Commodity Derivative Pricing under the Benchmark Approach

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A thesis submitted for the degree of
Doctor of Philosophy
at the University of Technology, Sydney.
27 February 2013.

UNIVERSITY OF TECHNOLOGY, SYDNEY
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Acknowledgements

First and foremost, I would like to thank Professor Eckhard Platen, my supervisor, for providing the opportunity to undertake this PhD and for his consistent and enthusiastic support. I would also like to thank Professor Erik Schlögl and Professor Tony He for their help throughout my research. This thesis would not have been possible without their support and encouragement. Further, I wish to express gratitude to my best friends, Rei Chen, Jan Baldeaux, Hardy Hulley, Boda Kang, Renata Rendek and Nicholas Yap. They have given me lots of good advice on various issues from my research to career development. I am also grateful to the staff and students of the Finance Discipline Group for their support and for providing a friendly atmosphere. Last but not least, I would like to thank my parents and my wife, Li Zeng, for their moral support and undying love. Without them, I would not have been able to accomplish anything.

Abstract

This thesis models commodity prices and derivatives, written on commodity prices, under the benchmark approach. Under this approach, the commodity prices are modeled under the real world probability measure while the corresponding numéraire is the numéraire portfolio (NP), which is the growth optimal portfolio that maximizes expected logarithmic utility. The existence of an equivalent risk neutral probability measure is not required. Under the proposed new concept of benchmarked risk minimization, the minimal price for a nonhedgeable contingent claim is identified, and the fluctuations of the benchmarked profit and loss, when denominated in units of the NP, are minimized. The resulting real world pricing formula generalizes the classical risk neutral pricing formula. The NP will be approximated by a well-diversified stock index. New forward and futures price formulas will be derived, which generalize their classical counterparts. Stylized empirical facts for the dynamics of the NP in a selected commodity denomination will be identified. These lead to a model which falls outside classical no-arbitrage assumptions, but is covered by the benchmark approach. Under this model, some long dated derivatives will be shown to be less expensive than under the classical paradigm.

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Chapter 1

Introduction

1.1 Brief Survey of Results

The objective of this thesis is to provide a general approach to the pricing of not fully replicable contingent claims and to the modeling of commodity prices and their derivatives under the benchmark approach, proposed in Platen [2002], Platen [2006] and Platen and Heath [2010]. Within this approach, asset prices are modeled under the real world probability measure, where the corresponding numéraire is chosen to be the numéraire portfolio (NP), which is also the growth optimal portfolio (GOP) that maximizes expected logarithmic utility. The GOP was first studied by Kelly [1956] and later by Latane [1959], Markowitz [1959], Breiman [1960] and others. When the NP is taken as numéraire, see Long [1990], Platen [2002] and Platen and Heath [2010], pricing is performed under the real world probability measure.

This thesis introduces the new concept of benchmarked risk minimization for pricing and hedging not fully replicable contingent claims, which generalizes classical risk minimization in the sense of Föllmer and Sondermann [1986] and Schweizer [1999]. Under benchmarked risk minimization, a change of probability measure, in particular, the existence of an equivalent risk neutral probability measure is not required. This makes the presented approach on pricing and hedging more general than the classical risk neutral approach and classical risk minimization. New, more general forward and futures price formulas will be derived. Realistic models will be presented that allow to obtain explicit or almost explicit formulas for a range of commodity derivatives. In practice, the NP can be approximated by a well-diversified portfolio, see Platen [2005], Platen and Heath [2010] and Platen and Rendek [2012a], which provides practicality to the benchmark approach when modeling commodity prices and their derivatives. Stylized empirical facts for the dynamics of the NP in a selected commodity denomination will be established and a respective parsimonious model proposed, which leads us beyond the classical no-arbitrage paradigm.

The following three specific topics will be discussed in the three main chapters of this thesis:

- The pricing of not fully replicable contingent claims.
- The pricing of forward and futures contracts on commodities.
- The identification of stylized empirical facts on oil price dynamics.

1.2 Commodity Price Modeling

The commodity market trades large volumes of primary commodity products. Millions of dollars worth of transactions in copper, gold, crude oil and other raw materials take place globally every day. The London Metal Exchange (LME) is an important example of such a commodity market (industrial metals). According to LME [2012], in 2011 the LME traded \$15.4 trillion notional, 3.5 billion tonnes or 146 million lots. This indicates more than 80% of non-ferrous metal futures transactions in the world. Trading volumes in commodity derivatives, such as futures and options, have risen extremely fast, resulting in commodity price risk becoming particularly important for the risk management of many companies.

Convenience Yield

Since commodity price risk plays an increasing role in the risk management of the global financial market, the modeling of commodity prices is of special importance. To price and hedge contingent claims written on commodities, one crucial problem is to derive and explain the shape of the commodity forward curve. For example, when considering crude oil, several empirical studies argue that most of the time, the forward curve is in 'backwardation', i.e. the forward price is lower than the spot price. On the other hand, in times of crises, the forward curve may be in 'contango', i.e. the forward price is higher than the spot price, see e.g. Litzenberger and Rabinowitz [1995] and Geman [2005]. This leads to the well-known theory of normal backwardation, e.g. for energy markets, see Geman [2005].

An important theory, which aims to explain facts observed for commodity prices, is the theory of storage, which dates back to the work of Kaldor [1939] and Working [1948]. The theory of storage was introduced in commodity price modeling by a classical argument, which states that the holders of inventory of commodities rely, in some sense, on 'time value' that is reflected in the convenience yield. The convenience yield describes the benefit obtained by physically holding the commodity itself, rather than holding a corresponding forward or futures contract. The convenience yield is widely believed to be closely related to manufacturing and storage procedures in industry, which are associated with the most elementary properties of the commodity itself. For example, an oil refinery needs crude oil to produce gasoline. For the company, it is convenient to store sufficient amounts of crude oil so that the production process is unlikely to be interrupted. Once the production is stopped due to some lack of supply of oil, a significant cost would typically be incurred to restore the production. As the company stores the commodity for some time, the company will benefit from the 'convenience' of crude oil being available when arranging the production procedure. This 'convenience' can be measured, as will be discussed later. If one denominates the benefit from the availability of the commodity in units of the commodity itself, then one finds that the resulting commodity savings account typically grows. That is, by storing crude oil for some time, the company gains usually in comparison to the situation when it does not have with enough certainty crude oil conveniently available. The storage cost for some commodities can be higher than the benefit from the convenience to have the commodity available at any time. Consequently, the resulting convenience yield for holding the commodity can have positive or negative values. Furthermore, if one retains and accumulates the net benefit of the convenience to have the commodity available in an account, denominated in units of the commodity, then one obtains a commodity account with the convenience yield as growth rate, see Brennan and Schwartz [1985]. The resulting account can be viewed as the commodity savings account of the given commodity in analogy to a currency savings account. When denominated in units of the commodity it can be assumed to be locally riskless, which means that it has a time derivative. A commodity savings account describes in units of the commodity how the wealth of an investor would grow if he or she invests in the commodity and reinvests the net benefits from the convenience of having the commodity available by taking benefits and also the storage cost into account.

Commodity Short Rate Models

The more recent literature on commodity price modeling dates back to the pioneering work of Black [1976], who was followed by Gibson and Schwartz [1990], Bessembinder and Seguin [1993], Schwartz [1997], Eydeland and Geman [1998], Pindyck [2001] and many others. Inspired by the celebrated work of Black and Scholes [1973], Gibson and Schwartz [1990] used geometric Brownian motion to model the spot price of crude oil. Motivated by an empirical study on crude oil in Gibson and Schwartz [1989], a mean reverting stochastic process, similar to the one described in Vasicek [1977], was introduced to model the convenience yield. Related studies can be found in Nielsen and Schwartz [2004] and Ribeiro and Hodges [2004]. In these models, the interest rate is supposed to be a constant.

It has been shown under classical assumptions, that if the interest rate is a constant, then the forward price and the futures price of commodities are identical. We will see under the more general benchmark approach that they may differ. To address the problem of stochastic interest rates and stochastic convenience yields, Schwartz [1997] introduced a stochastic short term interest rate into the Gibson and Schwartz [1990] model, creating their well-known three-factor model. In this model, the interest rate is supposed to follow a mean reverting stochastic process, as described in Vasicek [1977]. The Vasicek [1977] interest rate model is often considered to have a drawback, because its interest rate can become negative. An alternative choice is the Cox, Ingersoll, and Ross [1985]

(CIR) model. In this research, we will at some stages assume that the short term interest rate follows a time-transformed square root process, or more generally, an affine process, where the short rate is positive and linear mean reverting. On the other hand, in the case of commodities, the negativity of the convenience yield, which is the short rate of a commodity, is a reasonable possibility. In Gibson and Schwartz [1990], the convenience yield is defined as the difference between the benefit from the convenience to have the commodity available and the storage cost per unit of time. If the storage cost is relatively high, then some negative commodity short rate could easily arise. This situation is associated with a, so called, contango or backwardation in the commodity term structure. Studies in this direction include Litzenberger and Rabinowitz [1995], Geman [2005] and Geman [2007]. Furthermore, it can also be shown that the interest rate has some connection with inflation and thus with commodity prices, see Geman [2005]. Bruti-Liberati, Nikitopoulos-Sklibosios, and Platen [2007] suggest a framework for interest rate term structure modeling involving a time-transformed squared Bessel process, which could be useful in handling the dynamics of convenience yields. Research related to the Schwartz [1997] three-factor model comprises papers by Cortazar and Schwartz [1997], Geman [2001], Miltersen and Schwartz [1998] and Hilliard and Reis [1998].

Exchange Price of a Commodity

In this thesis, the spot price of a commodity will be modeled similarly as the exchange price of currencies (that is their exchange rate). This approach has been suggested in a study on the multi-currency minimal market model in Platen and Heath [2010]. The physical commodity itself is interpreted as a 'foreign currency' and the convenience yield, see Gibson and Schwartz [1990] and Miltersen and Schwartz [1998], is interpreted as the locally riskless 'interest rate' of this 'foreign currency', see Platen and Heath [2010]. Meanwhile, the monetary currency is considered as the 'domestic currency', and the domestic interest rate is modeled by appropriate interest rate models, see e.g. Platen and Heath [2010] and Bruti-Liberati, Nikitopoulos-Sklibosios, and Platen [2007] and Filipović [2001]. In a multi-currency minimal market model, first suggested in Platen [2001], the exchange price is modeled by the ratio of the NP in domestic currency denomination over the NP in foreign currency denomination. In this model, the exchange price volatility is determined by both the volatilities of the NP in foreign currency denomination and the NP in domestic currency denomination. The aggregate volatility is, therefore, separated into one part linked to the domestic currency and one part related to the foreign currency.

When applying this kind of exchange price modeling to the commodity situation, the commodity exchange price volatility is determined by both the volatility of the NP in domestic currency denomination, as well as, the volatility of the NP in commodity denomination. This makes the presented research rather different to most previous studies on commodity price modeling, where the exchange price is modeled as a single stochastic process. It will be shown that the proposed approach allows disentangling the key factors driving commodity prices. This type of new commodity model will allow us to improve pricing and hedging of commodity derivatives.

1.3 A Benchmark Approach to Commodities

The classical approach to asset pricing is built on the arbitrage pricing theory (APT), proposed in the pioneering work of Ross [1976], and developed further by Harrison and Kreps [1979], Harrison and Pliska [1981], Föllmer and Sondermann [1986], Föllmer and Schweizer [1991], Delbaen and Schachermayer [1994], Delbaen and Schachermayer [1998] and Karatzas and Shreve [1998]. The APT is related to the Fundamental Theorem of Asset Pricing, with its most general versions derived in Delbaen and Schachermayer [1994] and Delbaen and Schachermayer [1998]. More precisely, the absence of a free lunch with vanishing risk is equivalent to the existence of an equivalent risk neutral probability measure. This is an important mathematical result, which makes the APT precise. However, there seems to be no economic reason to exclude some weak forms of arbitrage that may exist in reality, as has been pointed out in Loewenstein and Willard [2000] and Platen [2002]. Some free lunches with vanishing risk can easily be allowed without creating an unreasonable financial market model. From the perspective of empirical evidence and economic reasoning, an equivalent risk neutral probability measure does not need to exist for realistic models of the financial market. As we will see, going beyond APT does not create any serious problem in pricing, hedging and risk management.

The benchmark approach, proposed by Platen [2002], Platen [2006] and Platen and Heath [2010], provides a general methodology to pricing and hedging, which is still applicable for models where the traditional risk neutral approach, that is the APT, fails. Under the benchmark approach the numéraire portfolio (NP), which is the growth optimal portfolio (GOP) and benchmark, is taken as the central building block. The GOP or Kelly portfolio was discovered in Kelly [1956]. It is defined as the positive, self-financing portfolio that is maximizing the expected logarithmic utility from terminal wealth.

Numéraire Portfolio

Long [1990] suggested to employ the Kelly portfolio as numéraire for pricing. In this case the pricing measure becomes the real world probability measure. One benefit of introducing the concept of the numéraire portfolio (NP) is that it allows us to interpret the Radon-Nikodym derivative in economic terms, see Bajeux-Besnainou and Portait [1997]. The Radon-Nikodym derivative, defined for the change between two different probability measures, can be interpreted as the ratio of the numéraires of these two respective probability measures, see Geman, El Karoui, and Rochet [1995]. As a result, if the equivalent risk neutral probability measure exists for a given model, then the Radon-Nikodym derivative for the change from the real world probability measure to the risk neutral probability measure is the ratio between the corresponding numéraires, that is, between the savings account and the NP. If this change of probability measure shall work properly, then this ratio has to form a martingale under the real world probability measure, see Karatzas and Shreve [1998]. However, under the benchmark approach, even in a jump diffusion financial market, see Platen [2002], Platen [2006] and Platen and Heath [2010], the ratio of the savings account over the NP needs only to be a nonnegative local martingale, which is a supermartingale, see Karatzas and Shreve [1991]. In Long [1990], Bajeux-Besnainou and Portait [1997] and Becherer [2001] it has been assumed that an equivalent risk neutral probability measure exists, where as Platen [2002] and Karatzas and Kardaras [2007] do no longer insist on such restrictive property. As mentioned above, for a continuous, complete market the Radon-Nikodym derivative process is formed by the ratio of the savings account over the NP, normalized to one at time zero, see e.g. Karatzas and Shreve [1998]. The classical APT with its risk neutral pricing approach assumes that this process is a martingale under the real world probability measure. However, as we will see later and has been empirically demonstrated in Platen [2009] and Platen and Bruti-Liberati [2010], in the long term this is not a realistic assumption. In this thesis we will take the view that the Radon-Nikodym derivative of the putative risk neutral measure can not be realistically modeled as a martingale. It appears to be necessary to model this process as a strict supermartingale. This means that its expected future values can be significantly below its current value.

Real World Pricing

The strict supermartingale property makes the change of probability measure impossible, and, thus, the classical Law of One Price to fail. Instead, the Law of the Minimal Price was proposed in Platen [2002], which yields the real world pricing formula. In the real world pricing formula, the conditional expectation is taken under the real world probability measure with the NP as corresponding numéraire, when appropriate assumptions are additionally imposed. This real world pricing formula generalizes the concepts of pricing kernel, deflator, state price density and stochastic discount factor studied by Constantinides [1992], Duffie [2001], Rogers [1997] and Cochrane [2001], respectively. For electricity price derivatives Platen and West [2011] applied real world pricing in a commodity context. Platen [2002] showed that if the equivalent risk neutral probability measure exists, then the real world pricing formula, with the NP taken as numéraire, yields the classical risk neutral pricing formula. The benchmark approach allows one to relax the key assumptions of the classical risk neutral approach, that is, the APT. As a result, the benchmark approach is more general and allows a wider modeling world and more general contingent claims to be considered, as we will demonstrate in the thesis.

Benchmarked Risk Minimization

This thesis proposes the new concept of benchmarked risk minimization, see Du and Platen [2012a], which allows one to price not fully replicable contingent claims in a simple manner. In some sense it combines classical risk minimization, in the sense of Föllmer and Sondermann [1986] and Föllmer and Schweizer [1991], with the benchmark approach. As we will see, under benchmarked risk minimization one determines the minimal possible price for a given contingent claim and obtains a profit and loss that when denominated in units of the NP, is in some sense orthogonal to all tradable portfolios when these are denominated in units of the NP. The assumptions one needs to impose to apply benchmarked risk minimization are extremely weak. No square integrability is required. Only a real world expectation has to be computed for identifying the minimal price and some drift has to be set to zero for determining the respective hedging strategy. Interestingly, when an equivalent risk neutral probability measure exists, then the real world price and the risk neutral price coincide. However, the hedging strategies usually differ. Benchmarked risk minimization takes any evolving information about the nonhedgeable part of a contingent claim into account, whereas classical risk minimization ignores such information even when it is readily available. This property of benchmarked risk minimization is supported by common sense and can be very beneficial for risk management.

1.4 Forward-Futures Spread

Schwartz [1998] studied long dated commodity derivatives and provided a one-factor model. The Schwartz [1998] model is motivated by a comparative analysis of the Gibson and Schwartz [1990] two-factor model, and the Schwartz [1997] three-factor model. As mentioned previously, the most important difference between these two models is the appearance of the forward-futures spread, which is caused by the stochasticity of the interest rate in the Schwartz [1997] three-factor model. As a result, although these two models behave similarly in the short term, they diverge as the maturity increases, see Schwartz [1997]. However, in both models, the forward cost of carry, first studied in Cortazar and Schwartz

[1997], which is defined as the derivative of a futures price with respect to time to maturity and divided by the futures price itself, see Schwartz [1997], converges to a constant rate as the maturity goes to infinity. Thus, if the forward cost of carry is assumed to be a constant and carefully calibrated, then one stochastic factor for modeling the shadow spot price of the commodity, see Schwartz [1998], might be enough to capture both the forward and the futures price in the long term. The Schwartz [1998] one-factor model captures commodity prices in the long term reasonably well. However, this model focuses only on a long term approximation, which may not work satisfactorily when the time to maturity is short. This thesis aims to propose realistic models that reflect well the forward-futures spread for both short and long term contracts.

Long Term Contracts

The previously mentioned classical models under the APT generally work well in the short term, but there is a common drawback for their applicability in the long run. These classical models are built under the assumption of the existence of an equivalent risk neutral probability measure, which requires its Radon-Nikodym derivative to be a martingale under the real world probability measure so that the Girsanov Theorem and the Bayes rule can be applied. As we mentioned earlier, the existence of an equivalent risk neutral probability measure in the long run is questionable. This thesis will come to similar conclusions for the modeling of commodity prices and will, therefore, avoid the above mentioned classical assumption. It will instead apply benchmarked risk minimization, as described in Du and Platen [2012a]. The classical models under the APT are then still covered but with different hedging strategies that take evolving information about nonhedgeable parts of contingent claims into account.

1.5 Forward and Futures Prices

Previous commodity price models have been usually based on modeling the spot price process, such as in Gibson and Schwartz [1990], Schwartz [1997], Nielsen and Schwartz [2004] and in Hilliard and Reis [1998]. In these models, the modeling of the convenience yield is a necessity. However, as described in Eydeland and Wolyniec [2003], in specific markets such as the electricity market, the convenience yield is not even observable. Additionally, there may not be sufficient historical data for some newly emerging commodities, making it hard to perform reliable estimation, as described in Geman [2006]. To address this problem, models built under the Heath, Jarrow, and Morton [1992] framework have been introduced, which model the entire convenience yield term structure evolution. Related research includes Schwartz and Smith [2000], Geman [2006] and Cartea, Figueroa, and Geman [2009].

Forward Rate Term Structure

This thesis applies an alternative approach to the problem of modeling the commodity forward rate term structure. We call a price that is denominated in units of the NP a benchmarked price. With the application of the real world pricing formula of benchmarked risk minimization, we will show that the forward price of a commodity equals the ratio of the benchmarked zero coupon bond of the commodity over the benchmarked zero coupon bond of the monetary unit. It is, therefore, natural to model the convenience yield by the shortest forward rate of the zero coupon bond of one commodity unit rather than by modeling its commodity short rate directly. As this approach models some family of zero coupon bonds with different maturity dates, the whole term structure of commodity forward rates is modeled. This approach will allow us to obtain a connection to forward price models, similar as those described in Reismann [1992], Cortazar and Schwartz [1994], Amin and Jarrow [1992] and Carr and Jarrow [1995].

Forward and Futures

Several approaches on commodity price modeling take the forward price of the commodity as an input. The convenience yield is then automatically modeled. However, in reality, only a limited range of commodities, such as gold, have a quoted forward price. For some commodities, such as electricity, the spot price does not physically exist, and the only available data are obtained from the related futures price. In this case, a model for the forward-futures spread is needed to recover the forward price. Related research under classical assumptions includes Black [1976], Cox, Ingersoll, and Ross [1981], Richard and Sundaresan [1981], Duffie and Stanton [1992] and Shreve [2004]. Miltersen and Schwartz [1998] considered the forward-futures spread for commodities under the classical risk neutral approach. Murawski [2003] attached some default risk to the forward price, which adds more complexity. Most of the models on the forward-futures spread derive the forward price from a classical hedging argument. If an equivalent risk neutral probability measure exists, then, the classical forward price formula is the same as the one which the thesis will obtain under the benchmark approach. The standard literature derives the futures price through the classical risk neutral approach. This thesis studies the situation when an equivalent risk neutral probability measure may not exist. We will see that even in the case when the interest rate is assumed to be constant, the real world forward price may be different from the futures price and from the formally obtained "risk neutral" forward price, as shown in Du and Platen [2012b].

1.6 Diversified Indices as Proxy for the NP

To price in practice contingent claims under the real world pricing formula of the benchmark approach, one has to approximate the numéraire portfolio, that is the benchmark, first.

Proxy of the Numéraire Portfolio

Platen [2005] derived a Diversification Theorem in a continuous financial market, which states that under some regularity property, any diversified portfolio approximates the NP when the fractions invested are vanishing sufficiently fast as the number of constituents increases. In practice, as a result of diversification, most well diversified portfolios, e.g. a total return stock index, approximate the NP. In Platen and Redenk [2012] it has been shown under very weak assumptions that naive diversification already leads asymptotically to the NP. Consequently, the problem to model the NP is reduced to the selection and modeling of a well-diversified stock index such as the total return MSCI World Stock Index or an equi-weighted stock index.

Modeling the Dynamics of the NP

There exists some literature on models that attempt to capture the dynamics of a diversified stock index. Important is the identification of stylized empirical facts. In Du, Platen, and Rendek [2012] the denomination of a proxy of the NP in oil prices is empirically analyzed and subsequentially modeled in a parsimonious way. The typical model of the NP in this thesis will be the minimal market model (MMM), originally proposed in Platen [2001]. The MMM generates the NP dynamics such that the savings account, when benchmarked by the NP, forms a strict supermartingale. The MMM and its generalizations in this thesis use only few parameters to provide realistic long term dynamics of the NP, which capture well stylized empirical facts, see Platen [2002], Platen [2006] and Platen and Heath [2010], Platen and Rendek [2012a] and Du, Platen, and Rendek [2012]. By contrast with other models, which mostly focus on characterizing the volatility of an index, the MMM takes the drift of the discounted NP as the key parameter process to be modeled. With such a drift parameterization, the dynamics of the discounted NP can generally be interpreted as those of a time transformed squared Bessel process of dimension four. It is well known that the inverse of a squared Bessel process of dimension four is a strict local martingale and, thus, a supermartingale, see Revuz and Yor [1999]. As a result, the Radon-Nikodym derivative of the putative risk neutral measure is not a martingale under the MMM and the APT can not be applied. However, the long term dynamics of the well-diversified proxy of the discounted NP is realistically captured by the resulting strict supermartingale.

Exclusion of Strong Arbitrage

Since the Fundamental Theorem of Asset Pricing, see Delbaen and Schachermayer [1998], does not apply for the MMM, this type of model permits some weaker forms of arbitrage, as described e.g. in Loewenstein and Willard [2000], Platen [2002], Platen [2006] and Platen and Heath [2010]. These weaker forms of arbitrage cannot be practically exploited to accumulate unlimited wealth in the market because some collateral will always have to be posted by the investor in his or her portfolio of total wealth, which can then become negative. Under the benchmark approach, so-called strong arbitrage, in the sense of Platen and Bruti-Liberati [2010], is automatically excluded because a nonnegative supermartingale that starts at zero can never reach any strictly positive value. Thus, there is no chance to generate any strictly positive wealth out of zero initial capital by a benchmarked nonnegative portfolio since it forms a supermartingale. This is an economically sensible statement, whereas, the request on the existence of an equivalent risk neutral probability measure is more a mathematically convenient and from the modeling point of view a far too restrictive assumption when aiming for capturing long term contracts as common for some commodity derivatives.

Volatility

The MMM automatically generates a stochastic volatility process, reflecting well the typically observed leverage effect for currency denominations of stock indices like proxies of the NP. No additional stochastic factor to characterize stochastic volatility is used for the stylized version of the MMM. In this thesis, the MMM and a two-component generalization of this model, see Du, Platen, and Rendek [2012], will be applied to model the NP in both its domestic currency denomination and its commodity spot price denomination. The commodity spot price in domestic currency will be obtained as the ratio of the NP value in currency denomination over the NP value in commodity denomination. The thesis will confirm that stylized empirical facts for an oil spot price denomination of the NP will be well reflected, also in the long term, by the proposed two-component model. It will be demonstrated how to model commodity prices and how to obtain the real world price of a contingent claim in a benchmark setting.

1.7 Main Contributions of the Thesis

The five main contributions of this thesis are the following:

(i) The thesis models commodity prices and their short and long dated derivatives under the benchmark approach, which generalizes the classical risk neutral approach.

(ii) The new concept of benchmarked risk minimization is proposed, which yields the real world pricing formula also for not fully replicable contingent claims, and combines the ideas of classical risk minimization, in the sense of Föllmer-Sondermann-Schweizer, with those of the benchmark approach.

(iii) The introduction of an exchange price model for commodity spot prices, where the physical commodity is interpreted as the 'foreign currency' and the convenience yield is interpreted as the 'interest rate' of the 'foreign currency', is shown to disentangle the factors driving commodity and currency dynamics.

(iv) It is demonstrated that even in the case when the interest rate is constant, the real world futures price may differ from the respective forward price in the proposed wider modeling world.

(v) The identification of stylized empirical facts for a proxy of the NP in the denomination of the crude oil spot price, is used to construct a realistic parsimonious model. The estimated model parameters show that it is not covered under the classical no-arbitrage paradigm.

The results of the thesis represent the content of the following three working papers: Du and Platen [2012a], Du and Platen [2012b] and Du, Platen, and Rendek [2012]. The first paper has been submitted to Mathematical Finance and received excellent referee reports and is under revision. The other two manuscripts will be updated and soon submitted to high ranking journals.

Chapter 2

Benchmarked Risk Minimization

For the commodity market there are two rather important issues to study that are problematic. The first one is that, in the commodity market, derivatives with rather long time to maturity (up to 30 years) are common. Accordingly, pricing and hedging of long dated derivatives is required, which is a challenging task for both practitioners and academics. The benchmark approach of Platen [2002] provides a robust way to price long dated commodity derivatives using the real world probability measure and the numéraire portfolio (NP) as numéraire. The second important issue is that contingent claims in the commodity market are usually not perfectly hedgeable and the market is incomplete. For example, Trolle and Schwartz [2009] observe and model in the crude oil market the unspanned stochastic volatility, which is defined as the volatility that cannot be perfectly hedged away by trading futures contracts only. One intuitively appealing and practically feasible methodology to price and hedge commodity derivatives under such circumstances is given by the classical risk minimization approach of Föllmer and Sondermann [1986], Föllmer and Schweizer [1989], Schweizer [1991] and Schweizer [1999]. However, this approach relies on the existence of an equivalent risk neutral probability measure for the underlying model, which we later explain to be a restrictive assumption, which should be better avoided.

This chapter provides a methodology, which merges the ideas of the classical risk minimization approach of Föllmer-Sondermann-Schweizer and the benchmark approach. Symmetry with respect to all primary security accounts, including the domestic savings account, will be achieved. Second moment conditions will be avoided. The resulting pricing rule will be that of real world pricing with the numéraire portfolio (NP) as numéraire and the real world probability measure as pricing measure. The minimal possible price for a contingent claim will be identified. The remaining benchmarked profit and loss will form a local martingale, that is, it will be driftless. This local martingale starts at zero and is orthogonal to all benchmarked self-financing portfolios, in the sense that its product with such a portfolio forms a local martingale, that is, this product becomes driftless. Consequently, the total benchmarked profit and loss process of a trading book, with an increasing number of sufficiently different benchmarked profit and loss processes, becomes then in total asymptotically negligible. This practically important property is highly desirable and crucial from a risk management point of view. We will see that benchmarked risk minimization is the least expensive pricing and hedging method. Moreover, based on this concept, the resulting total benchmarked profit and loss of an increasing number of sufficiently different, not fully replicable benchmarked contingent claims can be removed asymptotically via diversification. Furthermore, we will see that a benchmarked risk minimizing hedging strategy minimizes the fluctuations of the benchmarked nonhedgeable part of a benchmarked contingent claim in various intuitively appealing ways. Remarkably, it takes evolving information about the nonhedgeable part of a nonhedgeable claim into account, whereas classical risk minimization ignores such information.

In the following, the concept of benchmarked risk minimization will be presented for semimartingale markets. When applied, it is naturally separated into two steps: First, the calculation of the real world conditional expectation of the benchmarked contingent claim provides the minimal possible price. Second, the drift of the product of the benchmarked profit and loss with each benchmarked portfolio process is set to zero, which makes the fluctuations of the benchmarked profit and loss process minimal, in the sense that traded benchmarked uncertainty is orthogonal to the benchmarked profit and loss. An equivalent risk neutral probability measure is not required, as well as, square integrability and the martingale property for benchmarked profit and loss or its product with benchmarked portfolios are not requested. This opens a considerably wider modeling world than available under the classical approach and allows one to handle very general contingent claims.

This chapter is organized as follows: Section 2.1 presents a general semimartingale market. In Section 2.2 the real world pricing formula is derived. Section 2.3 considers benchmarked profit and loss processes. The new concept of benchmarked risk minimization is proposed in Section 2.4. Section 2.5 links martingale representations to benchmarked risk minimization. In Section 2.6 a jump diffusion market is introduced. Section 2.7 discusses illustrative examples, covering also the case of random jump size and infinite jump intensity. Section 2.8 concludes this chapter.

2.1 Financial Market

Within this chapter we model a semimartingale financial market in continuous time. Consider a filtered probability space $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$ that satisfies the usual conditions, as described in Protter [2005]. Here, the sigma field $\mathcal{F}_t$ models the information available at time $t \in [0, \infty)$. The filtration $\underline{\mathcal{F}} = (\mathcal{F}_t)_{t \in [0,\infty)}$ describes the evolution of market information over time. P denotes the real world probability measure. In this market we consider $d \in \{1, 2, \ldots\}$ adapted, non-negative assets, which we call primary security accounts, where all interests and dividends are reinvested. We assume that a numéraire portfolio (NP) exists such that every nonnegative primary security account process $\hat{S}^j = \{\hat{S}^j_t, t \in [0, \infty)\}$, $j \in \{1, \ldots, d\}$, when expressed in units of the NP, forms an $(\underline{\mathcal{F}}, P)$-local martingale and, thus, an $(\underline{\mathcal{F}}, P)$-supermartingale, see e.g. Platen and Heath [2010]. Karatzas and Kardaras [2007] provide general conditions for the existence of a NP. Hereafter, we refer to prices, when denominated in units of the NP, as benchmarked prices. It is worthwhile to note that the NP, when benchmarked, is trivially the constant one.

We denote by $S^{i,i}_t$ the ith primary security account value at time $t \in [0, \infty)$, when denominated in units of the ith security itself, $i \in \{1, \ldots, d\}$. In the case of the ith currency as security, $S^{i,i}_t$ denotes its currency savings account in units of this currency. In the case of the ith stock as security, $S^{i,i}_t$ denotes its share savings account in units of its shares. Then the NP $S^{i,\delta*}_t$, when denominated in units of the ith security, can be expressed by the ratio

$$S_t^{i,\delta*} = \frac{S_t^{i,i}}{\hat{S}_t^i} \quad , \tag{2.1}$$

for $t \in [0, \infty)$, $i \in \{1, \dots, d\}$. Consequently, the jth primary security account $S_t^{i,j}$, when denominated at time t in units of the ith security, can be obtained as the product

$$S_t^{i,j} = \hat{S}_t^j S_t^{i,\delta*} \quad , \tag{2.2}$$

for $i, j \in \{1, \dots, d\}$, $t \in [0, \infty)$.

The market participants can combine primary security accounts to form portfolios. Denote by $\boldsymbol{\delta} = \{\boldsymbol{\delta}_t = (\delta_t^1, \dots, \delta_t^d)^\top, t \in [0, \infty)\}$ the strategy, where δ_t^j, $j \in \{1, \dots, d\}$, represents the number of units of the jth primary security account that are held at time t in a corresponding portfolio. When denominated in units of the NP, this benchmarked portfolio is denoted by the process $\hat{S}^\delta = \{\hat{S}_t^\delta, t \in [0, \infty)\}$, where

$$\hat{S}_t^\delta = \sum_{j=1}^d \delta_t^j \hat{S}_t^j \tag{2.3}$$

for $t \in [0, \infty)$. As already mentioned, the benchmarked NP is the constant one.

If changes in the value of a portfolio are only due to changes in the values of the primary security accounts, then no extra funds flow in or out of the portfolio, and the corresponding strategy is called *self-financing*. This can be expressed as follows:

$$\hat{S}_t^\delta = \hat{S}_0^\delta + \sum_{j=1}^d \int_0^t \delta_s^j d\hat{S}_s^j \ , \tag{2.4}$$

for all $t \in [0, \infty)$. Since each benchmarked primary security account process $\hat{S}^j$, $j \in \{1, \dots, d\}$, is a local martingale, the benchmarked self-financing portfolio $\hat{S}^\delta$ is also a local martingale. A benchmarked non-negative, self-financing portfolio is, therefore, a supermartingale by Fatou's Lemma. This reflects the defining property of the NP of being the strictly positive portfolio that when used as benchmark makes all benchmarked nonnegative portfolios supermartingales, see Long [1990], Becherer [2001] and Karatzas and Kardaras [2007].

Dynamic trading strategies that may not be self-financing are crucial for risk management. Obviously, not all strategies can be allowed. It is sensible to focus on strategies that acknowledge the fact that the NP is the "best" performing portfolio, in the sense that they yield benchmarked nonnegative price processes that are supermartingales. Furthermore, denote by $[\hat{\mathbf{S}}] = \{[\hat{\mathbf{S}}]_t = ([\hat{S}^i, \hat{S}^j]_t)_{i,j=1}^d, t \in [0, \infty)\}$ the matrix valued optional covariance process, see Protter [2005], of the vector process of benchmarked primary security accounts $\hat{\mathbf{S}} = \{\hat{\mathbf{S}}_t = (\hat{S}_t^1, \dots, \hat{S}_t^d)^\top, t \in [0, \infty)\}$. Now, let us introduce a class of strategies that can form non-self-financing portfolios.

Definition 2.1.1. *A* dynamic trading strategy $\boldsymbol{v}$, *initiated at time* $t = 0$, *is an* $\mathbf{R}^{d+1}$*-valued stochastic process* $\boldsymbol{v} = \{\boldsymbol{v}_t = (\eta_t, \vartheta_t^1, \dots, \vartheta_t^d)^\top, t \in [0, \infty)\}$, *where* $\boldsymbol{\vartheta} = \{\boldsymbol{\vartheta}_t = (\vartheta_t^1, \dots, \vartheta_t^d)^\top, t \in [0, \infty)\}$ *describes the number of units invested in the benchmarked primary security accounts. The benchmarked price process* $\hat{V}^{\boldsymbol{v}} = \{\hat{V}_t^{\boldsymbol{v}}, t \in [0, \infty)\}$ *of the associated portfolio is given by*

$$\hat{V}_t^{\boldsymbol{v}} = \boldsymbol{\vartheta}_t^\top \hat{\mathbf{S}}_t + \eta_t \tag{2.5}$$

at time $t \in [0, \infty)$*. Here* $\boldsymbol{\vartheta}$ *is assumed to be an* $\mathbf{R}^d$*-valued, predictable process satisfying*

$$\int_0^t \boldsymbol{\vartheta}_u^\top d[\hat{\mathbf{S}}]_u \boldsymbol{\vartheta}_u < \infty \tag{2.6}$$

for all $t \in [0, \infty)$*. The adapted, scalar process* $\eta = \{\eta_t, t \in [0, \infty)\}$*, starting with* $\eta_0 = 0$*, monitors the benchmarked non-self-financing part of the right-continuous benchmarked price process* $\hat{V}^{\boldsymbol{v}}$*, so that*

$$\hat{V}_t^{\boldsymbol{v}} = \hat{V}_0^{\boldsymbol{v}} + \int_0^t \boldsymbol{\vartheta}_s^\top d\hat{\mathbf{S}}_s + \eta_t \quad , \tag{2.7}$$

for $t \in [0, \infty)$*. Here* $\hat{V}^{\boldsymbol{v}}$ *is assumed to be a supermartingale and the stochastic integral in (2.7) is interpreted as a vector Itô integral.*

With the above notion of dynamic trading strategy one can generate a wide range of benchmarked price processes. Later we will restrict the class of admissible dynamic trading strategies for the purpose of benchmarked risk minimization.

The benchmarked gains from trade during the time interval $[0,t]$, $t \in [0,\infty)$, from holding ϑ_s^j units of the jth primary security account for $j \in \{1,\ldots,d\}$ at time s, $s \in [0,t]$, are given by the vector Itô integral

$$\int_0^t \boldsymbol{\vartheta}_s^\top d\hat{\mathbf{S}}_s \quad . \tag{2.8}$$

We emphasize that a dynamic trading strategy generates via its self-financing part (2.8) benchmarked gains from trade in a manner that does not require outside funds and also does not generate extra funds. However, in general, capital has to be added or removed from a portfolio so that its benchmarked value matches the value of the benchmarked price process $\hat{V}^{\boldsymbol{v}}$. We will see that for risk management purposes it is enough to monitor the units of the NP that have to be added or removed from the portfolio to match a desired price process without being required to hold these units physically.

The predictability of the integrand in the benchmarked gains from trade, (2.8), expresses the real informational constraint that the allocation expressed in ϑ^j is not allowed to anticipate the movements of $\hat{\mathbf{S}}^j$. This predictability is also theoretically needed (but not sufficient) for the integrand in (2.8) to yield a proper vector Itô integral with respect to the vector of benchmarked primary security account processes. The process η needs only to be adapted, which is less restrictive than the predictability required for the components of the process $\boldsymbol{\vartheta}$. This adaptivity will be a feature of the concept of benchmarked risk minimization, to be introduced in Section 5.

Via the process η the investor monitors the adapted, cumulative "virtual" capital inflow and outflow from the portfolio. In previous work by Föllmer and Sondermann [1986] and Schweizer [1999], a similar adapted process was employed for describing the holdings in their numéraire, the domestic savings account. This choice of numéraire creates some asymmetry in the requested measurability properties among all primary security accounts. The dynamic trading strategies introduced in Definition 2.1.1 monitor the inflow and outflow of extra capital in units of the NP. This choice of numéraire brings all primary security accounts, including the domestic savings account, into comparable positions.

Note, if there is no inflow or outflow of capital in a dynamic trading strategy, then one deals with a self-financing portfolio, as described in (2.4). More generally, when allowing extra capital inflows and outflows, one obtains directly from Definition 2.1.1 the following result:

Corollary 2.1.2. *For a dynamic trading strategy $\boldsymbol{v} = \{\boldsymbol{v}_t = (\eta_t, \vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0, \infty)\}$, as introduced in Definition 2.1.1, the benchmarked portfolio value is given by*

$$\hat{V}_t^{\boldsymbol{v}} = \hat{S}_t^{\delta} = \sum_{j=1}^{d} \delta_t^j \hat{S}_t^j \tag{2.9}$$

with

$$\delta_t^j = \vartheta_t^j + \eta_t \delta_{*,t}^j \quad . \tag{2.10}$$

Here $\delta_{*,t}^j$, $j \in \{1, \ldots, d\}$, denotes the number of units of the jth primary security account in the NP at time t. In particular, we have

$$1 = \hat{S}_t^{\delta *} = \sum_{j=1}^{d} \delta_{*,t}^j \hat{S}_t^j \quad , \tag{2.11}$$

for $t \in [0, \infty)$.

Note that δ^j is not predictable, in general, since η is only adapted. This represents a departure from classical risk minimizing strategies. Under classical risk minimization the dynamic trading strategy for the same contingent claim resulting from, say, a US dollar denominated risk neutral price process, would involve measurability properties different to those of a Euro denominated risk neutral price process. From a theoretical point of view, this asymmetry is undesirable, and hence avoided here.

2.2 Real World Pricing

The main aim of hedging is risk minimization for the delivery of a targeted payoff via some dynamic trading strategy. Fix a bounded stopping time $T > 0$, and let $\mathcal{L}^1(\mathcal{F}_T)$ denote the set of integrable $\mathcal{F}_T$-measurable random variables.

Definition 2.2.1. *For a bounded stopping time $T \in (0, \infty)$ a nonnegative payoff $\hat{H}_T \in \mathcal{L}^1(\mathcal{F}_T)$, denominated in units of the NP, is called a* benchmarked contingent claim.

Since one can decompose a general payoff into its nonnegative and negative part, there is no real restriction imposed when considering in Definition 2.2.1 nonnegative payoffs.

Definition 2.2.2. *We say, a dynamic trading strategy*

$$\boldsymbol{v} = \{\boldsymbol{v}_t = (\eta_t, \vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0, \infty)\}$$

delivers *the benchmarked contingent claim $\hat{H}_T$ if*

$$\hat{V}_T^{\boldsymbol{v}} = \hat{S}_T^\delta = \hat{H}_T \tag{2.12}$$

P-a.s. A benchmarked contingent claim is called replicable *if there exists a self-financing dynamic trading strategy $\boldsymbol{v}$ that delivers the claim.*

There may exist several self-financing strategies that deliver a given benchmarked contingent claim. Examples can be found in Platen and Heath [2010]. The defining property of the NP ensures that all non-negative, self-financing portfolios when benchmarked are supermartingales. Since the minimal nonnegative supermartingale is a martingale, see Revuz and Yor [1999], one obtains the following result:

Proposition 2.2.3. *If for a given benchmarked contingent claim $\hat{H}_T$ a self-financing benchmarked portfolio $\hat{S}^{\delta_{\hat{H}_T}}$ exists, which equals the martingale satisfying the respective* real world pricing formula

$$\hat{S}_t^{\delta_{\hat{H}_T}} = E(\hat{H}_T | \mathcal{F}_t) \tag{2.13}$$

for all $t \in [0, T]$ P-a.s., then this portfolio yields the least expensive replication of $\hat{H}_T$.

Note that equation (2.13) provides the minimal price. In general, contingent claims may be not fully replicable. We will see in Section 5 that the above real world pricing formula (2.13) also makes sense for non-replicable claims.

2.3 Benchmarked Profit and Loss

Risk can be reduced by hedging and diversification. Hedging a non-replicable contingent claim usually results in a hedge error. The current paper aims to identify the least expensive way of delivering contingent claims through hedging, while asymptotically reducing the total hedge error in a large trading book. The following notion allows us to keep track of hedge errors.

Definition 2.3.1. *For a dynamic trading strategy $\boldsymbol{v} = \{\boldsymbol{v}_t = (\eta_t, \vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0, \infty)\}$, with benchmarked price $\hat{V}_t^{\boldsymbol{v}} = \hat{S}_t^\delta = \sum_{j=1}^d \delta_t^j \hat{S}_t^j$ at time $t \in [0, \infty)$, the* benchmarked profit and loss *(P&L) process $\hat{C}^\delta = \{\hat{C}_t^\delta, t \in [0, \infty)\}$ is defined by*

$$\hat{C}_t^\delta = \hat{S}_t^\delta - \sum_{j=1}^d \int_0^t \vartheta_u^j d\hat{S}_u^j - \hat{S}_0^\delta \tag{2.14}$$

for $t \in [0, \infty)$.

One obtains directly from Definition 2.3.1 with Definition 2.1.1 the following statement:

Corollary 2.3.2. *For a dynamic trading strategy $\boldsymbol{v} = \{\boldsymbol{v}_t = (\eta_t, \vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0, \infty)\}$ the corresponding benchmarked P&L process $\hat{C}^\delta = \{\hat{C}_t^\delta, t \in [0, \infty)\}$ coincides with the adapted process $\eta = \{\eta_t, t \in [0, \infty)\}$ that monitors the cumulative inflow and outflow of extra capital.*

For simplicity, in the current paper the hedging and, thus, the benchmarked P&L process $\hat{C}^\delta$ for a given benchmarked portfolio $\hat{S}^\delta$ is assumed to start at the initial time $t = 0$. Therefore, the benchmarked P&L has initial value $\hat{C}_0^\delta = \eta_0 = 0$ and monitors at time t with $\hat{C}_t^\delta = \eta_t$ the adapted accumulated benchmarked capital that flew in or out of the benchmarked portfolio process $\hat{V}^{\boldsymbol{v}} = \hat{S}^{\delta}$ until this time. In other words, $\hat{C}^{\delta}_t$ represents the benchmarked external costs incurred by the portfolio $\hat{S}^{\delta}$ over the time period $[0, t]$ after the hedge was set up at the initial time zero. Intuitively, the adapted process η can be interpreted as benchmarked hedge error.

If one has to deliver a general claim, one faces a fluctuating P&L process and, thus, an intrinsic risk that needs to be controlled. For implementing systematically such a control one can introduce a criterion to obtain a desirable behavior of the benchmarked P&L process. The question is, what criterion would be most appropriate from a risk management point of view when aiming for reasonable generality?

To get an idea, consider the case when benchmarked P&Ls are pooled in the large trading book of a global financial institution or insurance company and form independent, square integrable martingales that start at zero. By increasing the number of benchmarked P&Ls in such a trading book, it follows by the Law of Large Numbers, which refers to the real world probability measure, that the resulting total benchmarked P&L process will become asymptotically negligible. In this manner, the benchmarked total P&L of the large trading book can be asymptotically removed via diversification. The insight that such removal is possible will be crucial. We aim to capture this insight by the following remark:

Remark 2.3.3. *Benchmarked P&Ls should preferably be driftless and, thus, local martingales, starting at initiation with value zero.*

We will see later that benchmarked P&Ls will be automatically a local martingale because benchmarked traded wealth forms always local martingales and the minimal possible benchmarked price processes are martingales.

According to Remark 2.3.3, a benchmarked P&L should be locally in the mean self-financing. Mean-self-financing turns out to be an extremely useful notion, which was introduced in Schweizer [1991] when using the savings account as numéraire and employing an assumed risk neutral probability measure as pricing measure. In the current paper we have a different setting. Under the benchmark approach we use the NP as numéraire and the real world probability measure for taking expectations. Hence, the following notion will be important for the concept of benchmarked risk minimization:

Definition 2.3.4. *A dynamic trading strategy* $\boldsymbol{v} = \{\boldsymbol{v}_t = (\eta_t, \vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0, \infty)\}$ *is called* locally real world mean-self-financing *if its adapted process* η *is a local martingale starting at zero.*

This notion maintains symmetry with respect to all primary security accounts, including the domestic savings account. It avoids the restrictive assumption on the existence of an equivalent risk neutral probability measure since it uses the real world probability measure P. We will formalize the insight of Remark 2.3.3 in the next section when introducing the concept of benchmarked risk minimization by focusing on strategies that are locally real world mean-self-financing.

2.4 Benchmarked Risk Minimization

It is not immediately obvious how to price and hedge a general contingent claim, even when taking into account the observations made above. Conceptually, there exist many ways to hedge a non-replicable claim, and a wide range of literature has emerged. An intuitively appealing and practically useful concept is classical risk minimization, pioneered by Föllmer and Sondermann [1986] and further developed in Föllmer and Schweizer [1989] and Schweizer [1991, 1995]. Schweizer [1999] provides an excellent survey.

Under classical risk minimization the hedging strategy is implemented via a savings account discounted portfolio, which is assumed to form a square integrable martingale under an assumed equivalent risk neutral probability measure. The fluctuations of discounted P&L processes are measured by employing a quadratic criterion on discounted price processes, where a "good" strategy turns out to be mean-self-financing under the assumed risk neutral probability measure.

Most importantly, Föllmer and Sondermann [1986] and Schweizer [1991, 1995] linked the optimization problem of risk minimization to the well-known Kunita-Watanabe decomposition, see Schweizer [1999]. This crucial decomposition became known as *Föllmer-Schweizer decomposition* in the context of pricing and hedging in incomplete markets. The Föllmer-Schweizer decomposition has been extensively studied by several authors in the literature. Conditions for its existence have been given, for instance, in Buckdahn [1993], Schweizer [1994], Stricker [1996], Delbaen, Monat, Schachermayer, Schweizer, and Stricker [1997] and Pham, Rheinländer, and Schweizer [1998].

It should be mentioned that Schweizer [1991, 1995] introduced the interesting concept of local risk minimization, which employs, a local in time quadratic criterion, see Schweizer [1999]. Other authors, including Biagini, Guasoni, and Pratelli [2000], generalized the concept of local risk minimization to markets with event risk. Biagini, Cretarola, and Platen [2011] proposed a version of local risk minimization that uses the real world probability measure as pricing measure and considers pricing under partial information. Earlier, Bouleau and Lamberton [1989] derived a pricing and hedging methodology for Markovian asset price processes under a quadratic criterion, which is, in some sense, related to local risk minimization. Further results in similar directions can be found, in Duffie and Richardson [1991] and Schweizer [1994].

The current paper is of a conceptual nature, and proposes a pricing and hedging approach in the spirit of classical risk minimization, but under the real world probability measure with the NP as numéraire. It avoids the requirement of the existence of an equivalent risk neutral probability measure, and aims for minimal price processes. Furthermore, in a large trading book it will allow for the asymptotic removal of nonhedgeable risk via diversification, as indicated prior to Remark 2.3.3. It also will provide symmetry with respect to all primary security accounts, and square integrability assumptions will be avoided. Finally, evolving information about nonhedgeable uncertainty will be taken into account during hedging, which classical risk minimization ignores.

Since the new concept will require only very weak assumptions, it will permit the handling of more general financial market models and more general contingent claims than covered under classical risk minimization. The main requirement is the existence of the NP, which is a very weak assumption, as can be seen e.g. in Karatzas and Kardaras [2007].

Recall from Definition 2.1.1 that dynamic trading strategies form benchmarked nonnegative price processes that are consistent with the fact that the NP is the "best" performing portfolio in the sense that benchmarked price processes form supermartingales. Furthermore, note at this stage that for a given benchmarked price process a corresponding dynamic trading strategy remains potentially exposed to some ambiguity concerning what forms its self-financing part and what constitutes its non-self-financing part, see equation (2.10). This ambiguity will be removed by focusing below on benchmarked P&Ls with fluctuations that are "orthogonal" to those of any self-financing benchmarked portfolio, and thus, intuitively have no chance to be removed via hedging. To formalize this we introduce the following notion:

Definition 2.4.1. *A dynamic trading strategy* $\boldsymbol{v} = \{\boldsymbol{v}_t = (\eta_t, \vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0,\infty)\}$ *has an* orthogonal *benchmarked P&L* $\eta = \{\eta_t, t \in [0,\infty)\}$ *if* η *is orthogonal to benchmarked traded wealth in the sense that* $\eta_t \int_0^t \bar{\boldsymbol{\vartheta}}_s^\top d\hat{\mathbf{S}}_s$ *is a local martingale for every predictable self-financing strategy* $\bar{\boldsymbol{\vartheta}} = \{\bar{\boldsymbol{\vartheta}}_t = (\bar{\vartheta}_t^1, \ldots, \bar{\vartheta}_t^d)^\top, t \in [0,\infty)\}$ *satisfying (2.6).*

In some sense, from an orthogonal benchmarked P&L all hedgeable uncertainty is removed. To fix the so far identified desirable properties of dynamic trading strategies, let us define the following set:

Definition 2.4.2. *Let* $\hat{\mathcal{V}}_{\hat{H}_T}$ *be the set of locally real world mean-self-financing dynamic trading strategies* $\boldsymbol{v}$*, which deliver* $\hat{H}_T$ *with orthogonal benchmarked P&L and corresponding benchmarked price* $\hat{V}_t^{\boldsymbol{v}} = \hat{S}_t^\delta$ *for all* $t \in [0,T]$*, satisfying (2.7)-(2.10).*

Note that we will later see that e.g. for Markovian jump diffusion markets $\hat{\mathcal{V}}_{\hat{H}_T}$ is not empty.

There may exist several nonnegative benchmarked hedge portfolios that could deliver a given benchmarked contingent claim. The following concept of *benchmarked risk minimization* selects the most economical benchmarked price process, which is the least expensive possible price process with the above identified desirable properties:

Definition 2.4.3. *A dynamic trading strategy* $\widetilde{\boldsymbol{v}} = \{\widetilde{\boldsymbol{v}}_t = (\widetilde{\eta}_t^1, \widetilde{\vartheta}_t^1, \ldots, \widetilde{\vartheta}_t^d)^\top, t \in [0,T]\} \in \hat{\mathcal{V}}_{\hat{H}_T}$ *with corresponding benchmarked price process* $\hat{V}^{\widetilde{\boldsymbol{v}}} = \hat{S}^{\widetilde{\delta}}$ *is called* benchmarked risk minimizing *(BRM) if for all dynamic trading strategies* $\boldsymbol{v} \in$

$\hat{\mathcal{V}}_{\hat{H}_T}$, *with* $\hat{S}^{\delta}_t$ *satisfying* (2.9), *the price* $\hat{S}^{\tilde{\delta}}_t$ *is minimal in the sense that*

$$\hat{S}^{\tilde{\delta}}_t \leq \hat{S}^{\delta}_t \tag{2.15}$$

P*-a.s. for all* $t \in [0, T]$.

As required by relation (2.15) and similarly as in Section 3, we can exploit the fact that the martingale among the nonnegative supermartingales contained in $\hat{\mathcal{V}}_{\hat{H}_T}$ yields the minimal possible benchmarked price process, see Revuz and Yor [1999]. We specify below this intuitive and practically useful insight.

2.5 Regular Benchmarked Contingent Claims

To utilize efficiently the above introduced concept of BRM strategies for hedging, it will be extremely useful to have access to corresponding martingale representations, similar as in the classical case, see Schweizer [1999]. Such representations would have to be analogous to the previously mentioned Föllmer-Schweizer decomposition. We emphasize, in the current paper we will use martingale representations for benchmarked continent claims under the real world probability measure and not under some assumed risk neutral probability measure.

Unfortunately, martingale representations cannot be easily mathematically guaranteed for general semimartingale markets. Systematic results in this direction can be found, for instance, in Jacod, Meleard, and Protter [2000]. Fortunately, martingale representations exist for most integrable benchmarked contingent claims in most Markovian market models, as will be demonstrated in Section 2.7.1. Since a martingale representation of a benchmarked contingent claim, which separates the hedgeable and the orthogonal nonhedgeable part, is crucial for practical hedging, we introduce the following notion:

Definition 2.5.1. *We call a benchmarked contingent claim* $\hat{H}_T \in \mathcal{L}^1(\mathcal{F}_T)$ regular *if it has for all* $t \in [0, T]$ *the following representation:*

$$\hat{H}_T = E(\hat{H}_T | \mathcal{F}_t) + \sum_{j=1}^{d} \int_t^T \vartheta^j_{\hat{H}_T}(s) d\hat{S}^j_s + \eta_{\hat{H}_T}(T) - \eta_{\hat{H}_T}(t) \tag{2.16}$$

P-a.s. with some predictable vector process

$$\boldsymbol{\vartheta}_{\hat{H}_T} = \{\boldsymbol{\vartheta}_{\hat{H}_T}(t) = (\vartheta^1_{\hat{H}_T}(t), \ldots, \vartheta^d_{\hat{H}_T}(t))^\top,\ t \in [0,T]\}$$

satisfying (2.6), and some local martingale $\eta_{\hat{H}_T} = \{\eta_{\hat{H}_T}(t), t \in [0,T]\}$ with

$$\eta_{\hat{H}_T}(0) = 0 \quad .$$

Furthermore, for any predictable process $\boldsymbol{\vartheta} = \{\boldsymbol{\vartheta}_t = (\vartheta^1_t, \ldots, \vartheta^d_t)^\top, t \in [0,T]\}$, satisfying (2.6), the product process $Z^{\boldsymbol{\vartheta},\hat{H}_T} = \{Z^{\boldsymbol{\vartheta},\hat{H}_T}_t, t \in [0,T]\}$ with

$$Z^{\boldsymbol{\vartheta},\hat{H}_T}_t = \eta_{\hat{H}_T}(t) \sum_{j=1}^{d} \int_0^t \vartheta^j_s d\hat{S}^j_s \ , \tag{2.17}$$

$t \in [0,T]$, is a local martingale.

It will be illustrated in Section 2.7.1 that by employing Markovian factor models one can systematically obtain martingale representations for benchmarked contingent claims, which can be specified such that these are regular. By combining Definition 2.4.3 and Definition 2.5.1, the above introduced concept of benchmarked risk minimization allows us to obtain in a straightforward manner the following statement:

Corollary 2.5.2. *For a regular benchmarked contingent claim $\hat{H}_T \in \mathcal{L}^1(\mathcal{F}_T)$ there exists a BRM strategy $\boldsymbol{v} = \{\boldsymbol{v}_t = (\eta_{\hat{H}_T}(t), \vartheta^1_{\hat{H}_T}(t), \ldots, \vartheta^{d+1}_{\hat{H}_T}(t))^\top, t \in [0,T]\} \in \hat{\mathcal{V}}_{\hat{H}_T}$ with corresponding benchmarked portfolio process $\hat{S}^{\delta_{\hat{H}_T}} = \hat{V}^{\boldsymbol{v}}$, satisfying (2.9) that delivers the benchmarked contingent claim $\hat{S}^{\delta_{\hat{H}_T}}_T = \hat{V}^{\boldsymbol{v}}_T = \hat{H}_T$ P-a.s. The benchmarked price at time $t \in [0,T]$ is determined by the real world pricing formula*

$$\hat{S}^{\delta_{\hat{H}_T}}_t = \hat{V}^{\boldsymbol{v}}_t = E(\hat{H}_T | \mathcal{F}_t) \quad , \tag{2.18}$$

yielding within the set of strategies $\hat{\mathcal{V}}_{\hat{H}_T}$ the minimal possible price process. The resulting benchmarked P&L at time $t \in [0,T]$ is given by

$$\hat{C}^{\delta_{\hat{H}_T}}_t = \eta_{\hat{H}_T}(t) \quad . \tag{2.19}$$

This process is orthogonal to benchmarked traded wealth in the sense that the product $\hat{C}_t^{\delta_{\hat{H}_T}} \int_0^t \boldsymbol{\vartheta}_s^\top d\hat{\mathbf{S}}_s$ is an $(\underline{\mathcal{F}}, P)$-local martingale for every predictable self-financing strategy $\boldsymbol{\vartheta} = \{\boldsymbol{\vartheta}_t = (\vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0,T]\}$, satisfying (2.6). In terms of the martingale representation (2.16), the components of the strategy $\boldsymbol{v}$ are obtained by the number $\eta_{\hat{H}_T}(t)$ of units of the NP to be monitored in the nonhedgeable part of $\hat{H}_T$, and the number of units $\vartheta^j_{\hat{H}_T}(t)$ of the jth primary security account, $j \in \{1, 2, \ldots, d\}$, $t \in [0,T]$, to be held at time t in the self-financing hedgeable part of $\hat{H}_T$.

The self-financing hedgeable part of the benchmarked price process $\hat{S}_t^{\delta_{\hat{H}_T}}$ is a local martingale and has at time $t \in [0,T]$ the benchmarked value

$$E(\hat{H}_T|\mathcal{A}_t) - \eta_{\hat{H}_T}(t) = \sum_{j=1}^{d} \int_0^t \vartheta^j_{\hat{H}_T}(s) d\hat{S}^j_s \quad . \tag{2.20}$$

The vector of units $\boldsymbol{\vartheta}_{\hat{H}_T}(t) = (\vartheta^1_{\hat{H}_T}(t) \ldots, \vartheta^d_{\hat{H}_T}(t))^\top$ to be held in the primary security accounts follows by making the benchmarked P&L orthogonal to benchmarked traded wealth, as will be illustrated in Section 8. Due to the possible presence of redundant primary security accounts, the self-financing strategy $\boldsymbol{\vartheta}_{\hat{H}_T}$ may not be unique.

We emphasize that BRM strategies yield for not fully replicable contingent claims the real world pricing formula (2.18), which we have derived earlier for replicable contingent claims in (2.13). It can be shown, similarly as in Platen and Heath [2010], that utility indifference pricing for not fully replicable benchmarked contingent claims, in the sense of Davis [1997], also yields the real world pricing formula (2.18). As demonstrated in Platen and Heath [2010], actuarial pricing can be shown to follow directly from real world pricing.

The above results demonstrate that, based on the existence of a martingale representation for a regular benchmarked contingent claim $\hat{H}_T$, one obtains via benchmarked risk minimization a unique minimal price process together with a hedging strategy that makes its benchmarked price process an $(\underline{\mathcal{F}}, P)$-martingale. The benchmarked P&L of a regular benchmarked contingent claim is a local martingale and orthogonal to any benchmarked self-financing portfolio, in the sense that their product becomes a local martingale.

We emphasize that by avoiding quadratic criteria, BRM strategies do not request square integrability of benchmarked quantities. The benchmarked self-financing hedgeable part and also the benchmarked P&L do only need to be local martingales. Together with the removal of the request on the existence of an equivalent risk neutral probability measure, this widens further the applicability of the fundamental ideas on risk minimization, originally formulated in Föllmer and Sondermann [1986] and Schweizer [1991].

We will illustrate later in a continuous financial market with an equivalent minimal martingale measure, see Schweizer [1999], that the corresponding Radon-Nikodym derivative is the normalized benchmarked savings account, and a BRM strategy yields the same price process as follows under classical risk minimization. However, the hedging strategy is typically different after initiation of the hedge. Classical risk minimization singles out the equivalent minimal martingale measure as pricing measure and minimizes the respective second moment of the discounted hedge error. This makes the discounted hedge error orthogonal to discounted traded wealth in a risk neutral sense. BRM strategies rely on the NP as numéraire, which makes the real world probability measure the pricing measure, and choose the hedge such that the benchmarked hedge error, when multiplied with traded wealth, does not create locally any trend for the product process. This rather general form of orthogonality of benchmarked hedge errors allows one to remove in a large trading book the total benchmarked hedge error via diversification.

2.6 A Jump Diffusion Market

To illustrate the proposed concept of benchmarked risk minimization, it is best to study a particular financial market and some examples. For this purpose, we introduce in the following a rather general jump diffusion market, similar to the one described in Christensen and Platen [2005]. Here the continuous part of the corresponding uncertainty is modeled by independent Brownian motions and the corresponding jumps are modeled by a jump measure.

To be specific, consider on the given filtered probability space $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$ an

$\underline{\mathcal{F}}$-adapted, m'-dimensional Wiener process

$$\mathbf{W} = \{\mathbf{W}_t = (W_t^1, \dots, W_t^m, W_t^{m+1}, \dots, W_t^{m'})^\top, t \in [0, \infty)\} \ ,$$

$m \in \{0, 1, \dots, m'-1\}$, which models all continuous uncertainty. The event driven uncertainty is modeled by the Poisson type jump measure $p(dv, dt)$ with corresponding intensity measure $\phi(dv, t)dt$ for the given mark space $(\mathcal{E}, B(\mathcal{E}))$ with $\mathcal{E} \subseteq R \setminus \{0\}$ and $t \in [0, \infty)$. Here the jump intensity measure $\phi(A, t)$, $t \in [0, \infty)$, is for all $A \subseteq \mathcal{E}$ assumed to form a predictable process. The corresponding jump martingale measure $q(\cdot, \cdot)$ is then characterized by the equation $q(dv, dt) = p(dv, dt) - \phi(dv, t)dt$.

Assume that there exist d primary security accounts. The ith primary security account $S_t^{i,i}$ at time t, when denominated in units of the ith security itself, is assumed to form the locally riskless ith savings account in ith security denomination, given by the expression

$$S_t^{i,i} = S_0^{i,i} \exp\left\{\int_0^t r_s^i ds\right\} \tag{2.21}$$

for all $t \in [0, \infty)$, with initial value $S_0^{i,i} > 0$, $i \in \{1, \dots, d\}$. Here the adapted ith short rate process $r^i = \{r_t^i, t \in [0, \infty)\}$ models the dividend rate in the case of a stock, the interest rate in the case of a currency and the corresponding convenience yield in the case of a commodity. We assume, similarly as in Christensen and Platen [2005], that for $i \in \{1, \dots, d\}$ the numéraire portfolio (NP) exists with strictly positive, finite value $S_t^{i,\delta*}$ at time t, and this portfolio is tradable. When denominated in units of the ith security, its dynamics are given by the stochastic differential equation (SDE)

$$\begin{aligned} dS_t^{i,\delta*} = S_{t-}^{i,\delta*} \Bigg(& r_t^i dt + \sum_{k=1}^{m} (\theta_t^{i,k})(\theta_t^{i,k} dt + dW_t^k) \\ & + \int_{\mathcal{E}} \frac{\psi^i(v,t)}{1-\psi^i(v,t)} (\psi^i(v,t)\phi(dv,t)dt + q(dv,dt)) \Bigg) \end{aligned} \tag{2.22}$$

for all $t \in [0, \infty)$ and with $S_0^{i,\delta*} > 0$. Note that the Wiener processes $W^{m+1}, \dots, W^{m'}$ do not drive traded benchmarked wealth. This indicates that we model here also nontraded uncertainty.

The time t value $S_t^{i,j}$ of the jth primary security account, when denominated in units of the ith security, satisfies by (2.1) and (2.2) the equation

$$S_t^{i,j} = \frac{S_t^{j,j}}{S_t^{j,\delta*}} S^{i,\delta*} \tag{2.23}$$

for $t \in [0, \infty)$ and $i, j \in \{1, \dots, d\}$. Note that the ratio $\frac{S_t^{i,\delta*}}{S_t^{j,\delta*}}$ represents the exchange rate from the jth into the ith security denomination. It then follows by the Itô formula the SDE

$$\begin{aligned} dS_t^{i,j} &= S_{t-}^{i,j} \Bigg(r_t^i dt + \sum_{k=1}^{m} (\theta_t^{i,k} - \theta_t^{j,k})(\theta_t^{i,k} dt + dW_t^k) \\ &\quad + \int_{\mathcal{E}} \frac{(\psi^i(v,t) - \psi^j(v,t))}{1 - \psi^i(v,t)} \left(\psi^i(v,t)\phi(dv,t)dt + q(dv,dt) \right) \Bigg) \end{aligned} \tag{2.24}$$

for $t \in [0, \infty)$, $i, j \in \{1, 2, \dots, d\}$. The adapted ith short rate process $r^i = \{r_t^i, t \in [0, \infty)\}$, the predictable (i, k)th market price of risk process $\theta^{i,k} = \{\theta_t^{i,k}, t \in [0, \infty)\}$ and the ith predictable vth relative market price of event risk process $\psi^i(v, \cdot) = \{\psi^i(v, t), t \in [0, \infty)\}$ are assumed to satisfy the conditions

$$\int_0^t \left(|r_s^i| + |\theta_s^{ik}| + (\theta_s^{ik})^2 \right) ds < \infty \tag{2.25}$$

and

$$\int_0^t \left(\int_{\mathcal{E}} \left| \frac{\psi^i(v,t) - \psi^j(v,t)}{1 - \psi^i(v,t)} \right| \phi(dv,s) + \int_{\mathcal{E}} |\psi^i(v,s)| \phi(dv,s) \right) ds < \infty \tag{2.26}$$

P-a.s. for all $i, j \in \{1, \dots, d+1\}$, $k \in \{1, \dots, m\}$, $t \in [0, \infty)$. For all $i \in \{1, \dots, d\}$ and $v \in \mathcal{E}$ the process $\psi^i(v, \cdot)$ is assumed to be predictable and to satisfy the relation

$$\psi^i(v,t) < 1 \tag{2.27}$$

for Lebesgue-almost-every $t \in [0, \infty)$. Condition (2.27) avoids the explosion of the NP in the ith denomination.

Under the above parametrization no denomination in units of any security or currency gets preference. This is different to classical modeling, which focuses on the denomination of securities in units of the domestic currency or domestic savings account. It is worth mentioning that Flesaker and Hughston [2000] proposed a similar approach, where the notion of "natural numéraire" is introduced, which leads to prices as ratios.

In the resulting jump diffusion market the above d primary security accounts can model cum-dividend stocks, savings accounts of currencies or commodity accounts. All dividends, interests, costs or earnings are reinvested. We allow jump processes with random jump size and infinite activity such as Lévy processes.

For the jth benchmarked primary security account, $\hat{S}_t^j = \frac{S_t^{i,j}}{S^{i,\delta*}}$, $j, i \in \{1, \ldots, d\}$, it follows by the Itô formula the SDE

$$d\hat{S}_t^j = -\hat{S}_{t-}^j \left(\sum_{k=1}^m \theta_t^{j,k} dW_t^k + \int_{\mathcal{E}} \psi^j(v,t) q(dv,dt) \right) \quad . \tag{2.28}$$

Note that the benchmarked primary security accounts are local martingales, as required in Section 2. For any benchmarked self-financing portfolio $\hat{S}_t^\delta$ with $\boldsymbol{\delta} = \{\boldsymbol{\delta}_t = (\delta_t^1, \ldots, \delta_t^d)^\top, t \in [0,\infty)\}$, see (2.4), it follows the SDE

$$d\hat{S}_t^\delta = -\sum_{j=1}^d \delta_t^j \hat{S}_{t-}^j \left(\sum_{k=1}^m \theta_t^{j,k} dW_t^k + \int_{\mathcal{E}} \psi^j(v,t) q(dv,dt) \right) \quad . \tag{2.29}$$

Note that the SDE (2.29) is driftless and, thus, benchmarked self-financing portfolios form $(\underline{\mathcal{F}}, P)$-local martingales. Since nonnegative local martingales are supermartingales this confirms that the portfolio identified in (2.22) is the NP of the given jump diffusion market in the denomination of the ith security.

The above market model is rather general. It covers many jump diffusion markets driven by Wiener processes and Poisson processes with state dependent intensities, as well as, markets where jump measures model random jump sizes and infinite jump activity. From the modeling point of view, the above market description together with (2.1), (2.2) and (2.3) is focused on the modeling of the benchmarked primary security accounts and the respective short rates. This market characterization is sufficient for modeling many traded securities.

2.7 Illustrative Comments and Examples

We have so far not specified the evolution of nontraded uncertainty. This becomes necessary when modeling benchmarked contingent claims. It can be conveniently achieved when the state variables follow Markovian dynamics. As we will see, the concept of benchmarked risk minimization can be conveniently applied to benchmarked contingent claims in the above introduced jump diffusion market. The presence of infinite jump activities and random jump sizes does not create any limitation.

2.7.1 Regular Claims in Markovian Markets

For pricing and hedging it is not sufficient to only secure the existence of a martingale representation for a given benchmarked contingent claim. A martingale representation, as discussed e.g. in Karatzas and Shreve [1991] for payoffs in markets that are purely driven by Wiener processes, is theoretically valuable. However, it does not solve the practical problem of identifying prices and deriving hedge ratios because it is rarely readily exploitable for computations. To derive pricing functions and hedging strategies, one needs a complete quantitative description of a martingale representation, preferably with properties as given in Definition 2.5.1 for regular benchmarked contingent claims.

Multi-factor Markovian jump diffusion market models appear to be naturally suited for obtaining martingale representations with necessary quantitative details. A vector of factor processes, obtained as solution of a system of SDEs, characterizes typically a multi-factor Markovian jump diffusion market model. Such models can be constructed, for instance, by assuming that market prices of risk, relative market prices of event risk, short rates and intensity measures are twice differentiable functions of Markovian factor processes. By application of the Itô formula the mentioned financial quantities together with the benchmarked primary security prices become the solution of a corresponding Markovian system of SDEs. This type of structure is sufficient to characterize for a given benchmarked contingent claim, via the real world pricing formula, the corresponding benchmarked pricing function. The latter can be determined in some cases explicitly.

The well-known Feynman-Kac formula, links the conditional expectation appearing in the real world pricing formula (2.18) with a corresponding partial integro-differential equation. For instance, in Platen and Heath [2010], Section 9.7, a rather general version of the Feynman-Kac formula for a jump diffusion market is given. The application of the Itô formula to the benchmarked pricing function yields a martingale representation of the benchmarked contingent claim. Determining the benchmarked pricing function and the resulting martingale representation represents the first step in applying benchmarked risk minimization. In the second step, as will be demonstrated in examples, a regular benchmarked contingent claim is naturally obtained by selecting the benchmarked hedgeable part such that the benchmarked P&L of the benchmarked price process becomes orthogonal to all benchmarked self-financing portfolios.

Once a martingale representation for a regular benchmarked contingent claim is established, it is straightforward to deduce by Corollary 2.5.2 the respective BRM strategy. We will give below examples that illustrate the naturally emerging two-step methodology.

2.7.2 BRM Strategies and a Quadratic Criterion

The concept of benchmarked risk minimization avoids deliberately restrictive assumptions. We will see below that, intuitively, one can interpret it as the result of the minimization of the expected square of the benchmarked P&L if appropriate square integrability and martingale properties were in place. In this case orthogonality is directly linked to the minimization of some "distance" between the benchmarked price process and the benchmarked traded wealth, where the expected square of the benchmarked P&L measures this "distance". Many models and contingent claims we cover do not permit such direct and simple interpretation.

To support the above intuition let us consider a regular benchmarked contingent claim $\hat{H}_T$, with $T \in (0, \infty)$ fixed, and martingale representation given in (2.16). For illustration we assume that in (2.16) the terms $\sum_{j=1}^{d} \int_0^t \vartheta^j_{\hat{H}_T}(s) d\hat{S}^j_s$ and $\eta_{\hat{H}_T}(t)$ form independent, square integrable martingales. Additionally assume, for simplicity, that also $\hat{S}^1_t, \ldots, \hat{S}^d_t$ and $\eta_{\hat{H}_T}(t)$ are independent, square integrable martingales. The latter property guarantees conveniently that $\eta_{\hat{H}_T}$ is orthogonal to any traded benchmarked wealth, in the sense of Definition 2.4.1.

Assume now that $\hat{H}_T$ is square integrable so that we have a square integrable benchmarked price process $\hat{V}_t^{\boldsymbol{v}} = \hat{S}_t^{\delta}$, which delivers $\hat{H}_T$ at maturity T and is following a dynamic trading strategy $\boldsymbol{v} = \{\boldsymbol{v}_t = (\eta_t, \vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0, \infty)\}$ as in (2.9)-(2.10), where the self-financing part $\sum_{j=1}^d \int_0^t \vartheta_s^j d\hat{S}_s^j$ forms a square integrable martingale, independent of $\eta_{\hat{H}_T}$. The above mentioned quadratic criterion is minimizing the second moment of the benchmarked P&L. This quantity can be interpreted as a measure for the risk of the hedge. It captures some "distance" between the benchmarked P&L and traded wealth. This "distance" would be zero if the claim could be perfectly hedged. Analogous as in its geometric interpretation this "distance" would be minimal when the benchmarked P&L becomes "orthogonal" to the benchmarked traded wealth as captured by Definition 2.4.1. This is in spirit similar to classical risk minimization. However, we use in this paper the NP as numéraire and the real world probability measure when taking expectations. More precisely, we minimize, if possible, the distance

$$E\left((\hat{C}_T^{\delta})^2\right) = E\left((\hat{H}_T - \sum_{j=1}^d \int_0^T \vartheta_s^j d\hat{S}_s^j - \hat{S}_0^{\delta})^2\right) ,$$

which can be interpreted as a measure for the fluctuations of the benchmarked P&L. By exploiting the martingale representation (2.16) and the assumed independence and square integrability properties, it follows that

$$\begin{aligned} E\left((\hat{C}_T^{\delta})^2\right) &= E\left(\left(E(\hat{H}_T|\mathcal{F}_0) - \hat{S}_0^{\delta} + \sum_{j=1}^d \int_0^T (\vartheta_{\hat{H}_T}^j(s) - \vartheta_s^j) d\hat{S}_s^j + \eta_{\hat{H}_T}(T)\right)^2\right) \\ &= E\left(E(\hat{H}_T|\mathcal{F}_0) - \hat{S}_0^{\delta}\right)^2 + \sum_{j=1}^d E\left(\int_0^T (\vartheta_{\hat{H}_T}^j(s) - \vartheta_s^j)^2 d[\hat{S}^j]_s\right) + E((\eta_{\hat{H}_T}(T))^2). \end{aligned}$$

When minimizing the right hand side of the above equation it is obvious that the minimum can only be obtained when setting the benchmarked initial price to $\hat{S}_0^{\delta} = E(\hat{H}_T|\mathcal{F}_0)$, which is the price obtained by the real world pricing formula (2.18). Furthermore, taking the minimum requires choosing the self-financing part of the strategy such that $\vartheta_t^j = \vartheta_{\hat{H}_T}^j(t)$ for all $t \in [0, T]$ and $j \in \{1, \ldots, d\}$. The minimal possible second moment for the benchmarked P&L becomes then the second moment of the benchmarked nonhedgeable part $\eta_{\hat{H}_T}(T)$, which can not be reduced.

Note that in a general semimartingale market a BRM strategy allows one to achieve the effect of the above minimization of the fluctuations of the benchmarked P&L even when the above assumed square integrability, independence and martingale properties may not be given. Obviously, the concept of benchmarked risk minimization can be applied more generally than the above quadratic criterion. In a more general sense than under the above quadratic criterion, the fluctuations of the benchmarked P&L of a BRM strategy become minimal. This property permits the reduction or potential asymptotic removal of the total benchmarked P&L in a large trading book, as discussed prior to Remark 2.3.3.

2.7.3 An Example with Nonhedgeable Contingent Claim

To illustrate similarities and also an important difference between the pricing and hedging under benchmarked risk minimization and classical risk minimization, we consider a simple continuous market model as specification of the jump diffusion market given in the previous section. For simplicity, we consider as traded securities only a constant domestic savings account $S_t^{1,1} = 1$ and the numéraire portfolio (NP) $S_t^{1,2} = S_t^{1,\delta*}$, denominated in domestic currency. The latter can be interpreted as traded market index. Additionally, we consider a contingent claim H_T denominated in units of the domestic currency with fixed maturity $T \in [0, \infty)$. For simplicity, its conditional expectation $H_t = E(H_T|\mathcal{F}_t)$, $t \in [0, T]$, is assumed to be independent from all benchmarked primary security accounts and supposed to form a continuous process.

The first step of benchmarked risk minimization requires the application of the real world pricing formula (2.18) to obtain the benchmarked price $\hat{S}_t^{\delta_{\hat{H}}}$ of the benchmarked claim $\hat{H}_T = H_T \hat{S}_T^1 = H_T (S_T^{1,\delta*})^{-1}$ at time $t \in [0, T]$. That is, we have by the assumed independence of H_t and $\hat{S}_T^1$ that

$$\hat{S}_t^{\delta_{\hat{H}}} = E(\hat{H}_T|\mathcal{F}_t) = E(H_T|\mathcal{F}_t) E(\hat{S}_T^1|\mathcal{F}_t) = H_t \hat{P}(t, T) \; . \tag{2.30}$$

Here $\hat{S}_t^1 = \frac{S_t^{1,1}}{S_t^{1,\delta*}}$ denotes the benchmarked savings account, which we assume, for simplicity, to form a scalar diffusion. Furthermore, $\hat{P}(t,T) = E(\hat{S}_T^1|\mathcal{F}_t)$ is the real world price at time t of a zero coupon bond with maturity T. Since $\hat{P}(t,T)$ is a martingale under the real world probability measure and $\hat{P}(t,T)$ is here a function of time and $\hat{S}_t^1$, it satisfies the SDE

$$d\hat{P}(t,T) = \frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1} d\hat{S}_t^1 \quad .$$

Note that the benchmarked NP is simply the constant $\hat{S}_t^2 = \frac{S_t^{1,2}}{S_t^{1,\delta*}} = 1$.

When denoting by $\vartheta^1_{\hat{H}_T}(t)$ the number of units of the savings account that the BRM strategy holds at time $t \in [0,T]$, then the benchmarked P&L satisfies by (2.3.1) and the product rule for $H_t\hat{P}(t,T)$ the SDE

$$d\hat{C}_t^{\delta_{\hat{H}_T}} = d\hat{S}_t^{\delta_{\hat{H}_T}} - \vartheta^1_{\hat{H}_T}(t) d\hat{S}_t^1 = \left(H_t \frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1} - \vartheta^1_{\hat{H}_T}(t) \right) d\hat{S}_t^1 + \hat{P}(t,T) dH_t \quad .$$

The second step of benchmarked risk minimization requires that, the benchmarked P&L has to be made orthogonal to benchmarked traded wealth. This means, we identify $\vartheta^1_{\hat{H}_T}(t)$ such that $\hat{C}^{\delta_{\hat{H}_T}}$ is orthogonal to any benchmarked self-financing portfolio in the sense of Definition 2.4.1. This means, the product of $\hat{C}^{\delta_{\hat{H}_T}}$ and any benchmarked self-financing portfolios has to be a local martingale, that is, it has to be driftless. More precisely, a benchmarked self-financing portfolio $\hat{S}_t^\vartheta$, which invests at time t the predictable number ϑ_t^1 of units of the savings account and the remainder of its wealth in the NP, satisfies the SDE

$$d\hat{S}_t^\vartheta = \vartheta_t^1 d\hat{S}_t^1 \quad .$$

Since the processes $\hat{S}^\vartheta$, $\hat{S}^1$ and $\hat{C}^{\delta_{\hat{H}_T}}$ are $(\underline{\mathcal{F}}, P)$-local martingales, the product $\hat{C}^{\delta_{\hat{H}_T}} \hat{S}^\vartheta$ satisfies by the product rule in the given continuous market an SDE with zero drift if the derivative of the covariation of $\hat{C}^{\delta_{\hat{H}_T}}$ and $\hat{S}^\vartheta$ vanishes for all

$t \in [0, \infty)$, that is,

$$0 = \frac{d[\hat{S}^{\vartheta}, \hat{C}^{\delta_{\hat{H}_T}}]_t}{dt} = \left(H_t \frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1} - \vartheta^1_{\hat{H}_T}(t) \right) \vartheta^1_t \frac{d[\hat{S}^1]_t}{dt} \ .$$

Therefore, the benchmarked P&L is orthogonal to traded wealth, in the sense of Definition 2.4.1 if

$$\vartheta^1_{\hat{H}_T}(t) = H_t \frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1} \quad . \tag{2.31}$$

Thus, the respective BRM strategy monitors in this example the benchmarked P&L

$$\eta_{\hat{H}_T}(t) = \hat{C}^{\delta_{\hat{H}_T}}_t = \int_0^t \hat{P}(s,T) dH_s \quad , \tag{2.32}$$

which is a local martingale that starts at zero, as requested in Definition 2.5.1. Note that we have identified by the equations (2.30), (2.31) and (2.32) the key quantities for the characterization of the martingale representation of a regular benchmarked contingent claim $\hat{H}_T$, as required in (2.16).

We can now apply Corollary 2.5.2 to conclude that the self-financing benchmarked hedgeable part of the resulting BRM strategy invests at time t in

$$\vartheta^2_{\hat{H}_T}(t) = \hat{S}^{\delta_{\hat{H}_T}}_t - \eta_{\hat{H}_T}(t) - \vartheta^1_{\hat{H}_T}(t) \hat{S}^1_t$$

units of the NP.

In this example, it is easy to see by checking the derivative of the quadratic variation of the above benchmarked P&L that the same strategy would have emerged if one would have minimized this quadratic variation instead of requiring the orthogonality of the benchmarked P&L to benchmarked traded wealth. Indeed the minimization of the quadratic variation of the benchmarked P&L is another way of interpreting BRM strategies in a continuous setting. It allows avoiding square integrability, independence and martingale conditions, similarly as discussed in the previous section. However, such a concept would not have worked conveniently in the case with jumps. We will see in Section 2.7.8 that the suggested type of orthogonality between the benchmarked P&L and the benchmarked traded wealth is better suited for the case with jumps.

2.7.4 Classical and BRM Prices

It is straightforward to notice in the above example that the candidate for the Radon-Nikodym derivative process for the putative equivalent minimal martingale measure under classical risk minimization, see Schweizer [1995], equals the benchmarked savings account process $\hat{S}^1$. If the benchmarked savings account is not a true martingale, then classical risk minimization cannot be applied. In such a case, the BRM strategy can still be determined. When formally applying classical risk minimization by ignoring the assumption on the existence of an equivalent minimal martingale measure, it follows by the supermartingale property of benchmarked nonnegative price processes that the obtained price will be typically higher than the minimal possible price obtained via the real world pricing formula. We refer to Platen and Heath [2010] and Bruti-Liberati, Nikitopoulos-Sklibosios, and Platen [2010] for illustrative examples.

Let us now discuss the special case when $\hat{S}^1$ is a true martingale and classical risk minimization can be applied. The real world pricing formula (2.18) can then be rewritten for the discounted price process $\bar{S}_t^{\delta_{\hat{H}_T}} = \frac{\hat{S}_t^{\delta_{\hat{H}_T}}}{\hat{S}_t^1}$ in the form

$$\begin{aligned}\bar{S}_t^{\delta_{\hat{H}_T}} &= \frac{1}{\hat{S}_t^1} E\left(\frac{H_T}{S_T^{1,\delta *}} | \mathcal{F}_t\right)\\ &= E\left(\frac{\hat{S}_T^1}{\hat{S}_t^1}\frac{H_T}{S_T^{1,1}} | \mathcal{F}_t\right) = E\left(\frac{\Lambda_T}{\Lambda_t}\frac{H_T}{S_T^{1,1}} | \mathcal{F}_t\right) = E^Q\left(\frac{H_T}{S_T^{1,1}} | \mathcal{F}_t\right),\end{aligned}$$

where $\hat{S}_t^1 = \frac{S_t^{1,1}}{S_t^{1,\delta *}}$ denotes the benchmarked savings account and $S_T^{1,1}$ the savings account in units of the domestic currency. The respective prices under classical and BRM strategies coincide in this case because the Radon-Nikodym derivative for the equivalent minimal martingale measure Q is here characterized by the normalized benchmarked savings account $\Lambda_t = \frac{dQ}{dP}|_{\mathcal{F}_t} = \frac{\hat{S}_t^1}{\hat{S}_0^1}$. We summarize this observation as follows:

Remark 2.7.1. *Under sufficiently strong assumptions, which secure the existence* *of the equivalent minimal martingale measure, see Schweizer [1995], classical risk minimization yields the same prices as benchmarked risk minimization.*

The next section will reveal an important departure of BRM strategies from classical risk minimization even when the latter can be applied.

2.7.5 Evolving Information on Nonhedgeable Claims

In our example of the previous section with H_T and $S_T^{1,\delta*}$ as independent random variables, classical risk minimization would invest at initialization of the hedge the total value of the initial price in the savings account and would keep all invested wealth in the savings account until maturity. Our above example reveals that the BRM strategy would be different: During the hedge the number of units (2.31) held in the savings account at time t will change according to the evolving information given by the expected value $H_t = E(H_T|\mathcal{F}_t)$ of the contingent claim H_T. In general, the number $\vartheta^1_{\hat{H}_T}(t) = H_t$ of units to be held in the savings account changes over time. The above formula follows in the special case when $\hat{S}^1$ is a true martingale, and one has $\hat{P}(t,T) = \hat{S}^1_t = E(\hat{S}^1_T|\mathcal{F}_t)$ and, thus, $\frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1_t} = 1$. The BRM strategy of this example is an intuitive and practically appealing strategy. It encapsulates naturally the evolving information about the nonhedgeable claim via its conditional expectation. The BRM strategy is based on the best forecast of the nonhedgeable claim when using the respective martingale as the hedge ratio $\vartheta^1_{\hat{H}_T}$. We summarize this important observation in the following remark:

Remark 2.7.2. *During hedging under classical risk minimization, evolving information about the nonhedgeable part of a contingent claim has no impact, whereas a BRM strategy takes such information into account.*

The above observed important difference signals that a BRM strategy manages the benchmarked P&L such that, in some sense, it fluctuates less than the comparable classical risk minimizing strategy by taking evolving information into account.

2.7.6 Minimal Market Model with Gamma Type Jumps

So far we have not exploited much the wider modeling world that the benchmark approach provides.This subsection discusses a model that can be handled using BRM strategies. In the following, let us generalize the minimal market model (MMM), see Platen and Heath [2010], to a model with gamma type jumps. This model will allow us to demonstrate that BRM strategies can conveniently handle models with random jump size and infinite jump intensity, which seems to be more difficult under classical risk minimization.

To begin with, we set d and m equal to one in the jump diffusion market of Section 2.6, assume zero interest rate and specify the mark space as the interval $\mathcal{E} = (0, \ln 2)$. Furthermore, we specify the SDE (2.28) for the benchmarked primary security account value $\hat{S}_t^1$ in the form

$$d\hat{S}_t^1 = -\hat{S}_{t-}^1 \left(\theta_t^{1,1} dW_t^1 + \int_{\mathcal{E}} \psi^1(v,t) q(dv,dt) \right) \quad , \tag{2.33}$$

with $\theta_t^{1,1} = \sqrt{\frac{1}{Y_t}}$, where $Y = \{Y_t, t \in [0,\infty)\}$ is a square root process satisfying the SDE

$$dY_t = (1 - \xi Y_t)dt + \sqrt{Y_t} dW_t^1 \tag{2.34}$$

for $t \in [0,\infty)$, with net growth rate $\xi > 0$ and initial value $Y_0 > 0$. Let $\alpha_t = \alpha exp\{\xi t\}$ denote an exponential function of time with scaling parameter $\alpha > 0$. The product $\alpha_t Y_t$ forms then a time transformed squared Bessel process of dimension four and its inverse is a strict local martingale, see Revuz and Yor [1999]. Furthermore, we set in (2.33) the relative market price of event risk to

$$\psi^1(v,t) = 1 - e^v \tag{2.35}$$

for $v \in \mathcal{E}$. Finally, we use the intensity measure

$$\phi(dv,t) = \gamma \frac{e^{-\lambda v}}{v} dv \ , \tag{2.36}$$

$v \in \mathcal{E}$, $t \geq 0$, of a gamma process with shape parameter $\gamma > 0$ and rate parameter

$\lambda > 0$ to form the jump martingale measure $q(dv, dt) = p(dv, dt) - \phi(dv, t)dt$, see e.g. Cont and Tankov [2003].

Our second factor process $L = \{L_t, t \in [0, \infty)\}$, with

$$L_t = \exp\left\{-\int_0^t \int_{\mathcal{E}} (e^v - 1)\gamma \frac{e^{-\lambda v}}{v} dv ds + \int_0^t \int_{\mathcal{E}} v p(dv, ds)\right\} \quad , \tag{2.37}$$

forms a square integrable martingale. The constructed gamma type process L has only positive jumps, which due to (2.36) have unbounded intensity for vanishing jump size. It is straightforward to show that

$$\hat{S}_t^1 = \frac{1}{S_t^{1,\delta *}} = \frac{L_t}{\alpha_t Y_t} \tag{2.38}$$

satisfies the SDE (2.33).

Under the above MMM with gamma type jumps, the benchmarked savings account $\hat{S}_t^1$ equals the product of a positive strict local martingale with an independent positive jump martingale L_t and, thus, forms a positive strict local martingale. Obviously, the Radon-Nikodym derivative

$$\Lambda_t = \frac{dQ}{dP}|_{\mathcal{F}_t} = \frac{\hat{S}_t^1}{\hat{S}_0^1} = \frac{L_t \alpha_0 Y_0}{L_0 \alpha_t Y_t} \tag{2.39}$$

of the respective putative risk neutral measure Q is in this case a strict supermartingale and not a martingale. This property prevents the application of classical risk minimization. However, benchmarked risk minimization with its real world pricing, as described in (2.18), can be readily applied. In (2.39) Λ_t forms a strict supermartingale and we have $E\left(\frac{\Lambda_T}{\Lambda_t}|\mathcal{F}_t\right) < 1$ for $0 \leq t < T$. By applying formally the typical pricing rule of classical risk minimization, expressed via the Bayes formula in the form

$$\frac{E\left(\frac{\Lambda_T}{\Lambda_t} H_T|\mathcal{F}_t\right)}{E\left(\frac{\Lambda_T}{\Lambda_t}|\mathcal{F}_t\right)} \, ,$$

it follows by the above identified strict supermartingale property of Λ that this ratio yields for $0 \leq t < T$ a higher value than the real world price $E\left(\frac{\Lambda_T}{\Lambda_t} H_T|\mathcal{F}_t\right)$

given by the real world pricing formula. As will be shown below, and has been discussed in Chapter 3 of Platen and Bruti-Liberati [2010], the difference between the prices can be substantial for long dated derivatives.

2.7.7 Pricing of Real World Zero Coupon Bond

To illustrate a key property of a BRM strategy, which is the identification of the minimal possible price, consider for the model of the previous section the pricing at time $t \in [0, T]$ of a contract that pays one unit of the currency at maturity $T \in (0, \infty)$. Recall that the savings account $S_t^{1,1} = 1$ is constant for $t \in [0, \infty)$. Consequently, if one would formally apply the classical risk neutral pricing rule to value this contract, then one would obtain the constant "risk neutral" contract price or savings bond price

$$P^*(t, T) = \frac{S_t^{1,1}}{S_T^{1,1}} = 1 \quad . \tag{2.40}$$

Obviously, the classical hedge for the savings bond, yielding one unit of the currency at time T, invests without any change over time one monetary unit in the savings account.

In contrast, when applying the real world pricing formula (2.18) to value at time t the delivery of one unit of the currency at maturity T, this provides the benchmarked real world zero coupon bond price $\hat{P}(t, T) = \frac{P(t,T)}{S_t^{1,\delta*}}$ at time $t \in [0, T)$, forming a martingale. When denominating this price in domestic currency, one obtains

$$P(t, T) = \hat{P}(t, T) S_t^{1,\delta*} = E\left(\frac{S_t^{1,\delta*}}{S_T^{1,\delta*}} \middle| \mathcal{F}_t\right) = E\left(\frac{\hat{S}_T^1}{\hat{S}_t^1} \middle| \mathcal{F}_t\right) = E\left(\frac{\Lambda_T}{\Lambda_t} \middle| \mathcal{F}_t\right) . \tag{2.41}$$

If the Radon-Nikodym derivative process $\Lambda = \{\Lambda_t = \frac{\hat{S}_T^1}{\hat{S}_t^1}, t \geq 0\}$ were an $(\underline{\mathcal{F}}, P)$-martingale, then real world pricing would equal risk neutral pricing. However, as already pointed out, Λ_t forms in the given example an $(\underline{\mathcal{F}}, P)$-strict supermartingale. By the well-known formula for the negative first moment of a squared Bessel process of dimension four, see e.g. equation (8.7.17) in Platen and Heath [2010], it follows by the independence of the martingale L of the square root process Y

that

$$P(t,T) = E\left(\frac{\Lambda_T}{\Lambda_t}\bigg|\mathcal{F}_t\right) = E\left(\frac{L_T}{L_t}\bigg|\mathcal{F}_t\right) E\left(\frac{\alpha_t Y_t}{\alpha_T Y_T}\bigg|\mathcal{F}_t\right)$$
$$= 1 - \exp\left\{-\left(\frac{2\xi Y_t}{e^{\xi(T-t)}-1}\right)\right\} < 1 = P^*(t,T) \tag{2.42}$$

for $t \in [0,T)$.

Obviously, the real world zero coupon bond price $P(t,T)$ is less expensive than the corresponding constant savings bond price $P^*(t,T) = 1$. Note that the benchmarked real world zero coupon bond price process forms by the real world pricing formula (2.18) the martingale that delivers at maturity one benchmarked unit of the currency. On the other hand, the benchmarked savings bond is a strict supermartingale and, therefore, more expensive despite delivering, in a self-financing manner, the same terminal payoff at maturity T.

2.7.8 Hedging of Real World Zero Coupon Bond

Recall that under the benchmark approach there may exist several self-financing portfolios that hedge a given payoff. In our example of the previous section, holding until maturity one unit of the savings account is one possible self-financing strategy. Hedging the real world zero coupon bond such that its benchmarked value becomes a martingale, provides another one. By continuing the previous example, this leads us to the question of hedging a real world zero coupon bond. Observe that the benchmarked real world zero coupon bond price $\hat{P}(t,T)$ at time $t \in [0,T]$ satisfies by (2.38) and (2.42) the explicit formula

$$\hat{P}(t,T) = \frac{P(t,T)}{S_t^{1,\delta*}}$$
$$= \frac{L_t}{\alpha_t Y_t}\left(1 - \exp\left\{-\frac{2\xi Y_t}{e^{\xi(T-t)}-1}\right\}\right) \quad . \tag{2.43}$$

Due to the Markovianity of the underling two-factor model we obtain the benchmarked real world zero coupon bond price as a function $\hat{P}_T(\cdot,\cdot,\cdot)$ of time t and the two factors Y_t and L_t at time t, that is, we have

$$\hat{P}(t,T) = \hat{P}_T(t, Y_t, L_t) \quad .$$

Recall that the factor Y_t is characterized by the SDE (2.34), and the martingale L_t satisfies by (2.37) the SDE

$$dL_t = -L_{t-} \int_{\mathcal{E}} \psi^1(v,t) q(dv,dt) \quad . \tag{2.44}$$

By applying the Itô formula to the above benchmarked real world zero coupon bond price one obtains the SDE

$$\begin{aligned} d\hat{P}_T(t, Y_t, L_t) &= \frac{\partial \hat{P}_T(t, Y_t, L_t)}{\partial t} + \frac{\partial \hat{P}_T(t, Y_t, L_t)}{\partial Y} dY_t + \frac{\partial \hat{P}_T(t-, Y_{t-}, L_{t-})}{\partial L}(dL_t - \Delta L_t) \\ &+ \frac{1}{2}\frac{\partial^2 \hat{P}_T(t, Y_t, L_t)}{(\partial Y)^2} d[Y]_t \\ &+ \hat{P}_T(t, Y_t, L_t) - \hat{P}_T(t-, Y_{t-}, L_{t-}) \quad , \end{aligned}$$

where $\Delta L_t = L_t - L_{t-}$. Observe by (2.43) that one has

$$\hat{P}_T(t, Y_t, L_t) - \hat{P}_T(t-, Y_{t-}, L_{t-}) = \frac{\partial \hat{P}_T(t-, Y_{t-}, L_{t-})}{\partial L} \Delta L_t \quad .$$

After straightforward calculations one obtains for $t \in [0,T)$ the following martingale representation for the benchmarked real world zero coupon bond:

$$\begin{aligned} \hat{P}_T(T, Y_T, L_T) = \hat{P}_T(t, Y_t, L_t) + \int_t^T \frac{\partial \hat{P}_T(s, Y_s, L_s)}{\partial Y} \sqrt{Y_s} dW_s^1 \\ + \int_t^T \hat{P}_T(s-, Y_{s-}, L_{s-}) \frac{dL_s}{L_{s-}} \quad . \end{aligned} \tag{2.45}$$

Note that (2.45) does not represent the martingale representation of a regular benchmarked contingent claim in the form (2.16). Now, our task is to identify the number ϑ_t^1 of units invested in the savings account, which makes the benchmarked P&L orthogonal to traded benchmarked wealth. The benchmarked P&L η_t for given ϑ_t^1 satisfies by (2.14), Corollary 2.3.2, (2.33) and (2.44) the SDE

$$\begin{aligned} d\eta_t &= d\hat{P}_T(t, Y_t, L_t) - \vartheta_t^1 d\hat{S}_t^1 \\ &= \left(\frac{\partial \hat{P}_T(t, Y_t, L_t)}{\partial Y} \sqrt{Y_t} + \vartheta_t^1 \hat{S}_t^1 \sqrt{\frac{1}{Y_t}} \right) dW_t^1 \\ &\quad + \left(\hat{P}_T(t-, Y_{t-}, L_{t-}) - \vartheta_t^1 \hat{S}_{t-}^1 \right) \frac{dL_t}{L_{t-}}, \end{aligned} \tag{2.46}$$

with $\eta_0 = 0$, as defined in Definition 2.3.4. The required orthogonality between the benchmarked P&L process η and all benchmarked self-financing portfolio processes $\hat{S} = \{\hat{S}_t = \int_0^t \bar{\vartheta}_s d\hat{S}_s^1, t \in [0, T]\}$ with predictable process $\bar{\vartheta} = \{\bar{\vartheta}_t, t \in [0, T]\}$ will allow us to identify the hedge ratio ϑ_t^1 of a BRM strategy. Essentially we have to make the product $\eta_t \hat{S}_t$ a local martingale. This is achieved by making its drift zero. The SDE for the product $Z_t = \eta_t \hat{S}_t = \eta_t \int_0^t \bar{\vartheta}_s d\hat{S}_s^1$ is obtained by (2.33) and (2.46) as

$$\begin{aligned} dZ_t = \eta_{t-} \bar{\vartheta}_t d\hat{S}_t^1 &+ \left(\int_0^{t-} \bar{\vartheta}_s d\hat{S}_s^1 \right) d\eta_t \\ -\bar{\vartheta}_t \hat{S}_t^1 \sqrt{\frac{1}{Y_t}} &\left(\frac{\partial \hat{P}_T(t, Y_t, L_t)}{\partial Y} \sqrt{Y_t} + \vartheta_t^1 \hat{S}_t^1 \sqrt{\frac{1}{Y_t}} \right) dt \\ + \int_{\mathcal{E}} &(\hat{P}_T(t-, Y_{t-}, L_{t-}) - \vartheta_t^1 \hat{S}_{t-}^1) \bar{\vartheta}_t \hat{S}_{t-}^1 \psi^1(v, t)^2 p(dv, dt) \quad . \end{aligned}$$

We have to find ϑ_t^1 such that the drift of Z_t vanishes for all values of $\bar{\vartheta}_t$. Thus, we obtain the following condition:

$$\begin{aligned} -\sqrt{\frac{1}{Y_t}} &\left(\frac{\partial \hat{P}_T(t, Y_t, L_t)}{\partial Y} \sqrt{Y_t} + \vartheta_t^1 \hat{S}_t^1 \sqrt{\frac{1}{Y_t}} \right) \\ &+ \int_{\mathcal{E}} (\hat{P}_T(t, Y_t, L_t) - \vartheta_t^1 \hat{S}_t^1) \psi^1(v, t)^2 \phi(dv, t) = 0 \quad , \end{aligned}$$

which gives us because of $\frac{\partial \hat{P}_T(t,Y,L)}{\partial Y} = -\frac{\hat{S}^1}{Y} \frac{\partial \hat{P}_T(t,Y,L)}{\partial \hat{S}^1}$ by setting

$$\bar{\psi}_t = \int_{\mathcal{E}} \psi^1(v, t)^2 \phi(dv, t)$$

the hedge ratio

$$\vartheta^1_t = \left(\frac{\partial \hat{P}_T(t-, Y_{t-}, L_{t-})}{\partial \hat{S}^1_t} + \hat{P}_T(t-, Y_{t-}, L_{t-})\frac{Y_{t-}}{\hat{S}^1_{t-}}\bar{\psi}^1_t\right)(1 + Y_{t-}\bar{\psi}^1_t)^{-1} \quad . \tag{2.47}$$

When the jumps of $\frac{dL_t}{L_{t-}}$ would vanish, the benchmarked P&L would vanish too and the hedge would deliver the claim perfectly. Note that we have a regular benchmarked contingent claim $\hat{H}_T = \frac{1}{\hat{S}^{1,\delta *}_T}$ with martingale representation

$$\hat{H}_T = E(\hat{H}_T|\mathcal{F}_t) + \int_t^T \vartheta^1_{\hat{H}_T}(s)d\hat{S}^1_s + \eta_{\hat{H}_T}(T) - \eta_{\hat{H}_T}(t) \ ,$$

as required in (2.16). The hedge ratio is given in (2.47), where $\vartheta^1_{\hat{H}_T}(t) = \vartheta^1_t$, and it results the benchmarked P&L $\eta_{\hat{H}_T}(t) = \eta_t$, where $d\eta_t$ is given in (2.46). This martingale representation determines by Corollary 2.5.2 the BRM strategy. Note that the benchmarked P&L involves here some elements of traded uncertainty, which follows from the fact that the underlying security is driven by several sources of uncertainty.

In the case with jumps one can say that the predictable projection of the quadratic variation of the benchmarked P&L is minimized in the above calculations, as we will illustrate below: Consider the quadratic variation of the benchmarked P&L, which can be expressed by (2.46) as

$$\begin{aligned}
[\eta]_t &= \int_0^t \left(\frac{\partial \hat{P}_T(s, Y_s, L_s)}{\partial Y}\sqrt{Y_s} + \vartheta^1_s \hat{S}^1_s \sqrt{\frac{1}{Y_s}}\right)^2 ds \\
&\quad + \int_0^t \int_{\mathcal{E}} \left(\hat{P}_T(s-, Y_{s-}, L_{s-}) - \vartheta^1_s \hat{S}^1_{s-}\right)^2 \psi^1(v,s)^2 p(dv, ds) \ .
\end{aligned}$$

Substituting on the right hand side of the above equation the jump measure $p(dv, dt)$ by its predictable projection $\phi(dv, t)dt$, in the sense of Dellacherie and Meyer [1982] and Revuz and Yor [1999], it follows that at time $t \in [0, T)$ the resulting predictable projection of $[\eta]_t$ increases per unit of time with the rate

$$\gamma_t = \left(\frac{\partial \hat{P}_T(t, Y_t, L_t)}{\partial Y}\sqrt{Y_t} + \vartheta^1_t \hat{S}^1_t \sqrt{\frac{1}{Y_t}}\right)^2 + \left(\hat{P}_T(t, Y_t, L_t) - \vartheta^1_t \hat{S}^1_t\right)^2 \bar{\psi}^1_t \ ,$$

where ψ_t^1 was defined for (2.47). In the presence of jumps, one can interpret γ_t as a measure for the risk in the benchmarked P&L at time t. It is straightforward to show that the previously calculated BRM strategy minimizes for all $t \in [0, T]$ the rate γ_t, and thus, the predictable projection of the quadratic variation of the benchmarked P&L. It is straightforward to show that this holds generally for the previously introduced jump diffusion market model and regular benchmarked contingent claims.

We emphasize that the proposed concept of benchmarked risk minimization works for multi-factor and multi-asset models. It does not require to check square integrability or martingale properties. In its first step, the minimal price is obtained by calculating the respective real world conditional expectation. In its second step, the hedging strategy is determined by making the drift of a product to zero. These two straightforward calculations make benchmarked risk minimization an easily applicable and very general concept for pricing and hedging of not fully replicable contingent claims.

2.8 Conclusions on Benchmarked Risk Minimization

This chapter proposed the concept of benchmarked risk minimization for pricing and hedging of not fully replicable contingent claims. Benchmarked risk minimization goes beyond classical risk minimization, originally developed by Föllmer, Sondermann and Schweizer. A wider range of contingent claims can be priced and hedged in a richer modeling world. It uses the numéraire portfolio as numéraire, a real world conditional expectation for identifying the minimal possible price, and a zero drift condition for capturing the hedge ratio. It remains applicable when no equivalent risk neutral probability measure exists, and does not require square integrability or martingale properties related to the hedge error. The main assumption is extremely weak and requires that a numéraire portfolio exists. The numéraire portfolio is then employed as numéraire and benchmark. The benchmarked profit and loss of a benchmarked risk minimizing strategy is a local martingale. It is orthogonal to benchmarked traded wealth in the sense that the product of benchmarked profit and loss with all benchmarked portfolios is a local martingale.

When using benchmarked risk minimization, the total benchmarked profit and loss of a large trading book with increasing number of sufficiently different contingent claims vanishes asymptotically. In this sense, benchmarked risk minimization yields the minimal possible price and allows one to remove nonhedgeable risk via diversification. In two steps, by first calculating a real world expectation, and second setting a drift to zero, it can be straightforwardly applied to determine for a given benchmarked contingent claim the minimal possible price and the corresponding benchmarked risk minimizing strategy. This strategy minimizes the predictable projection of the quadratic variation of the benchmarked profit and loss.

Benchmarked risk minimization takes evolving information about the nonhedgeable part of a contingent claim into account. In the case when classical risk minimization can be applied, benchmarked risk minimization yields the same price process as classical risk minimization, however, it employs a hedging strategy which takes evolving information about the nonhedgeable uncertainty into account. Classical risk minimization does not exploit such information despite the fact that it is freely available.

Chapter 3

Commodity Forward and Futures Contracts

Commodity derivatives play a crucial role in risk management of financial markets. The most commonly used derivative instruments on commodities are their corresponding forward and futures contracts, as well as, options written on futures. Previous literature typically relates forward contracts to a static hedging argument and futures contracts to a dynamic hedging argument. This line of research usually yields the argument that the forward price in units of the respective zero coupon bond forms a martingale under the corresponding forward measure, while the futures price forms a martingale under an assumed risk neutral probability measure, see Black [1976], Cox, Ingersoll, and Ross [1981], Miltersen and Schwartz [1998] and Geman [2005].

As pointed out in Chapter 2 of this thesis and will be empathized in Chapter 4, the classical approach is too narrow for long term commodity derivative modeling. In particular, when pricing long dated contracts, which are frequently traded in commodity markets, one needs a more general approach than the classical one. In the current chapter, we model commodity derivative prices under the benchmark approach by combining benchmarked risk minimization, as introduced in Chapter 2, and the well developed line of research on short rate models, including Gibson and Schwartz [1990], Schwartz [1997], Hilliard and Reis [1998], Schwartz and Smith [2000], Richter and Sørensen [2002], Nielsen and Schwartz [2004], Casassus and Collin-Dufresne [2005] and others. The employed model provides the ability to capture dependence structures in commodity markets. It does not have an equivalent risk neutral probability measure and, therefore, requests the application of the benchmark approach. It still provides some computationally tractable formulas for commodity derivatives.

When we apply the benchmark approach to study the commodity derivative market, several research questions arise naturally among others:

1. Does the savings account of a commodity, when denominated by the NP, form a martingale under the real world probability measure?

2. Does the benchmarked futures price of a commodity form a real world martingale?

3. Does the benchmarked value of a forward contract form a real world martingale?

4. Does the benchmarked value of the margin account of a futures contract form a martingale?

The first two research questions will lead us to real world forward and futures price formulas, whereas the latter questions are important for hedging and risk management purposes.

This chapter is organized as follows: Section 3.1 studies properties of forward and futures contracts under the benchmark approach. Section 3.2 proposes an alternative model for real world futures prices of commodities. This model provides computationally tractable formulas for futures prices and European call and put options written on futures prices of commodities. Section 3.3 concludes the chapter.

3.1 Real World Forward and Futures Contracts

The current chapter derives new formulas for forward and futures prices by using the benchmark approach. It models commodity spot prices by building on the well developed line of research on short rate commodity models, including Gibson and Schwartz [1990], Schwartz [1997], Hilliard and Reis [1998], Schwartz and

Smith [2000], Richter and Sørensen [2002], Nielsen and Schwartz [2004], Casassus and Collin-Dufresne [2005] and others. However, instead of assuming a risk neutral probability measure, under the benchmark approach we model jointly under the real world probability measure the dynamics of the NP in domestic currency denomination and in commodity denomination. The ratio of these two processes models then the spot price process of the commodity. Note that this methodology disentangles the key factors driving the commodity spot price. Using the benchmark approach in the above indicated manner makes the research in this paper distinct from approaches in the previous literature.

Intuitively, one can say that the current chapter interprets one unit of the physical commodity as 'foreign currency'. Therefore, the convenience yield in the commodity savings account, as described in Gibson and Schwartz [1990], can be interpreted as the 'interest rate' of the 'foreign currency', which is here, the commodity.

3.1.1 Spot Price

We define the spot price $X_t^{i,j}$ as the number of units of the ith security that one pays for one unit of the jth security at time $t \in [0, \infty)$, $i, j \in \{1, \ldots, d+1\}$. Note that when $i = j$ we have trivially the spot price $X_t^{i,i} = 1$, for $t \in [0, \infty)$, $i \in \{1, \ldots, d+1\}$.

As already indicated, the spot price will be expressed as the ratio between the values of a given strictly positive self-financing portfolio, denominated in units of the two respective securities. By selecting the NP as this strictly positive portfolio one uses a portfolio that is globally extremely well diversified, see Platen and Rendek [2012a]. In principle, one can observe the dynamics of a primary security in a least perturbed manner when denominating the well diversified NP in units of this security. This is an important advantage for the modeling of security dynamics under the benchmark approach, see Du, Platen, and Rendek [2012]. The exchange price $X_t^{i,j}$, in our case the spot price, is then modeled at time t as the ratio

$$X_t^{i,j} = \frac{V_t^{i,\delta*}}{V_t^{j,\delta*}} \tag{3.1}$$

for $t \in [0, \infty)$, $i, j \in \{1, \ldots, d+1\}$. We remark by (2.21) that $S_t^{j,j}$ denotes the jth savings account denominated in units of the jth security. Consequently, the jth primary security account $S_t^{i,j}$ in ith security denomination is obtained as

$$S_t^{i,j} = X_t^{i,j} S_t^{j,j} = \frac{V_t^{i,\delta*}}{V_t^{j,\delta*}} S_t^{j,j}$$

for $t \in [0, \infty)$, $i, j \in \{1, \ldots, d+1\}$. This formula explains that one needs only to model denominations of the NP in different securities and the locally riskless savings accounts in the respective denominations.

3.1.2 Forward Price

A forward contract with delivery at the maturity date $T \in [0, \infty)$ is an agreement between two parties for receiving one unit of a security. It has value zero when initiated at time $t_0 \in [0, T]$. According to the specification of a forward contract, the short position has the obligation to deliver one unit of, say, the jth primary security, $j \in \{1, \ldots, d+1\}$, at the delivery date T for a fixed forward price $F_{t_0}^{i,j,T}$, denominated in units of the ith primary security, $i \in \{1, \ldots, d+1\}$. The jth primary security is here interpreted as commodity, while the ith primary security is interpreted as domestic currency. Furthermore, the forward price $F_{t_0}^{i,j,T}$ for one unit of the jth security is determined at the initiation time t_0 of the forward contract and is, thus, $\mathcal{A}_{t_0}$-measurable. By observing that one enters a forward contract for free at the initiation time t_0, in conjunction with the fact that at the maturity date T its payoff equals

$$H_T^{i,j} = X_T^{i,j} - F_{t_0}^{i,j,T} \quad , \tag{3.2}$$

one obtains by applying the real world pricing formula (2.13) to the benchmarked contingent claim $\hat{H}_T = \frac{H_T^{i,j}}{V_T^{i,\delta*}}$ the real world forward price $F_{t_0}^{i,j,T}$ as follows:

Theorem 3.1.1. *The real world forward price at initiation time $t_0 \in [0, T]$ in units of the ith currency, for one unit of the jth commodity to be delivered at time*

$T \in [0, \infty)$, $i, j \in \{1, \ldots, d+1\}$, equals

$$F_{t_0}^{i,j,T} = \frac{\hat{P}^j(t_0, T)}{\hat{P}^i(t_0, T)} = X_{t_0}^{i,j} \frac{P^j(t_0, T)}{P^i(t_0, T)} \quad , \tag{3.3}$$

where the zero coupon bond price

$$P^j(t_0, T) = E\left(\frac{V_{t_0}^{j,\delta*}}{V_T^{j,\delta*}} | \mathcal{F}_{t_0} \right) \tag{3.4}$$

denotes the fair value for the delivery of one unit of the jth commodity at maturity T, denominated in units of the jth commodity itself.

Proof. By substituting the benchmarked payoff $\hat{H}_T = \frac{H_T^{i,j}}{V_T^{i,\delta*}}$, see (3.2), into the real world pricing formula (2.13), one obtains the benchmarked value $\hat{U}_{t_0}^{i,j,T}$ of the forward contract at the initiation time t_0 as

$$\hat{U}_{t_0}^{i,j,T} = E\left(\frac{H_T^{i,j}}{V_T^{i,\delta*}} \middle| \mathcal{A}_{t_0} \right) \tag{3.5}$$

for $t_0 \in [0, T]$, $T \in [0, \infty)$, $i, j \in \{1, \ldots, d+1\}$. Note that the value $\hat{U}_{t_0}^{i,j,T}$ represents the benchmarked price to be paid at time t to enter the long position of a forward contract. Since one enters a forward contract for free at the initiation time t_0, the benchmarked value of the forward contract equals at this time zero, that is,

$$0 = \hat{U}_{t_0}^{i,j,T} \quad . \tag{3.6}$$

By substituting (3.6) and (3.2) into (3.5), one obtains, after rearranging, the real world forward price

$$F_{t_0}^{i,j,T} = \frac{E\left(\frac{1}{V_T^{i,\delta*}} X_T^{i,j} \middle| \mathcal{A}_{t_0} \right)}{\hat{P}^i(t_0, T)} \tag{3.7}$$

for $t_0 \in [0, T]$, $T \in [0, \infty)$, $i, j \in \{1, \ldots, d+1\}$. By substituting the ratio (3.1) for the exchange rate into (3.7), using (2.41) and rearranging, one obtains

$$F_{t_0}^{i,j,T} = \frac{1}{\hat{P}^i(t_0,T)} E\left(\frac{1}{V_T^{i,\delta*}} \frac{V_T^{i,\delta*}}{V_T^{j,\delta*}} \Big| \mathcal{A}_{t_0} \right) = \frac{1}{\hat{P}^i(t_0,T)} E\left(\frac{1}{V_T^{j,\delta*}} \Big| \mathcal{A}_{t_0} \right) = \frac{\hat{P}^j(t_0,T)}{\hat{P}^i(t_0,T)}.$$

for $t_0 \in [0,T]$, $T \in [0,\infty)$, $i,j \in \{1,\ldots,d+1\}$, which yields with the notation (3.4) the equation (3.3). $\square$

It is important to note that the price (3.4) for the delivery of one unit of the commodity has a similar mathematical structure as the zero coupon bond of a currency, see (2.41). However, the zero coupon bond (2.41) delivers a unit of a commodity rather than that of a currency at maturity. Therefore, we call the price $P^j(t,T)$, given in (3.4), the "zero coupon bond" at time t with maturity T of the jth commodity.

Note that the above expression for a forward price generalizes the typical forward price formula derived under the classical risk neutral approach; see e.g. Miltersen and Schwartz [1998]. In the case when there exist respective equivalent risk neutral probability measures for the ith and jth security denominations, the forward price, obtained under the more general benchmark approach, recovers the forward price of the classical risk neutral approach. In general, when an equivalent risk neutral probability measure fails to exist, the formally taken classical risk neutral price may be different from the above real world forward price, as will be illustrated in Section 3.2.

Suppose a forward contract was initiated at time $t_0 \in [0,T]$ with maturity time $T \in [0,\infty)$. The value of a forward contract at the initiation time t_0 is zero. After some time period, say, at time $t \in [t_0,T]$, the value of this forward contract, which shall be denoted by $U_t^{i,j,t_0,T}$, is in general no longer equal to zero. One can apply the real world pricing formula (2.13) to obtain the fair value of a forward contract:

Theorem 3.1.2. *The fair value $U_t^{i,j,t_0,T}$ of a forward contract at time t for one unit of the jth commodity, with initiation time t_0 and maturity date T, equals*

$$U_t^{i,j,t_0,T} = P^i(t,T)(F_t^{i,j,T} - F_{t_0}^{i,j,T}) \quad , \tag{3.8}$$

when denominated in units of the ith currency, $t_0 \in [0, T]$, $t \in [t_0, T]$, $T \in [0, \infty)$, $i, j \in \{1, \ldots, d+1\}$.

Proof. To obtain the fair value of the forward contract at time t, we substitute the payoff $X_T^{i,j} - F_{t_0}^{i,j,T}$, see (3.2), of this forward contract into the real world pricing formula (2.13), such that

$$\hat{U}_t^{i,j,t_0,T} = \frac{U_t^{i,j,t_0,T}}{V_t^{i,\delta *}} = E\left(\frac{1}{V_T^{i,\delta *}}(X_T^{i,j} - F_{t_0}^{i,j,T})\Big|\mathcal{F}_t\right) \tag{3.9}$$

for $t_0 \in [0, T]$, $t \in [t_0, T]$, $T \in [0, \infty)$, $i, j \in \{1, \ldots, d+1\}$. By substituting the ratio (3.1) of the exchange rate into (3.9) and rearranging using (3.4), one obtains

$$U_t^{i,j,t_0,T} = P^i(t,T)\left(X_t^{i,j}\frac{P^j(t,T)}{P^i(t,T)} - F_{t_0}^{i,j,T}\right).$$

Observe by (3.3) that $X_t^{i,j}\frac{P^j(t,T)}{P^i(t,T)}$ represents the forward price $F_t^{i,j,T}$ with initiation time t and maturity T. This yields directly (3.8). □

Note that formula (3.8) generalizes the classical risk neutral formula for the value of a forward contract, as given, for instance in Geman [2005]. However, it involves here the fair zero coupon bond price, which is, in general, lower than the risk neutral bond price, see Platen and Bruti-Liberati [2010] and Section 4. In the case when there exist respective equivalent risk neutral probability measures for the ith and jth security denominations, the value of a forward contract obtained under the more general benchmark approach coincides with its risk neutral value.

By (3.8), one obtains the benchmarked value of a forward contract as

$$\hat{U}_t^{i,j,t_0,T} = \frac{U_t^{i,j,t_0,T}}{V_t^{i,\delta *}} = E\left(\frac{1}{V_T^{i,\delta *}}(X_T^{i,j} - F_{t_0}^{i,j,T})\Big|\mathcal{F}_t\right) \tag{3.10}$$

for $t_0 \in [0, T]$, $t \in [t_0, T]$, $T \in [0, \infty)$, $i, j \in \{1, \ldots, d+1\}$. Since the random variable under the real world conditional expectation on the right hand side of (3.10) does not depend on t, this leads to the following statement:

Proposition 3.1.3. *The benchmarked value of a forward contract is an* $(\underline{\mathcal{F}}, P)$*-martingale.*

This practically important result answers Question 3. It states that under the benchmark approach the current benchmarked value of a forward contract is the best forecast of its future benchmarked values.

Equation (3.8) also confirms that the value of a forward contract at initiation t_0 is zero. However, as soon as the contract is signed the value $U_t^{i,j,t_0,T}$ of the contract departs, in general, from zero for $t \in (t_0, T]$. We emphasize that its value process is fair. At the maturity date T, the value of the forward contract converges to the difference

$$U_T^{i,j,t_0,T} = X_T^{i,j} - F_{t_0}^{i,j,T} = H_T^{i,j} \ ,$$

see (3.2), $t_0 \in [0, T]$, $T \in [0, \infty)$, $i, j \in \{1, \dots, d+1\}$, which is, in general, not zero. Since the value of a forward contract is after initiation, in general, not zero, a forward contract gives rise to counterparty risk for both parties of the contract. To reduce or avoid counterparty risk, futures contracts emerged, which will be discussed in the next section.

3.1.3 Futures Price

Similar to a forward contract, a futures contract ensures the exchange of a commodity at a fixed future delivery time. The difference is that a futures contract is traded on a futures exchange and is linked to a margin account. By the close of each trading day, the clearing house adjusts the value of the futures contract to zero for all parties involved by employing a margin deposit system. As a result, the value of a futures contract always remains close to zero. This avoids the build up of any significant counterparty risk. Below, we will refer to details of the margin deposit system, which will clarify the futures price evolution in more detail.

Previous research, such as the one described in Black [1976], Cox, Ingersoll, and Ross [1981] and Shreve [2004], has used classical no-arbitrage arguments to derive the result that the futures price forms a martingale under an assumed risk neutral probability measure. As mentioned previously, the current paper uses the more general benchmark approach, which does not rely on the existence of a risk neutral probability measure. Below we will study at first futures prices in a discrete time setting, and discuss later the results that emerge asymptotically for the case of continuous time.

3.1.3.1 Discrete Time Futures Price and Margin Account

Consider a time interval $[t, T]$ with $t \in [0, T]$ and $T \in [0, \infty)$. Within this time interval we define a sequence of trading periods $\{[t_k, t_{k+1}),\ k \in \{0, \ldots, n-1\}\}$, where $t = t_0 < t_1 < t_2 \cdots < t_n = T$. Here $[t_k, t_{k+1})$ denotes the kth trading period, which for simplicity, has fixed length $\Delta = t_{k+1} - t_k > 0$ for $k \in \{0, \ldots, n-1\}$.

First, consider one trading period only, say, the kth trading period $[t_k, t_{k+1})$, and the delivery of one unit of the jth security at time T. Similarly as for a forward contract one enters for free at the beginning of the kth trading period the long and short positions in a futures contract for the exchange of one unit of the jth security at time T. Therefore, the respective values of the long and short positions at the beginning of the trading period, when denominated in units of the ith currency, should equal zero. Denote by $Y^{i,j,T}_{t_k}$ the respective futures price in units of the ith currency at the beginning of the trading period $[t_k, t_{k+1})$ for one unit of the jth commodity to be delivered at time T, that is, a long position in the futures contract. Due to changes in the futures price over the trading period $[t_k, t_{k+1})$, a payment of $Y^{i,j,T}_{t_{k+1}} - Y^{i,j,T}_{t_k}$ into the futures exchange will be requested under such a futures contract at the end of this trading period. The corresponding benchmarked payoff $\hat{H}_{t_{k+1}}$ in the ith currency denomination at time t_{k+1} equals then

$$\hat{H}_{t_{k+1}} = \frac{Y^{i,j,T}_{t_{k+1}} - Y^{i,j,T}_{t_k}}{V^{i,\delta*}_{t_{k+1}}}. \tag{3.11}$$

By the application of the real world pricing formula (2.13) one obtains the real world futures price as follows:

Theorem 3.1.4. *The real world futures price $Y^{i,j,T}_{t_k}$ at the start of the kth trading* *period is given as*

$$Y_{t_k}^{i,j,T} = X_{t_k}^{i,j} \mathbb{E}\left(\frac{V_{t_k}^{j,\delta*}}{V_{t_n}^{j,\delta*}} \prod_{m=k}^{n-1} \frac{1}{P^i(t_m, t_{m+1})} \Big| \mathcal{F}_{t_k} \right) \tag{3.12}$$

for $k \in \{0, 1, \ldots, n-1\}$, $i, j \in \{1, \ldots, d+1\}$. *It turns out that the product*

$$\hat{Y}_{t_k}^{i,j,T} \prod_{m=0}^{k-1} \frac{1}{P^i(t_m, t_{m+1})} \tag{3.13}$$

of the benchmarked futures price with the product of a sequence of inverted zero coupon bond prices forms a discrete time martingale, where

$$\hat{Y}_{t_k}^{i,j,T} = \frac{Y_{t_k}^{i,j,T}}{V_{t_k}^{i,\delta*}} \tag{3.14}$$

for $k \in \{0, 1, \ldots, n\}$, $i, j \in \{1, \ldots, d+1\}$.

Proof. Due to the payment into the margin deposit system and by substituting the payoff (3.11) into the real world pricing formula (2.13) one obtains that

$$0 = E\left(\frac{1}{V_{t_{k+1}}^{i,\delta*}} (Y_{t_{k+1}}^{i,j,T} - Y_{t_k}^{i,j,T}) \Big| \mathcal{F}_{t_k} \right). \tag{3.15}$$

After rearranging this formula, one obtains the equality

$$E\left(\hat{Y}_{t_{k+1}}^{i,j,T} \Big| \mathcal{F}_{t_k} \right) = \hat{Y}_{t_k}^{i,j,T} P^i(t_k, t_{k+1}) , \tag{3.16}$$

where the benchmarked futures price $\hat{Y}_{t_k}^{i,j,T}$ is given in (3.14). Observe in (3.16) that the zero coupon bond price $P^i(t_k, t_{k+1})$ is $\mathcal{F}_{t_k}$-measurable. By multiplying both sides of (3.16) by the $\mathcal{F}_{t_k}$-measurable product of inverses of ith zero coupon bonds,

$$\prod_{m=0}^{k} \frac{1}{P^i(t_m, t_{m+1})} \quad ,$$

one obtains the equation:

$$E\left(\hat{Y}^{i,j,T}_{t_{k+1}}\prod_{m=0}^{k}\frac{1}{P^i(t_m,t_{m+1})}\middle|\mathcal{F}_{t_k}\right)=\hat{Y}^{i,j,T}_{t_k}\prod_{m=0}^{k-1}\frac{1}{P^i(t_m,t_{m+1})}\quad. \tag{3.17}$$

This shows that the expression

$$\hat{Y}^{i,j,T}_{t_k}\prod_{m=0}^{k-1}\frac{1}{P^i(t_m,t_{m+1})}$$

forms for $t_k \in \{t_0, t_1, \ldots, t_{n-1}\}$ a real world discrete time martingale, which proves the second part of the theorem.

At the maturity date $t_n = T$, the futures price equals the spot price, that is, one has

$$\hat{Y}^{i,j,T}_{t_n}=\frac{X^{i,j}_{t_n}}{V^{i,\delta*}_{t_n}}=\frac{V^{i,\delta*}_{t_n}}{V^{j,\delta*}_{t_n}V^{i,\delta*}_{t_n}}=\frac{1}{V^{j,\delta*}_{t_n}}$$

for $i, j \in \{1, \ldots, d+1\}$. By substituting this relationship into equation (3.17) for $k = n-1$, taking conditional expectation and rearranging this equation one obtains (3.12). □

The product $\prod_{m=0}^{k-1}\frac{1}{P^i(t_m,t_{m+1})}$ in (3.13) reflects the fact that at time t_m one invests the amount held in the margin account in the zero coupon bond that matures at the end t_{m+1} of the trading period. As soon as this bond matures, one rolls over the amount invested into an investment using the zero coupon bond that is covering the next trading period, etc. This is equivalent to investments into a roll-over short-term bond account, as described in Platen and Heath [2010].

The above proposition allows us to answer Question 3, at the beginning of this chapter, for the discrete time case. We conclude that the discrete time benchmarked futures price does not form a martingale. However, the product given in (3.13) forms a discrete time martingale along the sequence of time points when the investments are rolled over.

Recall that a futures contract is entered for free. Therefore, similarly as for a forward contract, the value of a futures contract is at initiation zero. The gains or losses over time of the long position of a futures contract are reflected in the value of the margin account of the holder of the futures contract. The margin account accumulates the margin payments $Y^{i,j,T}_{t_{k+1}} - Y^{i,j,T}_{t_k}$ at the close t_{k+1} of every trading period $[t_k, t_{k+1})$, $k \in \{0, 1, \ldots, n-1\}$. One can interpret these margin payments as being borrowed from or invested into an ith currency roll-over short term bond account.

Theorem 3.1.5. *The benchmarked value of the margin account of a futures contract, observed at the end of each trading period, forms a discrete time martingale.*

Proof. Recall the formula of the futures price (3.12). If one invests the margin payment $Y^{i,j,T}_{t_{l+1}} - Y^{i,j,T}_{t_l}$ received at time t_{l+1}, $l < k$, which could be either positive or negative, into the ith roll-over short-term zero coupon bond account, then the value at time t_{k+1} of the t_{l+1}-th margin payment, just after the kth margin payment, would be

$$(Y^{i,j,T}_{t_{l+1}} - Y^{i,j,T}_{t_l}) \prod_{m=l+1}^{k} \frac{1}{P^i(t_m, t_{m+1})}. \tag{3.18}$$

Denote by $M^{i,j,t_0,T}_{t_{k+1}}$ the value of the resulting *margin account* process at the date t_{k+1} just after the $(k+1)$-th margin payment with initiation date t_0, closing date $T = t_n$, and in ith currency denomination. Its value is then

$$M^{i,j,t_0,T}_{t_{k+1}} = \sum_{l=0}^{k} \frac{(Y^{i,j,T}_{t_{l+1}} - Y^{i,j,T}_{t_l})}{\prod_{m=l+1}^{k} P^i(t_m, t_{m+1})} \quad ,$$

where we set $\prod_{m=k+1}^{k} P^i(t_m, t_{m+1}) = 1$. By benchmarking $M^{i,j,t_0,T}_{t_{k+1}}$ and taking the conditional expectation of the benchmarked value $\hat{M}^{i,j,t_0,T}_{t_{k+1}} = \frac{M^{i,j,t_0,T}_{t_{k+1}}}{V^{i,\delta*}_{t_{k+1}}}$ under the information available at time t_k, one obtains the relationship

$$\begin{aligned} \mathbb{E}(\hat{M}^{i,j,t_0,T}_{t_{k+1}} | \mathcal{F}_{t_k}) &= E\left(\frac{1}{V^{i,\delta*}_{t_{k+1}}} \sum_{l=0}^{k} \frac{(Y^{i,j,T}_{t_{l+1}} - Y^{i,j,T}_{t_l})}{\prod_{m=l+1}^{k} P^i(t_m, t_{m+1})} \Big| \mathcal{F}_{t_k} \right) \\ = \; & E\left(\frac{1}{V^{i,\delta*}_{t_{k+1}}} \sum_{l=0}^{k-1} \frac{(Y^{i,j,T}_{t_{l+1}} - Y^{i,j,T}_{t_l})}{\prod_{m=l+1}^{k} P^i(t_m, t_{m+1})} \Big| \mathcal{F}_{t_k} \right) + E\left(\frac{Y^{i,j,T}_{t_{k+1}} - Y^{i,j,T}_{t_k}}{V^{i,\delta*}_{t_{k+1}}} \Big| \mathcal{F}_{t_k} \right). \end{aligned}$$

By application of (3.15) one obtains that

$$E\left(\frac{Y^{i,j,T}_{t_{k+1}} - Y^{i,j,T}_{t_k}}{V^{i,\delta*}_{t_{k+1}}}\Big|\mathcal{F}_{t_k}\right) = 0 \quad .$$

Thus, the conditional expectation becomes

$$\begin{aligned}
\mathbb{E}(\hat{M}^{i,j,t_0,T}_{t_{k+1}}|\mathcal{F}_{t_k}) &= E\left(\frac{1}{V^{i,\delta*}_{t_{k+1}}}\sum_{l=0}^{k-1}\frac{(Y^{i,j,T}_{t_{l+1}} - Y^{i,j,T}_{t_l})}{\prod_{m=l+1}^{k} P^i(t_m, t_{m+1})}\Big|\mathcal{F}_{t_k}\right) \\
&= \sum_{l=0}^{k-1}\frac{(Y^{i,j,T}_{t_{l+1}} - Y^{i,j,T}_{t_l})}{\prod_{m=l+1}^{k} P^i(t_m, t_{m+1})} E\left(\frac{1}{V^{i,\delta*}_{t_{k+1}}}\Big|\mathcal{F}_{t_k}\right) \\
&= \frac{1}{V^{i,\delta*}_{t_k}}\sum_{l=0}^{k-1}\frac{(Y^{i,j,T}_{t_{l+1}} - Y^{i,j,T}_{t_l})}{\prod_{m=l+1}^{k} P^i(t_m, t_{m+1})} P^i(t_k, t_{k+1}) \\
&= \frac{1}{V^{i,\delta*}_{t_k}}\sum_{l=0}^{k-1}\frac{(Y^{i,j,T}_{t_{l+1}} - Y^{i,j,T}_{t_l})}{\prod_{m=l+1}^{k-1} P^i(t_m, t_{m+1})} = \hat{M}^{i,j,t_0,T}_{t_k}.
\end{aligned}$$

This yields the discrete time martingale property of the benchmarked margin account at the close of each trading period. □

This proposition answers Question 3, at the beginning of this chapter, for the discrete time case. Observe that since the margin account evolves during the trading period proportional to the zero coupon bond covering this period, its benchmarked value forms during the trading period a continuous time martingale. This does not mean that the benchmarked value of the margin account is a continuous time martingale for the entire time period from initiation until maturity. However, as we will see below, this is true in an asymptotic sense.

3.1.3.2 Continuous Time Approximation

As detailed above, the futures exchange sets the value of a futures contract to zero at the close of each trading period by adjusting the margin account. As soon as trading starts, the value of a futures contract diverges, in general, from zero. At the close of the trading period, the margin deposit system again sets the value of the futures contract to zero by adjusting the margin account. Here, it can be seen that although at the end of each trading period the value of a futures contract is set to zero, within a trading period the value of a futures contract diverges from zero, similarly as the value of a forward contract diverges from zero during its life time. As a result, futures contracts avoid counterparty risk in an approximate sense only since the length of each trading period is short but not zero.

This consideration leads us to discuss the continuous time limit of the futures price process when letting the time between the close of trading periods converge to zero. Recall that the formula of a discrete time futures price is described in equation (3.12). The term $\prod_{m=0}^{n-1} \frac{1}{P^i(t_m,t_{m+1})}$ in this formula represents a roll-over short-term zero coupon bond account. As discussed in Platen and Heath [2010], this account value converges almost surely towards the savings account value $S_T^{i,i}$, see (2.21), if the length of the time Δ between the close of consecutive trading periods converges to zero. This yields the futures price in continuous time as follows:

Corollary 3.1.6. *The futures price $Y_t^{i,j,T}$ at time t in ith denomination for the delivery of one unit of the jth commodity at maturity T is approximated by its almost sure continuous time limit given in the form*

$$Y_t^{c,i,j,T} = X_t^{i,j} E\left(\frac{V_t^{j,\delta*}}{V_T^{j,\delta*}} \frac{S_T^{i,i}}{S_t^{i,i}} \middle| \mathcal{F}_t \right) = E\left(\frac{V_t^{i,\delta*}}{V_T^{j,\delta*}} \frac{S_T^{i,i}}{S_t^{i,i}} \middle| \mathcal{F}_t \right) \tag{3.19}$$

for $t \in [0,T]$, $T \in [0,\infty)$, $i,j \in \{0,1,\cdots,d\}$, where $S^{i,i} = \{S_t^{i,i}, t \in [0,\infty)\}$ denotes the ith savings account process.

Formula (3.19) is convenient because it avoids any time discretization. It allows us to observe that the product of the continuous time limit of the futures price with the benchmarked ith savings account, that is, $Y_t^{c,i,j,T} \hat{S}_t^i = \frac{Y_t^{c,i,j,T}}{V_t^{i,\delta*}} S_t^{i,i} = E\left(\frac{S_T^{i,i}}{V_T^{j,\delta*}} | \mathcal{F}_t\right)$ is a martingale, with value $\frac{S_T^{i,i}}{V_T^{j,\delta*}}$ at maturity.

As pointed out in Platen and Heath [2010], the normalized benchmarked savings account

$$\Lambda_t^i = \frac{S_t^{i,i}}{S_0^{i,i}} \frac{V_0^{i,\delta*}}{V_t^{i,\delta*}} = \frac{\hat{S}_t^i}{\hat{S}_0^i} \tag{3.20}$$

can be interpreted as the Radon-Nikodym derivative of the putative risk neutral probability measure. If an equivalent risk neutral probability measure exists, then the futures price is a martingale under this probability measure. This is a classical result, see e.g. Geman [2005]. Note additionally that in the case when an equivalent risk neutral probability measure exists, it follows from Theorem 3.1.5 that the discounted value of the margin account is a discrete time martingale under the risk neutral measure.

Note that in reality we are not dealing with a savings account as shown in (2.21). This may lead to distorted results when using the continuous time limit of a futures price in long term risk management.

Consider now the special case, where the ith short rate process is a deterministic function of time. We emphasize that the convenience yield of the jth commodity, that is, the jth short rate process, may still be stochastic. When assuming additionally that the benchmarked ith savings account $\hat{S}_t^i = \frac{S_t^{i,i}}{V_t^{i,\delta*}}$ is a martingale, one has by (2.41) the zero coupon bond price

$$P^i(t,T) = \frac{S_t^{i,i}}{S_T^{i,i}} E\left(\frac{\Lambda_T^i}{\Lambda_t^i}\middle|\mathcal{F}_t\right) = \frac{S_t^{i,i}}{S_T^{i,i}} \quad .$$

In this special case, the almost sure continuous time limit of the futures price equals by (3.3) and (3.4) the forward price $F_t^{i,j,T}$, that is,

$$Y_t^{c,i,j,T} = X_t^{i,j} \frac{S_T^{i,i}}{S_t^{i,i}} E\left(\frac{V_t^{j,\delta*}}{V_T^{j,\delta*}}\middle|\mathcal{F}_t\right) = X_t^{i,j} \frac{P^j(t,T)}{P^i(t,T)} = F_t^{i,j,T} \quad . \tag{3.21}$$

This relation generalizes a result for a similar special case in Miltersen and Schwartz [1998], which was derived under classical risk neutral assumptions. We emphasize that, in general, one has only the property that the benchmarked savings account forms an $(\underline{\mathcal{F}}, P)$-supermartingale. Thus, even in models that assume deterministic interest rates forward and futures prices can be different under the benchmark approach.

3.1.4 Options on Futures

There exist various types of options on commodities. Amongst, the most important and heavily traded ones are the options written on futures prices. The European call option $C_t^{Y^c,T_0,T_1}(K)$ written on the futures price $Y_{T_0}^{c,i,j,T_1}$ at time t, $0 < t < T_0 < T_1$, with strike price K and maturity T_0 can be written, with the real world pricing formula (2.13), as

$$C_t^{Y^c,T_0,T_1}(K) = E\left(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}\left[Y_{T_0}^{c,i,j,T_1} - K\right]^+ |\mathcal{F}_t\right) ,$$

whereas the corresponding put option can be written as

$$P_t^{Y^c,T_0,T_1}(K) = E\left(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}\left[K - Y_{T_0}^{c,i,j,T_1}\right]^+ |\mathcal{F}_t\right) .$$

Accordingly, the real world put-call parity for commodity futures prices can be obtained, by applying (2.41) and (3.19), as

$$\begin{aligned} C_t^{Y^c,T_0,T_1}(K) - P_t^{Y^c,T_0,T_1}(K) &= E\left(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}Y_{T_0}^{c,i,j,T_1}|\mathcal{F}_t\right) - KE\left(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}|\mathcal{F}_t\right) \\ &= E\left(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}X_{T_0}^{i,j}E\left(\frac{V_{T_0}^{j,\delta*}}{V_{T_1}^{j,\delta*}}\frac{S_{T_1}^{i,i}}{S_{T_0}^{i,i}}|\mathcal{F}_{T_0}\right)|\mathcal{F}_t\right) - KP^i(t,T_0) \\ &= X_t^{i,j}E\left(\frac{V_t^{j,\delta*}}{V_{T_1}^{j,\delta*}}\frac{S_{T_1}^{i,i}}{S_{T_0}^{i,i}}|\mathcal{F}_t\right) - KP^i(t,T_0) \quad . \end{aligned} \tag{3.22}$$

In the special case where the interest rate r_t^i is deterministic, the real-world put-call parity can be simplified as

$$C_t^{Y^c,T_0,T_1}(K) - P_t^{Y^c,T_0,T_1}(K) = e^{-\int_t^T r_s^i ds}\left(Y_t^{c,i,j,T_1} - KE\left(\frac{\Lambda_T^i}{\Lambda_t^i}\middle|\mathcal{F}_t\right)\right) ,$$

where Λ^i is given in (3.20). By assuming further that the ith benchmarked savings account is a martingale, one has $E\left(\frac{\Lambda_T^i}{\Lambda_t^i}|\mathcal{F}_t\right) = 1$ and the real-world put-call parity thus can be written as

$$C_t^{Y^c,T_0,T_1}(K) - P_t^{Y^c,T_0,T_1}(K) = e^{-\int_t^T r_s^i ds} \left(Y_t^{c,i,j,T_1} - K \right) .$$

The above expression is identical to the put-call parity obtained under the classical risk neutral approach, see, e.g. Miltersen and Schwartz [1998]. We emphasize that, in general, the benchmarked savings account forms only a real world supermartingale and one can only guarantee $E\left(\frac{\Lambda_T^i}{\Lambda_t^i}|\mathcal{F}_t\right) \leq 1$. This indicates that even in models where the interest rate is deterministic, the put-call parity obtained from the benchmark approach is different from the one derived under the classical risk neutral approach.

3.2 An Alternative Model for Commodity Prices

To demonstrate how to exploit the rich modeling world of the benchmark approach, we present in this section an alternative model for commodities. The proposed model represents a generalization of the minimal market model (MMM) of Platen [2001] for the discounted NP in both the domestic currency denomination and the commodity denomination. Furthermore, it applies affine short rate models of the type described in Duffie and Kan [1996], Dai and Singleton [2000] and Filipović [2001] for the interest rate of the domestic currency and the convenience yield of the commodity. We emphasize that the MMM violates classical assumptions. More precisely, an equivalent risk neutral probability measure does not exist for this model. In particular, the corresponding Radon-Nikodym derivative, which is the normalized benchmarked savings account, forms under the MMM the inverse of a time transformed squared Bessel process of dimension four. This process is known to be a strict local martingale and, therefore, it is a strict supermartingale, see Revuz and Yor [1999]. Weaker forms of classical arbitrage exist in the proposed model, see e.g. Platen and Heath [2010]. However, economically relevant arbitrage is automatically excluded through the supermartingale property of benchmarked nonnegative portfolios. Under the benchmark approach one obtains for this model commodity derivative prices by applying the real world pricing formula (2.13). We will see that computationally tractable formulas for commodity futures prices result for this model. This allows us to demonstrate in this case explicitly differences between the formally applied classical risk neutral approach and the benchmark approach. Moreover, the model appears to be quite reasonable in practice for pricing and hedging commodity derivatives, in particular, when the time to maturity extends over many years.

3.2.1 Minimal Market Model

To begin with, let us introduce a drift parameterization of the NP. We write the SDE (2.22) for the NP, when discounted by the ith savings account, that is, for $\bar{V}_t^{i,\delta*} = \frac{V_t^{i,\delta*}}{S_t^{i,i}}$, as

$$\begin{aligned} d\bar{V}_t^{i,\delta*} &= \alpha_t^i dt + \sqrt{\bar{V}_t^{i,\delta*}\alpha_t^i}\, d\widetilde{W}_t^i \\ &= \sqrt{\bar{V}_t^{i,\delta*}\alpha_t^i}\left(d\widetilde{W}_t^i + \sqrt{\frac{\alpha_t^i}{\bar{V}_t^{i,\delta*}}}dt\right) \end{aligned} \tag{3.23}$$

where $\alpha_t^i = \bar{V}_t^{i,\delta*}\sum_{k=1}^d(\theta_t^{i,k})^2$ represents the trend, and $\theta_t^{i,k}$ the market price of risk with respect to the kth Wiener process in ith denomination, $i \in \{1, \dots, d+1\}$. Here, $\widetilde{W}_t^i$ is the respective driving Wiener process under the real world probability measure, with stochastic differential

$$d\widetilde{W}_t^i = \frac{1}{\sqrt{\sum_{k=1}^d(\theta_t^{i,k})^2}}\sum_{k=1}^d \theta_t^{i,k} dW_t^k \, .$$

For simplicity, we assume that the Wiener processes $\widetilde{W}^1, \dots, \widetilde{W}^{d+1}$ are mutually independent so that the processes $\bar{V}^{1,\delta*}, \dots, \bar{V}^{d+1,\delta*}$ become independent.

A stylized version of the MMM, see Platen and Rendek [2012a], assumes that the discounted NP for the ith security can be expressed by the product

$$\bar{V}_t^{i,\delta*} = A_t^i Y_t^i \tag{3.24}$$

for $t \in [0,\infty)$, $i \in \{1, \dots, d+1\}$. Here A_t^i is an exponential function of time, which models the long term average growth of $\bar{V}_t^{i,\delta*}$ as

$$A_t^i = A^i \exp\{\xi^i t\} \ , \tag{3.25}$$

with the positive net growth rate parameter $\xi^i > 0$ and the positive scaling parameter $A^i > 0$, for $t \geq 0$. Furthermore, $Y_t^i = \frac{\bar{V}_t^{i,\delta*}}{A_t^i}$ models the normalized discounted NP at the time t, which satisfies by the Itô formula the SDE

$$dY_t^i = \xi^i(1 - Y_t^i)dt + \sqrt{\xi^i Y_t^i} d\widetilde{W}_t^i \ , \tag{3.26}$$

for $t \in [0, \infty)$ with $Y_0^i > 0$, $\alpha_t^i = A_t^i \xi^i$. Note that, the square root processes $Y^1, \ldots, Y^{d+1}$ are mutually independent.

To capture the dynamics of commodity spot, forward and futures prices, we also have to model the savings accounts of the currencies and the commodities. The current chapter studies the savings account of securities by modeling the interest rates and convenience yields via affine models, see Duffie and Kan [1996] and Filipović [2001]. With this type of models, the dependence between commodities and equities, as described e.g. in Geman [2005], can be studied.

Now, assume that the short rate processes r^i and r^j, $i, j \in \{1, \ldots, d+1\}$, $i \neq j$, are affine combinations of Y^i and Y^j such that

$$\begin{cases} r_t^i = a^i + b^{ii} Y_t^i + b^{ij} Y_t^j \\ r_t^j = a^j + b^{ji} Y_t^i + b^{jj} Y_t^j \end{cases} . \tag{3.27}$$

Here Y^i and Y^j are independent square root processes as described in (3.26). The parameters a^i, a^j are constants, which influence the average level of the short rates. The variables b^{ii}, b^{ij}, b^{ji} and b^{jj} are constants, which describe the dependence structure between the short rates and the discounted NP in different denominations. The savings account of the ith security, $i \in \{1, \ldots, d+1\}$, is given as

$$S_t^{i,i} = S_0^{i,i} \exp\left\{ \int_0^t r_s^i ds \right\} \tag{3.28}$$

with initial value $S_0^{i,i} > 0$.

The spot price of the jth commodity under the ith currency denomination, as described in (3.1), can be expressed by (3.24) and (3.28) as

$$X_t^{i,j} = \frac{V_t^{i,\delta*}}{V_t^{j,\delta*}} = \frac{\bar{V}_t^{i,\delta*} S_t^{i,i}}{\bar{V}_t^{j,\delta*} S_t^{j,j}} = \frac{S_t^{i,i} Y_t^i A_t^i}{S_t^{j,j} Y_t^j A_t^j}$$

for $t \in [0, \infty)$, $i, j \in \{1, \ldots, d+1\}$. Furthermore, for $T \geq t$, one has that

$$\begin{aligned} X_T^{i,j} &= \frac{X_t^{i,j}}{\frac{S_t^{i,i} Y_t^i A_t^i}{S_t^{j,j} Y_t^j A_t^j}} \frac{S_T^{i,i} Y_T^i A_T^i}{S_T^{j,j} Y_T^j A_T^j} \\ &= X_t^{i,j} \frac{Y_t^j}{Y_t^i} \frac{Y_T^i}{Y_T^j} e^{-\left((a^j - a^i) + (\xi^j - \xi^i)\right)(T-t) - (b^{ji} - b^{ii}) \int_t^T Y_s^i ds - (b^{jj} - b^{ij}) \int_t^T Y_s^j ds}. \end{aligned} \tag{3.29}$$

An equivalent risk neutral probability measure does not exist for this model, as was mentioned earlier and will be discussed in Section 3.2.4.

3.2.2 Forward and Futures Prices under the MMM

In the following, we apply the real world pricing formula (2.13) to obtain real world or fair derivative prices of commodities. As we will see, under the proposed MMM, zero coupon bond, commodity forward and futures prices can be explicitly obtained.

To begin with, we introduce the conditional expectations

$$R(Y_t^i, c, \xi^j, T-t) = E\left(e^{-\int_t^T c Y_s^i ds} \middle| \mathcal{F}_t\right) \tag{3.30}$$

and

$$G(Y_t^i, c, \xi^i, T-t) = E\left(\frac{1}{Y_T^i} e^{-\int_t^T c Y_s^i ds} \middle| \mathcal{F}_t\right) \quad , \tag{3.31}$$

where Y^i is the square root process introduced in (3.26), and c is a positive constant.

Here, the first conditional expectation $R(Y_t^i, c, \xi^i, T-t)$ can be explicitly calculated by applying standard results for affine processes, see e.g. Filipović

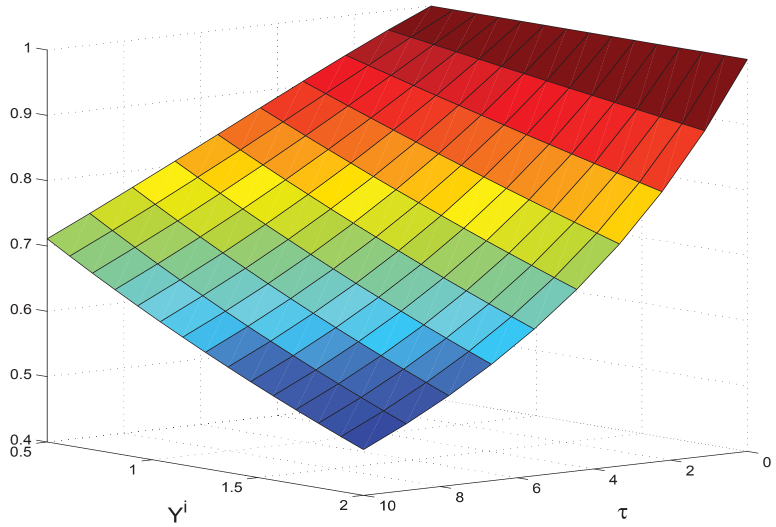

Figure 3.1: $R(Y_t^i, c, \xi^j, \tau)$ *as a function of* Y^i *and* τ*, with* $c = 0.05$, $\xi^i = 0.02$

[2001], yielding

$$R(Y_t^i, c, \xi^i, T-t) = \exp\left\{\widetilde{A}(c, \xi^i, T-t) - \widetilde{B}(c, \xi^i, T-t)Y_t^i\right\} \ ,$$

where

$$\widetilde{A}(c, \xi^i, \tau) = 2\ln\left(\frac{L_3(\tau)}{L_2(\tau)}\right)$$

and

$$\widetilde{B}(c, \xi^i, \tau) = c\frac{L_1(\tau)}{L_2(\tau)}$$

with

$$\begin{aligned} L_1(\tau) &= 2(e^{w\tau} - 1) \\ L_2(\tau) &= w(e^{w\tau} + 1) + \xi^i(e^{w\tau} - 1) \\ L_3(\tau) &= 2we^{(w+\xi^i)\tau/2} \\ w &= \sqrt{(\xi^i)^2 + 2c\xi^i} \ , \end{aligned}$$

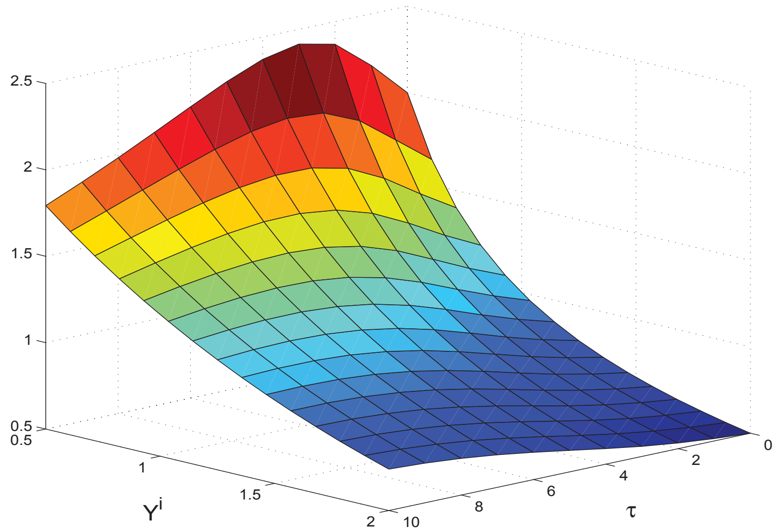

Figure 3.2: $G(Y_t^i, c, \xi^i, \tau)$ *as a function of* Y^i *and* τ*, with* $c = 0.05$*,* $\xi^i = 0.12$

for $\tau \in [0, T]$. We illustrate in Figure 3.1 the function $R(Y_t^i, c, \xi^j, \tau)$ for $c = 0.05$ and $\xi^i = 0.02$.

According to Craddock and Lennox [2007], the second conditional expectation $G(Y_t^i, c, \xi^i, T - t)$ given in (3.30) can also be explicitly obtained as

$$
\begin{aligned}
&G(Y_t^i, c, \xi^i, T-t) \\
&= \frac{\xi^i \left(\frac{2\sqrt{Y_t^i}}{\sinh(\frac{\xi^i(T-t)}{2})}\right)^{\sqrt{1+8\frac{c}{\xi^i}}}}{(Y_t^i)^{1/7}\sinh(\frac{\xi^i(T-t)}{2})} \exp\left\{(\xi^i)^2(T-t) - \frac{Y_t^i}{\tanh(\frac{\xi^i(T-t)}{2})} + Y_t^i\right\} \\
&\quad \times \frac{\Gamma(1+\frac{1}{2}(1+\sqrt{1+8\frac{c}{\xi^i}}))}{\Gamma(1+\sqrt{1+8\frac{c}{\xi^i}})} \left(\frac{\xi^i + \tanh(\frac{\xi^i(T-t)}{2})}{\tanh(\frac{\xi^i(T-t)}{2})}\right)^{-(1+\frac{1}{2}(1+\sqrt{1+8\frac{c}{\xi^i}}))} \\
&\quad \times {}_1F_1\left((1+\frac{1}{2}(1+\sqrt{1+8\frac{c}{\xi^i}})), 1+\sqrt{1+8\frac{c}{\xi^i}}, \frac{\left(\frac{2\sqrt{Y_t^i}}{\sinh(\frac{\xi^i(T-t)}{2})}\right)^2}{4\left(\frac{\xi^i+\tanh(\frac{\xi^i(T-t)}{2})}{\tanh(\frac{\xi^i(T-t)}{2})}\right)}\right) \quad ,
\end{aligned}
$$

where $\Gamma(\cdot)$ denotes the gamma function and ${}_1F_1(\cdot)$ a hypergeometric function, see Koev and Edelman [2006]. For illustration, we display in Figure 3.2 the function

$G(Y_t^i, c, \xi^i, \tau)$ as a function of Y^i and τ, with $c = 0.05$ and $\xi^i = 0.12$.

Furthermore, according to Broadie and Kaya [2006] the Laplace transform of $\int_t^T Y_s^i ds$ given Y_t^i and Y_T^i can also be explicitly obtained, such that for any complex value $a \in C$ one has

$$
\begin{aligned}
M(Y_t^i, Y_T^i, \xi^i, a) &= E\left(e^{-a\int_t^T Y_s^i ds} | Y_t^i, Y_T^i\right) \\
&= \frac{\gamma(a) e^{-\frac{1}{2}(\gamma(a)-\xi^i)(T-t)}(1-e^{-\xi^i(T-t)})}{\xi^i(1-e^{-\gamma(a)(T-t)})} \\
&\times \exp\left\{\frac{Y_t^i + Y_T^i}{\xi^i}\left[\frac{\xi^i(1+e^{-\xi^i(T-t)})}{1-e^{-\xi^i(T-t)}} - \frac{\gamma(a)(1+e^{-\gamma(a)(T-t)})}{1-e^{-\gamma(a)(T-t)}}\right]\right\} \\
&\times \frac{I_1\left[\frac{4(\gamma(a)\sqrt{Y_T^i Y_t^i})}{\xi^i}\frac{e^{-\frac{1}{2}\gamma(a)(T-t)}}{(1-e^{-\gamma(a)(T-t)})}\right]}{I_1\left[4\sqrt{Y_T^i Y_t^i}\frac{e^{-\frac{1}{2}\xi^i(T-t)}}{1-e^{-\xi^i(T-t)}}\right]} \quad ,
\end{aligned}
$$

where $\gamma(a) = \sqrt{(\xi^i)^2 + 2\xi^i a}$, $I_v(\cdot)$ is the modified Bessel function of the first kind, see Bayin [2006].

It follows by (2.41), (3.24)-(3.28) and Platen and Heath [2010] that the fair zero coupon bond price for the ith security, $i \in \{1, \ldots, d+1\}$, can be obtained as follows:

$$
\begin{aligned}
P^i(t,T) &= E\left(\frac{V_t^{i,\delta*}}{V_T^{i,\delta*}}\Big|\mathcal{F}_t\right) = E\left(\frac{A_t^i Y_t^i S_t^{i,i}}{A_T^i Y_T^i S_T^{i,i}}\Big|\mathcal{F}_t\right) \\
&= \frac{A_t^i Y_t^i}{A_T^i} E\left(\frac{1}{Y_T^i}\exp\left\{-\int_t^T (a^i + b^{ii}Y_s^i + b^{ij}Y_s^j)ds\right\}\Big|\mathcal{F}_t\right) \\
&= Y_t^i e^{-(\xi^i + a^i)(T-t)} G(Y_t^i, b^{ii}, \xi^i, T-t) R(Y_t^j, b^{ij}, \xi^j, T-t) \quad . \qquad (3.32)
\end{aligned}
$$

Consequently, the resulting explicit formula for the real world forward price, see (3.3), initiated at time t with delivery at $T \in [t, \infty)$ can be written as

$$
F_t^{i,j,T} = \frac{\hat{P}^j(t,T)}{\hat{P}^i(t,T)} = X_t^{i,j}\frac{P^j(t,T)}{P^i(t,T)}
$$

$$= X_t^{i,j}\frac{Y_t^j}{Y_t^i}e^{-((\xi^j-\xi^i)+(a^j-a^i))(T-t)}\frac{G(Y_t^j,b^{jj},\xi^j,T-t)R(Y_t^i,b^{ji},\xi^i,T-t)}{G(Y_t^i,b^{ii},\xi^i,T-t)R(Y_t^j,b^{ij},\xi^j,T-t)}. \tag{3.33}$$

Similarly, the continuous time limit of the commodity futures price, as in (3.19), can also be explicitly obtained as

$$\begin{aligned}
Y_t^{c,i,j,T} &= X_t^{i,j}E\left(\frac{V_t^{j,\delta*}}{V_T^{j,\delta*}}\frac{S_T^{i,i}}{S_t^{i,i}}\middle|\mathcal{F}_t\right) = X_t^{i,j}E\left(\frac{A_t^jY_t^jS_t^{j,j}}{A_T^jY_T^jS_T^{j,j}}\frac{S_T^{i,i}}{S_t^{i,i}}\middle|\mathcal{F}_t\right)\\
&= X_t^{i,j}\frac{A_t^jY_t^j}{A_T^j}\\
&\quad\times E\left(\frac{1}{Y_T^j}\exp\left\{-\int_t^T\left((a^j-a^i)+(b^{ji}-b^{ii})Y_s^i+(b^{jj}-b^{ij})Y_s^j\right)ds\right\}\middle|\mathcal{F}_t\right)\\
&= X_t^{i,j}Y_t^je^{-(\xi^j+(a^j-a^i))(T-t)}\\
&\quad\times G(Y_t^j,b^{jj}-b^{ij},\xi^j,T-t)R(Y_t^i,b^{ji}-b^{ii},\xi^i,T-t).
\end{aligned} \tag{3.34}$$

The continuous time limit of the real world forward-futures spread of commodities, $S_t^{c,i,j,T}$, under the proposed MMM can, thus, be obtained as

$$\begin{aligned}
S_t^{c,i,j,T} &= F_t^{i,j,T} - Y_t^{c,i,j,T}\\
&= X_t^{i,j}\frac{Y_t^j}{Y_t^i}e^{-((\xi^j-\xi^i)+(a^j-a^i))(T-t)}\\
&\quad\times\left(\frac{G(Y_t^j,b^{jj},\xi^j,T-t)R(Y_t^i,b^{ji},\xi^i,T-t)}{G(Y_t^i,b^{ii},\xi^i,T-t)R(Y_t^j,b^{ij},\xi^j,T-t)}\right.\\
&\qquad\left.-Y_t^ie^{-\xi^i(T-t)}G(Y_t^j,b^{jj}-b^{ij},\xi^j,T-t)R(Y_t^i,b^{ji}-b^{ii},\xi^i,T-t)\right).
\end{aligned} \tag{3.35}$$

3.2.3 Options on Commodities

In this section, we study European call options written on zero coupon bond, forward and futures prices. We will see that under the proposed MMM, with the explicit formulas (3.32)-(3.34), almost explicit formulas for the corresponding option prices can be obtained where some numerical integration has still to be performed.

3.2.3.1 Option on Zero Coupon Bond

For illustration, let us consider the European call option $C_t^{P,T_0,T_1}(K)$ written on the zero coupon bond price $P^i(T_0,T_1)$ at time t, $0<t<T_0<T_1$ with strike price K and maturity T_0. By applying the real world pricing formula (2.13) and the explicit formula for the bond price (3.32), one obtains

$$\begin{aligned} C_t^{P,T_0,T_1}(K) &= E\left(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}[P^i(T_0,T_1)-K]^+|\mathcal{F}_t\right) \\ &= E\Bigg(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}\Big[Y_{T_0}^i e^{-(\xi^i+a^i)(T_1-T_0)}G(Y_{T_0}^i,b^{ii},\xi^i,T_1-T_0)R(Y_{T_0}^j,b^{ij},\xi^j,T_1-T_0) \\ &\quad -K\Big]^+|\mathcal{F}_t\Bigg). \end{aligned} \tag{3.36}$$

After some obvious calculations and simplifications, equation (3.36) becomes

$$\begin{aligned} C_t^{P,T_0,T_1}(K) &= E\left(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}[P^i(T_0,T_1)-K]^+|\mathcal{F}_t\right) \\ &= Y_t^i e^{-(\xi^i+a^i)(T_1-t)}E\left(e^{-\int_t^{T_0}(b^{ii}Y_s^i+b^{ij}Y_s^j)ds}L(Y_{T_0}^i,Y_{T_0}^j)|\mathcal{F}_t\right) \quad , \end{aligned}$$

where

$$L(Y_{T_0}^i,Y_{T_0}^j)=\frac{1}{Y_{T_0}^i}\left[Y_{T_0}^i G(Y_{T_0}^i,b^{ii},\xi^i,T_1-T_0)R(Y_{T_0}^j,b^{ij},\xi^j,T_1-T_0)-\widetilde{K}\right]^+$$

with

$$\widetilde{K}=Ke^{(\xi^i+a^i)(T_1-T_0)} \quad .$$

Denote by $\mathcal{F}_t^i$ the sigma algebra generated by $\mathcal{F}_t$ and the entire path of Y^i until time T_0, and then, denote by $\sigma(Y_{T_0}^j)$ the sigma algebra generated by $Y_{T_0}^j$. By applying the properties of conditional expectations, the above formula becomes

$$\begin{aligned} & C_t^{P,T_0,T_1}(K) \\ &= Y_t^i e^{-(\xi^i+a^i)(T_1-t)}E\left(E\left(e^{-\int_t^{T_0}(b^{ii}Y_s^i+b^{ij}Y_s^j)ds}L(Y_{T_0}^i,Y_{T_0}^j)|\mathcal{F}_t^i\cup\sigma(Y_{T_0}^j)\right)|\mathcal{F}_t\right) \end{aligned}$$

$$= Y_t^i e^{-(\xi^i+a^i)(T_1-t)} E\left(e^{-\int_t^{T_0} b^{ii} Y_s^i ds} L(Y_{T_0}^i, Y_{T_0}^j) E\left(e^{-\int_t^{T_0} b^{ij} Y_s^j ds} | \mathcal{F}_t^i \cup \sigma(Y_{T_0}^j) \right) | \mathcal{F}_t \right)$$
$$= Y_t^i e^{-(\xi^i+a^i)(T_1-t)} E\left(e^{-\int_t^{T_0} b^{ii} Y_s^i ds} L(Y_{T_0}^i, Y_{T_0}^j) M(Y_t^j, Y_{T_0}^j, \xi^j, b^{ij}) | \mathcal{F}_t \right) \quad .$$

Similarly, denote by $\mathcal{F}_t^j$ the sigma field generated by $\mathcal{F}_t$ and the entire path of Y^j until time T_0, and then, denote by $\sigma(Y_{T_0}^i)$ the sigma field generated by random variable $Y_{T_0}^i$. By applying the properties of conditional expectations, the above formula becomes

$$\begin{aligned}
&C_t^{P,T_0,T_1}(K) \\
&= Y_t^i e^{-(\xi^i+a^i)(T_1-t)} \\
&\quad \times E\left(L(Y_{T_0}^i, Y_{T_0}^j) M(Y_t^j, Y_{T_0}^j, \xi^j, b^{ij}) E\left(e^{-\int_t^{T_0} b^{ii} Y_s^i ds} | \mathcal{F}_t^j \cup \sigma(Y_{T_0}^i) \right) | \mathcal{F}_t \right) \\
&= Y_t^i e^{-(\xi^i+a^i)(T_1-t)} E\left(L(Y_{T_0}^i, Y_{T_0}^j) M(Y_t^j, Y_{T_0}^j, \xi^j, b^{ij}) M(Y_t^i, Y_{T_0}^i, \xi^i, b^{ii}) | \mathcal{F}_t \right) \quad .
\end{aligned}$$

Furthermore, according to Platen and Heath [2010], the square-root process Y^i has the explicitly known transition density

$$\begin{aligned}
&p(x, Y_t^i, \xi^i) \\
&= \frac{2\exp\left\{-2\left(\frac{x+Y_t^i e^{-\xi^i(T-t)}}{1-e^{-\xi^i(T-t)}}\right)\right\}}{(1-e^{-\xi^i(T-t)})} \left(\frac{x e^{\xi^i(T-t)}}{Y_t^i}\right)^{\frac{1}{2}} I_1\left(4\frac{\sqrt{x Y_t^i e^{-\xi^i(T-t)}}}{1-e^{-\xi^i(T-t)}}\right). \quad (3.37)
\end{aligned}$$

By observing that $Y_{T_0}^i$ and $Y_{T_0}^j$ are independent, the option price can be expressed as

$$\begin{aligned}
&C_t^{P,T_0,T_1}(K) \\
&= Y_t^i e^{-(\xi^i+a^i)(T_1-t)} \\
&\times \int_0^\infty \int_0^\infty L(x,y) M(Y_t^j, y, \xi^j, b^{ij}) M(Y_t^i, x, \xi^i, b^{ii}) p(x, Y_t^i, \xi^i) p(y, Y_t^j, \xi^j) dx dy.
\end{aligned}$$

Thus, the option price (3.36) is reduced to a double integral with explicit integrand, see Filipović [2001]. Such an integral can be solved by using numerical methods and we call it, therefore, an almost explicit formula.

3.2.3.2 Option on Forward Price

The European call option $C_t^{F,T_0,T_1}(K)$ written on the forward price $F_{T_0}^{i,j,T_1}$ at time t, $0 < t < T_0 < T_1$, with strike price K and maturity T_0 can be written, with the real world pricing formula (2.13), as

$$C_t^{F,T_0,T_1}(K) = E\left(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}\left[F_{T_0}^{i,j,T_1} - K\right]^+ |\mathcal{F}_t\right) \quad . \tag{3.38}$$

By applying the explicit formula (3.33) for the forward price, the option price (3.38) becomes

$$\begin{aligned}
&C_t^{F,T_0,T_1}(K) \\
&= E\left(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}\left[X_{T_0}^{i,j}\frac{Y_{T_0}^j}{Y_{T_0}^i}e^{-((\xi^j-\xi^i)+(a^j-a^i))(T_1-T_0)}L^i(Y_{T_0}^i)L^j(Y_{T_0}^j) - K\right]^+ |\mathcal{F}_t\right),
\end{aligned}$$

where

$$L^i(Y_{T_0}^i) \quad = \quad \frac{R(Y_{T_0}^i, b^{ji}, \xi^i, T_1 - T_0)}{G(Y_{T_0}^i, b^{ii}, \xi^i, T_1 - T_0)}$$

and

$$L^j(Y_{T_0}^j) \quad = \quad \frac{G(Y_{T_0}^j, b^{jj}, \xi^j, T_1 - T_0)}{R(Y_{T_0}^j, b^{ij}, \xi^j, T_1 - T_0)} \quad .$$

By applying (3.29) and (3.34) and then, after some calculations, the option price (3.38) becomes

$$\begin{aligned}
&C_t^{F,T_0,T_1}(K) \\
&= \frac{A_t^i Y_t^i}{A_{T_0}^i} E\Bigg(\frac{1}{Y_{T_0}^i}\frac{S_t^{ii}}{S_{T_0}^{ii}} \\
&\quad \times \left[X_t^{i,j}\frac{Y_t^j}{Y_t^i}\frac{Y_{T_0}^i}{Y_{T_0}^j}e^{-\left((a^j-a^i)+(\xi^j-\xi^i)\right)(T_0-t)-(b^{ji}-b^{ii})\int_t^{T_0} Y_s^i ds-(b^{jj}-b^{ij})\int_t^{T_0} Y_s^j ds}\right. \\
&\quad \left.\times \frac{Y_{T_0}^j}{Y_{T_0}^i}e^{-((\xi^j-\xi^i)+(a^j-a^i))(T_1-T_0)}L^i(Y_{T_0}^i)L^j(Y_{T_0}^j) - K\right]^+ |\mathcal{F}_t\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A_t^i Y_t^i}{A_{T_0}^i} E\Bigg(\frac{1}{Y_{T_0}^i} e^{-a^i(T_0-t) - b^{ii}\int_t^{T_0} Y_s^i ds - b^{ij}\int_t^{T_0} Y_s^j ds} \\
&\quad \times \Bigg[X_t^{i,j} \frac{Y_t^j}{Y_t^i} e^{-\left((a^j-a^i)+(\xi^j-\xi^i)\right)(T_1-t)} L^i(Y_{T_0}^i) L^j(Y_{T_0}^j) \\
&\quad \times e^{-(b^{ji}-b^{ii})\int_t^{T_0} Y_s^i ds - (b^{jj}-b^{ij})\int_t^{T_0} Y_s^j ds} - K \Bigg]^+ |\mathcal{F}_t \Bigg) \\
&= Y_t^i e^{-(\xi^i+a^i)(T_0-t)} X_t^{i,j} \frac{Y_t^j}{Y_t^i} e^{-\left((a^j-a^i)+(\xi^j-\xi^i)\right)(T_1-t)} \\
&\quad \times E\Bigg(\frac{1}{Y_{T_0}^i} e^{-b^{ii}\int_t^{T_0} Y_s^i ds - b^{ij}\int_t^{T_0} Y_s^j ds} \\
&\quad \times \Bigg[L^i(Y_{T_0}^i) L^j(Y_{T_0}^j) e^{-(b^{ji}-b^{ii})\int_t^{T_0} Y_s^i ds - (b^{jj}-b^{ij})\int_t^{T_0} Y_s^j ds} \\
&\quad - \frac{K}{X_t^{i,j} \frac{Y_t^j}{Y_t^i} e^{-((a^j-a^i)+(\xi^j-\xi^i))(T_1-t)}} \Bigg]^+ |\mathcal{F}_t \Bigg).
\end{aligned}
$$

Thus, the option price (3.38) becomes

$$
\begin{aligned}
&C_t^{F,T_0,T_1}(K) \\
&= X_t^{i,j} Y_t^j \exp\left\{ -((\xi^j-\xi^i)+(a^j-a^i))(T_1-T_0) - (\xi^j+a^j)(T_0-t) \right\} \\
&\times E\left(\frac{1}{Y_{T_0}^i} e^{-\int_t^{T_0}(b^{ii}Y_s^i + b^{ij}Y_s^j)ds} \right. \\
&\times \left. \left[L^i(Y_{T_0}^i) L^j(Y_{T_0}^j) \exp\left\{ -\int_t^{T_0} \left((b^{ji}-b^{ii})Y_s^i + (b^{jj}-b^{ij})Y_s^j \right) ds \right\} - \widetilde{K} \right]^+ |\mathcal{F}_t \right),
\end{aligned}
$$

where

$$
\widetilde{K} = \frac{K}{X_t^{i,j}} \frac{Y_t^i}{Y_t^j} \exp\left\{ ((\xi^j-\xi^i)+(a^j-a^i))(T_1-t) \right\} \quad .
$$

According to Filipović [2001], for any $w > 1$, $K > 0$, $y \in R$, one has the following relationship

$$
\frac{1}{2\pi} \int_R e^{(w+\mathrm{i}\lambda)y} \frac{K^{-(w-1+\mathrm{i}\lambda)}}{(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)} d\lambda = (e^y - K)^+ \quad .
$$

By setting

$$y = \ln(L^i(Y^i_{T_0})L^j(Y^j_{T_0})) - (b^{ji} - b^{ii}) \int_t^{T_0} Y^i_s ds - (b^{jj} - b^{ij}) \int_t^{T_0} Y^j_s ds \ ,$$

we can express the option price as

$$\begin{aligned}
& C_t^{F,T_0,T_1}(K) \\
& = X_t^{i,j} Y_t^j \exp\left\{-((\xi^j - \xi^i) + (a^j - a^i))(T_1 - T_0) - (\xi^j + a^j)(T_0 - t)\right\} \\
& \quad \times E\left(\frac{1}{Y^i_{T_0}} e^{-\int_t^{T_0}(b^{ii}Y^i_s + b^{ij}Y^j_s)ds} \ \frac{1}{2\pi}\int_R e^{(w+\mathrm{i}\lambda)\ln(L^i(Y^i_{T_0})L^j(Y^j_{T_0}))} \right. \\
& \quad \left. \times e^{-(w+\mathrm{i}\lambda)\left((b^{ji}-b^{ii})\int_t^{T_0} Y^i_s ds + (b^{jj}-b^{ij})Y^j_s ds\right)} \frac{\widetilde{K}^{-(w-1+\mathrm{i}\lambda)}}{(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)} d\lambda | \mathcal{F}_t \right)
\end{aligned}$$

By applying Fubini's theorem, the option price can be expressed as

$$\begin{aligned}
& C_t^{F,T_0,T_1}(K) \\
& = X_t^{i,j} Y_t^j \exp\left\{-((\xi^j - \xi^i) + (a^j - a^i))(T_1 - T_0) - (\xi^j + a^j)(T_0 - t)\right\} \\
& \times \frac{1}{2\pi}\int_R E\left(\frac{1}{Y^i_{T_0}} e^{(w+\mathrm{i}\lambda)\ln(L^i(Y^i_{T_0}))} e^{(w+\mathrm{i}\lambda)\ln(L^j(Y^j_{T_0}))} \right. \\
& \times \left. \exp\left\{-\widetilde{b}^i(\lambda)\int_t^{T_0} Y^i_s ds - \widetilde{b}^j(\lambda)\int_t^{T_0} Y^j_s ds\right\} | \mathcal{F}_t \right) \frac{\widetilde{K}^{-(w-1+\mathrm{i}\lambda)}}{(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)} d\lambda \quad ,
\end{aligned}$$

with $\widetilde{b}^i(\lambda) = b^{ii} + (w+\mathrm{i}\lambda)(b^{ji} - b^{ii})$, $\widetilde{b}^j(\lambda) = b^{ij} + (w+\mathrm{i}\lambda)(b^{jj} - b^{ij})$.

Denote by $\Psi^F(Y^i_t, Y^j_t, \lambda)$ the conditional expectation

$$\begin{aligned}
& \Psi^F(Y^i_t, Y^j_t, \lambda) \\
& = E\left(\frac{1}{Y^i_{T_0}} e^{(w+\mathrm{i}\lambda)\ln(L^i(Y^i_{T_0})) + (w+\mathrm{i}\lambda)\ln(L^j(Y^j_{T_0})) - \widetilde{b}^i(\lambda)\int_t^{T_0} Y^i_s ds - \widetilde{b}^j(\lambda)\int_t^{T_0} Y^j_s ds} | \mathcal{F}_t \right) \quad .
\end{aligned}$$

By applying the property of conditional expectations similarly to the pricing of bond options, and then, observing that $Y^i_{T_0}$ and $Y^j_{T_0}$ are independent, one has that

$$\Psi^F(Y^i_t, Y^j_t, \lambda) = E\left(\frac{1}{Y^i_{T_0}} e^{(w+\mathrm{i}\lambda)\ln(L^i(Y^i_{T_0}))} M(Y^i_t, Y^i_{T_0}, \xi^i, \widetilde{b}^i(\lambda)) | \mathcal{F}_t \right)$$

$$\times E\left(e^{(w+\mathrm{i}\lambda)\ln(L^j(Y^j_{T_0}))}M(Y^j_t, Y^j_{T_0}, \xi^j, \widetilde{b}^j(\lambda))|\mathcal{F}_t\right)$$

Observe that the square-root process of dimension four has an explicit transition density (3.37). Thus, the conditional expectation $\Psi^F(Y^i_t, Y^j_t, \lambda)$ can be obtained almost explicitly as

$$\begin{aligned}
&\Psi^F(Y^i_t, Y^j_t, \lambda)\\
&= \left(\int_0^\infty \frac{1}{y} e^{(w+\mathrm{i}\lambda)\ln(L^i(y))} M(Y^i_t, y, \xi^i, \widetilde{b}^i(\lambda)) p(y, Y^i_t, \xi^i) dy\right)\\
&\quad \times \left(\int_0^\infty e^{(w+\mathrm{i}\lambda)\ln(L^j(y))} M(Y^j_t, y, \xi^j, \widetilde{b}^j(\lambda)) p(y, Y^j_t, \xi^j) dy\right). \qquad (3.39)
\end{aligned}$$

Accordingly, we obtain the almost explicit formula for the option on the forward price as

$$\begin{aligned}
&C_t^{F,T_0,T_1}(K)\\
&= X_t^{i,j} Y_t^j \exp\left\{-((\xi^j - \xi^i) + (a^j - a^i))(T_1 - T_0) - (\xi^j + a^j)(T_0 - t)\right\}\\
&\times \frac{1}{2\pi}\int_R \Psi^F(Y^i_t, Y^j_t, \lambda) \frac{\widetilde{K}^{-(w-1+\mathrm{i}\lambda)}}{(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)} d\lambda \quad ,
\end{aligned}$$

where $\Psi^F(Y^i_t, Y^j_t, \lambda)$ is given in (3.39).

3.2.3.3 Option on Futures Prices

Similar to the case of an option on a forward price, the European call option written on the continuous limit of the futures price can also be obtained in an almost explicit form with a similar technique. The European call option $C_t^{Y^c,T_0,T_1}(K)$ written on the continuous time limit of the futures price $Y_{T_0}^{c,i,j,T_1}$ at time t, $0 < t < T_0 < T_1$, with strike price K and maturity T_0 can be obtained almost explicitly as

$$\begin{aligned}
C_t^{c,Y,T_0,T_1}(K) &= E\left(\frac{V_t^{i,\delta*}}{V_{T_0}^{i,\delta*}}\left[Y_{T_0}^{c,i,j,T_1} - K\right]^+ |\mathcal{F}_t\right)\\
&= X_t^{i,j} Y_t^j \exp\left\{-(\xi^j + (a^j - a^i))(T_1 - t) - a^i(T_0 - t)\right\}
\end{aligned}$$

$$\times\frac{1}{2\pi}\int_R \Psi^Y(Y_t^i, Y_t^j,\lambda)\frac{\widetilde{K}^{-(w-1+\mathrm{i}\lambda)}}{(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)}d\lambda \quad , \tag{3.40}$$

where

$$\begin{aligned}
&\widetilde{K} = \frac{K}{X_t^{i,j}}\frac{Y_t^i}{Y_t^j}\exp\left\{((\xi^j-\xi^i)+(a^j-a^i))(T_1-t)+\xi^i(T_1-T_0)\right\}\\
&\Psi^Y(Y_t^i,Y_t^j,\lambda) =\\
&\quad\left(\int_0^\infty \frac{1}{y}e^{(w+\mathrm{i}\lambda)\ln(yR(y,b^{ji}-b^{ii},\xi^i,T_1-T_0))}M(Y_t^i,y,\xi^i,\widetilde{b}^i(\lambda))q^i(y,Y_t^i,\xi^i)dy\right)\\
&\quad\times\left(\int_0^\infty e^{(w+\mathrm{i}\lambda)\ln(G(y,b^{jj}-b^{ij},\xi^j,T_1-T_0))}M(Y_t^j,y,\xi^j,\widetilde{b}^j(\lambda))q^j(y,Y_t^j,\xi^j)dy\right)
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{b}^i(\lambda) &= b^{ii}+(w+\mathrm{i}\lambda)(b^{ji}-b^{ii})\\
\widetilde{b}^j(\lambda) &= b^{ij}+(w+\mathrm{i}\lambda)(b^{jj}-b^{ij}) \quad .
\end{aligned}$$

The corresponding put option $P_t^{Yc,T_0,T_1}(K)$ can, thus, be obtained by applying the real world put-call parity (3.22) as

$$P_t^{c,Y,T_0,T_1}(K) = C_t^{c,Y,T_0,T_1}(K) - X_t^{i,j}E\left(\frac{V_t^{j,\delta*}}{V_{T_1}^{j,\delta*}}\frac{S_{T_1}^{i,i}}{S_{T_0}^{i,i}}\bigg|\mathcal{F}_t\right)+KP^i(t,T_0)$$

where the call option price $C_t^{c,Y,T_0,T_1}(K)$ is given in (3.40), the zero coupon bond price $P^i(t,T_0)$ is given by (3.32) and the conditional expectation $E\left(\frac{V_t^{j,\delta*}}{V_{T_1}^{j,\delta*}}\frac{S_{T_1}^{i,i}}{S_{T_0}^{i,i}}|\mathcal{F}_t\right)$ can be obtained as

$$\begin{aligned}
E\left(\frac{V_t^{j,\delta*}}{V_{T_1}^{j,\delta*}}\frac{S_{T_1}^{i,i}}{S_{T_0}^{i,i}}\bigg|\mathcal{F}_t\right) &= E\left(\frac{Y_t^j}{Y_{T_1}^j}\frac{A_t^j}{A_{T_1}^j}e^{-\int_t^{T_0}r_s^j ds-\int_{T_0}^{T_1}(r_s^j-r_s^i)ds}\bigg|\mathcal{F}_t\right)\\
= Y_t^j\frac{A_t^j}{A_{T_1}^j}e^{-a^j(T_1-t)+a^i(T_1-T_0)}&E\left(\frac{1}{Y_{T_1}^j}e^{-\int_t^{T_0}b^{jj}Y_s^j ds-\int_{T_0}^{T_1}(b^{jj}-b^{ij})Y_s^j ds}\bigg|\mathcal{F}_t\right)\\
&\times E\left(e^{-\int_t^{T_0}b^{ji}Y_s^i ds-\int_{T_0}^{T_1}(b^{ji}-b^{ii})Y_s^i ds}\bigg|\mathcal{F}_t\right)
\end{aligned}$$

$$
\begin{aligned}
&= Y_t^j \frac{A_t^j}{A_{T_1}^j} e^{-a^j(T_1-t)+a^i(T_1-T_0)} \\
&\quad \times E\left(\frac{1}{Y_{T_1}^j} M(Y_t^j, Y_{T_0}^j, \xi^j, b^{jj}) M(Y_{T_0}^j, Y_{T_1}^j, \xi^j, b^{jj} - b^{ij}) \Big| \mathcal{F}_t \right) \\
&\quad \times E\left(M(Y_t^i, Y_{T_0}^i, \xi^i, b^{ji}) M(Y_{T_0}^i, Y_{T_1}^i, \xi^i, b^{ji} - b^{ii}) \Big| \mathcal{F}_t \right) \quad .
\end{aligned}
$$

The above conditional expectations can be obtained by applying numerical methods such as exact simulation schemes, see Platen and Rendek [2009].

3.2.4 Non-existence of a Risk Neutral Probability Measure

An important feature of the proposed MMM model is that it does not have an equivalent risk neutral probability measure. Recall that the savings account $S_t^{i,i}$ satisfies equation (2.21), $i \in \{1, \ldots, d+1\}$. In the given complete market, the Radon-Nikodym derivative Λ_t^i for the putative risk neutral measure for the ith currency denomination is given by the ratio of the ith savings account over the NP in ith denomination, and normalized to one at the initial time, that is, one has

$$
\Lambda_t^i = \frac{S_t^{i,i} V_0^{i,\delta*}}{S_0^{i,i} V_t^{i,\delta*}} = \frac{\hat{S}_t^i}{\hat{S}_0^i} \quad .
$$

In case the ith benchmarked savings account $\hat{S}_t^i$ and, thus, Λ_t^i were $(\underline{\mathcal{F}}, P)$-martingales, an equivalent risk neutral probability measure would exist and the risk neutral pricing formula could be shown to follow from the real world pricing formula. More precisely, for a contingent claim $H_T^i = \hat{H}_T V_T^{i,\delta*}$ in ith security denomination, it follows from the real world pricing formula (2.13) that

$$
\hat{V}_t^{i,\delta_{\hat{H}_T}} = E\left(\frac{1}{V_T^{i,\delta*}} H_T^i | \mathcal{F}_t \right) \quad .
$$

This means, one has

$$V_t^{i,\delta_{\hat{H}_T}} = V_t^{i,\delta *} \hat{V}_t^{i,\delta_{\hat{H}_T}} = E\left(\frac{\Lambda_T^i}{\Lambda_t^i} \frac{S_t^{i,i} H_T^i}{S_T^{i,i}} | \mathcal{F}_t\right) \ .$$

By using the Bayes rule and exploiting the fact that under classical assumptions Λ^i forms an $(\underline{\mathcal{F}}, P)$-martingale, one obtains in this special case from the real world pricing formula the risk neutral pricing formula

$$V_t^{i,\delta_{\hat{H}_T}} = E^Q\left(\frac{S_t^{i,i}}{S_T^{i,i}} H_T^i | \mathcal{F}_t\right) \ , \tag{3.41}$$

where E^Q denotes the conditional expectation under the assumed risk neutral measure Q. However, for the above MMM this line of arguments breaks down because in this case, one has for $i \in \{1, \ldots, d+1\}$ and $t \in [0, T)$, that

$$E\left(\frac{\Lambda_T^i}{\Lambda_t^i} | \mathcal{F}_t\right) = \left(1 - \exp\left\{-\frac{2Y_t^i}{e^{\xi^i (T-t)} - 1}\right\}\right) < 1 \ , \tag{3.42}$$

see, Platen and Heath [2010]. This shows that the Radon-Nikodym derivative Λ^i of the putative risk neutral measure is here a strict supermartingale and not a martingale. Consequently, under the MMM an equivalent risk neutral probability measure does not exist. Despite the fact that risk neutral pricing cannot be applied, one can still apply the real world pricing formula (2.13). If one formally applies risk neutral pricing, then a systematic pricing error arises, as we discuss below.

"Risk Neutral" Zero Coupon Bond

For example, we assume that a "risk neutral" agent assumes the existence of equivalent risk neutral probability measures, however, the market dynamics follows the above MMM. Furthermore, this agent chooses for pricing the minimal martingale measure of Föllmer and Schweizer, which changes the drift of the ith Wiener process but keeps the drift of the jth Wiener process unchanged, see Schweizer [1999]. The SDE (3.23) under the new "risk neutral" measure Q

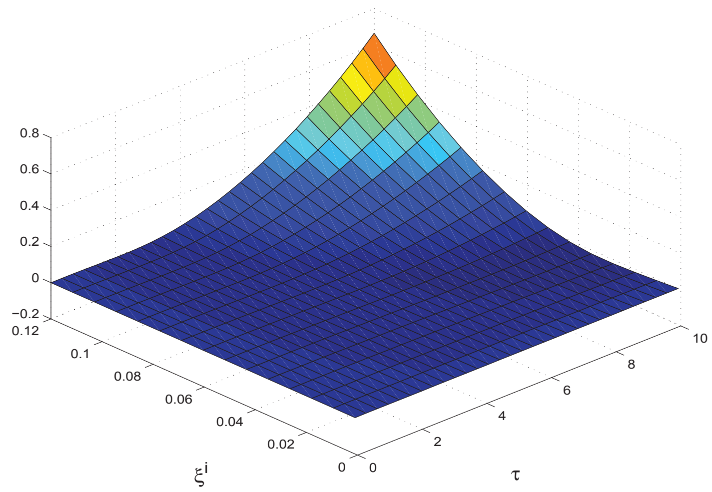

Figure 3.3: *Relative pricing error of "risk neutral" zero coupon bond*

becomes

$$d\bar{V}_t^{i,\delta *} = \sqrt{\bar{V}_t^{i,\delta *}\alpha_t^i}dW_t^{i,Q} \quad . \tag{3.43}$$

Note that Q is not a true probability measure. Recall that $\alpha_t^i = A_t^i\xi^i$ and $Y_t^i = \frac{\bar{V}_t^{i,\delta *}}{A_t^i}$. By applying the Itô formula, the SDE of Y_t^i can be obtained under the Q measure as

$$dY_t^i = -\xi^i Y_t^i dt + \sqrt{Y_t^i\xi^i}dW_t^{i,Q} \quad .$$

Observe that here Y^i is a square-root process of dimension 0. The Q-probability of this process to reach zero does not vanish, whereas its P-probability vanishes for this event. Obviously, Q and P are not equivalent measures. By applying formally the risk neutral pricing formula (3.41), the "risk neutral" zero coupon bond price of the ith currency is obtained as

$$\begin{aligned} P^{i,*}(t,T) &= E^Q\left(\frac{S_t^{i,i}}{S_T^{i,i}}|\mathcal{F}_t\right) \\ &= E^Q\left(e^{-\int_t^T r_s^i ds}|\mathcal{F}_t\right) = E^Q\left(e^{-\int_t^T (a^i+b^{ii}Y_s^i+b^{ij}Y_s^j)ds}|\mathcal{F}_t\right) \end{aligned}$$

$$
\begin{aligned}
&= e^{-a^i(T-t)} E^Q \left(e^{-\int_t^T b^{ii} Y_s^i ds} | \mathcal{F}_t \right) E^Q \left(e^{-\int_t^T b^{ij} Y_s^j ds} | \mathcal{F}_t \right) \\
&= e^{-a^i(T-t)} L^P(Y_t^i, b^{ii}, \xi^i, T-t) L^P(Y_t^j, b^{ij}, \xi^j, T-t),
\end{aligned}
$$

where

$$
L^P(Y_t^i, b^{ii}, \xi^i, T-t) = E^Q \left(e^{-b^{ii} \int_t^T Y_s^i ds} | \mathcal{F}_t \right) \quad .
$$

Here, the conditional expectation $L^P(Y_t^i, b^{ii}, \xi^i, T-t)$ can be solved by applying numerical methods, see Platen and Bruti-Liberati [2010].

The relative pricing error of the "risk neutral" zero coupon bond price in the sense of this example can be obtained as

$$
\begin{aligned}
\epsilon &= \frac{P^{i,*}(t,T) - P^i(t,T)}{P^i(t,T)} \\
&= \frac{L^P(Y_t^i, b^{ii}, \xi^i, T-t) - Y_t^i e^{-\xi^i(T-t)} G(Y_t^i, b^{ii}, \xi^i, T-t)}{Y_t^i e^{-\xi^i(T-t)} G(Y_t^i, b^{ii}, \xi^i, T-t)} \quad .
\end{aligned}
$$

In Figure 3.3, we plot the relative pricing error ϵ as a function of ξ^i and time to maturity $\tau = T-t$. The initial value Y_t^i and Y_t^j are set to be 1. We have also used $b^{ii} = 0.02$. Figure 3.3 shows that the relative pricing error of "risk neutral" zero coupon bond in the sense of (3.43) is rather small for very short time to maturity. However, as the time to maturity becomes large, the systematic mispricing of the "risk neutral" approach emerges strongly, and can not be neglected anymore.

In the next section, we visualize the benchmarked US dollar savings account for the period from 1973 until 2012, where one can see that the Radon-Nikodym derivative of the putative risk neutral measure should be most likely modeled by a strict supermartingale and not by a martingale. Furthermore, in Section 3.2.5 we show benchmarked savings accounts for commodity spot prices, where a similar conclusion can be drawn.

3.2.5 Empirical Observations and Model Fitting

In the following, we apply market data of commodity spot prices and a well diversified stock index to visualize a proxy of the NP and several benchmarked

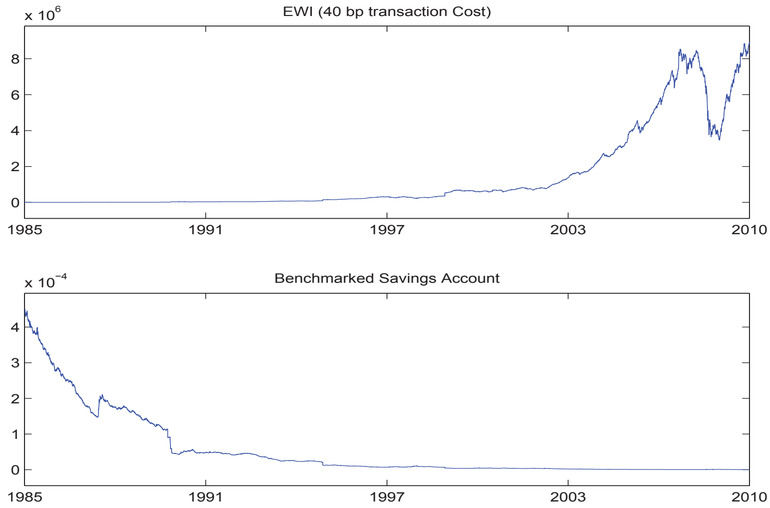

Figure 3.4: *The EWI in US dollar denomination (upper panel) and benchmarked US savings account (lower panel)*

savings accounts.

To begin with, let us approximate the NP in US dollar denomination. Here, we apply an equi-weighted index (EWI) with 40 basis point proportional transaction costs as described in Platen and Rendek [2012b]. The EWI is an extremely broadly diversified stock index, which is daily adjusted by, in principle, equal value weighting available market capitalization weighted Datastream country subsector indices. According to Platen [2005] and Platen and Rendek [2012b], such a well diversified stock index can be interpreted as an approximation of the NP. It is exhibited in US dollar denomination in the upper panel of Figure 3.4.

Furthermore, we approximate the interest rate of the US currency by the yield of the three months US Treasury Bill. The benchmarked US savings account

$$\hat{S}_t^i = \frac{S_t^{i,i}}{V_t^{i,\delta*}}$$

can, thus, be obtained from observed data and is displayed in the lower panel of Figure 3.4. Observe that the benchmarked US savings account $\hat{S}_t^i$ behaves more likely as a real world strict supermartingale than as a martingale. Note

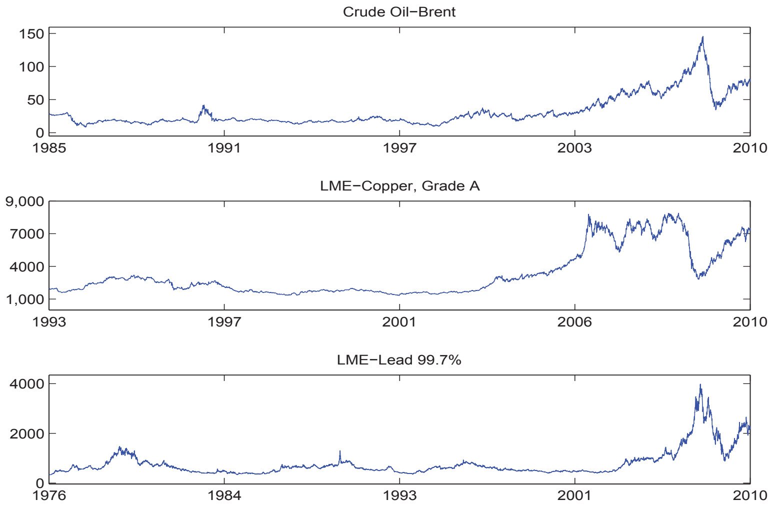

Figure 3.5: *Spot prices of three commodities: crude oil (upper panel), copper (middle panel), aluminium (lower panel)*

that a complete market model under the classical no-arbitrage paradigm assumes the benchmarked savings account to be a martingale. This classical assumption would create for long term derivatives potentially a significant model error in view of Figure 3.4. Closer to reality are models with benchmarked savings accounts that are strict supermartingales, as is the case for the previously discussed MMM and the model we will derive in the next chapter.

We also obtained daily data for several commodity spot prices and three months commodity futures prices from Datastream. These include prices for crude oil (Brent), copper (LME, grade A), and Aluminium (LME). In Figure 3.5, we plot the spot prices of these three commodities.

By applying formula (3.1) for the commodity spot price, the NP in commodity denomination can be obtained from its US dollar denomination by the equation

$$V_t^{j,\delta*} = \frac{V_t^{i,\delta*}}{X_t^{i,j}} \ .$$

Figure 3.6 displays the resulting denominations of the NP in units of the above mentioned three commodities. In all three cases we note that the NP rises on

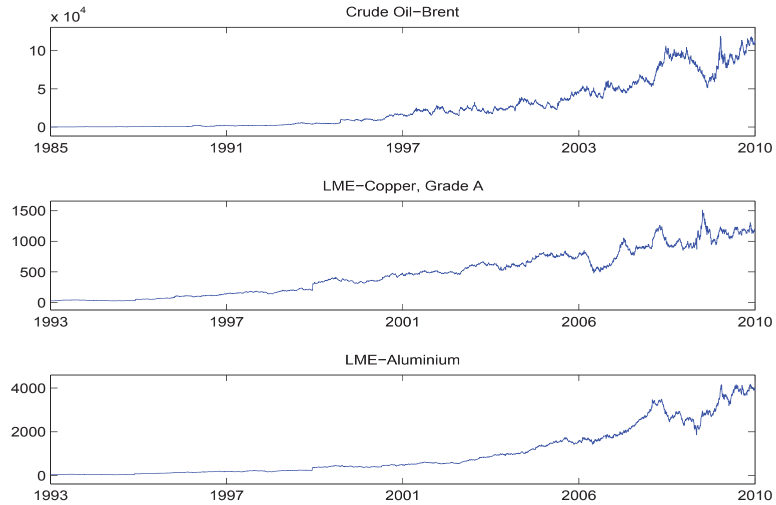

Figure 3.6: *EWI in three commodity denominations: crude oil (upper panel), copper (middle panel), aluminium (lower panel).*

average substantially over the observation period.

The next step is to construct empirically the commodity savings account

$$S_t^{j,j} = e^{\int_0^t r_s^j ds} ,$$

which is, in general, different from the spot price. This construction is similar to what one does to obtain the savings account of a currency. For determining the commodity savings account, we have to estimate the whole trajectory of the convenience yield r_s^j, $s \in [0,T]$. Recall that we have approximated the interest rate by the yield of three months US Treasury Bills. In the case of commodities, we can use a similar argument as follows:

We assume that for a rather short time period Δ, in our case three months, the continuous time limit $Y_t^{c,i,j,t+\Delta}$ of the futures price (3.19) approximates the futures price $Y_t^{i,j,t+\Delta}$ sufficiently well, and we can approximately write

$$Y_t^{i,j,t+\Delta} \approx Y_t^{c,i,j,t+\Delta} = X_t^{i,j} E\left(\frac{\hat{S}_{t+\Delta}^j}{\hat{S}_t^j} \exp\left\{ -\int_t^{t+\Delta} \left(r_s^j - r_s^i\right) ds \right\} \Big| \mathcal{F}_t \right)$$

$$\approx X_t^{i,j} e^{-(r_t^j - r_t^i)\Delta} E\left(\frac{\hat{S}_{t+\Delta}^j}{\hat{S}_t^j} \bigg| \mathcal{F}_t \right) \; .$$

We assume further that for the rather short time period of three months, the benchmarked savings account $\hat{S}_t^j$ of the jth commodity has not changed much and one can set

$$E\left(\frac{\hat{S}_{t+\Delta}^j}{\hat{S}_t^j} \bigg| \mathcal{F}_t \right) \approx 1 \; .$$

This does not mean that $\hat{S}_t^j$ is assumed to form a martingale, however, it indicates that it is close to a local martingale. The above approximation is similar to the one used for currencies, see Platen and Heath [2010].

The futures price $Y_t^{i,j,t+\Delta}$ can, thus, be approximated as

$$Y_t^{i,j,t+\Delta} \approx X_t^{i,j} e^{-(r_t^j - r_t^i)\Delta}$$

for a small time period Δ. Therefore, the convenience yield r_t^j of the jth commodity at time $t \geq 0$ can be estimated according to the formula

$$r_t^j \approx -\frac{\ln\left(\frac{Y_t^{i,j,t+\Delta}}{X_t^{i,j}} \right)}{\Delta} + r_t^i \; .$$

This is similar to the classical estimation of convenience yields, as e.g. described in Geman [2005]. Note, in the above situation we do not assume the existence of a risk neutral probability measure. Using three months futures prices provided by Datastream, the estimated convenience yields for the three commodities are plotted in Figure 3.7. One notes in all three cases significant fluctuations but also a mean-reverting behaviour.

Now, the benchmarked savings account $\hat{S}_t^j$ can be constructed as

$$\hat{S}_t^j = \frac{S_t^{j,j}}{V_t^{j,\delta *}} \; .$$

By employing the observed quantities we calculate the respective three benchmarked savings accounts and display these in Figure 3.8. Observe that the bench-

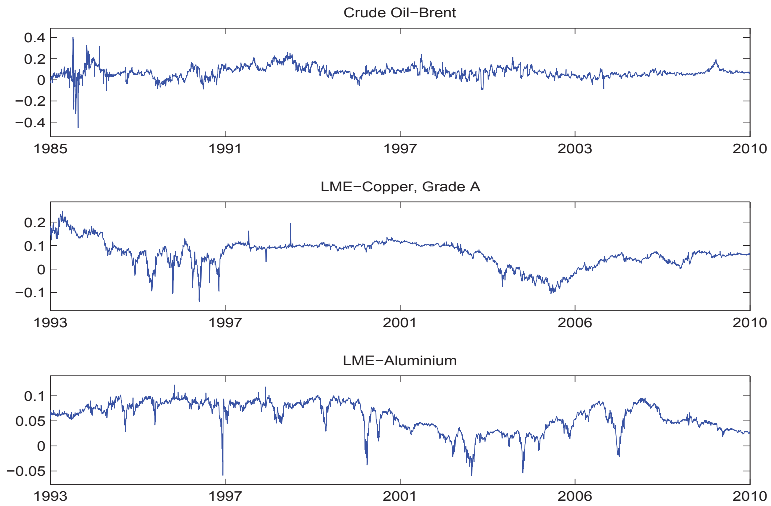

Figure 3.7: *Estimated convenience yields of three commodities: crude oil (upper panel), copper (middle panel), aluminium (lower panel)*

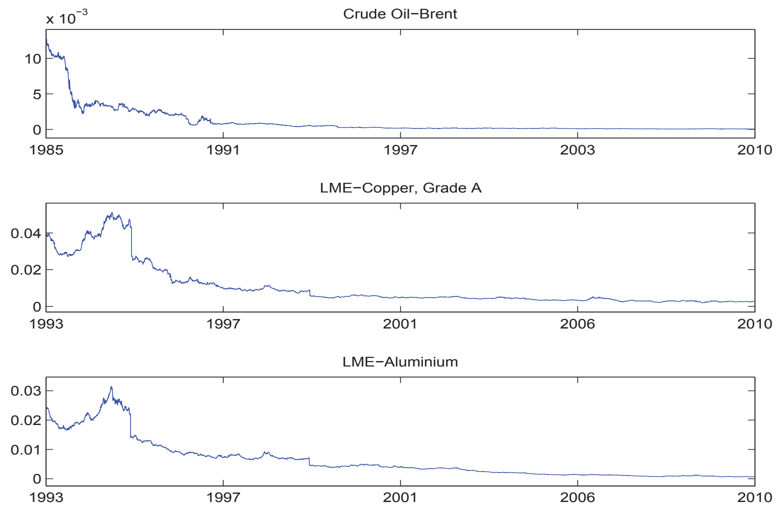

Figure 3.8: *Benchmarked savings accounts of three commodities: crude oil (upper panel), copper (middle panel), aluminium (lower panel)*

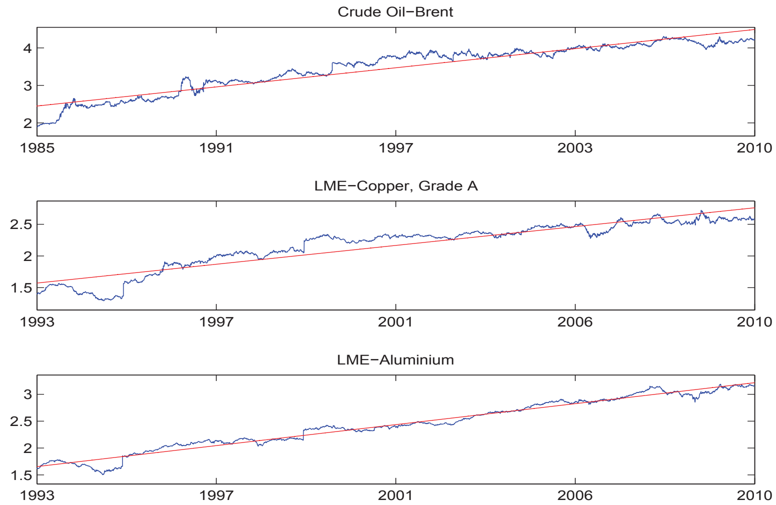

Figure 3.9: *Logarithm of discounted EWI in commodities denominations of three commodities: crude oil (upper panel), copper (middle panel), aluminium (lower panel)*

marked savings accounts for all these three commodities behave more like strict supermartingales than martingales. This observation suggests that it is more realistic for long term modeling to employ models that reflect this strict supermartingale property. As we discussed earlier, such models may be not covered under the classical no-arbitrage paradigm. However, they are accommodated by the benchmark approach.

Figure 3.9 plots the logarithm of the NP when denominated in the respective commodity savings account together with a linearly regressed trend line. The slope of the trend line yields the estimate for the net growth rate ξ^j under the MMM. The normalized NP Y_t^j of the jth commodity is then obtained by dividing the discounted NP, $\bar{V}_t^{j,\delta*} = \frac{V_t^{j,\delta*}}{S_t^{j,j}}$, by the exponential function $A_t^j = A^j e^{\xi^j t}$, that is,

$$Y_t^j = \frac{\bar{V}_t^{j,\delta*}}{A_t^j} \ ,$$

for $t \geq 0$, where the scale parameter A^j in (3.25) is chosen such that the mean of Y_t^j amounts to one. The resulting normalized NP for different commodities

Security	A^j	ξ^j	b^{jj}	b^{ji}	a^j	Y_0^j
Crude Oil	11.58311	0.078358	0.228957	0.777764	-0.93401	0.57379
Copper	4.811697	0.068302	0.060352	0.105778	-0.10143	0.84818
Aluminium	5.224492	0.089741	0.022486	0.038163	-0.00249	0.972092
Security	A^i	ξ^i	b^{ii}	b^{ij}	a^i	Y_0^i
US Dollar	31.7658	0.116598	0.00000	0.060447	-0.01279	0.895317

Table 3.1: *Estimated parameters for three commodities and the US dollar.*

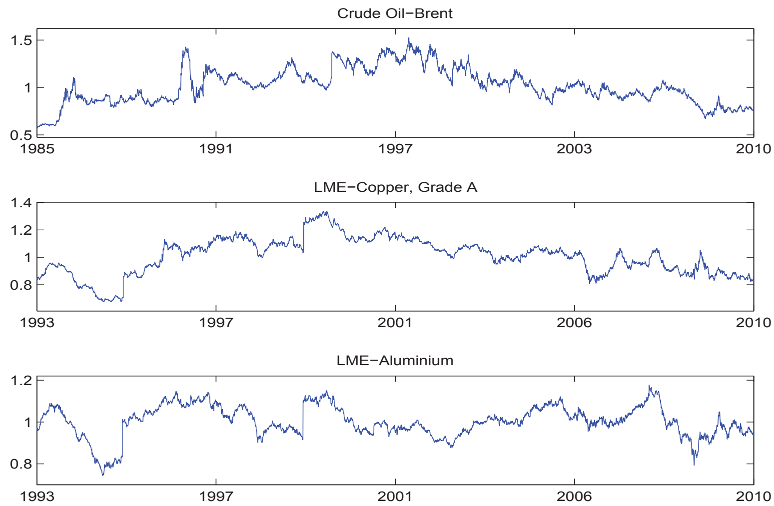

Figure 3.10: *Normalized NP of three commodities: crude oil (upper panel), copper (middle panel), aluminium (lower panel).*

is shown in Figure 3.10. All three processes seem to show a mean-reverting behaviour and potentially have a stationary density. When using our estimated parameters together with the observed initial values, we obtain with the MMM a model for these processes. In Table 3.1 we show for the three commodities and the US dollar the estimated parameters A^j and ξ^j. Note that the US dollar has also a normalized NP for which we obtained the respective parameters similarly and included these in Table 3.1.

We can now analyze the dependence of the jth convenience yield on the jth normalized NP. In Figure 3.11, we display the quadratic covariation between the convenience yield and the respective normalized NP for each of the three

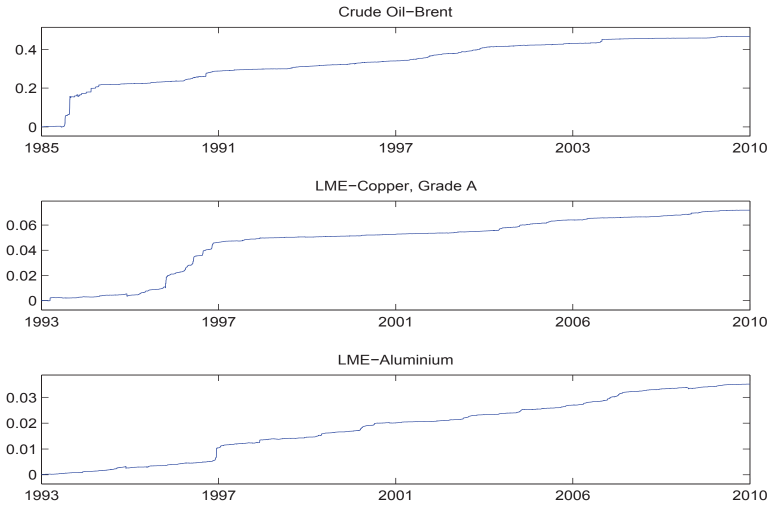

Figure 3.11: *Quadratic covariation between convenience yield and respective normalized NP of three commodities: crude oil (upper panel), copper (middle panel), aluminium (lower panel)*

commodities. The figure indicates that the convenience yields and the normalized NP for all these three commodities have a positive correlation. To be precise, we estimate the corresponding parameters in (3.27) for the convenience yields and the US short rate as follows:

From (3.26) and (3.27) it follows that the quadratic covariation of r_t^j and Y_t^j yields

$$[r^j, Y^j]_T = b^{jj}[Y^j]_T = b^{jj}\xi^j \int_0^T Y_s^j ds \quad ,$$

$j \in \{1, \ldots, d+1\}$. Therefore, by using the fact that the mean of the square root process Y^j equals one, one obtains approximately

$$b^{jj} \approx \frac{[r^j, Y^j]_T}{\xi^j T} \quad ,$$

$j \in \{1, \ldots, d+1\}$. Furthermore, From (3.27) it follows that the quadratic variation of r_t^j yields

$$[r^j]_T = (b^{jj})^2 \xi^j \int_0^T Y_s^j ds + (b^{ji})^2 \xi^i \int_0^T Y_s^i ds \ .$$

By using the fact that the mean of the square root processes Y^i and Y^j equals one, one obtains approximately

$$b^{ji} \approx \left(\frac{1}{\xi^i T} [r^j]_T - (b^{jj})^2 \frac{\xi^j}{\xi^i} \right)^{\frac{1}{2}} \ . \tag{3.44}$$

The estimated parameters b^{jj} and b^{ji} are also shown in Table 3.1, $j, i \in \{1, \ldots, d+1\}$, $j \neq i$.

Finally, we notice that the average $\bar{r}^j = \frac{1}{T} \int_0^T r_s^j ds$ of r_t^j equals by (3.27) approximately

$$\bar{r}^j \approx a^j + b^{ji} + b^{jj} \quad ,$$

which yields

$$a^j \approx \bar{r}^j - b^{ji} - b^{jj} \ ,$$

$j, i \in \{1, \ldots, d+1\}$, $j \neq i$. The resulting parameter estimates are shown in Table 3.1, together with the initial values Y_0^j. Note that for US dollar the short rate appears, in principle, to be independent of the normalized NP in US dollar denomination.

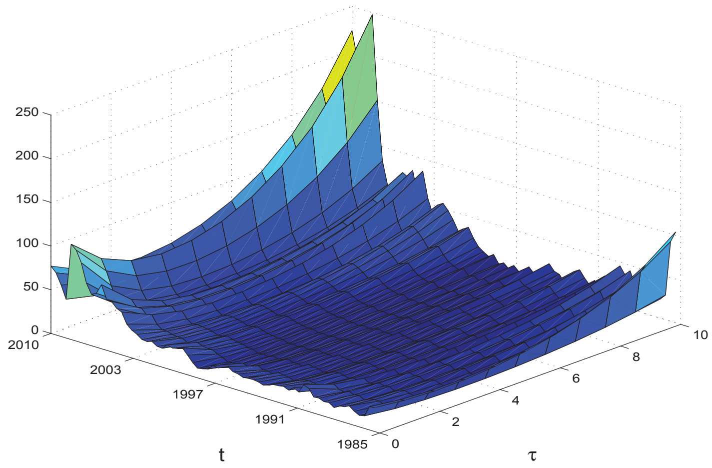

Figure 3.12: *Calculated term structure of the real world futures price of crude oil*

With the estimated parameters, we can now calculate the real world prices for commodity derivatives. In Figure 3.12, we display the term structure of the calculated real world crude oil continuous time futures prices.

3.3 Conclusions on Commodity Forward and Futures Contracts

The chapter studied forward and futures contracts of commodities under the benchmark approach. The commodity prices are modeled under the real world probability measure while the corresponding numéraire is chosen to be the numéraire portfolio. By applying the real world pricing formula, rather than the classical risk neutral pricing formula, several interesting properties of benchmarked forward and futures prices emerge, which help to understand their differences to classical prices.

An alternative model for real world futures prices of commodities is employed. This model is a hybrid of the minimal market model and affine short rate models. It does not have an equivalent risk neutral probability measure but still provides computationally tractable formulas for futures prices and prices of European call and put options written on futures prices of commodities.

Chapter 4

Modeling of Oil Prices

In Chapter 3, we have denominated the numéraire portfolio (NP) in units of three commodities (crude oil, copper and aluminium). Several important empirical facts, in particular, the strict supermartingale property of the benchmarked savings account can be observed in all of these three denominations, as well as, the US dollar denomination. Furthermore, a version of minimal market model has been studied as an example, which captures the strict supermartingale property, and provides tractable formulas for forward, futures and option prices on commodities. In this chapter, we move one step further by focusing in more detail on the stylized empirical facts of the discounted numéraire portfolio of crude oil. The following study is motivated by the possibility that oil prices behave, in some sense, similarly to the US dollar because oil is traded in this currency. Furthermore, we will see that log-returns of our proxy for the NP in oil price denomination appear to follow a Student-t distribution. For these reasons we model oil prices by a similar methodology as was proposed in Platen and Rendek [2012a] for currencies, which derives a parsimonious two-component affine diffusion model with one driving Brownian motion. The observable state variables of the model are the normalized NP and the inverse of the stochastic market activity, both modeled as square root processes.

The square root process in market activity time for the normalized aggregate wealth emerges from the affine nature of aggregate wealth dynamics, which has been derived in Platen and Rendek [2012a] under basic assumptions and does not contain any parameters that have to be estimated. The proposed two-component model for the normalized NP in oil price denomination reproduces a list of major stylized empirical facts, including Student-t distributed log-returns, typical volatility properties and the strict local martingale property of the benchmarked saving account.

This chapter is organized as follows: Section 4.1 extracts a list of stylized empirical facts for the observed normalized NP dynamics for oil price denomination. Section 4.2 proposes for these dynamics a parsimonious, tractable model involving the power of a time transformed affine diffusion. It also discusses the volatility and market activity dynamics arising from the proposed model. Section 4.3 describes a robust step-by-step methodology for fitting the proposed model. It also visualizes volatility and market activity as they emerge under the model. Section 4.4 describes for the model an almost exact simulation method, which allows us to confirm that the empirical properties of the model match the list of stylized empirical facts of Section 4.1. Finally, Section 4.5 summarizes the model described in Platen and Rendek [2012a] for the denomination of the numéraire portfolio in currency denomination. This allows us to model the oil price in currency denomination.

4.1 Empirical Observations

Platen and Rendek [2012a] observed seven stylized empirical facts pertaining to diversified world stock indices in currency denomination. Below we check whether similar or different properties can be identified for the oil savings account denominated NP, that is, the proxy of the NP, here called the EWI, of the previous chapter denominated in units of an oil savings account, which we call below oil discounted EWI.

(i) Uncorrelated Returns

Figure 4.1 displays the autocorrelation function for the log-returns of the oil discounted EWI with 95% confidence bounds. Similarly, to the log-returns of this index in currency denominations, the autocorrelation of log-returns of the oil discounted EWI is close to zero.

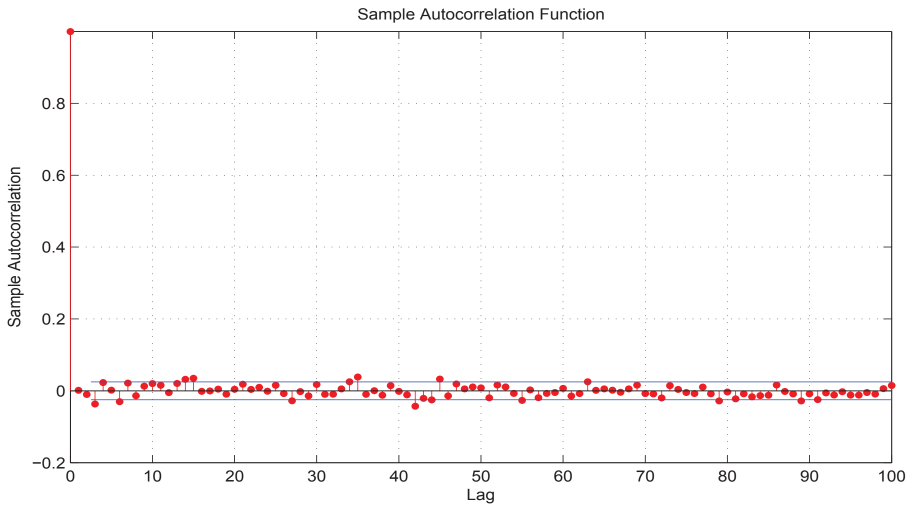

Figure 4.1: Autocorrelation function for log-returns of the oil discounted EWI.

(ii) Correlated Absolute Returns

Figure 4.2 plots the autocorrelation function of the absolute log-returns of the oil discounted EWI. Even for large lags the autocorrelation is non-negligible and does not seem to show an exponential decline.

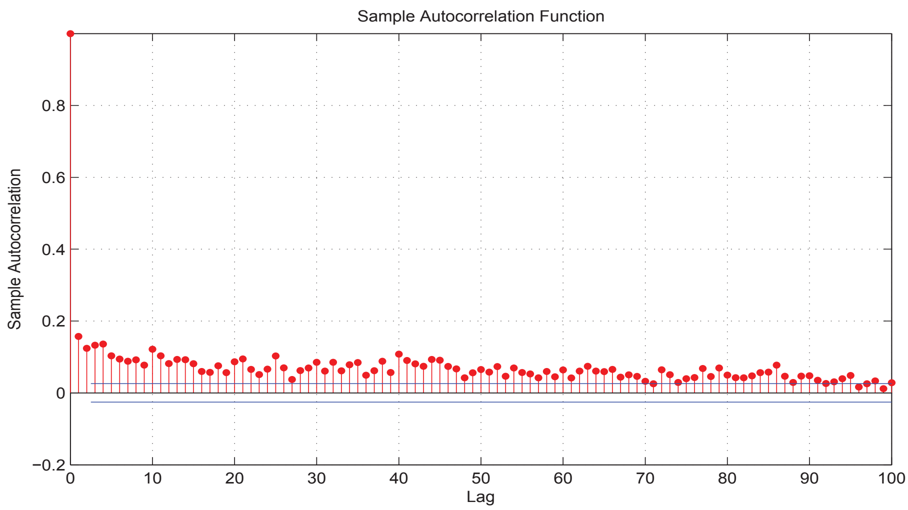

Figure 4.2: Autocorrelation function for the absolute log-returns of the oil discounted EWI.

(iii) Student-t Distributed Returns

Figure 4.3 displays the log-histogram of normalized log-returns of the oil discounted EWI with the logarithm of the Student-t density with 3.13 degrees of freedom. We refer to the last column in Table 4.1 for the estimated degrees of freedom. Visually the fit seems to be very good. In order to further quantify the fit of the Student-t distribution we perform a log-likelihood ratio test in the family of the symmetric generalized hyperbolic (SGH) distributions, see Rao [1973] and Platen and Rendek [2008]. Table 4.1 reports the test statistics calculated for four special cases of the SGH distribution. These are: the Student-t distribution, the normal inverse Gaussian (NIG) distribution, the hyperbolic distribution and the variance gamma (VG) distribution. The test statistics are here distributed according to the chi-square distribution with one degree of freedom. Therefore, the hypothesis of the Student-t distribution being the best candidate distribution in the family of the SGH distributions cannot be rejected at the 99.9% level of significance, since $0.00000002 < \chi^2_{0.001,1} \approx 0.000002$.

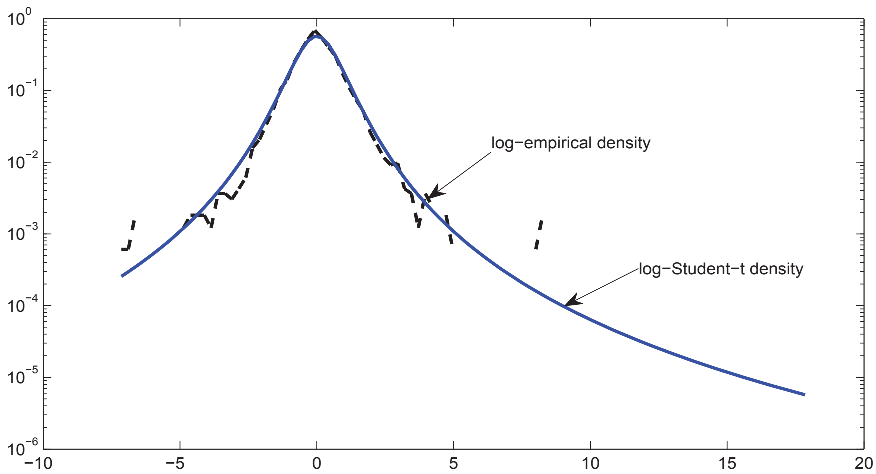

Figure 4.3: Logarithms of empirical density of normalized log-returns of the oil discounted EWI and Student-t density with 3.13 degrees of freedom.

(iv) Volatility Clustering

Table 4.1: Log-Maximum likelihood test statistic for the log-returns of the oil discounted EWI.

Commodity	Student-t	NIG	Hyperbolic	VG	df.
Crude Oil	**0.00000002**	61.168568	182.120161	181.189575	3.13

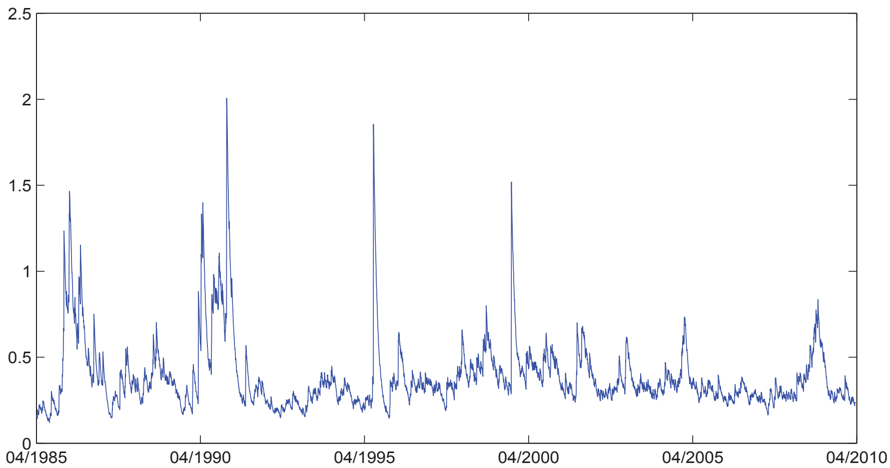

Figure 4.4: Estimated volatility from log-returns of the oil discounted EWI.

Figure 4.4 illustrates the estimated annualized volatility V_{t_i} of the oil discounted EWI. The squared volatility $V_{t_i}^2$ at time t_i is obtained from squared log-returns via exponential smoothing. For the discretization time $t_i = \Delta i$, for $i \in \{0, 1, 2, \ldots\}$, the exponential smoothing is applied to squared log-returns $R_{t_i}^2$ in the following way:

$$V_{t_{i+1}}^2 = \alpha\sqrt{\Delta}R_{t_i}^2 + (1 - \alpha\sqrt{\Delta})V_{t_i}^2, \tag{4.1}$$

for $i \in \{0, 1, 2, \ldots\}$. Here the smoothing parameter λ is assumed to equal $\alpha = 0.92$. This choice works well and has been used in Platen and Rendek [2012a].

The volatility of the oil discounted EWI in Figure 4.4 exhibits periods of

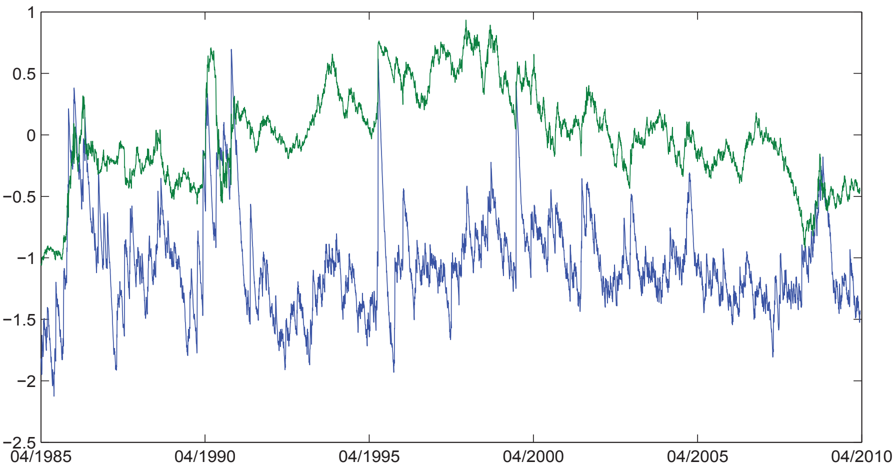

Figure 4.5: Logarithms of normalized EWI for oil (upper graph) and its volatility (lower graph).

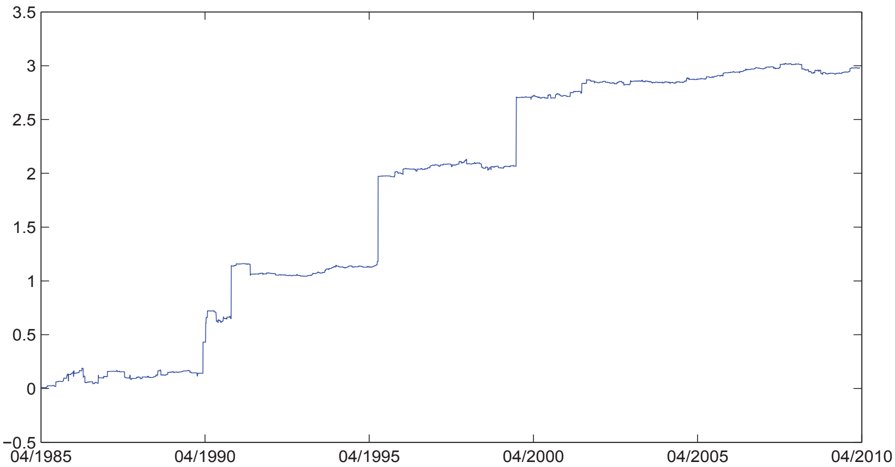

Figure 4.6: Quadratic covariation between the logarithms of normalized EWI for oil and its volatility.

low volatility and periods of high volatility. It is reasonable to conjecture that such volatility is potentially a stationary stochastic process.

(v) Long Term Exponential Growth

Figure 4.7 illustrates the logarithm of the oil discounted EWI with a trend line fitted by linear regression. The logarithm of the oil discounted EWI exhibits consistent long term linear growth, which in turn results in the long term exponential growth for the oil discounted EWI.

(vi) Anti-Leverage Effect

A leverage effect is typically observed for the currency discounted world stock index, see Platen and Rendek [2012a]. This empirical property is, however, not observed for the oil discounted EWI and its normalized version, shown in Figure 4.8. In Figure 4.8, the average long term growth of the oil discounted EWI is taken out by dividing with a respective exponential function of time. In Figure 4.5 we plot the logarithms of the normalized EWI for oil and its volatility. We observe that when the normalized EWI for oil moves upwards, in general, the volatility increases, and vice versa. This implies an anti-leverage effect for the oil discounted EWI and its volatility. In fact, the covariation function between the normalized EWI for oil and its volatility, displayed in Figure 4.6, indicates a mostly positive correlation between the increments for the normalized EWI for oil and its volatility.

(vii) Extreme Volatility at Major Oil Discounted EWI Moves

Extreme volatility at major index downturns was observed in Platen and Rendek [2012a] for the discounted EWI in currency denominations. Figure 4.5 and Figure 4.6, however, indicate that for the normalized EWI for oil the volatility increases when the index moves strongly up and the increase is more substantial when compared to "normal" moves of the index.

4.2 Modeling of Oil Prices

This section derives a parsimonious two-component model for the oil discounted EWI. It follows to some extent the methodology described in Platen and Rendek [2012a] with some important differences in the design of the dependencies in the two-component model.

Discounted Index

The discounted index $\bar{S}_t^{\delta *}$, which is the oil discounted EWI introduced previously, is modeled by the product

$$\bar{S}_t^{\delta *} = A_{\tau_t}(Y_{\tau_t})^q \tag{4.2}$$

for $t \geq 0$, see Platen and Rendek [2012a], $q > 0$. An exponential function A_{τ_t} of a given τ-time, the market activity time (to be specified below), models the long term average growth of the discounted index as

$$A_{\tau_t} = A \exp\{a\tau_t\} \tag{4.3}$$

for $t \geq 0$.

We use in (4.3) the initial parameter $A > 0$ and the long term average net growth rate $a \in R$ with respect to market activity time.

Normalized Index

As a consequence of equation (4.2), the ratio $(Y_{\tau_t})^q = \frac{\bar{S}_t^{\delta *}}{A_{\tau_t}}$ denotes the normalized index, that is, the normalized index for oil, at time t. This normalized index is assumed to form an ergodic diffusion process evolving according to τ-time. We assume that it satisfies the SDE

$$dY_\tau = \left(\frac{\delta}{4} - \frac{1}{2} \left(\frac{\Gamma\left(\frac{\delta}{2} + q\right)}{\Gamma\left(\frac{\delta}{2}\right)} \right)^{\frac{1}{q}} Y_\tau \right) d\tau + \sqrt{Y_\tau}\, dW(\tau), \tag{4.4}$$

for $\tau \geq 0$ with $Y_0 > 0$. Only the two parameters $\delta > 2$ and $q > 0$ enter the SDE (4.4) together with its initial value $Y_0 > 0$.

Market Activity Time

We model the evolution of the market activity time τ_t via the ordinary differential equation

$$d\tau_t = M_t dt \tag{4.5}$$

for $t \geq 0$ with $\tau_0 \geq 0$. Here we call the derivative of τ-time with respect to calendar time t the market activity $\frac{d\tau_t}{dt} = M_t$ at time $t \geq 0$. In Platen and Rendek [2012a] market activity has been modeled by the inverse of a square root process. Similarly, but different, the process $\frac{1}{M} = \{\frac{1}{M_t}, t \geq 0\}$ is assumed to be a fast moving square root process in t-time with the dynamics

$$d\left(\frac{1}{M_t}\right) = \left(\frac{\nu}{4}\gamma - \epsilon\frac{1}{M_t}\right)dt - \sqrt{\frac{\gamma}{M_t}}dW_t, \tag{4.6}$$

for $t \geq 0$ with $M_0 > 0$, where $\gamma > 0$, $\nu > 2$ and $\epsilon > 0$. Note the negative sign in front of the diffusion term which indicates the main difference of the here proposed model to the one suggested in Platen and Rendek [2012a] for a currency savings account discounted EWI. The Brownian motion $W(\tau)$, which models in market activity time the long term nondiversifiable uncertainty with respect to oil denomination, is driving the normalized index Y_τ in the SDE (4.4). This process is linked to the standard Brownian motion $W = \{W_t, t \geq 0\}$ in t-time through the market activity M via the stochastic differential

$$dW(\tau_t) = \sqrt{\frac{d\tau_t}{dt}}dW_t = \sqrt{M_t}dW_t \tag{4.7}$$

for $t \geq 0$ with $W_0 = 0$, see Revuz and Yor [1999]. The Brownian motion $W = \{W_t, t \geq 0\}$ in (4.7) is the one driving in the equation (4.6). The above setup produces a two-component model with only one source of uncertainty. Note that the increments of the inverse of market activity are positively correlated to the increments of the normalized index as empirically observed.

Expected Rate of Return and Volatility

By application of the Itô formula one obtains from (4.2), (4.3), (4.4), (4.5) and (4.7) for the oil discounted EWI $\bar{S}_t^{\delta *}$ the stochastic differential equation (SDE)

$$d\bar{S}_t^{\delta *} = \bar{S}_t^{\delta *} \left(\mu_t dt + \sigma_t dW_t \right) \tag{4.8}$$

for $t \geq 0$, with initial value $\bar{S}_0^{\delta *} = A_0 (Y_0)^q$ and expected rate of return

$$\mu_t = \left(\frac{a}{M_t} - \frac{q}{2} \left(\frac{\Gamma\left(\frac{\delta}{2} + q\right)}{\Gamma\left(\frac{\delta}{2}\right)} \right)^{\frac{1}{q}} + \left(\frac{\delta}{4} q + \frac{1}{2} q(q-1) \right) \frac{1}{M_t Y_{\tau_t}} \right) M_t. \tag{4.9}$$

The volatility with respect to t-time emerges in the form

$$\sigma_t = q \sqrt{\frac{M_t}{Y_{\tau_t}}}. \tag{4.10}$$

Covered by the Benchmark Approach

Due to the SDE (4.8) and the Itô formula, the dynamics for the benchmarked savings account $\hat{S}_t^1 = (\bar{S}_t^{\delta *})^{-1} = \frac{S_t^1}{S_t^{\delta *}}$, which is the inverse of the oil discounted NP, is characterized by the SDE

$$d\hat{S}_t^1 = \hat{S}_t^1 \left(\left(-\mu_t + \sigma_t^2 \right) dt - \sigma_t dW_t \right), \tag{4.11}$$

for $t \geq 0$, see (4.9) and (4.10). It follows if for all $t \geq 0$ one has

$$\sigma_t^2 \leq \mu_t \tag{4.12}$$

then the benchmarked savings account $\hat{B}_t$ forms an $(\underline{\mathcal{F}}, P)$-supermartingale. This is the key property needed to accommodate the model under the benchmark approach, see Platen and Heath [2010].

To guarantee almost surely in the proposed model the inequality (4.12), one has by (4.9) and (4.10) to satisfy the following two conditions:

Assumption 4.2.1. *First, the dimension δ of the square root process Y needs to* *satisfy the equality*

$$\delta = 2(q + 1). \tag{4.13}$$

Assumption 4.2.2. *The long term average net growth rate a with respect to τ-time has to satisfy the inequality*

$$\frac{q}{2}\left(\frac{\Gamma\left(2q+1\right)}{\Gamma\left(q+1\right)}\right)^{\frac{1}{q}} \leq a. \tag{4.14}$$

When equality holds in (4.14) for the proposed model the benchmarked savings account is a local martingale, as assumed in the version of the benchmark approach formulated in Platen and Heath [2010]. For a more general version of the benchmark approach, where there is no equality in (4.14), we refer to Platen [2011] and Platen and Rendek [2012a].

4.3 Model Fitting

Let us now describe the model fitting procedure to the oil discounted EWI. In the simplified version of the model we assume $q = 1$ in (4.2), therefore $\delta = 4$ in (4.4) and $\nu = 4$ in (4.6). For the fitting of more general cases, we refer to Platen and Rendek [2012a]. The main reason for assuming $q = 1$ is the fact that it is empirically extremely difficult to give a sufficiently precise estimate for the degrees of freedom of the observed Student-t distributed log-returns, see also (iii) in Section 4.4. On the other hand, we may employ arguments from Platen and Rendek [2012a], which suggest theoretically for currency denominated log-returns a Student-t distribution with four degrees of freedom. The data indicate with the estimated 3.13 degrees of freedom for the index log-returns that four degrees of freedom would work well for a realistic model and would make it very tractable.

Step 1: Normalization of Index

By the fact that M_t seems to have an inverse gamma density with ν degrees of freedom the mean of M_t is explicitly known. By the ergodic theorem this mean

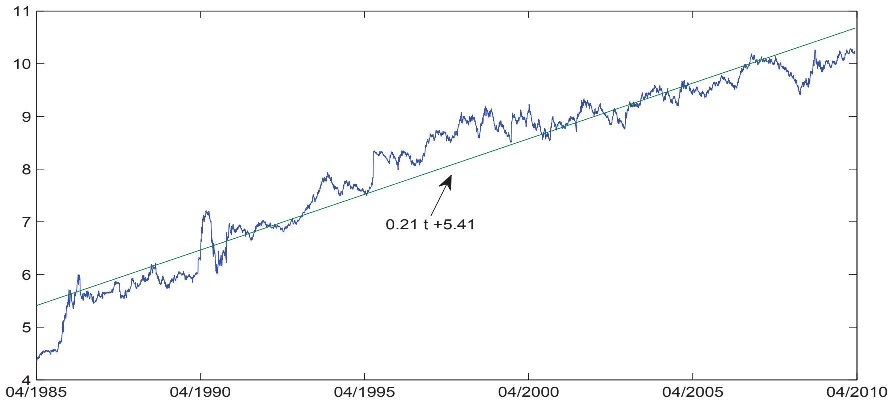

Figure 4.7: Logarithm of the oil discounted EWI and linear fit.

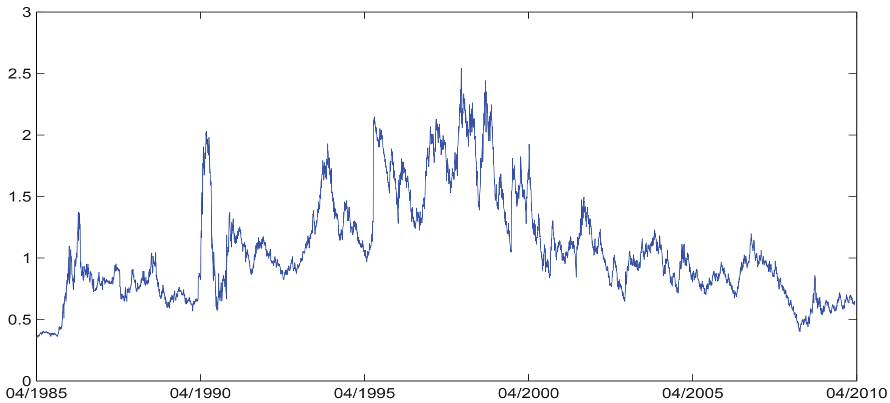

Figure 4.8: Normalized EWI for oil.

amounts to

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t M_s ds = \frac{4}{\nu - 2} \frac{\epsilon}{\gamma} \quad \text{P-a.s.} \tag{4.15}$$

Therefore, it is possible to approximate (4.3) by the following expression

$$A_{\tau_t} \approx A \exp\left\{ \frac{4a\epsilon}{\gamma(\nu - 2)} t \right\}, \tag{4.16}$$

for $t \geq 0$. Therefore, since a line can be fitted to the logarithm of the discounted EWI of oil, see Figure 4.7, it is straightforward to calculate $A = 223.32$ and $\frac{4a\epsilon}{\gamma(\nu-2)} \approx 0.21$. Figure 4.8 exhibits the normalized EWI of oil obtained as the ratio of the oil discounted EWI over the function in (4.16).

Step 2: Observing Market Activity

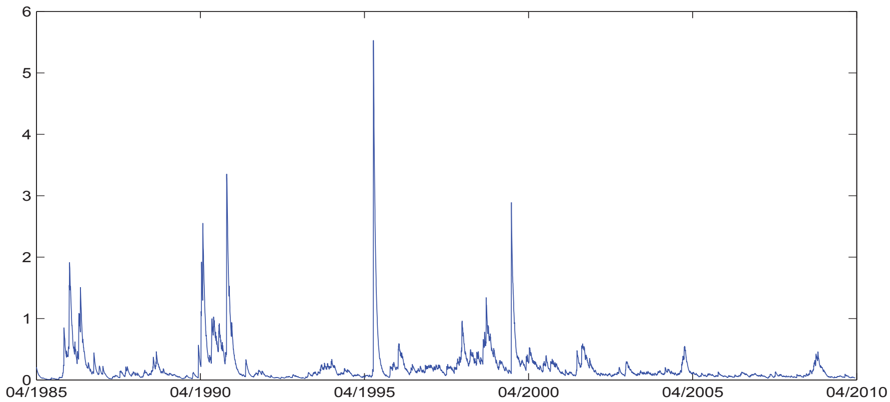

Figure 4.9: Market activity.

By (4.4), (4.5) and an application of the Itô formula, one obtains as time derivative of the quadratic variation for $\sqrt{Y_{\tau_t}}$ the expression

$$\frac{d[\sqrt{Y}]_{\tau_t}}{dt} = \frac{1}{4}\frac{d\tau_t}{dt} = \frac{M_t}{4}, \tag{4.17}$$

which is proportional to market activity. The estimation of the trajectory of the market activity process M is performed using daily observations. First, the "raw" time derivative $Q_t = \frac{d[\sqrt{Y}]_{\tau_t}}{dt}$ at the ith observation time $t = t_i$ is estimated from the finite difference

$$\hat{Q}_{t_i} = \frac{[\sqrt{Y}]_{\tau_{t_{i+1}}} - [\sqrt{Y}]_{\tau_{t_i}}}{t_{i+1} - t_i} \tag{4.18}$$

for $i \in \{0, 1, \ldots\}$. Second, exponential smoothing is applied to the observed finite differences according to the recursive standard moving average formula

$$\tilde{Q}_{t_{i+1}} = \alpha\sqrt{t_{i+1} - t_i}\hat{Q}_{t_i} + (1 - \alpha\sqrt{t_{i+1} - t_i})\tilde{Q}_{t_i}, \tag{4.19}$$

$i \in \{0, 1, \ldots\}$, with weight parameter $\alpha > 0$.

Figure 4.9 displays the resulting trajectory of M_t for daily observations, when interpreting this value as estimate of $4\frac{d}{dt}[\sqrt{Y}]_{\tau_t}$, for $t \geq 0$. Here an initial value of $M_0 \approx 0.21$ emerged and the time average of the trajectory of $(M_t)^{-1}$ amounted to 11.98.

Step 4: Parameter γ

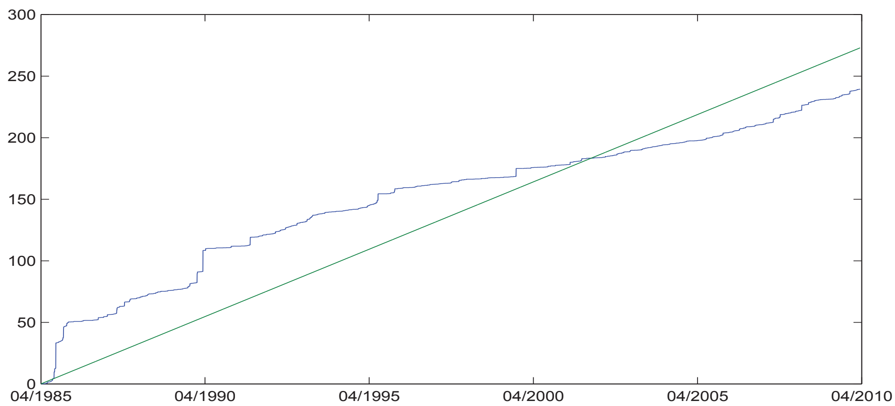

Figure 4.10: Quadratic variation of the square root of $\frac{1}{M}$.

Figure 4.10 plots the quadratic variation of the square root of the estimated process $\frac{1}{M}$. Our estimate for the slope equals here 10.94. Since under the proposed model we have $\frac{d}{dt}\left[\sqrt{\frac{1}{M}}\right]_t = \frac{1}{4}\gamma$, we obtain $\gamma \approx 43.76$.

Step 5: Parameters ν and ϵ

Figure 4.11 displays the histogram of market activity with inverse gamma fit with $\nu = 2.80$ degrees of freedom. Since we consider a simplified version of the model, we have chosen the same degrees of freedom for $\delta = 4$ and $\nu = 4$, we obtain from the average value of the market activity $\frac{4\epsilon}{\gamma(\nu-2)} \approx 0.21$ the estimate for $\epsilon = 4.57$.

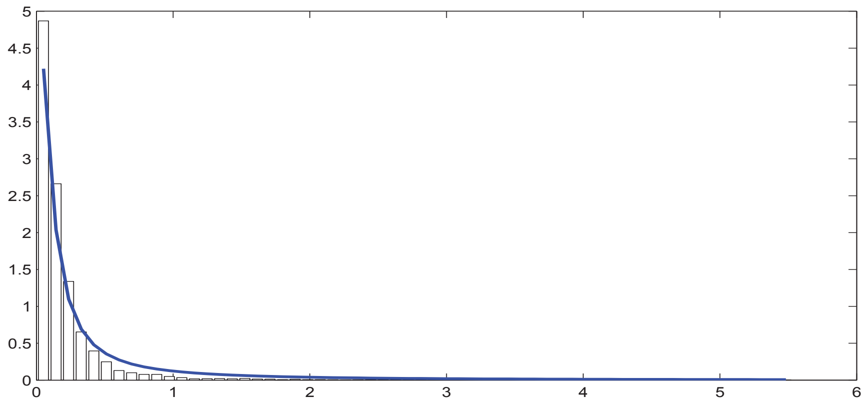

Figure 4.11: Histogram of market activity with inverse gamma fit.

Step 6: Long Term Average Net Growth Rate

Finally, we obtain the long term average net growth rate with an approximate value of $a \approx 1$, since the average value of the market activity is $\frac{4\epsilon}{\gamma(\nu-2)} \approx 0.21$. This indicates, the condition (4.14) is approximately satisfied as an equality. Therefore, the benchmark approach can be applied, as described in Platen and Heath [2010] and Platen and Rendek [2012a].

4.4 Simulation Study

The aim of this section is to describe an almost exact simulation method for the model introduced in Section 4.2. As indicated before, both of the processes $\frac{1}{M}$ and Y are square root processes of dimension $\delta = \nu = 4$ in the stylized version of the model, which we propose. The transition density of the square root process is the non-central chi-square density, therefore, the simulation can be considered to be almost exact when sampling from this transition density. More precisely, it will be exact for the process $\frac{1}{M}$ and almost exact for Y. The following four steps describe the simulation of the normalized index and its volatility:

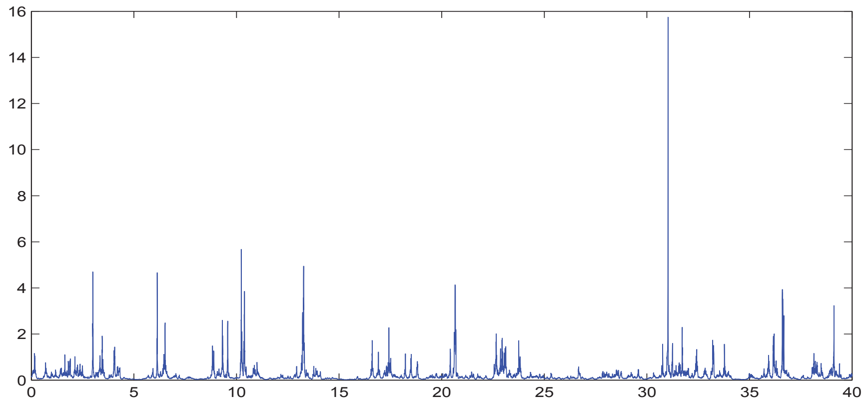

Figure 4.12: Simulated path of M.

1. Simulation of the Process $\frac{1}{M}$

First, we describe the simulation of the inverse $\frac{1}{M}$ of the market activity process. It is characterized by the SDE (4.6) and represents a square root process of dimension $\nu = 4$.

This process can be sampled exactly due to its non-central chi-square transition density of dimension $\nu = 4$. That is, we have

$$\begin{aligned} \frac{1}{M_{t_{i+1}}} &= \frac{\gamma(1 - e^{-\epsilon(t_{i+1}-t_i)})}{4\epsilon} \\ &\quad \times \left(\chi^2_{3,i} + \left(\sqrt{\frac{4\epsilon e^{-\epsilon(t_{i+1}-t_i)}}{\gamma(1 - e^{-\epsilon(t_{i+1}-t_i)})} \frac{1}{M_{t_i}}} - Z_i \right)^2 \right), \end{aligned} \tag{4.20}$$

for $t_i = \Delta i, i \in \{0, 1, \ldots\}$; see also Broadie and Kaya [2006] and Platen and Rendek [2012a]. Here Z_i is an independent standard Gaussian distributed random variable and $\chi^2_{3,i}$ is an independent chi-square distributed random variable with three degrees of freedom. Then the right hand side of (4.20) becomes a non-central chi-square distributed random variable with the requested non-centrality and four degrees of freedom.

Figure 4.12 plots the simulated path of the market activity M. The market activity displayed in this figure has more pronounced spikes when compared to the estimated market activity shown in Figure 4.9. We will see later that when the market activity is estimated from the path of the simulated index it resembles closely the historically observed path in Figure 4.9.

2. Calculation of τ-Time

The next step of the simulation generates the market activity time, the τ-time. By (4.5) one aims for the increment

$$\tau_{t_{i+1}} - \tau_{t_i} = \int_{t_i}^{t_{i+1}} M_s ds \approx M_{t_i}(t_{i+1} - t_i), \tag{4.21}$$

$i \in \{0, 1, \ldots\}$.

Figure 4.13 plots the simulated τ-time, which is the market activity time obtained from the path of the simulated market activity in Figure 4.12 with the use of the approximation (4.21).

3. Calculation of the Y Process

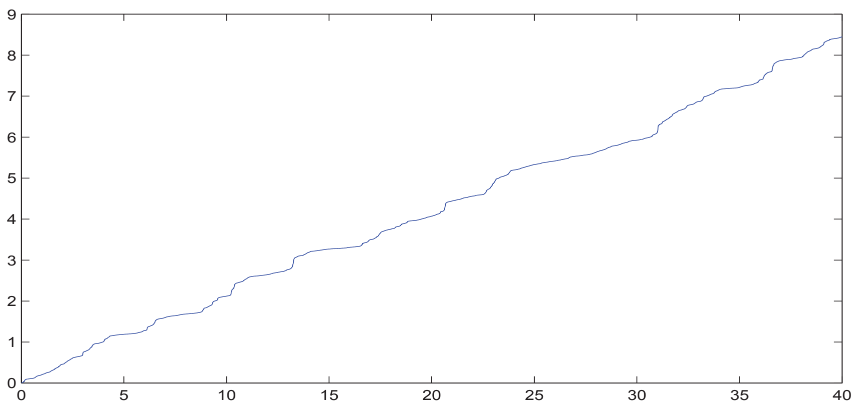

Figure 4.13: Simulated τ-time, the market activity time.

The simulation of the Y process is very similar to the simulation of the square root process $\frac{1}{M}$. Both processes are square root processes of dimension four and both are driven by the same source of uncertainty. We, therefore, employ in each time step the same Gaussian random variable Z_i and the same chi-square

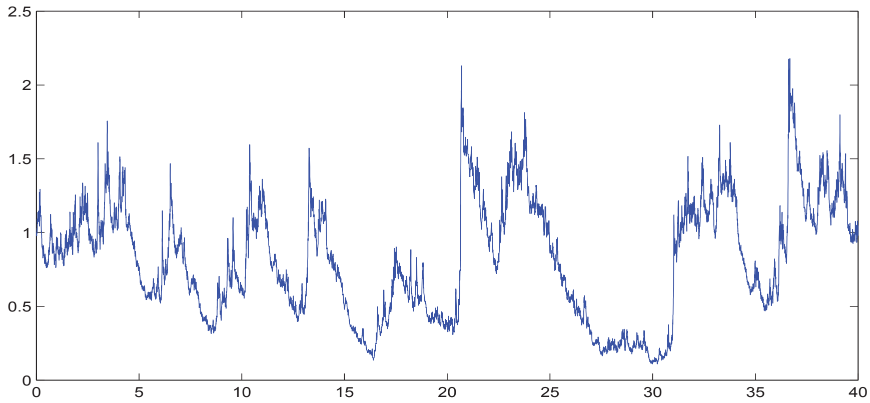

Figure 4.14: Simulated trajectory of the normalized index Y_{τ_t}.

distributed random variable $\chi^2_{3,i}$, as in (4.20). This leads us to the new value of the Y process,

$$\begin{aligned} Y_{\tau_{t_{i+1}}} &= \frac{1-e^{-(\tau_{t_{i+1}}-\tau_{t_i})}}{4} \\ &\quad \times \left(\chi^2_{3,i} + \left(\sqrt{\frac{4e^{-(\tau_{t_{i+1}}-\tau_{t_i})}}{1-e^{-(\tau_{t_{i+1}}-\tau_{t_i})}} Y_{\tau_{t_i}}} + Z_i \right)^2 \right), \end{aligned} \tag{4.22}$$

for $t_i = \Delta i, i \in \{0, 1, \ldots\}$. Note that the difference $\tau_{t_{i+1}} - \tau_{t_i}$ was approximated by using in (4.21) the market activity of the previous step.

Figure 4.14 displays the simulated trajectory of the normalized index Y obtained by the formula (4.22). This trajectory resembles the normalized EWI for oil displayed in Figure 4.8.

By analyzing the increments of the two processes $\frac{1}{M}$ and Y for vanishing time step size, one can show by using arguments as employed in Diop [2003] and Alfonsi [2005] that the pair of the simulated solutions (4.20) and (4.22) converges weakly to the solution of the two dimensional SDE given by equations (4.6) and (4.4). Note that in a weak sense the simulation of $\frac{1}{M}$ can be interpreted as being exact and that of Y_τ as being almost exact.

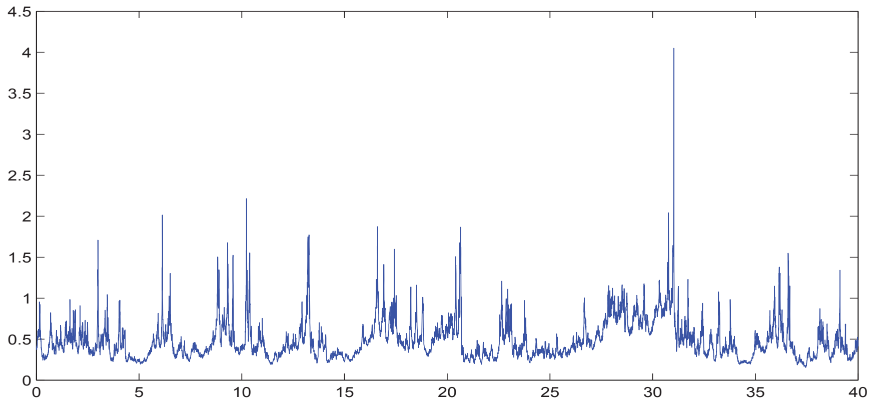

Figure 4.15: Simulated volatility of the index.

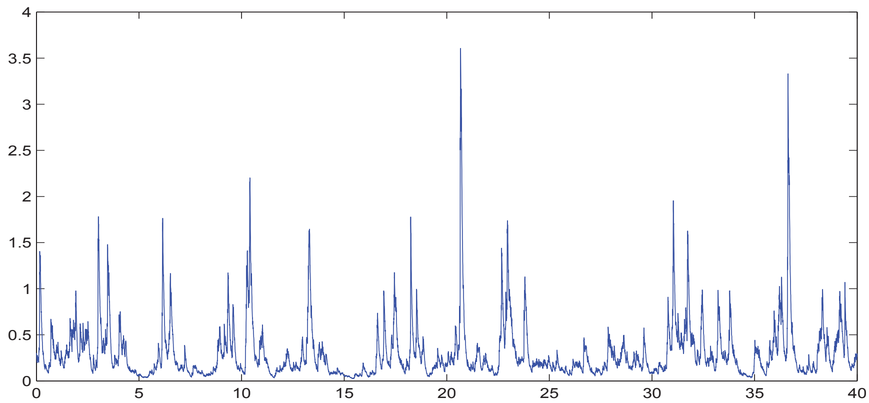

Figure 4.16: Estimated market activity of the simulated index.

4. Calculating the Volatility Process

The volatility process at time t_i is calculated under the stylized model with $q = 1$ as

$$\sigma_{t_i} = \sqrt{\frac{M_{t_i}}{Y_{\tau_{t_i}}}} \tag{4.23}$$

for $i \in \{0, 1, 2, \ldots\}$, see (4.10). The simulated volatility, obtained by (4.23) from the trajectory of the simulated market activity, displayed in Figure 4.12, and the

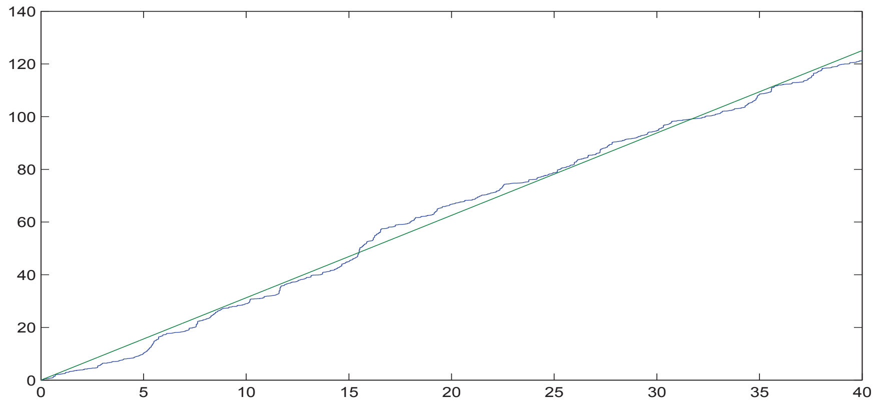

Figure 4.17: Quadratic variation of the square root of the inverse of estimated market activity.

trajectory of the simulated normalized index, plotted in Figure 4.14, is shown in Figure 4.15. It again exhibits more pronounced spikes when compared to the estimated volatility of the oil discounted EWI in Figure 4.4. These spikes are practically removed when estimating from the simulated trajectory. Figure 4.16 plots the estimated market activity of the simulated index. Note that smoothing removes most spikes of the simulated market activity shown in Figure 4.12. Moreover, the quadratic variation of the square root of the observed inverse of the estimated market activity is more in line with the quadratic variation of the inverse of market activity obtained from the normalized EWI for oil, see Figure 4.17 and Figure 4.10.

Empirical Properties of the Proposed Model

Let us now check for the proposed model the seven empirical stylized facts identified in Section 4.1. The estimation methods of Section 4.1 are now applied to the simulated trajectory of the index.

(i) Uncorrelated Returns

Figure 4.18 displays the autocorrelation function for log-returns of the simulated index. Similarly as in Figure 4.1, the autocorrelation function decreases fast to zero and stays at zero for large lags. In fact, it is mostly

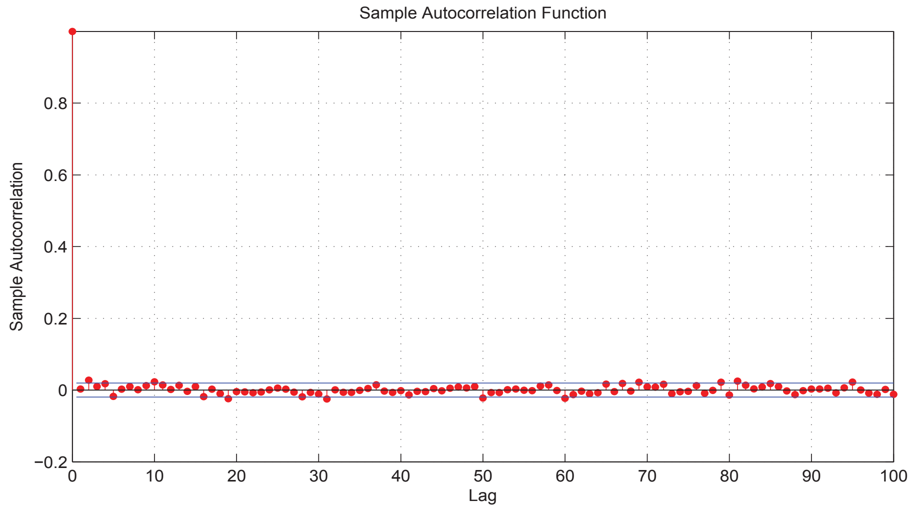

Figure 4.18: Autocorrelation function for log-returns of the simulated index.

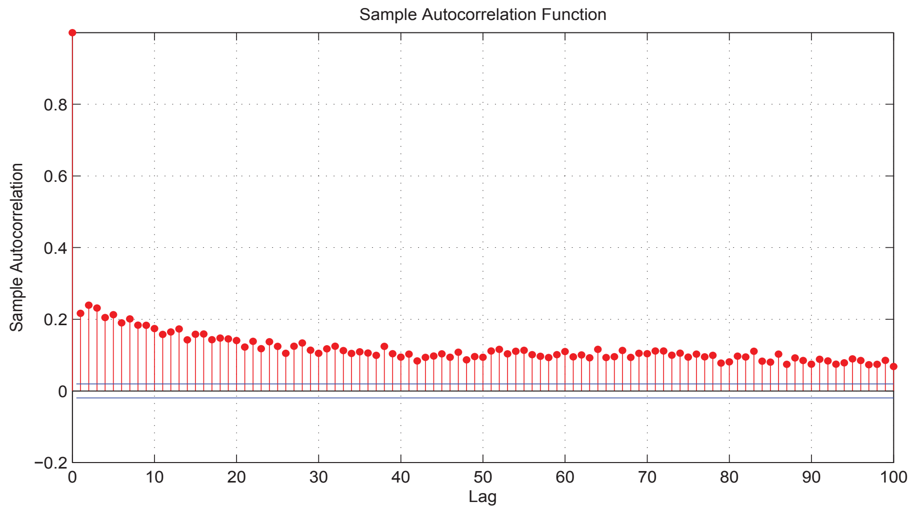

Figure 4.19: Autocorrelation function for absolute log-returns of the simulated index.

located between the 95% confidence bounds.

(ii) Correlated Absolute Returns

Figure 4.19 plots the autocorrelation function for the absolute log-returns of the simulated index. Such autocorrelation of absolute log-returns does not decrease to zero. It is located outside the 95% confidence bounds even for rather large lags. This is in line with the autocorrelation function of the absolute log-returns of the oil discounted EWI displayed in Figure 4.2.

(iii) Student-t Distributed Returns

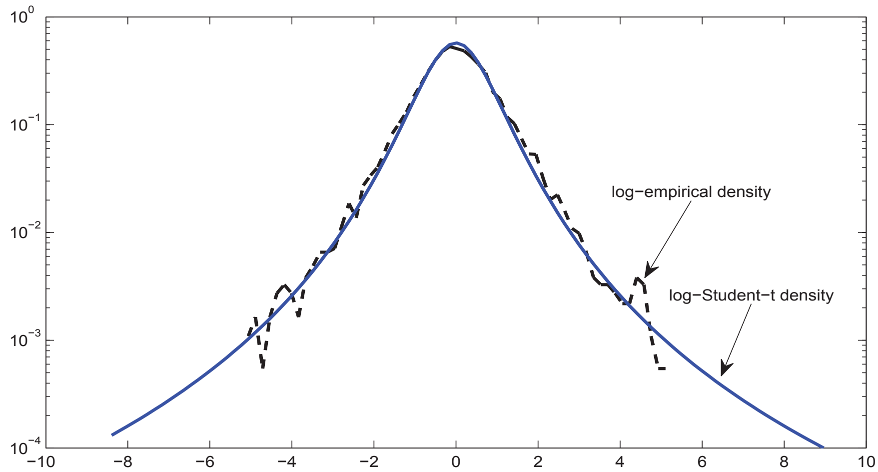

Figure 4.20: Logarithms of the empirical distribution of the normalized log-returns of the simulated index and Student-t density with four degrees of freedom.

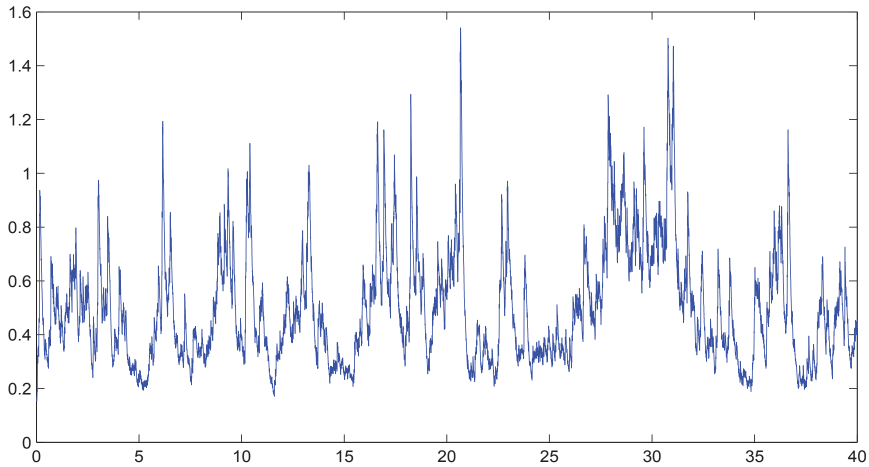

Figure 4.21: Estimated volatility of the simulated index.

Figure 4.20 illustrates the logarithms of the empirical distribution of the normalized log-returns of the simulated index and Student-t density with four degrees of freedom. As expected from the design of the model in Section 4.2, the distribution of log-returns of the simulated index is a Student-t distribution with about four degrees of freedom. Note that the estimated degrees of freedom may vary significantly for the simulated trajectories, as was illustrated in Platen and Rendek [2012a]. Such deviations can easily reach one degree of freedom. This is also one of the reasons why we fixed the parameters δ and ν to four in the proposed stylized version of the model.

(iv) Volatility Clustering

As expected from the model design, the estimated volatility of the simulated index, plotted in Figure 4.21, has periods of higher and periods of lower values. The estimated squared volatility was obtained by exponential smoothing (4.1) with $\alpha = 0.92$ applied to the squared log-returns of the simulated index.

(v) Long Term Exponential Growth

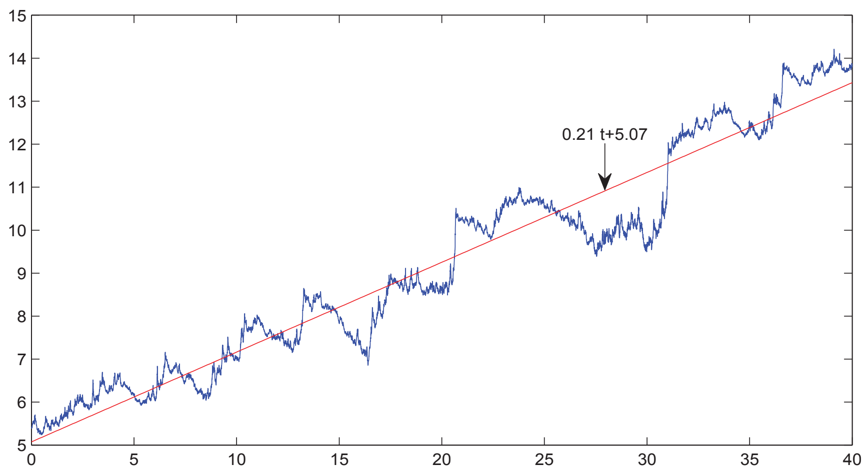

Figure 4.22: Logarithm of simulated index with linear fit.

Given the simulated normalized index in Figure 4.14 it is straightforward to calculate the index values by multiplication of the normalized index with the exponential function given in (4.16). The logarithm of the simulated index is displayed in Figure 4.22 with the respective least squares linear fit. The model clearly recovers the long term exponential growth of the EWI for oil.

(vi) Anti-Leverage Effect

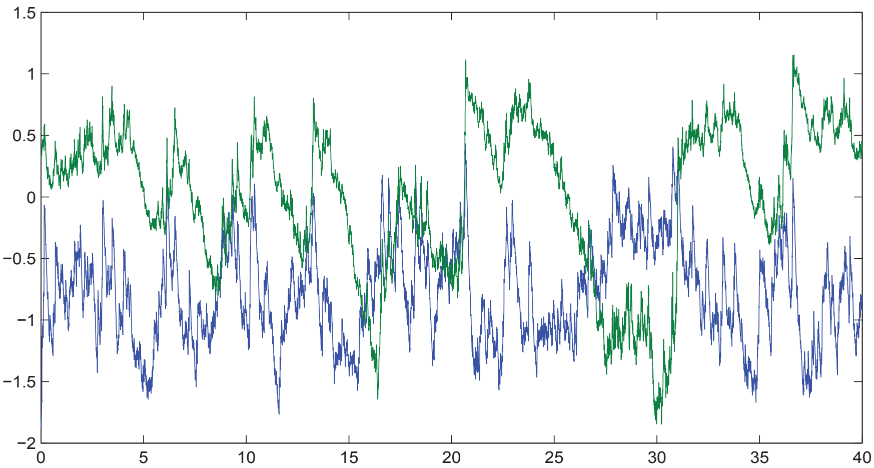

Figure 4.23: Logarithms of simulated normalized index (upper graph) and its estimated volatility (lower graph).

It has been noticed in Section 4.1 that the normalized EWI for oil is mostly positively correlated to its volatility. When sudden upward moves in the simulated normalized index for oil are observed, the volatility spikes up. This means that the market activity increases when the prices of oil are low relative to the NP. This anti-leverage effect for the EWI of oil is also recovered by the proposed model. The logarithms of the simulated normalized index and its estimated volatility are illustrated in Figure 4.23. The positive correlation is here clearly noticeable. Such positive correlation is also observed when comparing the simulated market activity in Figure 4.12 and the simulated normalized index in Figure 4.14.

Additionally, Figure 4.24 plots the quadratic covariation between the logarithms of simulated index and its estimated volatility. It resembles the

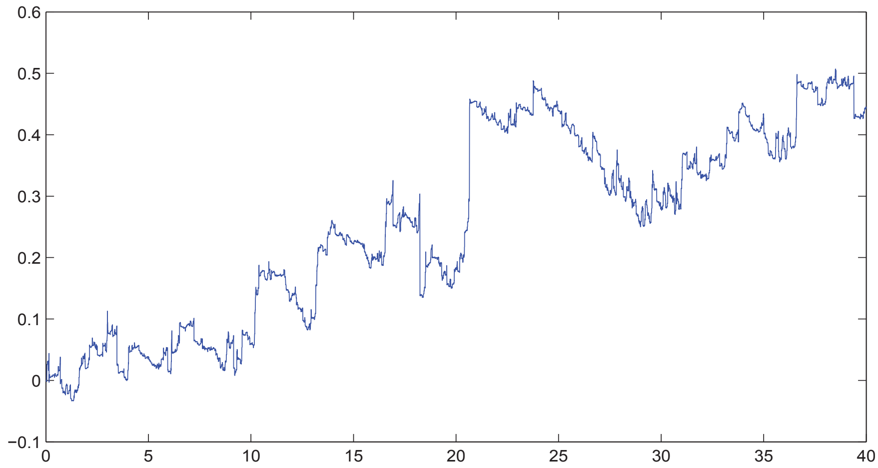

Figure 4.24: Quadratic covariation between the logarithms of simulated normalized index and its estimated volatility.

corresponding quadratic covariation for the normalized EWI for oil and its estimated volatility in Figure 4.6.

(vii) Extreme Volatility at Major Index Moves

Finally, the model produces extreme volatility at major index upward moves. This was already visible in Figure 4.23. During sudden upward moves in the index the volatility jumps up. This models the fact that the market is more active when the normalized EWI for oil moves strongly upward. This corresponds usually with a strong downward move of the oil price.

In summary, one can say that the proposed model captures well all seven stylized empirical facts listed in Section 4.1. It cannot be easily falsified on these grounds in the sense of Popper [1934]. The paper has shown that it is possible to identify a parsimonious two-component model for a diversified equity index denominated in oil. This stylized model has only one driving Brownian motion, three initial parameters and three structural parameters.

4.5 Modeling the Spot Price of Oil

We can model the oil denominated NP in the way as proposed previously in this chapter. On the other hand, we can model the currency denominated NP, as described in Platen and Rendek [2012a]. Therefore, it is possible to express the oil spot price dynamics by exploiting the SDEs derived for these quantities.

By (3.1) we can express the spot price of oil as the ratio of domestic currency denominated (US dollar) NP, $S_t^{i,\delta *}$, over the oil denominated NP, $S_t^{j,\delta *}$, that is

$$X_t^{i,j} = \frac{S_t^{i,\delta *}}{S_t^{j,\delta *}}, \tag{4.24}$$

for $t \geq 0$.

We model the currency denominated NP as in Platen and Rendek [2012a], and use an analogous notation to the oil denomination. Therefore, we set

$$S_t^{i,\delta *} = \bar{S}_t^{i,\delta *} S_t^{i,i}, \tag{4.25}$$

where

$$S_t^{i,i} = \exp\left\{ \int_0^t r_s^i ds \right\} \tag{4.26}$$

for $t \geq 0$. Here r_t^i is the short rate of the domestic currency.

The discounted NP in the currency denomination is modeled as in Platen and Rendek [2012a], and resembles the model described in this chapter. The domestic savings account discounted NP $\bar{S}_t^{i,\delta *}$ at time t is equal to

$$\bar{S}_t^{i,\delta *} = A_{\tau_t^i}^i Y_{\tau_t^i}^i, \tag{4.27}$$

with

$$A_{\tau_t^i}^i = A^i \exp\{a^i \tau_t^i\} \tag{4.28}$$

for $t \geq 0$. Additionally, the normalized NP $Y_{\tau^i}^i$ in τ^i-time can be characterized as square root process of dimension four via the SDE:

$$dY^i_{\tau^i} = (1 - Y^i_{\tau^i})d\tau^i + \sqrt{Y^i_{\tau^i}}\ dW^i(\tau^i). \tag{4.29}$$

For simplicity, W^i is assumed to be an independent Brownian motion in τ^i-time. The τ^i-time is given by an ordinary differential equation involving the ith currency market activity M^i. That is,

$$d\tau^i_t = M^i_t dt, \tag{4.30}$$

where the inverse of ith currency market activity satisfies the SDE

$$d\left(\frac{1}{M^i_t}\right) = (\gamma^i - \epsilon^i \frac{1}{M^i_t})dt + \sqrt{\frac{\gamma^i}{M^i_t}} dW^i_t \tag{4.31}$$

for $t \geq 0$. Here the Brownian motion W^i_t in t-time is related to the Brownian motion $W^i(\tau^i_t)$ in τ^i-time by equation

$$dW^i(\tau^i_t) = \sqrt{\frac{d\tau^i}{dt}} dW^i_t = \sqrt{M^i_t} dW^i_t. \tag{4.32}$$

It is important to note the difference in the sign in front of the diffusion term of the SDE (4.31), which is opposite to the one in the SDE (4.6) for the commodity oil. The fit of the model to the US dollar denomination of the discounted EWI provided the parameters: $A^i = 2922.08$, $Y^i_0 = 0.76$, $M^i_0 = 0.044$, $\gamma^i = 511.33$, $\epsilon^i = 11.31$ and $a^i = 6.31$.

In this manner we have constructed an oil spot price model, which separately models the movements of the oil price relative to the index and the currency relative to the same index. This disentangles the impact of the two main factors that drive the oil price in US dollar denomination. One notes also the influence of the oil convenience yield and the US interest rate on the long term evolution of the oil price under the proposed model. This model permits a more realistic pricing of oil derivatives then previous models, in particular for long dated derivatives, as explained in Du and Platen [2012b] and Chapter 3.

4.6 Conclusion on Modeling of Oil Prices

This chapter derived a parsimonious two-component affine diffusion model with one driving Brownian motion to capture the dynamics of oil prices. We observed that the oil price behaves in some sense similarly to the US dollar. However, there are also clear differences. To identify these difference we studied the empirical features of an extremely well diversified world stock index, which is a proxy of the numéraire portfolio, in the denomination of the oil price. Using a diversified index in oil price denomination allowed us to disentangle the factors driving the oil price. This chapter reveals that the volatility of the numéraire portfolio denominated in crude oil, increases at major oil price upward moves. Furthermore, the log-returns of the index in oil price denomination appear to follow a Student-t distribution. These and further stylized empirical properties guide us to the proposed tractable diffusion model, which has the normalized numéraire portfolio and the market activity as components. This chapter has also described an almost exact simulation technique which allows us to illustrate the properties of the proposed model. These confirm that the proposed two-component model matches well the observed stylized empirical facts.

Chapter 5

Conclusions and Further Directions of Research

This thesis studies the pricing and hedging of commodity derivatives under the benchmark approach. The new concept of benchmarked risk minimization has been proposed for semimartingale markets. Using this new concept, the thesis demonstrated how to price and hedge not fully replicable contingent claims in commodity markets under the benchmark approach. Additionally, new models have been proposed, which allow one to capture the dependence structure between the interest rate, the convenience yield and the numéraire portfolio in currency and commodity denomination. These models maintain the strict supermartingale property of benchmarked currency and commodity savings accounts and provide computationally tractable formulas for forward, futures and option prices for commodities. In addition, empirical studies have been undertaken which confirm visually the strict supermartingale property of the benchmarked currency and the benchmarked commodity savings accounts. For crude oil, Student-t distributed log-returns and the anti-leverage volatility property have been identified. A parsimonious two-component affine diffusion model has been proposed, which matches well the observed stylized empirical facts of the crude oil price dynamics.

There are several possible directions of future research: First, because of their physical properties, different commodities can have extremely distinct empirical features. The stylized empirical facts of commodity prices other than those for crude oil should be studied and modeled. A second interesting direction of future study is to apply the hybrid model proposed in Chapter 3 for foreign exchange markets and also for volatility indices. This kind of models would most likely have the ability to capture observed dependence structures.

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