on subsequent integration and simplification yields

$$\begin{cases} \int_t^T \tilde{\sigma}^f_i(t,s)ds = \sqrt{V_i(t)}\bar{\beta}_{1i}(t,T), & \text{for } i = 1,\dots,n, \\ \int_t^T \tilde{\sigma}^\lambda_i(t,s)ds = \sqrt{V_i(t)}\bar{\beta}_{2i}(t,T) & \text{for } i = 1,\dots,n, \\ \int_t^T \tilde{\sigma}^\lambda_i(t,s)ds = \sqrt{V_i(t)}\bar{\beta}_{3i}(t,T) & \text{for } i = n+1,\dots,2n. \end{cases} \tag{B.5.4}$$

Substituting the equations in (B.5.4) into (B.5.1) yields the result. ♦

B.6 Proof of Proposition 3.11 on the Integral Transform.

Using the expectations[47]

$$\tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r^d(u)du}\Big|\mathcal{F}_t\Big] = \bar{P}^d(t,T_0) \quad \text{and} \quad \tilde{\mathbb{E}}_t\Big[e^{-\int_t^{T_0} r^d(u)du+\bar{X}^d(T_0,T)}\Big|\mathcal{F}_t\Big],$$

(3.93) can be written as

$$\begin{cases} \mathcal{P}(t,r^d,T;T_0,K) = K\bar{P}^d(t,T_0)\tilde{\mathbb{E}}_t\left[\dfrac{e^{-\int_t^{T_0} r^d(u)du}\mathbb{1}_{\{\bar{X}^d(T_0,T)<\xi\}}}{\bar{P}^d(t,T_0)}\middle|\mathcal{F}_t\right] \\ -\tilde{\mathbb{E}}_t\Big[e^{-\int_t^{T_0} r^d(u)du+\bar{X}^d(T_0,T)}\Big|\mathcal{F}_t\Big]\tilde{\mathbb{E}}_t\left[\dfrac{e^{-\int_t^{T_0} r^d(u)du+\bar{X}^d(T_0,T)}\mathbb{1}_{\{\bar{X}^d(T_0,T)<\xi\}}}{\tilde{\mathbb{E}}_t\Big[e^{-\int_t^{T_0} r^d(u)du+\bar{X}^d(T_0,T)}\Big]}\middle|\mathcal{F}_t\right]. \end{cases} \tag{B.6.1}$$

This can further be simplified to yield

$$\mathcal{P}(t,r^d,T;T_0,K) = K\bar{P}^d(t,T_0)F_1(t,T_0,T) - G(t,T_0,T)F_2(t,T_0,T), \tag{B.6.2}$$

where

$$G(t,T_0,T) = \tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r^d(u)du+\bar{X}^d(T_0,T)}\Big|\mathcal{F}_t\Big], \tag{B.6.3}$$

represents the time t price of discounted forward pseudo bond price $\bar{P}^d(T_0,T)$ whereas

$$\begin{cases} F_1(t,T_0,T) = \tilde{\mathbb{E}}\left[\dfrac{e^{-\int_t^{T_0} r^d(u)du}\mathbb{1}_{\{\bar{X}^d(T_0,T)<\xi\}}}{\bar{P}^d(t,T_0)}\middle|\mathcal{F}_t\right], \\ F_2(t,T_0,T) = \tilde{\mathbb{E}}\left[\dfrac{e^{-\int_t^{T_0} r^d(u)du+\bar{X}^d(T_0,T)}\mathbb{1}_{\{\bar{X}^d(T_0,T)<\xi\}}}{\tilde{\mathbb{E}}\Big[e^{-\int_t^{T_0} r^d(u)du+\bar{X}^d(T_0,T)}\Big|\mathcal{F}_t\Big]}\middle|\mathcal{F}_t\right], \end{cases} \tag{B.6.4}$$

[47]This approach was followed in Bouziane [2008] and Bakshi and Madan [2000] from where we borrow the notation.

are some contingent claims, the Arrow-Debreu securities which can be interpreted as probabilities since $F_j(t,T_0,T) \in (0,1)$ for $j = 1,2$. [48]

If we choose a Radon-Nikodymn derivative $\frac{d\mathbb{P}^1}{d\tilde{\mathbb{P}}} = \frac{\exp\Big(-\int_t^{T_0} r^d(u)du\Big)}{\bar{P}^d(t,T_0)}$, then $F_1(t,T_0,T)$ is the price of an Arrow-Debreu security under a transformed equivalent measure and satisfies $F_1(t,T_0,T) = \mathbb{E}^{\mathbb{P}^1}\left[\mathbb{1}_\chi \middle| \mathcal{F}_t\right]$. Similarly, we can define another Radon-Nikodymn derivative

$$\frac{d\mathbb{P}^2}{d\tilde{\mathbb{P}}} = \frac{\exp\Big(-\int_t^{T_0} r^d(u)du + \bar{X}^d(T_0,T)\Big)}{G(t,T_0,T)},$$

such that the second Arrow-Debreu security has a price $F_2(t,T_0,T) = \mathbb{E}_t^{\mathbb{P}_2}\left[\mathbb{1}_\chi\right]$, under another transformed measure $\mathbb{P}_2$.

The price of a European put option that is knocked out on default of the underlying defaultable bond can then be expressed as

$$\mathcal{P}(t,r^d,T;T_0,K) = K\bar{P}^d(t,T_0)\mathbb{E}^{\mathbb{P}_1}\left[\mathbb{1}_{\{\bar{X}^d(T_0,T)<\xi\}}\middle|\mathcal{F}_t\right] - G(t,T_0,T)\mathbb{E}^{\mathbb{P}_2}\left[\mathbb{1}_{\{\bar{X}^d(T_0,T)<\xi\}}\middle|\mathcal{F}_t\right]. \tag{B.6.9}$$

[48] Equivalently, we can express (B.6.4) in integral form as

$$F_1(t,T_0,T) \equiv \frac{\int_\chi \exp\Big(-\int_t^{T_0} r^d(u)du\Big)\vartheta(v)dv}{\int_\Omega \exp\Big(-\int_t^{T_0} r^d(u)du\Big)\vartheta(v)dv}, \tag{B.6.5}$$

$$F_2(t,T_0,T) \equiv \frac{\int_\chi \exp\Big(-\int_t^{T_0} r^d(u)du + \bar{X}^d(T_0,T)\Big)\vartheta(v)dv}{\int_\Omega \exp\Big(-\int_t^{T_0} r^d(u)du + \bar{X}^d(T_0,T)\Big)\vartheta(v)dv}, \tag{B.6.6}$$

whereas the discounted forward price in (B.6.3) and pseudo bond price can also be written as

$$G(t,T_0,T) = \int_\chi \exp\Big(-\int_t^{T_0} r^d(u)du + \bar{X}^d(T_0,T)\Big)\vartheta(v)dv, \tag{B.6.7}$$

$$\bar{P}^d(t,T_0) = \int_\Omega \exp\Big(-\int_t^{T_0} r^d(u)du\Big)\vartheta(v)dv, \tag{B.6.8}$$

respectively, where the set $\chi \equiv \{\bar{X}^d(T_0,T) < \xi\}$ and Ω represents the set $\{\bar{X}^d(T_0,T) > 0\}$.

Equivalently, this can be written as

$$\mathcal{P}(t, r^d, T; T_0, K) = K\bar{P}^d(t, T_0)\big[1 - \Pi_1(\xi)\big] - G(t, T_0, T)\big[1 - \Pi_2(\xi)\big] \tag{B.6.10}$$

where the exercise probabilities Π_1 and Π_2 are given by

$$\Pi_1(\xi) = \mathbb{E}^{\mathbb{P}_1}\big[\mathbb{1}_{\{\bar{X}^d(T_0,T)>\xi\}}\big|\mathcal{F}_t\big] \quad \text{and} \quad \Pi_2(\xi) = \mathbb{E}^{\mathbb{P}_2}\big[\mathbb{1}_{\{\bar{X}^d(T_0,T)>\xi\}}\big|\mathcal{F}_t\big],$$

respectively. We therefore observe that knowing the prices of four 'primitive' securities; the matching discount bond, the scaled forward price and the two Arrow-Debreu securities is equivalent to solving the option valuation problem.

The characteristic function for the exercise probability $\Pi_j(\xi, \phi)$, $j = 1, 2$ can be represented by $f_j(\xi) = \mathbb{E}_t^{\mathbb{P}_j}\big(e^{i\phi\bar{X}^d(T_0,T)}\big)$. In particular, we observe that

$$\begin{aligned} f_1(\xi, \phi) &= \mathbb{E}^{\mathbb{P}_1}\big[e^{i\phi\bar{X}^d(T_0,T)}\big|\mathcal{F}_t\big] = \tilde{\mathbb{E}}\Big[\frac{e^{-\int_t^{T_0} r^d(u)du}}{\bar{P}^d(t, T_0)}e^{i\phi\bar{X}^d(T_0,T)}\Big|\mathcal{F}_t\Big] \\ &= \bar{P}^d(t, T_0)^{-1}\tilde{\mathbb{E}}\big[e^{-\int_t^{T_0} r^d(u)du + i\phi\bar{X}^d(T_0,T)}\big|\mathcal{F}_t\big]. \end{aligned} \tag{B.6.11}$$

Similarly, the characteristic function of the 2^{nd} exercise probability can be expressed as

$$f_2(\xi, \phi) = G(t, T_0, T)^{-1}\tilde{\mathbb{E}}\big[e^{-\int_t^{T_0} r^d(u)du + (1+i\phi)\bar{X}^d(T_0,T)}\big|\mathcal{F}_t\big]. \tag{B.6.12}$$

It can then be shown using Gil-Pelaez Inversion Theorem[49] that the time t price of each Arrow-Debreu security is given by

$$\Pi_j(\xi) = \frac{1}{2} + \frac{1}{\pi}\int_{0+}^{\infty} Re\Big[\frac{e^{-i\phi\xi}f_j(\xi), \phi}{i\phi}\Big]d\phi, \tag{B.6.13}$$

for $j = 1, 2$. Hence the proof of Proposition 3.11 follows. ♦

[49] See Bouziane [2008, Theorem 4.2.2].

B.7 Proof of Proposition 3.12

Application of Itô's theorem to (3.101) results in

$$\frac{d\Upsilon_t(z)}{\Upsilon_t(z)} = \left[\partial_t A(t,z) + \sum_{i=1}^{n} \partial_t C_i(t,z) V_i(t)\right] dt + z d\hat{X}(t) + \frac{1}{2} z\bar{z} d[\hat{X}]_t + \sum_{i=1}^{n} C_i(t,z) dV_i(t)$$
$$+ \frac{1}{2}(C_i(t,z))^2 d[V_i]_t + z \sum_{i=1}^{n} C_i(t,z) d[\hat{X}, V_i]_t, \tag{B.7.1}$$

given that $[\hat{X}]$ is the quadratic variation of $\hat{X}(t,T_0,T)$, $\bar{z}$ is the conjugate of z and $\partial_t A(t,z)$, $\partial_t C_i(t,z)$ are the derivatives with respect to t. Under the T_0-measure, the log-bond price process can be expressed as

$$d\hat{X}(t,T_0,T) = -\frac{1}{2}\sum_{i=1}^{2n}\Big(d_i(t,T) - d_i(t,T_0)\Big)^2 V_i(t)dt - \sum_{i=1}^{2n}\Big(d_i(t,T) - d_i(t,T_0)\Big)\sqrt{V_i(t)}d\tilde{W}_i^{T_0}(t), \tag{B.7.2}$$

while stochastic volatility process follows the dynamics

$$dV_i(t) = \kappa_i^{\mathbb{V}}(\bar{V}_i - V_i)dt + \sum_{j=1}^{3}\bar{\sigma}_{ij}^{\mathbb{V}}\sqrt{V_i(t)}d\tilde{W}_{(j-1)n+i}^{T_0}(t). \tag{B.7.3}$$

Substituting (B.7.2) and (B.7.3) back into (B.7.1) yields

$$\begin{aligned}\frac{d\Upsilon_t(z)}{\Upsilon_t(z)} = &\Big[\partial_t A(t,z) + \sum_{i=1}^{2n} C_i(t,z)\kappa_i^{\mathbb{V}}\bar{V}_i + \Big(\partial_t C_i(t,z) - \frac{z}{2}\sum_{i=1}^{2n}\left(d_i(t,T) - d_i(t,T_0)\right)^2 \\ &+\frac{z\bar{z}}{2}\sum_{i=1}^{2n}\left(d_i(t,T) - d_i(t,T_0)\right)^2 - \sum_{i=1}^{2n} C_i(t,z)\kappa_i^{\mathbb{V}} + \frac{1}{2}\sum_{i=1}^{2n}\sum_{j=1}^{3}\left(C_i(t,z)\bar{\sigma}_{ij}^{\mathbb{V}}\right)^2 \\ &-z\sum_{i=1}^{2n}\sum_{j=1}^{3} C_i(t,z)\left(d_i(t,T) - d_i(t,T_0)\right)\bar{\sigma}_{ij}^{\mathbb{V}}\Big)V_i(t)\Big]dt \\ &+\sum_{i=1}^{2n}\left(d_i(t,T) - d_i(t,T_0)\right)\sqrt{V_i(t)}d\tilde{W}_i^{T_0}(t) + \sum_{j=1}^{3}\bar{\sigma}_i^{\mathbb{V}}\sqrt{V_i(t)}d\tilde{W}_{(j-1)n+i}^{T_0}(t).\end{aligned} \tag{B.7.4}$$

The stochastic process $d\Upsilon_t(z)$ is driftless and a martingale if the deterministic functions $A(t,z)$ and $C_i(t,z)$ solve the system of ordinary differential equations

$$\begin{cases} \dfrac{dC_i(t,z)}{dt} = -\dfrac{1}{2}\displaystyle\sum_{j=1}^{3}(\bar{\sigma}_{ij}^{\mathbb{V}})^2 C_i^2(t,z) + \left(\kappa_i^{\mathbb{V}} + z\sum_{j=1}^{3}(d_i(t,T) - d_i(t,T_0))\bar{\sigma}_{ij}^{\mathbb{V}}\right)C_i(t,z) \\ \qquad + \dfrac{1}{2}\big(z - z\bar{z}\big)\big(d_i(t,T) - d_i(t,T_0)\big)^2, \\ \dfrac{dA(t,z)}{dt} = -\displaystyle\sum_{i=1}^{2n} C_i(t,z)\kappa_i^{\mathbb{V}}\bar{V}_i, \end{cases} \tag{B.7.5}$$

subject to the boundary conditions $\bar{P}^d(T_0,T_0) = 1$ and $A(T_0,z) = C_i(T_0,z) = 0$. By change of variable technique with $\bar{t} = T_0 - t$, this system of differential equations can be written as

$$\begin{cases} \dfrac{dC_i(\bar{t},z)}{d\bar{t}} = \dfrac{1}{2}(\bar{\sigma}_i^{\mathbb{V}})^2 C_i^2(\bar{t},z) - \left(\kappa_i^{\mathbb{V}} + z\displaystyle\sum_{j=1}^{3}\bar{d}_i(\bar{t})\bar{\sigma}_{ij}^{\mathbb{V}}\right)C_i(\bar{t},z) - \dfrac{1}{2}\big(z - z\bar{z}\big)\bar{d}_i(\bar{t})^2, \\ \dfrac{dA(\bar{t},z)}{d\bar{t}} = \displaystyle\sum_{i=1}^{2n} C_i(\bar{t},z)\kappa_i^{\mathbb{V}}\bar{V}_i, \end{cases} \tag{B.7.6}$$

subject to the initial conditions $A(0,z) = C_i(0,z) = 1$ with $\bar{d}_i(\bar{t}) = d_i(T - T_0 + \bar{t}) - d_i(\bar{t})$.

Hence the proof follows. ♦

Appendix III

C.1 Proof of Proposition 4.4

For notational simplicity, we suppress the dependency of the forward rate, short rate and bond price dynamics on the Markov chain $X(t)$.

Starting from the stochastic integral equation (4.36), we integrate the terms in the drift to obtain

$$\begin{aligned}
f^d(t,T) &= f^d(\tau_k,T) - \frac{\sigma^f(X_{\tau_k})^2}{2\kappa_f(X_{\tau_k})^2}\Big(\big(1-e^{-\kappa_f(X_{\tau_k})(T-t)}\big)^2 - \big(1-e^{-\kappa_f(X_{\tau_k})(T-\tau_k)}\big)^2\Big) \\
&- \frac{\sigma^\lambda(X_{\tau_k})^2}{2\kappa_\lambda(X_{\tau_k})^2}\Big(\big(1-e^{-\kappa_\lambda(X_{\tau_k})(T-t)}\big)^2 - \big(1-e^{-\kappa_\lambda(X_{\tau_k})(T-\tau_k)}\big)^2\Big) \\
&- \frac{\sigma^f(X_{\tau_k})\sigma^\lambda(X_{\tau_k})}{\kappa_f(X_{\tau_k})\kappa_\lambda(X_{\tau_k})}\sqrt{1-\rho^2}\Big[e^{-\kappa_f(X_{\tau_k})(T-t)}\Big(1+\frac{\kappa_f(X_{\tau_k})}{\kappa_f(X_{\tau_k})+\kappa_\lambda(X_{\tau_k})}e^{-\kappa_\lambda(X_{\tau_k})(T-t)}\Big) \\
&- e^{-\kappa_\lambda(X_{\tau_k})(T-\tau_k)}\Big(1+\frac{\kappa_f(X_{\tau_k})}{\kappa_\lambda(X_{\tau_k})+\kappa_f(X_{\tau_k})}e^{-\kappa_f(X_{\tau_k})(T-\tau_k)}\Big) \\
&+ e^{-\kappa_\lambda(X_{\tau_k})(T-t)}\Big(1+\frac{\kappa_\lambda(X_{\tau_k})}{\kappa_f(X_{\tau_k})+\kappa_\lambda(X_{\tau_k})}e^{-\kappa_f(X_{\tau_k})(T-t)}\Big) \\
&- e^{-\kappa_f(X_{\tau_k})(T-\tau_k)}\Big(1+\frac{\kappa_\lambda(X_{\tau_k})}{\kappa_\lambda(X_{\tau_k})+\kappa_f(X_{\tau_k})}e^{-\kappa_\lambda(X_{\tau_k})(T-\tau_k)}\Big)\Big] \\
&+ \int_{\tau_k}^t \Big(\sigma^f(X_{\tau_k})e^{-\kappa_f(X_{\tau_k})(T-u)} + \rho\sigma^\lambda(X_{\tau_k})e^{-\kappa_\lambda(X_{\tau_k})(T-u)}\Big)d\tilde{W}_1(u) \\
&+ \sqrt{1-\rho^2}\int_{\tau_k}^t \sigma^\lambda(X_{\tau_k})e^{-\kappa_\lambda(X_{\tau_k})(T-u)}d\tilde{W}_2(u). \qquad \text{(C.1.1)}
\end{aligned}$$

Setting $T=t$ in (C.1.1), the defaultable short rate dynamics between any two jump times

$t \in [\tau_k, \tau_{k+1}[$ can be written as

$$r^d(t) = \mu^d(t,x) + \int_{\tau_k}^t \Big(\sigma^f(X_{\tau_k})e^{-\kappa_f(X_{\tau_k})(t-u)} + \rho\sigma^\lambda(X_{\tau_k})e^{-\kappa_\lambda(X_{\tau_k})(t-u)}\Big)d\tilde{W}_1(u)$$
$$+ \sqrt{1-\rho^2}\int_{\tau_k}^t \sigma^\lambda(X_{\tau_k})e^{-\kappa_\lambda(X_{\tau_k})(t-u)}d\tilde{W}_2(u), \tag{C.1.2}$$

where

$$\mu^d(t,x) = f^d(\tau_k,t) + \frac{\sigma^f(X_{\tau_k})^2}{2\kappa_f(X_{\tau_k})^2}\big(1-e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\big)^2 + \frac{\sigma^\lambda(X_{\tau_k})^2}{2\kappa_\lambda(X_{\tau_k})^2}\big(1-e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\big)^2$$
$$- \frac{\sigma^f(X_{\tau_k})\sigma^\lambda(X_{\tau_k})}{\kappa_f(X_{\tau_k})\kappa_\lambda(X_{\tau_k})}\sqrt{1-\rho^2}\Big[3 - e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\Big(1+\frac{\kappa_f(X_{\tau_k})}{\kappa_\lambda(X_{\tau_k})+\kappa_f(X_{\tau_k})}e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\Big)$$
$$- e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\Big(1+\frac{\kappa_\lambda(X_{\tau_k})}{\kappa_\lambda(X_{\tau_k})+\kappa_f(X_{\tau_k})}e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\Big)\Big].$$

Taking differentials yields the stochastic differential equation

$$dr^d(t) = \Big[\frac{\partial\mu^d(t,x)}{\partial t} - \int_{\tau_k}^t \sigma^f(X_{\tau_k})\kappa_f(X_{\tau_k})e^{-\kappa_f(X_{\tau_k})(t-u)}d\tilde{W}_1(u)$$
$$-\rho\int_{\tau_k}^t \sigma^\lambda(X_{\tau_k})\kappa_\lambda(X_{\tau_k})e^{-\kappa_\lambda(X_{\tau_k})(t-u)}d\tilde{W}_1(u) - \sqrt{1-\rho^2}\int_{\tau_k}^t \sigma^\lambda(X_{\tau_k})\kappa_\lambda(X_{\tau_k})e^{-\kappa_\lambda(X_{\tau_k})(t-u)}d\tilde{W}_2(u)\Big]dt$$
$$+ \big(\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big)d\tilde{W}_1(t) + \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k})d\tilde{W}_2(t). \tag{C.1.3}$$

We define the coefficients μ_f and μ_λ as

$$\mu_f(t,x) = \frac{\sigma^f(X_{\tau_k})^2}{2\kappa_f(X_{\tau_k})^2}\big(1-e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\big)^2,$$
$$\mu_\lambda(t,x) = \frac{\sigma^\lambda(X_{\tau_k})^2}{2\kappa_\lambda(X_{\tau_k})^2}\big(1-e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\big)^2 - \frac{\sigma^f(X_{\tau_k})\sigma^\lambda(X_{\tau_k})}{\kappa_f(X_{\tau_k})\kappa_\lambda(X_{\tau_k})}$$
$$\times\sqrt{1-\rho^2}\Big[2 - e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\Big(1+\frac{\kappa_f(X_{\tau_k})}{\kappa_\lambda(X_{\tau_k})+\kappa_f(X_{\tau_k})}e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\Big)$$
$$- e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\Big(1+\frac{\kappa_\lambda(X_{\tau_k})}{\kappa_\lambda(X_{\tau_k})+\kappa_f(X_{\tau_k})}e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\Big)\Big],$$

so that the short rate and short-term credit spread processes follow the stochastic integral equations

$$r(t) = \mu_f(t,x)dt + \int_{\tau_k}^{t} \sigma^f(X_{\tau_k})e^{-\kappa_f(X_{\tau_k})(t-s)}d\tilde{W}_1(s),$$
$$\lambda(t) = \mu_\lambda(t,x)dt + \rho\sigma^\lambda(X_{\tau_k})\int_{\tau_k}^{t} e^{-\kappa_\lambda(X_{\tau_k})(t-s)}d\tilde{W}_1(s) + \sqrt{1-\rho^2}\int_{\tau_k}^{t} \sigma^\lambda(X_{\tau_k})e^{-\kappa_\lambda(X_{\tau_k})(t-s)}d\tilde{W}_2(s),$$

respectively. Given that $\mu^d(t,x) = \mu^f(t,x) + \mu^\lambda(t,x)$, then the partial derivative of $\mu^d(t,x)$ with respect to t can be explicitly given by

$$\begin{aligned}\frac{\partial \mu^d(t,x)}{\partial t} &= f_2^d(\tau_k,t) - \frac{\sigma^f(X_{\tau_k})^2}{\kappa_f(X_{\tau_k})}\left(1-e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\right) - \frac{\sigma^\lambda(X_{\tau_k})^2}{\kappa_\lambda(X_{\tau_k})}\left(1-e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\right)\\ &- \frac{\sigma^f(X_{\tau_k})\sigma^\lambda(X_{\tau_k})}{\kappa_f(X_{\tau_k})\kappa_\lambda(X_{\tau_k})}\sqrt{1-\rho^2}\Big[\kappa_f(X_{\tau_k})e^{-\kappa_f(X_{\tau_k})(t-\tau_k)} + \kappa_\lambda(X_{\tau_k})e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\\ &+ \left(\kappa_f(X_{\tau_k}) + \kappa_\lambda(X_{\tau_k})\right)e^{-\left(\kappa_f(X_{\tau_k})+\kappa_\lambda(X_{\tau_k})\right)(t-\tau_t)}\Big].\end{aligned}$$

It then follows that the SDE for the defaultable short rate process between two general jump times $t \in [\tau_k, \tau_{k+1}[$ of the chain can be written as

$$dr^d(t) = \Big[\theta(t,x) - \kappa_f(X_{\tau_k})r(t) - \kappa_\lambda(X_{\tau_k})\lambda(t)\Big]dt + \sum_{i=1}^{2}\sigma_i^d(X_{\tau_k})d\tilde{W}_i(t), \tag{C.1.5}$$

where the coefficients in the drift are given by

$$\theta(t,x) = \frac{\partial \mu^d(t,x)}{\partial t} + \kappa_f(X_{\tau_k})\mu_f(t,x) + \kappa_\lambda(X_{\tau_k})\mu_\lambda(t,x),$$

$$\sigma_1^d(X_{\tau_k}) = \sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k}) \quad \text{and} \quad \sigma_2^d(X_{\tau_k}) = \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k}).$$

Rearranging the terms in the drift of (C.1.5) yields the system of stochastic differential equations in Proposition 4.4. ♦

C.2 Proof of Proposition 4.5 on the two-factor Hull-White type model

We recall from (4.43a) that the default-free short rate dynamics between two jump times $t \in [\tau_k, \tau_{k+1}[$ follow

$$r(t) = f(\tau_k, t) + S_1(t,x) + \varphi_1(t,x). \tag{C.2.1}$$

Given that

$$d\varphi_1(t,x) = -\kappa_f(X_{\tau_k})\varphi_1(t,x)dt + \sigma^f(X_{\tau_k})d\tilde{W}_1(t),$$

it follows that stochastic differential equation for the short rate process satisfies

$$dr(t) = \big[f_2(\tau_k, t) + \frac{\partial}{\partial t}S_1(t,x) - \kappa_f(X_{\tau_k})\varphi_1(t,x)\big]dt + \sigma^f(X_{\tau_k})d\tilde{W}_1(t). \tag{C.2.2}$$

Substituting $\varphi_1(t,x) = r(t) - f(\tau_k,t) - S_1(t,x)$ into (C.2.2) then yields

$$dr(t) = \Big[\Theta_f(t,x) - \kappa_f(X_{\tau_k})r(t)\Big]dt + \sigma^f(X_{\tau_k})d\tilde{W}_1(t), \tag{C.2.3}$$

where

$$\Theta_f(t,x) = f_2(\tau_k,t) + \kappa_f(X_{\tau_k})f(\tau_k,t) + \kappa_f(X_{\tau_k})S_1(t,x) + \frac{\partial}{\partial t}S_1(t,x).$$

Similarly, taking the differentials of the $S_i(t,x)$, $(i = 1,2,3,4)$ in (4.44) we observe that

$$\begin{aligned}
dS_1(t,x) &= \frac{\sigma^f(X_{\tau_k})^2}{\kappa_f(X_{\tau_k})}e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\Big(e^{-\kappa_f(X_{\tau_k})(t-\tau_k)} - 1\Big)dt,\\
dS_2(t,x) &= \frac{\sigma^\lambda(X_{\tau_k})^2}{\kappa_\lambda(X_{\tau_k})}e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\Big(e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)} - 1\Big)dt,\\
dS_3(t,x) &= \sqrt{1-\rho^2}\frac{\sigma^f(X_{\tau_k})\sigma^\lambda(X_{\tau_k})}{\kappa_\lambda(X_{\tau_k})}e^{-\kappa_f(X_{\tau_k})(t-\tau_k)}\Big(e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)} - 1\Big)dt,\\
dS_4(t,x) &= \sqrt{1-\rho^2}\frac{\sigma^f(X_{\tau_k})\sigma^\lambda(X_{\tau_k})}{\kappa_{f(X_{\tau_k})}}e^{-\kappa_\lambda(X_{\tau_k})(t-\tau_k)}\Big(e^{-\kappa_f(X_{\tau_k})(t-\tau_k)} - 1\Big)dt.
\end{aligned}$$

In addition, the differentials for the $\varphi_i(t,x)$, $(i=2,3)$ satisfy

$$d\varphi_2(t,x) = -\kappa_\lambda(X_{\tau_k})\varphi_2(t,x)dt + \rho\sigma^\lambda(X_{\tau_k})d\tilde{W}_2(t), \tag{C.2.4}$$

$$d\varphi_3(t,x) = -\kappa_\lambda(X_{\tau_k})\varphi_3(t,x)dt + \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k})d\tilde{W}_3(t). \tag{C.2.5}$$

Then the stochastic differential equation for the defaultable short rate dynamics in (4.43b) satisfies

$$\begin{aligned} dr^d(t) = &\Big[f_2(\tau_k,t) + \frac{\partial}{\partial t}\sum_{i=1}^{4} S_i(t,x) - \kappa_f(X_{\tau_k})\varphi_1(t,x) - \kappa_\lambda(X_{\tau_k})\big(\varphi_2(t,x)+\varphi_3(t,x)\big)\Big]dt \\ &+ \big[\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big]d\tilde{W}_1(t) + \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k})d\tilde{W}_2(t). \end{aligned} \tag{C.2.6}$$

But from (4.43a), we observe that

$$\varphi_2(t,x) + \varphi_3(t,x) = r^d(t) - f^d(\tau_k,t) - \sum_{i=1}^{4} S_i(t,x) - \varphi_1(t,x). \tag{C.2.7}$$

Substituting (4.43a) and (C.2.7) into (C.2.6) then yields

$$\begin{aligned} dr^d(t) = &\Big[f_2(\tau_k,t) + \frac{\partial}{\partial t}\sum_{i=1}^{4} S_i(t,x) - \kappa_f(X_{\tau_k})\varphi_1(t,x) - \kappa_\lambda(X_{\tau_k})[r^d(t) - f^d(\tau_k,t) - \sum_{i=1}^{4} S_i(t,x) \\ &- \varphi_1(t,x)]\Big]dt + \big(\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big)d\tilde{W}_1(t) + \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k})d\tilde{W}_2(t), \end{aligned}$$

which on further simplification yields

$$\begin{aligned} dr^d(t) = &\Big[f_2(\tau_k,t) + \frac{\partial}{\partial t}\sum_{i=1}^{4} S_i(t,x) + \kappa_\lambda(X_{\tau_k})\sum_{i=1}^{4} S_i(t,x) + \kappa_\lambda(X_{\tau_k})f(\tau_k,t) \\ &- \big(\kappa_f(X_{\tau_k}) - \kappa_\lambda(X_{\tau_k})\big)\varphi_1(t,x) - \kappa_\lambda(X_{\tau_k})r^d(t)\Big]dt + \big[\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big]d\tilde{W}_1(t) \\ &+ \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k})d\tilde{W}_2(t). \end{aligned} \tag{C.2.8}$$

If we define the coefficients

$$\Theta_d(t,x) = f_2^d(\tau_k,t) + \kappa_\lambda(X_{\tau_k})f^d(\tau_k,t) + \frac{\partial}{\partial t}\sum_{i=1}^{4} S_i(t,x) + \kappa_\lambda(X_{\tau_k})\sum_{i=1}^{4} S_i(t,x),$$

and

$$g(t,x) = (\kappa_f(X_{\tau_k}) - \kappa_\lambda(X_{\tau_k}))\varphi_1(t,x),$$

then (C.2.8) reduces to a system of two-factor mean-reverting defaultable short rate model

$$\begin{aligned} dr^d(t) &= \Big[\Theta_d(t,x) - g(t,x) - \kappa_\lambda(X_{\tau_k})r^d(t)\Big]dt + \big(\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big)d\tilde{W}_1(t) \\ &\quad + \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k})d\tilde{W}_2(t), \qquad \text{(C.2.9)} \\ dg(t,x) &= -\kappa_f(X_{\tau_k})g(t,x)dt + \big[\kappa_f(X_{\tau_k}) - \kappa_\lambda(X_{\tau_k})\big]\sigma^f(X_{\tau_k})d\tilde{W}_1(t). \qquad \text{(C.2.10)} \end{aligned}$$

It turns out that

$$g(t,x) = (\kappa_f(X_{\tau_k}) - \kappa_\lambda(X_{\tau_k}))\big(r(t) - f(\tau_k,t) - S_1(t,x)\big).$$

It then follows that

$$\begin{aligned} dr^d(t) &= \Big[\Theta_d(t,x) + \kappa^d(X_{\tau_k})\big(f(\tau_k,t) + S_1(t,x)\big) - \kappa^d(X_{\tau_k})r(t) - \kappa_\lambda(X_{\tau_k})r^d(t)\Big]dt \\ &\quad + \big(\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big)d\tilde{W}_1(t) + \sqrt{1-\rho^2}\sigma^\lambda(X_{\tau_k})d\tilde{W}_2(t), \qquad \text{(C.2.11)} \end{aligned}$$

where we recall that $\kappa_d(X_{\tau_k}) = (\kappa_f(X_{\tau_k}) - \kappa_\lambda(X_{\tau_k}))$. Rearranging the coefficients in the drift of equation (C.2.11)

$$\bar{\Theta}_d(t,x) = \frac{\Theta_d(t,x) + \kappa_d(X_{\tau_k})\big(f(\tau_k,t) + S_1(t,x)\big)}{\kappa_\lambda(X_{\tau_k})}$$

and in (C.2.3)

$$\bar{\kappa}_d(X_{\tau_k}) = \frac{\kappa_d(X_{\tau_k})}{\kappa_\lambda(X_{\tau_k})}, \quad \bar{\Theta}_f(t,x) = \frac{\Theta_f(t,x)}{\kappa_f(X_{\tau_k})},$$

then yields the result in Proposition 4.5. ♦

C.3 Proof of Proposition 4.7

For notational simplicity, we let $\bar{P} = \bar{P}^d$. Then from the partial differential equations in (4.53) we observe the linearity of the coefficients $\frac{\partial \bar{P}}{\partial r}$ and $\frac{\partial \bar{P}}{\partial r^d}$ in terms of r and r^d. From (4.49), it can be seen that

$$\frac{\partial \bar{P}}{\partial t} = \left[\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r - \frac{\partial C}{\partial t} r^d\right]\bar{P}, \quad \frac{\partial \bar{P}}{\partial r} = -B\bar{P}, \quad \frac{\partial^2 \bar{P}}{(\partial r)^2} := B^2 \bar{P},$$

$$\frac{\partial \bar{P}}{\partial r^d} = -C\bar{P}, \quad \frac{\partial^2 \bar{P}}{(\partial r^d)^2} = C^2 \bar{P} \quad \text{and} \quad \frac{\partial^2 \bar{P}}{\partial r \partial r^d} = BC\bar{P}.$$

Then, we can write (4.53) as

$$\begin{aligned}\Big[\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r - \frac{\partial C}{\partial t} r^d - \kappa_f(X_{\tau_k})\big(\bar{\theta}_f(X_{\tau_k}) - r\big)B - \kappa_\lambda(X_{\tau_k})\big(\bar{\theta}_d(X_{\tau_k}) - \bar{\kappa}_d(X_{\tau_k})r - r^d\big)C \\ + \frac{1}{2}\sigma^f(X_{\tau_k})^2 B^2 + \frac{1}{2}\Big(\big(\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big)^2 + (1-\rho^2)\sigma^\lambda(X_{\tau_k})^2\Big)C^2 \\ + \sigma^f(X_{\tau_k})\big(\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big)BC\Big]\bar{P} + < \bar{P}, \tilde{H}X(t) > - r^d \bar{P} = 0. \qquad \text{(C.3.1)}\end{aligned}$$

Matching the coefficients of r^d, r and the constant term we obtain

$$\begin{aligned}&\Big[-\frac{\partial C}{\partial t} + \kappa_\lambda(X_{\tau_k})C - 1\Big] r^d \bar{P} \\ &+ \Big[-\frac{\partial B}{\partial t} + \kappa_f(X_{\tau_k})B + \kappa_\lambda(X_{\tau_k})\bar{\kappa}_d(X_{\tau_k})C\Big] r\bar{P} \\ &+ \Big(\frac{\partial A}{\partial t} - \kappa_f(X_{\tau_k})\bar{\Theta}_f(t,x)B - \kappa_\lambda(X_{\tau_k})\bar{\Theta}_d(t,x)C \\ &+ \frac{1}{2}\sigma^f(X_{\tau_k})^2 B^2 + \sigma^f(X_{\tau_k})\big(\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big)BC \\ &+ \frac{1}{2}\big[\sigma^f(X_{\tau_k})^2 + 2\rho\sigma^f(X_{\tau_k})\sigma^\lambda(X_{\tau_k}) + \sigma^\lambda(X_{\tau_k})^2\big]\Big)\bar{P} - < \bar{P}, \tilde{H}X(t) > = 0. \qquad \text{(C.3.2)}\end{aligned}$$

We note that the last term in equation (C.3.2) is shown to reduce to

$$< \bar{P}, \tilde{H}X(t) > = \sum_{i,j=1}^{N} e^{A_j - Br - Cr^d} h_{ij}^X, \tag{C.3.3}$$

$$= e^{-Br-Cr^d} \sum_{i,j=1}^{N} e^{A_j} h_{ij}^X = e^{A_i - Br - Cr^d} \sum_{i,j=1}^{N} e^{A_j - A_i} h_{ij}^X. \tag{C.3.4}$$

That is, for a fixed i

$$< \bar{P}, \tilde{H}X(t) >= \bar{P} \sum_{j=1}^{N} e^{A_j - A_i} h_{ij}^X. \tag{C.3.5}$$

Substituting (C.3.5) into (C.3.2) and solving, the latter reduces to

$$\begin{aligned}
&\Big[-\frac{\partial C}{\partial t} + \kappa_\lambda(X_{\tau_k})C - 1 \Big] r^d \\
&+ \Big[-\frac{\partial B}{\partial t} + \kappa_f(X_{\tau_k})B + \kappa_\lambda(X_{\tau_k})\bar{\kappa}_d(X_{\tau_k})C \Big] r \\
&+ \Big[\frac{\partial A}{\partial t} - \kappa_f(X_{\tau_k})\bar{\Theta}_f(t,x)B - \kappa_\lambda(X_{\tau_k})\bar{\Theta}_d(t,x)C \\
&+ \frac{1}{2}\sigma^f(X_{\tau_k})^2 B^2 + \sigma^f(X_{\tau_k})\big(\sigma^f(X_{\tau_k}) + \rho\sigma^\lambda(X_{\tau_k})\big)BC \\
&+ \frac{1}{2}\big[\sigma^f(X_{\tau_k})^2 + 2\rho\sigma^f(X_{\tau_k})\sigma^\lambda(X_{\tau_k}) + \sigma^\lambda(X_{\tau_k})^2\big] - \sum_{i,j=1}^{N} e^{A_j - A_i} h_{ij}^X \Big] = 0.
\end{aligned} \tag{C.3.6}$$

If we define the states that our Markov chains visits by $S = \{s_1, s_2, ...s_N\}$, the Markov chain dependent parameters switch between a set of N values such that $\kappa_f(X_{\tau_k}) = \{\kappa_f^1, \kappa_f^2, ..., \kappa_f^N\}$, $\kappa_\lambda(X_{\tau_k}) = \{\kappa_\lambda^1, \kappa_\lambda^2, ..., \kappa_\lambda^N\}$, $\sigma^f(X_{\tau_k}) = \{\sigma_1^f, \sigma_2^f, ..., \sigma_N^f\}$ and $\sigma^\lambda(X_{\tau_k}) = \{\sigma_1^\lambda, \sigma_2^\lambda, ..., \sigma_N^\lambda\}$. Substituting these values into the partial differential equation (C.3.2) and separating the common terms yields the system of ordinary differential equations in Proposition 4.7. Hence the proof follows. ♦

Bibliography

K. Amin and R. Morton. Implied Volatility Functions in Heath-Jarrow-Morton Models. *Journal of Financial Economics*, 35(2):141–180, 1995.

A. Andersson and P. Vanini. Credit Migration Risk modelling. *Journal of Credit Risk*, 6(1):3–30, 2010.

A. Ang and G. Bekaert. Regime Switches in Interest Rates. *Journal of Business and Economic Statistics*, 20(2):163–182, 2002.

P. Artzner and F. Delbaen. Default Risk Insurance and Incomplete Markets. *Mathematical Finance*, 5:187–195, 1995.

G. Bakshi and D. Madan. Spanning and Derivative Security Valuation. *Journal of Financial Economics*, 55(2):205–238, 2000.

C. A. Ball and W. N. Torous. The Stochastic Volatility of Short-term Interest Rates: Some International Evidence. *Journal of Finance*, 54(6):2339–2359, 1999.

R. Bansal and H. Zhou. Term structure of interest rates with regime shifts. *Journal of Finance*, 57 (5):1997–2043, 2002.

A. Berndt, P. Ritchken, and Z. Sun. On Correlation and Default Clustering in Credit Markets. *Review of Financial Studies*, 23(7):2680–2729, 2010.

R. Bhar and C. Chiarella. Transformation of Heath-Jarrow-Morton Models to Markovian Systems. *European Journal of Finance*, 3:1–26, 1997.

T.R. Bielecki and M. Rutkowski. Multiple Ratings model of Defaultable Term Structure. *Mathematical Finance*, 10:125–139, 2000b.

T.R. Bielecki and M. Rutkowski. *Credit Risk: Modeling, Valuation and Hedging.* Springer, 2002.

T.R. Bielecki and M. Rutkowski. Modelling of Defaultable Term Structure: Conditionally Markov Approach. *IEEE Transactions on Automatic Control*, 49:361–373, 2004.

T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. Replication of Defaultable Claims within the Reduced-Form framework. *Paris-Princeton Lectures on Mathematical Finance, Springer*, 2004.

T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. Hedging of Basket Credit Derivatives in Credit Swap Market. *Journal of Credit Risk*, 3(1):91–132, 2007.

T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. Hedging of Credit Default Swaptions in a Hazard Process Model. *Working Paper*, 2008.

T. Björk and A. Gombani. Minimal Realisations for Forward Rates. *Finance and Stochastics*, 3 (4):413–432, 1999.

T. Björk and C. Landèn. On the Construction of Finite Dimensional Realisations for Nonlinear Forward Rate Models. *Finance and Stochastics*, 6(3):303–329, 2002.

T. Björk and L. Svensson. On the Existence of Finite Dimensional Realisations for Nonlinear Forward Rate Models. *Mathematical Finance*, 11(2):205–243, 2001.

T. Björk, Y. Kabanov, and W. Runggaldier. Bond Market Structure in the Presence of Marked Point Processes. *Mathematical Finance*, 7(2):211–239, 1997.

T. Björk, C. Landén, and L. Svensson. Finite-dimensional Markovian realizations for stochastic volatility forward-rate models. *Proceedings of Royal Society London*, 460,(2041):53–83, 2004.

F. Black and J. Cox. Valuing Corporate Securities: Some Effects of Bond Indenture Provisions. *Journal of Finance*, pages 351–367, 1976.

F. Black and M. Scholes. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81:637–54, 1973.

M. Bouziane. *Pricing Interest-Rate Derivatives: A Fourier-Transform Based Approach.* Lecture Notes in Economics and Mathematical Systems, Springer, 2008.

A. Brace, D. Gatarek, and M. Musiela. The Market Model of Interest Rate Dynamics. *Mathematical Finance*, 7:127–155, 1997.

M. Brennan and E. Schwartz. Analysing Convertible Bonds. *Journal of Financial and Quantitative Analysis*, 15:907–929, 1980.

D. Brigo and M. Morini. CDS Market Formulas and Models. *Invited Presentation at XVIII Warwick Option Conference*, 2005.

British-Bankers-Association. Credit Derivatives Report 2006. *London: BBA Publications*, 2006.

E. Briys and F. de Varenne. Valuing Risky Fixed Rate Debt: An Extension. *Journal of Financial and Quantitative Analysis*, 32(2):239–248, 1997.

J. Casassus, P. Collin-Dufresne, and R. Goldstein. Unspanned Stochastic Volatility and Fixed Income Derivatives Pricing. *Journal of Banking and Finance*, 29:2723–2749, 2005.

L. Cathcart and G. El-Jahel. Valuation of defaultable bonds. *Journal of Fixed Income*, 8(1):65–78, 1998.

K. C. Chan, G. A. Karolyi, F. A. Longstaff, and A. B. Sanders. An Empirical Comparison of Alternative Models of the Short-Term Interest Rate. *Journal of Finance*, 47:1209–1227, 1992.

L. Chen and D. Filipovic. Credit Derivatives in an Affine Framework. *Asia-Pacific Financial Markets*, 14:123–140, 2007.

C. Chiarella and O. Kwon. A Complete Markovian Stochastic Volatility Model in the HJM Framework. *Asia Pacific Financial Markets*, 7(4):293–304, 2000a.

C. Chiarella and O. Kwon. A Class of Heath-Jarrow-Morton Term Structure Models with Stochastic Volatility. *QFRC Working Paper No. 34, School of Finance and Economics, University of Technology, Sydney*, 2000b.

C. Chiarella and O. Kwon. Classes of Interest Rate Models under the HJM Framework. *Asia-Pacific Financial Markets*, 8(1):1–22, 2001.

C. Chiarella and O. Kwon. Finite Dimensional Affine Realisations of HJM Models in Terms of Forward Rates and Yields. *Review of Derivatives Research*, 6(2):129–155, 2003.

C. Chiarella and C. Nikitopoulos-Sklibosios. A Class of Jump-Diffusion Bond Pricing Models Within the HJM Framework with State Dependent Volatilities. *Asia-Pacific Financial Markets*, 10:87–127, 2003.

C. Chiarella, V. Fanelli, and S. Musti. Modelling the Evolution of Credit Spreads Using the Cox Process within the HJM Framework: A CDS Option Pricing Model. *European Journal of Operational Research*, 2011.

P. Collin-Dufresne and R. Goldstein. Do Bonds Span the Fixed Income Markets? Theory and Evidence for Unspanned Stochastic Volatility. *Journal of Finance*, 57:1685–1730, 2002.

P. Cotton, J. P. Fouque, G. Papanicolaou, and R. Sircar. Stochastic Volatility Corrections for Interest Rate Derivatives. *Mathematical Finance*, 14(2):173–200, 2004.

J. Cox, J. E. Ingersoll, and S. A. Ross. A Theory of the Term Structure of Interest Rates. *Econometrica*, 53:385–407, 1985.

S. R. Das and P. Tufano. Pricing Credit-Sensitive Debt when Interest Rates, Credit Ratings and Credit Spreads are Stochastic. *The Journal of Financial Engineering*, 5(2):161–198, 1996.

M. Davis. Piecewise-deterministic Markov Processes: A General class of non-diffusion Stochastic Models. *Journal of Royal Statistical Society*, 46:353–388, 1984.

D. D'Souza, K. Amir-Atefi, and B. Racheva-Jotova. Valuation of a credit spread put option: The stable paretian model with copulas. In S.T. Rachev, editor, *Handbook of Computational and Numerical Methods in Finance*, pages 15 – 69. Birkhauser Boston, 2004.

G. R. Duffee. The Relationship of Treasury Yields and Corporate Bond Yield Spreads. *Journal of Finance*, 53(6):2225–2241, 1998.

D. Duffie. Defaultable Term Structure Models with Fractional Recovery of Par. *Graduate School of Business, Stanford University*, 1998. Working Paper.

D. Duffie and M. Huang. Swap Rates and Credit Quality. *Journal of Finance*, 51(3):921–949, 1996.

D. Duffie and R. Kan. A Yield-Factor Model of Interest Rates. *Mathematical Finance*, 6(4):379–406, 1996.

D. Duffie and D. Lando. Term Structures of Credit Spreads with Incomplete Accounting Information. *Econometrica*, 69(3):633–664, 2001.

D. Duffie and K. J. Singleton. An Econometric Model of the Term Structure of Interest Rate Swap Yields. *The Journal of Finance*, 52(4):1287–1322, 1997.

D. Duffie and K. J. Singleton. Modeling Term Structures of Defaultable Bonds. *The Review of Financial Studies*, 12(4):687–720, 1999.

D. Duffie and K. J. Singleton. *Credit Risk: Pricing, Measurement and Management.* Princeton University Press, 2003.

D. Duffie, L. Pedersen, and K. Singleton. Modeling Sovereign Yield Spreads: A Case Study of Russian Debt. *The Journal of Finance*, 58:119160, 2003.

G. S. Durham. Monte Carlo Methods for Estimating, Smoothing, and Filtering One and Two-Factor Stochastic Volatility Models. *Journal of Econometrics*, 133(1):273 – 305, 2006.

E. Eberlein and F. Özkan. The Defaultable Lévy Term Structure: Ratings and Restructuring. *Mathematical Finance*, 13(2):277–300, 2003.

M. Elhouar. Finite Dimensional Realizations of Regime-Switching HJM models. *Applied Mathematical Finance*, 15(4):1–24, 2008.

R. J. Elliott and T. K. Siu. On Markov-Modulated Exponential-Affine Bond Price Formulae. *Applied Mathematical Finance*, 16(1):1–15, 2009.

R. J. Elliott and C. A. Wilson. *The Term Structure of Interest Rates in a Hidden Markov Setting: Hidden Markov Models in Finance.* International Series in Operations Research & Management Science, Springer US, 2007.

R. J. Elliott, L. Aggoun, and J. B. Moore. *Hidden Markov Models: Estimation and Control.* Application in Mathematics 29, Springer Verlag, Berlin-Heidelberg-New York, 1994.

D. Filipovic. *Term Structure Models: A Graduate Course.* Springer Finance, 2009.

D. Filipovic and J. Teichmann. On Finite-Dimensional Term Structure Models. Technical report, Princeton University, 2002. Working paper.

D. Filipovic and J. Teichmann. Existence of Invariant Manifolds for Stochastic Equations in Infinite Dimension. *Journal of Functional Analysis*, 197:398–432, 2003.

D. Filipovic, S. Tappe, and J. Teichmann. Term Structure Models Driven by Wiener Processes and Poisson Measures: Existence and Positivity. *SIAM Journal on Financial Mathematics*, 1:523 – 554, 2010.

J. P. Fouque and M. J Lorig. Short-Maturity Asymptotics for a Fast Mean-Reverting Correction to Heston's Stochastic Volatility Model. *SIAM Journal on Financial Mathematics*, 1:126 – 141, 2010.

J. P. Fouque, R. Sircar, and K. Solna. Stochastic Volatility Effects on Defaultable Bonds. *Applied Mathematical Finance*, 13(3):215–244, 2006.

J. P. Fouque, B.C. Wignall, and X. Zhou. Modeling Correlated Defaults: First Passage Model under Stochastic Volatility. *Journal of Computational Finance*, 11(3):43–78, 2008.

J. R. Frank and W. N. Torous. An Empirical Investigation of U.S Firms in Reorganisation. *Journal of Finance*, 44:747–769, 1989.

J. R. Frank and W. N. Torous. A Comparison of Financial Recontracting in Distressed Exchanges and Reorganizations. *Journal of Financial Economics*, 35:349–370, 1994.

R. Garcia and P. Perron. "An Analysis of the Real Interest Rate under Regime Shifts. *The Review of Economics and Statistics*, 78:111–125, 1996.

R. Geske. The Valuation of Corporate Liabilities as Compound Options. *Journal of Financial and Quantitative Analysis*, 12:541–552, 1977.

R. Geske. The Valuation of Compound Options. *Journal of Financial Economics*, 7:63–81, 1979.

K. Giesecke. Correlated Default with Incomplete Information. *Journal of Banking and Finance*, 28(7):1521–1545, 2004.

R. J. Grossman and M. Hansen. Credit Spreads and Default Risk: Interpreting the Signals. *Fitch Ratings, Special Report*, 2010.

J. Hamilton. A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle. *Econometrica*, 57:357–384, 1989.

A. T. Hansen and R. Poulsen. A Simple Regime Switching Term Structure Model. *Finance and Stochastics*, 4(4):409–429, 2000.

D. Heath, R. A. Jarrow, and A. Morton. Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica*, 60:77–105, 1992.

S. Heston. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies*, 6:327–343, 1993.

B. Hilberink and L.C.G. Rogers. Optimal Capital Structure and Endogenous Default. *Finance and Stochastics*, 6(2):237–263, 2002.

G. Hobson and L.C.G. Rogers. Complete Models with Stochastic Volatility. *Mathematical Finance*, 8(1):27–48, 1998.

P. Houweling and T. Vorst. Pricing Default Swaps: Empirical Evidence. *Journal of International Money and Finance*, 24:1200–1225, 2005.

J. C. Hull and A. White. The Pricing of Options with Stochastic Volatilities. *Journal of Finance*, 42:281–300, 1987.

J. C. Hull and A. White. Pricing Interest Rate Derivative Securities. *The Review of Financial Studies*, 3(4):573–592, 1990.

J. C. Hull and A. White. The Impact of Default Risk on the Prices of Options and other Derivative Securities. *Journal of Banking and Finance*, 19:299–322, 1995.

J. C. Hull and A. White. Valuing Credit Default Swaps II: Modelling Default Correlation. *Journal of Derivatives*, 8:12–22, 2001.

J. C. Hull and A. White. The Valuation of Credit Default Swap Options. *Journal of Derivatives*, 10(3):40–50, 2003.

J. C. Hull and A. White. Merton's Model, Credit Risk and Volatility Skews. *Journal of Credit Risk*, 1(1):3 – 28, 2004/2005.

J. C. Hull, M. Predescu, and A. White. Bond Prices, Default Probabilities and Risk Premiums. *Journal of Credit Risk*, 1(2), 2005.

T. R. Hurd and A. Kuznetsov. Fast CDO Computations in the Affine Markov Chain Model. *Working Paper, McMaster University*, 2006.

T. R. Hurd and A. Kuznetsov. Affine Markov Chain model of Multifirm Credit Migration. *Journal of Credit Risk*, 3:3–29, 2007.

K. Inui and M. Kijima. A Markovian Framework in Multi-Factor Heath-Jarrow-Morton Models. *Journal of Financial and Quantitative Analysis*, 33(3):423–440, 1998.

G. M. Jabbour, F. El-masri, and S. D. Young. On the Lognormality of Forward Credit Default Swap Spreads. *Journal of Financial Transformation*, 22:41–48, 2008.

J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes.* Springer Verlag, 2003. Second Edition.

F. Jamshidian. An Exact Bond Option Formula. *Journal of Finance*, 44:205–209, 1989.

F. Jamshidian. Valuation of Credit Default Swaps and Swaptions. *Finance and Stochastics*, 8: 343–371, 2004.

R. A. Jarrow. *Modelling Fixed Income Securities and Interest Rate Options.* McGraw-Hill, 1996.

R. A. Jarrow and S.M. Turnbull. Pricing Derivatives on Financial Securities subject to Credit Risk. *The Journal of Finance*, L(1):53–85, 1995.

R. A. Jarrow and S.M. Turnbull. Intersection of Market and Credit Risk. *Journal of Banking and Finance*, pages 271–299, 2000.

R. A. Jarrow and F. Yu. Counterparty Risk and the Pricing of Defaultable Securities. *The Journal of Finance*, 56:1765–1799, 2001.

R. A. Jarrow, D. Lando, and S.M. Turnbull. A Markov Model for the Term Structure of Credit Risk Spreads. *The Review of Financial Studies*, 10(2):481–523, 1997.

M. Jeanblanc and S. Valchev. Default-Risky Prices with Jumps, Liquidity risk and Heterogenous Investors. *National Centre of Competence in Research Financial Valuation and Risk Management*, Working Paper No. 149, 2004.

M. Jeanblanc, M. Yor, and M. Chesney. *Mathematical Methods for Financial Markets.* Springer, 2009.

H. Johnson and R. Stulz. The Pricing of Options with Default Risk. *Journal of Finance*, 42, 1987.

E. Jones, S. Mason, and E. Rosenfeld. Contingent Claim Analysis of Corporate Capital Structures: An Empirical Investigation. *Journal of Finance*, 39, 1984.

M. Kalimipalli and R. Susmel. Regime-Switching Stochastic Volatility and Short-Term Interest Rates. *The Journal of Empirical Finance*, 11:309–329, 2004.

O. Kallenberg. *Foundations of Modern Probability.* Springer-Verlag, 1997.

O. Keith and A. Stuart. *Kendall's Advanced Theory of Statistics, 6th Edition.* Wiley Publishers, 1994.

A. Q. M. Khaliq and R. H. Liu. New Numerical Scheme for Pricing American Option with Regime Switching. *International Journal of Theoretical and Applied Finance*, 12(3):319–340, 2009.

M. Kijima and K. Komoribayashi. A Markov Chain Model for Valuing Credit Risk Derivatives. *Journal of Derivatives*, 6:97–108, 1998.

I. J. Kim, K. Ramaswamy, and S. Sundaresan. The Valuation of Corporate Fixed Income Securities. *Rodney L. White Center for Financial Research Working Papers, Wharton School*, 32-89, 1993a.

I. J. Kim, K. Ramaswamy, and S. Sundaresan. Does Default Risk in Coupons Affect the Valuation of Corporate Bonds. *Financial Management*, 22:117–131, 1993b.

F. C. Klebaner. *Introduction to Stochastic Calculus with Applications, 2th Edition.* Imperial College Press, 2005.

M. Krekel and J. Wenzel. A Unified Approach to Credit Default Swaption and Constant Default Swap Valuation. *Berichte des Fraunhofer ITWM*, 96, 2006.

M. K. Lam and W. Li. A Stochastic Volatility Model with Markov Switching. *Journal of Business and Economic Statistics*, 16(2):244–253, 1998.

C. Landén. Bond Pricing in a Hidden Markov Model of the Short Rate. *Finance and Stochastics*, 4(4):371–389, 2000.

D. Lando. Three Essays on Contingent Claims Pricing. *Cornell University*, 1994. Ph.D. Dissertation.

D. Lando. On Cox Processes and Credit Risky Securities. *Review of Derivatives Research*, 2(2/3): 99–120, 1998.

H. E. Leland and K. B. Toft. Optimal Capital Structure, Endogenous Bankruptcy and the Term Structure of Credit Spreads. *Journal of Finance*, 51(3):987–1019, 1996.

H. Li and F. Zhao. Unspanned Stochastic Volatility: Evidence from Hedging Interest Rate Derivatives. *Journal of Finance*, 61:341–378, 2006.

A. Lipton and A. Rennie. *The Oxford Handbook of Credit Derivatives.* Oxford University Press, 2011.

R. Litterman and T. Iben. Corporate Bond Valuation and the Term Structure of Credit Spreads. *Journal of Portfolio Management*, Spring:52–64, 1991.

F. Longstaff and E. Schwartz. Interest Rate Volatility and the Term Structure: a Two-Factor General Equilibrium Model. *Journal of Finance*, 47(4):1259–1282, 1992.

F. Longstaff and E. Schwartz. The Pricing of Credit Risk Derivatives. *Journal of Fixed Income*, 5 (1):6–14, 1995a.

F. Longstaff and E. Schwartz. A Simple Approach to Valuing Risky Fixed and Floating Rate Debt. *Journal of Finance*, 50(3):789–819, 1995b.

D. B. Madan and H. Unal. Pricing the Risks of Default. *Review of Derivatives Research*, 2:121–160, 1998.

D. B. Madan and H. Unal. A Two-Factor Hazard Rate Model for Pricing Risky Debt and the Term Structure of Credit Spreads. *Journal of Financial and Quantitative Analysis*, 35:43–65, 2000.

R. Maksymiuk and D. Gatarek. Applying HJM to Credit Risk. *Risk*, April:67–68, 1999.

F. Mercurio and J. M. Moraleda. An Analytically Tractable Interest Rate Model with Humped Volatility. *European Journal of Operational Research*, 120:205–214, 2000.

Merrill-Lynch. Credit Derivatives Handbook 2006 - Vol. 1: A Guide to Single-Name and Index CDS Products. *Credit Derivatives Strategy, Merrill Lynch*, 2006a.

Merrill-Lynch. Credit Derivatives Handbook 2006 - Vol. 2: A Guide to the Exotics Credit Derivatives Market. *Credit Derivatives Strategy, Merrill Lynch*, 2006b.

C.R. Merton. On the pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance*, 29:449–470, 1974.

Moody's. Recovery Rates on Defaulted Corporate Bonds and Preferred Stocks, 1982 - 2003. *Moody's Investors Service, Global Credit Research*, 2003.

V. Naik and M. H. Lee. Yield Curve Dynamics with Discrete Shifts in Economic Regimes: Theory and Estimation. *University of British Columbia*, 1997. Working Paper.

J. Pan and K. K. Singleton. Default and Recovery Implicit in the Term Structure of Sovereign CDS Spreads. *Journal of Finance*, 63(5):2345–2384, 2008.

D. Pugachevsky. Generalising with HJM. *Risk*, August:103–105, 1999.

G. Pye. Gauging the Default Premium. *Financial Analysts Journal*, January-February:49–52, 1974.

R. Reno and A. Uboldi. On the Presence of Unspanned Volatility in European Interest Rate Options. *Applied Financial Economics Letters*, 1:15–18, 2005.

D. Repplinger. *Pricing of Bond Options: Unspanned Stochastic Volatility and Random Fields.* Lecture Notes in Economics and Mathematical Systems, Springer, 2008.

P. Ritchken and L. Sankarasubramanian. Volatility Structures of Forward Rates and the Dynamics of Term Structure. *Mathematical Finance*, 5(1):55–73, 1995.

T. Rolski, H. Schmidli, V. Schmidt, and J. Teuggels. *Stochastic Processes for Insurance and Finance.* John Wiley & Sons, 1998.

W. J. Runggaldier. Jump-diffusion models. In S.T Rachev, editor, *Handbook of Heavy Tailed Distributions in Finance*, volume 1, pages 169–209. Elesevier/North-Holland, 2003.

M. Rutkowski and A. Armstrong. Valuation of Credit Default Swaptions and Credit Default Index Swaptions. *International Journal of Theoretical and Applied Finance*, 12(7):1027–1053, 2009.

G. Sandmann and S. J. Koopman. Estimation of Stochastic Volatility Models via Monte Carlo Maximum Likelihood. *Journal of Econometrics*, 87(2):271 – 301, 1998.

P. J. Schönbucher and D. Schubert. Copula-Dependent Default Risk in Intensity Models. *University of Bonn*, 2001. Working Paper, December.

P.J. Schönbucher. Term Structure Modelling of Defaultable Bonds. *Review of Derivatives Research*, 2:161–192, 1998.

P.J. Schönbucher. Credit Risk Modelling and Credit Derivatives. *Department of Statistics, University of Bonn, Germany*, 2000. PhD Thesis.

P.J. Schönbucher. *Credit Derivatives Pricing Models: Model, Pricing and Implementation.* John Wiley & Sons, 2003.

P.J. Schönbucher. A Measure of Survival. *Risk*, January, 2004.

L. Scott. Option Pricing when the Variance Changes Randomly: Theory, Estimators, and Applications. *Journal of Financial and Quantitative Analysis*, 22:419–438, 1987.

L. Scott. Pricing Stock options in a Jump-Diffusion Model with Stochastic Volatility and Interest Rates: Application of Fourier Inversion Methods. *Mathematical Finance*, 7(4):413–426, 1997.

H. Shirakawa. Evaluation of yield spread for credit risk. In R. Anderson, editor, *Advances in Mathematical Economics*, volume 1, chapter 5, pages 83 – 97. Springer, 1999.

M. Sola and J. Driffill. Testing the Term Structure of Interest Rates using a Stationary Vector Autoregression with Regime Switching. *Journal of Economic Dynamics and Control*, 18:601–628, 1994.

E. Stein and J. Stein. Stock Price Distributions with Stochastic Volatility. *Review of Financial Studies*, 4:727–752, 1991.

N. Tahani. Valuing Credit Derivatives Using Gaussian Quadrature: A Stochastic Volatility Framework. *Journal of Futures Markets*, 24:3–35, 2004.

J. Topper. *Financial Engineering with Finite Elements.* John Wiley & Sons, 2005.

A. B. Trolle and E. S. Schwartz. A General Stochastic Volatility Model for the Pricing of Interest Rate Derivatives. *The Review of Financial Studies*, 22(5):2007–2057, 2009.

S. Valchev. Stochastic Volatility Gaussian Heath-Jarrow-Morton Models. *Applied Mathematical Finance*, 11:347–369, 2004.

S. Valchev and R. Elliott. Libor Market Model with Regime-Switching Volatility. *National Centre of Competence in Research Financial Valuation and Risk Management*, Working Paper No. 228, 2004.

O. Vasicek. An Equilibrium Characterisation of the Term Structure. *Journal of Financial Economics*, 5:177–188, 1977.

H. Y. Wong and T. L. Wong. Reduced-form Models with Regime Switching: An Empirical Analysis for Corporate Bonds. *Asia Pacific Financial Markets*, 14(3):229–253, 2007.

S. Wu and Y. Zeng. A General Equilibrium Model of the Term Structure of Interest Rates under Regime-Switching Risk. *International Journal of Theoretical and Applied Finance*, 8(7):839–869, 2005.

S. Wu and Y. Zeng. The Term Structure of Interest Rates under Regime Shifts and Jumps. *Economic Letters*, 93:215 – 221, 2006.

C. Zhou. A Jump-Diffusion Approach to Modelling Credit Risk and Valuing Defaultable Securities. *Finance and Economics Discussion Paper Series*, 15, 1997. Board of Governors of the Federal Reserve System.

C. Zhou. The Term Structure of Credit Spreads with Jump Risk. *Journal of Banking and Finance*, 25:2015–2040, 2001.