

# MODELING DIVERSIFIED EQUITY INDICES

By  
Renata Rendek

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Author: **Renata Rendek**

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# Abstract

The objective of this thesis is to study and model the dynamics of aggregate wealth, that is, the dynamics of the market capitalization weighted world stock index in different currency denominations. In order for the considered model to be valid over long time periods, it turns out that it needs to be formulated in a general financial modeling framework, the benchmark approach. In order to visualize and test the proposed aggregate wealth dynamics, exact and almost exact simulation techniques for multi-dimensional stochastic processes have been developed. Moreover, the model specification is preceded by a detailed study of the distribution of log-returns of world stock indices in different currency denominations. Various types of world equity indices are constructed and systematically studied, in particular, equi-weighted indices. When the number of constituents is increasing and the given investment universe is well securitized the Naive Diversification Theorem states that a sequence of equi-weighted indices approximates the growth optimal portfolio, which is also the numéraire portfolio.

Finally, by conjecturing for the normalized world stock index the dynamics of a time transformed square root process, and by establishing a list of stylized empirical facts, a two-component index model has been proposed. This model is very parsimonious and driven only by the non-diversifiable risk of the market. Via almost exact simulation this model is shown to reflect well all listed empirical stylized facts and is difficult to falsify.