# INVESTIGATIONS INTO EIFFICIENT IMPLICIT LOD-FDTD USING ORTHOGONAL AND NON- 

## ORTHOGONAL MESHES

By

## Md. Masud Rana



Faculty of Engineering and Information Technology
University of Technology Sydney

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## CERTIFICATE

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.


#### Abstract

In this thesis we aim to develop efficient, enhanced versions of locally one dimensional finite difference time domain (LOD-FDTD) using orthogonal and nonorthogonal meshes with convolutional perfectly matched layered (CPML) absorbing boundary condition (ABC) for solving a range of electromagnetic (EM) and microwave problems. To solve many real world propagation problems related to electrically large structures and compute the EM response from resonant and curved structures both in two dimensional (2-D) and three dimensional (3-D) employing orthogonal and nonorthogonal meshes, novel LOD-FDTD with CPML ABC are presented to render the problem manageable and treatable with available resources within a reasonable time frame and without placing an unrealistic burden on the computational resources.

In the first part of the thesis, a segmented (S)-LOD-FDTD method has been developed for EM propagation modelling in electrically large symmetric structures. After modifying 3-D symmetric structures to two dimensional (2-D) structures, the segmentation approach is applied. The developed S-LOD-FDTD method has been validated through propagation prediction inside large straight, branched and curved tunnels. The predictions on path loss agree reasonably well with the results obtained using segmented alternating direction implicit finite difference time domain (S-ADIFDTD) method as well as with published measured data. The results indicate higher signal attenuation for the junction/transition regions as compared to regions away from such abrupt transitions. A performance comparison of the proposed method has also been described in terms of CPU time and memory. It was found that by dividing the domain into more segments, both execution time and memory usage can be reduced.

Subsequently, a non-orthogonal LOD-FDTD (LOD-NFDTD) method is presented for EM scattering from 2-D structures. Formulations of scattered field and CPML ABC in generalised non-orthogonal curvilinear grids for 2-D LOD-NFDTD are also presented. The non-orthogonal grids are used to fully mesh the computational domain, which leads to efficient computation. Moreover, the proposed technique requires fewer arithmetic operations than the nonorthogonal ADI-FDTD (ADI-NFDTD) method, leading to a reduction of CPU time. The numerical dispersion of the proposed method as a function of Courant-Friedrich-Lewy (CFL) number (CFLN) is also discussed. Computational


results for EM scattering from 2-D conducting, dielectric, and coated cylinders are presented. The proposed method is unconditionally stable and the numerical results agree reasonably well with the results in the literature, as well as with the ADI-NFDTD results. Compared to ADI-NFDTD, the proposed method is characterised by a lighter calculation burden and higher accuracy.

We also propose a dispersion controlled rotationally symmetric LOD-FDTD (D-RS-LOD-FDTD) method for analysing rotationally symmetric (RS) microwave structures and antennas. First, the formulation for conventional RS-LOD-FDTD with CPML ABC is presented. Then D-RS-LOD-FDTD algorithm with CPML is derived and utilised to reduce the dispersion that may result from modelling RS microwave structures. As a preliminary calculation, the open tip monopole (OTM) antenna has been analysed. The dispersion control parameters contribute to the improvement in accuracy even with a large time step beyond the CFL limit. Computational results for the return loss and specific absorption rate from OTM and expanded tip wire (ETW) antennas embedded inside a tissue-like phantom media are presented. The use of the dispersion control parameters not only reduces the resultant dispersion effectively but also enables us to employ a large time step for efficient computations, so that the computation time can be reduced to about half of that required for its explicit counterpart (RS-FDTD).

We also present a two sub-step CPML ABC for the conventional (C)-LOD-FDTD method for both orthogonal and non-orthogonal curvilinear meshes for analysing 3-D microwave structures. Numerical results on three dimensional (3-D) microwave structures using the proposed methods are also presented. A fundamental scheme based LOD-FDTD (F-LOD-FDTD) for both orthogonal and non-orthogonal meshes are proposed to minimise the resultant computational load for solving 3-D microwave structures, in addition to freeing the right-hand side of the resultant update equations of matrix operations. Numerical stability of the F-LOD-FDTD for both orthogonal and non-orthogonal meshes is also presented to demonstrate the unconditional stability of the proposed methods. Numerical results are presented to illustrate the significance of the proposed approaches. A comparison with the C-LOD-FDTD-CPML in terms of CPU time and memory requirements reveals the merits of the proposed F-LOD-FDTD CPML method for both orthogonal and non-orthogonal curvilinear meshes in terms of lighter calculation burden and higher efficiency.

## DEDICATION

To my parents, wife, and family

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## List of Symbols

| E | vector electric field intensity (volts/meters) |
| :---: | :---: |
| $\mathrm{E}_{\mathrm{x}, \mathrm{y}, \mathrm{z}}$ | electric field intensity in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions (volts/meter) |
| $E^{\text {inc }}$ | incident electric field intensity (volts/meter) |
| $\mathrm{E}^{\text {scat }}$ | scattered electric field intensity (volts/meter) |
| $\mathrm{E}^{\text {tot }}$ | total electric field intensity (volts/meter) |
| H | vector magnetic field intensity (amperes/meters) |
| $\mathrm{H}_{\mathrm{x}, \mathrm{y}, \mathrm{z}}$ | magnetic field intensity in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions (amperes/meter) |
| $\mathrm{H}^{\text {inc }}$ | incident magnetic field intensity (amperes /meter) |
| $\mathrm{H}^{\text {scat }}$ | scattered magnetic field intensity (amperes/meter) |
| $\mathrm{H}^{\text {tot }}$ | total magnetic field intensity (amperes /meter) |
| $\Delta \mathrm{x}, \Delta \mathrm{y}, \Delta \mathrm{z}$ | space increment in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions (meters) |
| i, j, k | mesh index in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions |
| $\Delta \mathrm{t}$ | time increment (seconds) |
| $\varepsilon$ | permittivity (farads/meters) |
| $\varepsilon_{0}$ | free space permittivity (farads/meters) |
| $\varepsilon_{r}$ | relative permittivity (farads/meters) |
| $\varepsilon_{\mathrm{x}, \mathrm{y}, \mathrm{z}}$ | permittivity components in in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions defined in stretched |
|  | coordinate method (farads/meters) |
| $\varepsilon_{x t y, y t, z t}$ | permittivity components in in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions defined in traditionally |
|  | (farads/meters) |
| $\mu$ | permeability (henries/meters) |
| $\mu_{0}$ | free space permeability (henries /meters) |
| $\mu_{\mathrm{r}}$ | relative permeability (henries /meters) |
|  | xxvi |


| $\mu_{\mathrm{x}, \mathrm{y}, \mathrm{z}}$ | permeability components in in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions defined in stretched |
| :---: | :---: |
|  | coordinate method (henries/meters) |
| $\sigma$ | electrical conductivity (siemens/meters) |
| $\sigma_{\mathrm{x}, \mathrm{y}, \mathrm{z}}$ | electrical conductivity components in in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions defined in |
|  | stretched coordinate method (siemens/meters) |
| $\sigma_{\text {xt,yt,zt }}$ | electrical conductivity components in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions defined in |
|  | traditionally (siemens/meters) |
| $\sigma^{*}$ | magnetic conductivity |
| $\sigma^{*}{ }_{x, y, z}$ | magnetic conductivity components in in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions defined in |
|  | stretched coordinate method |
| $\sigma^{*}{ }^{\text {xt, yt, zt }}$ | magnetic conductivity components in in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions defined in |
|  | traditionally |
| k | vector wave propagation constant (radian/meter) |
| $\mathrm{k}_{\mathrm{x}, \mathrm{y}, \mathrm{z}}$ | wave propagation constant components in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions defined in |
| $\eta$ | electromagnetic wave impedance (ohm) |
| $\eta_{0}$ | free space electromagnetic wave impedance (ohm) |
| $\mathrm{c}_{0}$ | free space electromagnetic wave velocities (meter/second) |
| f | frequency (1/second) |
| $\omega$ | angular frequency (radians/second) |
| $\mathrm{J}_{\text {s }}$ | equivalent tangential electric currents density (amperes/meter) |
| M | equivalent tangential magnetic currents density (volts/meter) |
| R | normal reflection coefficient of PML region backed by a perfect |
|  | conducting wall |
| $\mathrm{r} \varphi$ | radius and angles in cylindrical coordinate |

## Acronyms and Abbreviations

| 1-D, 2-D | one dimensional, two dimensional |
| :---: | :---: |
| 3D | three dimensional |
| ABC | absorbing boundary condition |
| ADI | alternating direction implicit |
| BOR | body of revolution |
| CPML | convolutional perfectly matched layer |
| C | conventional |
| DFT | direct Fourier transform |
| EM | electromagnetic |
| FDTD | finite difference time domain |
| F | fundamental scheme |
| FFT | fast Fourier transform |
| LOD | locally one dimensional |
| MoM | Method of moment |
| NF-FF | near field to far field |
| PEC | perfect electric conductor |
| PMC | perfect magnetic conductor |
| PML | perfectly matched layer |
| RS | rotationally symmetry |
| SF | scattered field |
| TEM | transverse electromagnetic |
| TE | transverse electric |
| TM | transverse magnetic |

## Chapter 1

## Introduction

This thesis investigates into efficient implicit locally one dimensional finite difference time domain (LOD-FDTD) method using orthogonal and nonorthogonal meshes for solving electromagnetic (EM) problems. This chapter presents a brief overview of implicit FDTD techniques, and more specifically presents a review of implicit LOD-FDTD technique found in the literature for the solution of EM problems. Then, the motivation is presented following the scope of the thesis. Finally, a list of publications resulting from this thesis is presented.

### 1.1 Background Theory

Computational electromagnetics (CEM) seeks to solve Maxwell's equations in both time and frequency domain to obtain numerical solution of real life problems. The recent development of faster and more powerful computers has allowed more advanced time domain CEM techniques. A classification of CEM techniques is summarised in Fig. 1.1. However, the standard explicit FDTD [1] method has been used extensively for solving varieties of electromagnetic problems due to its versatility and simplicity [2][5], but the explicit FDTD has some limitations. FDTD needs intensive memory and CPU time requirements for solving electromagnetic wave propagation due to following two modelling constraints: a) the spatial increment step which must be small enough in comparison with the smallest wavelength (usually 10-20 steps per wavelength) in order to make the numerical dispersion error negligible, and b) the time step must be small enough so that it satisfies the Courant-Friedrich-Lewy (CFL) stability condition. Moreover, for solving many real world problems, it is often necessary to employ enhanced versions of the FDTD which also increase the computational burden. To circumvent or relax the above constraints, various time domain techniques have been developed [6]-[7].


Fig. 1.1 Categories within computational electromagnetics

Mainly, to overcome the CFL stability constraint, firstly the unconditionally stable alternating direction implicit finite-difference time-domain (ADI-FDTD) was proposed [8]-[12]. More recently, other unconditionally stable implicit methods such as the SplitStep (SS) [13]-[15], Crank Nicolson (CN) [16]-[18] and the locally one-dimensional (LOD) FDTD methods [19]-[22] have been proposed.

### 1.2 An Overview of Implicit FDTD Methods

Methods which are unconditionally stable and have no restrictions on the time step are called implicit methods. An implicit method finds a solution by solving an equation involving both the current state of the system and the later one. But an explicit method computes the state of a system at a later time from the current state of the system. For many problems such as electrically small and high Q-structures, use of explicit method needs impractically small time steps $\Delta t$ to keep the error in the result within a bound. To analyse this type of problems use of implicit method can be advantageous as it can achieve a given accuracy with much less computational time employing larger time
steps. A briefly review of the formulations of implicit FDTD methods and the development associated with these methods, including explicit FDTD is provided next.

### 1.2.1 Explicit FDTD Method

The starting point for the construction of the finite difference time domain (FDTD) algorithm is Maxwell's curl equations. For a source free region of space that is linear, isotropic and non-dispersive, the differential form of Maxwell's equations which includes magnetic and electrical conductivity are given as follows

$$
\begin{gather*}
\frac{\partial \vec{E}}{\partial t}=\frac{1}{\varepsilon} \nabla \times \vec{H}-\frac{\sigma}{\varepsilon} \vec{E}  \tag{1.1a}\\
\frac{\partial \vec{H}}{\partial t}=-\frac{1}{\mu} \nabla \times \vec{E}-\frac{\sigma_{m}}{\varepsilon} \vec{H} \tag{1.1b}
\end{gather*}
$$

where $\vec{E}$ is the electric field strength vector in volts per meter, $\vec{H}$ is the magnetic field strength vector in amperes per meter and $\sigma, \sigma_{m}$ are electrical and magnetic conductivity respectively. Equation (1.1) is consisted of two vector equations, and each vector equation can be composed into three scalar equations for three dimensional spaces. Therefore, these vector curl equations (1.1) can be expanded into six coupled scalar equations in a Cartesian coordinate system $(x, y, z)$ as follows:

$$
\begin{align*}
& \frac{\partial E_{x}}{\partial t}=\frac{1}{\varepsilon}\left(\frac{\partial H_{z}}{d y}-\frac{\partial H_{y}}{d z}-\sigma E_{x}\right)  \tag{1.2a}\\
& \frac{\partial E_{y}}{\partial t}=\frac{1}{\varepsilon}\left(\frac{\partial H_{x}}{d z}-\frac{\partial H_{z}}{d x}-\sigma E_{y}\right)  \tag{1.2b}\\
& \frac{\partial E_{z}}{\partial t}=\frac{1}{\varepsilon}\left(\frac{\partial H_{y}}{d x}-\frac{\partial H_{x}}{d y}-\sigma E_{z}\right)  \tag{1.2c}\\
& \frac{\partial H_{x}}{\partial t}=\frac{1}{\mu}\left(\frac{\partial E_{y}}{d z}-\frac{\partial E_{z}}{d y}-\sigma_{m} H_{x}\right)  \tag{1.2d}\\
& \frac{\partial H_{y}}{\partial t}=\frac{1}{\mu}\left(\frac{\partial E_{z}}{d x}-\frac{\partial E_{x}}{d z}-\sigma_{m} H_{y}\right)  \tag{1.2e}\\
& \frac{\partial H_{z}}{\partial t}=\frac{1}{\mu}\left(\frac{\partial E_{x}}{d y}-\frac{\partial E_{y}}{d x}-\sigma_{m} H_{z}\right) \tag{1.2f}
\end{align*}
$$

## Chapter 1: Introduction

Therefore, the compact form of the time domain Maxwell's equations from (1.2a)-(1.2f) in lossy media can be written as follows

$$
\begin{equation*}
\frac{\partial \mathbf{U}}{\partial t}=([A]+[B]+[L]) \mathbf{U} \tag{1.3}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mathbf{A}=\left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & \frac{1}{\varepsilon} \partial_{y} \\
0 & 0 & 0 & \frac{1}{\varepsilon} \partial_{z} & 0 & 0 & \\
0 & 0 & 0 & 0 & \frac{1}{\varepsilon} \partial_{x} & 0 \\
0 & \frac{1}{\mu} \partial_{z} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\mu} \partial_{x} & 0 & 0 & 0 \\
\frac{1}{\mu} \partial_{y} & 0 & 0 & 0 & 0 & 0
\end{array}\right] \\
& \mathbf{B}=\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & \frac{-1}{\varepsilon} \partial_{z} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-1}{\varepsilon} \partial_{y} \\
0 & 0 & 0 & \frac{-1}{\varepsilon} \partial_{y} & 0 & 0 \\
0 & 0 & \frac{-1}{\mu} \partial_{y} & 0 & 0 & 0 \\
\frac{-1}{\mu} \partial_{z} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{\mu} \partial_{x} & 0 & 0 & 0 & 0
\end{array}\right]
\end{aligned}
$$

$$
\mathbf{L}=\left[\begin{array}{cccccc}
\frac{-1}{\varepsilon} \sigma & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{\varepsilon} \sigma & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{\varepsilon} \sigma & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1}{\varepsilon} \sigma_{m} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-1}{\varepsilon} \sigma_{m} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-1}{\varepsilon} \sigma_{m}
\end{array}\right]
$$

where $\mathbf{U}=\left[E_{x}, E_{y}, E_{z}, H_{x}, H_{y}, H_{z}\right]^{T}$.
where $\vec{E}$ is the electric field strength vector in volts per metre, $\vec{H}$ is the magnetic field strength vector in amperes per metre, and $\sigma, \sigma_{m}$ are electrical and magnetic conductivities respectively. The explicit FDTD method based on (1.3) has been widely used in electromagnetic modelling due to its simple updating equations and easiness for numerical implementation since 1966 [1]. Since then, the explicit FDTD method has been assessed from various aspects in the microwave community in [2]-[5], and has also been used for modelling objects having curved features. Holland [23] first introduced the non-orthogonal FDTD (NFDTD) algorithm which was later refined by many researchers including Lee et al. [24], Hao et al. [25], Fusco [26], Kantartzis et al. [27] and Armenta et al. [28]. For analysing rotationally symmetric (RS) resonant structures, RS-FDTD method has also been used effectively in [29]-[33]. For simulating problems in open region domains, the computational domain of the FDTD method is terminated by the absorbing boundary condition (ABC) to suppress spurious reflections from the grid terminations [1]. Various ABCs have been developed for the explicit FDTD method [34]-[40]. More recently, explicit FDTD has been applied to the modelling of more complex media in [41]-[45].

In the explicit FDTD method, however, the spatial increment steps ( $\Delta x, \Delta y, \Delta z$ ) must be small enough relative to the wavelength (usually 10-20 steps per wavelength) and time steps $\Delta t$ must satisfy the CFL condition as shown in (1.4). Otherwise, the FDTD method becomes unstable.

$$
\begin{equation*}
\Delta t \leq \Delta t_{\max }=\frac{1}{c \sqrt{\frac{1}{\Delta x^{2}}+\frac{1}{\Delta y^{2}}+\frac{1}{\Delta z^{2}}}}=\left.\frac{\Delta}{c \sqrt{3}}\right|_{\substack{\text { cub ic- cell } \\ \text { sp ace lattice }}} \tag{1.4}
\end{equation*}
$$

Due to the CFL stability constraint (1.4), the space and time increments of the FDTD method must be no larger than a small fraction of the smallest wavelength and temporal period of interest, typically, 10 to 20 samples per cycle depending on the type of application. When the size of the geometrical features is much smaller than $1 / 10$ th or $1 / 20$ th of the smallest wavelength of interest, the use of explicit FDTD methods requires a super-excessive computational load. To alleviate this difficulty, unconditionally stable implicit FDTD techniques can be applied which permit accurate and numerical stable operation for values of $\Delta t$ exceeding the CFL limit. The unconditionally stable ADI-FDTD, SS-FDTD, CN-FDTD and LOD-FDTD are discussed next.

### 1.2.2 ADI-FDTD Method

While the upper limit of the time step for the explicit FDTD method is determined by the CFL constraint, the application of the implicit scheme can eliminate the CFL conditions. By applying the Crank-Nicolson (CN) algorithm

$$
\begin{equation*}
\partial \mathbf{u}^{(n+1 / 2)} / \partial t \approx\left(\mathbf{u}^{n+1}-\mathbf{u}^{n}\right) / \Delta t, \mathbf{u}^{(n+1 / 2)} \approx\left(\mathbf{u}^{n+1}+\mathbf{u}^{n}\right) / 2 \tag{1.5}
\end{equation*}
$$

to (1.3) at $t=(n+1 / 2) \Delta t$, we obtain

$$
\begin{equation*}
\mathbf{u}^{n+1}=\frac{\left([I]+\frac{\Delta t}{2}[A]+\frac{\Delta t}{2}[B]-\frac{\Delta t}{2}[L]\right)}{\left([I]-\frac{\Delta t}{2}[A]-\frac{\Delta t}{2}[B]+\frac{\Delta t}{2}[L]\right)} \mathbf{u}^{n} \tag{1.6}
\end{equation*}
$$

Factoring (1.6) results in

$$
\begin{equation*}
\mathbf{u}^{n+1}=\frac{\left([I]+\frac{\Delta t}{2}[B]-\frac{\Delta t}{4}[L]\right)\left([I]+\frac{\Delta t}{2}[A]-\frac{\Delta t}{4}[L]\right)}{\left([I]-\frac{\Delta t}{2}[B]+\frac{\Delta t}{4}[L]\right)\left([I]-\frac{\Delta t}{2}[A]+\frac{\Delta t}{4}[L]\right)} \mathbf{u}^{n} \tag{1.7}
\end{equation*}
$$

The elements of $\mathbf{u},[A],[B]$ and $[L]$ are the same as mentioned in Section 1.2.1. According to the ADI scheme [8]-[11], half time step $n+1 / 2$ can be introduced in between the adjacent time steps $n$ and $n+1$, and using this fact, (1.7) can be derived as:

Sub-step 1:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{2}[A]+\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1 / 2}=\left([I]+\frac{\Delta t}{2}[B]-\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n} \tag{1.8a}
\end{equation*}
$$

Sub-step 2:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{2}[B]+\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1}=\left([I]+\frac{\Delta t}{2}[A]-\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1 / 2} \tag{1.8b}
\end{equation*}
$$

By substituting $\mathbf{u},[A]$ and $[B]$ in Section 1.2.1 into (1.8a)-(1.8b), the updating equations of ADI-FDTD are obtained. Thus, the CFL constraint can be overcome with ADI-FDTD [8]-[9]. The ADI-FDTD has also been assessed from various aspects by the research community [46]-[49]. To reduce the numerical dispersion error, the envelope ADI-FDTD has been developed [50]-[51]. For the ADI-FDTD, the anisotropy of the numerical dispersion with respect to the wave propagation angle increases with increase in the time step. To compensate the anisotropy, the artificial permittivity has been introduced [52]-[55]. As an alternative, higher order ADI-FDTD methods have also been employed to reduce the numerical dispersion error [56]-[62]. To reduce additional splitting error, an iteration procedure including perfectly matched layer (PML) ABC [66]-[68] have been developed for the ADI-FDTD. For modelling objects having curved features, the unconditionally stable alternating direction implicit (ADI) technique was introduced for nonorthogonal grids called ADI-NFDTD method [69][72]. Kantartzis et al. [69] implemented the dispersionless ADI-NFDTD algorithm that optimizes the dispersion. However, their approach needs calculation of the higher order terms twice for each time step and, thus leading to additional computational burden. Zheng et al [70]-[71] proposed 2-D and 3-D ADI-NFDTD method where they employed non-orthogonal grids locally only to model the curved/complex regions of the scatterer but used conventional orthogonal grids for the other regions of the scatterers. Further, their use of conventional PML boundary condition to truncate the computational domain would not lead to improved accuracies. In addition, their method [70]-[71] requires calculation of Jacobian coordinate transformation to convert the curvilinear coordinates into conventional FDTD lattice where CFL constraint must be satisfied, and as a result increases computational burden. For analysing the rotationally symmetric resonant structures, unconditionally stable ADI-FDTD method has also been developed in the cylindrical coordinate system [73]. But the cylindrical ADI-FDTD
presented in [73] requires a much larger memory because it solves the problems in a three dimensional domain. To overcome this difficulty, ADI-BOR-FDTD method was developed in [74]. To compute the radiation or scattering from the complex bodies of revolution, PML was presented for the ADI-BOR-FDTD [75]. The ADI-BOR-FDTD method has also been extended to model dispersive media [76]. However, the ADI-BOR-FDTD formulation is only second order accurate in time. In addition, it requires more arithmetic operations and gives large dispersion error at larger time steps. More recently, ADI-FDTD has been extended by many authors [77]-[83] to fine-tune its performance. Due to the enormity of the available literature on the ADI-FDTD method, only selective papers have been reviewed here.

### 1.2.3 SS-FDTD Method

A comparison between (1.6) and (1.8) reveals the addition of the splitting error term in (1.8), i.e. $\Delta t^{2}[A][B] / 4$ which leads to second order accuracy in time. Although the time step size in the ADI-FDTD simulation is no longer bounded by the CFL criterion, the method exhibits a splitting error associated with the square of the time step size, which limits the accuracy of the ADI-FDTD method. The ADI-FDTD method is computationally more expensive than the conventional FDTD method. The unconditionally-stable FDTD method based on the split-step scheme was consequently developed [13]-[15], which may consume less CPU time than the ADI-FDTD method. The split-step approach is based on the Strang splitting formulae [13]-[14] and involves three updating procedures. For this, (1.7) can be rewritten in three sub-steps as follows: Sub-step 1:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{4}[A]+\frac{\Delta t}{6}[L]\right) \mathbf{u}^{n+1 / 4}=\left([I]+\frac{\Delta t}{4}[A]-\frac{\Delta t}{6}[L]\right) \mathbf{u}^{n} \tag{1.9a}
\end{equation*}
$$

Sub-step 2:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{2}[B]+\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+3 / 4}=\left([I]+\frac{\Delta t}{2}[B]-\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1 / 4} \tag{1.9b}
\end{equation*}
$$

Sub-step 3:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{4}[A]+\frac{\Delta t}{6}[L]\right) \mathbf{u}^{n+1}=\left([I]+\frac{\Delta t}{4}[A]-\frac{\Delta t}{6}[L]\right) \mathbf{u}^{n+3 / 4} \tag{1.9c}
\end{equation*}
$$

By substituting $\mathbf{u},[A]$ and $[B]$ in (1.9a)-(1.9c), the updating equations for the split-step FDTD method can be obtained. The SS-FDTD in [13] consumes lesser CPU time compared to ADI-FDTD but its temporal accuracy is only first order. To improve the temporal accuracy, a modified SS-FDTD was introduced in [14]-[15]. Subsequently, other SS-FDTD methods were proposed in [84]-[86]. To reduce the dispersion error, an efficient six-stage split-step unconditionally-stable FDTD method was proposed in [87]. Furthermore, to improve the accuracy, unconditionally-stable FDTD methods with high-order accuracy and low dispersion error in 2-D domains were proposed in [88][90]. Then the method in [88] was extended to 3-D domains, and high-order split-step unconditionally-stable FDTD known as SS4-FDTD was proposed in [91]. The stability and numerical error of the extended SS4-FDTD method including lumped inductors were systematically investigated in [92]. However, SS-FDTD method presents large numerical dispersion error when the time step is large and large number of arithmetic operations are involved which reduces the computational efficiency [90]-[92].

### 1.2.4 CN-FDTD Method

The CN scheme applied to the 2-D FDTD technique was first introduced in [16][17]. The CN scheme solves the discretised Maxwell's equations by a full time-step size with one marching procedure and averages the right hand sides of the discretised Maxwell's equations at $\mathrm{n}+1$ and n time steps. So, the CN-FDTD scheme from (1.6) can be written as follows:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{2}[A]-\frac{\Delta t}{2}[B]+\frac{\Delta t}{2}[L]\right) \mathbf{u}^{n+1}=\left([I]+\frac{\Delta t}{2}[A]+\frac{\Delta t}{2}[B]-\frac{\Delta t}{2}[L]\right) \mathbf{u}^{n} \tag{1.10}
\end{equation*}
$$

The CN scheme is a well-known implicit algorithm for solving partial differential equations with second-order accuracy in both time and space [93]-[94]. An unconditionally stable CN-FDTD method in 2-D and 3-D cases has been developed in [16]-[18], [95]-[96]. The Crank-Nicolson-Douglas-Gunn (CNDG) method as 2-D [16]-[17], and as 3-D [95]-[96] as well as the Crank-Nicolson approximate-factorization-splitting (CNAFS) method in 3-D [97]-[98] have all got small anisotropy, but anisotropy can be zero for some combinations of the mesh density and time-step
size. The 3-D CN-FDTD method for frequency-dependent media (FD-CN-FDTD) has been presented by Rouf et al. in [99]. Recently, implicit isotropic dispersion finite difference time domain (ID-FDTD) algorithm has been proposed in [100], but the scheme is applicable only for the orthogonal rectangular grid. The CN-FDTD scheme is believed to have higher numerical accuracy than the ADI-FDTD method [17]-[18], but it also has a huge sparse irreducible matrix. Directly solving this matrix by Gaussian elimination or an iterative method is so CPU intensive that the CN scheme is hardly useful for practical problems. Since our main interest for the thesis is LOD-FDTD method, so the details on the development of LOD-FDTD methods will be reviewed next.

### 1.2.5 LOD-FDTD Method

Similar to ADI-FDTD method, equation (1.7) can be solved in two sub-steps for LOD-FDTD method. But unlike the ADI-FDTD, in each half step of the LOD-FDTD, we move forward only in individual $\mathrm{X}, \mathrm{Y}$, or Z directions. So, according to the LOD principle, equation (1.7) can be written as follows:

Sub-step 1:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{2}[A]+\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1 / 2}=\left([I]+\frac{\Delta t}{2}[A]-\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n} \tag{1.11a}
\end{equation*}
$$

Sub-step 2:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{2}[B]+\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1}=\left([I]+\frac{\Delta t}{2}[B]-\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1 / 2} \tag{1.11b}
\end{equation*}
$$

By substituting $\mathbf{u},[A]$ and $[B]$ in the equations (1.11a)-(1.11b), the updating equations for the LOD-FDTD method can be obtained. It has been proved that LOD-FDTD reduces the number of arithmetic operations compared to ADI-FDTD [20]-[22]. From the equation (1.11), it can also be seen that LOD formulation yields an additional error term whose coefficient is $\Delta t^{2}([A][B]-[B][A]) / 4$. If $[\mathrm{A}]$ and $[\mathrm{B}]$ commute, the error term can be eliminated, leading to the second order accuracy in time for the LOD formulations, as in the case of its ADI counterparts. Strictly speaking, the present LOD formulations are first order accurate in time, due to the absence of commutivity of [A] and $[B]$. It was also justify that the numerical results obtained from LOD-FDTD are in
good agreement with those from ADI-FDTD. This is probably true because the value of ( $[A][B]-[B][A]$ ) is negligible small, and the subsequent additional error hardly affects the numerical results. However, the LOD-FDTD method applied local splitting of the operators and is claimed to be more efficient compared to ADI-FDTD and SS-FDTD methods. The LOD-FDTD requires lesser number of arithmetic operations and lower execution time than the ADI-FDTD and SS-FDTD methods [20]-[22]. Moreover, in each half step of the LOD-FDTD, it is necessary to deal with 1-D which also simplifies implementation and eases the computationally burden. Before providing the outline of this thesis, a brief selective literature review is presented below, showing developments associated with the LOD-FDTD method which is relevant to this thesis over the years.

### 1.3 Literature Review of the LOD-FDTD Method

The locally one dimensional (LOD) scheme for the FDTD method was first proposed in [19], which is free from CFL constraints. It was shown in [19] that LOD-FDTD provides simple implementation of the algorithm and reduced execution time compared to ADI-FDTD method. The method was then extended to two dimensions by many authors [101]-[107]. Various ABCs for the 2-D LOD-FDTD to terminate the computational domain were proposed in [108]-[111]. Nasimento et al. [108] developed two split-field perfectly matched layer (PML) implementation for 2-D LOD-FDTD and proved that split field PML implementation in LOD-FDTD yields superior results compared to traditional split PML applied to ADI-FDTD for similar set of parameters. But Ahmed et al. [110] developed the convolutional perfectly matched layer (CPML) for the 2-D LOD-FDTD and showed that it provides less reflection error compared to other PML implementation [108]-[109]. The numerical dispersion error of the 2-D LOD-FDTD was investigated in [112]-[114]. The effects of spatial and temporal steps on the numerical dispersion were studied and it was found that larger time step results in higher numerical dispersion. To reduce the dispersion error of the 2-D LOD-FDTD method, higher order scheme was proposed [114].

Most of the above mentioned studies employed LOD-FDTD with 2-D formulation. The natural extension, therefore, was to use 2-D approximation to analyse RS 3-D structures using LOD-FDTD method. Thus a body of revolution (BOR) locally one
dimensional FDTD (LOD-BOR-FDTD) was proposed in [115]. The usefulness of the LOD-BOR-FDTD method was investigated through the analysis of circular cavity resonators with and without a dielectric disc, and compared them with the explicit BOR-FDTD and ADI-BOR-FDTD. However, no formulation of PML for the LOD-BOR-FDTD was available. For the analysis of RS structures, it was shown that the LOD-BOR-FDTD efficiently provides numerical results comparable to its ADI counterpart [115]. The LOD-BOR-FDTD algorithm was also extended for analysing Debye dispersive media using bilinear Z transform [116]. Also, to reduce the number of matrix-operators and thus reducing the computational load, LOD-BOR-FDTD was formulated using the fundamental scheme [117]. However, the CPML was not available for the LOD-BOR-FDTD with or without fundamental scheme [115]-[117].

For analysing three-dimensional structures using LOD-FDTD, a two sub-step [21], [118]-[120] and three sub-step procedures [22] were proposed. Both the two-step method and the three step method were extended analysing various microwave structures [118]-[123], [126]. The 3-D LOD-FDTD method was also used for the analysis of the semiconductor devices in [124]-[125]. To investigate the dispersion error of the 3-D LOD-FDTD method, an arbitrary order locally one dimensional 3-D LOD-FDTD method was proposed in [118]. The investigation suggested that the dispersion errors could be reduced using either higher order or employing a denser grid. To further reduce the dispersion error, various techniques of dispersion control for LOD-FDTD were proposed [127]-[129]. A parameter optimisation approach was also developed [128] to improve the numerical dispersion performance of the three sub-step 3-D LOD-FDTD method. The two sub-step 3-D LOD-FDTD method presented in [21] requires special input and output processing procedures for computing the field components of interest which might decrease the computational efficiency of the method. Later two sub-step LOD-FDTD method was extended considering lossless and frequency dependent media [118]-[126]. The three sub-steps 3-D LOD-FDTD of [22] was also applied for the analysis of 3-D structures in [123]. Various ABCs viz. Mur's, PML and CPML for use with LOD-FDTD to truncate the computational domain were also derived [130]-[133]. Among them the Mur's and PML approaches [130]-[133] show absorption errors at low frequencies and for evanescent waves. On the other hand, the CPML can be completely independent of the host medium without requiring any
modifications in formulation when applied for analysing lossy, dispersive, anisotropic, nonlinear, and inhomogeneous media. The use of CPML can also provide significant saving in memory [2]. In addition, the CPML permits an easy implementation of the complex frequency shifted (CFS) stretching factor that allows the reflection of the low frequency evanescent waves to be significantly reduced [2]. Ahmed et al. extended CPML for the 3-D LOD-FDTD with three sub-step [131]. It is clear that the two substeps LOD-FDTD can reduce the number of required arithmetic operations as compared to the three sub-step LOD-FDTD. However, no CPML is so far available for the two sub-step LOD-FDTD method [22], [118]-[122].

Use of conventional LOD-FDTD with existing ABCs [130]-[133] for modelling realistic 3-D microwave structures require solution of large numbers of matrix operations since the right hand sides of resultant update equations contain many field variables and matrix operators. To lessen some of the computational burden, Tan [134] proposed a novel scheme known as fundamental scheme suitable for all implicit FDTD methods to improve the computational efficiency. Later, the fundamental scheme was extended to LOD-FDTD [136], [139]-[140]. However, for 3-D fundamental scheme based LOD-FDTD, so far in the literature, only Mur's [137], PML [126] and PEC and PMC boundary condition [138] have been reported. Also for analysing threedimensional curved geometries, non-orthogonal curvilinear mesh LOD-FDTD based on fundamental scheme is not available in the literature so far.

### 1.4 Motivations

So far in the literature LOD-FDTD method has been developed in 2-D, body of revolution (BOR) and 3-D for analysing various microwave structures. But, 2-D and 3D LOD-FDTD method based on curvilinear coordinates, CPML for two sub-step 3-D LOD-FDTD, CPML for F-LOD-FDTD or LOD-FDTD using orthogonal and nonorthogonal meshes are not available in the literature. For solving many real world problems related to electrically large structures, 2-D, resonant (body of revolution) and 3-D structures using orthogonal and nonorthogonal meshes, it is often necessary to employ enhanced versions of the LOD-FDTD in order to render the problem manageable and treatable with the available resource within a reasonable time frame. So, in this thesis, efficient enhanced versions of LOD-FDTD using orthogonal and non-
orthogonal curvilinear meshes have been proposed and investigated for solving EM problems. A brief outline of this thesis is given below.

### 1.5 Thesis Outline

This thesis structurally consists of five parts:

- Development and implementation of segmented LOD-FDTD (S-LOD-FDTD) technique (Chapter 2)
- Development and implementation of nonorthogonal LOD-FDTD (LODNFDTD) method with CPML ABC for predicting EM scattering from 2-D conducting, dielectric and mixed structures (Chapter 3)
- Development and implementation of dispersion controlled rotationally symmetric LOD-FDTD (D-RS-LOD-FDTD) CPML for analysing resonant structures and antennas that have RS geometries (Chapter 4)
- Development and implementation of C-LOD-FDTD CPML and F-LOD-FDTD CPML for analysing 3-D structures using orthogonal meshes (Chapter 5)
- Development of 3-D C-LOD-NFDTD CPML as well as F-LOD-NFDTD CPML approaches for analysing 3-D curved structures using nonorthogonal curvilinear meshes (Chapter 6)


### 1.5.1 Summary of Chapters

In some detail, we now outline the relevant material in each chapter.

## Chapter 1

In this chapter, we present a review of the implicit FDTD methods and also a review of literature on the LOD-FDTD methods. The scope of the thesis is outlined and a list of publications arising from this research is provided.

## Chapter 2

The second chapter is devoted to the theoretical investigation of the locally one dimensional FDTD (LOD-FDTD) method. Derivation of 2-D LOD-FDTD and the CPML ABC of the 2-D LOD-FDTD method are presented. A novel segmented LOD-

FDTD approach is proposed. The results of S-LOD-FDTD technique being employed to model wave propagation in electrically large symmetric tunnels are presented.

## Chapter 3

This chapter describes the mathematical formulation of the proposed 2-D LOD-NFDTD method for both TE and TM cases. A non-orthogonal curvilinear mesh generation technique is discussed. The CPML ABC formulation for LOD-NFDTD for both TE and TM cases to terminate the computational domain is described. The near-field to far-field transformation and scattered field formulation for the LOD-NFDTD method is discussed. Numerical stability and dispersion analysis of the proposed method are also presented. Numerical results for many 2-D cylindrical structures that include conducting, dielectric and coated conducting and mixed structures (overfilled dielectric cavity and bent perfect electric conducting (PEC) cavity) obtained using the proposed LOD-NFDTD method are provided and compared with the results obtained from other exact and numerical methods to validate our LOD-NFDTD method in 2-D.

## Chapter 4

In this chapter, a conventional RS-LOD-FDTD along with CPML ABC is presented for both $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ cases. A dispersion control (D)-RS-LOD-FDTD technique for $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ cases is proposed. The results on numerical reflection are also presented for the D-RS-LOD-FDTD for $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ cases. Also the methods to calculate S parameters, specific absorption rate (SAR) using both RS-LOD-FDTD and D-RS-LODFDTD are provided. Numerical analysis of various RS structures is demonstrated and compared with other methods.

## Chapter 5

This chapter proposes a two sub-step conventional (C)-LOD-FDTD method along with an efficient two sub-step CPML ABC. The stability analysis of the C-LOD-FDTD is presented. Then a F-LOD-FDTD is described. The chapter also proposes CPML ABC of F-LOD-FDTD. Numerical stability analysis of the F-LOD-FDTD method has also been presented. Pure scattered field and near-field to far field formulation for both C -

LOD-FDTD and F-LOD-FDTD are presented. Computational results on various 3-D microwave devices are provided along with detailed comparisons.

## Chapter 6

To improve the computational efficiency of the LOD-FDTD method for analysing curved 3-D structures, this chapter presents both 3-D C-LOD-NFDTD and F-LODNFDTD based on curvilinear meshes. The CPML ABC for both the proposed methods is also derived. Theoretical stability analysis of the C-LOD-NFDTD and F-LODNFDTD has also been derived to prove the unconditional stability of both methods. Pure scattered field and near-field to far field formulation for both C-LOD-NFDTD and F-LOD-NFDTD are presented. Computational performance for both methods has also been discussed.

## Chapter 7

This chapter highlights the major contributions that have arisen from Chapter 2 to Chapter 6. A brief scope for future work is also indicated.

## Appendix A

In the appendix, we present derivation of ADI-FDTD and ADI-NFDTD along with CPML ABC.

### 1.6 Publications Arising from this Research

## Refereed Journal Articles:

1. Md. Masud Rana and Ananda S. Mohan, "Convolutional perfectly matched layer ABC for 3-D LOD-FDTD using fundamental scheme," IEEE Microwave and Wireless Components Letters., vol. 23, no. 8, pp. 388-390, Aug., 2013.
2. Md. Masud Rana and Ananda S. Mohan, "Non-orthogonal LOD-FDTD method for EM scattering from two dimensional structures," IEEE Transactions on Electromagnetic Compatibility, vol. 55, no. 4, pp. 764-772, Aug., 2013.
3. Md. Masud Rana and Ananda S. Mohan, "Segmented-Locally-one-dimensional FDTD method for EM propagation inside large complex tunnel environments," IEEE Transactions on Magnetics, vol. 48, no. 2, pp. 223-226, Feb., 2012.

## Refereed Conference Papers:

4. Md. Masud Rana and Ananda S. Mohan, "An efficient two-step CPML for 3-D LOD-FDTD convolutional perfectly matched layer ABC for 3-D LOD-FDTD using fundamental scheme," Accepted by 2013 IEEE International Workshop on Electromagnetics (iWEM, 2013), (Acceptance notified on the $10^{\text {th }}$ June, 2013), 13 August, Hong Kong, 2013. (to be published in IEEE xplore)
5. Md. Masud Rana and Ananda S. Mohan, "Rotationally symmetric (RS)-LODFDTD with CPML for analysing resonant structures," in Proceedings of 2012 International Symposium on Antenna and Propagation (ISAP), pp. 930-933, Oct. 29-Nov. 02, 2012, Nagoya, Japan,. (available in IEEE xplore)
6. Md. Masud Rana and Ananda S. Mohan, "Locally one dimensional FDTD for EM scattering from two-dimensional structures", in Proceedings of 2011 IEEE Applied Electromagnetics Conference (AEMC), pp. 1-4, 18-22, Dec. 2011, Kolkata, India. (available in IEEE xplore)
7. Md. Masud Rana and Ananda S. Mohan, "Nonorthogonal locally one dimensional FDTD (LOD-FDTD) method," in Proceedings of 2011 IEEE APS/URSI International Symposium, pp. 2303-2306, 3-8 July, 2011, Spokane, USA. (available in IEEE xplore)
8. Md. Masud Rana and Ananda S. Mohan, "Locally one dimensional FDTD (LODFDTD) method for EM wave propagation modelling", in Proceedings of Australia Symposium on Antenna (ASA), 16-17 Feb, 2011, Sydney, Australia. (ISBN 978-0-643-10168-5)
9. Md. Masud Rana and Ananda S. Mohan, "EM wave propagation modelling in complex tunnel environments", in Proceedings of Asia Pacific Symposium on Applied Electromagnetics and Mechanics (APSAEM), pp. 396-401, 28-30 July, 2010, Kuala Lumpur, Malaysia. (ISBN 978-4-931455-16-0)
10. Md. Masud Rana and Ananda S. Mohan, "Propagation modelling in complex tunnel environments using FDTD method", in Proceedings of Workshop on Applications of Radio Science (WARS), 11-12 Feb, 2010, Canberra, Australia.

## Chapter 2

## 2-D LOD-FDTD for EM Propagation Modelling in Electrically Large Symmetric Structures

### 2.1 Introduction

The standard explicit finite difference time domain (FDTD) method has been used extensively for modelling radio wave propagation in indoor environments [141], [143]. When the standard explicit FDTD method is applied to model EM propagation in electrically large problems at microwave frequencies, its computational load in terms of memory and CPU execution time can become excessive. To ease the computational burden for solving large scale problems using FDTD, many techniques have been proposed [143]-[145]. Chevalier and Inan [144] proposed the segmented long path propagation (SLP) technique to reduce the computational load for long ionospheric propagation using FDTD and FDFD. Most recently, the segmented finite difference time domain (S-FDTD) method was introduced to reduce computational requirements and enhance the feasibility of solving electrically large problems on a standard personal computer [145]. However, the S-FDTD method, which is based on the explicit FDTD method inherits the limitations of the FDTD method because it needs to satisfy the CFL stability constraint; hence it would still require a large number of segments which might increase the computational burden when solving electrically large scale propagation problems. To overcome some of these limitations, the alternating direction implicit FDTD (ADI-FDTD) method was proposed [146]-[148]. It has also been found that the ADI-FDTD technique requires more execution time for higher CFLN [10]-[11], [148]. To overcome some of the limitations of ADI-FDTD, LOD-FDTD was proposed [19][22] which requires lesser number of arithmetic operations and lower execution time [20]. Nevertheless, the direct application of the LOD-FDTD method for solving electrically large scale problems would still require increased computational resources

## Chapter 2: 2-D LOD-FDTD for EM Propagation Modelling in Electrical Large Symmetric Structures

due to the solution of sets of simultaneous equations that may become too large to efficiently compute for problems with dense mesh. Therefore, methods to overcome the limitations of the LOD-FDTD method to solve for EM propagation in electrically large structures are required.

In this chapter, we first introduce the theory of conventional LOD-FDTD and then provide the derivations of the 2-D LOD-FDTD method. The CPML ABC for 2-D LODFDTD is also discussed. We develop the segmented-LOD-FDTD (S-LOD-FDTD) method for solving the EM wave propagation inside electrically large symmetric tunnel environments. To make a comparison with the S-LOD-FDTD method, the S-ADIFDTD method is next discussed briefly. For performance validation of the proposed method, EM propagation inside various large symmetric tunnels is analyzed followed by a discussion of error analysis. Finally, a brief summary of this chapter is provided.

### 2.2 Introduction to Locally One Dimensional Finite Difference Time Domain (LOD-FDTD) Method

To remove the CFL constraint, a novel unconditionally stable alternating direction implicit finite difference time domain (ADI-FDTD) method was developed [8]-[12]. The ADI-FDTD method is a popular implicit FDTD method that is second order accurate in both space and time. However, ADI-FDTD exhibits a splitting error [13[14] that depends not only on the time step size but also on the magnitude of the spatial derivatives. Many alternative implicit FDTD techniques have since been reported in the literature [13]-[22]. Among them, LOD-FDTD is claimed to be more efficient because it requires fewer arithmetic operations [19]-[22]. Moreover, the LOD-FDTD formulation is a simple type of split-step approach and is first order accurate in time [19]. The LOD technique involves the solution of each partial differential equation using a two sub-step scheme in time [19]. For each half step of the LOD method, it is needed to move forward only in the $\mathrm{X}, \mathrm{Y}$, or Z direction.

This makes the scheme attractive computationally. Applying the LOD principle to the Maxwell equations (1.1) of Chapter 1, two sub-step LOD-FDTD equations can be written as follows:

Sub-step 1:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{2}[A]+\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1 / 2}=\left([I]+\frac{\Delta t}{2}[A]-\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n} \tag{2.1a}
\end{equation*}
$$

Sub-step 2:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{2}[B]+\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1}=\left([I]+\frac{\Delta t}{2}[B]-\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1 / 2} \tag{2.1b}
\end{equation*}
$$

where $\mathbf{u},[A],[B]$, and $[L]$ are the same as previously mentioned in Chapter 1. By substituting $\mathbf{u},[A],[B]$ and $[\mathrm{L}]$ in (2.1a)-(2.1b), and moving forward for each half time step separately in the $X, Y$ or $Z$ direction, we obtain the LOD-FDTD equations. The field component in 3-D computational space is summarised in Fig. 2.1. The LODFDTD cyclic permutations form a complete time-step iteration which is shown in the Fig. 2.2. The derivations of the updating equations for LOD-FDTD are described in the next section.


Fig. 2.1 Space lattice for LOD-FDTD


Fig. 2.2 FDTD is a leapfrog scheme using explicit updates; whereas LOD-FDTD uses sequential implicit and explicit updates for $E$ and $H$ fields, respectively, each half time step

### 2.2.1 Derivation of LOD-FDTD Method

By substituting $\mathbf{u},[A]$ and $[B]$ in (2.1a) and (2.1b), the updating equations for the LOD-FDTD method in two sub-steps can be derived as follows.

Sub-step 1:

$$
\begin{align*}
& E_{x}^{n+1 / 2}=\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{x}^{n}+\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times\left(\frac{\partial H_{z}^{n+1 / 2}}{d y}+\frac{\partial H_{z}^{n}}{d y}\right)  \tag{2.2a}\\
& E_{y}^{n+1 / 2}=\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{y}^{n}+\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times\left(\frac{\partial H_{x}^{n+1 / 2}}{d z}+\frac{\partial H_{x}^{n}}{d z}\right)  \tag{2.2b}\\
& E_{z}^{n+1 / 2}=\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{z}^{n}+\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times\left(\frac{\partial H_{y}^{n+1 / 2}}{d x}+\frac{\partial H_{y}^{n}}{d x}\right)  \tag{2.2c}\\
& H_{x}^{n+1 / 2}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{x}^{n}+\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times\left(\frac{\partial E_{y}^{n+1 / 2}}{d z}+\frac{\partial E_{y}^{n}}{d z}\right)  \tag{2.2d}\\
& H_{y}^{n+1 / 2}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{y}^{n}+\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times\left(\frac{\partial E_{z}^{n+1 / 2}}{d x}+\frac{\partial E_{z}^{n}}{d x}\right) \tag{2.2e}
\end{align*}
$$

$$
\begin{equation*}
H_{z}^{n+1 / 2}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{z}^{n}+\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times\left(\frac{\partial E_{x}^{n+1 / 2}}{d y}+\frac{\partial E_{x}^{n}}{d y}\right) \tag{2.2f}
\end{equation*}
$$

Sub-step 2:

$$
\begin{align*}
& E_{x}^{n+1}=\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{x}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times\left(\frac{\partial H_{y}^{n+1}}{d z}+\frac{\partial H_{y}^{n+1 / 2}}{d z}\right)  \tag{2.3a}\\
& E_{y}^{n+1}=\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{y}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times\left(\frac{\partial H_{z}^{n+1}}{d x}+\frac{\partial H_{z}^{n+1 / 2}}{d x}\right)  \tag{2.3b}\\
& E_{z}^{n+1}=\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{z}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times\left(\frac{\partial H_{x}^{n+1}}{d y}+\frac{\partial H_{x}^{n+1 / 2}}{d y}\right)  \tag{2.3c}\\
& H_{x}^{n+1}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{x}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times\left(\frac{\partial E_{z}^{n+1}}{d y}+\frac{\partial E_{z}^{n+1 / 2}}{d y}\right)  \tag{2.3d}\\
& H_{y}^{n+1}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{y}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times\left(\frac{\partial E_{x}^{n+1}}{d z}+\frac{\partial E_{z}^{n+1 / 2}}{d z}\right)  \tag{2.3e}\\
& H_{z}^{n+1}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{z}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times\left(\frac{\partial E_{y}^{n+1}}{d x}+\frac{\partial E_{y}^{n+1 / 2}}{d x}\right) \tag{2.3f}
\end{align*}
$$

where $\vec{E}, \vec{H}$ are electric and magnetic field intensities and $\sigma_{e}, \sigma_{m}$ are electric conductivity and equivalent magnetic loss, respectively. It can be observed from (2.2)(2.3) that the LOD-FDTD method involves the solution of each partial differential equation using a two sub-step scheme in time.
Several things can be noted here for the LOD-FDTD method:

- Two terms in the right-hand side of sub-step 1 are discretised on the new ' $n+1 / 2$ ' and ' $n$ ' old time steps and the two terms of the right-hand side of sub-step 2 are discretised on the new ' $n+1$ ' and ' $n+1 / 2$ ' old time steps.
- On the time step ' $n+1 / 2$ ' of sub-step 1 and time step ' $n+1$ ' of sub-step 2 , tridiagonal linear system can be obtained for $E_{x}, E_{y}$ and $E_{z}$ field components. In substep 1, by substituting (2.2f) into (2.2a), (2.2e) into (2.2c) and (2.2d) into (2.2b), simultaneous linear system can be written for $E_{x}, E_{y}$ and $E_{z}$ in tri-diagonal matrix form which is solved implicitly using the Thomas algorithm [149]. Using these electric field component values with (2.2d) to (2.2f), explicit magnetic field components $H_{x}, H_{y}$ and $H_{z}$ are then obtained. Similarly, in the sub-step 2 on time step ' $n+1$ ', by placing (2.3e) into (2.3a), (2.3f) into (2.3b) and (2.3d) into (2.3c),
simultaneous linear equations are obtained for $E_{x}, E_{y}$ and $E_{z}$ in tri-diagonal matrix form which are solved implicitly using the Thomas algorithm [149]. Using these electric field $E_{x}, E_{y}$ and $E_{z}$ component values with (2.3d) to (2.3f), explicit magnetic field components $H_{x}, H_{y}$ and $H_{z}$ are then obtained.

The flowchart of the LOD-FDTD method is shown in Fig. 2.3.


Fig. 2.3 Flowchart of the LOD-FDTD method

## Chapter 2: 2-D LOD-FDTD for EM Propagation Modelling in Electrical Large Symmetric Structures

The updating equations for the LOD-FDTD in 3-D and 2-D can be derived using (2.2) and (2.3) which are used throughout the thesis. The updating equations for LOD-FDTD in 2-D are provided below.

### 2.3 Derivation of 2-D LOD-FDTD Method

### 2.3.1 $\mathrm{TE}_{\mathrm{z}}$ Case

Equations (2.2) to (2.3) can be further simplified for analysing 2-D EM problem using LOD-FDTD method. The updating equations for the 2-D LOD-FDTD method for the $\mathrm{TE}_{z}$ case are formulated into two sub-steps which are shown below.

Sub-step 1:

$$
\begin{align*}
\left.E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}=C_{e x e} \times\left. E_{x}\right|_{i+1 / 2, j} ^{n} & +C_{e x k z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right) \\
& +C_{e x h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)  \tag{2.4a}\\
\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=C_{h z h} \times\left. H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n} & +C_{h z e x} \times\left(\left.E_{x}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}\right) \\
& +C_{h z e x} \times\left(\left.E_{x}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}\right) \tag{2.4b}
\end{align*}
$$

Sub-step 2:

$$
\begin{gather*}
\left.E_{y}\right|_{i, j+1 / 2} ^{n+1}=C_{\text {eye }} \times\left. E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}-C_{e y h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
-C_{e y h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1}\right)  \tag{2.5a}\\
\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=C_{h z h} \times\left. H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-C_{h z e y} \times\left(\left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.E_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2}\right) \\
-C_{h z e y} \times\left(\left.E_{y}\right|_{i, j+1 / 2} ^{n+1}-\left.E_{y}\right|_{i+1, j+1 / 2} ^{n+1}\right)  \tag{2.5b}\\
\text { where } C_{\text {exe }}=C_{e y e}=\left(4 \varepsilon-\sigma_{e} \Delta t\right) /\left(4 \varepsilon+\sigma_{e} \Delta t\right), C_{e x h z}=2 \Delta t / \Delta y\left(4 \varepsilon+\sigma_{e} \Delta t\right) \\
C_{\text {eyhz }}=2 \Delta t / \Delta x\left(4 \varepsilon+\sigma_{e} \Delta t\right) C_{h z h}=\left(4 \mu-\sigma_{m} \Delta t\right) /\left(4 \mu+\sigma_{m} \Delta t\right) \\
C_{h z e x}=2 \Delta t / \Delta y\left(4 \mu+\sigma_{m} \Delta t\right), C_{h z e y}=2 \Delta t / \Delta x\left(4 \mu+\sigma_{m} \Delta t\right),
\end{gather*}
$$

In (2.4a) and (2.4b) of sub-step $1, E_{x}$ and $H_{z}$ field components can be defined as synchronous variables. Equation (2.4a) cannot be directly solved. Placing (2.4b) into (2.4a) results in a tri-diagonal matrix equation (2.6) which can be solved efficiently.

$$
\begin{align*}
& -\left.\alpha_{1} E_{x}\right|_{i+1 / 2, j-1} ^{n+1 / 2}+\left.\left(1+2 \alpha_{1}\right) E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}-\left.\alpha_{1} E_{x}\right|_{i+1 / 2, j+1} ^{n+1 / 2} \\
& \quad= \\
& \left.\quad C_{e x e} E_{x}\right|_{i+1 / 2, j} ^{n}+C_{h z h} \times C_{e x h z}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)  \tag{2.6}\\
& \quad+\alpha_{1}\left(\left.E_{x}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}+\left.E_{x}\right|_{i+1 / 2, j-1} ^{n}\right) \\
& \quad \quad+C_{e x h z}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)
\end{align*}
$$

where $\alpha_{1}=C_{\text {exhz }} \times C_{\text {ehex }}$. Similarly, from sub-step 2, by placing (2.5b) into (2.5a), we obtain a simultaneous linear equation in tri-diagonal matrix form, as shown in (2.7).

$$
\begin{align*}
-\left.\beta_{1} E_{y}\right|_{i-1, j+1 / 2} ^{n+1} & +\left.\left(1+2 \beta_{1}\right) E_{y}\right|_{i, j+1 / 2} ^{n+1}-\left.\beta_{1} E_{y}\right|_{i+1, j+1 / 2} ^{n+1} \\
= & \left.C_{e y e} E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}-C_{h z h} \times C_{e y h z}\left(\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
& +\beta_{1}\left(\left.E_{y}\right|_{i-1, j+1 / 2} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}+\left.E_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2}\right)  \tag{2.7}\\
& \quad-C_{e y h z}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)
\end{align*}
$$

where $\beta_{1}=C_{\text {eyhz }} \times C_{\text {hzey }}$. The resultant equations (2.6) and (2.7) can be solved using numerical techniques and packages that are readily available (such as the Thomas algorithm [149]). In the following sub-section, we will describe the approaches used to solve the equations involving tri-diagonal matrix.

### 2.3.2 Approach for Solving Equation Involving Tri-diagonal Matrix

The developed efficient approach can be applied for the solution of the simultaneous linear equation involving tri-diagonal matrix obtained for 2-D and 3-D cases throughout the thesis. We start with (2.2) and (2.3) which can be written in a simple matrix form as

$$
\begin{align*}
& \mathbf{M}_{1} \mathbf{X}^{n+1 / 2}=\mathbf{P}_{1} \mathbf{X}^{n} \quad \text { For the first half time step }  \tag{2.8a}\\
& \mathbf{M}_{2} \mathbf{X}^{n+1}=\mathbf{P}_{2} \mathbf{X}^{n+1 / 2} \text { For the second half time step } \tag{2.8b}
\end{align*}
$$

where $\mathbf{M}_{1}, \mathbf{P}_{1}, \mathbf{M}_{2}$ and $\mathbf{P}_{2}$ are the matrix coefficients of the update equations (2.2) and (2.3), and $\mathbf{X}$ is a vector composed of the field components at each spatial location in the grid, hence, quite a long vector. The coefficient matrices are both sparse, so (2.8a) and ( 2.8 b ) can be combined into a single time step equation as follows:

$$
\left.\begin{array}{l}
\mathbf{X}^{n+1 / 2}=\mathbf{M}_{1}^{-1} \mathbf{P}_{1} \mathbf{X}^{n}  \tag{2.9}\\
\mathbf{X}^{n+1}=\mathbf{M}_{2}^{-1} \mathbf{P}_{2} \mathbf{X}^{n+1 / 2}
\end{array}\right\} \rightarrow \mathbf{X}^{n+1}=\mathbf{M}_{2}^{-1} \mathbf{P}_{2} \mathbf{M}_{1}^{-1} \mathbf{P}_{1} \mathbf{X}^{n}=\Lambda \mathbf{X}^{n}
$$

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Where $\Lambda=\mathbf{M}_{2}^{-1} \mathbf{P}_{2} \mathbf{M}_{1}{ }^{-1} \mathbf{P}_{1}$. Although the above approach is simple and easy to implement, its solution requires large computational resources. By using the above approach, equations (2.6) and (2.7) can be solved. By analysing the first sub-step in equation (2.4), it is found that after the electric field components are obtained from (2.6), magnetic field component can be updated using (2.4b). Thus the magnetic field component can be solved in a simple way without matrix multiplications or inversion. In a practical simulation, however, a more efficient approach is applied to solve the equation involving tri-diagonal matrix. To compute $E_{x}$ from (2.6), a more efficient procedure can be used that can also be applied for the solution of the equation involving tri-diagonal matrix obtained for 3-D LOD-FDTD. This procedure is described below. Consider (2.6) which can be written in simplified form as

$$
\begin{equation*}
a_{j} u_{j-1}+b_{j} u_{j}+c_{j} u_{j+1}=d_{j} \tag{2.10}
\end{equation*}
$$

where $u_{j}$ represents $\left.E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}$ and $a_{j}, b_{j}, c_{j}$, and $d_{j}$ represents the corresponding known coefficient values in (2.6). Assume that $j$ sweeps from 0 to $\mathrm{N}+1$, and $u_{0}=0$, and $u_{N+1}=0$ on the boundary. The application of (2.10) in the order of ascending $j$ leads to a set of N simultaneous equations

$$
\left\{\begin{array}{l}
b_{1} u_{1}+c_{1} u_{2}=d_{1}  \tag{2.11}\\
a_{2} u_{1}+b_{2} u_{2}+c_{2} u_{3}=d_{2} \\
\quad \ldots \ldots \ldots \\
a_{r} u_{r-1}+b_{r} u_{r}+c_{r} u_{r+1}=d_{r} \\
\quad \ldots \ldots \ldots \\
a_{N} u_{N-2}+b_{N-1} u_{N-1}+c_{N-1} u_{N}=d_{N-1} \\
a_{N} u_{N-1}+b_{N} u_{N}=d_{N}
\end{array}\right.
$$

The matrix expression (2.11) can be written as

$$
\begin{equation*}
\mathbf{A u}=\mathbf{d} \tag{2.12}
\end{equation*}
$$

where

$$
\begin{gathered}
\mathbf{A}=\left[\begin{array}{ccccccc}
b_{1} & \mathrm{c}_{1} & 0 & & \cdots & & 0 \\
a_{2} & \mathrm{~b}_{2} & \mathrm{c}_{2} & 0 & & \cdots & \\
& & \cdots & & & \\
& & & \mathrm{a}_{r} & \mathrm{~b}_{\mathrm{r}} & \mathrm{c}_{\mathrm{r}} & \\
& & & & & \\
& & \cdots & 0 & \mathrm{a}_{N-1} & \mathrm{~b}_{\mathrm{N}-1} & \mathrm{c}_{\mathrm{N}-1} \\
0 & & \cdots & & 0 & \mathrm{a}_{N} & \mathrm{~b}_{\mathrm{N}}
\end{array}\right] \\
0 \\
\\
\\
\mathbf{u}=\left[\begin{array}{llllll}
u_{1} & \mathrm{u}_{2} & \cdots & \mathrm{u}_{\mathrm{N}}
\end{array}\right]^{T} \\
\mathbf{d}=\left[\begin{array}{lllll}
d_{1} & \mathrm{~d}_{2} & \cdots & \mathrm{~d}_{\mathrm{N}}
\end{array}\right]^{T}
\end{gathered}
$$

Two different approaches can be used to solve (2.12).

## A) Inverse matrix

It can be observed that the size of $\mathbf{A}$ is smaller than that of $\mathbf{M}_{1}, \mathbf{M}_{2}$ since it is in one dimension (i.e. j in this case). For a $100 \times 100 \times 100$ problem, the size of $\mathbf{A}$ is $100 \times 100$ which is $1 / 10^{8}$ of the size of $\mathbf{M}_{1}$ or $\mathbf{M}_{2}$. Therefore, less memory is required for the computation of the inverse of the matrix. Furthermore, $\mathbf{A}^{-1}$ may be used once only to calculate the field values at the left-most grid point $u_{1}$. Once $u_{1}$ is obtained, forward substitution can be applied in (2.10) to find the other components; more specifically, the second leftmost value at $j=1$, at the $(n+1 / 2)$-th time step can be obtained directly from

$$
\begin{equation*}
u_{2}=\frac{1}{c_{1}}\left(d_{1}-b_{1} u_{1}\right) \tag{2.13}
\end{equation*}
$$

The remaining $u$ can be computed by applying (2.10)

$$
\begin{equation*}
u_{j+1}=\frac{1}{c_{j}}\left(d_{j}-b_{j} u_{j}-a_{j} u_{j-1}\right) \tag{2.14}
\end{equation*}
$$

with a sequence of ascending $j$ that permits one to find $u_{j+1}$ from $u_{j}$ and $u_{j-1}$. In this way, we avoid the application of $\mathbf{A}^{-1}$ which is not necessarily very sparse for most of the computations. Thus, the computation efficiency is improved.

## B) Inverse matrix

Alternatively, by using the Gaussian elimination method, ' $u$ ' can be obtained without the involvement of $\mathbf{A}^{-1}$. Hence, the time-consuming calculation of $\mathbf{A}^{-1}$ can be totally avoided.

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Our experiences show that, for a linear, lossless and isotropic medium, approach A) is more efficient because it is only calculated once, can be used in each time step advance and only the first row of $\mathbf{A}^{-1}$ is needed. Wherever, simultaneous linear systems with tridiagonal matrix are obtained in this thesis, approach A) has been used to solve the equation. The updating equations of 2-D LOD-FDTD for the $\mathrm{TM}_{\mathrm{z}}$ case are discussed below.

### 2.3.3 $\mathrm{TM}_{\mathrm{z}}$ Case

The formulations of LOD-FDTD for the 2-D $\mathrm{TM}_{\mathrm{z}}$ wave are derived in the same way as in the case of the $\mathrm{TE}_{\mathrm{z}}$ wave. The updating equations for both sub-steps 1 and 2 are given next.

Sub-step 1:

$$
\begin{align*}
\left.H_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}=C_{h y h} \times\left. H_{y}\right|_{i, j+1 / 2} ^{n} & +C_{h y e z} \times\left(\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
& +C_{h y e z} \times\left(\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.E_{z}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)  \tag{2.15a}\\
\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=C_{e z e} \times\left. E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n} & +C_{e z h y} \times\left(\left.H_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.H_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}\right) \\
& +C_{e z h y} \times\left(\left.H_{y}\right|_{i+1, j+1 / 2} ^{n}-\left.H_{y}\right|_{i, j+1 / 2} ^{n}\right) \tag{2.15b}
\end{align*}
$$

Sub-step 2:

$$
\begin{array}{r}
\left.H_{x}\right|_{i+1 / 2, j} ^{n+1}=C_{h x h} \times\left. H_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}-C_{h x e z} \times\left(\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right) \\
- \\
\quad C_{h x e z} \times\left(\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.E_{z}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right) \\
\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=C_{e z e} \times\left. E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-C_{e z h x} \times\left(\left.H_{x}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.H_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)  \tag{2.16b}\\
\\
-C_{e z h x} \times\left(\left.H_{x}\right|_{i+1 / 2, j+1} ^{n+1}-\left.H_{x}\right|_{i+1 / 2, j} ^{n+1}\right)
\end{array}
$$

where $C_{e z e}=\left(4 \varepsilon-\sigma_{e} \Delta t\right) /\left(4 \varepsilon+\sigma_{e} \Delta t\right), C_{\text {hyyz }}=2 \Delta t / \Delta x\left(4 \varepsilon+\sigma_{m} \Delta t\right)$

$$
\begin{gathered}
C_{e z h y}=2 \Delta t / \Delta x\left(4 \varepsilon+\sigma_{e} \Delta t\right) C_{h x h}=C_{h y h}=\left(4 \mu-\sigma_{m} \Delta t\right) /\left(4 \mu+\sigma_{m} \Delta t\right) \\
C_{h y e z}=2 \Delta t / \Delta y\left(4 \mu+\sigma_{m} \Delta t\right), C_{e z h x}=2 \Delta t / \Delta y(4 \varepsilon+\sigma \Delta t),
\end{gathered}
$$

Unlike (2.4a) and (2.42b), the equations of sub-step 1 and 2 of the $\mathrm{TM}_{\mathrm{z}}$ case cannot be used for direct numerical calculation. Placing (2.15a) in (2.15b) of sub-step 1 and (2.16a) into (2.16b) of sub-step 2 yields the simultaneous linear equations for

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$\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}$ and $\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}$, that result in the tri-diagonal matrix form which can be solved by using approach A) that has been described in Section 2.3.2.

### 2.4 Absorbing Boundary Conditions

In numerical modelling, it is often necessary to simulate objects located within an open region which is assumed to extend to infinity, meanwhile, it is almost impossible to handle an open region problem directly due to the limitations of computer resources (i.e. the data storage capability of a computer is limited by the amount of memory). To alleviate this difficulty, an ABC is utilised to truncate the computational domains which suppresses the spurious reflections of the outgoing waves to an acceptable level. Numerous absorbing boundary conditions (ABCs) have been developed to truncate open structures for analysis with FDTD and ADI-FDTD methods [34]-[40], [66]-[67]. Among them, the CPML ABC is very popular because it is completely independent of the host medium and there is no need for modifications in formulation, when it is applied to lossy, dispersive, nonlinear, inhomogeneous and anisotropic media. The use of CPML provides a significant saving in memory [39]. In addition, the CPML permits the easy implementation of the complex frequency shifted (CFS) stretching factor that allows the reflection of low frequency evanescent waves to be significantly reduced [39].

Similar to ADI-FDTD, various ABCs such as Mur's ABC [130], [132], [138] split field and unsplit PML [108]-[110] have been developed for the LOD-FDTD method. Although these ABC draw much attention, due to their superior effectiveness and robustness, they are not highly effective at absorbing evanescent waves and signals with long time signature etc. For 2-D and 3-D LOD-FDTD methods, CPML is developed in [110], [131]. Ahmed et al. [131] developed CPML for the 3-D LOD-FDTD using three sub-steps, but the updating equation using three sub-steps increases the arithmetic operations and takes more computational time. To improve the computational time for the 2-D and 3-D LOD-FDTD method, two-step procedure for the LOD-FDTD CPML using orthogonal and nonorthogonal meshes has been developed in this thesis. The formulations of CPML, which are discussed in generalised form, can be used for 2-D and 3-D LOD-FDTD method.

### 2.4.1 Formulation of CPML ABC for LOD-FDTD

The starting point of the CPML formulation for the LOD-FDTD is a PML medium assumed to terminate a finite space occupied by a host medium, as shown in Fig. 2.4 (2D case).


Fig. 2.4 Structure of a two dimensional TE $_{z}$ LOD-FDTD employing the CPML ABC

Within this region, Maxwell's equations can be expressed in the stretch coordinate space [39]-[40]. Without loss of generality, the PML equations for lossy medium are posed in the stretch coordinate space [39]-[40] as

$$
\begin{align*}
& j \omega \varepsilon E_{x}+\sigma E_{x}=\frac{1}{s_{e y}} \frac{\partial}{\partial y} H_{z}-\frac{1}{s_{e z}} \frac{\partial}{\partial z} H_{y}  \tag{2.17a}\\
& j \omega \varepsilon E_{y}+\sigma E_{y}=\frac{1}{s_{e z}} \frac{\partial}{\partial z} H_{x}-\frac{1}{s_{e x}} \frac{\partial}{\partial x} H_{z}  \tag{2.17b}\\
& j \omega \varepsilon E_{z}+\sigma E_{z}=\frac{1}{s_{e x}} \frac{\partial}{\partial x} H_{y}-\frac{1}{s_{e y}} \frac{\partial}{\partial y} H_{x} \tag{2.17c}
\end{align*}
$$

where $s_{e x}, s_{e y}$, and $s_{e z}$ are the stretched coordinates metrics, and the stretched coordinate metric is chosen to be

$$
\begin{equation*}
s_{e i}=\kappa_{e i}+\frac{\sigma}{\alpha_{e i}+j \omega \varepsilon_{0}} \tag{2.18}
\end{equation*}
$$

where $i=x, y$ or $z$. Similarly the magnetic field equation can be written as follows:

$$
\begin{align*}
& j \omega \mu H_{x}+\sigma_{m} H_{x}=-\frac{1}{s_{m y}} \frac{\partial}{\partial y} E_{z}+\frac{1}{s_{m z}} \frac{\partial}{\partial z} E_{y}  \tag{2.19a}\\
& j \omega \mu H_{y}+\sigma_{m} H_{y}=-\frac{1}{s_{m z}} \frac{\partial}{\partial z} E_{x}+\frac{1}{s_{m x}} \frac{\partial}{\partial x} E_{z}  \tag{2.19b}\\
& j \omega \mu H_{z}+\sigma_{m} H_{z}=-\frac{1}{s_{m x}} \frac{\partial}{\partial x} E_{y}+\frac{1}{s_{m y}} \frac{\partial}{\partial y} E_{x} \tag{2.19c}
\end{align*}
$$

where $s_{m i}$ are the stretched coordinate matrices. For the sake of generality, the stretched coordinate metric is chosen to be

$$
\begin{equation*}
s_{m i}=\kappa_{i}+\frac{\sigma_{m i}}{\alpha_{i}+j \omega \mu_{0}} \tag{2.20}
\end{equation*}
$$

where $\kappa_{i}, \sigma_{i}$ and $\alpha_{i}$ are positive real. The choice of the metric is referred to here as the complex frequency shifted (CFS) PML stretched coordinate metric [39]. For $\alpha_{i}>0$, this choice has the distinct advantage that it allows the PML to be highly absorptive of evanescent and low frequency waves [39]-[40]. Next (2.17) and (2.19) are transformed into the time domain. To this end, the terms on the right-hand side become convolutions. Next, following the LOD principle outlined in Section 2.2, (2.17) and (2.19) are mapped into the discrete space. To this end, (2.17a) is expressed as

$$
\begin{equation*}
E_{x}^{n+1 / 2}=\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{x}^{n}+\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right) \times \kappa_{y i}}\left(\frac{\partial H_{z}^{n+1 / 2}}{d y}+\frac{\partial H_{z}^{n}}{d y}+\left.\psi_{e x y}\right|^{n}+\left.\psi_{e x y}\right|^{n+1 / 2}\right) \tag{2.21}
\end{equation*}
$$

where the auxiliary variables $\left.\psi_{\text {exy }}\right|^{n}$ and $\left.\psi_{\text {exy }}\right|^{n+1 / 2}$ satisfy the recursive relations [2]

$$
\begin{equation*}
\psi_{e x y_{i+12}, j, k}^{n+1 / 2}=c_{y_{j}} \psi_{e x y_{i+12}, j, k}^{n}+\frac{d_{y_{j}}}{2 \Delta y}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right) \tag{2.22a}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{s}=e^{-\left(\left(\sigma_{s} / \kappa_{s}\right)+\alpha_{s}\right)\left(\Delta t / \varepsilon_{0}\right)} \tag{2.22b}
\end{equation*}
$$

$$
\begin{gather*}
d_{s}=\frac{\sigma_{s}}{\kappa_{s}\left(\sigma_{s}+\kappa_{s} \sigma_{s}\right)} \times\left(c_{s}-1\right), \quad(s=x, y, \text { or } z)  \tag{2.22c}\\
\sigma_{s}(s)=\frac{\sigma_{s_{\max }}\left|s-s_{0}\right|^{m}}{\delta^{m}}  \tag{2.22d}\\
\kappa_{s}(s)=1+\left(\kappa_{\max }-1\right) \frac{\left|s-s_{0}\right|^{m}}{\delta^{m}} \tag{2.22e}
\end{gather*}
$$

Similarly, from (2.19a) the magnetic field component in the CPML region can be written as follows
$H_{x}^{n+1 / 2}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{x}^{n}+\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right) \kappa_{y j}}\left(\frac{\partial E_{y}^{n+1 / 2}}{d z}+\frac{\partial E_{y}^{n}}{d z}+\left.\psi_{h z y}\right|^{n}+\left.\psi_{h z y}\right|^{n+1 / 2}\right)$
where the auxiliary variables $\left.\psi_{h z y}\right|^{n}$ and $\left.\psi_{h z y}\right|^{n+1 / 2}$ satisfy the recursive relations similar to (2.22). From (2.21) and (2.23), it can be observed that the auxiliary terms $\left.\psi_{e x y}\right|^{n+1 / 2}$ and $\left.\psi_{h z y}\right|^{n+1 / 2}$ contain the same time index $(n+1 / 2)$ as that of the field on the left side. This additional variable at $(n+1 / 2)$ does not affect numerical results significantly except that it contributes to additional complexities in the computation. Therefore, for the sake of simplification, both auxiliary variables will be considered at the same time index " $n$ " and as a result (2.21) and (2.23) reduce to (2.24a) and (2.24b). The auxiliary variables are responsible for wave absorption in the absorbing boundary conditions. These features obtain improved efficiency and ease of programming for the LODFDTD CPML.

$$
\begin{align*}
E_{x}^{n+1 / 2} & =\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{x}^{n}+\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right) \times \kappa_{e y}}\left(\frac{\partial H_{z}^{n+1 / 2}}{d y}+\frac{\partial H_{z}^{n}}{d y}+\left.2 \psi_{e x y}\right|^{n}\right)  \tag{2.24a}\\
H_{x}^{n+1 / 2}= & \frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{x}^{n}+\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right) \times \kappa_{m z}}\left(\frac{\partial E_{y}^{n+1 / 2}}{d z}+\frac{\partial E_{y}^{n}}{d z}+\left.2 \psi_{h x z}\right|^{n}\right) \tag{2.24d}
\end{align*}
$$

Similarly, other field components from (2.2)-(2.3) can be written as follows.
Sub-step 1:

$$
\begin{gather*}
E_{y}^{n+1 / 2}=\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{y}^{n}+\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right) \kappa_{e z}} \times\left(\frac{\partial H_{x}^{n+1 / 2}}{d z}+\frac{\partial H_{x}^{n}}{d z}+\left.2 \psi_{e y z}\right|^{n}\right)  \tag{2.24b}\\
E_{z}^{n+1 / 2}=\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{z}^{n}+\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right) \times k_{e x}} \times\left(\frac{\partial H_{y}^{n+1 / 2}}{d x}+\frac{\partial H_{y}^{n}}{d x}+\left.2 \psi_{e z x}\right|^{n}\right)  \tag{2.24c}\\
H_{y}^{n+1 / 2}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{y}^{n}+\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right) \times \kappa_{m x}} \times\left(\frac{\partial E_{z}^{n+1 / 2}}{d x}+\frac{\partial E_{z}^{n}}{d x}+\left.2 \psi_{h y x}\right|^{n}\right)  \tag{2.24e}\\
H_{z}^{n+1 / 2}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{z}^{n}+\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right) \times \kappa_{m y}} \times\left(\frac{\partial E_{x}^{n+1 / 2}}{d y}+\frac{\partial E_{x}^{n}}{d y}+\left.2 \psi_{h z y}\right|^{n}\right) \tag{2.24f}
\end{gather*}
$$

Sub-step 2:

$$
\begin{align*}
E_{x}^{n+1} & =\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{x}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right) \times \kappa_{e z}} \times\left(\frac{\partial H_{y}^{n+1}}{d z}+\frac{\partial H_{y}^{n+1 / 2}}{d z}+\left.2 \psi_{e x z}\right|^{n+1 / 2}\right)  \tag{2.25a}\\
E_{y}^{n+1} & =\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{y}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right) \times \kappa_{e x}} \times\left(\frac{\partial H_{z}^{n+1}}{d x}+\frac{\partial H_{z}^{n+1 / 2}}{d x}+\left.2 \psi_{e y x}\right|^{n+1 / 2}\right) \tag{2.25b}
\end{align*}
$$

$$
\begin{align*}
& E_{z}^{n+1}=\frac{\left(4 \varepsilon-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)} \times E_{z}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \varepsilon+\sigma_{e} \Delta t\right) \times \kappa_{e y}} \times\left(\frac{\partial H_{x}^{n+1}}{d y}+\frac{\partial H_{x}^{n+1 / 2}}{d y}+\left.2 \psi_{e z y}\right|^{n+1 / 2}\right)  \tag{2.25c}\\
& H_{x}^{n+1}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{x}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right) \times \kappa_{m y}}\left(\frac{\partial E_{z}^{n+1}}{d y}+\frac{\partial E_{z}^{n+1 / 2}}{d y}+\left.2 \psi_{h x y}\right|^{n+1 / 2}\right)  \tag{2.25d}\\
& H_{y}^{n+1}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{y}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right) \kappa_{m z}}\left(\frac{\partial E_{x}^{n+1}}{d z}+\frac{\partial E_{z}^{n+1 / 2}}{d z}+\left.2 \psi_{h y z}\right|^{n+1 / 2}\right)  \tag{2.25e}\\
& H_{z}^{n+1}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} \times H_{z}^{n+1 / 2}-\frac{2 \Delta t}{\left(4 \mu+\sigma_{m} \Delta t\right) \kappa_{m x}}\left(\frac{\partial E_{y}^{n+1}}{d x}+\frac{\partial E_{y}^{n+1 / 2}}{d x}+\left.2 \psi_{h z x}\right|^{n+1 / 2}\right) \tag{2.25f}
\end{align*}
$$

The updating equations for the CPML of the LOD-FDTD method in 3-D and 2-D can be derived from (2.24a)-(2.24f) and (2.25a)-(2.25f). The updating equations for the CPML ABC for the 3-D LOD-FDTD method with orthogonal and nonorthogonal meshes are described in Chapters 5 and 6, and the CPML updating equations for the 2D LOD-FDTD method are given below.

### 2.5 Updating Equations for 2-D LOD-FDTD CPML ABC

The accuracy of the LOD-FDTD method can be degraded if proper boundary condition is not considered. By following the CPML theory described in Section 2.4, the updating equations for the 2-D LOD-FDTD CPML for both $\mathrm{TE}_{z}$ and $\mathrm{TM}_{z}$ are derived as follows.

### 2.5.1 CPML ABC for TE Case

The CPML equations for the 2-D LOD-FDTD method for the $\mathrm{TE}_{z}$ case are formulated into two sub-steps which are shown below.

Sub-step 1:

$$
\begin{align*}
&\left.E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}=\left.a E_{x}\right|_{i+1 / 2, j} ^{n}+\frac{b}{\Delta y}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)  \tag{2.26a}\\
&+\frac{b}{\Delta y}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)+\left.\left.C_{\psi_{e x}}\right|^{n} \psi_{e_{x y}}\right|_{\mid i+1 / 2, j} ^{n} \\
&\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.c H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}+\frac{d}{\Delta y}\left(\left.E_{x}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)  \tag{2.26b}\\
&+\frac{d}{\Delta y}\left(\left.E_{x}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}\right)+\left.\left.C_{\psi_{h z}}\right|^{n} \psi_{h_{z y}}\right|_{i+1 / 2, j+1 / 2} ^{n}
\end{align*}
$$

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Sub-step 2:

$$
\begin{align*}
& \begin{aligned}
\left.E_{y}\right|_{i, j+1 / 2} ^{n+1}=\left.a E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}- & -\frac{b}{\Delta x}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
& -\frac{b}{\Delta x}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1}\right)-\left.\left.C_{\psi_{e y}}\right|^{n+1 / 2} \psi_{e_{y x}}\right|_{i, j+1 / 2} ^{n+1 / 2}
\end{aligned} \\
& \begin{aligned}
&\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.c H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\frac{d}{\Delta x}\left(\left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.E_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2}\right) \\
&-\frac{d}{\Delta x}\left(\left.E_{y}\right|_{i, j+1 / 2} ^{n+1}-\left.E_{y}\right|_{i+1, j+1 / 2} ^{n+1}\right)-\left.\left.C_{\psi_{h z}}\right|^{n+1 / 2} \psi_{h_{z x}}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}
\end{aligned}  \tag{2.27a}\\
& \text { where, } a=\left(4 \varepsilon-\sigma_{e} \Delta t\right) /\left(4 \varepsilon+\sigma_{e} \Delta t\right), b=2 \Delta t / \kappa(j)\left(4 \varepsilon+\sigma_{e} \Delta t\right), \\
& \begin{aligned}
c=\left(4 \mu-\sigma_{m} \Delta t\right) /\left(4 \mu+\sigma_{m} \Delta t\right), d=2 \Delta t / \kappa(j)(4 \mu+\sigma \Delta t) C_{\psi_{h z}}=\frac{\Delta t}{\kappa(j)(4 \mu+\sigma \Delta t)} \\
\text { where }\left.\quad \psi_{e_{x y}}\right|_{i+1 / 2, j} ^{n}=b_{r} \psi_{e_{x y}} l_{i+1 / 2, j}^{n-1 / 2}+a_{r}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \\
\left.\psi_{h_{z y}}\right|_{i+1 / 2, j+1 / 2} ^{n}=\left.b_{r} \psi_{h_{z y}}\right|_{i+1 / 2, j+1 / 2} ^{n-1 / 2}+a_{r}\left(\left.E_{z}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{z}\right|_{i+1 / 2, j} ^{n}\right)
\end{aligned}  \tag{2.27b}\\
& b_{r}=e^{-\left(\left(\sigma_{r} / \kappa_{r}\right)+\alpha_{r}\right)\left(\Delta t / \varepsilon_{0}\right)}, a_{r}=\frac{\sigma_{r}}{\kappa_{r}\left(\sigma_{r}+\kappa_{r} \sigma_{r}\right)} \times\left(b_{r}-1\right),(\mathrm{r}=x, y)
\end{align*}
$$

Here, subscriptse, and $h$ indicate the coefficients for the electric and magnetic fields. $\psi_{h x y}, \psi_{h y x}, \psi_{e z x}$ and $\psi_{e z y}$ are discrete variables that have non-zero values only in some CPML regions and are necessary for the implementation of the absorbing boundary. To avoid reflections between the computational domain and the CPML boundary as a result of discontinuities, the losses due to the CPML must be zero at the interface of the computational domain. Equations (2.26a) and (2.26b) of sub-step 1 cannot be used for direct numerical calculation, but simultaneous linear equations are obtained from (2.26a) and (2.26b) that result in the tri-diagonal matrix form. Similarly, the equations for the second sub-step (2.27a) and (2.27b) give the tri-diagonal matrix which can be solved using approach A ) which has been described in Section 2.3.2.

### 2.5.2 CPML ABC for TM Case

The formulations of LOD-FDTD CPML for 2-D TM wave are derived in the same way as in the case of the TE wave. The updating equations for both sub-steps 1 and 2 are given next.

Sub-step1:

$$
\begin{align*}
\left.H_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}= & \left.c H_{y}\right|_{i, j+1 / 2} ^{n}+\frac{d}{\Delta x}\left(\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
& +\frac{d}{\Delta x}\left(\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.E_{z}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)+\left.C_{\psi_{n y}} \psi_{h_{y x}}\right|_{i, j+1 / 2} ^{n}  \tag{2.30a}\\
\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}= & \left.a E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}+\frac{b}{\Delta x}\left(\left.H_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.H_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}\right) \\
& +\frac{b}{\Delta x}\left(\left.H_{y}\right|_{i+1, j+1 / 2} ^{n}-\left.H_{y}\right|_{i, j+1 / 2} ^{n}\right)+\left.C_{\psi_{e z}} \psi_{e_{2 x}}\right|_{i+1 / 2, j+1 / 2} ^{n} \tag{2.30b}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.H_{x}\right|_{i+1 / 2, j} ^{n+1}=\left.c H_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}-\frac{d}{\Delta y}\left(\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right) \\
& -\frac{d}{\Delta y}\left(\left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.E_{z}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right)-\left.C_{\psi_{h x}} \psi_{h_{x y}}\right|_{i+1 / 2, j} ^{n+1 / 2}  \tag{2.31a}\\
& \left.E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.a E_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\frac{b}{\Delta y}\left(\left.H_{x}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.H_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}\right) \\
& -\frac{b}{\Delta y}\left(\left.H_{x}\right|_{i+1 / 2, j+1} ^{n+1}-\left.H_{x}\right|_{i+1 / 2, j} ^{n+1}\right)-\left.C_{\psi_{e 2}} \psi_{e_{2 y}}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}  \tag{2.31b}\\
& \text { where } \\
& \left.\psi_{e_{x x}}\right|_{i+1 / 2, j} ^{n}=\left.b_{r} \psi_{e_{x x}}\right|_{i+1 / 2, j} ^{n-1 / 2}+a_{r}\left(\left.H_{y}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{y}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)  \tag{2.32}\\
& \left.\psi_{h_{x x}}\right|_{i+1 / 2, j+1 / 2} ^{n}=b_{r} \psi_{h_{y x}}{ }_{i+1 / 2, j+1 / 2}^{n-1 / 2}+a_{r}\left(\left.E_{z}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{z}\right|_{i+1 / 2, j} ^{n}\right)  \tag{2.33}\\
& b_{r}=e^{-\left(\left(\sigma_{r} / \kappa_{r}\right)+\alpha_{r}\right)\left(\Delta t / \varepsilon_{0}\right)}  \tag{2.34a}\\
& a_{r}=\frac{\sigma_{r}}{\kappa_{r}\left(\sigma_{r}+\kappa_{r} \sigma_{r}\right)} \times\left(c_{r}-1\right), \quad(r=x, y, \text { or } z)  \tag{2.34b}\\
& \sigma_{r}(r)=\frac{\sigma_{r_{\max }}\left|r-r_{0}\right|^{m}}{\delta^{m}}  \tag{2.34c}\\
& \kappa_{r}(r)=1+\left(\kappa_{\max }-1\right) \frac{\left|r-r_{0}\right|^{m}}{\delta^{m}} \tag{2.34d}
\end{align*}
$$

where $\delta$ is the thickness of the PML absorber, $r_{0}$ is the PML interface, $m$ is the order of the polynomial. To minimise the reflection error, the following parameters for CPML are considered.

$$
\begin{equation*}
\sigma_{\text {opt }}=\frac{m+1}{150 \pi \Delta x}=11.21(\mathrm{~S} / \mathrm{m}) \tag{2.35}
\end{equation*}
$$

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where, $\sigma_{s_{\max }}=0.7 \sigma_{\mathrm{opt}}(\mathrm{S} / \mathrm{m}), k_{s_{\max }}=15$ and $m=4, \psi_{h_{p_{x}}}$ and $\psi_{e_{e_{x x}}}$ are discrete variables which may have non-zero values only in some CPML regions but are necessary to implement the absorbing boundary [2]. Like (2.26) and (2.27), the equations of sub-step 1 and 2 of the $\mathrm{TM}_{\mathrm{z}}$ case cannot be used for direct numerical calculation. Placing (2.30a) in (2.30b) yields the simultaneous linear equations that result in the tri-diagonal matrix form. However, from the CPML updating equations of both the TE and TM case, it can be observed that each equation contains one auxiliary " $\psi$ " term, and also that there are two " $\psi$ " terms in each sub-step. For ADI-FDTD CPML four auxiliary terms in the first step and four in the second are needed for the updating equations, thus LOD-FDTD CPML requires smaller number of auxiliary equations than ADI-FDTD CPML, thus resulting in higher computational efficiency.

### 2.6 Source Functions for LOD-FDTD

The LOD-FDTD algorithm given by (2.4)-(2.5) and (2.15)-(2.16) for 2-D LODFDTD was derived for the source-free Maxwell's curl equation. To extend this algorithm to include an electric and magnetic current source, special care is required, as discussed in this section. Similar to the wave source conditions of ADI-FDTD [63][65], there are two possible ways to implement a point-wise wave source condition for the LOD-FDTD method. First, if the source condition is to be implemented as a magnetic field excitation, a standard explicit formulation for a hard source or current source is used. Second, if the source condition is to be implemented as an electric field excitation (such as a current source or a resistive voltage source), an implicit formulation is used. For clarification, the electric field excitation for the implicit formulation is described below. The electric field component of (2.4a) for the $\mathrm{TE}_{\mathrm{z}}$ case in sub-step 1 can be written with a soft source excitation as follows:

$$
\begin{align*}
\left.E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}=C_{e x e} \times\left. E_{x}\right|_{i+1 / 2, j} ^{n} & +C_{e x h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)  \tag{2.36}\\
& +C_{e x h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)+C_{e x h z} \times\left. J_{x}\right|_{i+1 / 2, j} ^{n+p 1}
\end{align*}
$$

Similarly, the electric field component of (2.15a) for the $\mathrm{TE}_{z}$ case in sub-step 2 can be written with a soft source excitation as follows:

$$
\begin{align*}
&\left.E_{y}\right|_{i, j+1 / 2} ^{n+1}=C_{e y e} \times\left. E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}-C_{e y h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
& \quad-C_{e y h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1}\right)-C_{e y h z} \times\left. J_{y}\right|_{i, j+1 / 2} ^{n+p 2} \tag{2.37}
\end{align*}
$$

These equations are the standard LOD-FDTD equations in 2-D modified by the inclusion of an electric source term. The indexes $\left(n+p_{1}\right)$ and $\left(n+p_{2}\right)$ denote discrete time indexes to be determined carefully. The solution of the implicit equation (2.6) for the first sub-step leads to the following tri-diagonal equation system.

$$
\begin{align*}
& -\left.\alpha_{1} E_{x}\right|_{i+1 / 2, j-1} ^{n+1 / 2}+\left.\left(1+2 \alpha_{1}\right) E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}-\left.\alpha_{1} E_{x}\right|_{i+1 / 2, j+1} ^{n+1 / 2} \\
& =\left.C_{e x e} E_{x}\right|_{i+1 / 2, j} ^{n}+C_{h z h} \times C_{e x h z}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \\
& \quad+\alpha_{1}\left(\left.E_{x}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}+\left.E_{x}\right|_{i+1 / 2, j-1} ^{n}\right)  \tag{2.38}\\
& \quad \quad+C_{e x h z}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)+C_{e x h z} \times\left. J_{x}\right|_{i+1 / 2, j} ^{n+p 1}
\end{align*}
$$

It is seen from (2.38) that the $\left.E_{x}\right|_{i+1 / 2, j-1} ^{n+1 / 2}$ fields along a particular y-directed line are updated simultaneously and repeated for each $i$-coordinate and for 3-D LOD-FDTD case, each $k$ and $i$ - coordinate are used. A similar expression can be written for the electric field component of sub-step 2. Note that the presence of electric current sources in the model will affect the known column vector on the right-hand side of those tridiagonal matrix systems for $x, y$ or $z$-directed lines that pass through the locations of the current sources. Therefore, the electric current source information must be embedded within the affected tri-diagonal matrix systems and cannot be implemented as a separate explicit update. Similar approaches of wave source conditions have also been considered for the ADI-FDTD method [63]-[65]. Similar to the ADI-FDTD [63], the excitation is applied in both procedures not just in first procedure. Furthermore, the above wave source approaches can also be applied for the 3-D LOD-FDTD.

### 2.7 Segmented Technique for EM Propagation Modelling in Large Symmetric Structures

To model the EM propagation over electrically large structure (typically hundreds of $\lambda$ ), the entire computational domain is represented by a single static grid which can be computationally prohibitive (in terms of memory and/or CPU hours). For example,

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modelling the problem space ( $\mathrm{IE} \times \mathrm{JE}$ ) of a dimension of $2.4 \times 10^{5}$ cells using 2-D FDTD takes two days in personal computer (PC) and requires a huge amount of memory to obtain result. The implicit ADI-FDTD and LOD-FDTD can be applied to reduce the computational time, but the direct application of the LOD-FDTD method for solving for an electrically large structure such as a tunnel still requires increased computational load. Therefore, to overcome the limitations of the LOD-FDTD method for EM propagation modelling in large sized tunnels, efficient methods are required. We will show that use of the segmented technique results in a large reduction in computation time and memory. Before applying the segmentation technique in electrically large symmetric structures, the actual 3-D structures are modified into a 2-D approach following the theory outlined in [2] to further increase computational efficiency. The conversion process of 3-D real structure to a 2-D structure is described below.

According to [2] and [150], if a structure is modelled to extend to infinity in the x or y or $z$-direction, and if the incident field is also uniform in the x or y or $z$-direction, then all partial derivatives of the field with respect to $z$ must equal zero. As a result, the 3-D structures act as 2-D structure. Depending on the orientation of the electric and magnetic field lines relative to the surface of the structure, two modes TE and TM can be obtained. Note that the TE mode sets up E field lines in a plane perpendicular to the infinitely long axis (the $z$ axis) of the structure, and the TM mode sets up $E$ field lines only that are parallel the $z$-axis. Here, we take a straight rectangular tunnel (as shown in Fig. 2.5 (a)) as an example to illustrate the 2-D LOD-FDTD model construction.

(a)

(b)

Fig. 2.5 A rectangular tunnel (a) three-dimensional view and (b) two-dimensional view

The 3-D view of the tunnel is shown in Fig. 2.5 (a) and the corresponding 2-D view which is obtained following the above principle is shown in Fig. 2.5 (b). After obtaining the 2-D model, the segmentation approach has been applied to further improve the computational performance of solving the electrically large structure which is described in the following section.

### 2.7.1 Segmented (S)-LOD-FDTD Technique

To apply the proposed S-LOD-FDTD method, we break up the computational space into segments and use LOD-FDTD as described in Section 2.3.1 and 2.3.3 and convolutional perfectly matched layer (CPML) ABC as described in Section 2.5 for each segment and solve them sequentially. Fig. 2.6 schematically represents the 2-D computational space for a general rectangular tunnel with a branch for solving using proposed S-LOD-FDTD method where electromagnetic fields from a transmitter to a receiver are calculated. In general, the computational domain is divided into a number of individual segments each with equal or variable segment lengths depending on the shape of the tunnel so that different types of tunnels such as branched or curved tunnels can be modelled. Whenever there are abrupt changes in tunnel geometries (see Fig. 2.6), the boundaries must be carefully treated to ensure accurate computation of the fields. The proposed S-LOD-FDTD algorithm is summarised as follows:

1. Start the conventional LOD-FDTD iteration with CPML absorbing boundary condition in the first Segment when the signal source S 0 is provided.

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2. When the fields of Segment 1 reach at each unit cell on interface 1, save the electric and magnetic fields from each unit cell recorded at interface 1 for use in the next segment.
3. Referring to Fig. 2.6, the fields from ' $\mathrm{S} 1_{\mathrm{L}}$ ' on the left side of interface 1 are saved from Segment 1 and used as input fields in ' $\mathrm{S} 1_{\mathrm{R}}$ ', that is lying on the right side of the interface which then feeds Segment 2 and so on.
4. Whenever an abrupt change or branching junction falls within a segment, its effect on the fields needs to be considered before propagating the signal into the next segment.
5. Sequentially propagate the extracted fields at the interface into the next Segment.

Repeating steps 2-5 to complete the simulations in Segments 2, 3, $4 \ldots \ldots$ n.


Fig. 2.6 Computational domain in S-LOD-FDTD for branched tunnel

For the implementation of the CPML ABC in each segment of S-LOD-FDTD technique, special care must be taken because data are exchanged spatially between segments where each contains part of the CPML boundary condition. The LOD-FDTD iteration comprises two sub-steps which each require the solution of a tri-diagonal matrix system of equations along either $x$ or $y$ directions respectively. For clarification, we present in (2.39) the $E_{x}$ implicit update equations in the stretched coordinate CPML

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region. The update equations in non-PML regions are attained by simply neglecting the convolution terms of the right-hand side of (2.39).

$$
\begin{align*}
& -\left.\alpha_{1} E_{x}\right|_{i+1 / 2, j-1} ^{n+1 / 2}+\left.\left(1+2 \alpha_{1}\right) E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}-\left.\alpha_{1} E_{x}\right|_{i+1 / 2, j+1} ^{n+1 / 2} \\
& =\left.C_{e x e} E_{x}\right|_{i+1 / 2, j} ^{n}+C_{h z h} \times C_{e x h z}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \\
& \quad+\alpha_{1}\left(\left.E_{x}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}+\left.E_{x}\right|_{i+1 / 2, j-1} ^{n}\right)  \tag{2.39}\\
& \quad+C_{e x h z}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \\
& \quad \quad+\left.C_{\psi_{e x}} \psi_{e_{x y}}\right|_{i+1 / 2, j} ^{n}+\left.2 \alpha_{1} \psi_{h_{z y}}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.2 \alpha_{1} \psi_{h_{z y}}\right|_{i+1 / 2, j-1 / 2} ^{n}
\end{align*}
$$

The coefficients of (2.39) are same as mentioned after (2.27). Consider the Segment 1 with CPML ABC in Fig. 2.6 where all explicit and implicit equations of electric field are employed along $x$ and $y$ direction. When the right-hand side data along the $y$ direction of the implicit update from Segment 1 are sent to Segment 2, right-hand side are first updated as for non-PML regions and the convolution terms are subsequently added in separate loops. Every cell $(i+1 / 2, j+1 / 2)$ belonging to a PML area is associated with a convolution data buffer element. However, the $\psi_{h_{z y}}, \psi_{e_{x y}}$ terms related to the ( $i, j$ 1/2) cell appear in (2.39). Convolution data PML buffer elements associated with the neighbouring process are required for cell updates at the -y segment boundary, but the EM field elements necessary for calculating the convolution terms for -y boundary cells already exist in the local domain because the EM field boundary data between neighbouring segments is shared (Fig. 2.6). Therefore, the respective PML data at the boundary is calculated twice, once in each of the neighbouring segments. This avoids the increase in the communication overhead that would arise if the CPML buffer data was calculated only once and communicated to the neighbouring sub-process.

When there is a junction in the tunnel, two conditions are satisfied. The first condition is that the sums of all the currents entering a junction equal the sum of currents leaving the junction i.e the enforcement of Kirchoff's current law. The second condition is that the tangential component of the electric field must be continuous across the structure surfaces at the junction [151]-[152].

By using the above segmentation technique, we can analyse various electrically large symmetric tunnel structures which are described in Section 2.8.

### 2.7.2 Segmented (S)-ADI-FDTD Technique

To make comparison with the proposed S-LOD-FDTD method, the S-ADI-FDTD has also been implemented to model the EM propagation inside electrically large symmetric structures. To apply the S-ADI-FDTD method, after breaking up the computational space into segments use ADI-FDTD and CPML ABC for each segment and solve them sequentially. The S-ADI-FDTD technique follows the same procedure as used for the S-LOD-FDTD technique, which has been described in the previous section. For the S-ADI-FDTD technique, conventional ADI-FDTD with CPML absorbing boundary condition is used to start the iteration when the signal source S 0 is provided. The same process 2-5 of Section 2.7.1, which has been described in the S-LOD-FDTD technique, is then followed. The ADI-FDTD with CPML ABC used for each segment is provided in Appendix A.

### 2.8 Numerical Analysis Using S-LOD-FDTD and S-ADIFDTD Methods

### 2.8.1 Straight Tunnel

In this section, we first consider the Roux tunnel [153] which has a perfectly straight geometry as shown in Fig. 2.7 (a). The origin of the tunnel-coordinate system is arranged at the centre of the tunnel cross section. The transverse section is semicircular, as shown in Fig. 2.7 (b), and has a diameter of 8.3 m ; the maximum height is 5.8 m at the centre of the tunnel with material properties $\varepsilon_{r}=2.5$ and $\sigma=0.05 \mathrm{~S} / \mathrm{m}$. Before applying the segmentation technique in the Roux tunnel [153], the actual 3-D structures have been modified into a 2-D approach following the theory outlined in section 2.7.1. The 2-D model of the straight Roux tunnel with the segmentation is shown in the Fig. 2.7 (c). The straight tunnel with 500 m length has been divided into 20 segments and each segment length is 25 m ; then the segmentation technique has been applied to predict the path loss in the tunnel. The predicted path loss in this tunnel (for a length of 500 m ) at 2.4 GHz computed using the proposed S-LOD-FDTD for CFLN $=2$ and 10 is shown in Fig. 2.8 (a) and (b) along with the measured data extracted from the publication by E. Masson et al. [153].

(a)

(c)

Fig. 2.7 (a) Roux tunnel [153] (b) Profile of the Roux tunnel (c) 2-D segmented problem space of the Roux tunnel [153] for S-LOD-FDTD simulation


Fig. 2.8 Comparison of pathloss with measured data of E. Masson et al. [153] (a) CFLN=2, (b) CFLN=10

(a)

(b)

Fig. 2.9 Comparison with measured data of E. Masson et al. [153] (a) CFLN=2, and (b) CFLN=10

Fig. 2.9 (a)-(b) shows the computed path loss using S-ADI-FDTD compared with the measured results obtained from [153]. It is observed from Figs. 2.8 and 2.9 that the predicted path loss agrees reasonably well with the results obtained by other methods even for higher CFLN, as well as with the measured result. Fig. 2.10 shows a comparison of the averaged path loss obtained for the Roux tunnel in which the measured results extracted from the published graphs in [153] were averaged over 50 m length and compared with the averaged simulated data for $\mathrm{CFLN}=10$. The measured path loss of the Roux tunnel published in [153] has been cited by many authors and has been taken as the reference result to compare our simulated results.


Fig. 2.10 Comparison of averaged (over 50 m ) path loss for Roux Tunnel for CFLN=10

However, our main aim here is to reduce the computational time for propagation modelling in a large symmetric structure. For the large tunnel structure [153], the execution time required by the S-ADI-FDTD method was 4.32 hrs with a memory over 1090 MB , whereas the S-LOD-FDTD took 3.5 hrs with a memory of 1010 MB . Comparison in terms of execution time and memory shows that S-LOD-FDTD is more effective because it requires less time and memory. The cell size chosen for the Roux tunnel is $\lambda / 10$.

### 2.8.2 Branched Tunnel

We now consider a branched tunnel [154] to predict the path loss using our proposed method as shown in Fig. 2.11 (a). The origin of the branched tunnel-coordinate system is arranged at the centre of the tunnel cross section. The branched tunnel has 3 m height and 4.2 m width at the centre of the tunnel with material properties $\varepsilon_{r}=10$ and $\sigma=0.01$ $\mathrm{S} / \mathrm{m}$. The transmitter is positioned in the main section at a distance of 10 m away from the tunnel junction. The junction is formed by joining the branch and main sections of the tunnel at an angle of $15^{\circ}$. The receiver is assumed to move away from the transmitter into the branch section assuming TM incidence. For the junction of the tunnel (Fig. 2.11), two conditions are satisfied.

(a)

(b)

Fig. 2.11 (a) Branch tunnel [154] (b) 2-D segmented problem space of the branched tunnel [154] for S-LOD-FDTD simulation

The first condition is that the sums of all the currents entering the junction equal the sum of currents leaving the junction i.e. the enforcement of Kirchoff's current law. The second condition is that the tangential component of the electric field must be continuous across the structure surfaces at the junction.

(a)


Fig. 2.12 (a)-(c) Comparison with the measured data of Zhang et al. for CFLN=2, 10 and 12 respectively

Before applying the segmentation technique in the branched tunnel [154], the actual 3-D structure has been modified into a 2-D approach, following the theory outlined in Section 2.7.1. The 2-D model of the branched tunnel with segmentation is shown in the Fig. 2.11 (b). The main section of the tunnel with 100 m length and the branched section with 100 m length (as shown in Fig 2.11) have been considered to compare our simulated results with the measured results. The branched and main section of the tunnel has been divided into $10(5+5)$ segments and each segment length is 20 m . The segmentation technique has then been applied to predict the path loss in the branched

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tunnel. The predicted path loss in this tunnel (for a length of 100 m ) at 900 MHz computed using the proposed S-LOD-FDTD for CFLN=2, 10 and 12 are shown in Fig. 2.12(a) and (b) along with the measured data extracted from the publication by Zhang et al. [154], as well as the results obtained using the S-ADI-FDTD method. The execution time and memory required for S-LOD-FDTD is 2.86 hrs and 900 MB respectively whereas execution time 3.45 hrs and memory 1010 MB required for S-ADI-FDTD for a CFLN=10.

### 2.8.3 Curved Tunnel

Finally, we present results for a curved tunnel with a straight entrance using the proposed method and compare them with the published measured data given in [155]. The width and height of the tunnel cross section are 8 m and 6 m respectively and the transmitter and receiver are centred in the tunnels, with their height above ground being 3 and 1.5 m , respectively. The electrical characteristics of the tunnel wall are $\sigma=0.01$ $\mathrm{s} / \mathrm{m}$ and $\varepsilon_{\mathrm{r}}=5$. Uniform segmentation was considered for the straight segment of the curved tunnel, but for the curved segment, the slope at each segment interface has to be determined by multiplying with the fields calculated for the previous segment before they are propagated into the next segment. The axial length of the curved tunnel is 400 m and the length of the curved section is 200 m ; a segment length of 25 m was considered. To apply the segmentation technique for this tunnel, we first convert the 3D model to a 2-D model following the theory outlined in Section 2.7.1. The 2-D model of the tunnel with segmentation is shown in the 2.13 (b).



Fig. 2.13 (a) Geometry of curved tunnel [155] (b) Segmented problem space of the curved tunnel

The predicted path loss in this tunnel (for a length of 400 m ) at 1 GHz computed using the proposed S-LOD-FDTD for CFLN=2, 10 and 12 is shown in Fig. 2.14 (a)-(c) along with the measured data extracted from [155] and the results obtained using the S-ADIFDTD method. It can be observed from Fig. 2.14 (a)-(c) that the predicted path loss agrees reasonably well with the results obtained by other methods, even for higher CFLN, as well as with the measured result.

(a)


Fig. 2.14 (a)-(c) Comparison with measured received power along the propagation axis of curved tunnel for $\mathrm{CFLN}=2,10$ and 15

### 2.8.4 Error Analysis of the Proposed Method

Fig. 2.15 shows the relative errors of the S-LOD-FDTD method with respect to CFLN. The relative error is calculated using the following formula:

$$
\begin{equation*}
\frac{A_{\text {ref }}-A_{\text {simulated }}}{A_{\text {ref }}} \times 100 \% \tag{2.40}
\end{equation*}
$$

where $A_{\text {ref }}$ is the reference value obtained from the measured result and $A_{\text {simulated }}$ is the calculated value using the proposed S-LOD-FDTD and S-ADI-FDTD methods. From Fig. 2.15, it can be observed that the relative errors increase with the increasing CFLN, but that the maximum error is only around $4 \%$.


Fig. 2.15 Relative error with respect to CFLN CFLN

The results confirm the ability of the proposed S-LOD-FDTD method to accurately predict the propagation parameters of two dimensionally approximated large tunnels.

### 2.8.5 Evaluation of Computational Performance of S-LODFDTD Method

Before computing any simulation for the prediction of the path loss in the tunnel environment, the total number of time steps for each segment that is essential to achieve a steady state of the signal has to be determined first. The required number of total time steps can differ from environment to environment, not only as a result of the different distance of interest, but also because of different multipath effects in a particular environment [145]. The number of iteration time steps for each individual segment is considered to be six times the ratio between the distance of interest and the cell dimension $\Delta \mathrm{x}(6 \times$ ratio time schemes). The points of interest are considered eight cells far from the absorbing boundaries so that the signal can easily interface between two segments. During the interface between two segments, it was observed that the signal in vertical directions had most significant effect, whereas the signal in horizontal directions had little effect in the simulation. It was also observed that the S-LOD-FDTD approach produced high accuracy results at close range regardless of the segment size chosen, but showed more variability at longer range, particularly for smaller segment sizes. Table 2.1 summarises the computational performance in terms of execution time and memory of the proposed S-LOD-FDTD method for the Roux tunnel for different segment sizes. Clearly, for a certain time scheme, an optimal segment size exists in

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terms of CPU execution time and memory performance. The relationship between the total CPU execution time and segment size can be defined as:

$$
\begin{equation*}
\text { Total_CPU_Time }=n . d t . N \tag{2.41}
\end{equation*}
$$

where $n$ is the number of time steps (iterations) for each segment to reach its steady state, $d t$ is the CPU time required for each single time step (iteration) in the segment, and $N$ is the number of segments into which the problem space is divided [145].

Table 2.1
Required Computational Time and Memory for Roux Tunnel (CFLN=10)

| Segment <br> Numbers | S-LOD-FDTD |  | S-ADI-FDTD |  |
| :---: | :---: | :---: | :---: | :---: |
|  | CPU Times <br> $(\mathrm{hrs})$ | Memory <br> $(\mathrm{MB})$ | CPU Times <br> $(\mathrm{hrs})$ | Memory <br> $(\mathrm{MB})$ |
| $250 \mathrm{~m} \times 2$ | 26.646 | 3500 | 27.646 | 3550 |
| $125 \mathrm{~m} \times 4$ | 20.590 | 2500 | 21.590 | 2800 |
| $50 \mathrm{~m} \times 10$ | 6.745 | 1650 | 8.745 | 1850 |
| $25 \mathrm{~m} \times 20$ | 3.5 | 1010 | 4.32 | 1090 |
| $10 \mathrm{~m} \times 50$ | 1.860 | 0.950 | 1.960 | 10 |

The tabulated results indicate that by dividing the domain into more segments, both execution time and memory usage can be reduced. However, the segment size cannot be reduced to be arbitrarily small in order to obtain further improvements. In the case of Roux tunnel, it was observed that when the size of the segment fell below 5 m , the results became unstable. Because the total number of time steps ( $6 \times$ ratio time scheme) iterated in each segment is not sufficient for the solution to reach its steady state for chosen segment length. Stable results can be obtained by increasing the total number of time steps from $6 \times$ ratio time scheme to $10 \times$ ratio time schemes. For an electrically large Roux tunnel, considering the stability of the solution, the S-LOD-FDTD obtained maximum time and memory savings for 20 equal sized segments where the length of each segment was 25 metres. The maximum time and memory saving for the branched and curved tunnel was obtained for 5 and 8 equal sized segments of 20 m and 25 m respectively.

### 2.9 Discussion

In this chapter, we initially introduced the theory of the LOD-FDTD method. The updating equations for the 2-D LOD-FDTD method was provided and the derivation of CPML ABC for the 2-D LOD-FDTD method was discussed. Wave source conditions for the LOD-FDTD simulations were discussed and the developed new segmented-LOD-FDTD (S-LOD-FDTD) method for EM propagation modelling in electrically large scale problems was presented. The S-ADI-FDTD method, which has been used for comparison with the S-LOD-FDTD method, was also discussed. After modifying the 3-D symmetric structures to 2-D structures, the segmentation approach was applied. The developed S-LOD-FDTD method was validated through propagation prediction inside large straight, branched and curved tunnels. The predictions on path loss agree reasonably well with the results obtained using S-ADI-FDTD, as well as with published measured data. The results indicate higher signal attenuation for the junction/transition regions compared to regions away from such abrupt transitions.

From the error analysis of the proposed S-LOD-FDTD method, it is confirmed that the proposed S-LOD-FDTD method is computationally more efficient compared to S-ADI-FDTD and provides reasonably accurate results for propagation predictions inside large tunnels. A performance comparison of the proposed method was also described in terms of CPU time and memory, and it was found that by dividing the domain into more segments, both execution time and memory usage can be reduced. The results reveal that the proposed segmentation approach can help to reduce computational resources and hence can be extended for EM modelling of any large scale propagation problems.

## Chapter 3

# Nonorthogonal LOD-FDTD Method for EM Scattering from Two Dimensional (2-D) Structures 

### 3.1 Introduction

With the the advent of powerful and inexpensive computers, the explicit FDTD algorithm has been used to solve numerous EM scattering problems in such diverse areas as optics, biomedicine, oceanography, and radar remote sensing, and in many other problems in electromagnetic compatibility (EMC) [2]. This can be confirmed by various noteworthy contributions, especially in Cartesian coordinates [3]-[4]. However, for modelling objects having curved features using orthogonal grids in Cartesian coordinates, very fine meshes are required, which results in smaller time steps, leading to a huge increase in memory and CPU time. To overcome these difficulties, Holland [23] first introduced the non-orthogonal FDTD (NFDTD) algorithm which was later refined by many researchers [24]-[28]. Conventional explicit NFDTD schemes suffer from the CFL constraint and as a result, finer grid sizes and smaller time steps are necessary to retain the stability of the method, which causes the processing time to increase drastically. To eliminate the dependence on the CFL stability constraint, an implicit method such as the unconditionally stable alternating direction implicit (ADI) technique was introduced to the non-orthogonal co-ordinates [69]-[72]. Kantartzis et al. [69] implemented the dispersionless ADI-NFDTD algorithm that optimises the dispersion. However, their approach requires the calculation of the higher order terms twice for each time step, thus leading to additional computational burden. Zheng et al. [70]-[71] proposed 2-D and 3-D ADI-NFDTD methods in which they employed nonorthogonal grids locally only to model the curved/complex regions of the scatterer but used conventional orthogonal grids for the other regions of the scatterers. They used only the conventional PML ABC to truncate the computational domain, and in addition,
their method [70]-[71] requires calculation of the Jacobian coordinate transformation to convert the curvilinear coordinates into a conventional FDTD lattice where the CFL constraint must be satisfied. As a result, the method by Zhang et al. [70]-[71] requires increased computational resources. It has also been found that the ADI-FDTD technique requires more execution time for higher CFLN [19], [108]. To overcome some of these limitations, a novel implicit method known as locally 1-D FDTD (LOD-FDTD) was proposed [19]-[22], [108], [110] which requires fewer arithmetic operations and less execution time than the ADI-FDTD method. Although curvilinear non-orthogonal grid methods were developed for the standard explicit FDTD and implicit ADI-FDTD methods, thus far, only orthogonal grids have been used for the LOD-FDTD method. In this chapter, our aim is to extend the 2-D LOD-FDTD for generalised non-orthogonal grids to solve EM scattering from 2-D conducting, dielectric and mixed structures.

This chapter is organised as follows: Section 3.2 describes the mathematical formulation of the proposed 2-D LOD-NFDTD method for both TE and TM cases. The mesh generation technique is discussed in Section 3.3. In Section 3.4, the CPML ABC for LOD-NFDTD for both TE and TM cases is described. The near field to far field transformation and scattered field formulations for the LOD-NFDTD method are presented in Section 3.5 and 3.6 respectively. The numerical stability and dispersion analysis of the proposed method are also discussed in Sections 3.7 and 3.8. Numerical results for many 2-D cylindrical structures that include conducting, dielectric and coated conducting and mixed structures obtained using the proposed LOD-NFDTD method are provided and compared with the results obtained from other exact and numerical methods to validate our technique in Section 3.9. Finally, discussion is provided in Section 3.10

### 3.2 Mathematical Formulation of 2-D LOD-NFDTD Method

In this section, 2-D LOD-NFDTD formulation is developed for electromagnetic scattering for generalised non-orthogonal meshes following [23]-[28]. For rectangular structures, the nonorthogonal curvilinear meshes are equivalent to the rectangular orthogonal meshes. But for representing the curved or oblique boundary of EM
structures, fewer meshes are required when nonorthogonal meshes are employed. For example, orthogonal rectangular and nonorthogonal curvilinear meshes of circular dielectric cylinder are shown in Fig. 3.1 (b) and (c). Fig. 3.1 (c) clearly shows that fewer meshes are required when nonorthogonal meshes are employed.


Fig. 3.1 (a) Circular dielectric cylinder, (b) Orthogonal rectangular meshes of the circular cylinder, (c) Nonorthogonal curvilinear meshes of the circular cylinder.

Hence our interest in this chapter is to propose nonorthogonal LOD-NFDTD method for analysing curved microwave 2-D structures. The 2-D LOD-NFDTD is described here and 3-D LOD-NFDTD is discussed in Chapter 6. Before discussing the derivation of the 2-D LOD-NFDTD method, the curvilinear coordinate systems are discussed briefly below.

### 3.2.1 Curvilinear Coordinate Systems

An oblique coordinate system $\left(u^{1}, u^{2}, u^{3}\right)$ may be characterised by its unitary vectors ' $a_{i}$ ' (as shown in Fig. 3.2), where the differential length vector is described as:

$$
\begin{equation*}
d r=\sum_{i=1}^{3} \frac{d r}{\partial u^{i}} d u^{i}=\sum_{i=1}^{3} a_{i} d u^{i} \tag{3.1}
\end{equation*}
$$

Equivalently, it may be characterised by its reciprocal unitary vectors,

$$
\begin{equation*}
a^{i}=a_{j} \times a_{k} / \sqrt{g} \tag{3.2}
\end{equation*}
$$

where $g$ is the determinant of the metric with element

(a)

(b)

Fig. 3.2 Nonorthogonal curvilinear coordinate system $\left(\mathbf{u}^{1}, \mathbf{u}^{2}, \mathbf{u}^{3}\right)$; (a) unitary vectors ( $\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}$ ) (b) reciprocal unitary vectors $\left(\mathbf{a}^{1}, \mathbf{a}^{2}, \mathbf{a}^{\mathbf{3}}\right)$

$$
\begin{equation*}
g_{i j}=\sum_{k=1}^{3} \frac{\partial x^{k}}{\partial u^{i}} \cdot \frac{\partial x^{k}}{\partial u^{j}}=a_{i} \cdot a_{j} \tag{3.3}
\end{equation*}
$$

and $x^{i}$ are Cartesian coordinates. The $a^{i}$ point along coordinate lines and $a_{i}$ are normal to surface of constant $u^{i}$. For orthogonal coordinates $a_{i}$ and $a^{i}$ are sometimes normalized to unit vectors.

$$
\begin{equation*}
i_{i}=\frac{a_{i}}{\sqrt{a_{i} \cdot a_{i}}}, i_{i}=\frac{a^{i}}{\sqrt{a^{i} \cdot a^{i}}} \tag{3.4}
\end{equation*}
$$

The $a_{i}$ and $a^{i}$ have the properties: $a^{i} \cdot a_{j}=\delta_{i j}, a_{i} \cdot a_{j}=g_{i j}, a^{i} \cdot a^{j}=g^{i j}$
where $g_{i j}$ is the inverse of $g^{i j}$. In general, a vector may be represented by its covariant and contra-variant components as follows:

$$
\begin{equation*}
E=e_{i} \cdot a^{i} \text { and } E=e^{i} \cdot a_{i} \tag{3.5}
\end{equation*}
$$

The associated components having the usual dimensions are defined by

$$
\begin{equation*}
\mathrm{E}=E_{i} \cdot i^{i}=E^{i} \cdot i_{i} \tag{3.6}
\end{equation*}
$$

where $\mathrm{E}^{1}=\sqrt{g_{11}} e^{1}, \mathrm{E}_{1}=\sqrt{g^{11}} e_{1}$
The covariant and contra-variant components are related by

$$
\begin{equation*}
e_{i}=g_{i j} e^{j}, e^{i}=g^{i j} e_{j} \tag{3.7}
\end{equation*}
$$

Thus the physically real $E^{i}$ and $E_{j}$ are related by

$$
\begin{equation*}
E_{i}=G_{i j} E^{j}, E^{i}=G^{i j} E_{j} \tag{3.8}
\end{equation*}
$$

and

$$
\begin{equation*}
E^{i}=G^{i j} E_{j}, E^{i}=G_{i j} E^{j} \tag{3.9}
\end{equation*}
$$

where $G^{i j}=\sqrt{\frac{g_{i i}}{g_{i j}}} g^{i j}, G_{i j}=\sqrt{\frac{g_{i j}}{g_{i i}}} g_{i j}$. Based on the above discussion, Maxwell's curl equations can be derived using the covariant and contra-variant projections. The derivation of 2-D LOD-NFDTD method is described next.

### 3.2.2 Derivation of the 2-D LOD-NFDTD Method

Consider Maxwell's equations in an isotropic lossy medium:

$$
\begin{align*}
\varepsilon \frac{\partial \vec{E}}{\partial t}+\sigma_{e} \vec{E} & =\nabla \times \vec{H}  \tag{3.10a}\\
-\mu \frac{\partial \vec{H}}{\partial t}-\sigma_{m} \vec{H} & =\nabla \times \vec{E} \tag{3.10b}
\end{align*}
$$

where $\vec{E}, \vec{H}$ are electric and magnetic field intensities and $\sigma_{e}, \sigma_{m}$ are electric conductivity and equivalent magnetic loss, respectively. The above equations will be cast in a generalised curvilinear co-ordinates (non-orthogonal) system with variables $\left(u^{1}, u^{2}, u^{3}\right)$. To deal with the electric and magnetic field quantities on such grids, two different local co-ordinate systems (covariant and contra-variant) are used. Equation (3.10) can be applied for non-orthogonal co-ordinates as shown in [2]. Denoting the covariant electric and magnetic field components which represent the flow of field along the grid as $E_{m}, H_{m}(m=1,2,3)$, and the contra-variant electric and magnetic field components which represent the flow going through facets of the grid as $E^{m}, H^{m}(m=1,2,3)$, the NFDTD differential equations in a lossy medium from (3.10) can be derived in the generalised curvilinear coordinate system as follows:

$$
\begin{align*}
& \varepsilon \frac{\partial E^{1}}{\partial t}+\sigma E^{1}=\frac{1}{\sqrt{g}}\left(\frac{\partial H_{3}}{d u^{2}}-\frac{\partial H_{2}}{d u^{3}}\right)  \tag{3.11a}\\
& \varepsilon \frac{\partial E^{2}}{\partial t}+\sigma E^{2}=\frac{1}{\sqrt{g}}\left(\frac{\partial H_{1}}{d u^{3}}-\frac{\partial H_{3}}{d u^{1}}\right) \tag{3.11b}
\end{align*}
$$

$$
\begin{gather*}
\varepsilon \frac{\partial E^{3}}{\partial t}+\sigma E^{3}=\frac{1}{\sqrt{g}}\left(\frac{\partial H_{2}}{d u^{1}}-\frac{\partial H_{1}}{d u^{2}}\right)  \tag{3.11c}\\
\mu \frac{\partial H^{1}}{\partial t}+\sigma_{m} H^{1}=-\frac{1}{\sqrt{g}}\left(\frac{\partial E_{3}}{d u^{2}}-\frac{\partial E_{2}}{d u^{3}}\right)  \tag{3.11d}\\
\mu \frac{\partial H^{2}}{\partial t}+\sigma_{m} H^{2}=-\frac{1}{\sqrt{g}}\left(\frac{\partial E_{1}}{d u^{3}}-\frac{\partial E_{3}}{d u^{1}}\right)  \tag{3.11e}\\
\mu \frac{\partial H^{3}}{\partial t}+\sigma_{m} H^{3}=-\frac{1}{\sqrt{g}}\left(\frac{\partial E_{2}}{d u^{1}}-\frac{\partial E_{1}}{d u^{2}}\right) \tag{3.11f}
\end{gather*}
$$

where $g$ is the metric tensor calculated using (3.3). The updating equations (3.11a)(3.11f) of explicit NFDTD can be applied to any arbitrary structure with curved boundary or oblique surface for accurate modelling, without employing the staircase approximation required for Yee's algorithm. The explicit NFDTD method has been successfully applied to analyse optical dielectric waveguide, dielectric-loaded resonant cavity, microstrip discontinuities etc. [2]. However, conventional explicit NFDTD schemes suffer from the CFL constraint and as a result, finer grid sizes and smaller time steps are required to retain stability, causing the processing time to increase drastically. To eliminate the dependence on the CFL stability constraint and reduce the computational burden, the implicit LOD-FDTD will be used here to model the arbitrary curved structures. By applying the LOD principle to (3.11a)-(3.11f), the updating equations for the LOD-NFDTD method can be obtained. In this chapter, the 2-D nonorthogonal LOD-FDTD (LOD-NFDTD) is described, and the 3-D LOD-NFDTD is discussed in Chapter 6. The formulation of the 2-D LOD-NFDTD method for both the TE and TM cases is described below.

### 3.2.2.1 Formulation of LOD-NFDTD Method for TE Case

The placement of the TE field components of the NFDTD method is shown in Fig. 3.3 (a). Similarly, following the LOD principle, the placement of the TE field components of the LOD-NFDTD method within one cell is shown in Fig. 3.3 (b). By applying the LOD principle to (3.11a)-(3.11f), the electric and magnetic fields for substeps 1 and 2 of the LOD-NFDTD method for the 2-D TE case are given below.

(a)

(b)

Fig. 3.3 Contravariant field components in generalised curvilinear coordinates (TE case); (a) NFDTD (b) LOD-NFDTD

Sub-step 1:

$$
\begin{align*}
\left.E^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}= & \left.a E^{1}\right|_{i+1 / 2, j} ^{n} \\
& +\frac{b}{d u^{2}}\left\{\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)+\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)\right\}  \tag{3.12a}\\
\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} & =\left.c H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& +\frac{d}{d u^{2}}\left(\left.E_{1}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{1}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)+\frac{d}{d u^{2}}\left(\left.E_{1}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{1}\right|_{i+1 / 2, j} ^{n}\right) \tag{3.12b}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E^{2}\right|_{i, j+1 / 2} ^{n+1}=\left.a E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2} \\
& \quad-\frac{b}{d u^{1}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)-\frac{b}{d u^{1}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1}\right)  \tag{3.13a}\\
& \left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}= \\
& \left.c H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}  \tag{3.13b}\\
& \\
& \quad-\frac{d}{d u^{1}}\left(\left.E_{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.E_{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}\right)-\frac{d}{d u^{1}}\left(\left.E_{2}\right|_{i, j+1 / 2} ^{n+1}-\left.E_{2}\right|_{i+1, j+1 / 2} ^{n+1}\right) \\
& \text { where } a=\left(4 \varepsilon-\sigma_{e} \Delta t\right) /\left(4 \varepsilon+\sigma_{e} \Delta t\right), b=2 \Delta t / \sqrt{g}\left(4 \varepsilon+\sigma_{e} \Delta t\right), \\
& c=\left(4 \mu-\sigma_{m} \Delta t\right) /\left(4 \mu+\sigma_{m} \Delta t\right), d=2 \Delta t / \sqrt{g}(4 \mu+\sigma \Delta t)
\end{align*}
$$

From (3.12) and (3.13), it can be observed that compared to the ADI-NFDTD [70] method, the formulation of the LOD-NFDTD method is simple as it requires moving forward only in one dimension in each half step. The covariant $E_{m}, H_{m}$ and contravariant $E^{m}, H^{m}(m=1,2,3)$, together with $g$, are all defined in [23] and this can also be calculated using (3.3). The relationship between covariant fields $H_{m}$, and contra-
variant fields $H^{m}(m=1,2,3)$ are given by $H_{m}=g_{m 1} H^{1}+g_{m 2} H^{2}+g_{m 3} H^{3}$ and $H^{m}=g^{m 1} H_{1}+g^{m 2} H_{2}+g^{m 3} H_{3}$, where $g_{m l}$ and $g^{m l} \quad(m, l=1,2,3)$, are tensors defined in (3.3). A similar relation holds for $E_{m}$ and $E^{m}$. Here $H^{m}$ and $E^{m}(m=1,2,3)$, are true fields components. For 2-D TEz case, $H_{3}=H^{3},\left.E^{1}\right|^{n+1 / 2},\left.H_{3}\right|^{n+1 / 2}$ in (3.12a) and $\left.H^{3}\right|^{n+1 / 2},\left.E_{1}\right|^{n+1 / 2}$ in (3.12b) are defined as synchronous variables. Since (3.12b) cannot be calculated directly, simultaneous linear equations have to be formed from (3.12a) and (3.12b) by eliminating the synchronous variables $H_{3}{ }_{i+1 / 2, j+1 / 2}^{n+1 / 2},\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}$. Since $H_{3}=H^{3}$ for the 2-D TEz case, we can obtain the expression of the $H_{3}{ }_{i+1 / 2, j-1 / 2}^{n+1 / 2}$, from (3.12b). Equation (3.12b) is placed in (3.12a), then, according to [23], the desired covariant field components are averaged by known contra-variant fields to give a second order accurate approximation, as given below.

$$
\begin{align*}
& -\left.\alpha_{11} E^{1}\right|_{i+1 / 2, j-1} ^{n+1 / 2}+\left.\left(1+2 \alpha_{11}\right) E^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}-\left.\alpha_{11} E^{1}\right|_{i+1 / 2, j+1} ^{n+1 / 2} \\
& =\left.a E^{1}\right|_{i+1 / 2, j} ^{n}+\frac{b c}{d u^{2}}\left(\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H^{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \\
& \quad+  \tag{3.14}\\
& \quad \alpha_{11}\left(\left.E_{1}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{1}\right|_{i+1 / 2, j} ^{n}-\left.E_{1}\right|_{i+1 / 2, j} ^{n}+\left.E_{1}\right|_{i+1 / 2, j-1} ^{n}\right) \\
& \quad+\frac{b}{d u^{2}}\left(\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H^{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)
\end{align*}
$$

where $\alpha_{11}=g_{11}\left(b d / \Delta u^{2} \Delta u^{2}\right)$. Equation (3.12b) can be calculated directly. Similarly, from (3.13a) and (3.13b), by eliminating $\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}$ and then omitting the higher order approximation, we have

$$
\begin{align*}
&-\left.\beta_{22} E^{2}\right|_{i-1, j+1 / 2} ^{n+1}+\left.\left(1+2 \beta_{22}\right) E^{2}\right|_{i, j+1 / 2} ^{n+1}-\left.\beta_{22} E^{2}\right|_{i+1, j+1 / 2} ^{n+1} \\
&=\left.a E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\frac{b c}{d u^{1}}\left(\left.H^{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
&+\beta_{22}\left(\left.E^{2}\right|_{i-1, j+1 / 2} ^{n+1 / 2}-\left.E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}+\left.E^{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}\right)  \tag{3.15}\\
& \quad-\frac{b}{d u^{1}}\left(\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H^{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)
\end{align*}
$$

where $\beta_{22}=\left(b d / \Delta u^{1} \Delta u^{1}\right) g_{22}$. Equations (3.14) and (3.15) are the simultaneous linear system with tri-diagonal matrix; which can be solved using the approach A) as described in Section 2.3.2 of Chapter 2. The formulation of the 2-D LOD-NFDTD method for TM case is described next.

### 3.2.2.2 Formulation of LOD-NFDTD Method for TM Case

Similar to the TE case, the field components for the TM case are placed in the computational cell as shown in Fig. 3.4. The calculation of the electric and magnetic fields for the 2-D TM case for both sub-steps 1 and 2 are derived in the same way as that of the TE case, given by the following:

Sub-step 1:

$$
\begin{align*}
& \left.H^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.c H^{2}\right|_{i, j+1 / 2} ^{n} \\
& +\frac{d}{d u^{1}}\left\{\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)+\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.E_{3}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)\right\}  \tag{3.16a}\\
& \left.E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.a E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& \quad \quad \frac{b}{d u^{1}}\left(\left.H_{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i, j+1 / 2} ^{n+1 / 2}\right)+\frac{b}{d u^{1}}\left(\left.H_{2}\right|_{i+1, j+1 / 2} ^{n}-\left.H_{2}\right|_{i, j+1 / 2} ^{n}\right) \tag{3.16b}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.H^{1}\right|_{i+1 / 2, j} ^{n+1}=\left.c H^{2}\right|_{i+1 / 2, j} ^{n+1 / 2} \\
& -\frac{d}{d u^{2}}\left\{\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)+\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.E_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right)\right\}  \tag{3.17a}\\
& \begin{aligned}
\left.E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1} & =\left.a E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \\
& -\frac{b}{d u^{2}}\left\{\left(\left.H_{1}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.H_{1}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)+\left(\left.H_{1}\right|_{i+1 / 2, j+1} ^{n+1}-\left.H_{1}\right|_{i+1 / 2, j} ^{n+1}\right)\right\}
\end{aligned}
\end{align*}
$$

where $a, b, c, d$ are same as those mentioned previously. Similar to the TE case, the covariant and contra-variant together with $g$, are taken to be the same as those given in [23] and this can be calculated using (3.3). For 2-D $\mathrm{TM}_{\mathrm{z}}$ case, $E_{3}=E^{3}, E_{3}, H^{2}$ in equation (3.16a) and $H_{2}$ and $E^{3}$ in equation (3.16b) are defined as synchronous variables.


Fig. 3.4 Contra-variant field components in generalised curvilinear coordinates (TM case) for the LODNFDTD

Since equation (3.16a) is not directly solvable, by placing (3.16b) in (3.16a), and as per [23], the desired covariant field components are averaged by known contra-variant fields to give a second order accurate approximation, leading to the following equation.

$$
\begin{align*}
& -\left.\gamma_{22} H^{2}\right|_{i-1, j+1 / 2} ^{n+1 / 2}+\left.\left(1+2 \gamma_{22}\right) H^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.\gamma_{22} H^{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2} \\
& \quad=\left.c H^{2}\right|_{i, j+1 / 2} ^{n}+\frac{a d}{d u^{1}}\left(\left.E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.E^{3}\right|_{i-1 / 2, j+1 / 2} ^{n}\right) \\
& +  \tag{3.18}\\
& \quad \gamma_{22}\left\{\left(\left.H^{2}\right|_{i+1, j+1 / 2} ^{n}-\left.H^{2}\right|_{i, j+1 / 2} ^{n}\right)+\left(\left.H^{2}\right|_{i, j+1 / 2} ^{n}-\left.H^{2}\right|_{i-1, j+1 / 2} ^{n}\right)\right\} \\
& \quad+\frac{d}{d u^{1}}\left(\left.E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.E^{3}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)
\end{align*}
$$

where $\gamma_{22}=g_{22} b d / d u^{1} d u^{1}$. Equation (3.16b) can be calculated directly. Since the simultaneous linear equation (3.18) can be written in tri-diagonal matrix form, it can be solved efficiently using the approach A) as described in Section 2.3.3 in Chapter 2. Similarly, from (3.17a) and (3.17b), by eliminating $\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}$ then omitting the higher order approximation, we have

$$
\begin{align*}
\delta_{11} H^{1} & \left.\right|_{i+1 / 2, j-1} ^{n+1}+\left.\left(1-2 \delta_{11}\right) H^{1}\right|_{i+1 / 2, j} ^{n+1}+\left.\delta_{11} H^{1}\right|_{i+1 / 2, j+1} ^{n+1} \\
= & \left.c H^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}-\frac{a d}{d u^{2}}\left(\left.E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E^{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right) \\
& -\delta_{11}\left\{\left(\left.H^{1}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.H^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)-\left(\left.H^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}-\left.H^{1}\right|_{i+1 / 2, j-1} ^{n+1 / 2}\right)\right\}  \tag{3.19}\\
& -\frac{a d}{d u^{2}}\left(\left.E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E^{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)
\end{align*}
$$

where $\delta_{11}=g_{11} b d / d u^{2} d u^{2}$. Compared to 2-D ADI-NFDTD [70], the formulation of 2-D LOD-NFDTD is less complex, which will help to reduce computational time. The LOD-FDTD formulation that is based on the curvilinear system can be easily applied to any complex arbitrary structures. The nonorthogonal meshing which has been used with LOD-NFDTD method for analysing 2-D structures is discussed next.

### 3.3 Nonorthogonal Meshing Technique

A good mesh generation technique is essential because it makes the method more general, efficient and more accurate which leads to reduced numerical errors and increased computational efficiency. Different types of grid generation techniques are available in the literature [24]-[27]. We employ the grid topologies that are defined and
classified in [26]-[28]. In this thesis, structured nonorthogonal gridding technique has been used for LOD-NFDTD following [26]-[28]. A coordinate transformation is used to map an arbitrary nonorthogonal structured grid onto a rectangular grid with uniformly spaced cells so that any numerical method can be applied. The proposed approach is flexible enough to handle any type of curved material boundary. To implement the proposed mapping, two different coordinate systems, denoted here by $x^{i}$ and $u^{j}$ for all $i, j=1,2,3$ are used. The first set $x^{i}$ is simply the Cartesian coordinate system (i.e., $x^{1}=x, x^{2}=y$, and $x^{3}=z$ ).

$$
\begin{equation*}
x^{i}=x^{i}\left(u^{j}\right) \quad \text { for all } i, j=1,2,3 \tag{3.20}
\end{equation*}
$$

The second set $u^{j}$ is a completely arbitrary set of curvilinear coordinates. An invertible coordinate transformation is created so that the $u^{j}$ coordinate surfaces, which are drawn by keeping each of the three $u^{j}$ coordinates constant, follow all of the material boundaries of the given problem in $x^{i}$ coordinates one at a time. In this way, a uniform discretisation of Maxwell's equations in the $u^{j}$ coordinate system will be automatically mapped onto a conformal discretisation in the $x^{i}$ coordinate system. Here, the coordinate transformation for the case of a circular dielectric cylinder is described. Consider an arbitrary point $(x c, y c)$ in the computational domain which is mapped to the point $\left(x^{2}, y^{2}\right)$ in the physical domain. To map the computational domain to a circle of radius ' $r_{1}$ ' the following mapping equations are required.

$$
\begin{gather*}
y^{1}=\left\{\begin{array}{c}
\left(\begin{array}{c}
\left.(d)+\sqrt{R(d)^{2}-\left(\frac{D(d)|x c|}{d}\right)^{2}}\right),|y c| \geq|x c| \\
\frac{D(d)|y c|}{d},
\end{array}|y c|<|x c|\right.
\end{array}\right.  \tag{3.21}\\
\text { where } C(d)=D(d)-\sqrt{R(d)^{2}-D(d)^{2}}, d=\max \{|x c|,|y c|\}, \\
D(d)=r_{1} d / \sqrt{2}, R(d)=r_{1} d
\end{gather*}
$$

$$
x^{1}=\left\{\begin{array}{l}
\left(C(d)+\sqrt{R(d)^{2}-\left(\frac{D(d)|y c|}{d}\right)^{2}}\right),|x c| \geq|y c|  \tag{3.22}\\
\frac{D(d)|x c|}{d}, \quad|x c|<|y c|
\end{array}\right.
$$

Equations (3.21) and (3.22) are derived based on the coordinate transformation in [28], [165], and finally mapping equations are given by

$$
\begin{gather*}
x^{2}=\operatorname{sign}(x c)^{*} x^{1}  \tag{3.23}\\
y^{2}=\operatorname{sign}(y c)^{*} y^{1} \tag{3.24}
\end{gather*}
$$



Fig. 3.5 Nonorthogonal meshes of the circular dielectric cylinder

The nonorthogonal grids for a circular dielectric cylinder generated by following the above equations are given in Fig. 3.5. From the figure, it can be observed that the boundaries of the dielectric cylinder are indeed traced by the coordinate lines so the boundary conditions of the dielectric can easily be applied in the computational coordinates. The above technique has been applied to generate nonorthogonal meshes of other 2-D structures which have been analysed in this chapter. This has also been extended for generating 3-D nonorthogonal meshes as will be shown in Chapter 6. The development of convolutional PML for the 2-D LOD-NFDTD is discussed next.

### 3.4 CPML ABC for 2-D LOD-NFDTD

To date, only the PML ABC has been considered in the literature [69]-[72] whenever non-orthogonal grids with implicit FDTD methods are employed. However, PML is not highly effective in absorbing evanescent waves and cannot be placed closer to the objects in the problem space. The advantages of the CPML ABC are provided in Section 2.4 of Chapter 2. Here, to overcome the limitations of PML, we derive the CPML absorbing boundary condition for the 2-D LOD-NFDTD method. The CPML is highly effective in absorbing evanescent waves with a long time signature and can be placed closer to the objects within the problem space to gain time and memory savings. The CPML method maps Maxwell's equation into a complex stretched coordinate space by making use of the complex frequency shifted (CFS) tensor.

$$
\begin{equation*}
S_{e i}=\kappa_{e i}+\frac{\sigma_{p e i}}{\alpha_{e i}+j \omega \varepsilon_{0}}, S_{n i}=\kappa_{m i}+\frac{\sigma_{p m i}}{\alpha_{m i}+j \omega \varepsilon_{0}} i=x, y, z \tag{3.25}
\end{equation*}
$$

where $S_{e i}, S_{m i}$ are the stretched coordinate metrics, and $\sigma_{p e i}$ and $\sigma_{p m i}$ are the electric and magnetic conductivities of the terminating media. By following the theory of CPML as described in Chapter 2, the CPML formulation for the 2-D LOD-NFDTD method for both the TE and TM cases are derived as follows.

### 3.4.1 Derivation of CPML ABC of LOD-NFDTD for TE Case

For the explanation of the method for LOD-NFDTD CPML, here, we provide the updating equations for sub-steps 1 and 2 from (3.12) and (3.13) as follows:

Sub-step 1:

$$
\begin{align*}
& \left.E^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}=\left.a E^{1}\right|_{i+1 / 2, j} ^{n}
\end{aligned}+\frac{b}{d u^{2}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right), ~ \begin{aligned}
&\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.c H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}+\frac{d}{d u^{2}}\left(\left.E_{1}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{1}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)  \tag{3.26a}\\
&\left.\quad+\frac{d}{d u^{2}}\left(\left.E_{1}\right|_{i+1 / 2, j+j+1} ^{n}-\left.E_{1}\right|_{i+1 / 2, j} ^{n}\right)+\left.C_{\psi_{h^{3}}} \psi_{h_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n}-H_{i+1 / 2, j-1 / 2}^{n}\right)+\left.C_{\psi_{e}} \psi_{e_{12}}\right|_{i+1 / 2, j} ^{n}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
&\left.E^{2}\right|_{i, j+1 / 2} ^{n+1}=\left.a E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}- \frac{b}{d u^{1}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
& \quad-\frac{b}{d u^{1}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1}\right)-\left.C_{\psi_{e^{2}}} \psi_{e_{21}}\right|_{i, j+1 / 2} ^{n+1 / 2}  \tag{3.27a}\\
&\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.c H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\frac{d}{d u^{1}}\left(\left.E_{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.E_{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}\right) \\
&-\frac{d}{d u^{1}}\left(\left.E_{2}\right|_{i, j+1 / 2} ^{n+1}-\left.E_{2}\right|_{i+1, j+1 / 2} ^{n+1}\right)-\left.C_{\psi_{h^{3}}} \psi_{h_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \tag{3.27b}
\end{align*}
$$

where $b=2 \Delta t / k(j) \sqrt{g}\left(4 \varepsilon+\sigma_{e} \Delta t\right) C_{\psi_{e^{\prime}}}=\Delta t / k(j) \sqrt{g}\left(4 \varepsilon+\sigma_{e} \Delta t\right) d u^{2}$

$$
\begin{align*}
& d=2 \Delta t / k(j) \sqrt{g}\left(4 \mu+\sigma_{m} \Delta t\right), C_{\psi_{h^{3}}}=\Delta t / k(j) \sqrt{g}\left(4 \mu+\sigma_{m} \Delta t\right) d u^{1} \\
& \left.\psi_{e_{12}}\right|_{i+1 / 2, j} ^{n}=b_{r} \psi_{e_{12}} n_{i+1 / 2, j}^{n-1 / 2}+a_{r}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)  \tag{3.28a}\\
& \left.\psi_{l_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n}=\left.b_{r} \psi_{h_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n-1 / 2}+a_{r}\left(\left.E_{1}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{1}\right|_{i+1 / 2, j} ^{n}\right) \tag{3.28b}
\end{align*}
$$

Unlike (3.12a) and (3.12b), (3.26a) and (3.26b) as well as (3.27a) and (3.27b) cannot be used for direct numerical calculation. Placing (3.26b) in (3.26a) and (3.27b) in (3.27a) yields the simultaneous linear equations (3.29) and (3.30) that result in the tri-diagonal matrix.

$$
\begin{align*}
& -\left.\alpha_{11} E^{1}\right|_{i+1 / 2, j-1} ^{n+1 / 2}+\left.\left(1+2 \alpha_{11}\right) E^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}-\left.\alpha_{11} E^{1}\right|_{i+1 / 2, j+1} ^{n+1 / 2} \\
& =\left.a E^{1}\right|_{i+1 / 2, j} ^{n}+\frac{b c}{d u^{2}}\left(\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H^{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \\
& +  \tag{3.29}\\
& +\alpha_{11}\left(\left.E_{1}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{1}\right|_{i+1 / 2, j} ^{n}-\left.E_{1}\right|_{i+1 / 2, j} ^{n}+\left.E_{1}\right|_{i+1 / 2, j-1} ^{n}\right) \\
& \\
& \quad+\frac{b}{d u^{2}}\left(\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H^{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \\
& +\left.C_{\psi_{e^{\prime}}} \psi_{e_{12}}\right|_{i+1 / 2, j} ^{n}+\left.2 \alpha_{11} \psi_{h_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n}+\left.C_{\psi_{h^{3}}} \psi_{h_{32}}\right|_{i+1 / 2, j-1 / 2} ^{n}
\end{align*}
$$

where $\alpha_{11}=g_{11}\left(b d / \Delta u^{2} \Delta u^{2}\right)$.

$$
\begin{align*}
& -\left.\beta_{22} E^{2}\right|_{i-1, j+1 / 2} ^{n+1}+\left.\left(1+2 \beta_{22}\right) E^{2}\right|_{i, j+1 / 2} ^{n+1}-\left.\beta_{22} E^{2}\right|_{i+1, j+1 / 2} ^{n+1} \\
& =\left.a E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\frac{b c}{d u^{1}}\left(\left.H^{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
& +\beta_{22}\left(\left.E^{2}\right|_{i-1, j+1 / 2} ^{n+1 / 2}-\left.E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}+\left.E^{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}\right)  \tag{3.30}\\
& -\frac{b}{d u^{1}}\left(\left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H^{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
& -\left.C_{\psi_{e^{2}}} \psi_{e_{21}}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.2 \beta_{22} \psi_{h_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.C_{\psi_{h^{3}}} \psi_{h_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}
\end{align*}
$$

where $\beta_{22}=\left(b d / \Delta u^{1} \Delta u^{1}\right) g_{22}$. Equations (3.29) and (3.30) are the simultaneous linear system with tri-diagonal matrix, which can be solved using approach A) as described in Section 2.3.2 of Chapter 2. However, $\psi_{b_{32}}$ and $\psi_{e_{12}}$ are discrete variables which may have non-zero values only in some CPML regions but are necessary to implement the absorbing boundary [2].

### 3.4.2 Derivation of CPML ABC of LOD-NFDTD for TM Case

The formulations of LOD-NFDTD CPML for the 2-D TM case are derived in the same way as in the case of the TE wave. The updating equations for both sub-steps 1 and 2 are given next.

Sub-step 1:

$$
\begin{align*}
\left.H^{2}\right|_{i, j+1 / 2} ^{n+1 / 2} & =\left.c H^{2}\right|_{i, j+1 / 2} ^{n}+\frac{d}{d u^{1}}\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)  \tag{3.31a}\\
& +\frac{d}{d u^{1}}\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.E_{3}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)+\left.C_{\psi_{n^{2}}} \psi_{h_{21}}\right|_{i, j+1 / 2} ^{n} \\
\left.E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} & =\left.a E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}+\frac{b}{d u}\left(\left.H_{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i, j+1 / 2} ^{n+1 / 2}\right) \\
& +\frac{b}{d u^{1}}\left(\left.H_{2}\right|_{i+1, j+1 / 2} ^{n}-\left.H_{2}\right|_{i, j+1 / 2} ^{n}\right)+\left.C_{\psi_{e^{e}}} \psi_{e_{31}}\right|_{i+1 / 2, j+1 / 2} ^{n} \tag{3.31b}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
\left.H^{1}\right|_{i+1 / 2, j} ^{n+1}= & \left.c H^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}-\frac{d}{d u^{2}}\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)  \tag{3.32a}\\
& -\frac{d}{d u^{2}}\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.E_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right)-\left.C_{\psi_{h_{1}}} \psi_{h_{12}}\right|_{i+1 / 2, j} ^{n+1 / 2} \\
\left.E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}= & \left.a E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\frac{b}{d u^{2}}\left(\left.H_{1}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.H_{1}\right|_{i+1 / 2, j} ^{n+1 / 2}\right) \\
& -\frac{b}{d u^{2}}\left(\left.H_{1}\right|_{i+1 / 2, j+1} ^{n+1}-\left.H_{1}\right|_{i+1 / 2, j} ^{n+1}\right)-\left.C_{\psi_{e^{3}}} \psi_{e_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \tag{3.32b}
\end{align*}
$$

where

$$
\begin{gather*}
\left.\psi_{e_{31} \mid}\right|_{i+1 / 2, j} ^{n}=\left.b_{r} \psi_{e_{31}}\right|_{i+1 / 2, j} ^{n-1 / 2}+a_{r}\left(\left.H_{2}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{2}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)  \tag{3.33a}\\
\left.\psi_{h_{21}}\right|_{i+1 / 2, j+1 / 2} ^{n}=\left.b_{r} \psi_{h_{21}}\right|_{i+1 / 2, j+1 / 2} ^{n-1 / 2}+a_{r}\left(\left.E_{3}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{3}\right|_{i+1 / 2, j} ^{n}\right)  \tag{3.33b}\\
c_{s}=e^{-\left(\left(\sigma_{s} / \kappa_{s}\right)+\alpha_{s}\right)\left(\Delta t / \varepsilon_{0}\right)}  \tag{3.34a}\\
d_{s}=\frac{\sigma_{s}}{\kappa_{s}\left(\sigma_{s}+\kappa_{s} \sigma_{s}\right)} \times\left(c_{s}-1\right), \quad(s=x, y, \text { or } z) \tag{3.34b}
\end{gather*}
$$

$$
\begin{align*}
\sigma_{s}(s) & =\frac{\sigma_{s_{\max }}\left|s-s_{0}\right|^{m}}{\delta^{m}}  \tag{3.34c}\\
\kappa_{s}(s) & =1+\left(\kappa_{\max }-1\right) \frac{\left|s-s_{0}\right|^{m}}{\delta^{m}} \tag{3.34d}
\end{align*}
$$

where $\delta$ is the thickness of the PML absorber, $s_{0}$ is the PML interface, $m$ is the order of the polynomial. To avoid reflections between the computational domain and the CPML boundary due to the discontinuity, the losses due to the CPML must be made zero at the interfaces of the computational domain. The number of auxiliary equations is small which results in a higher computational efficiency for the non-orthogonal LOD-FDTD CPML. The reflection error of the LOD-NFDTD CPML increases with higher CFLN. Now, we perform the reflection test on the proposed CPML ABC, where the reflections are calculated from PML layers used by the scattered field obtained using plane wave scattering from a perfect electric conducting cylinder of radius $0.795 \lambda$. The 8 cell CPML region is extended $0.5 \lambda$ away from the surface of the scatterer. The cell size chosen for this problem is $\lambda / 10$. To minimise the reflection error, the following parameters $\sigma_{\text {opt }}=(m+1) /(150 \pi \Delta x)=11.21(\mathrm{~S} / \mathrm{m}), \quad m=4, \quad \sigma_{s_{\max }}=0.7 \sigma_{\text {opt }}(\mathrm{S} / \mathrm{m}) \quad$ and $k_{s_{\text {max }}}=15$ are considered for the CPML region. The discrete variables $\psi_{h_{21}}$ and $\psi_{e_{31}}$ may have non-zero values only in some CPML regions but are necessary for implementing the absorbing boundary [2]. The observed scattered field is considered two cells away from the cylinder in $y$-direction. The reflection error is calculated using the following equations:

$$
\begin{equation*}
\text { error }=20 \log 10\left(\frac{\left|H^{3}-H_{r e f}^{3}\right|}{\max \left|H_{r e f}^{3}\right|}\right) \tag{3.35}
\end{equation*}
$$



Fig. 3.6 Reflection error of the LOD-NFDTD CPML method

Fig. 3.6 shows the reflection error for different CFLN. It can be observed from the figure that error increases with the increase in CFLN due to dispersion.

### 3.5 Near-Field to Far-Field Transformation for 2-D LODNFDTD Method

The near-field to far-field (NF-FF) transformation algorithm is used to calculate the radar cross-section (RCS) of the scatterer due to an incident plane wave. In this case, an imaginary surface is first selected to enclose the electromagnetic object. The currents $\vec{J}$ and $\vec{M}$ on the surface are determined by $\vec{E}$ and $\vec{H}$ fields computed from the LODNFDTD method inside the computational domain. These currents are transformed into the frequency domain while being captured. After completing for all the time steps, the far field terms $L_{\theta}, L_{\phi}, N_{\theta}$ and $N_{\phi}$ are calculated. These far field terms are calculated in the same way as given in [2]. Bistatic RCS can then be calculated using the following equation.

$$
\begin{equation*}
R C S_{\theta}=\frac{k^{2}}{8 \pi \eta_{0} P_{\text {inc }}}\left|L_{\phi}+\eta_{0} N_{\theta}\right|^{2} \tag{3.36}
\end{equation*}
$$

The $P_{\text {inc }}$ can be calculated as:

$$
\begin{equation*}
P_{\text {inc }}=\frac{1}{2 \eta_{0}}\left|E_{\text {inc }}(\omega)\right|^{2} \tag{3.37}
\end{equation*}
$$

where $E_{\text {inc }}(\omega)$ is the discrete Fourier transform (DFT) of the incident electric field waveform, at the frequency for which RCS calculation is required.

### 3.6 Pure Scattered Field Formulation for 2-D LOD-NFDTD Method

In this section, we present the pure scattered field formulation of the 2-D LODNFDTD method. A schematic of the problem space where a plane wave interacts with the scatterer is shown in Fig. 3.7. To simulate a plane wave excitation in 2D LODNFDTD, the problem space is divided into two regions, the total field region and the scattered field region. The vectorial sum of incident and scattered fields present within a
given space provide the total fields. The total fields satisfy Maxwell's equation, which for a region free of current sources can be written as:

$$
\begin{gather*}
\frac{\partial H_{\text {total }}}{\partial t}=-\frac{1}{\mu} \nabla \times E_{\text {total }}-\sigma^{*} H_{\text {total }}  \tag{3.38a}\\
\frac{\partial E_{\text {total }}}{\partial t}=\frac{1}{\varepsilon} \nabla \times H_{\text {total }}-\sigma E_{\text {total }} \tag{3.38b}
\end{gather*}
$$

The scattered field formulation for the LOD-NFDTD method for both TE and TM waves is given below.


Fig. 3.7 Total field/scattered field region for 2-D problem space in LOD-NFDTD

### 3.6.1 2-D TE Case

By applying the LOD principle (3.38a)-(3.38b), the scattered field formulation of 2-D LOD-NFDTD for the TE wave can be written using two sub-step procedures as follows: Sub-step 1:

$$
\begin{align*}
& \left.E_{\text {scat }}^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}=\left.a E_{\text {scat }}^{1}\right|_{i+1 / 2, j} ^{n} \\
& \quad+\frac{b}{d u^{2}}\left(\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right) \\
&  \tag{3.39a}\\
& \quad+\frac{b}{d u^{2}}\left(\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \\
& \quad+\left.C_{\text {eleic }} E_{\text {inc }}^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}+\left.C_{\text {eleip }} E_{i n c}^{1}\right|_{i+1 / 2, j} ^{n}
\end{align*}
$$

$$
\begin{align*}
\left.H_{\text {scat }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}= & \left.c H_{\text {scat }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& +\frac{d}{d u^{2}}\left(\left.E_{s c a t, 1}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{s c a t, 1}\right|_{i+1 / 2, j} ^{n+1 / 2}\right) \\
& +\frac{d}{d u^{2}}\left(\left.E_{\text {scat }, 1}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{\text {scat }, 1}\right|_{i+1 / 2, j} ^{n}\right)  \tag{3.39b}\\
& +\left.C_{h 3 h i c} H_{i n c}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}+\left.C_{h 3 h i p} H_{i n c}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E_{\text {scat }}^{2}\right|_{i, j+1 / 2} ^{n+1}=\left.a E_{\text {scat }}^{2}\right|_{i, j+1 / 2} ^{n+1 / 2} \\
& -\frac{b}{d u^{1}}\left(\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{s c a t, 3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)  \tag{3.40a}\\
& -\frac{b}{d u^{1}}\left(\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{\text {scat }, 3}\right|_{i-1 / 2, j+1 / 2} ^{n+1}\right) \\
& +\left.C_{e 2 e i c} E_{i n c}^{2}\right|_{i, j+1 / 2} ^{n+1}+\left.C_{e 2 e i p} E_{i n c}^{2}\right|_{i, j+1 / 2} ^{n+1 / 2} \\
& \left.H_{\text {scat }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.c H_{\text {scat }}^{3}\right|_{{ }_{i+1 / 2, j+1 / 2}^{n+1 / 2}} \\
& -\frac{d}{d u^{1}}\left(\left.E_{\text {scat }, 2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.E_{\text {scat }, 2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{\text {scat }, 2}\right|_{i, j+1 / 2} ^{n+1}+\left.E_{\text {scat }, 2}\right|_{i+1, j+1 / 2} ^{n+1}\right)  \tag{3.40b}\\
& +\left.C_{h 3 h i c} H_{\text {inc }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}+\left.C_{h 3 h i p} H_{i n c}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}
\end{align*}
$$

where $a, b, c, d$ are the same as mentioned before, and

$$
\begin{aligned}
& C_{\text {eleic }}=\frac{\left(4\left(\varepsilon_{0} / \sqrt{g}-\varepsilon\right)-\sigma_{e} \Delta t\right)}{\left(4 \varepsilon+\sigma_{e} \Delta t\right)}, C_{\text {eleip }}=-\frac{\left(4\left(\varepsilon_{0} / \sqrt{g}-\varepsilon\right)+\sigma_{e} \Delta t\right)}{(4 \varepsilon+\sigma \Delta t)} \\
& C_{h 3 h i c}=\frac{\left(4\left(\mu_{0} / \sqrt{g}-\mu\right)-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)} C_{h 3 h i p}=-\frac{\left(4\left(\mu_{0} / \sqrt{g}-\mu\right)+\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)}
\end{aligned}
$$

For the 2-D $\mathrm{TE}_{\mathrm{Z}}$ case, $H_{3}=H^{3}, E^{1}$ and $H_{3}$ in equation (3.39a) are defined as synchronous variables since equation (3.39b) is not directly solved.

$$
\begin{align*}
& -\left.\alpha_{11} E_{\text {scat }}^{1}\right|_{i+1 / 2, j-1} ^{n+1}+\left.\left(1+2 \alpha_{11}\right) E_{\text {scat }}^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}-\left.\alpha_{11} E_{\text {scat }}^{1}\right|_{i+1 / 2, j+1} ^{n+1 / 2} \\
& =\left.a E_{\text {scat }}^{1}\right|_{i+1 / 2, j} ^{n}+\frac{b c}{d u^{2}}\left(\left.H_{\text {scat }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{\text {scat }}^{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \\
& \quad+\alpha_{11}\left(\left.E_{\text {scat }}^{1}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{\text {scat }}^{1}\right|_{i+1 / 2, j} ^{n}-\left.E_{\text {scat }}^{1}\right|_{i+1 / 2, j} ^{n}+\left.E_{\text {scat }}^{1}\right|_{i+1 / 2, j-1} ^{n}\right) \\
& +\frac{b}{d u^{2}}\left(\left.H_{\text {scat }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{\text {scat }}^{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)+\left.C_{\text {eleic }} E_{\text {inc }}^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}+\left.C_{\text {eleip }} E_{\text {inc }}^{1}\right|_{i+1 / 2, j} ^{n}  \tag{3.41}\\
& \quad+\frac{b}{d u^{2}}\left(\left.C_{h 3 h i c} H_{i n c}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}+\left.C_{h 3 h i p} H_{\text {inc }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}\right) \\
& \quad-\frac{b}{d u^{2}}\left(\left.C_{h 3 h i c} H_{\text {inc }}^{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}+\left.C_{h 3 h i p} H_{\text {inc }}^{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)
\end{align*}
$$

Placing (3.39b) in (3.39a), and following [23], the desired covariant field components are averaged by the known contra-variant fields to give a second order accurate approximation, leading to the simultaneous linear equation (3.41) with tri-diagonal matrix where $\alpha_{11}=\left(b d / \Delta u^{2} \Delta u^{2}\right) g_{11}$. Similarly, from (3.40a) and (3.40b) of substep 2, eliminating $\left.H_{\text {scat, } 3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}$, the simultaneous linear system with tri-diagonal matrix is obtained as follows:

$$
\begin{align*}
& -\left.\beta_{22} E_{\text {scat }}^{2}\right|_{i-1, j+1 / 2} ^{n+1}+\left.\left(1+2 \beta_{22}\right) E_{\text {scat }}^{2}\right|_{i, j+1 / 2} ^{n+1}-\left.\beta_{22} E_{\text {scat }}^{2}\right|_{i+1, j+1 / 2} ^{n+1} \\
& =\left.a E_{\text {scat }}^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\frac{b c}{d u^{1}}\left(\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{s c a t, 3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
& +\beta_{22}\left(\left.E_{\text {scat }}^{2}\right|_{i-1, j+1 / 2} ^{n+1 / 2}-\left.E_{s c a t}^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.E_{s c a t}^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}+\left.E_{\text {scat }}^{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}\right) \\
& \quad-\frac{b}{d u^{1}}\left(\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{\text {scat }, 3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)+\left.C_{e 2 e i c} E_{i n c}^{2}\right|_{i, j+1 / 2} ^{n+1}  \tag{3.42}\\
& \quad+\left.C_{e 2 e i p} E_{i n c}^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\frac{b}{d u^{1}}\left(\left.C_{h 3 h i c} H_{i n c}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}+\left.C_{h 3 h i p} H_{i n c}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}\right) \\
& \quad+\frac{b}{d u^{1}}\left(\left.C_{h 3 h i c} H_{i n c}^{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1}+\left.C_{h 3 h i p} H_{i n c}^{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)
\end{align*}
$$

### 3.6.2 2-D TM Case

The scattered field formulations for the 2-D TM wave are derived in the same way for the TE wave. The updating equations for sub-steps 1 and 2 are given next. Sub-step 1:

$$
\begin{align*}
& \left.H_{\text {scat }}^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.c H_{\text {scat }}^{2}\right|_{i, j+1 / 2} ^{n} \\
& +\frac{d}{d u^{1}}\left\{\left(\left.E_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{\text {scat }, 3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)\right.  \tag{3.43a}\\
& \left.+\left(\left.E_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.E_{\text {scat }, 3}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)\right\} \\
& +\left.C_{h 2 h i c} H_{i n c}^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}+\left.C_{h 2 h i p} H_{i n c}^{2}\right|_{i, j+1 / 2} ^{n} \\
& \left.E_{\text {scat }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.a E_{\text {scat }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& +\frac{b}{d u^{1}}\left\{\left(\left.H_{\text {scat }, 2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.H_{\text {scat }, 2}\right|_{i, j+1 / 2} ^{n+1 / 2}\right)\right.  \tag{3.43b}\\
& \left.+\left(\left.H_{\text {scat }, 2}\right|_{i+1, j+1 / 2} ^{n}-\left.H_{\text {scat }, 2}\right|_{i, j+1 / 2} ^{n}\right)\right\} \\
& +\left.C_{e 3 e i c} E_{\text {inc }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}+\left.C_{e 3 e i p} E_{\text {inc }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
&\left.H_{\text {scat }}^{1}\right|_{i+1 / 2, j} ^{n+1}=\left.c H_{\text {scat }}^{1}\right|_{i+1 / 2, j} ^{n+1 / 2} \\
&-\frac{d}{d u^{2}}\left(\left.E_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{\text {scat }, 3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right) \\
& \quad-\frac{d}{d u^{2}}\left(\left.E_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.E_{\text {scat }, 3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)  \tag{3.44a}\\
& \quad+\left.C_{h x h i c} H_{i n c}^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}+\left.C_{h x h i p} H_{i n c}^{1}\right|_{i+1 / 2, j} ^{n} \\
&\left.E_{\text {scat }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.a E_{\text {scat }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \\
&-\frac{b}{d u^{2}}\left(\left.H_{\text {scat }, 1}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.H_{\text {scat }, 1}\right|_{i+1 / 2, j} ^{n+1 / 2}\right) \\
&+\frac{b}{d u^{2}}\left(\left.H_{\text {scat }, 1}\right|_{i+1 / 2, j+1} ^{n+1}-\left.H_{\text {scat }, 1}\right|_{i+1 / 2, j} ^{n+1}\right)  \tag{3.44b}\\
& \quad+\left.C_{\text {ezeic }} E_{\text {inc }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}+\left.C_{\text {ezeip }} E_{\text {inc }}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}
\end{align*}
$$

Since equation (3.43a) of sub-step 1 is not directly solved, by placing (3.43b) in (3.43a), and as per [23], the desired covariant field components are averaged by the known contra-variant fields to give a second order accurate approximation, leading to the following equation.

$$
\begin{align*}
& -\left.\gamma_{22} H_{s c a t}^{2}\right|_{i-1, j+1 / 2} ^{n+1 / 2}+\left.\left(1+2 \gamma_{22}\right) H_{\text {scat }}^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-\left.\gamma_{22} H_{\text {scat }}^{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2} \\
& =\left.c H_{s c a t}^{2}\right|_{i, j+1 / 2} ^{n}+\frac{a d}{d u^{1}}\left(\left.E_{s c a t, 3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.E_{\text {scat }, 3}\right|_{i-1 / 2, j+1 / 2} ^{n}\right) \\
& +\gamma_{22}\left(\left.H_{s c a t}^{2}\right|_{i+1, j+1 / 2} ^{n}-\left.H_{s c a t}^{2}\right|_{i, j+1 / 2} ^{n}+\left.H_{s c a t}^{2}\right|_{i, j+1 / 2} ^{n}-\left.H_{s c a t}^{2}\right|_{i-1, j+1 / 2} ^{n}\right) \\
& +\frac{d}{d u^{1}}\left(\left.E_{s c a t, 3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.E_{s c a t, 3}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)  \tag{3.45}\\
& +\left.C_{h 2 h i c} H_{i n c}^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}+\left.C_{h 2 h i p} H_{i n c}^{2}\right|_{i, j+1 / 2} ^{n} \\
& +\frac{d}{d u^{1}}\left(\left.C_{e 3 e i c} E_{i n c}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}+\left.C_{e 3 e i p} E_{i n c}^{3}\right|_{i+1 / 2, j+1 / 2} ^{n}\right) \\
& -\frac{d}{d u^{1}}\left(\left.C_{e 3 e i c} E_{i n c}^{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}+\left.C_{e 3 e i p} E_{i n c}^{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)
\end{align*}
$$

A similar simultaneous linear equation with tri-diagonal matrix is obtained from substep 2. The simultaneous linear equation with tri-diagonal matrix can be solved using approach A) as described in Section 2.3.2 in Chapter 2. For a given incident wave, the above equations can be used to calculate the scattered field. The total field can be obtained by adding the scattered field to the incident field. Equations (3.39)-(3.45) are
suitable for any incident wave involving magnetic as well as non-magnetic materials. Note that incident fields are applied only in the internal region. In the CPML region only scattered fields exist, which are absorbed by CPML. In the next section, stability analysis of the 2-D LOD-NFDTD method will be described.

### 3.7 Stability Analysis of the 2-D LOD-NFDTD Method

To check the stability of the proposed approach, (3.12), (3.13) and other field components are expressed in matrix form following Von Neumann [166] method. Updating the fields from the $\mathrm{n}^{\text {th }}$ to the $(\mathrm{n}+1 / 2)^{\text {th }}$ time step, we have

$$
\begin{equation*}
\left.\mathbf{X}\right|^{n+1 / 2}=\left.\mathbf{M}_{1} \cdot \mathbf{X}\right|^{n} \tag{3.46a}
\end{equation*}
$$

Similarly, when the fields are updated from the $n+1 / 2$ to $n+1{ }^{\text {th }}$ time step, we have

$$
\begin{equation*}
\left.\mathbf{X}\right|^{n+1}=\left.\mathbf{M}_{2} \cdot \mathbf{X}\right|^{n+1 / 2} \tag{3.46b}
\end{equation*}
$$

where the composite vector $\mathbf{X}$ has three field components. For the TE case, the following result can be obtained:

$$
\left.\mathbf{X}\right|^{n}=\left[\begin{array}{lll}
\left.E^{1}\right|^{n} & \left.E^{2}\right|^{n} & \left.H^{3}\right|^{n} \tag{3.47}
\end{array}\right]^{T}
$$

where $T$ denotes a transpose and both $\mathbf{M}_{1}$ and $\mathbf{M}_{2}$ are $3 \times 3$ matrices that will be addressed next. It is assumed that the spatial frequencies can be expressed by $k_{1}$ and $k_{2}$ along the directions of $u^{1}$ and $u^{2}$ respectively. The numerical stability of this method is studied using the approach by Von Neumann [166]. In the spatial spectral domain, a 2D TE field component can be written as

$$
\begin{equation*}
\mathbf{X}(n, i, j)=\left.\mathbf{X}_{0}\right|^{n} \varsigma_{q} \exp \left[-J\left(k_{1} i \Delta u^{1}+k_{2} j \Delta u^{2}\right)\right] \tag{3.48}
\end{equation*}
$$

where $J=\sqrt{-1}$ and $\mathrm{q}=1$ and 2 for the first and second procedures, respectively. Substituting (3.48) into (3.12a) and (3.12b), we have

$$
\left[\begin{array}{lcc}
\varsigma_{1}-1 & 0 & J\left(2 b / \Delta u^{2}\right)\left(\varsigma_{1}+1\right) S_{2}  \tag{3.49}\\
0 & \varsigma_{1}-1 & 0 \\
-J\left(2 b / \Delta u^{2}\right)\left(\varsigma_{1}+1\right) S_{1} & 0 & \varsigma_{1}-1
\end{array}\right]\left[\begin{array}{l}
E_{0}^{1} \\
E_{0}^{2} \\
H_{0}^{3}
\end{array}\right]=0
$$

where $S_{q}=\sin \left(k_{q} \Delta u^{q} / 2\right)$ for $q=1,2$. To achieve a non-trivial solution, the determinant of the matrix in (3.49) has to be zero, leading to

$$
\begin{equation*}
P \varsigma_{1}^{2}-2 \varsigma_{1}+Q=0 \tag{3.50}
\end{equation*}
$$

where

$$
\begin{aligned}
& P=1+\left(4(\Delta t)^{2} / g \mu \varepsilon \Delta u^{2} \Delta u^{2}\right) \sin ^{2}\left(k_{2} \Delta u^{2} / 2\right) \\
& Q=1-\left(4(\Delta t)^{2} / g \mu \varepsilon \Delta u^{2} \Delta u^{2}\right) \sin ^{2}\left(k_{2} \Delta u^{2} / 2\right)
\end{aligned}
$$

The solution of (3.50) is given by

$$
\begin{equation*}
\varsigma_{1}=(1 \pm J \sqrt{P Q-1}) / P \tag{3.51}
\end{equation*}
$$

Similarly from the sub-step 2 , we obtain

$$
\begin{equation*}
\varsigma_{2}=(1 \pm J \sqrt{P Q-1}) / Q \tag{3.52}
\end{equation*}
$$

From (3.51) and (3.52), we can write

$$
\begin{equation*}
\left|\varsigma_{1}\right| \cdot\left|\varsigma_{2}\right|=1 \tag{3.53}
\end{equation*}
$$

This proves that the proposed 2-D LOD-NFDTD algorithm is inherently stable. Numerically, we also demonstrate the stability and accuracy of the proposed LODNFDTD algorithm. Consider a 2-D PEC cylinder excited by a sine modulated cosine pulse. The radius of the cylinder is 0.15 m . The cell size chosen for this problem is $\lambda / 10$. The time step size for the cylinder is calculated by the relation $\Delta t=C F L N \Delta t_{C F L}$. For the 2D LOD-NFDTD method, the CFL condition is given below.

$$
\begin{equation*}
C_{0} \Delta t \leq \min J\left\|\Delta u^{1} \Delta u^{2} /\left(\sum_{i=1, j=1}^{2} g_{i j} \Delta u^{i} \Delta u^{j}\right)\right\|_{l m} \tag{3.54}
\end{equation*}
$$

where $J$ is the Jacobian [157] and $(l, m)$ are grid locations. The symbol $g_{i j}$ can be calculated using (3.3). The observation point is recorded at ten cells away from the outer surface of the conductor.

(a)

(b)

Fig. 3.8 (a) Time domain $H_{z}$ recorded at the observation point ( 10 cells away from the outer surface of PEC conductor) for CFLN=2 (b) Recorded $H_{z}$ at the observation point for CFLN=8

Fig. 3.8 (a) shows the time domain waveforms of the magnetic field recorded at the observation point for CFLN=2. More simulations were carried out for higher CFLN values to study the stability and the accuracy of the LOD-NFDTD algorithm. Fig. 3.8 (b) shows the time domain waveforms of the magnetic field recorded at the observation point for CFLN=8 with 12000 time steps. From Figs. 3.8 (a) and (b), it can be observed that the LOD-NFDTD is stable.

### 3.8 Dispersion Analysis of the 2-D LOD-NFDTD Method

The numerical dispersion effects exist regardless of the coordinate system and are a consequence of replacing the differential equations with difference equations. Equation (3.12a)-(3.12b) and (3.13a)-(3.13b) are expressed in matrix form to quantify the numerical dispersion for the proposed method. To simplify the problem, a lossless medium is considered. Updating the fields from the $n$ to $n+1 / 2^{\text {th }}$ time step, we have

$$
\begin{equation*}
\left.\mathbf{X}\right|^{n+1 / 2}=\left.\mathbf{M}_{1} \cdot \mathbf{X}\right|^{n} \tag{3.55}
\end{equation*}
$$

Similarly, when the fields are updated from the $n+1 / 2$ to $n+1{ }^{\text {th }}$ time step, we have

$$
\begin{equation*}
\left.\mathbf{X}\right|^{n+1}=\left.\mathbf{M}_{2} \cdot \mathbf{X}\right|^{n+1 / 2} \tag{3.56}
\end{equation*}
$$

In the 2-D LOD-NFDTD method, the relationship between the field components at $n$ and $n+1$ time steps can be obtained by (3.55) and (3.56), leading to

$$
\begin{equation*}
\left.\mathbf{X}\right|^{n+1}=\left.\mathbf{M}_{1} \cdot \mathbf{M}_{2} \cdot \mathbf{X}\right|^{n}=\left.\mathbf{M} \cdot \mathbf{X}\right|^{n} \tag{3.57}
\end{equation*}
$$

To obtain the analytical expression for the dispersion, the matrix form is expressed as

$$
\begin{equation*}
\left.\mathbf{M . X}\right|^{n}=0 \tag{3.58}
\end{equation*}
$$

where $\mathbf{M}$ is a $3 \times 3$ matrix to be derived as follows. Initially, a trial solution of the fields is assumed which is a monochromatic wave with angular frequency $\omega$

$$
\begin{equation*}
\mathbf{X}(n, i, j)=\left.\mathbf{X}_{0}\right|^{n} \exp \left[J\left(\omega n \Delta t-k_{1} i \Delta u^{1}-k_{2} j \Delta u^{2}\right)\right] \tag{3.59}
\end{equation*}
$$

where $n$ is the time index and $i, j$ are space indexes. Insertion of (3.59) into (3.12a)(3.12b), and (3.13a)-(3.13b), then after simplifying the resultant equations, the following equations are obtained.

$$
\begin{gather*}
\left.E^{1}\right|^{n} \cdot \sin \left(\frac{\omega \Delta t}{2}\right)=-\left.\frac{\Delta t}{\Delta u^{2} \varepsilon \sqrt{g}} H^{3}\right|^{n} \cdot \sin \left(\frac{k_{2} \Delta u^{2}}{2}\right) \cdot\left(e^{-\frac{J \omega \Delta t}{2}}+1\right)  \tag{3.60a}\\
\left.E^{2}\right|^{n} \cdot \sin \left(\frac{\omega \Delta t}{2}\right)=\left.\frac{\Delta t}{\varepsilon \Delta u^{1} \sqrt{g}} \cdot \sin \left(\frac{k_{1} \Delta u^{1}}{2}\right) \cdot\left\{1+e^{\frac{J \omega \Delta t}{2}}\right\} \cdot H^{3}\right|^{n}  \tag{3.60b}\\
\left.H^{3}\right|^{n} \sin \left(\frac{\omega \Delta t}{2}\right)=-\left.\frac{\Delta t \cdot g_{11}}{\Delta u^{2} \mu \sqrt{g}} E^{1}\right|^{n} \sin \left(\frac{k_{2} \Delta u^{2}}{2}\right)\left(e^{-(J \omega / 2) \Delta t)} \cdot+1\right)  \tag{3.60c}\\
+\left.\frac{\Delta t g_{22}}{\mu \Delta u^{1} \sqrt{g}} \cdot E^{2}\right|^{n} \cdot \sin \left(\frac{k_{1} \Delta u^{1}}{2}\right)\left(e^{J \omega \Delta t / 2}+1\right)
\end{gather*}
$$

Finally, the matrix $\mathbf{M}$ can be obtained as follows

$$
\mathbf{M}=\left[\begin{array}{lcl}
\sin \left(\frac{\omega \Delta t}{2}\right) & 0 & \frac{\mathrm{~W}^{2}}{\varepsilon}\left(1+\mathrm{e}^{\frac{\mathrm{j} \omega \Delta t}{2}}\right)  \tag{3.61}\\
0 & \sin \left(\frac{\omega \Delta t}{2}\right) & -\frac{\mathrm{W}^{2}}{\varepsilon}\left(1+\mathrm{e}^{\frac{\mathrm{j} \omega \Delta t}{2}}\right) \\
\chi_{11} & -\chi_{22} & \sin \left(\frac{\omega \Delta t}{2}\right)
\end{array}\right]
$$

where

$$
\begin{gathered}
\chi_{11}=\frac{g_{11} W^{2}}{\varepsilon}\left(1+e^{\frac{-j \omega \Delta t}{2}}\right), \quad \chi_{22}=\frac{g_{22} W^{1}}{\varepsilon}\left(1+e^{\frac{j \omega \Delta t}{2}}\right) \\
W^{p}=\frac{\Delta t}{\sqrt{g} \Delta u^{p}} \sin \left(\frac{k_{p} \Delta u^{p}}{2}\right) \quad(p=1,2)
\end{gathered}
$$

By equating the determinant of the matrix $\mathbf{M}$ to zero, the dispersion relationship can be obtained as:

$$
\begin{equation*}
\sin ^{2}\left(\frac{\omega \Delta t}{2}\right)=\frac{1}{\varepsilon}\left(W^{1} \chi_{22}+W^{2} \chi_{11}\right)+\frac{1}{\varepsilon} \cos \left(\frac{\omega \Delta t}{2}\right)\left(W^{1} \chi_{22}+W^{2} \chi_{11}\right) \tag{3.62}
\end{equation*}
$$

Equation (3.62) is also applicable to the TM case. The simplest case for the TE wave with Cartesian mesh is considered with $k_{1}=k_{x}, k_{2}=k_{y}, \Delta u^{1}=\Delta x$ and $\Delta u^{2}=\Delta y$.

Equation (3.62) will be converted into the theoretical dispersion relation as $\Delta t, \Delta u^{1}$, $\Delta u^{2}$ tend to zero:

$$
\begin{equation*}
k_{x}^{2}+k_{y}^{2}=(\omega / c)^{2}=k^{2} \tag{3.63}
\end{equation*}
$$

The dispersion error of the LOD-NFDTD increases with the increase in CFLN. The numerical dispersion characteristics of the proposed LOD-NFDTD method are derived analytically as above. Now, we will discuss the numerical analysis of the dispersion error of the proposed method. From Fig. 3.9, it can be observed that dispersion errors increase with the increasing CFLN. Dispersion characteristics also change significantly due to the non-orthogonality of the grid chosen. The CFL limit and the effect of grid size on dispersion error are studied here in detail following [128]. The LOD-NFDTD method allows larger time steps than the standard explicit FDTD method. The effect of the CFL number (CFLN) and grid size on dispersion error will be described below.
a) Effect of different CFLN on numerical dispersion

Fig. 3.9 shows the normalised phase velocity with respect to incident wave angle $\varphi$ for different CFL numbers. From Fig. 3.9 it can be observed that the numerical dispersion of the LOD-NFDTD method increases with increasing CFLN.


Fig. 3.9 Phase velocity versus wave angle

It can be seen from Fig. 3.10 that when CFLN tends towards zero, the numerical dispersion improves and approaches the analytical dispersion result.
b) Effect of number of cells per wavelength on numerical dispersion

Fig. 3.11 shows the normalised phase velocity with respect to wave angle $\varphi$ for different number of cells per wavelength. From Fig. 3.11, it can be seen that the dispersion error of the LOD-NFDTD method at higher CFLN can be reduced by increasing the number of cells per wavelength. In brief, a larger CFLN can be used by increasing the number of cell per wavelength to minimise dispersion error. Although a greater number of cells per wavelength increases the computational burden for the standard explicit FDTD compared to the LOD-NFDTD method, the computational cost in the proposed method can be compensated for using the larger CFLN. This approach has also been used with the 3-D LOD-FDTD method to minimise the dispersion error.


Fig. 3.10 Normalised phase velocity vs. wave angle for different CFLN


Fig. 3.11 Normalised phase velocity vs. wave angle for different number of cell per wavelength

### 3.9 ADI-NFDTD Method

To make comparison with the proposed LOD-NFDTD method, the ADI-NFDTD [70] has also been implemented for the same non-orthogonal meshes to calculate RCS for the 2-D cylindrical structures considered here. The CPML absorbing boundary conditions for ADI-NFDTD have also been derived and used to truncate the computational domain. The main aim is to draw a comparison between the LODNFDTD and ADI-NFDTD methods. The equation for the ADI-NFDTD CPML will be provided in Appendix A.

### 3.10 Computational Results on Scattering from 2-D Electromagnetic Structures

### 3.10.1 EM Scattering from Circular Conducting, Dielectric and Elliptic Dielectric Cylinders

In this section, computational results obtained using the proposed method are described for EM scattering problems in both the frequency domain and the time domain. First, we consider plane wave scattering from a PEC cylinder as shown in Fig. 3.12 (a). The non-orthogonal grid for the conducting circular cylinder is shown in Fig. 3.12 (b). The nonorthogonal meshes for this conducting cylinder are generated following the formulation as described in Section 3.3. The 8 layer CPML region is extended $0.5 \lambda$ away from the surface of the scatterer. The cell size chosen for this problem is $\lambda / 10$ and the same cell size is used.

(a)

(b)

Fig. 3.12 (a) Plane wave incidence on a conducting cylinder (b) Non-orthogonal mesh for the 2-D conducting circular cylinder


Fig. 3.13 (a) Bistatic scattering for CFLN=5, 8 (b) Comparison of bistatic scattering for CFLN=8 among different techniques for the circular conducting cylinder (radius $1.6 \lambda$ ) for $\mathrm{TE}_{\mathrm{z}}$ case

Fig. 3.13 (a) shows the results of bistatic radar cross section for a circular conducting cylinder of radius $1.6 \lambda$ for the $\mathrm{TE}_{z}$ wave for different CFLN=5, 8 and Fig. 3.13 (b) shows the comparison of the RCS for the same cylinder obtained using LOD-NFDTD, ADI-NFDTD and Mie series. Now, we present transient scattering from the circular conducting cylinder illuminated by a Gaussian plane wave using the LOD-NFDTD method. The results are compared with the data obtained from the literature as well as the results obtained using the ADI-NFDTD method. The incident field is a Gaussian plane wave of the form as:

$$
\begin{equation*}
\mathbf{E}^{i n c}(\mathbf{r}, t)=\mathbf{E}_{0} \frac{4}{T \sqrt{\pi}} \exp \left(-\left[\frac{4}{T}\left(c t-c t_{0}-\mathbf{r} \cdot \hat{\mathbf{k}}\right)\right]^{2}\right) \tag{3.64}
\end{equation*}
$$

where c is the velocity of propagation in the external medium, $\hat{\mathbf{k}}$ is a unit vector in the direction of propagation of the incident wave, $T$ is the pulse width of the Gaussian pulse, and at $t=t_{0}$ the Gaussian pulse reaches its maximum value. The results shown here were obtained with $E_{0}=1.0, T=2.0 \mathrm{LM}$ and $c t_{0}=3.0 \mathrm{LM}$. Note that 1 light metre (LM) is the unit of time taken by the electromagnetic wave to travel 1.0 m distance in space. Fig. 3.14 (a)-(b) shows the transient response obtained using LOD-NFDTD for CFLN $=2$ and 10 compared with ADI-NFDTD when the perfect conducting cylinder of radius $1.6 \lambda$ for $\mathrm{TE}_{z}$ wave is illuminated by a Gaussian pulse. From Figs. 3.13-3.14, it can be seen that the results on scattered fields and the transient scattering at higher CFLN agree well with the results obtained from ADI-NFDTD and Mie series for TE $_{z}$ waves. It can also be observed that the agreement between the LOD-NFDTD and ADINFDTD methods is fairly good both at early time and late time response.

(a)

(b)

Fig. 3.14 (a)-(b) Transient scattering at spatial location ( $0.05 \mathrm{~m}, 0.15 \mathrm{~m}$ ) for different CFLN for TE case

Similarly, the proposed method is also used for computing the scattering from the circular conducting cylinder for the $\mathrm{TM}_{\mathrm{z}}$ case in both the time and frequency domains.


Fig. 3.15 Bistatic scattering for the PEC cylinder (radius of $0.795 \lambda$ ) for $\mathrm{CFLN}=8$ for $\mathrm{TM}_{\mathrm{z}}$ case

(a)

(b)

Fig. 3.16 (a)-(b) Transient scattering at spatial location ( $0.05 \mathrm{~m}, 0.15 \mathrm{~m}$ ) for the circular conducting cylinder (radius of $0.795 \lambda$ ) for TM case for different CFLN for $\mathrm{TM}_{\mathrm{z}}$ case

Fig. 3.15 shows the bistatic radar cross section (RCS) for the $\mathrm{TM}_{\mathrm{z}}$ wave obtained using the LOD-NFDTD method for CFLN= 8 and compared with the results obtained by ADI-NFDTD as well as Mie series. Fig. 3.16 (a)-(b) shows the magnitude of the normalised transient current on a circular conducting cylinder of radius $0.795 \lambda$. From Figs. 3.15-3.16, it can be seen that the results on scattered fields and transient current at higher CFLN agree well with the results obtained from ADI-NFDTD and Mie series [156] for $\mathrm{TM}_{\mathrm{z}}$ case. Now we analyse the bistatic scattering from a circular dielectric cylinder. The non-orthogonal mesh for the dielectric cylinder is shown Fig. 3.17 (a). The parameters of the cylinder are $a=0.3 \lambda, \varepsilon_{r}=2.0, f=300 \mathrm{MHz}$. The computed bistatic RCS for the dielectric cylinder for CFLN $=8$ for both the TE and TM cases are shown in Fig. 3.17 (b). Figs. 3.18 (a)-(b) and 3.19 (a)-(b) show the transient response at spatial location $(0.01 \mathrm{~m}, 0.1 \mathrm{~m})$ for CFLN=10 and 12 respectively for the TE and TM cases for Gaussian pulse incidence.

(a)

(b)

Fig. 3.17 (a) Non-orthogonal mesh and (b) bistatic scattering from circular dielectric cylinder (radius $0.3 \lambda$ )


Fig. 3.18 (a)-(b) Transient scattering at spatial location $(0.1 \mathrm{~m}, 0.1 \mathrm{~m})$ for different CFLN for TE case


Fig. 3.19 (a)-(b) Transient scattering at spatial location ( $0.01 \mathrm{~m}, 0.1 \mathrm{~m}$ ) for different CFLN for TM case

The results are compared with the results obtained using the ADI-NFDTD method. From Fig. 3.17 (b), it can be seen that the results obtained by LOD-NFDTD with higher CFLN agrees reasonably well with the results available in the literature [158] as well as with the results obtained by the ADI-NFDTD method. As far as the transient response of the dielectric cylinder is concerned, it can also be observed that the agreement between LOD-NFDTD and ADI-NFDTD methods is fairly good both at early and late times. To further validate the proposed LOD-NFDTD method for EM scattering problems, we consider a dielectric elliptic cylinder with larger curvature. The nonorthogonal mesh for the dielectric elliptic cylinder is shown in Fig. 3.20 (a). The major and minor axes considered for the cylinders are $a=0.2 \lambda, b=0.15 \lambda$, respectively and other parameters are $\varepsilon_{r}=2$ and $\mu_{r}=1$, where $\lambda$ is the wavelength of the incident wave. The computed bistatic RCS is shown in Fig. 3.20 (b).

(b)

Fig. 3.20 (a) Non-orthogonal mesh (b) Bistatic scattering from the dielectric elliptic cylinder for TM incidence

As can be observed from the Fig. 3.20 (b), the results obtained by the LOD-NFDTD method with higher CFLN agrees reasonably well with the published result [159].

### 3.10.2 EM Scattering from Dielectric Coated Conducting and Two Layered Elliptic Cylinders

To validate the proposed method for a coated cylinder, a dielectric coated circular conducting cylinder is considered. Our numerical results on scattering from the dielectric coated circular conducting cylinder will be compared with the existing results in the literature obtained by using analytical approximations [160]. The parameters of the dielectric coated conducting cylinders are $a=50 \mathrm{~mm}, b=100 \mathrm{~mm}$, and $\varepsilon_{r}=9.8$. Fig. 3.21 (a) shows the non-orthogonal grids which are used to discretise the geometry.

(a)

(b)

Fig. 3.21 (a) Non-orthogonal mesh, (b) Bistatic scattering from circular dielectric coated conducting cylinder for $\mathrm{CFLN}=5,8$ for both TM and TE cases

The bistatic scattering of the coated cylinder for TE and TM waves is shown in Fig. 3.21 (b).

(a)

(b)

Fig. 3.22 (a)-(b) Transient scattering at spatial location $(10 \mathrm{~mm}, 110 \mathrm{~mm})$ at different CFLN for TE case
The results of the proposed LOD-NFDTD method agrees reasonably well with the available analytical results provided in [160] as well as with the results obtained by the ADI-NFDTD method.


Fig. 3.23 (a)-(b) Transient scattering at spatial location ( $10 \mathrm{~mm}, 110 \mathrm{~mm}$ ) at different CFLN for TM case

Figs. 3.22 (a)-(b) and 3.23 (a)-(b) show the transient scattering from the coated cylinder and the results are compared with the result obtained by the ADI-NFDTD method for both the TM and TE cases respectively. To further validate the proposed LOD-NFDTD method for EM scattering problems, we consider a two layered dielectric elliptic cylinder with larger radius of curvature. The parameters of the cylinder are chosen to be same as those given in [161]. The first and second layers of the dielectric elliptic cylinder are characterised by $\varepsilon_{r 1}=2.0, \varepsilon_{r 2}=1.4$ respectively. The nonorthogonal mesh for the two layered dielectric elliptic cylinder is shown in Fig. 3.24 (a). The computed bistatic RCS is shown in Fig. 3.24 (b).


Fig. 3.24 (a) Non-orthogonal mesh, (b) Bistatic scattering from the two layered elliptic dielectric cylinder for $\mathrm{CFLN}=8$ for TM wave.

Fig. 3.25 (a)-(b) shows the transient scattering from the two layered dielectric elliptic cylinder and the results are compared with the result obtained by the ADI-NFDTD method. From Fig. 3.24 (b), it can be seen that the result obtained by the LOD-NFDTD method agrees reasonably well with the analytical result. From Fig. 3.25 (a)-(b), it can be observed that the time domain solutions of the layered elliptic cylinder are stable in early time.


Fig. 3.25 (a)-(b) Transient scattering at point ' $T$ ' on the two layer dielectric elliptic cylinder at different CFLN for TM case for CFLN=4, 10

### 3.10.3 EM Scattering from Overfilled Dielectric Cavity Structures

To validate our proposed method for solving EM scattering for more complex mixed structures, a 2-D overfilled dielectric cavity embedded in the ground plane is considered. This type of structure is highly applicable for the design of cavity backed conformal antennas for civil and military use, and the characterisation of radar cross section (RCS) of vehicles with grooves [162]. The mixed-dielectric-conducting structure in the form of a 2-D overfilled dielectric cavity is embedded in a grounded bowl-like PEC structure [162] which is shown in Fig. 3.26. The bowl-like PEC circular cylindrical cavity structure is filled with a circular dielectric cylinder of radius 0.5 m having permittivity $\varepsilon_{1}=4$-i.


Fig. 3.26 Non-orthogonal mesh of the overfilled dielectric cavity

A dielectric semi-circular cylinder of radius 1 m and permittivity $\varepsilon_{2}=1.2$ encloses the cavity as shown in Fig 3.26 [162]. The non-orthogonal mesh employed is shown in Fig. 3.26.

(b)


Fig. 3.27 (a)-(c) RCS of the overfilled dielectric cavity for TE incidence wave for different CFLN

Fig. 3.27 (a)-(c) shows the bistatic radar cross section (RCS) for $\mathrm{TE}_{\mathrm{z}}$ incidence obtained using the LOD-NFDTD method for different CFLN and compared with the results obtained by the ADI-NFDTD as well as the results published in [162].

(a)

(b)


Fig. 3.28 (a)-(d) RCS of the overfilled dielectric cavity for TM incidence for different CFLN

Fig. 3.28 (a)-(d) shows the results on a bistatic radar cross section (RCS) of overfilled cavity for $\mathrm{TM}_{\mathrm{z}}$ wave. From Figs. 3.27-3.28, it can be seen that the results obtained from the proposed method agree reasonably well with the results obtained from the ADINFDTD method as well as the available analytical results provided in [163]. Using the analytical solution provided in [162], [163] for the TE and TM cases as reference, the maximum errors of the LOD-NFDTD and ADI-NFDTD methods for different CFLN are calculated as shown in Table 3.1 and Table 3.2. From the comparison of maximum errors, it can be concluded that the proposed LOD-NFDTD method provides less error compared to conventional ADI-NFDTD method.

Table 3.1
Comparison of the maximum errors in RCS calculation of overfilled dielectric cavity using LOD-NFDTD and ADI-NFDTD for $\mathrm{TE}_{z}$ incidence

| CFLN | \% Maximum error of <br> LOD-NFDTD Method | \% Maximum error of ADI- <br> NFDTD Method |
| :--- | :---: | :---: |
| 2 | 0.015 | 0.027 |
| 4 | 0.018 | 0.032 |
| 6 | 0.108 | 0.119 |
| 8 | 0.345 | 0.425 |
| 10 | 0.401 | 0.512 |

Table 3.2
Comparison of the maximum errors in RCS calculation of overfilled dielectric cavity using LOD-NFDTD and ADI-NFDTD for $\mathrm{TM}_{z}$ incidence

| CFLN | \% Maximum error of <br> LOD-NFDTD Method | \% Maximum error of ADI- <br> NFDTD Method |
| :--- | :---: | :---: |
| 2 | 0.002 | 0.008 |
| 4 | 0.017 | 0.025 |
| 6 | 0.104 | 0.151 |
| 8 | 0.345 | 0.515 |
| 10 | 0.423 | 0.672 |


(a)


Fig. 3.29 (a)-(d) Transient scattering from the structure shown in Fig. 3.26 at spatial location ( -0.4 m , 0.1 m ) obtained using LOD-NFDTD method at different CFLN for TE incidence

The transient scattering from the overfilled cavity as obtained by the proposed LODNFDTD method is compared with the results obtained using the ADI-NFDTD methods as shown in Fig. 3.29 (a)-(d).

### 3.10.4 EM Scattering from Overfilled Cavity for Different Dielectric Filling

We include EM scattering for an overfilled cavity having different dielectric filling using the LOD-NFDTD method. Referring to Fig. 3.26, in this example we assume $\varepsilon_{1}=4$-i and $\varepsilon_{2}=2$. Fig. 3.30 (a)-(b) shows the results of RCS for this case. Here, we consider only TE incidence but vary CFLN. The results indicate the suitability of the proposed method for analysing the scattering from complex mixed structures.


Fig. 3.30 (a)-(b) RCS of the overfilled cavity for different dielectric filling for $\mathrm{TE}_{z}$ case


Fig. 3.31 (a)-(b) Transient scattering at spatial location ( $-0.4 \mathrm{~m}, 0.1 \mathrm{~m}$ ) from the overfilled cavity with different dielectric filling obtained using LOD-NFDTD method. TE illumination considered

We have computed the results using the LOD-NFDTD method on transient scattering from the overfilled cavity (refer to Fig. 3.26) when it is filled with different dielectric media $\varepsilon_{1}=4$-i and $\varepsilon_{2}=2$. The results for different CFLN are shown in Fig. 3.31 (a)-(b).

### 3.10.5 EM Scattering from 2-D Bent PEC Cavity Structures

The next complex problem that we investigate is EM scattering using the LODNFDTD method at 10 GHz from the bent PEC cavity structure shown in Fig. 3.32 (a). The non-orthogonal mesh of the structure is shown in Fig. 3.32 (b). The bistatic radar
cross section of the bent cavity for both TE and TM incidence are shown in Figs. 3.33 (a)-(b) and 3.34 (a)-(b) respectively. From these figures, it can be observed that the results obtained using the LOD-NFDTD method are in good agreement with the results obtained using the ADI-NFDTD method and also with the available published results that are obtained by the MoM/EFIE approaches [164]. Using the analytical solution provided in [164] for the TM case as a reference, the maximum errors of the LODNFDTD and ADI-NFDTD method for different CFLN are calculated and tabulated in Table 3.3.

(a)

(b)

Fig. 3.32 Geometry of the bent cavity (unit in cm ) (b) Non-orthogonal meshes of the bent cavity structure


(b)

Fig. 3.33 (a)-(b) RCS obtained using LOD-NFDTD method at different CFLN for perpendicular polarisation


Fig. 3.34 (a)-(b) RCS obtained using LOD-NFDTD method at different CFLN for parallel polarisation

From the comparison of maximum errors, it is seen that the proposed LOD-NFDTD method offers lesser error compared to ADI-NFDTD method.

Table 3.3
Comparison of maximum errors for calculating of RCS of bent cavity using LOD-
NFDTD and ADI-NFDTD for $\mathrm{TM}_{\mathrm{z}}$ wave

| CFLN | \% Maximum error of <br> LOD-NFDTD Method | \% Maximum error of <br> ADI-NFDTD Method |
| :--- | :---: | :---: |
| 2 | 0.003 | 0.007 |
| 4 | 0.011 | 0.023 |
| 6 | 0.114 | 0.155 |
| 8 | 0.364 | 0.612 |
| 12 | 0.762 | 0.982 |

Figs. 3.35 (a)-(b) and 3.36 (a)-(b) show the transient scattering at point ' P ' and the results are compared with the results obtained by the ADI-NFDTD method. From Fig. 3.35 (a)-(b), it can be seen that the solution is stable in the early time, but small ripples can be observed in the late time at location ' P ' where there is transition from the straight region of the cavity to the bent region. For the larger time steps, however, these ripples disappear fully as can be observed from the Fig. 3.35 (b) and 3.36 (b).

(a)

(b)

Fig. 3.35 (a)-(b) Transient scattering at point ' P ' on the bent cavity obtained using LOD-NFDTD method at different CFLN for TE incidence


Fig. 3.36 (a)-(b) Transient scattering at point ' P ' on the bent cavity obtained using LOD-NFDTD method at different CFLN for TM incidence

### 3.11 Discussion

This chapter deals with the investigation of a new implicit nonorthogonal mesh LODFDTD (LOD-NFDTD) for solving 2-D EM scattering problems. This technique overcomes some of the limitations of the standard explicit FDTD and ADI-NFDTD methods. A theoretical study of the LOD-NFDTD has been performed for both TE and TM cases. It is found that nonorthogonal meshes are more advantageous for modelling objects with curved features and can reduce computational burden than using orthogonal meshes. CPML absorbing boundary conditions for the LOD-NFDTD method have also been developed to improve the efficiency. A new pure scattered field formulation of the 2-D LOD-NFDTD has also been presented and the stability and dispersion of the method have been analysed.

The new LOD-NFDTD has been applied to EM scattering from circular conducting, dielectric, coated conducting and layered elliptic cylinders as well as overfilled dielectric cavity and bent PEC cavity structures. From the analysis, it is observed that the proposed method is unconditionally stable and the numerical results correspond closely to the results available in the literature as well as with the ADI-NFDTD results. Compared with ADI-NFDTD, the proposed method is characterised by a lighter calculation burden and higher accuracy.

## Chapter 4

# Rotationally Symmetric LOD-FDTD with Dispersion Control Parameters 

### 4.1 Introduction

The purpose of this chapter is to present a simplified LOD-FDTD algorithm to analyse various rotationally symmetric (RS) three-dimensional (3-D) structures with improved dispersion control. 3-D microwave structures with RS are commonly encountered in many electromagnetic and wireless applications in the form of antennas, cylindrical cavity, and microwave filters [29]-[30], [178]. However, among the most popular numerical techniques for analysing the RS resonant structures, the mode matching method, integral equation technique, and the finite element method are well known [2], [29]-[30]. The RS standard explicit finite-difference time domain (RSFDTD) method has also been used effectively for treating electromagnetic problems in the time domain, involving structures with circular symmetry [29]-[33]. However, explicit RS-FDTD suffers from the CFL stability constraint and as a result, finer grid sizes and smaller time steps are required to retain stability which causes a significant increase in computational time. To eliminate the dependence on the CFL stability constraint, an unconditionally stable ADI-FDTD method has been developed in the cylindrical coordinate system [73], but the cylindrical ADI-FDTD presented in [73] requires a much larger memory because it solves problems in a three dimensional domain. To overcome this difficulty, the BOR/(RS)-ADI-FDTD method was developed in [74]. To compute the radiation or scattering from the complex body of revolution, PML was presented for the RS-ADI-FDTD [75] method. The RS-ADI-FDTD method was then extended to dispersive media in [76]. However, the RS-ADI-FDTD may require more arithmetic operations and shows large dispersion error at larger time steps.

To overcome some of these difficulties in analysing rotationally symmetric structures, RS locally one dimensional FDTD (RS-LOD-FDTD) was developed in [115]. The usefulness of the RS-LOD-FDTD method was investigated through the analysis of circular cavity resonators with and without a dielectric disc, along with the explicit RS-FDTD and RS-ADI-FDTD. However, only the analysis of the circular cavity resonator is given in [115], and no ABCs were mentioned with the RS-LODFDTD method. The RS-LOD-FDTD algorithm was also extended for Debye dispersive media in [116]. The RS-LOD-FDTD based on fundamental scheme was also introduced in [117].

However, RS-LOD-FDTD exhibits dispersion error at larger time steps similar to RS-ADI-FDTD for analysing resonant structures [115]. To improve the dispersion error and reduce execution time, dispersion control parameters with RS-LOD-FDTD have been introduced in this chapter, which we call D-RS-LOD-FDTD. To date, only conventional ABCs such as MURs and PML ABC have been proposed with the RS-LOD-FDTD method in the literature [115]-[117] for modelling bodies with RS. It has been established that employing conventional ABCs such as PML etc. do not lead to improved solution accuracies; hence, there is a need to develop an efficient technique with a CPML ABC for analysing RS structures using D-RS-LOD-FDTD method.

In this chapter, we first present conventional RS-LOD-FDTD and CPML ABC for RS-LOD-FDTD in Sections 4.2, and 4.3 respectively for both the $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ polarization. Next, the theory of D-RS-LOD-FDTD for both $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ cases is discussed in Section 4.4. The formulation of the CPML ABC for the D-RS-LOD-FDTD for $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ cases is presented in Sub-sections 4.5.1 and 4.5.2. Methods to obtain S-parameters and specific absorption rate (SAR) using the D-RS-LOD-FDTD methods are provided in Sections 4.6 and 4.7. Computational results on various RS structures are demonstrated in Section 4.8. Performance analysis of the D-RS-LOD-FDTD CPML is discussed in Section 4.9. Finally, a brief discussion is provided in Section 4.10.

### 4.2 Conventional Rotationally Symmetric LOD-FDTD for $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{\mathbf{0 n}}$ Mode Analysis

### 4.2.1 Maxwell's Curl Equations in Cylindrical Coordinates

The problems considered in this chapter are symmetric about an axis. RS structures allow the analytical extraction of the known azimuthal behaviour of the fields around the axis of symmetry and the projection of the original three-dimensional (3-D) problem to a numerically solvable two-dimensional (2-D) plane. However, under this condition, original three-dimensional (3-D) problems can be represented in two and a half dimensional (2.5-D) forms. In a $2.5-\mathrm{D}$ problem, there are six components of electromagnetic field distributed in the $(r, z)$ space. The starting point for the derivation of the RS-LOD-FDTD method is Maxwell's equations in cylindrical coordinates, which include magnetic and electrical conductivity for a source free region of space that is linear, isotropic and non-dispersive is given by:

$$
\begin{array}{r}
\varepsilon \frac{\partial \vec{E}}{\partial t}+\sigma \vec{E}=\nabla \times \vec{H} \\
\mu \frac{\partial \vec{H}}{\partial t}+\sigma_{m} \vec{H}=-\nabla \times \vec{E} \tag{4.1b}
\end{array}
$$

The fields in circularly symmetric structures can be expanded in a Fourier series of sine and cosine form as:

$$
\begin{array}{r}
E=\sum_{m=0}^{\infty}\left(E_{u} \cos m \phi+E_{v} \sin m \phi\right) \\
H=\sum_{m=0}^{\infty}\left(H_{u} \cos m \phi+H_{v} \sin m \phi\right) \tag{4.2b}
\end{array}
$$

Substituting (4.2) into (4.1), we obtain the following equations:

$$
\begin{gather*}
\pm \frac{m}{r} \hat{\phi} \times H_{v, u}+\nabla \times H_{u, v}=\varepsilon \frac{\partial}{\partial t} E_{u, v}+\sigma E_{u, v}  \tag{4.3a}\\
\pm \frac{m}{r} \hat{\phi} \times E_{v, u}+\nabla \times E_{u, v}=-\mu \frac{\partial}{\partial t} H_{u, v}-\sigma_{m} H_{u, v} \tag{4.3b}
\end{gather*}
$$

Expanding these vector curl equations (4.3) into six coupled scalar equations in the cylindrical coordinate system ( $r, \phi, z$ ) gives:

$$
\begin{gather*}
\varepsilon \frac{\partial E_{r}}{\partial t}=\frac{1}{\rho} \frac{\partial H_{z}}{\partial \phi}-\frac{\partial H_{\phi}}{\partial z}-\sigma_{z} E_{r}  \tag{4.4a}\\
108
\end{gather*}
$$

$$
\begin{gather*}
\varepsilon \frac{\partial E_{\phi}}{\partial t}=\frac{\partial H_{r}}{\partial z}-\frac{\partial H_{z}}{\partial r}-\sigma E_{\phi}  \tag{4.4b}\\
\varepsilon \frac{\partial E_{z}}{\partial t}=\frac{1}{r} \frac{\partial}{\partial r}\left(r H_{\phi}\right)-\frac{1}{r} \frac{\partial H_{r}}{\partial \phi}-\sigma E_{z}  \tag{4.4c}\\
\mu \frac{\partial H_{r}}{\partial t}=\frac{\partial E_{\phi}}{\partial z}-\frac{1}{r} \frac{\partial E_{z}}{\partial \phi}-\sigma_{m} H_{\rho}  \tag{4.4d}\\
\mu \frac{\partial H_{\phi}}{\partial t}=\frac{\partial E_{z}}{\partial r}-\frac{\partial E_{r}}{\partial z}-\sigma_{m} H_{\phi}  \tag{4.4e}\\
\mu \frac{\partial H_{z}}{\partial t}=\frac{1}{r} \frac{\partial E_{r}}{\partial \phi}-\frac{1}{r} \frac{\partial}{\partial r}\left(r E_{\phi}\right)-\sigma_{m} H_{z} \tag{4.4f}
\end{gather*}
$$

The rotationally symmetric $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ equations can be derived from the equations (4.4a)-(4.4f). In the RS-LOD-FDTD method, it is assumed that the angular variation of the electromagnetic fields has either $\sin (m \varphi)$ or $\cos (m \varphi)$ which may be factored out of Maxwell's equations. As a result, the fields at any arbitrary $\phi=\phi_{0}$ plane may be related to the corresponding field value in the reference $\phi$ plane, viz. $\phi=0$. This enables us to reduce the original 3-D problem to an equivalent 2-D problem, with reference to say the $\phi=0$ plane. We begin analysing using the 3-D cylindrical coordinate system as shown in Fig. 4.1 (a), and project it onto the r-z plane to obtain a 2-D lattice as shown in Fig. 4.1 (b). Note that in the 2-D cell $\left(E_{z}, H_{r}\right)$ and ( $E_{r}, H_{z}$ ) share the same positions. According to the LOD-FDTD principle [20]-[22] described in chapter 2, the field components for the RS-LOD-FDTD method can thus be written as follows. For simplicity, we will examine the case for $m=0$, so that the RS-LOD-FDTD equations are derived as:
First step:

$$
\begin{gather*}
E_{r}^{n+1 / 2}=E_{r}^{n}  \tag{4.5a}\\
\frac{E_{\phi}^{n+1 / 2}-E_{\phi}^{n}}{\Delta t / 2}+\frac{\sigma}{\varepsilon} \frac{E_{\phi}^{n+1 / 2}+E_{\phi}^{n}}{2}=\frac{1}{\varepsilon}\left(\frac{\partial H_{r}^{n+1 / 2}}{\partial z}+\frac{\partial H_{r}^{n}}{\partial z}\right)  \tag{4.5b}\\
\frac{E_{z}^{n+1 / 2}-E_{z}^{n}}{\Delta t / 2}+\frac{\sigma}{\varepsilon} \frac{E_{z}^{n+1 / 2}+E_{z}^{n}}{2}=\frac{1}{\varepsilon r} \frac{\partial\left(r H_{\phi}^{n+1 / 2}\right)}{\partial r}+\frac{1}{\varepsilon r} \frac{\partial\left(r H_{\phi}^{n}\right)}{\partial r}  \tag{4.5c}\\
\frac{H_{r}^{n+1 / 2}-H_{r}^{n}}{\Delta t / 2}+\frac{\sigma_{m}}{\mu} \frac{H_{r}^{n+1 / 2}+H_{r}^{n}}{2}=\frac{1}{\mu} \frac{\partial E_{\phi}^{n+1 / 2}}{\partial z}+\frac{1}{\mu} \frac{\partial E_{\phi}^{n}}{\partial z} \tag{4.5d}
\end{gather*}
$$



Fig. 4.1 (a) A dielectric loaded cavity defined in 3-D cylindrical co-ordinates and (b) its projection onto the r-z plane for the analysis of rotationally symmetric $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ modes.

$$
\begin{gather*}
\frac{H_{\phi}^{n+1 / 2}-H_{\phi}^{n}}{\Delta t / 2}+\frac{\sigma_{m}}{\mu} \frac{H_{\phi}^{n+1 / 2}+H_{\phi}^{n}}{2}=\frac{1}{\mu} \frac{\partial E_{z}^{n+1 / 2}}{\partial r}+\frac{1}{\mu} \frac{\partial E_{z}^{n}}{\partial r}  \tag{4.5e}\\
H_{z}^{n+1 / 2}=H_{z}^{n} \tag{4.5f}
\end{gather*}
$$

Second step:

$$
\begin{gather*}
\frac{E_{r}^{n+1}-E_{r}^{n+1 / 2}}{\Delta t / 2}+\frac{\sigma}{\varepsilon} \frac{E_{r}^{n+1}+E_{r}^{n+1 / 2}}{2}=-\frac{1}{\varepsilon} \frac{\partial H_{\phi}^{n+1}}{\partial z}-\frac{1}{\varepsilon} \frac{\partial H_{\phi}^{n+1 / 2}}{\partial z}  \tag{4.6a}\\
\frac{E_{\phi}^{n+1}-E_{\phi}^{n+1 / 2}}{\Delta t / 2}+\frac{\sigma}{\varepsilon} \frac{E_{\phi}^{n+1}+E_{\phi}^{n+1 / 2}}{2}=-\frac{1}{\varepsilon}\left(\frac{\partial H_{z}^{n+1}}{\partial r}+\frac{\partial H_{z}^{n+1 / 2}}{\partial r}\right)  \tag{4.6b}\\
E_{z}^{n+1}=E_{z}^{n+1 / 2}  \tag{4.6c}\\
H_{r}^{n+1}=H_{r}^{n+1 / 2}  \tag{4.6d}\\
\frac{H_{\phi}^{n+1}-H_{\phi}^{n+1 / 2}}{\Delta t / 2}+\frac{\sigma_{m}}{\mu} \frac{H_{\phi}^{n+1}+H_{\phi}^{n+1 / 2}}{2}=-\frac{1}{\mu} \frac{\partial E_{r}^{n+1}}{\partial z}-\frac{1}{\mu} \frac{\partial E_{r}^{n+1 / 2}}{\partial z}  \tag{4.6e}\\
\frac{H_{z}^{n+1}-H_{z}^{n+1 / 2}}{\Delta t / 2}+\frac{\sigma_{m}}{\mu} \frac{H_{z}^{n+1}-H_{z}^{n+1 / 2}}{2}=-\frac{1}{\mu r} \frac{\partial\left(r E_{\phi}^{n+1}\right)}{\partial r}-\frac{1}{\mu r} \frac{\partial\left(r E_{\phi}^{n+1 / 2}\right)}{\partial r} \tag{4.6f}
\end{gather*}
$$

For $m=0$, we must pay attention to handling the singular point at $r=0$. Instead of (4.5c), we adopt the following equation in accordance with LOD implementation:

$$
\begin{equation*}
E_{z}^{n+1 / 2}=E_{z}^{n}+\frac{4 \Delta t}{\varepsilon \Delta r} H_{\phi}^{n+1 / 2} \tag{4.7}
\end{equation*}
$$

The derivation for the $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ modes of the RS-LOD-FDTD method are given next.

### 4.2.2 Formulation of RS LOD-FDTD Algorithm for TE $_{0 \mathrm{n}}$ Mode Analysis

Fig. 4.1 (b) shows the projection of a 3-D rotationally symmetric object onto the r-z plane. Assume that the fields may be expressed as a linear combination of $\mathrm{TE}_{0 \mathrm{n}}$ modes. The resulting fields will have no variation with angular variable $\phi$ and hence may be expressed as:

$$
\begin{equation*}
H_{r}(r, z, t), \quad H_{z}(r, z, t), \quad E_{\phi}(r, z, t) \tag{4.8}
\end{equation*}
$$

Note that in (4.8), the fields in three dimensions are represented by only two spatial dimensions. For the TE mode the equation of the RS-LOD-FDTD method can be derived as follows:

Sub-step 1:

$$
\begin{align*}
&\left.E_{\phi}\right|_{i+1 / 2, j} ^{n+1 / 2}=\left.C_{e \phi e} E_{\phi}\right|_{i+1 / 2, j} ^{n} \\
&+\frac{C_{e \phi \phi r}}{\Delta z}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)+\frac{C_{e \phi h r}}{\Delta z}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)  \tag{4.9a}\\
&\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.C_{h r h} H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
&+\frac{C_{h r e \phi}}{\Delta z}\left(\left.E_{\phi}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{\phi}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)+\frac{C_{h r e \phi}}{\Delta z}\left(\left.E_{\phi}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{\phi}\right|_{i+1 / 2, j} ^{n}\right)  \tag{4.9b}\\
&\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n} \tag{4.9c}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E_{\phi}\right|_{i, j+1 / 2} ^{n+1}=\left.C_{e \phi e} E_{\phi}\right|_{i, j+1 / 2} ^{n+1 / 2} \\
& \quad-\frac{C_{e \phi h z}}{\Delta r}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1}\right)-\frac{C_{e \phi h z}}{\Delta r}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)  \tag{4.10a}\\
& \qquad\left.H_{r}\right|_{i, j+1 / 2} ^{n+1}=\left.H_{r}\right|_{i, j+1 / 2} ^{n+1 / 2} \tag{4.10b}
\end{align*} H_{\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.C_{h z h} H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}}^{-\frac{C_{h z e \phi}}{r_{i} \Delta r}\left(\left.r_{i+1} E_{\phi}\right|_{i+1, j+1 / 2} ^{n+1}-\left.r_{i} E_{\phi}\right|_{i, j+1 / 2} ^{n+1}\right)-\frac{C_{h z e \phi}}{r_{i} \Delta r}\left(\left.r_{i+1} E_{\phi}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.r_{i-1 / 2} E_{\phi}\right|_{i, j+1 / 2} ^{n+1 / 2}\right)}
$$

where $C_{\text {epe }}=\frac{(4 \varepsilon-\sigma \Delta t)}{(4 \varepsilon+\sigma \Delta t)}, C_{\text {ephr }}=\frac{2 \Delta t}{(4 \varepsilon+\sigma \Delta t)}, C_{\text {hrh }}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)}, C_{\text {hreq }}=\frac{2 \Delta t}{(4 \mu+\sigma \Delta t)}$
The equations of sub-step 1 cannot be solved directly, so substituting (4.9b) into (4.9a) yields the simultaneous linear equations (4.11) for $\left.E_{\phi}\right|^{n+1 / 2}$ that result in the tri-diagonal matrix which can be solved implicitly, and (4.9b) can then be explicitly solved for $\left.H_{r}\right|^{n+1 / 2}$.

$$
\begin{align*}
&-\left.\frac{C_{e \phi h r} C_{h r e \phi}}{\Delta z^{2}} E_{\phi}\right|_{i+1 / 2, j+1} ^{n+1 / 2}+\left.\left(1+2 \frac{C_{e \phi h r} C_{h r e \phi}}{\Delta z^{2}}\right) E_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.\frac{C_{e \phi h r} C_{\text {hre }}}{\Delta z^{2}} E_{\phi}\right|_{i+1 / 2, j-1} ^{n+1 / 2} \\
&=\left.C_{e \phi e} E_{\phi}\right|_{i+1 / 2, j} ^{n}  \tag{4.11}\\
&+\frac{C_{e \phi h r} C_{h r h}}{\Delta z}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \\
& \Delta z^{2}\left.\left.\mathrm{E}_{\phi}\right|_{i+1 / 2, j+1} ^{n}-\left.\mathrm{E}_{\phi}\right|_{i+1 / 2, j} ^{n}-\left.\mathrm{E}_{\phi}\right|_{i+1 / 2, j} ^{n}+\left.\mathrm{E}_{\phi}\right|_{i+1 / 2, j-1} ^{n}\right) \\
&+\frac{C_{e \phi h r}}{\Delta z}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)
\end{align*}
$$

Similarly, in sub-step 2 we substitute (4.10c) into (4.10a) and implicitly solve the resultant tri-diagonal system for $\left.E_{\phi}\right|^{n+1}$, and then explicitly solve (4.10c) for $\left.H_{z}\right|^{n+1}$. The derivations for the $\mathrm{TM}_{0 \mathrm{n}}$ mode of RS-LOD-FDTD are discussed next.

### 4.2.3 Formulation of RS LOD-FDTD Algorithm for TM $_{0 \mathrm{n}}$ Mode Analysis

The equations for RS-LOD-FDTD for the TM case are derived in the same way as that of the TE case. The electric and magnetic field components for sub-steps 1 and 2 are shown next.

Sub-step 1:

$$
\begin{equation*}
\left.E_{r}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.E_{r}\right|_{i, j+1 / 2} ^{n} \tag{4.12a}
\end{equation*}
$$

$$
\begin{align*}
\left.E_{z}\right|_{i, j+1 / 2} ^{n+1 / 2}=C_{e z e} & \left.E_{z}\right|_{i, j+1 / 2} ^{n} \\
& +\frac{C_{e z h \phi}}{r_{i} \Delta r}\left(\left.r_{i+1 / 2} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.r_{i-1 / 2} H_{\phi}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)  \tag{4.12b}\\
& +\frac{C_{e z h \phi}}{r_{i} \Delta r}\left(\left.r_{i+1 / 2} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.r_{i-1 / 2} H_{\phi}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)
\end{align*}
$$

Here, also for $m=0$, we must pay attention to handling the singular point at $r=0$. Instead of (4.12b), we adopt the following equation in accordance with LOD implementation.

$$
\begin{equation*}
\left.E_{z}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.E_{z}\right|_{i, j+1 / 2} ^{n}+\left.\frac{2 C_{e z h \phi}}{\Delta r} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \tag{4.12c}
\end{equation*}
$$

The other magnetic field component can be obtained as follows:

$$
\begin{align*}
& \left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.C_{h \phi h} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& \quad+\frac{C_{h \phi e z}}{\Delta r}\left(\left.E_{z}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i, j+1 / 2} ^{n+1 / 2}\right)+\frac{C_{h \phi e z}}{\Delta r}\left(\left.E_{z}\right|_{i+1, j+1 / 2} ^{n}-\left.E_{z}\right|_{i, j+1 / 2} ^{n}\right) \tag{4.12d}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E_{r}\right|_{i+1 / 2, j} ^{n+1}=\left.C_{e r e} E_{r}\right|_{i+1 / 2, j} ^{n+1 / 2} \\
& -\frac{C_{e r h \phi}}{\Delta z}\left(\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{\phi}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right)-\frac{C_{e r h \phi}}{\Delta z}\left(\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{\phi}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)  \tag{4.13a}\\
& \qquad\left.E_{z}\right|_{i+1 / 2, j} ^{n+1}=\left.E_{z}\right|_{i+1 / 2, j} ^{n+1 / 2}  \tag{4.13b}\\
& \left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.C_{h \phi h} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \\
& \quad-\frac{C_{h \phi e r}}{\Delta z}\left(\left.E_{r}\right|_{i+1 / 2, j+1} ^{n+1}-\left.E_{r}\right|_{i+1 / 2, j} ^{n+1}\right)-\frac{C_{h \phi e r}}{\Delta z}\left(\left.E_{r}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{r}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)  \tag{4.13c}\\
& \text { where } C_{e z e}=\frac{(4 \varepsilon-\sigma \Delta t)}{(4 \varepsilon+\sigma \Delta t)}, C_{e z h \phi}=\frac{2 \Delta t}{(4 \varepsilon+\sigma \Delta t)}, C_{h \phi h}=\frac{\left(4 \mu-\sigma_{m} \Delta t\right)}{\left(4 \mu+\sigma_{m} \Delta t\right)}, C_{h \phi e z}=\frac{2 \Delta t}{(4 \mu+\sigma \Delta t)}
\end{align*}
$$

Similar to the TE case, placing (4.12d) into (4.12b) yields the simultaneous linear equations that result in the tri-diagonal matrix in sub-step 1. Similarly, substituting (4.13c) into (4.13a) results in tri-diagonal matrix equations in sub-step 2. The CPML absorbing boundary conditions with RS-LOD-FDTD are formulated below.

### 4.3 CPML ABC for RS-LOD-FDTD Method

It is known that CPML ABC is highly effective in absorbing evanescent waves with long time signature and can be placed closer to objects within the problem space to gain time and memory savings. Chapters 2 and 3 discussed in detail the development of CPML ABCs for both the Cartesian and Curvilinear space lattices. This section describes the CPML ABCs to terminate the rotational symmetric (RS)-LOD-FDTD grid
in axial and radial directions. The theory discussed in this section will later be used to analyse RS structures.

### 4.3.1 Formulation of CPML ABC for TE $_{0 \mathrm{n}}$ RS-LOD-FDTD

Similar to the CPML ABC for Cartesian space lattice, and following the same procedure described in Chapters 2, we can derive the CPML equations for $\mathrm{TE}_{0 \mathrm{n}}$ RS-LOD-FDTD as follows.

Sub-step 1:

$$
\begin{align*}
\left.E_{\phi}\right|_{i+1 / 2, j} ^{n+1 / 2}= & \left.C_{e \phi e} E_{\phi}\right|_{i+1 / 2, j} ^{n}+\frac{1}{\kappa_{e z}} \frac{C_{e \phi \phi r}}{\Delta z}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right) \\
& +\frac{1}{\kappa_{e z}} \frac{C_{e \phi h r}}{\Delta z}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)+\left.C_{\psi e \phi z} \psi_{e \phi z}\right|_{i+1 / 2, j} ^{n}  \tag{4.14a}\\
\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}= & \left.C_{h r h} H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n}+\frac{1}{\kappa_{e z}} \frac{C_{h r e \phi}}{\Delta z}\left(\left.E_{\phi}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{\phi}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)  \tag{4.14b}\\
& +\frac{1}{\kappa_{e z}} \frac{C_{h r e \phi}}{\Delta z}\left(\left.E_{\phi}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{\phi}\right|_{i+1 / 2, j} ^{n}\right)+\left.C_{\psi h r z} \psi_{h r z}\right|_{i+1 / 2, j+1 / 2} ^{n}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
\left.E_{\phi}\right|_{i, j+1 / 2} ^{n+1}= & \left.C_{e \phi e} E_{\phi}\right|_{i, j+1 / 2} ^{n+1 / 2}-\frac{1}{\kappa_{e r}} \frac{C_{e \phi h z}}{\Delta r}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1}\right)  \tag{4.15a}\\
& -\frac{1}{\kappa_{e r}} \frac{C_{e \phi h z}}{\Delta r}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)-\left.C_{\psi e \phi r}\right|_{e \phi r} r_{i, j+1 / 2}^{n+1 / 2} \\
\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}= & \left.C_{h z h} H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\frac{1}{\kappa_{e r}} \frac{C_{h z e \phi}}{r_{i} \Delta r}\left(\left.r_{i+1} E_{\phi}\right|_{i+1, j+1 / 2} ^{n+1}-\left.r_{i} E_{\phi}\right|_{i, j+1 / 2} ^{n+1}\right)  \tag{4.15b}\\
= & \frac{1}{\kappa_{e r}} \frac{C_{h z e \phi}}{r_{i} \Delta r}\left(\left.r_{i+1} E_{\phi}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.r_{i-1 / 2} E_{\phi}\right|_{i, j+1 / 2} ^{n+1 / 2}\right)-\left.C_{\psi h z r} \psi_{h z r}\right|_{i_{i+1 / 2, j+1 / 2}^{n+1 / 2}}
\end{align*}
$$

where $C_{\psi e \phi z}=\frac{1}{\kappa_{e z}} \Delta z C_{e \phi h r}, C_{\psi e \phi r}=\frac{1}{\kappa_{e r}} \Delta r C_{e \phi h z}$.

$$
\begin{align*}
& \left.\psi_{e \phi z}\right|_{i+1 / 2, j} ^{n}=\left.b_{r} \psi_{e \phi z}\right|_{i+1 / 2, j} ^{n-1 / 2}+a_{r}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)  \tag{4.16a}\\
& \left.\psi_{h r z}\right|_{i+1 / 2, j+1 / 2} ^{n}=b_{r} \psi_{h r z}{ }_{i+1 / 2, j+1 / 2}^{n-1 / 2}+a_{r}\left(\left.E_{\phi}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{\phi}\right|_{i+1 / 2, j} ^{n}\right) \tag{4.16b}
\end{align*}
$$

Similar equations can be derived for $\psi_{\text {e申p }} r_{i, j+1 / 2}^{n+1 / 2}$, and $\left.\psi_{h z r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}$. Note that the discrete coefficients $b_{r}$ and $a_{r}$ are nonzero in the PML regions. These coefficients are
computed using similar equations to those presented in Section 2.4 of Chapter 2. With the given form of updating equation coefficients, the electric and magnetic field components of (4.14) and (4.15) are updated using the first three terms on the right side in the entire domain. Then $\left.\psi_{e \phi z}\right|_{i+1 / 2, j} ^{n},\left.\psi_{h r z}\right|_{i+1 / 2, j+1 / 2} ^{n},\left.\psi_{e \phi r r}\right|_{i, j+1 / 2} ^{n+1 / 2}$ and $\left.\psi_{h z r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}$ are calculated using their previous time step values and the new values for the fields. The same procedure also applies even for updating the other electric and magnetic field components by using their respective updating equations and CPML parameters. It should be noted that equations (4.14) and (4.15) have similar form as those given in Section 2.4 of Chapter 2 and Section 3.4 of Chapter 3, but the subscript notations have been modified to indicate the electric and magnetic parameters separately. This notation facilitates the establishment of a connection between the parameters in the formulations and their counterparts in program implementation. Fig. 4.2 shows the structure of the 2D $\mathrm{TE}_{0_{\mathrm{n}}}$ rotationally symmetric LOD-FDTD grid with CPML ABCs, which can be used even for $\mathrm{TM}_{0 \mathrm{n}}$ cases.

The theoretical reflection coefficient, conductivity profile and its implementation have been described in Section 2.4 of Chapter 2 and are not repeated here. To validate the equations derived for the $\mathrm{TE}_{0 \mathrm{n}}$ rotationally symmetric CPML, we consider a free space domain as shown in Fig. 4.3. The CPML parameters for RS-LOD-FDTD are considered to be same as those mentioned in Chapters 3 and 4.


Fig. 4.2 A rotational symmetric LOD-FDTD grid for $\mathrm{TE}_{0 \mathrm{n}}$ mode enclosed by CPML ABC.


Fig. 4.3 Simulation model for free space


Fig. 4.4 Reflection error for RS-LOD-FDTD $\mathrm{TE}_{0 \mathrm{n}}$ case in free space model

The overall computational domain is defined by $40 \times 40$ cells with $\Delta r=\Delta z=1 \mathrm{~mm}$. An axial symmetric boundary condition is applied to the left boundary; the other boundaries are terminated by 10 CPML layers. The source located at grid point $(0,20)$ is a differentiated Gaussian pulse given by

$$
\begin{equation*}
H_{r}(t)=-2 \frac{\left(t-t_{0}\right)}{\tau} \exp \left(-\frac{\left(t-t_{0}\right)^{2}}{\tau^{2}}\right) \tag{4.17}
\end{equation*}
$$

The observation point is placed close to the CPML boundary at grid point ( 30,30 ). The reflection error is calculated as follows.

$$
\begin{equation*}
H_{e r r}(t)=20 \log \left(\frac{\left|H_{o b r}(t)-H_{r e f}(t)\right|}{\max \left|H_{r e f}(t)\right|}\right) \tag{4.18}
\end{equation*}
$$

where $H_{r e f}(t)$ is the reference result calculated in a grid large enough that any reflection from the boundaries is isolated. Fig. 4.4 shows the calculated reflection error for CFLN=2, 8. From the figure, it is clearly seen that a lower reflection error can be obtained for lower CFLN. It is also seen that more evanescent energy is absorbed by the proposed CPML.

### 4.3.2 Formulation of CPML ABC for TM $_{0 \mathrm{n}}$ RS-LOD-FDTD

The derivation of CPML for the $\mathrm{TM}_{0 \mathrm{n}}$ RS-LOD-FDTD is similar to that of the $\mathrm{TE}_{0 \mathrm{n}}$ case presented in Section 4.3.1. Similar to the $\mathrm{TE}_{0 \mathrm{n}}$ case, the CPML updating equation for the $\mathrm{TM}_{0 \mathrm{n}}$ RS-LOD-FDTD method can be written for the sub-steps 1 and 2 as follows:

Sub-step 1:

$$
\begin{align*}
\left.E_{z}\right|_{i, j+1 / 2} ^{n+1 / 2}= & \left.C_{e z e} E_{z}\right|_{i, j+1 / 2} ^{n} \\
+ & \frac{1}{\kappa_{e r}} \frac{C_{e z h \phi}}{r_{i} \Delta r}\left(\left.r_{i+1 / 2} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.r_{i-1 / 2} H_{\phi}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)  \tag{4.19a}\\
& +\frac{1}{\kappa_{e r}} \frac{C_{e z h \phi}}{r_{i} \Delta r}\left(\left.r_{i+1 / 2} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.r_{i-1 / 2} H_{\phi}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)+\left.C_{\psi e z r} \psi_{e z r}\right|_{i, j+1 / 2} ^{n} \\
\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} & =\left.C_{h \phi h} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n}+\frac{1}{\kappa_{e r}} \frac{C_{h \phi e z}}{\Delta r}\left(\left.E_{z}\right|_{i+1 / j+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i, j+1 / 2} ^{n+1 / 2}\right)  \tag{4.19b}\\
& +\frac{1}{\kappa_{e r}} \frac{C_{h \phi \phi z}}{\Delta r}\left(\left.E_{z}\right|_{i+1, j+1 / 2} ^{n}-\left.E_{z}\right|_{i, j+1 / 2} ^{n}\right)+\left.C_{\psi h \phi r} \psi_{h \phi r}\right|_{i+1 / 2, j+1 / 2} ^{n}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
\left.E_{r}\right|_{i+1 / 2, j} ^{n+1}= & \left.C_{e r e} E_{r}\right|_{i+1 / 2, j} ^{n+1 / 2}-\frac{1}{\kappa_{e z}} \frac{C_{e r h \phi}}{\Delta z}\left(\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{\phi}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right)  \tag{4.20a}\\
& -\frac{1}{\kappa_{e z}} \frac{C_{e r h \phi}}{\Delta z}\left(\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{\phi}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)-\left.C_{\psi e r z} \psi_{e r z}\right|_{i+1 / 2, j} ^{n+1 / 2} \\
\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1}= & \left.C_{h \phi h} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\frac{1}{\kappa_{e z}} \frac{C_{h \phi e r}}{\Delta z}\left(\left.E_{r}\right|_{i+1 / 2, j+1} ^{n+1}-\left.E_{r}\right|_{i+1 / 2, j} ^{n+1}\right)  \tag{4.20b}\\
& -\frac{1}{\kappa_{e z}} \frac{C_{h \phi e r}}{\Delta z}\left(\left.E_{r}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{r}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)-\left.C_{\psi h \phi z} \psi_{h \phi z}\right|_{l_{i+1 / 2, j+1 / 2}^{n+1 / 2}}
\end{align*}
$$

where $C_{\psi e z r}=\frac{1}{\kappa_{e r}} \Delta r C_{e z h \phi}, C_{\psi h \phi r}=\frac{1}{\kappa_{e r}} \Delta r C_{h \phi e z}, C_{\psi e r z}=\frac{1}{\kappa_{e z}} \Delta z C_{e r h \phi}, C_{\psi h \phi z}=\frac{1}{\kappa_{e z}} \Delta z C_{h \phi e r}$

$$
\begin{align*}
& \left.\psi_{e z r}\right|_{i, j+1 / 2} ^{n}=\left.b_{r} \psi_{e z r}\right|_{i, j+1 / 2} ^{n-1 / 2}+a_{r}\left(\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{\phi}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)  \tag{4.21a}\\
& \left.\psi_{h \phi r}\right|_{i+1 / 2, j+1 / 2} ^{n}=\left.b_{r} \psi_{h \phi r}\right|_{i+1 / 2, j+1 / 2} ^{n-1 / 2}+a_{r}\left(\left.E_{z}\right|_{i+1, j+1 / 2} ^{n}-\left.E_{z}\right|_{i, j+1 / 2} ^{n}\right) \tag{4.21b}
\end{align*}
$$

Similar equations can be derived for $\left.\psi_{e r z}\right|_{i+1 / 2, j} ^{n+1 / 2}$ and $\left.\psi_{h \phi z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}$. Note that the discrete coefficients $b_{r}$ and $a_{r}$ are nonzero in the PML regions. These coefficients are computed using the equations similar to those presented in Section 4 of Chapter 2. The procedure is also similar to that of $\mathrm{TE}_{0 \mathrm{n}}$ case and the form of updating equation coefficients are also similar, where the electric and magnetic field components of (4.19) and (4.20) are updated using the first three terms on the right side in the entire domain. Then $\left.\psi_{e \phi z}\right|_{i+1 / 2, j} ^{n},\left.\psi_{h r z}\right|_{i+1 / 2, j+1 / 2} ^{n},\left.\psi_{e \phi r}\right|_{i, j+1 / 2} ^{n+1 / 2}$ and $\left.\psi_{h z r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}$ are calculated using their previous time step values and the new values of fields. To validate the CPML formulation for the $\mathrm{TM}_{0 \mathrm{n}}$ RS-LOD-FDTD, we consider the same grid size, discretisation and excitation as those used in Section 4.3.1. Similar to the $\mathrm{TE}_{0 \mathrm{n}}$ case, the reflection error for the $\mathrm{TM}_{0 \mathrm{n}}$ case is calculated by the following equation.


Fig.4.5 Reflection error for RS-LOD-FDTD $\mathrm{TM}_{0 \mathrm{n}}$ with CPML incorporated

$$
\begin{equation*}
E_{e r r}(t)=20 \log \left(\frac{\left|E_{\text {obr }}(t)-E_{r e f}(t)\right|}{\max \left|E_{r e f}(t)\right|}\right) \tag{4.22}
\end{equation*}
$$

where $E_{\text {ref }}(t)$ is the reference result calculated in a grid large enough that any reflection from the boundaries is isolated. Fig. 4.5 shows the reflection error for the $\mathrm{TM}_{0 \mathrm{n}}$ of CPML RS-LOD-FDTD for various CFLN.

### 4.4 Dispersion Control RS LOD-FDT(D-RS-LODFDTD) for $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ Mode Analysis

Although the conventional RS-LOD-FDTD method has been applied to analyse different types of resonant structures, but the numerical efficiency of the conventional RS-LOD-FDTD method is still limited by the numerical dispersion error [112]-[114]. The numerical phase velocity increases when the time step size becomes much larger than the CFL limit, and thus prohibits its applicability to electrically small objects with fine features. To reduce the numerical dispersion error, several methods viz. use of higher-order spatial difference equation [113], the introduction of artificial anisotropy [127] and use of parameter optimisation methods [128]-[129] etc. have been reported in the literature for unconditionally stable FDTD methods.

In this chapter, we employ dispersion control approach [112] to improve the numerical dispersion performance of the RS-LOD-FDTD. Only two additional coefficient parameters are added to the RS-LOD-FDTD equations and hence resultant additional computational burden is negligible. The unconditional stability of the method is investigated using numerical examples. The numerical dispersion relations are theoretically derived first and then analysed. The Maxwell's equation in an isotropic and lossless medium with dispersion control parameter can be written as follows:

$$
\begin{equation*}
\frac{\partial \mathbf{U}}{\partial t}=([A]+[B]) \mathbf{U} \tag{4.23}
\end{equation*}
$$

with

$$
\mathbf{A}=\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & -\frac{\gamma}{\varepsilon} \frac{\partial}{\partial z} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\alpha}{\varepsilon} \frac{\partial}{\partial \rho} \\
0 & 0 & 0 & -\frac{m}{\varepsilon \rho} & 0 & 0 \\
0 & 0 & \frac{m}{\mu \rho} & 0 & 0 & 0 \\
-\frac{\gamma}{\mu} \frac{\partial}{\partial z} 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{\alpha}{\mu \rho} \frac{\partial}{\partial \rho} & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\mathbf{B}=\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & \frac{m}{\varepsilon \rho} \\
0 & 0 & 0 & \frac{\gamma}{\varepsilon} \frac{\partial}{\partial z} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\alpha}{\varepsilon \rho} \frac{\partial(\rho)}{\partial \rho} & 0 \\
0 & \frac{\gamma}{\mu} \frac{\partial}{\partial z} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\alpha}{\mu} \frac{\partial}{\partial \rho} & 0 & 0 & 0 \\
-\frac{m}{\mu \rho} & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

where ' T ' denotes the matrix transpose. In the above equations, two new dispersion control parameters $\alpha$ and $\gamma$ are introduced to improve the numerical dispersion. The dispersion control parameters are determined by

$$
\begin{equation*}
\alpha=\frac{\Delta r}{c \Delta t} \frac{\tan (\mathrm{kc} \Delta \mathrm{t} / 2)}{\sin \left(\mathrm{k}_{\mathrm{r}} \Delta \mathrm{r} / 2\right)} \quad, \quad \gamma=\frac{\Delta z}{c \Delta t} \frac{\tan (\mathrm{kc} \Delta \mathrm{t} / 2)}{\sin \left(\mathrm{k}_{\mathrm{z}} \Delta \mathrm{z} / 2\right)} \tag{4.24}
\end{equation*}
$$

where $\mathrm{k}_{r}=\mathrm{k} \cos \phi, \mathrm{k}_{z}=\mathrm{k} \sin \phi \quad \phi=\left(0^{\circ}\right.$ or $\left.90^{\circ}\right)$ and k is the free space wavenumber. From (4.24), it can be observed that the dispersion control parameters mainly depend on the cell size and the time steps. So, by selecting the proper value of the cell size and time step, we obtain the dispersion control parameters which are used to improve the dispersion. For the special case of $\alpha=\gamma=1$, the proposed method reduces to the original RS-LOD-FDTD method. Based on the original RS-LOD-FDTD method which has been described in Section 4.2, (4.23) can be written as follows.

Sub-step 1:

$$
\begin{gather*}
E_{r}^{n+1 / 2}-\frac{m \Delta t}{2 \varepsilon r} H_{z}^{n+1 / 2}=E_{r}^{n}+\frac{m \Delta t}{2 \varepsilon r} H_{z}^{n}  \tag{4.25a}\\
E_{\phi}^{n+1 / 2}-\frac{\gamma \Delta t}{2 \varepsilon} \frac{\partial H_{r}^{n+1 / 2}}{\partial z}=E_{\phi}^{n}+\frac{\gamma \Delta t}{2 \varepsilon} \frac{\partial H_{r}^{n}}{\partial z}  \tag{4.25b}\\
E_{z}^{n+1 / 2}-\frac{\alpha \Delta t}{2 \varepsilon \rho} \frac{\partial\left(r H_{\phi}^{n+1 / 2}\right)}{\partial r}=E_{z}^{n}+\frac{\alpha \Delta t}{2 \varepsilon r} \frac{\partial\left(r H_{\phi}^{n}\right)}{\partial r} \tag{4.25c}
\end{gather*}
$$

$$
\begin{align*}
& H_{r}^{n+1 / 2}-\frac{\gamma \Delta t}{2 \mu} \frac{\partial E_{\phi}^{n+1 / 2}}{\partial z}=H_{r}^{n}+\frac{\gamma \Delta t}{2 \mu} \frac{\partial E_{\phi}^{n}}{\partial z}  \tag{4.25d}\\
& H_{\phi}^{n+1 / 2}-\frac{\alpha \Delta t}{2 \mu} \frac{\partial E_{z}^{n+1 / 2}}{\partial r}=H_{\phi}^{n}+\frac{\alpha \Delta t}{2 \mu} \frac{\partial E_{z}^{n}}{\partial r}  \tag{4.25e}\\
& H_{z}^{n+1 / 2}+\frac{m \Delta t}{2 \mu r} E_{r}^{n+1 / 2}=H_{z}^{n}-\frac{m \Delta t}{2 \mu r} E_{r}^{n} \tag{4.25f}
\end{align*}
$$

Sub-step 2:

$$
\begin{gather*}
E_{r}^{n+1}+\frac{\gamma \Delta t}{2 \varepsilon} \frac{\partial H_{\phi}^{n+1}}{\partial z}=E_{r}^{n+1 / 2}-\frac{\gamma \Delta t}{2 \varepsilon} \frac{\partial H_{\phi}^{n+1 / 2}}{\partial z}  \tag{4.26a}\\
E_{\phi}^{n+1}+\frac{\alpha \Delta t}{2 \varepsilon} \frac{\partial H_{z}^{n+1}}{\partial r}=E_{\phi}^{n+1 / 2}-\frac{\alpha \Delta t}{2 \varepsilon} \frac{\partial H_{z}^{n+1 / 2}}{\partial r}  \tag{4.26b}\\
E_{z}^{n+1}+\frac{m \Delta t}{2 \varepsilon r} H_{r}^{n+1}=E_{z}^{n+1 / 2}-\frac{m \Delta t}{2 \varepsilon r} H_{r}^{n+1 / 2}  \tag{4.26c}\\
H_{r}^{n+1}-\frac{m \Delta t}{2 \mu r} E_{z}^{n+1}=H_{r}^{n+1 / 2}+\frac{m \Delta t}{2 \mu r} E_{z}^{n+1 / 2}  \tag{4.26d}\\
H_{\phi}^{n+1}+\frac{\gamma \Delta t}{2 \mu} \frac{\partial E_{r}^{n+1}}{\partial z}=H_{\phi}^{n+1 / 2}-\frac{\gamma \Delta t}{2 \mu} \frac{\partial E_{r}^{n+1 / 2}}{\partial z}  \tag{4.26e}\\
H_{z}^{n+1}+\frac{\alpha \Delta t}{2 \mu r} \frac{\partial\left(r E_{\phi}^{n+1}\right)}{\partial r}=H_{z}^{n+1 / 2}-\frac{\alpha \Delta t}{2 \mu r} \frac{\partial\left(r E_{\phi}^{n+1 / 2}\right)}{\partial r} \tag{4.26f}
\end{gather*}
$$

The updating equations for the $\mathrm{TE}_{0 \mathrm{n}}$ and $\mathrm{TM}_{0 \mathrm{n}}$ modes of the D-RS-LOD-FDTD method are given next.

### 4.4.1 Formulation of D-RS-LOD-FDTD Algorithm for TE $_{0 n}$ Mode Analysis

In this section, the derivation of dispersion control TE $_{0 n}$ RS-LOD-FDTD is presented. Following the formulation of the conventional RS-LOD-FDTD method for the $\mathrm{TE}_{0 \mathrm{n}}$ mode presented in Section 4.2.1, the updating equation for the $\mathrm{TE}_{0 \mathrm{n}}$ mode of the $\mathrm{D}-\mathrm{RS}$ -LOD-FDTD method can be derived as follows.

Sub-step1:

$$
\begin{align*}
\left.E_{\phi}\right|_{i+1 / 2, j} ^{n+1 / 2} & =\left.E_{\phi}\right|_{i+1 / 2, j} ^{n} \\
& +\frac{\gamma \Delta t}{2 \varepsilon \Delta z}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)+\frac{\gamma \Delta t}{2 \varepsilon \Delta z}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n}\right) \tag{4.27a}
\end{align*}
$$

$$
\begin{align*}
& \left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& +\frac{\gamma \Delta t}{2 \mu \Delta z}\left(\left.E_{\phi}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{\phi}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)+\frac{\gamma \Delta t}{2 \mu \Delta z}\left(\left.E_{\phi}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{\phi}\right|_{i+1 / 2, j} ^{n}\right)  \tag{4.27b}\\
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n} \tag{4.27c}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E_{\phi}\right|_{i, j+1 / 2} ^{n+1}=\left.C_{e \phi e} E_{\phi}\right|_{i, j+1 / 2} ^{n+1 / 2} \\
& -\frac{\alpha \Delta t}{2 \varepsilon \Delta r}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1}\right)-\frac{\alpha \Delta t}{2 \varepsilon \Delta r}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)  \tag{4.28a}\\
& \left.H_{r}\right|_{i, j+1 / 2} ^{n+1}=\left.H_{r}\right|_{i, j+1 / 2} ^{n+1 / 2}  \tag{4.28b}\\
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.C_{h z h} H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \\
& -\frac{\alpha \Delta t}{2 \mu r_{i} \Delta r}\left(r_{i+1} E_{\phi}^{\left.\right|_{i+1, j+1 / 2} ^{n+1}}-\left.r_{i} E_{\phi}\right|_{i, j+1 / 2} ^{n+1}\right)-\frac{\alpha \Delta t}{2 \mu r_{i} \Delta r}\left(\left.r_{i+1} E_{\phi}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.r_{i-1 / 2} E_{\phi}\right|_{i, j+1 / 2} ^{n+1 / 2}\right) \tag{4.28c}
\end{align*}
$$

where two dispersion control parameters $\alpha$ and $\gamma$ are incorporated to reduce the numerical dispersion. For $\alpha=\gamma=1$, in the above equations (4.27)-(4.28) reduce to those of the conventional $\mathrm{TE}_{0 \mathrm{n}}$ RS-LOD-FDTD method. Since the equations of sub-step 1 (4.27b)-(4.27c) cannot be solved directly, substituting (4.27b) into (4.27a) yields the simultaneous linear equations for $\left.E_{\phi}\right|^{n+1 / 2}$ that result in the tri-diagonal matrix. This can be solved implicitly and (4.27b) can then be explicitly solved for $\left.H_{r}\right|^{n+1 / 2}$. Similarly, in sub-step 2 we substitute (4.28c) into (4.28a) and implicitly solve the resultant tridiagonal system for $\left.E_{\phi}\right|^{n+1}$, and then explicitly solve (4.28c) for $\left.H_{z}\right|^{n+1}$. The formulation of the D- RS-LOD-FDTD for the $\mathrm{TM}_{0 \mathrm{n}}$ is described next.

### 4.4.2 Formulation of D-RS-LOD-FDTD Algorithm for $\mathbf{T M}_{\mathbf{0 n}}$ Mode Analysis

The derivation of the dispersion control RS-LOD-FDTD TM $_{0 \mathrm{n}}$ is similar to that of the $\mathrm{TE}_{0 \mathrm{n}}$ case presented in Section 4.4.1. Similar to the $\mathrm{TE}_{0 \mathrm{n}}$ case, the updating equation for the $\mathrm{TM}_{0 \mathrm{n}}$ D-RS-LOD-FDTD method can be derived for the sub-steps 1 and 2 as follows:

Sub-step1:

$$
\begin{equation*}
\left.E_{r}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.E_{r}\right|_{i, j+1 / 2} ^{n} \tag{4.29a}
\end{equation*}
$$

$$
\left.\begin{align*}
\left.E_{z}\right|_{i, j+1 / 2} ^{n+1 / 2}= & E_{z}
\end{align*}\right|_{i, j+1 / 2} ^{n} .
$$

For $m=0$, we must pay attention to handling the singular point at $r=0$. Instead of (4.29b), we adopt the following equation in accordance with LOD implementation.

$$
\begin{equation*}
\left.E_{z}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.E_{z}\right|_{i, j+1 / 2} ^{n}+\left.\frac{\alpha 4 \Delta t}{\varepsilon \Delta r} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \tag{4.29c}
\end{equation*}
$$

The other magnetic field component can be written as follows:

$$
\begin{align*}
& \left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& \quad+\frac{\alpha \Delta t}{2 \mu \Delta r}\left(\left.E_{z}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i, j+1 / 2} ^{n+1 / 2}\right)+\frac{\alpha \Delta t}{2 \mu \Delta r}\left(\left.E_{z}\right|_{i+1, j+1 / 2} ^{n}-\left.E_{z}\right|_{i, j+1 / 2} ^{n}\right) \tag{4.29d}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E_{r}\right|_{i+1 / 2, j} ^{n+1}=\left.E_{r}\right|_{i+1 / 2, j} ^{n+1 / 2} \\
& -\frac{\gamma \Delta t}{2 \varepsilon \Delta z}\left(\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{\phi}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right)-\frac{\gamma \Delta t}{2 \varepsilon \Delta z}\left(\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{\phi}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)  \tag{4.30a}\\
& \qquad\left.E_{z}\right|_{i+1 / 2, j} ^{n+1}=\left.E_{z}\right|_{i+1 / 2, j} ^{n+1 / 2}
\end{aligned} \begin{aligned}
& \left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}  \tag{4.30b}\\
& \quad \quad-\frac{\gamma \Delta t}{2 \mu \Delta z}\left(\left.E_{r}\right|_{i+1 / 2, j+1} ^{n+1}-\left.E_{r}\right|_{i+1 / 2, j} ^{n+1}\right)-\frac{\gamma \Delta t}{2 \mu \Delta z}\left(\left.E_{r}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{r}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)
\end{align*}
$$

Similar to the $\mathrm{TE}_{0 \mathrm{n}}$ case, the equations of sub-step 1 (4.29b)-(4.29d) cannot be solved directly. Placing (4.29d) into (4.29b) yields the simultaneous linear equations for $\left.E_{z}\right|^{n+1 / 2}$ that result in the tri-diagonal matrix in the sub-step 1 and equation ( 4.29 d ) can be solved explicitly. Similarly, by substituting (4.30c) into (4.30a) the simultaneous linear equations for $\left.E_{z}\right|^{n+1}$ that result in the tri-diagonal matrix in sub-step 2 and (4.30c) can be solved explicitly.

### 4.5 Development of CPML ABC for D-RS-LOD-FDTD Method

### 4.5.1 Formulation of CPML ABC for TE $_{0 \mathrm{n}}$ D-RS-LOD-FDTD

Similar to the CPML ABC for the conventional RS-LOD-FDTD method, and following the procedure described in Chapters 2 and 3, we can derive the CPML equations for $\mathrm{TE}_{0 \mathrm{n}}$ D-RS-LOD-FDTD as follows.

Sub-step 1:

$$
\begin{align*}
\left.E_{\phi}\right|_{i+1 / 2, j} ^{n+1 / 2}= & \left.E_{\phi}\right|_{i+1 / 2, j} ^{n}+\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{2 \varepsilon \Delta z}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)  \tag{4.31a}\\
& +\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{2 \varepsilon \Delta z}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)+\left.C_{\psi e \phi z} \psi_{e \phi z}\right|_{i+1 / 2, j} ^{n} \\
\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}= & \left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n}+\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{2 \mu \Delta z}\left(\left.E_{\phi}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{\phi}\right|_{i+1 / 2, j} ^{n+1 / 2}\right) \\
& +\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{2 \mu \Delta z}\left(\left.E_{\phi}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{\phi}\right|_{i+1 / 2, j} ^{n}\right)+\left.C_{\psi h r z} \psi_{h r z}\right|_{i+1 / 2, j+1 / 2} ^{n} \tag{4.31b}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
\left.E_{\phi}\right|_{i, j+1 / 2} ^{n+1}= & \left.C_{e \phi e} E_{\phi}\right|_{i, j+1 / 2} ^{n+1 / 2}-\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{2 \varepsilon \Delta r}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1}\right)  \tag{4.32a}\\
& -\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{2 \varepsilon \Delta r}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)-\left.C_{\psi e \phi r} \psi_{e \phi r}\right|_{i, j+1 / 2} ^{n+1 / 2} \\
\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}= & \left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{2 \mu r_{i} \Delta r}\left(\left.r_{i+1} E_{\phi}\right|_{i+1, j+1 / 2} ^{n+1}-\left.r_{i} E_{\phi}\right|_{i, j+1 / 2} ^{n+1}\right)  \tag{4.32b}\\
& -\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{2 \mu r_{i} \Delta r}\left(\left.r_{i+1} E_{\phi}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.r_{i-1 / 2} E_{\phi}\right|_{i, j+1 / 2} ^{n+1 / 2}\right)-\left.C_{\psi h z r} \psi_{h z r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \\
\text { where } C_{\psi e \phi z}= & \frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{\varepsilon}, C_{\psi h r z}=\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{\mu}, C_{\psi e \phi r}=\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{\varepsilon}, C_{\psi h z r}=\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{r_{i} \mu} \\
& \psi_{e \phi z} l_{i+1 / 2, j}^{n}=\left.b_{r} \psi_{e \phi z}\right|_{i+1 / 2, j} ^{n-1 / 2}+a_{r}\left(\left.H_{r}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{r}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)  \tag{4.33a}\\
& \left.\psi_{h r z}\right|_{i+1 / 2, j+1 / 2} ^{n}=\left.b_{r} \psi_{h r z}\right|_{i+1 / 2, j+1 / 2} ^{n-1 / 2}+a_{r}\left(\left.E_{\phi}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{\phi}\right|_{i+1 / 2, j} ^{n}\right) \tag{4.33b}
\end{align*}
$$

Similar equation can be derived for $\left.\psi_{\text {eqr }}\right|_{i, j+1 / 2} ^{n+1 / 2}$ and $\left.\psi_{h z r}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}$. Note that the discrete coefficient $b_{r}$ and $a_{r}$ are nonzero in the PML regions. These coefficients are computed using the procedures presented in Section 2.4 of Chapter 2.

### 4.5.2 Formulation of CPML ABC for TM $_{0 n}$ D-RS-LOD-FDTD

The derivation of CPML for the $\mathrm{TM}_{0 \mathrm{n}} \mathrm{D}$-RS-LOD-FDTD is similar to that of the $\mathrm{TE}_{0 \mathrm{n}}$ case presented in Section 4.5.1. Similar to the $\mathrm{TE}_{0 \mathrm{n}}$ case, the CPML updating equation for the $\mathrm{TM}_{0 \mathrm{n}} \mathrm{D}$-RS-LOD-FDTD method can be written for sub-steps 1 and 2 as follows:

Sub-step 1:

$$
\begin{align*}
& \left.E_{z}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.E_{z}\right|_{i, j+1 / 2} ^{n} \\
& \quad+\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{2 \varepsilon r_{i} \Delta r}\left(\left.r_{i+1 / 2} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.r_{i-1 / 2} H_{\phi}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)  \tag{4.34a}\\
& \quad+\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{2 \varepsilon r_{i} \Delta r}\left(\left.r_{i+1 / 2} H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.r_{i-1 / 2} H_{\phi}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)+\left.C_{\psi e z r} \psi_{e z r}\right|_{i, j+1 / 2} ^{n} \\
& \left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}= \\
& \quad  \tag{4.34b}\\
& \left.\quad H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n}+\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{2 \mu \Delta r}\left(\left.E_{z}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i, j+1 / 2} ^{n+1 / 2}\right) \\
& \\
& \quad+\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{2 \mu \Delta r}\left(\left.E_{z}\right|_{i+1, j+1 / 2} ^{n}-\left.E_{z}\right|_{i, j+1 / 2} ^{n}\right)+\left.C_{\psi h \phi r} \psi_{h \phi r}\right|_{i+1 / 2, j+1 / 2} ^{n}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
&\left.E_{r}\right|_{i+1 / 2, j} ^{n+1}=\left.E_{r}\right|_{i+1 / 2, j} ^{n+1 / 2}-\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{2 \varepsilon \Delta z}\left(\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{\phi}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right)  \tag{4.35a}\\
&-\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{2 \varepsilon \Delta z}\left(\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{\phi}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right)-\left.C_{\psi e r z} \psi_{e r z}\right|_{i+1 / 2, j} ^{n+1 / 2} \\
&\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{2 \mu \Delta z}\left(\left.E_{r}\right|_{i+1 / 2, j+1} ^{n+1}-\left.E_{r}\right|_{i+1 / 2, j} ^{n+1}\right)  \tag{4.35b}\\
&-\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{2 \mu \Delta z}\left(\left.E_{r}\right|_{i+1 / 2, j+1} ^{n+1 / 2}-\left.E_{r}\right|_{i+1 / 2, j} ^{n+1 / 2}\right)-\left.C_{\psi h \phi z} \psi_{h \phi z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \\
& \text { where } C_{\psi e z r}= \frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{r_{i} \varepsilon}, C_{\psi h \phi r}=\frac{1}{\kappa_{e r}} \frac{\alpha \Delta t}{\mu}, C_{\psi e r z}=\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{\varepsilon}, C_{\psi h \phi z}=\frac{1}{\kappa_{e z}} \frac{\gamma \Delta t}{\mu} \\
&\left.\psi_{e z r}\right|_{i, j+1 / 2} ^{n}=\left.b_{r} \psi_{e z r}\right|_{i, j+1 / 2} ^{n-1 / 2}+a_{r}\left(\left.H_{\phi}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{\phi}\right|_{i-1 / 2, j+1 / 2} ^{n}\right)  \tag{4.36a}\\
&\left.\psi_{h \phi r}\right|_{i+1 / 2, j+1 / 2} ^{n}=\left.b_{r} \psi_{h \phi r}\right|_{i+1 / 2, j+1 / 2} ^{n-1 / 2}+a_{r}\left(\left.E_{z}\right|_{i+1, j+1 / 2} ^{n}-\left.E_{z}\right|_{i, j+1 / 2} ^{n}\right) \tag{4.36b}
\end{align*}
$$

Similar equations can be derived for $\left.\psi_{e r z}\right|_{i+1 / 2, j} ^{n+1 / 2}$ and $\left.\psi_{h \phi z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}$. Note that the discrete coefficients $b_{r}$ and $a_{r}$ are nonzero in the PML regions. These coefficients are computed using the method similar to that presented in Section 2.4 of Chapter 2. To further
validate the performance of the CPML, for the same configuration, the global ( $L^{2}$ norm) error for $E_{r}$ has been calculated as

$$
\begin{equation*}
L^{2}=\sum_{i=1}^{i \max } \sum_{j=1}^{j \max }\left(E_{r}(i, j)-E_{r}^{\text {ref }}(i, j)\right)^{2} \tag{4.37}
\end{equation*}
$$

where $E_{r}^{\text {ref }}$ is the field of the reference solution. Fig. 4.6 shows the $L^{2}$ norm error calculated using 10 layers CPML with the parameters given above.


Fig. 4.6 $L^{2}$ norm error as a function of time step for a $40 \times 40$ cell $\mathrm{TM}_{0 \mathrm{n}}$ D-RS-LOD-FDTD simulation having a 10 layer CPML

### 4.6 S-Parameter Extraction Technique for RS-LOD-FDTD

S-parameters are based on the concept of power wave. The incident and reflected power waves $a_{i}$ and $b_{i}$ associated with port $i$ are defined as

$$
\begin{equation*}
a_{i}=\frac{V_{i}+Z_{i} \times I_{i}}{2 \sqrt{\left|\operatorname{Re}\left\{Z_{i}\right\}\right|}}, b_{i}=\frac{V_{i}-Z_{i}^{*} \times I_{i}}{2 \sqrt{\left|\operatorname{Re}\left\{Z_{i}\right\}\right|}} \tag{4.38}
\end{equation*}
$$

where, $V_{i}$ and $I_{i}$ are the voltage and the current flowing into the $i^{\text {th }}$ port of a junction and $Z_{i}$ is the impedance looking out from the $i^{t h}$ port. In general $Z_{i}$ is complex; however, in most of microwave applications it is considered to be equal to $50 \Omega$. The S-parameters matrix can then be expressed as

$$
\left[\begin{array}{c}
b_{1}  \tag{4.39}\\
b_{2} \\
\vdots \\
b_{N}
\end{array}\right]=\left[\begin{array}{cccc}
S_{11} & S_{12} & \ldots & S_{1 N} \\
S_{21} & S_{22} & \ldots & S_{2 N} \\
\vdots & \vdots & \ddots & \vdots \\
S_{N 1} & S_{N 2} & \ldots & S_{N N}
\end{array}\right] \times\left[\begin{array}{c}
a_{1} \\
a_{2} \\
\vdots \\
a_{N}
\end{array}\right]
$$

By definition, the subscripts $m n$ indicate output port number $m$, and input port number $n$, of the scattering parameter $S_{m n}$. If only port $n$ is excited while all other ports are terminated by matched loads, the output power wave at port $m, b_{m}$, and the input power wave at port $n, a_{n}$, can be used to calculate $S_{m n}$ using

$$
\begin{equation*}
S_{m n}=\frac{b_{m}}{a_{n}} \tag{4.40}
\end{equation*}
$$

This technique can be applied for D-RS-LOD-FDTD or any LOD-FDTD method to obtain the S-parameters for a microwave multiport device.

### 4.7 Specific Absorption Rate (SAR) Calculation for RS-LOD-FDTD

The specific absorption rate (SAR) is a measure of energy absorption rate for biological media exposed to electromagnetic radiation. It can be computed using RS-LOD-FDTD coupled with the discrete Fourier transform (DFT) using the following equation:

$$
\begin{equation*}
\mathrm{SAR}=\frac{\sigma}{\rho}\left(\left|E_{r}\right|^{2}+\left|E_{z}\right|^{2}\right) \tag{4.41}
\end{equation*}
$$

where $E$ is the Fourier transform of the recorded electric field components and $\sigma$ is the conductivity, and $\rho$ is the density of the tissue or phantom of surrounding material. The SAR has an overall unit of $\mathrm{W} / \mathrm{kg}$.

### 4.8 Numerical Analysis of RS Microwave Structures Using RS-LOD-FDTD and D-RS-LOD-FDTD

To show the validity of the proposed RS-LOD-FDTD and D-RS-LOD-FDTD approaches, the resonant frequencies of a circular cavity with a dielectric disc ( $\varepsilon_{\mathrm{r}}$
$=35.74)$ placed at the centre of the cavity is first analysed. The cross section of the structure is shown in Fig. 4.7 (a). The size of the spatial meshes employed for the D-RS-LOD-FDTD and RS-LOD-FDTD are $\Delta r=0.17272 \mathrm{~mm}$ and $\Delta z=0.1524 \mathrm{~mm}$. Fig. 4.7 (b) shows the strength $E_{\varphi}$ versus frequency, in which CFLN=2,5 and 8 are used for RS-LOD-FDTD method. Table 4.1 shows a comparison of the results computed using RS-LOD-FDTD and the results obtained with RS-FDTD. It can be observed from the Table 4.1 and Fig. 4.7 (b) that the result calculated using RS-LOD-FDTD agrees reasonably well with the results obtained using RS-FDTD method.


Fig. 4.7 (a) Cross section of the cylindrical cavity with a dielectric disk filling ( $\varepsilon_{\mathrm{r}}=35.74$ and dimension in cm ), (b) Resonant frequency estimation for dielectric loaded cavity for $\mathrm{TE}_{01}$ at $\mathrm{CFLN}=2,5$ and 8

Table 4.1 Comparison of $\mathrm{TE}_{01}$ resonant frequencies of dielectric loaded cavity $\varepsilon_{\mathrm{r}}=35.74$

| Mode | Resonant frequency by <br> RS-LOD-FDTD |  | Resonant frequency <br> by RS-FDTD | \% Error |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{TE}_{01}$ | CFLN=2 | CFLN=8 | 3.439 | CFLN=2 | CFLN=8 |
|  | 3.436 | 3.451 |  | 0.087 | 0.349 |



Fig. 4.8 Resonant frequency estimation for dielectric loaded cavity for $\mathrm{TE}_{01}$ at $\mathrm{CFLN}=2,5$ and 8 with D -RS-LOD-FDTD

Table 4.2 Comparison of $\mathrm{TE}_{01}$ resonant frequencies of dielectric loaded cavity $\varepsilon_{\mathrm{r}}=35.74$

| Mode | Resonant frequency by <br> D-RS-LOD-FDTD |  | Resonant <br> frequency by RS- <br> FDTD | \% Error |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| TE $_{01}$ | CFLN=2 | CFLN=8 | 3.439 | CFLN=2 | CFLN=8 |
|  | 3.44 | 3.448 |  | 0.029 | 0.262 |

However, it can also be observed from Fig. 4.7 (b) that the results obtained by RS-LOD-FDTD deviates with an increase in CFLN. The \% error calculation from Table 4.1 also shows that the computed resonant frequency of the DR using RS-LOD-FDTD varies for larger CFLN. However, to improve the results, the strength $E_{\varphi}$ is computed
using dispersion control (D)-RS-LOD-FDTD as shown in Fig. 4.8 in which the control parameters are $\alpha=1.045$ and $\gamma=1.0346$ for CFLN $=5$ and $\alpha=1.084$ and $\gamma=1.0915$ for CFLN $=8$ are considered. For CFLN=5, the response improves well compared with the result obtained by RS-FDTD. Even though CFLN=8 is used, the response shows a good correspondence to the RS-FDTD result. Table 4.2 shows the \% error calculations with D-RS-LOD-FDTD which also provides the improvement compared to the calculated error with RS-LOD-FDTD as shown in Table 4.1.

Next, the accuracy of the dispersion control (D)-RS-LOD-FDTD method is investigated through the analysis of the open tip monopole (OTM) antenna as shown in Fig. 4.9. The result is compared with that obtained by conventional RS-LOD-FDTD as well as with the measured results available in the literature. The configuration of the monopole antenna inside the 3-D computational domain in cylindrical coordinates is shown in Fig. 4.9 (a). Fig. 4.9 (b) shows the projection of the 3-D rotationally symmetric object onto the r-z plane. The dimensions of the open tip monopole antenna are given in [167]. The cross sectional view of the OTM antenna for the simulation model is shown in Fig. 4.10. The open tip monopole antenna is immersed in $0.9 \%$ saline solution which has a dielectric constant of $\varepsilon_{\mathrm{r}}=75$ and conductivity of $\sigma=2.81 \mathrm{~S} / \mathrm{m}$ at 2.45 GHz.

(a)

(b)

Fig. 4.9 (a) A monopole antenna inside the 3-D computational domain in cylindrical coordinates, (b) its projection onto r-z plane for the analysis


Fig. 4.10 D-RS-LOD-FDTD simulation model of the OTM antenna in normal saline

Table 4.3 OTM antenna parameter dimensions (in mm)

| $\mathrm{r}_{1}$ | $\mathrm{r}_{2}$ | $\mathrm{~L}_{1}$ |
| :--- | :--- | :--- |
| 0.511 | 2.159 | 13 |

The parameters used to model the OTM antenna using D-RS-LOD-FDTD and CPML are given in Table 4.3. The S-parameters for the antenna are extracted from timedomain data using the modal extraction technique [167]. Fig. 4.11 (a)-(b) shows the computed S-parameters using RS-LOD-FDTD-CPML for different CFLN and RSFDTD for open tip antenna as a function of frequency. For RS-FDTD, the time step chosen is the upper limit determined by the CFL condition. CFLN represents the CFL number defined by $\Delta \mathrm{t} / \Delta \mathrm{t}_{\text {CFL }}$. Next, the SAR is also computed using RS-LOD-FDTD coupled with the discrete Fourier transform (DFT) using (4.41). The normalised SAR computed at a distance of $r=1.5 \mathrm{~mm}$ from the antenna is plotted for OTM antenna in Fig. 4.12. The result compares quite well with the result obtained using RS-FDTD method. It can be seen from Fig. 4.11 (a) that good agreement is obtained over a wide frequency range between the calculated and published measured values, thus validating the theoretical model. The 13 mm OTM is well matched ( $\mathrm{S}_{11}<-20 \mathrm{~dB}$ ) around 2.45 GHz.


Fig. 4.11 (a) Calculated and measured reflection coefficients of the open tip monopole antenna (b) calculated reflection coefficients of the open tip monopole antenna at different CFLN


Fig. 4.12 Computed SAR of OTM antenna using RS-LOD-FDTD for CFLN=2, 6, 10

From Fig. 4.11 (b), it is seen that S-parameter obtained by conventional RS-LODFDTD gradually shifts toward larger frequency with an increase in CFLN. Similarly, the computed SAR of the OTM antenna as shown in Fig. 4.12 by conventional RS-LOD-FDTD gradually shifts with an increase in CFLN. The accuracy degradation is created by the numerical dispersion error caused by the large CFLN; therefore, the conventional RS-LOD-FDTD can suffer from dispersion error when modelling antennas such as open tip monopole.

Now we will demonstrate the advantages of the proposed D-RS-LOD-FDTD. The calculated return loss and SAR from the OTM antenna using D-RS-LOD-FDTD CPML are plotted in Figs. 4.13 and 4.14 in which the control parameters used are $\alpha=1.0256$ and $\gamma=1.0256$ for $\operatorname{CFLN}=6$ and $\alpha=1.064$ and $\gamma=1.0725$ for CFLN=10. For CFLN=6, the response agrees well with the result obtained by conventional RS-FDTD.


Fig. 4.13 Reflection coefficients of the open tip monopole antenna using D-RS-LOD-FDTD


Fig. 4.14 Computed SAR of the open tip monopole antenna by D-RS-LOD-FDTD

Table 4.4 Comparison of computational time for computing return loss

| CFLN | CPU Time (Sec) |  |
| :---: | :---: | :---: |
|  | D-RS-LOD-FDTD | RS-LOD-FDTD |
| 2 | 100 | 110 |
| 4 | 70 | 85 |

Even though CFLN $=10$ is used, the response shows a close agreement with explicit RS-FDTD result. The computation time of the D-RS-LOD-FDTD method is shown in Table 4.4 compared with the computation time of RS-LOD-FDTD. As is shown, the computational time for D-RS-LOD-FDTD for CFLN=4 is reduced to $20 \%$ of the time of the conventional RS-LOD-FDTD. To further validate the proposed method, we analyse an expanded tip wire (ETW) antenna by the RS-LOD-FDTD and D-RS-LOD-FDTD methods. The configuration, structure and the parameters of the expanded tip wire antenna are shown in Fig. 4.15 (a), (b) and in Table 4.5. The antenna is constructed by modifying the shield of the coaxial cable and connecting an appropriate metallic tip to the inner conductor so that the antenna becomes an integral part of the cable. A coaxial choke is also placed near the antenna/cable junction. More description regarding this antenna is provided in [168]. The antenna is immersed in a myocardial (heart) tissue equivalent phantom. The myocardial tissue equivalent phantom has a measured permittivity $\varepsilon_{\mathrm{r}}=57$ and the loss tangent is $\delta=0.244$ at 2.45 GHz .



Fig. 4.15 (a) Structure of the ETW antenna (b) D-RS-LOD-FDTD simulation model of the ETW antenna in the tissue equivalent phantom

Table 4.5 ETW antenna parameter dimensions (in mm)

| $\mathrm{r}_{1}$ | $\mathrm{r}_{2}$ | $\mathrm{r}_{3}$ | $\mathrm{~T}_{1}$ | $\mathrm{~T}_{2}$ | $\mathrm{~L}_{1}$ | $\mathrm{~T}_{\mathrm{L}}=\mathrm{L}_{\mathrm{t}}$ | $\mathrm{C}_{1}$ | $\mathrm{C}_{\text {offset }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.255 | 0.816 | 1.071 | 0.816 | 0.255 | 8 | 1 | 15 | 7.5 |


(a)

(b)

Fig. 4.16 (a) Simulated and measured ETW antenna reflection coefficient (b) Computed return loss from ETW antenna at higher CFLN

The computed return loss compared with the measured return loss when the ETW antenna is immersed in a tissue equivalent phantom is shown in Fig. 4.16 (a)-(b).It can be observed from Fig. 4.16 (a) that the computed return loss by both RS-FDTD and RS-LOD-FDTD agrees well with the measured value [168], and at the resonant frequency of 2.45 GHz , both show lowest possible return loss. The computed S-parameters from the ETW antenna at larger CFLN are plotted in Fig. 4.16 (b). Similar to the OTM antenna in Fig. 4.16 (b), the response shifts toward larger frequency as CFLN is increased. The specific absorption rate (SAR) from the ETW antenna has also been calculated by the proposed method. Fig. 4.17 (a) shows the computed SAR from RS-LOD-FDTD compared with the measured value [168].

(a)

(b)

Fig. 4.17 (a) Normalised SAR at 1.5 mm away from the surface of the ETW antenna (b) Computed normalised SAR from the ETW antenna at larger CFLN

It can be observed from Fig. 4.17 (a) that the SAR values along the surface of the catheter are very low, indicating that when the antenna is attached to catheter then the heating on the catheter surface will be minimum. Fig. 4.17 (b) shows the calculated SAR from the ETW antenna at higher CFLN. The computed SAR of the ETW antenna by conventional RS-LOD-FDTD gradually shifts with an increase in CFLN indicating the dispersion effect. Since the return loss and SAR obtained by conventional RS-LODFDTD gradually shifts toward a larger frequency with an increase in CFLN, we need method to reduce the dispersion. Now we model the antenna using D-RS-LOD-FDTD and the results obtained on return loss and SAR are depicted in Figs. 4.18 and 4.19.


Fig. 4.18 Computed return loss from ETW antenna at higher CFLN by D-RS-LOD-FDTD


Fig. 4.19 Computed SAR from ETW antenna at higher CFLN by D-RS-LOD-FDTD

The control parameters used are $\alpha=1.0256$ and $\gamma=1.0256$ for $\mathrm{CFLN}=6$ and $\alpha=1.064$ and $\gamma=1.0725$ for CFLN=10. From Figs. 4.18 and 4.19, it can be observed that the response of the ETW antenna agrees well with the result obtained by explicit RS-FDTD for CFLN=6. Even for CFLN=10, the response shows a close agreement with RS-FDTD result. The computation time required for D-RS-LOD-FDTD and RS-LOD-FDTD methods are shown in Table 4.6. As is shown, the computational time required for the conventional RS-LOD-FDTD at CFLN=2 for calculating return loss is almost the same as D-RS-LOD-FDTD, but when higher CFLN=6 and 10 are used the computational times required for D-RS-LOD-FDTD are reduced to $15 \%$ and $30.33 \%$ respectively compared that required for conventional RS-LOD-FDTD.

Table 4.6
Comparison of computational time for calculating returns loss by different methods with larger CFLN

| CFLN | CPU Time (Sec) |  |
| :--- | :---: | :---: |
|  | D-RS-LOD-FDTD | RS-LOD-FDTD |
| 2 | 145 | 150 |
| 4 | 134 | 143 |
| 6 | 105 | 121 |
| 10 | 75 | 100 |

### 4.9 Discussion

In this chapter, the conventional RS-LOD-FDTD for both TE and TM cases were first presented. Subsequently, the CPML ABC for the RS-LOD-FDTD was developed. Next, to reduce the dispersion and to improve the computational efficiency of the conventional RS-LOD-FDTD method, dispersion control (D)-RS-LOD-FDTD was developed. CPML ABC for D-RS-LOD-FDTD was also presented. Various rotationally symmetric structures such as resonators, open tip monopole antenna as well as ETW antennas were analyzed by the proposed method to demonstrate the validation. From the calculated S-parameters and SAR values, it can be observed that the results obtained by D-RS-LOD-FDTD CPML agree well with the results obtained by RS-FDTD as well as with the measured results in the literature.

The computational performances of the proposed D-RS-LOD-FDTD method in terms of relative error and execution time have also been presented. The result demonstrates that the D-RS-LOD-FDTD method has lower error compared to conventional RS-LODFDTD. Also the method requires lower execution.

## Chapter 5

## Efficient LOD-FDTD Approaches for 3-D Bodies Using Orthogonal Meshes

### 5.1 Introduction

In the chapters 2, 3 and 4 of this thesis, we developed efficient LOD-FDTD methods for analysing 2-D structures using orthogonal and nonorthogonal meshes as well as rotationally symmetric (RS) structures using orthogonal meshes. In this chapter, we present efficient methods for 3-D LOD-FDTD approaches using orthogonal meshes for analysing 3-D microwave and antenna structures. First, we propose a modified two substep LOD-FDTD which we will note as a conventional (C)-LOD-FDTD method along with two sub-step CPML ABC , and we also present its stability analysis. We then develop an efficient fundamental scheme based LOD-FDTD (F-LOD-FDTD) method with CPML using orthogonal meshes and also present its stability analysis. As previously described, various implicit time domain techniques have been developed [8]-[22] to overcome the CFL constraints of the explicit FDTD method. The most popular among them is unconditionally stable locally one-dimensional (LOD)-FDTD method [19], [21]-[22] as it requires fewer arithmetic operations and has a shorter execution time [19] and [21] compared to other implicit FDTD methods. Moreover, in each half step of the LOD-FDTD, it is necessary to deal with only one dimension which also simplifies implementation and eases the computational burden. In the literature, two sub-step [21], [118]-[119] and three sub-step procedures [22], [127]-[128], [131] with 3-D LOD-FDTD are available for analysing 3-D structures. However, the two substep 3-D LOD-FDTD presented by E. L. Tan in [21] requires special input and output processing procedures for computing the field components of interest, which decrease the computational efficiency of the LOD-FDTD method. Later, two sub-step LODFDTD was extended in [118]-[119], [122] to consider lossless and frequency dependent media. Three sub-step 3-D LOD-FDTD by Ahmed et al. was developed in [22], and has
been applied for the analysis of 3-D structures in [127]-[128]. Various ABCs for use with LOD-FDTD to truncate the computational domain are available in the literature [110], [120], [130]-[131], [133]. Ahmed et al. [131] developed CPML ABC for the 3-D LOD-FDTD with three sub-step procedures. However, the two sub-step LOD-FDTD requires fewer arithmetic operations compared to the three sub-step LOD-FDTD, but the CPML ABC was not developed for the two sub-step LOD-FDTD method [21], [118]-[119]. Moreover, to the best of our knowledge, there are few papers that investigated the 3-D LOD-FDTD method for the analysis of microwave structures [118]-[122]. In this chapter, we develop the 3-D C-LOD-FDTD method by modifying Tan's two sub-step method [21] and also develop a new two sub-step CPML ABC. This method will henceforth be denoted as the C-LOD-FDTD CPML method. The proposed method successfully reduces the execution time and memory requirement. Numerical results on 3-D microwave structures will be presented to validate the proposed method. We will demonstrate that the C-LOD-FDTD method has a lower computational burden than the explicit FDTD method.

Even in the implementation of 3-D C-LOD-FDTD, it is necessary to deal with an increased number of complexities due to the presence of a tri-diagonal matrix. Substantial arithmetic operations and field variables are involved on the right-hand side of the update equations, leading to increase in computational time and memory usage. Moreover, the implementation of ABCs such as Mur's ABC [130], PML [120] or CPML [110], [131] into the LOD-FDTD algorithm make the update equations even more complicated, again leading to considerable increase in the number of field variables, arithmetic and memory indexing operations, and resulting in the further degradation of overall computational efficiency. For modelling 3-D microwave structures, in particular, the computational burden could increase drastically. To overcome this problem for implicit FDTD methods, Tan [134] proposed a fundamental scheme to improve computational efficiency. However, for the 3-D fundamental scheme based LOD-FDTD (F-LOD-FDTD), only Mur's [138] and PML [126] ABC have so far been reported in the literature. Hence, in this chapter, we will also present a fundamental scheme LOD-FDTD CPML (F-LOD-FDTD CPML) for analysing 3-D structures.

This chapter is organised as follows: in Section 5.2, C-LOD-FDTD with CPML ABCs is discussed. Stability analysis of the C-LOD-FDTD method will be discussed in Section 5.3. F-LOD-FDTD will be described in Section 5.4 and CPML ABC of F-LODFDTD will be presented in Section 5.4.2. Numerical stability analysis of the F-LODFDTD method will be discussed in Section 5.5. Near-field to far-field transformation for both methods is also presented in Section 5.6. Pure scattered field formulations applicable for both C-LOD-FDTD and F-LOD-FDTD are discussed in Section 5.7. Computational results obtained for the 3-D microwave and antenna structures using C -LOD-FDTD and F-LOD-FDTD methods are presented in Section 5.8. Finally, a brief discussion will be provided in Section 5.9.

### 5.2 Introduction of Conventional 3-D LOD-FDTD (C-LODFDTD) Method

There are two approaches available for formulating the updating equations for the 3D LOD-FDTD method [21]-[22]. Tan et al. [21] proposed a two sub-step approach, whereas Ahmed et al. [22] proposes a three sub-step approach. Ahmed et al. [22] also proposed a CPML ABC for three sub-step LOD-FDTD whereas the CPML for Tan's [21] two sub-step LOD-FDTD is not available in open literature. Clearly, the two substep approach saves computational resources when compared to the three sub-step approach. In this section, we will derive a modified two sub-step C-LOD-FDTD and also derive CPML ABC in generalised form. Before providing the derivation of the C -LOD-FDTD method, we briefly present below Tan's [21] two sub-step 3-D LODFDTD method. According to Tan's two sub-step 3-D LOD-FDTD [21], the implicit updating $\left.E_{x}\right|^{n+3 / 4}$ for the first procedure can be written as follows:

First procedure from $n+(1 / 4)$ to $n+(3 / 4)$ :

$$
\begin{gather*}
-\left.\frac{\Delta t^{2}}{4 \mu \varepsilon \Delta y^{2}} E_{x}\right|_{i+1 / 2, j-1, k} ^{n+3 / 4}+\left.\left(1+\frac{\Delta t^{2}}{2 \mu \varepsilon \Delta y^{2}}\right) E_{x}\right|_{i+1 / 2, j, k} ^{n+3 / 4}-\left.\frac{\Delta t^{2}}{4 \mu \varepsilon \Delta y^{2}} E_{x}\right|_{i+1 / 2, j+1, k} ^{n+3 / 4} \\
=\left.\left(1-\frac{\Delta t^{2}}{2 \mu \varepsilon \Delta y^{2}}\right) E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 4}+\frac{\Delta t^{2}}{4 \mu \varepsilon \Delta y^{2}}\left(\left.E_{x}\right|_{i+1 / 2, j+1, k} ^{n+1 / 4}+\left.E_{x}\right|_{i+1 / 2, j-1, k} ^{n+1 / 4}\right)  \tag{5.1a}\\
+\frac{\Delta t}{\varepsilon \Delta y}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 4}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n+1 / 4}\right)
\end{gather*}
$$

and the explicit updating $\left.H_{z}\right|^{n+3 / 4}$ for the first procedure can be written as follows:

$$
\begin{align*}
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+3 / 4}=\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 4}+\frac{\Delta t}{2 \mu \Delta y}  \tag{5.1b}\\
& \quad\left(\left.E_{x}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}+\left.E_{x}\right|_{i+1 / 2, j+1, k} ^{n+3 / 4}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+3 / 4}\right)
\end{align*}
$$

Similarly, for the second procedure from $n+(3 / 4)$ to $n+1(1 / 4)$, the implicit updating $\left.E_{x}\right|^{n+1(1 / 4)}$ can be written as follows:

$$
\begin{gather*}
-\left.\frac{\Delta t^{2}}{4 \mu \varepsilon \Delta z^{2}} E_{x}\right|_{i+1 / 2, j, k-1} ^{n+1(1 / 4)}+\left.\left(1+\frac{\Delta t^{2}}{2 \mu \varepsilon \Delta z^{2}}\right) E_{x}\right|_{i+1 / 2, j, k} ^{n+1(1 / 4)}-\left.\frac{\Delta t^{2}}{4 \mu \varepsilon \Delta z^{2}} E_{x}\right|_{i+1 / 2,2, k+1} ^{n+1(1 /)} \\
=\left.\left(1-\frac{\Delta t^{2}}{2 \mu \varepsilon \Delta z^{2}}\right) E_{x}\right|_{i+1 / 2, j, k} ^{n+3 / 4}+\frac{\Delta t^{2}}{4 \mu \varepsilon \Delta z^{2}}\left(\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{n+3 / 4}+\left.E_{x}\right|_{i+1 / 2, j, k-1} ^{n+3 / 4}\right)  \tag{5.2a}\\
-\frac{\Delta t}{\varepsilon \Delta z}\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+3 / 4}-\left.H_{y}\right|_{i+1 / 2, j, k-1 / 2} ^{n+3 / 4}\right)
\end{gather*}
$$

and the explicit updating $\left.H_{y}\right|^{n+3 / 4}$ for the second procedure can be written as follows:

$$
\begin{align*}
&\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1(1 / 4)}=\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+3 / 4}-\frac{\Delta t}{2 \mu \Delta z}  \tag{5.2b}\\
& \quad\left(\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{n+3 / 4}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+3 / 4}+\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{n+1(1 / 4)}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1(1 / 4)}\right)
\end{align*}
$$

From equations (5.1) and (5.2), it can be observed that the electric and magnetic fields are updated at quarter (or three-quarter) time steps; therefore it is necessary to relate the fields to an integer time step. This results in extra processing for the input and output field data as follows (the coefficient $1 / 16=1 / 4^{2}$ corresponds to the quarter-step updating). Input processing at $n=0$, the implicit $\left.E_{x}\right|^{1 / 4}$ can be written as follows:

$$
\begin{gather*}
-\left.\frac{\Delta t^{2}}{16 \mu \varepsilon \Delta z^{2}} E_{x}\right|_{i+1 / 2, j, k-1} ^{1 / 4}+\left.\left(1+\frac{\Delta t^{2}}{8 \mu \varepsilon \Delta z^{2}}\right) E_{x}\right|_{i+1 / 2, j, k} ^{1 / 4}-\left.\frac{\Delta t^{2}}{16 \mu \varepsilon \Delta z^{2}} E_{x}\right|_{i+1 / 2, j, k+1} ^{1 / 4} \\
=\left.\left(1-\frac{\Delta t^{2}}{8 \mu \varepsilon \Delta z^{2}}\right) E_{x}\right|_{i+1 / 2, j, k} ^{0}+\frac{\Delta t^{2}}{16 \mu \varepsilon \Delta z^{2}}\left(\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{0}+\left.E_{x}\right|_{i+1 / 2, j, k-1} ^{0}\right)  \tag{5.3a}\\
-\frac{\Delta t}{2 \varepsilon \Delta z}\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{0}-\left.H_{y}\right|_{i+1 / 2, j, k-1 / 2} ^{0}\right)
\end{gather*}
$$

and input processing for the explicit updating $\left.H_{y}\right|^{1 / 4}$ can be written as

$$
\begin{align*}
& \left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{1 / 4}=\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{0}-\frac{\Delta t}{4 \mu \Delta z}  \tag{5.3b}\\
& \quad\left(\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{0}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{0}+\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{1 / 4}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{1 / 4}\right)
\end{align*}
$$

Output processing at $\mathrm{n}+1$ for the implicit $\left.E_{x}\right|^{n+1}$ can be written as

$$
\begin{gather*}
-\left.\frac{\Delta t^{2}}{16 \mu \varepsilon \Delta z^{2}} E_{x}\right|_{i+1 / 2, j, k-1} ^{n+1}+\left.\left(1+\frac{\Delta t^{2}}{8 \mu \varepsilon \Delta z^{2}}\right) E_{x}\right|_{i+1 / 2, j, k} ^{n+1}-\left.\frac{\Delta t^{2}}{16 \mu \varepsilon \Delta z^{2}} E_{x}\right|_{i+1 / 2, j, k+1} ^{n+1} \\
=\left.\left(1-\frac{\Delta t^{2}}{8 \mu \varepsilon \Delta z^{2}}\right) E_{x}\right|_{i+1 / 2, j, k} ^{n+1(1 / 4)}+\frac{\Delta t^{2}}{16 \mu \varepsilon \Delta z^{2}}\left(\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{n+1(1 / 4)}+\left.E_{x}\right|_{i_{i+1 / 2, j, j-k-1}^{n+1(1 / 4)}} ^{16+1}\right)  \tag{5.4a}\\
- \\
-\frac{\Delta t}{2 \varepsilon \Delta z}\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1(1 / 4)}-\left.H_{y}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1(1 / 4)}\right)
\end{gather*}
$$

and the explicit updating $\left.H_{y}\right|^{n+1}$ for the output processing can be written as follows

$$
\begin{align*}
& \left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}=\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1(1 / 4)}-\frac{\Delta t}{4 \mu \Delta z}  \tag{5.4b}\\
& \quad\left(\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{n+1(1 / 4)}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1(1 / 4)}+\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{n+1}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1}\right)
\end{align*}
$$

However, from the above equation, it can be observed that Tan's two sub-step 3-D LOD-FDTD in [21] requires special input and output processing procedures for computing the field components of interest; as a result, they decrease the computational efficiency of the LOD-FDTD method. Two sub-step LOD-FDTD was modified in [118]-[119] considering lossless and frequency dependent media. In this chapter, we develop two sub-step 3-D LOD-FDTD (which we will call C-LOD-FDTD) in generalised form that can be used in both lossy and lossless media. Applying the LOD principle to the Maxwell equations (1.1) (available in Chapter 1), the C-LOD-FDTD in generalised form can be written as follows:

Sub-step 1:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{2}[A]+\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1 / 2}=\left([I]+\frac{\Delta t}{2}[A]-\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n} \tag{5.5a}
\end{equation*}
$$

Sub-step 2:

$$
\begin{equation*}
\left([I]-\frac{\Delta t}{2}[B]+\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1}=\left([I]+\frac{\Delta t}{2}[B]-\frac{\Delta t}{4}[L]\right) \mathbf{u}^{n+1 / 2} \tag{5.5b}
\end{equation*}
$$

where $\mathbf{u},[A],[B]$, and $[L]$ are the same as mentioned earlier in Chapter 1. By substituting the value of $\mathbf{u},[A],[B]$ and $[L]$ in (5.5a)-(5.5b), and moving forward for each half time step separately either in the $X, Y$ or $Z$ direction, we obtain the C-LODFDTD equations. It can be seen from (5.5) that the formulation is easy compared to Tan's procedure. We can easily apply the wave source in (5.5) without requiring extra processing time, which results in an efficient approach. The wave source conditions for the LOD-FDTD method have been described in Section 2.6 of Chapter 2. Equation (5.5)
also agrees with the two sub-steps LOD-FDTD provided in [118]-[119]. This approach can also be converted to Tan's method [21] by invoking $n=n+1 / 4$.

### 5.2.1 Updating Equation of 3-D C-LOD-FDTD Method

By placing the value of $\mathbf{u},[A]$ and $[B]$ in (5.5a) and (5.5b), we obtain equations (2.2a)-(2.2f) and (2.3a)-(2.3b) in Section 2.2.1 in Chapter 2 for the two sub-steps. The updating equations with indices for the 3-D C-LOD-FDTD method from (2.2)-(2.3) (from Chapter 2) in generalised form can be written into two sub-steps as follows:
Sub-step 1:

$$
\begin{align*}
& \left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}=C_{\text {exe }} \times\left. E_{x}\right|_{i+1 / 2, j, k} ^{n}+C_{e x h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n+1 / 2}\right) \\
& +C_{e x h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right)  \tag{5.6a}\\
& \left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}=C_{e y e} \times\left. E_{y}\right|_{i, j+1 / 2, k} ^{n}+C_{e y h x} \times\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{x}\right|_{i, j+1 / 2, k-1 / 2} ^{n+1 / 2}\right) \\
& +C_{e y h x} \times\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{x}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right)  \tag{5.6b}\\
& \left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}=C_{e z e} \times\left. E_{z}\right|_{i, j, k+1 / 2} ^{n}+C_{e z h y} \times\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{y}\right|_{i-1 / 2, j, k+1 / 2} ^{n+1 / 2}\right) \\
& +C_{e z h y} \times\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{y}\right|_{i-1 / 2, j, k+1 / 2} ^{n}\right)  \tag{5.6c}\\
& \left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}=C_{h x h} \times\left. H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n}+C_{h x e y} \times \\
& \left(\left.E_{y}\right|_{i, j+1 / 2, k+1} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right)+C_{h x e y} \times\left(\left.E_{y}\right|_{i, j+1 / 2, k+1} ^{n}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n}\right)  \tag{5.6d}\\
& \left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}=C_{h y h} \times\left. H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n}+C_{h y e z} \times \\
& \left(\left.E_{z}\right|_{i+1, j, k+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right)+C_{\text {hyez }} \times\left(\left.E_{z}\right|_{i+1, j, k+1 / 2} ^{n}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n}\right)  \tag{5.6e}\\
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}=C_{h z h} \times\left. H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}+C_{h z e x} \times \\
& \left(\left.E_{x}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right)+C_{h z e x} \times\left(\left.E_{x}\right|_{i+1 / 2, j+1, k} ^{n}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n}\right) \tag{5.6f}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1}=C_{e x e} \times\left. E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}+C_{e x h y} \times\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{y}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1}\right) \\
& +C_{e x h y} \times\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{y}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1 / 2}\right) \tag{5.7a}
\end{align*}
$$

$$
\begin{align*}
& \left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1}=C_{e y e} \times\left. E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}+C_{e y h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2, k} ^{n+1}\right) \\
& +C_{e y h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2, k} ^{n+1 / 2}\right)  \tag{5.7b}\\
& \left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1}=C_{e z e} \times\left. E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}+C_{e z h x} \times\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}-\left.H_{x}\right|_{i, j-1 / 2, k+1 / 2} ^{n+1}\right) \\
& +C_{e z h x} \times\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{x}\right|_{i, j-1 / 2, k+1 / 2} ^{n+1 / 2}\right)  \tag{5.7c}\\
& \left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}=C_{h x h} \times\left. H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}+C_{h x e z} \times \\
& \left(\left.E_{z}\right|_{i, j+1, k+1 / 2} ^{n+1}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1}\right)+C_{h x e z} \times\left(\left.E_{z}\right|_{i, j+1, k+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right)  \tag{5.7d}\\
& \left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}=C_{h y h} \times\left. H_{y}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}+C_{h y e x} \times \\
& \left(\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{n+1}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1}\right)+C_{h y e x} \times\left(\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{n+1 / 2}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right)  \tag{5.7e}\\
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}=C_{h z h} \times\left. H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}+C_{h z e y} \times \\
& \left(\left.E_{y}\right|_{i+1, j+1 / 2, k} ^{n+1}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1}\right)+C_{h z e y} \times\left(\left.E_{y}\right|_{i+1, j+1 / 2, k} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right)  \tag{5.7f}\\
& \text { where } C_{e x e}=C_{e y e}=C_{e z e}=\left(4 \varepsilon-\sigma_{e} \Delta t\right) /\left(4 \varepsilon+\sigma_{e} \Delta t\right), C_{e x h z}=2 \Delta t / \Delta y\left(4 \varepsilon+\sigma_{e} \Delta t\right), \\
& C_{e y h x}=2 \Delta t / \Delta z\left(4 \varepsilon+\sigma_{e} \Delta t\right) C_{e z h y}=2 \Delta t / \Delta x\left(4 \varepsilon+\sigma_{e} \Delta t\right) \\
& C_{h x h}=C_{h y h}=C_{h z h}=\left(4 \mu-\sigma_{m} \Delta t\right) /\left(4 \mu+\sigma_{m} \Delta t\right), C_{h x e y}=2 \Delta t / \Delta z\left(4 \mu+\sigma_{m} \Delta t\right) \\
& C_{h y e z}=2 \Delta t / \Delta x\left(4 \mu+\sigma_{m} \Delta t\right) C_{h z e x}=2 \Delta t / \Delta y\left(4 \mu+\sigma_{m} \Delta t\right)
\end{align*}
$$

From (5.6)-(5.7), it can be observed that the equations of sub-steps 1 and 2 are not directly solvable, so by placing (5.6f) into (5.6a), (5.6d) into (5.6b) and (5.6e) into (5.6c), a simultaneous linear system with tri-diagonal matrix can be obtained for $E_{x}$, $E_{y}$ and $E_{z}$ field components in sub-step 1 as follow:

$$
\begin{align*}
-\left.\alpha_{1} E_{x}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2}+ & \left.\gamma_{1} E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}-\left.\beta_{1} E_{x}\right|_{i+1 / 2, j-1, k} ^{n+1 / 2} \\
= & \left.C_{e x e} E_{x}\right|_{i+1 / 2, j, k} ^{n}+C_{e x h z} \times C_{h z h}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right)  \tag{5.8a}\\
& +\alpha_{1}\left(\left.E_{x}\right|_{i+1 / 2, j+1, k} ^{n}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n}+\left.E_{x}\right|_{i+1 / 2, j-1, k} ^{n}\right) \\
& +C_{e x h z}^{n}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right) \\
-\left.\alpha_{2} E_{y}\right|_{i, j+1 / 2, k+1} ^{n+1 / 2}+ & \left.\gamma_{2} E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}-\left.\beta_{1} E_{y}\right|_{i, j+1 / 2, k-1} ^{n+1 / 2} \\
= & \left.C_{e y e c} E_{y}\right|_{i, j+1 / 2, k} ^{n}+C_{e y h x}^{n} \times C_{h x h}\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{x}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right)  \tag{5.8b}\\
& +\alpha_{2}\left(\left.E_{y}\right|_{i, j+1 / 2, k+1} ^{n}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n}+\left.E_{y}\right|_{i, j+1 / 2, k-1} ^{n}\right) \\
& +C_{e y h x}\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{x}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right)
\end{align*}
$$

$$
\begin{align*}
-\left.\alpha_{3} E_{z}\right|_{i+1, j, j+k+1 / 2} ^{n+1 / 2}+ & \left.\gamma_{3} E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}-\left.\beta_{3} E_{z}\right|_{i-1, j, k+1 / 2} ^{n+1 / 2} \\
& =\left.C_{e z e} E_{z}\right|_{i, j, k+k+1 / 2} ^{n}+C_{e z h y} \times C_{h y h}\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{y}\right|_{i-1 / 2, j, j+k+1 / 2} ^{n}\right)  \tag{5.8c}\\
& +\alpha_{3}\left(\left.E_{z}\right|_{i+1, j, k+1 / 2} ^{n}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n}+\left.E_{z}\right|_{i-1, j, k+1 / 2} ^{n}\right) \\
& +C_{e z h y}\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{y}\right|_{i-1 / 2, j, k+1 / 2} ^{n}\right)
\end{align*}
$$

where $\alpha_{1}=C_{\text {exhz }} \times C_{\text {hzex }}, \beta_{1}=C_{\text {exhz }} \times C_{\text {hzex }}, \gamma_{1}=1-\alpha_{1}-\beta_{1}$. Similarly for sub-step 2, we obtain simultaneous linear system with tri-diagonal matrix for $E_{x}, E_{y}$ and $E_{z}$ field components, by placing (5.7f) into (5.7a), (5.7d) into (5.7b) and (5.7e) into (5.7c). The simultaneous linear equation with tri-diagonal matrix can be solved by following approach A) as described in Section 2.3.3 in Chapter 2. The derivations of CPML ABC for the C-LOD-FDTD method are described briefly in the next section.

### 5.2.2 CPML ABC for the 3D C-LOD-FDTD

The starting point of the CPML formulation for C-LOD-FDTD is a PML medium assumed to terminate a finite space occupied by a host medium, as shown in Fig. 5.1. The CPML for 3D LOD-FDTD that was originally derived by Ahmed et al. [131] uses a three sub-step approach.


Fig. 5.1 3-D structure for LOD-FDTD grid employing the CPML ABC

To make it more efficient, we have reformulated 3-D LOD-FDTD-CPML using only two sub-steps following the method given in [21] which will be labeled as conventional (C)-LOD-FDTD-CPML. The merits of CPML ABC have been described in Chapter 2 and 3 so will not be repeated here. Based on the generalised LOD splitting formulae, the two sub-steps of 3-D C-LOD-FDTD with CPML can be derived as follows:
Sub-step 1:

$$
\begin{align*}
\left(I-\frac{\Delta t}{2} \mathbf{A}+\frac{\Delta t}{4}[L]\right) \mathbf{U}^{n+1 / 2} & =\left(I+\frac{\Delta t}{2} \mathbf{A}-\frac{\Delta t}{4}[L]\right) \mathbf{U}^{n}+\frac{\Delta t}{2} \mathrm{X} \psi^{n}  \tag{5.9a}\\
\psi^{n} & =\mathrm{C} \psi^{n-1 / 2}+\mathrm{D} \mathbf{U}^{n} \tag{5.9b}
\end{align*}
$$

Sub-step 2:

$$
\begin{gather*}
\left(I-\frac{\Delta t}{2} \mathbf{B}+\frac{\Delta t}{4}[L]\right) \mathbf{U}^{n+1}=\left(I+\frac{\Delta t}{2} \mathbf{B}-\frac{\Delta t}{4}[L]\right) \mathbf{U}^{n+1 / 2}+\frac{\Delta t}{2} \mathrm{X} \psi^{n+1 / 2}  \tag{5.10a}\\
\psi^{n+1 / 2}=\mathrm{C} \psi^{n}+\mathrm{D} \mathbf{U}^{n+1 / 2} \tag{5.10b}
\end{gather*}
$$

where

$$
\begin{aligned}
& \mathrm{X}=\left[\begin{array}{ll}
\frac{1}{\varepsilon} \Theta & \mathrm{O}_{3 \times 6} \\
\mathrm{O}_{3 \times 6} & \frac{1}{\mu} \Theta
\end{array}\right] \Theta=\left[\begin{array}{rrrrrc}
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1
\end{array}\right] \\
& \mathrm{C}=\left[\begin{array}{ll}
C_{e} & \mathrm{O}_{6 \times 6} \\
\mathrm{O}_{6 \times 6} & C_{h}
\end{array}\right] \mathrm{D}=\left[\begin{array}{ll}
\mathrm{O}_{6 \times 6} & \mathrm{D}_{e} \\
\mathrm{D}_{h} & \mathrm{O}_{6 \times 6}
\end{array}\right] \\
& D_{e}=\left[\begin{array}{cccc}
0 & 0 & d_{y} \frac{\partial}{\partial y} \\
0 & d_{z} & \frac{\partial}{\partial z} & 0 \\
d_{z} & \frac{\partial}{\partial z} & 0 & 0 \\
0 & 0 & d_{x} \frac{\partial}{\partial x} \\
0 & d_{x} \frac{\partial}{\partial x} & 0 \\
d_{y} \frac{\partial}{\partial y} & 0 & 0
\end{array}\right] D_{h}=\left[\begin{array}{cccc}
0 & d_{z} \frac{\partial}{\partial z} & 0 \\
0 & 0 & d_{y} \frac{\partial}{\partial y} \\
0 & 0 & d_{x} \frac{\partial}{\partial x} \\
d_{z} \frac{\partial}{\partial z} & 0 & 0 \\
d_{y} \frac{\partial}{\partial y} & 0 & 0 \\
0 & d_{x} \frac{\partial}{\partial x} & 0
\end{array}\right]
\end{aligned}
$$

$$
C_{e}=\left[\begin{array}{llllll}
c_{y} & 0 & 0 & 0 & 0 & 0 \\
0 & c_{z} & 0 & 0 & 0 & 0 \\
0 & 0 & c_{z} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{x} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{x} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{y}
\end{array}\right], C_{h}=\left[\begin{array}{llllll}
c_{z} & 0 & 0 & 0 & 0 & 0 \\
0 & c_{y} & 0 & 0 & 0 & 0 \\
0 & 0 & c_{x} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{z} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{y} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{x}
\end{array}\right]
$$

By substituting the value of $\mathbf{u},[A],[B], X, C$ and $D$ in the (5.9a)-(5.9b) and (5.10a)(5.10b), the updating equations of the C-LOD-FDTD CPML can be written as follow: Sub-step 1:

$$
\begin{align*}
& \left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}=C_{e x e} \times\left. E_{x}\right|_{i+1 / 2, j, k} ^{n}+C_{e x h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n+1 / 2}\right) \\
& +C_{e x h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right)+\left.2 C_{\psi e x z} \psi_{E x z}\right|_{i+1 / 2, j, k} ^{n}  \tag{5.11a}\\
& \left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}=C_{e y e} \times\left. E_{y}\right|_{i, j+1 / 2, k} ^{n}+C_{e y h x} \times\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{x}\right|_{i, j+1 / 2, k-1 / 2} ^{n+1 / 2}\right) \\
& +C_{e y h x} \times\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{x}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right)+\left.2 C_{\psi e y x} \psi_{E y x}\right|_{i, j+1 / 2, k} ^{n}  \tag{5.11b}\\
& \left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}=C_{e z e} \times\left. E_{y}\right|_{i, j, k+1 / 2} ^{n}+C_{e z h y} \times\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{y}\right|_{i-1 / 2, j, k+1 / 2} ^{n+1 / 2}\right) \\
& +C_{e z h y} \times\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{y}\right|_{i-1 / 2, j, k+1 / 2} ^{n}\right)+\left.2 C_{\psi e z y} \psi_{E z y}\right|_{i, j, k+1 / 2} ^{n}  \tag{5.11c}\\
& \left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}=\left.C_{h x h} H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n} \\
& +C_{\text {hxey }}\left(\left.E_{y}\right|_{i, j+1 / 2, k+1} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right)  \tag{5.11d}\\
& +C_{\text {hxey }}\left(\left.E_{y}\right|_{i, j+1 / 2, k+1} ^{n}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n}\right)+\left.2 C_{\psi h x y} \psi_{H x y}\right|_{i, j+1 / 2, k+1 / 2} ^{n} \\
& \left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}=C_{h y h} \times\left. H_{y}\right|_{i, j+1 / 2, k+1 / 2} ^{n} \\
& +C_{\text {hyez }} \times\left(\left.E_{z}\right|_{i+1, j, k+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right)  \tag{5.11e}\\
& +C_{\text {hyez }} \times\left(\left.E_{z}\right|_{i+1, j, k+1 / 2} ^{n}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n}\right)+\left.2 C_{\psi h y z} \psi_{H y z}\right|_{i+1 / 2, j, k+1 / 2} ^{n} \\
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}=C_{h z h} \times\left. H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n} \\
& +C_{h z e x} \times\left(\left.E_{x}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right)  \tag{5.11f}\\
& +C_{h z e x} \times\left(\left.E_{x}\right|_{i+1 / 2, j+1, k} ^{n}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n}\right)+\left.2 C_{\psi h z x} \psi_{H z x}\right|_{i+1 / 2, j+1 / 2, k} ^{n}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1}=C_{\text {exe }} \times\left. E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}+C_{e x h y} \times\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{y}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1}\right) \\
& +C_{e x h y} \times\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{y}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1 / 2}\right)+\left.2 C_{\psi e x y} \psi_{E x y}\right|_{i+1 / 2, j, k} ^{n+1 / 2}  \tag{5.12a}\\
& \left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1}=C_{e y e} \times\left. E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}+C_{e y h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2, k} ^{n+1}\right) \\
& +C_{e y h z} \times\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2, k} ^{n+1 / 2}\right)+\left.2 C_{\psi e y z} \psi_{E y z}\right|_{i, j+1 / 2, k} ^{n+1 / 2}  \tag{5.12b}\\
& \left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1}=C_{e z e} \times\left. E_{y}\right|_{i, j, k+1 / 2} ^{n+1 / 2}+C_{e z h x} \times\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}-\left.H_{x}\right|_{i, j-1 / 2, k+1 / 2} ^{n+1}\right) \\
& +C_{e z h x} \times\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{x}\right|_{i, j-1 / 2, k+1 / 2} ^{n+1 / 2}\right)+\left.2 C_{\psi e z x} \psi_{E z x}\right|_{i, j, k+1 / 2} ^{n+1 / 2}  \tag{5.12c}\\
& \left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}=C_{h x h} \times\left. H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2} \\
& +C_{\text {hxez }} \times\left(\left.E_{z}\right|_{i, j+1, k+1 / 2} ^{n+1}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1}\right)  \tag{5.12d}\\
& +C_{h x e z} \times\left(\left.E_{z}\right|_{i, j+1, k+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right)+\left.2 C_{\psi h x z} \psi_{H x z}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2} \\
& \left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}=C_{h y h} \times\left. H_{y}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2} \\
& +C_{\text {hyex }} \times\left(\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{n+1}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1}\right)  \tag{5.12e}\\
& +C_{\text {hyex }} \times\left(\left.E_{x}\right|_{i+1 / 2, j, k+1} ^{n+1 / 2}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right)+\left.2 C_{\psi h y x} \psi_{H y x}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2} \\
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}=C_{h z h} \times\left. H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2} \\
& +C_{\text {hzey }} \times\left(\left.E_{y}\right|_{i+1, j+1 / 2, k} ^{n+1}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1}\right)  \tag{5.12f}\\
& +C_{\text {hzey }} \times\left(\left.E_{y}\right|_{i+1, j+1 / 2, k} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right)+\left.2 C_{\psi h z y} \psi_{H z y}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2} \\
& \text { where, } C_{e x h z}=1 / \kappa_{e y} \times C_{e x h z}, C_{e y h x}=1 / \kappa_{e z} \times C_{e y h x} C_{e z h y}=1 / \kappa_{e x} \times C_{e z h y} \\
& C_{h x e y}=1 / \kappa_{h y} \times C_{h x e y} \quad C_{h y e z}=1 / \kappa_{h z} \times C_{h y e z} C_{h z e x}=1 / \kappa_{h x} \times C_{h z e x} \text {, and } C_{\text {wexz }}=\Delta y \times C_{e x h z}
\end{align*}
$$

where the CPML terms $\psi_{e x_{i}+1 / 2, j / k}^{n+1 / 2}$ can be written as follows:

$$
\begin{equation*}
\psi_{e x y_{1+1 / 2, j, k}}^{n+1 / 2}=c_{y_{j}} \psi_{e x x_{i+1 / 2, j k}}^{n}+\frac{d_{y_{j}}}{2 \Delta y}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right) \tag{5.13a}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{s}=e^{-\left(\left(\sigma_{s} / \kappa_{s}\right)+\alpha_{s}\right)\left(\Delta t / \varepsilon_{0}\right)} \tag{5.13b}
\end{equation*}
$$

$$
\begin{gather*}
d_{s}=\frac{\sigma_{s}}{\kappa_{s}\left(\sigma_{s}+\kappa_{s} \sigma_{s}\right)} \times\left(c_{s}-1\right), \quad(s=x, y, \text { or } z)  \tag{5.13c}\\
\sigma_{s}(s)=\frac{\sigma_{s_{\max }}\left|s-s_{0}\right|^{m}}{\delta^{m}} \tag{5.13d}
\end{gather*}
$$

$$
\begin{equation*}
\kappa_{s}(s)=1+\left(\kappa_{\max }-1\right) \frac{\left|s-s_{0}\right|^{m}}{\delta^{m}} \tag{5.13e}
\end{equation*}
$$

where $\delta$ is the thickness of the PML absorber, $s_{0}$ is the PML interface, $m$ is the order of the polynomial. The other CPML terms can be written in a similar way. From (5.11)(5.12), it is seen that the equations of sub-steps 1 and 2 are not directly solvable, so by placing (5.11f) into (5.11a), (5.11d) into (5.11b) and (5.11e) into (5.11c), a tri-diagonal linear system can be obtained for the $E_{x}, E_{y}$ and $E_{z}$ field components in sub-step 1 which are similar to equations (5.8a)-(5.8c). Similarly for sub-step 2 , we obtain the tridiagonal linear system for $E_{x}, E_{y}$ and $E_{z}$ field components, by placing (5.12f) into (5.12a), (5.12d) into (5.12b) and (5.12e) into (5.12c). It can be observed that each of (5.11)-(5.12) contains only one auxiliary variable $\psi$. These auxiliary variables are used in the CPML region only and are zero in the remaining regions. Compared to FDTD CPML and ADI-FDTD CPML, where 12 auxiliary variables are needed for FDTDCPML, and 24 are needed for ADI-FDTD CPML, C-LOD-FDTD-CPML needs only one auxiliary variable in each equation, i.e. 12. Although the number of auxiliary variables for LOD-FDTD-CPML is the same as FDTD CPML, it is half of ADI-FDTD CPML. Only one auxiliary variable is enough for C-LOD-FDTD-CPML in each equation; the reasons are described below, following the procedure provided in [131].

### 5.2.2.1 Required Auxiliary Variables in C-LOD-FDTD CPML

For elucidation, (5.11b) is considered with two auxiliary variables, similar to FDTD CPML and ADI-FDTD CPML which can be written as

$$
\begin{equation*}
\left.E_{y}\right|^{n+1 / 2}=\left.E_{y}\right|^{n}+\frac{\Delta t}{2 \varepsilon}\left(\frac{\left.\partial H_{x}\right|^{n+1 / 2}}{d z}+\frac{\left.\partial H_{x}\right|^{n}}{d z}+\left.\psi_{E y z}\right|^{n+1 / 2}+\left.\psi_{E y z}\right|^{n}\right) \tag{5.14}
\end{equation*}
$$

In this equation, the auxiliary term $\left.\psi_{E y z}\right|^{n+1 / 2}$ contains the same time index $(n+1 / 2)$ as that of the field on the left side. This additional variable at $(n+1 / 2)$ does not affect numerical results significantly except to contribute to additional complexities in the computation. Therefore, for the sake of simplification, both auxiliary variables will be considered at the same time index " $n$ " and as a result (5.14) reduces to (5.11b). The auxiliary variable is responsible for wave absorption in the ABCs.

### 5.2.2.2 Validation

To validate the CPML implementation for the 3-D C-LOD-FDTD, we consider a free space domain as shown in Fig. 5.2 with $46 \times 46 \times 46$ cells including 8 layers of CPML in each direction. The observation point, where the $z$ directed electric field is recorded, is located at 8 cells away from the source. Gaussian pulse is used as a source.


Fig. 5.2 Free space domain employing the CPML ABC

To minimise the reflection error, the following parameters for CPML are considered. $\sigma_{\text {opt }}=\frac{m+1}{150 \pi \Delta x}=11.21(\mathrm{~S} / \mathrm{m}), \quad \sigma_{s_{\max }}=0.7 \sigma_{\text {opt }}(\mathrm{S} / \mathrm{m}), \quad k_{s_{\max }}=15$ and $m=4$. Fig. 5.3 shows the reflection error for C-LOD-FDTD CPML for CFLN=2, 6 and 10. Fig. 5.4 shows the reflection error for the 8 cell CPML layers of the C-LOD-FDTD method for CFLN=2 and 6 .


Fig. 5.3 Reflection error for different CFLN


Fig. 5.4 Reflection error for 8 CPML layer at CFLN=2 and 6

Note that CFLN is defined as the ratio between the time step in the LOD-FDTD and the maximum CFL limit in the standard FDTD. It can be observed from Fig. 5.3 and 5.4 that the proposed C-LOD-FDTD CPML has less reflection error. It can also be observed that the proposed approach provides less reflection error at lower CFLN. The CPML ABC has been used to analyse 3-D microwave structures in Section 5.8. .

### 5.3 Stability Analysis of the C-LOD-FDTD Method

### 5.3.1 Theoretical Stability Analysis

The Von Neumann method [166] has been used as a standard approach for the stability analysis of an unconditionally stable FDTD method where eigenvalues of the amplification matrix on the spectral domain are evaluated. If all the eigenvalues of the amplification matrix are no larger than unity in magnitude, the method is considered to be stable. In this section, we demonstrate the unconditional stability of the C-LODFDTD method using the Von Neumann method [166]. The equations (5.6)-(5.7) are first expressed in matrix form. Since the stability analysis of the C-LOD-FDTD requires detailed mathematical procedure, only the key results of the analysis are summarised below. The field components in the spatial spectral domain are assumed to be in the following form.

$$
\begin{align*}
& E_{r}^{n}(i, j, k)=E_{r}^{n} e^{-j\left(k_{x} i \Delta x+k_{y}, j \Delta y+k_{z} k \Delta z\right)}  \tag{5.15a}\\
& H_{r}^{n}(i, j, k)=H_{r}^{n} e^{-j\left(k_{i} \Delta \Delta x+k_{y}, j \Delta y+k_{z} k \Delta z\right)} \tag{5.15b}
\end{align*}
$$

where $r=x, y, z$ while $k_{x}, k_{y}$ and $k_{z}$ are wave numbers along the $x, y, z$ directions respectively. By substituting (5.15a) and (5.15b) into (5.6) and (5.7) and considering the lossless case, the following equations can be obtained for sub-steps 1 and 2.

Sub-step 1:

$$
\begin{align*}
U^{n+1 / 2}=\Lambda_{1} U^{n} & \text { where } \Lambda_{1} \text { is } 6 \times 6 \text { matrix }  \tag{5.16}\\
\Lambda_{1} & =\left[\begin{array}{cccccc}
\frac{G_{y}}{Q_{y}} & 0 & 0 & 0 & 0 & -\frac{i 2 A_{y}}{\varepsilon Q_{y}} \\
0 & \frac{G_{z}}{Q_{z}} & 0 & -\frac{i 2 A_{z}}{\varepsilon Q_{z}} & 0 & 0 \\
0 & 0 & \frac{G_{x}}{Q_{x}} & 0 & -\frac{i 2 A_{x}}{\varepsilon Q_{x}} & 0 \\
0 & -\frac{i 2 A_{z}}{\mu Q_{z}} & 0 & \frac{G_{z}}{Q_{z}} & 0 & 0 \\
0 & 0 & -\frac{i 2 A_{x}}{\mu Q_{x}} & 0 & \frac{G_{x}}{Q_{x}} & 0 \\
-\frac{i 2 A_{y}}{\mu Q_{y}} & 0 & 0 & 0 & 0 & \frac{G_{y}}{Q_{y}}
\end{array}\right]
\end{align*}
$$

Sub-step 2:

$$
\begin{gather*}
U^{n+1}=\Lambda_{2} U^{n+1 / 2}  \tag{5.17}\\
\Lambda_{2}=\left[\begin{array}{cccccc}
\frac{G_{z}}{Q_{z}} & 0 & 0 & 0 & \frac{i 2 A_{z}}{\varepsilon Q_{z}} & 0 \\
0 & \frac{G_{x}}{Q_{x}} & 0 & 0 & 0 & \frac{i 2 A_{x}}{\varepsilon Q_{x}} \\
0 & 0 & \frac{G_{y}}{Q_{y}} & \frac{i 2 A_{y}}{\varepsilon Q_{y}} & 0 & 0 \\
0 & 0 & \frac{i 2 A_{y}}{\mu Q_{y}} & \frac{G_{y}}{Q_{y}} & 0 & 0 \\
\frac{i 2 A_{z}}{\mu Q_{z}} & 0 & 0 & 0 & \frac{G_{z}}{Q_{z}} & 0 \\
0 & \frac{i 2 A_{x}}{\mu Q_{x}} & 0 & 0 & 0 & \frac{G_{x}}{Q_{x}}
\end{array}\right]
\end{gather*}
$$

By combining equations (5.16) and (5.17), we obtain the following equation,

$$
\begin{equation*}
U^{n+1}=\Lambda_{1} \Lambda_{2} U^{n}=\Lambda U^{n} \tag{5.18}
\end{equation*}
$$

where $\Lambda=\Lambda_{1} \Lambda_{2}$

$$
\Lambda=\left[\begin{array}{lccccc}
\frac{G_{y} G_{z}}{Q_{y} Q_{z}} & \frac{4 A_{x} A_{y}}{\mu \varepsilon Q_{x} Q_{y}} & 0 & 0 & \frac{i 2 A_{z} G_{y}}{\varepsilon Q_{y} Q_{z}} & -\frac{i 2 A_{y} G_{x}}{\varepsilon Q_{x} Q_{y}}  \tag{5.19}\\
0 & \frac{G_{x} G_{z}}{Q_{x} Q_{z}} & \frac{4 A_{y} A_{z}}{\mu \varepsilon Q_{y} Q_{z}} & -\frac{i 2 A_{z} G_{y}}{\varepsilon Q_{y} Q_{z}} & 0 & \frac{i 2 A_{x} G_{z}}{\varepsilon Q_{x} Q_{z}} \\
\frac{4 A_{x} A_{z}}{\mu \varepsilon Q_{x} Q_{z}} & 0 & \frac{G_{x} G_{y}}{Q_{x} Q_{y}} & \frac{i 2 A_{y} G_{x}}{\varepsilon Q_{x} Q_{y}} & -\frac{i 2 A_{x} G_{z}}{\varepsilon Q_{x} Q_{z}} & 0 \\
0 & -\frac{i 2 A_{z} G_{x}}{\mu Q_{x} Q_{z}} & \frac{i 2 A_{y} G_{z}}{\mu Q_{y} Q_{z}} & \frac{G_{y} G_{z}}{Q_{y} Q_{z}} & 0 & \frac{4 A_{x} A_{z}}{\mu \varepsilon Q_{x} Q_{z}} \\
\frac{i 2 A_{z} G_{x}}{\mu Q_{x} Q_{z}} & 0 & -\frac{i 2 A_{x} G_{y}}{\mu Q_{x} Q_{y}} & \frac{4 A_{x} A_{y}}{\mu \varepsilon Q_{x} Q_{y}} & \frac{G_{x} G_{z}}{Q_{x} Q_{z}} & 0 \\
-\frac{i 2 A_{y} G_{z}}{\mu Q_{y} Q_{z}} & \frac{i 2 A_{x} G_{y}}{\mu Q_{x} Q_{y}} & 0 & 0 & \frac{4 A_{y} A_{z}}{\mu \varepsilon Q_{y} Q_{z}} & \frac{G_{x} G_{y}}{Q_{x} Q_{y}}
\end{array}\right]
$$

Six eigenvalues are obtained from (5.19). Here, only the numerical computed eigenvalues are provided for the verification of the unconditional stability of the C -LOD-FDTD method. We compute the eigenvalues for the case of $k_{x}=0.2, k_{y}=0.2$ and $k_{z}=0.2$, and $\Delta x=\Delta y=\Delta z=1 \mathrm{~mm}$, and the computed eigenvalues are tabulated in Table 5.1. From Table 5.1, we observe that the magnitudes of the eigenvalues are never larger than unity. Hence, it can be considered that the C-LOD-FDTD method is unconditionally stable.

Table 5.1
Computed eigenvalues of C-LOD-FDTD

| Eigen values | CFLN=2 | CFLN=4 | CFLN=6 |
| :---: | :---: | :---: | :---: |
| $\lambda_{1}$ | 1.00000 | 1.00000 | 1.00000 |
| $\lambda_{2}$ | 0.99999 | 0.99989 | 0.99999 |
| $\lambda_{3}$ | 0.99997 | 0.99987 | 0.99997 |
| $\lambda_{4}$ | 0.99998 | 0.99986 | 0.99998 |
| $\lambda_{5}$ | 0.99997 | 0.99987 | 0.99997 |
| $\lambda_{6}$ | 0.99998 | 0.99985 | 0.99998 |

### 5.3.2 Numerical Stability Analysis

To check the stability of the proposed method numerically, the problem of the electromagnetic scattering by an elongated thin PEC plate is considered [2]. The C-LOD-FDTD CPML model is discretised spatially with uniform cubic cells spanning $\Delta x=\Delta y=\Delta z=1 \mathrm{~mm}$. The C-LOD-FDTD lattice is terminated with a 10 -cell CPML absorber. Excitation is provided by a $z$-directed electric dipole located 1 mm above one of the corners of the plate (as shown in Fig. 5.5). The time signature of the excitation is the differentiated Gaussian pulse with a half-width $t_{w}=53 \mathrm{ps}$ and a time delay $t_{0}=4 t_{w}$.


Fig. 5.5 Geometry of a vertical electric current dipole placed 1 mm above the corner of a $25 \times 100 \mathrm{~mm}$ thin PEC plate.

(a)

(b)

Fig. 5.6 Computed $E_{y}$ field in the plane of the plate at its opposite corner, in a direction normal to the plate edge at a distance of (a) 1 mm (b) 2 mm from the edge

The computed electric field in the plane of the plate at its opposite corner, in a direction normal to the plate edge at a distance of $1 \mathrm{~mm}(\mathrm{~A}), 2 \mathrm{~mm}(\mathrm{~B})$ from the edge is shown in Fig. 5.6 (a)-(b) respectively for $\mathrm{CFLN}=5$. The results demonstrate the numerical stability of the proposed C-LOD-FDTD CPML for a 3-D electromagnetic problem. The more efficient F-LOD-FDTD is described next.

### 5.4 3-D LOD-FDTD based on the Fundamental Scheme (F-LOD-FDTD) Method

Due to the presence of the tri-diagonal matrix in the update equations of C-LODFDTD CPML that was discussed in Section 5.2, a substantial number of arithmetic operations and field variables are involved in the right hand side of the update equations which reduce the computational efficiency of the method. To improve the computational efficiency, we derive LOD-FDTD using the fundamental scheme which we will call F-LOD-FDTD. The F-LOD-FDTD frees the matrix operators from righthand side of update equations in matrix form so that they become simpler and more concise for efficient implementation [134]. This also leads to a substantial reduction in the number of arithmetic operations required for computations. The underlying principle of the new algorithm as well as its significance will be discussed next.

### 5.4.1 Formulation of the F-LOD-FDTD

In this section, we present the generalised formulation of the efficient F-LOD-FDTD method. Starting with the C-LOD-FDTD scheme, its generalised matrix operator equations are reformulated in the simplest and most efficient forms. The new approach features convenient matrix-operator-free right-hand sides with the least number of terms, which leads to simplified computation. For simplicity and clarity, we have provided the derivation for lossless media. However, from (5.5a)-(5.5b), it can be observed that the C-LOD-FDTD involves matrix operators $\mathbf{A}$ and $\mathbf{B}$ on the right hand side which reduces the computational efficiency of the method. To maximise the efficiency, the C-LOD-FDTD can be cast into the fundamental form with matrix operator free right hand sides. Based on the principle of fundamental implicit schemes [134], the efficient LOD-FDTD using fundamental scheme can be modified into its simplest form with matrix operator free right hand sides as follows:
Sub-step 1:

$$
\begin{align*}
& \left(\frac{1}{2} I-\frac{\Delta t}{4} \mathbf{A}\right) \mathbf{V}^{n+1 / 2}=\mathbf{U}^{n}  \tag{5.20a}\\
& \mathbf{U}^{n+1 / 2}=\mathbf{V}^{n+1 / 2}-\mathbf{U}^{n} \tag{5.20b}
\end{align*}
$$

Sub-step 2:

$$
\begin{gather*}
\left(\frac{1}{2} I-\frac{\Delta t}{4} \mathbf{B}\right) \mathbf{V}^{n+1}=\mathbf{U}^{n+1 / 2}  \tag{5.21a}\\
\mathbf{U}^{n+1}=\mathbf{V}^{n+1}-\mathbf{U}^{n+1 / 2} \tag{5.21b}
\end{gather*}
$$

where $\mathbf{V}=\left[\begin{array}{llllll}e_{x} & e_{y} & e_{z} & h_{x} & h_{y} & h_{z}\end{array}\right]^{T}$ and $\mathbf{A}, \mathbf{B}, \mathbf{U}$ are same as mentioned in Chapter 1. From (5.20)-(5.21), it can be observed that the right-hand sides are in convenient matrix operator free form. By placing the value of $\mathbf{A}, \mathbf{B}, \mathbf{U}$ and $\mathbf{V}$ in (5.20) to (5.21), we can obtain the updating equation for the F-LOD-FDTD method. The updating equations of sub-step1 from (5.20) can be written as follows:
Sub-step 1: auxiliary implicit updating for electric and magnetic fields as:

$$
\begin{align*}
& \left.e_{x}\right|^{n+1 / 2}-\left.\frac{\Delta t}{2} \frac{1}{\varepsilon} \partial_{y} h_{z}\right|^{n+1 / 2}=\left.2 E_{x}\right|^{n}  \tag{5.22a}\\
& \left.e_{y}\right|^{n+1 / 2}-\left.\frac{\Delta t}{2} \frac{1}{\varepsilon} \partial_{z} h_{x}\right|^{n+1 / 2}=\left.2 E_{y}\right|^{n} \tag{5.22b}
\end{align*}
$$

$$
\begin{align*}
& \left.e_{z}\right|^{n+1 / 2}-\left.\frac{\Delta t}{2} \frac{1}{\varepsilon} \partial_{x} h_{y}\right|^{n+1 / 2}=\left.2 E_{z}\right|^{n}  \tag{5.22c}\\
& \left.h_{x}\right|^{n+1 / 2}-\left.\frac{\Delta t}{2} \frac{1}{\mu} \partial_{z} e_{y}\right|^{n+1 / 2}=\left.2 H_{x}\right|^{n}  \tag{5.22d}\\
& \left.h_{y}\right|^{n+1 / 2}-\left.\frac{\Delta t}{2} \frac{1}{\mu} \partial_{x} e_{z}\right|^{n+1 / 2}=\left.2 H_{y}\right|^{n}  \tag{5.22e}\\
& \left.h_{z}\right|^{n+1 / 2}-\left.\frac{\Delta t}{2} \frac{1}{\mu} \partial_{y} e_{x}\right|^{n+1 / 2}=\left.2 H_{z}\right|^{n} \tag{5.22f}
\end{align*}
$$

and the explicit updating for the electric and magnetic fields

$$
\begin{align*}
& \left.E_{\xi 1}\right|^{n+1 / 2}=\left.e_{\xi 1}\right|^{n+1 / 2}-\left.E_{\xi 1}\right|^{n}  \tag{5.23}\\
& \left.H_{\xi 1}\right|^{n+1 / 2}=\left.h_{\xi 1}\right|^{n+1 / 2}-\left.H_{\xi 1}\right|^{n} \tag{5.24}
\end{align*}
$$

where $\xi 1$ is $x, y$ or $z$. Since (5.22a)-(5.22f) are not directly solvable, we obtain the following tri-diagonal linear equations by placing (5.22f) into (5.22a), (5.22d) into (5.22b) and (5.22e) into (5.22c) as:

$$
\begin{align*}
& \left.e_{x}\right|^{n+1 / 2}-\left.b d \partial_{y}^{2} e_{x}\right|^{n+1 / 2}=\left.2 E_{x}\right|^{n}+\left.2 b \partial_{y} H_{z}\right|^{n}  \tag{5.25a}\\
& \left.e_{y}\right|^{n+1 / 2}-\left.b d \partial_{z}^{2} e_{y}\right|^{n+1 / 2}=\left.2 E_{y}\right|^{n}+\left.2 b \partial_{z} H_{x}\right|^{n}  \tag{5.25b}\\
& \left.e_{z}\right|^{n+1 / 2}-\left.b d \partial_{x}^{2} e_{z}\right|^{n+1 / 2}=\left.2 E_{z}\right|^{n}+\left.2 b \partial_{x} H_{y}\right|^{n} \tag{5.25c}
\end{align*}
$$

where $b=\frac{\Delta t}{2 \varepsilon}, d=\frac{\Delta t}{2 \mu}$. Similar to sub-step 1 , we can derive the update equations from (5.21) as follows:

Sub-step 2: auxiliary implicit updating for $e e^{n+1}$

$$
\begin{align*}
& \left.e_{x}\right|^{n+1}-\left.b d \partial_{z}^{2} e_{x}\right|^{n+1}=\left.2 E_{x}\right|^{n+1 / 2}-\left.2 b \partial_{z} H_{y}\right|^{n+1 / 2}  \tag{5.26a}\\
& \left.e_{y}\right|^{n+1}-\left.b d \partial_{x}^{2} e_{y}\right|^{n+1}=\left.2 E_{y}\right|^{n+1 / 2}-\left.2 b \partial_{x} H_{z}\right|^{n+1 / 2}  \tag{5.26b}\\
& \left.e_{z}\right|^{n+1}-\left.b d \partial_{y}^{2} e_{z}\right|^{n+1}=\left.2 E_{z}\right|^{n+1 / 2}-\left.2 b \partial_{y} H_{x}\right|^{n+1 / 2} \tag{5.26c}
\end{align*}
$$

and explicit updating for electric and magnetic field as

$$
\begin{align*}
& \left.E_{\xi 1}\right|^{n+1}=\left.e_{\xi 1}\right|^{n+1}-\left.E_{\xi 1}\right|^{n+1 / 2}  \tag{5.27}\\
& \left.H_{\xi 1}\right|^{n+1}=\left.h_{\xi 1}\right|^{n+1}-\left.H_{\xi 1}\right|^{n+1 / 2} \tag{5.28}
\end{align*}
$$

where $\xi 1$ is the same as mentioned after (5.24). For the solution of (5.22a)-(5.22f),
following the method in [134], the following steps are required. First (5.22f), (5.22d) and (5.22e) are substituted into (5.22a), (5.22b) and (5.22c) respectively and the resultant tri-diagonal equations are implicitly solved for the auxiliary variables $e_{x}^{n+1 / 2}, e_{y}^{n+1 / 2}$ and $e_{z}^{n+1 / 2}$. Equations (5.23) with these solutions yield $E_{x}^{n+1 / 2}, E_{y}^{n+1 / 2}$, and $E_{z}^{n+1 / 2}$. Furthermore $H_{x}^{n+1 / 2}, H_{y}^{n+1 / 2}$, and $H_{z}^{n+1 / 2}$ are explicitly obtained from (5.24) combined with (5.22d)-(5.22f), in which $h_{x}^{n+1 / 2}, h_{y}^{n+1 / 2}$, and $h_{z}^{n+1 / 2}$ are not required. Hence, when following Tan's formulation [134], we have to retain these variables for $e_{x}^{n+1 / 2}, e_{y}^{n+1 / 2}$ and $e_{z}^{n+1 / 2}$, and six field components. In our proposed approach, the above procedure is improved by substituting (5.22f) into (5.22a) and $e_{x}^{n+1 / 2}$ is implicitly obtained. Then explicitly $E_{x}^{n+1 / 2}$ from (5.23) and $H_{z}^{n+1 / 2}$ from (5.24) with (5.22f) are obtained. Now the array $e_{x}^{n+1 / 2}$ can be reused to calculate $e_{y}^{n+1 / 2}$ for the next implicit calculation with (5.22b) and (5.22d). A similar modification approach has been used in [136]. They claim more efficient computation can be achieved compared to Tan's approach [136]. Using this modified approach, we have derived the CPML ABC for the F-LOD-FDTD method which will be described next.

### 5.4.2 CPML ABC for 3D F-LOD-FDTD

So far in the literature, only Mur's ABC [138] and PML [126] have been developed for 3-D LOD-FDTD using the fundamental scheme, but these are not capable of absorbing low frequency and evanescent waves as well as preserving the unconditional stability of the method. In this section, we derive the CPML absorbing boundary condition for 3-D F-LOD-FDTD to obtain highly effective ABC that can absorb low frequency and evanescent waves. The theory presented in this section will later be used to analyse 3-D microwave structures. Based on the principle of fundamental implicit schemes [134] and the CPML principle described in Section 2.4 in Chapter 2, the efficient LOD-FDTD using fundamental scheme with CPML can be formulated as follows:

Sub-step 1:

$$
\begin{equation*}
\mathbf{U}^{n+1 / 2}=\mathbf{V}^{n+1 / 2}-\mathbf{U}^{n}+\frac{\Delta t}{2} \mathbf{X} \psi^{n} \tag{5.29a}
\end{equation*}
$$

$$
\begin{align*}
& \psi^{n}=\mathrm{C} \psi^{n-1 / 2}+\mathrm{D} \mathbf{U}^{n}  \tag{5.29b}\\
& \left(\frac{1}{2} I-\frac{\Delta t}{4} \mathbf{A}\right) \mathbf{V}^{n+1 / 2}=\mathbf{U}^{n} \tag{5.29c}
\end{align*}
$$

Sub-step2:

$$
\begin{align*}
& \mathbf{U}^{n+1}=\mathbf{V}^{n+1}-\mathbf{U}^{n+1 / 2}+\frac{\Delta t}{2} \mathbf{X} \psi^{n+1 / 2}  \tag{5.30a}\\
& \psi^{n+1 / 2}=\mathrm{C} \psi^{n}+\mathrm{D} \mathbf{U}^{n+1 / 2}  \tag{5.30b}\\
& \left(\frac{1}{2} I-\frac{\Delta t}{4} \mathbf{B}\right) \mathbf{V}^{n+1}=\mathbf{U}^{n+1 / 2} \tag{5.30c}
\end{align*}
$$

where $V=\left[\begin{array}{lllll}e_{x} & e_{y} & e_{z} & h_{x} & h_{y}\end{array} h_{z}\right]^{T}$ and $\mathbf{A}, \mathbf{B}, C, D$ and $X$ are the same as defined in Section 5.2. It can be seen from (5.29)-(5.30) that the proposed F-LOD-FDTD CPML algorithm has its right hand sides free of matrix operators $\mathbf{A}$ and $\mathbf{B}$, so this scheme results in a reduction of the number of update coefficients and field variables after all reduced arithmetic operations. The updating equations for the electric and magnetic field components of CPML ABC for the sub-steps 1 and 2 can be written as. Sub-step 1: auxiliary implicit updating for electric and magnetic fields as:

$$
\begin{align*}
& \left.\frac{1}{2} e_{x}\right|^{n+1 / 2}-\left.\frac{\Delta t}{4} \frac{1}{\varepsilon} \frac{1}{\kappa_{y_{j}}} \partial_{y} h_{z}\right|^{n+1 / 2}=\left.E_{x}\right|^{n}  \tag{5.31a}\\
& \left.\frac{1}{2} e_{y}\right|^{n+1 / 2}-\left.\frac{\Delta t}{4} \frac{1}{\varepsilon} \frac{1}{\kappa_{z_{k}}} \partial_{z} h_{x}\right|^{n+1 / 2}=\left.E_{y}\right|^{n}  \tag{5.31b}\\
& \left.\frac{1}{2} e_{z}\right|^{n+1 / 2}-\left.\frac{\Delta t}{4} \frac{1}{\varepsilon} \frac{1}{\kappa_{x_{i}}} \partial_{x} h_{y}\right|^{n+1 / 2}=\left.E_{z}\right|^{n}  \tag{5.31c}\\
& \left.\frac{1}{2} h_{x}\right|^{n+1 / 2}-\left.\frac{\Delta t}{4} \frac{1}{\mu} \frac{1}{\kappa_{z_{k+1 / 2}}} \partial_{z} e_{y}\right|^{n+1 / 2}=\left.H_{x}\right|^{n}  \tag{5.31d}\\
& \left.\frac{1}{2} h_{y}\right|^{n+1 / 2}-\left.\frac{\Delta t}{4} \frac{1}{\mu} \frac{1}{\kappa_{x_{i+1 / 2}}} \partial_{x} e_{z}\right|^{n+1 / 2}=\left.H_{y}\right|^{n}  \tag{5.31e}\\
& \left.\frac{1}{2} h_{z}\right|^{n+1 / 2}-\left.\frac{\Delta t}{4} \frac{1}{\mu} \frac{1}{\kappa_{y_{j+1 / 2}}} \partial_{y} e_{x}\right|^{n+1 / 2}=\left.H_{z}\right|^{n} \tag{5.31f}
\end{align*}
$$

and explicit updating for the electric and magnetic fields

$$
\begin{equation*}
\left.E_{\xi 1}\right|^{n+1 / 2}=\left.e_{\xi 1}\right|^{n+1 / 2}-\left.E_{\xi 1}\right|^{n}+2 a_{p} \psi_{E_{\xi 1 \xi 2}}^{n} \tag{5.32}
\end{equation*}
$$

$$
\begin{equation*}
\left.H_{\xi 1}\right|^{n+1 / 2}=\left.h_{\xi 1}\right|^{n+1 / 2}-\left.H_{\xi 1}\right|^{n}+2 b_{p} \psi_{H_{\xi 1 \xi 2}}^{n} \tag{5.33}
\end{equation*}
$$

where p is 1,2 , and $3, \xi 1$ is $x, y$ or $z$ and $\xi 2$ is $y, z$ or $x$. Since (5.31a)-(5.31f) are not directly solvable. By placing (5.31f) into (5.31a), (5.31d) into (5.31b) and (5.31e) into (5.31c), we obtain the following tri-diagonal linear systems:

$$
\begin{align*}
& \left.\frac{1}{2} e_{x}\right|^{n+1 / 2}-\left.\frac{b d}{2} \frac{1}{\kappa_{y_{j+1 / 2}}} \frac{1}{\kappa_{y_{j}}} \partial_{y}^{2} e_{x}\right|^{n+1 / 2}=\left.E_{x}\right|^{n}+\left.\frac{1}{\kappa_{y_{j}}} b \partial_{y} H_{z}\right|^{n}  \tag{5.34a}\\
& \left.\frac{1}{2} e_{y}\right|^{n+1 / 2}-\left.\frac{b d}{2} \frac{1}{\kappa_{z_{k+1 / 2}}} \frac{1}{\kappa_{z_{k}}} \partial_{z}^{2} e_{y}\right|^{n+1 / 2}=\left.E_{y}\right|^{n}+\left.\frac{1}{\kappa_{z_{k}}} b \partial_{z} H_{x}\right|^{n}  \tag{5.34b}\\
& \left.\frac{1}{2} e_{z}\right|^{n+1 / 2}-\left.\frac{b d}{2} \frac{1}{\kappa_{x_{i}}} \frac{1}{\kappa_{x_{i+1 / 2}}} \partial_{x}^{2} e_{z}\right|^{n+1 / 2}=\left.E_{z}\right|^{n}+\left.\frac{1}{\kappa_{x_{i}}} b \partial_{x} H_{y}\right|^{n} \tag{5.34c}
\end{align*}
$$

Similarly, for the second procedure we can derive the updating equations from (5.30) as follows:
Sub-step 2: auxiliary implicit updating for $\left.e\right|^{n+1}$

$$
\begin{align*}
& \left.\frac{1}{2} e_{x}\right|^{n+1}-\left.\frac{b d}{2} \frac{1}{\kappa_{z_{k+1 / 2}}} \frac{1}{\kappa_{z_{k+1}}} \partial_{z}^{2} e_{x}\right|^{n+1}=\left.E_{x}\right|^{n+1 / 2}-\left.\frac{1}{\kappa_{z_{k+1 / 2}}} b \partial_{z} H_{y}\right|^{n+1 / 2}  \tag{5.35a}\\
& \left.\frac{1}{2} e_{y}\right|^{n+1}-\left.\frac{b d}{2} \frac{1}{\kappa_{x_{i+1}}} \frac{1}{\kappa_{x_{i+1 / 2}}} \partial_{x}^{2} e_{y}\right|^{n+1}=\left.E_{y}\right|^{n+1 / 2}-\left.\frac{1}{\kappa_{x_{i+1 / 2}}} b \partial_{x} H_{z}\right|^{n+1 / 2}  \tag{5.35b}\\
& \left.\frac{1}{2} e_{z}\right|^{n+1}-\left.\frac{b d}{2} \frac{1}{\kappa_{y_{j+1 / 2}}} \frac{1}{\kappa_{y_{j+1}}} \partial_{y}^{2} e_{z}\right|^{n+1}=\left.E_{z}\right|^{n+1 / 2}-\left.\frac{1}{\kappa_{y_{j+1 / 2}}} b \partial_{y} H_{x}\right|^{n+1 / 2} \tag{5.35c}
\end{align*}
$$

and the explicit updating equations,

$$
\begin{gather*}
\left.E_{\xi 1}\right|^{n+1}=\left.e_{\xi 1}\right|^{n+1}-\left.E_{\xi 1}\right|^{n+1 / 2}+2 a_{p} \psi_{E_{\xi 1 \xi 2}}^{n+1 / 2}  \tag{5.36}\\
\left.H_{\xi 1}\right|^{n+1}=\left.h_{\xi 1}\right|^{n+1}-\left.H_{\xi 1}\right|^{n+1 / 2}+2 b_{p} \psi_{H_{\xi 1 \xi 2}}^{n+1 / 2} \tag{5.37}
\end{gather*}
$$

where p is 1,2 , and 3 . The updating equations with indices can be written as follows. Sub-step 1: auxiliary implicit updating for electric fields:

$$
\begin{align*}
& \left.\left.\alpha_{1}\right|_{i+1 / 2, j, k} e_{x}\right|_{i+1 / 2,2 j-1, k} ^{n+1 / 2}+\left.\left.\beta_{1}\right|_{i+1 / 2, j, k} e_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2} \\
& \quad+\left.\left.\gamma_{1}\right|_{i+1 / 2, j, k} e_{x}\right|_{i+1 / 2, j+1, k} ^{n+1}=\left.\left.C_{a 1}\right|_{i+1 / 2, j, k} E_{x}\right|_{i+1 / 2, j, k} ^{n}  \tag{5.38a}\\
& \quad+\left.C_{b 1}\right|_{i+1 / 2, j, k}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right)
\end{align*}
$$

$$
\begin{align*}
& \left.\left.\alpha_{2}\right|_{i, j+1 / 2, k} e_{y}\right|_{i, j+1 / 2, k-1} ^{n+1 / 2}+\left.\left.\beta_{2}\right|_{i, j+1 / 2, k} e_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2} \\
& \quad+\left.\left.\gamma_{2}\right|_{i, j+1 / 2, k} e_{y}\right|_{i, j+1 / 2, k+1} ^{n+1 / 2}=\left.\left.C_{a 2}\right|_{i, j+1 / 2, k} E_{y}\right|_{i, j+1 / 2, k} ^{n}  \tag{5.38b}\\
& \quad+\left.C_{b 2}\right|_{i, j+1 / 2, k}\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{x}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right)
\end{aligned} \quad \begin{aligned}
& \left.\left.\alpha_{3}\right|_{i, j, k+1 / 2} e_{z}\right|_{i-1, j, k+1} ^{n+1 / 2}+\left.\left.\beta_{3}\right|_{i, j, k+1 / 2} e_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2} \\
& \quad+\left.\left.\gamma_{3}\right|_{i, j, k+1 / 2} e_{z}\right|_{i+1, j, k+1 / 2} ^{n+1 / 2}=\left.\left.C_{a 3}\right|_{i, j, k+1 / 2} E_{z}\right|_{i, j, k+1 / 2} ^{n} \\
& \quad+\left.C_{b 3}\right|_{i, j, k+1 / 2}\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{y}\right|_{i-1 / 2, j, k+1 / 2} ^{n}\right) \tag{5.38c}
\end{align*}
$$

where $\left.\alpha_{1}\right|_{i+1 / 2, j, k}=-\frac{b d}{2} \frac{1}{\kappa_{y_{j+1 / 2}}} \frac{1}{\kappa_{y_{j}}} \frac{1}{\Delta y^{2}},\left.\gamma_{1}\right|_{i+1 / 2, j, k}=-\frac{b d}{2} \frac{1}{\kappa_{y_{j+1 / 2}}} \frac{1}{\kappa_{y_{j}}} \frac{1}{\Delta y^{2}}$
$\left.\beta_{1}\right|_{i+1 / 2, j, k}=1-\left.\alpha_{1}\right|_{i+1 / 2, j, k}-\left.\gamma_{1}\right|_{i+1 / 2, j, k}$, The electric field components from (5.32) can be written as follows:

$$
\begin{align*}
& \left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}=\left.e_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n}+2 a_{1} \psi_{E x y_{i+1 / 2, j, k}}^{n}  \tag{5.39a}\\
& \left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}=\left.e_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n}+2 a_{2} \psi_{E y z_{i, j+1 / 2, k}^{n}}^{n}  \tag{5.39b}\\
& \left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}=\left.e_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n}+2 a_{3} \psi_{E z x_{i, j, k+1 / 2}}^{n} \tag{5.39c}
\end{align*}
$$

The magnetic field components from (5.33) can be written as follows:

$$
\begin{align*}
& \left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+j / 2}=\left.h_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n}+2 b_{1} \psi_{H x y_{i j+j+1 /, k+1 / 2}^{n}}^{n}  \tag{5.40a}\\
& \left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}=\left.h_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n}+2 b_{2} \psi_{H y z_{i+1 / 2, j, j+1 / 2}^{n}}^{n}  \tag{5.40b}\\
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}=\left.h_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}+2 b_{3} \psi_{H z x_{i+1 / 2, j+1 / 2, k}^{n}}^{n} \tag{5.40c}
\end{align*}
$$

Similarly, updating equations for the second sub-step can be derived as follows. From (5.35a)-(5.35c), we can write the following equations:

Sub-step 2: auxiliary implicit updating for electric fields:

$$
\begin{align*}
& \left.\left.\alpha_{1}\right|_{i+1 / 2, j, k} e_{x}\right|_{i+1 / 2, j-1, k} ^{n+1}+\left.\left.\beta_{1}\right|_{i+1 / 2, j, k} e_{x}\right|_{i+1 / 2, j, k} ^{n+1} \\
& \quad+\left.\left.\gamma_{1}\right|_{i+1 / 2, j, k} e_{x}\right|_{i+1 / 2, j+1, k} ^{n+1}=\left.\left.C_{a 1}\right|_{i+1 / 2, j, k} E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}  \tag{5.41a}\\
& \quad+\left.C_{b 1}\right|_{i+1 / 2, j, k}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2, k} ^{n+1 / 2}\right) \\
& \begin{array}{l}
\left.\alpha_{2}\right|_{i, j+1 / 2, k}
\end{array} \quad \begin{array}{l}
\left.e_{y}\right|_{i, j+1 / 2, k-1} ^{n+1}+\left.\left.\beta_{2}\right|_{i, j+1 / 2, k} e_{y}\right|_{i, j+1 / 2, k} ^{n+1} \\
\quad+\left.\left.\gamma_{2}\right|_{i, j+1 / 2, k} e_{y}\right|_{i, j+1 / 2, k+1} ^{n+1}=\left.\left.C_{a 2}\right|_{i, j+1 / 2, k} E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2} \\
\quad+\left.C_{b 2}\right|_{i, j+1 / 2, k}\left(\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{x}\right|_{i, j+1 / 2, k-1 / 2} ^{n+1 / 2}\right)
\end{array}
\end{align*}
$$

$$
\begin{align*}
& \left.\left.\alpha_{3}\right|_{i, j, k+1 / 2} e_{z}\right|_{i-1, j, k+1} ^{n+1}+\left.\left.\beta_{3}\right|_{i, j, k+1 / 2} e_{z}\right|_{i, j, k+1 / 2} ^{n+1} \\
& \quad+\left.\left.\gamma_{3}\right|_{i, j, k+1 / 2} e_{z}\right|_{i+1, j, k+1 / 2} ^{n+1}=\left.\left.C_{a 3}\right|_{i, j, k+1 / 2} E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}  \tag{5.41c}\\
& \quad+\left.C_{b 3}\right|_{i, j, k+1 / 2}\left(\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{y}\right|_{i-1 / 2, j, k+1 / 2} ^{n+1 / 2}\right)
\end{align*}
$$

The electric field components for sub-step 2 from (5.36) can be written as follows:

$$
\begin{gather*}
\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1}=\left.e_{x}\right|_{i+1 / 2, j, k} ^{n+1}-\left.E_{x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}+2 a_{1} \psi_{E x y_{i+1 / 2, j, k}}^{n+1 / 2}  \tag{5.42a}\\
\left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1}=\left.e_{y}\right|_{i, j+1 / 2, k} ^{n+1}-\left.E_{y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}+2 a_{2} \psi_{E y z_{i, j+1 / 2, k}^{n+1 / 2}}^{n+1}  \tag{5.42b}\\
\left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1}=\left.e_{z}\right|_{i, j, k+1 / 2} ^{n+1}-\left.E_{z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}+2 a_{3} \psi_{E z x_{i, j, k+1 / 2}}^{n+1 / 2} \tag{5.42c}
\end{gather*}
$$

and the magnetic field components from (5.37) can be written as follows:

$$
\begin{align*}
& \left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}=\left.h_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}-\left.H_{x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}+2 b_{1} \psi_{H y_{i, j+1 / 2, k+1 / 2}^{n+1 / 2}}^{n+1}  \tag{5.43a}\\
& \left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}=\left.h_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}+2 b_{2} \psi_{H y y_{i+1 / 2, j+k+1 / 2}^{n+1 / 2}}^{n+1}  \tag{5.43b}\\
& H_{z}^{n+1 / 2, j+1 / 2, k}=\left.h_{z}^{n+1}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}+2 b_{3} \psi_{H x_{i+1 / 2, j+1 / 2, k}^{n+1 / 2}}^{n+1} \tag{5.43c}
\end{align*}
$$

where

$$
\begin{align*}
& \psi_{e \xi \xi \xi \sum_{i+1 / 2, j, k}}^{n}=c_{y_{j}} \psi_{e \xi \xi \xi \xi^{2}+1 / 2, j, k}^{n}+\frac{d_{y_{j}}}{2 \Delta y}\left(\left.H_{\xi \xi}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-H_{\xi 3}{ }_{i+1 / 2, j-1 / 2, k}^{n+1 / 2}\right)  \tag{5.44}\\
& c_{s}=e^{-\left(\left(\sigma_{s} / /_{s}\right)+\alpha_{s}\right)\left(\Delta t / \varepsilon_{0}\right)}, \quad \xi 3 \text { is } \mathrm{z}, x \text { or } y  \tag{5.45a}\\
& d_{s}=\sigma_{s} /\left(\kappa_{s}\left(\sigma_{s}+\kappa_{s} \sigma_{s}\right)\right) \times\left(c_{s}-1\right), \quad(s=x, y, \text { or } z)  \tag{5.45b}\\
& \sigma_{s}(s)=\left(\sigma_{s_{\max }}\left|s-s_{0}\right|^{m}\right) / \delta^{m}  \tag{5.45c}\\
& \kappa_{s}(s)=1+\left(\kappa_{\max }-1\right)\left(\left|s-s_{0}\right|^{m}\right) / \delta^{m} \tag{5.45d}
\end{align*}
$$

where $\delta$ is the thickness of the PML absorber, $s_{0}$ is the PML interface, $m$ is the order of the polynomial. The proposed F-LOD-FDTD-CPML requires only two sub-steps and obtains matrix operator-free right-hand sides. Compared to both the C-LOD-FDTDCPML and the original LOD-FDTD-CPML [131], the auxiliary variable ' $\psi$ ' is only required in the proposed approach to explicitly update the electric and magnetic field components given in (5.39) and (5.40) but not during the implicit updating process for auxiliary electric fields (5.38a)-(5.38c) for sub-step 1. A similar process occurs for substep 2 . From the updating equations of sub-steps 1 and 2 , it can also be observed that one variable $\psi$ is enough for the proposed approach; for the reason described in Section 5.2.

### 5.4.2.1 Validation

To validate the CPML implementation for the 3-D F-LOD-FDTD, we consider a free space domain with $46 \times 46 \times 46$ cells including 10 layers of CPML in each direction. The observation point, where the $z$ directed electric field is recorded, is located at 10 cells away from the source. Gaussian pulse is used as a source.


Fig. 5.7 Reflection error with respect to time for CFLN=2, 7 and 12


Fig. 5.8 Reflection error with respect to time for CFLN=8

To minimise the reflection error, the following parameters are considered: $\sigma_{\text {opt }}=(m+1) / 150 \pi \Delta x, \sigma_{s_{\max }}=0.7 \sigma_{\text {opt }}(\mathrm{S} / \mathrm{m}), k_{s_{\max }}=15$ and $m=4$ for CPML. Fig. 5.7
shows the reflection error for F-LOD-FDTD CPML for CFLN=2, 6 and 10. Fig. 5.8 shows the reflection error for CPML when used with both F-LOD-FDTD and C-LODFDTD methods for CFLN=8. It can be observed from Fig. 5.8 that the proposed approach has a lower reflection error than the C-LOD-FDTD with CPML.

To demonstrate the efficiency and accuracy benefits, a comparison of the number of required arithmetic operations between C-LOD-FDTD-CPML and the proposed F-LOD-FDTD-CPML over one complete time step are obtained and shown in Table 5.2.

Table 5.2 Comparison of the number of arithmetic operations per grid between C-LOD-FDTD-CPML and F-LOD-FDTD-CPML

| Arithmetic Operations |  | C-LOD- <br> FDTD- CPML | F-LOD-FDTD- <br> CPML |
| :---: | :---: | :---: | :---: |
| Implicit, RHS | $\mathrm{M} / \mathrm{D}$ | 54 | 18 |
|  | $\mathrm{~A} / \mathrm{S}$ | 72 | 12 |
| Explicit, RHS | $\mathrm{M} / \mathrm{D}$ | 36 | 24 |
|  | $\mathrm{~A} / \mathrm{S}$ | 30 | 30 |
|  | $\mathrm{M} / \mathrm{D}$ | 90 | 42 |
|  | $\mathrm{~A} / \mathrm{S}$ | 102 | 42 |
|  | $\mathrm{M} / \mathrm{D}+\mathrm{A} / \mathrm{S}$ | 192 | 84 |
| Tridiag. Matrices | $\mathrm{M} / \mathrm{D}$ | 18 | 18 |
|  | $\mathrm{~A} / \mathrm{S}$ | 12 | 12 |
|  | $\mathrm{M} / \mathrm{D}$ | 108 | 60 |
|  | $\mathrm{~A} / \mathrm{S}$ | 114 | 54 |
|  | $\mathrm{M} / \mathrm{D}+\mathrm{A} / \mathrm{S}$ | 222 | 114 |
|  | RHS | 1 | 2.29 |
|  | overall | 1 | 1.95 |

From Table 5.2 it can be observed that the total flops count for the right hand side of the resultant equations for the proposed F-LOD-FDTD CPML is 84 which are considerably
less than 192 for the C-LOD-FDTD CPML, thus offering an efficiency gain of 2.29. Table 5.2 also includes a comparison of the arithmetic operations needed for inverting tri-diagonal matrices because there is a cost involved in solving them. The results show that the F-LOD-FDTD CPML still offers superior performance and achieves an overall efficiency gain of 1.95 . In addition, we have introduced some operations for the F-LOD-FDTD for which implicit and explicit equations are alternatively calculated. It is thus necessary to retain only 7 field arrays rather than the 12 field arrays required by the C-LOD-FDTD and 9 field arrays required by the original formulation by Tan [134]. Thus the F-LOD-FDTD method can provide efficient operation. The stability analysis of the F-LOD-FDTD method is discussed next.

### 5.5 Stability Analysis for 3-D F-LOD-FDTD Method

### 5.5.1 Theoretical Stability Analysis

The Von Neumann method [166] has been used as a standard approach for the stability analysis. As stated earlier, if the magnitudes of the eigenvalues of the amplification matrix in the spectral domain are not larger than unity in magnitudes, the method will be considered to be unconditionally stable. Hence, we use the same method to demonstrate the unconditional stability of the F-LOD-FDTD. To check the stability of the proposed scheme, (5.20)-(5.24) and (5.26)-(5.28) are expressed in matrix form. Since the stability analysis of the F-LOD-FDTD requires detailed mathematical procedure, only the key results of the analysis are summarised below. The field components in the spatial spectral domain are assumed in the following form

$$
\begin{align*}
& E_{r}^{n}(i, j, k)=E_{r}^{n} e^{-j\left(k_{x} i \Delta x+k_{y} j \Delta y+k_{z} k \Delta z\right)}  \tag{5.46a}\\
& H_{r}^{n}(i, j, k)=H_{r}^{n} e^{-j\left(k_{i} i \Delta x+k_{y} j \Delta y+k_{z} k \Delta z\right)} \tag{5.46b}
\end{align*}
$$

where $r=x, y, z . k_{x}, k_{y}$ and $k_{z}$ are wave numbers along the $x, y, z$ directions respectively. By substituting (5.46a) and (5.46b) into (5.22)-(5.24) and (5.26)-(5.28), the following equation can be obtained:

Sub-step 1:

$$
\begin{equation*}
U^{n+1 / 2}=\Lambda_{1} U^{n} \tag{5.47}
\end{equation*}
$$

where $\Lambda_{1}$ is $12 \times 12$ matrix

$$
\Lambda_{1}=\left[\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{y}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{z}} & 0 & \frac{i W_{z}}{\varepsilon Q_{z}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{x}} & 0 & \frac{i W_{x}}{\varepsilon Q_{x}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{z}}{\mu Q_{z}} & 0 & \frac{1}{Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{x}}{\mu Q_{x}} & 0 & \frac{1}{Q_{x}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{y}}{\mu Q_{y}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{y}^{\prime}}{Q_{y}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{z}^{\prime}}{Q_{z}} & 0 & \frac{i W_{z}}{\varepsilon Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{x}^{\prime}}{Q_{x}} & 0 & \frac{i W_{x}}{\varepsilon Q_{y}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{z}}{\mu Q_{z}} & 0 & \frac{Q_{z}^{\prime}}{Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{x}}{\mu Q_{x}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{y}}{\mu Q_{y}} & 0 & 0 & 0 & 0 \\
0_{x} & \frac{Q_{x}^{\prime}}{Q_{y}}
\end{array}\right]
$$

Sub-step 2:

$$
\begin{equation*}
U^{n+1}=\Lambda_{2} U^{n+1 / 2} \tag{5.48}
\end{equation*}
$$

$$
\Lambda_{2}=\left[\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{z}} & 0 & 0 & 0 & \frac{i W_{z}}{\varepsilon Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{x}} & 0 & 0 & 0 & \frac{i W_{x}}{\varepsilon Q_{x}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{y}} & \frac{i W_{y}}{\varepsilon Q_{y}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{y}}{\mu Q_{y}} & \frac{1}{Q_{y}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{z}}{\mu Q_{z}} & 0 & 0 & 0 & \frac{1}{Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{x}}{\mu Q_{x}} & 0 & 0 & 0 & \frac{1}{Q_{x}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{z}^{\prime}}{Q_{z}} & 0 & 0 & 0 & \frac{i W_{z}}{\varepsilon Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{x}^{\prime}}{Q_{x}} & 0 & 0 & 0 & \frac{i W_{x}}{\varepsilon Q_{x}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{y}^{\prime}}{Q_{y}} & \frac{i W_{y}}{\varepsilon Q_{y}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{y}}{\mu Q_{y}} & \frac{Q_{y}^{\prime}}{Q_{y}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{z}}{\mu Q_{z}} & 0 & 0 & \frac{Q_{z}^{\prime}}{Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{x}}{\mu Q_{x}} & 0 & 0 & 0 & \frac{Q_{x}^{\prime}}{Q_{x}}
\end{array}\right]
$$

where $W_{x}=\frac{\Delta t}{\Delta x} \sin \left(\frac{k_{x} \Delta x}{2}\right), W_{y}=\frac{\Delta t}{\Delta y} \sin \left(\frac{k_{y} \Delta y}{2}\right), W_{z}=\frac{\Delta t}{\Delta z} \sin \left(\frac{k_{z} \Delta z}{2}\right)$
$Q_{x}^{\prime}=1-\frac{\left(W_{x}\right)^{2}}{\mu \varepsilon}, Q_{y}^{\prime}=1-\frac{\left(W_{y}\right)^{2}}{\mu \varepsilon}, Q_{z}^{\prime}=1-\frac{\left(W_{z}\right)^{2}}{\mu \varepsilon}$ and $\quad Q_{x}=1+\frac{\left(W_{x}\right)^{2}}{\mu \varepsilon}, \quad Q_{y}=1+\frac{\left(W_{y}\right)^{2}}{\mu \varepsilon}$,
$Q_{z}=1+\frac{\left(W_{z}\right)^{2}}{\mu \varepsilon}$
By combining (5.47) and (5.48), we obtain the following equations,

$$
\begin{equation*}
U^{n+1}=\Lambda_{1} \Lambda_{2} U^{n}=\Lambda U^{n} \tag{5.49}
\end{equation*}
$$

where $\Lambda=\Lambda_{1} \Lambda_{2}$

$$
\left.\Lambda=\left[\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{z}^{\prime}}{Q_{y} Q_{z}} & -\frac{W_{x} W_{y}}{\mu \varepsilon Q_{x} Q_{y}} & 0 & 0 & \frac{i W_{z}}{\varepsilon Q_{y} Q_{z}}
\end{array}\right] \frac{i W_{y} Q_{x}^{\prime}}{\varepsilon Q_{x} Q_{y}}\right]\left[\begin{array}{cccccccccccccc} 
 \tag{5.50}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{x}^{\prime}}{Q_{x} Q_{z}} & -\frac{W_{y} W_{z}}{\mu \varepsilon Q_{y} Q_{z}} & \frac{i W_{z} Q_{y}^{\prime}}{\varepsilon Q_{y} Q_{z}} & 0 & \frac{i W_{x}}{\varepsilon Q_{x} Q_{z}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{W_{x} W_{z}}{\mu \varepsilon Q_{x} Q_{z}} & \frac{Q_{y}^{\prime}}{Q_{x} Q_{y}} & \frac{i W_{y}}{\varepsilon Q_{x} Q_{y}} & \frac{i W_{x} Q_{z}^{\prime}}{\varepsilon Q_{x} Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{z} Q_{x}^{\prime}}{\mu Q_{x} Q_{z}} & \frac{i W_{y}}{\mu Q_{y} Q_{z}} & \frac{Q_{y}^{\prime}}{Q_{y} Q_{z}} & 0 & -\frac{W_{x} W_{z}}{\mu \varepsilon Q_{x} Q_{z}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{z}}{\mu Q_{x} Q_{z}} & \frac{i W_{x} Q_{y}^{\prime}}{\mu Q_{x} Q_{y}} & -\frac{W_{x} W_{y}}{\mu \varepsilon Q_{x} Q_{y}} & \frac{Q_{z}^{\prime}}{Q_{x} Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{y} Q_{z}^{\prime}}{\mu Q_{y} Q_{z}} & \frac{i W_{x}}{\mu Q_{x} Q_{y}} & 0 & 0 & -\frac{W_{y} W_{z}}{\mu \varepsilon Q_{y} Q_{z}} & \frac{Q_{x}^{\prime}}{Q_{x} Q_{y}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{y}^{\prime} Q_{z}^{\prime}}{Q_{y} Q_{z}} & -\frac{W_{y} W_{z}}{\mu \varepsilon Q_{x} Q_{y}} & 0 & 0 & 0 & \frac{i W_{z} Q_{y}^{\prime}}{\varepsilon Q_{y} Q_{z}} & \frac{i W_{y} Q_{x}^{\prime}}{\varepsilon Q_{x} Q_{y}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{x}^{\prime} Q_{z}^{\prime}}{Q_{x} Q_{z}} & -\frac{W_{y} W_{z}}{\mu \varepsilon Q_{y} Q_{z}} & \frac{i W_{z} Q_{y}^{\prime}}{\varepsilon Q_{y} Q_{z}} & 0 & \frac{i W_{x} Q_{z}^{\prime}}{\varepsilon Q_{x} Q_{z}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{W_{x} W_{z}}{\mu \varepsilon Q_{x} Q_{z}} & \frac{Q_{x}^{\prime} Q_{y}^{\prime}}{Q_{x} Q_{y}} & \frac{i W_{y} Q_{x}^{\prime}}{\varepsilon Q_{x} Q_{y}} & \frac{i W_{x} Q_{z}^{\prime}}{\varepsilon Q_{x} Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{z} Q_{x}^{\prime}}{\mu Q_{x} Q_{z}} & \frac{i W_{y} Q_{z}^{\prime}}{\mu Q_{y} Q_{z}} & \frac{Q_{y}^{\prime} Q_{z}^{\prime}}{Q_{y} Q_{z}} & 0 & -\frac{W_{x} W_{z}}{\mu \varepsilon Q_{x} Q_{z}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i W_{z} Q_{x}^{\prime}}{\mu Q_{x} Q_{z}} & \frac{i W_{x} Q_{y}^{\prime}}{\mu Q_{x} Q_{y}} & -\frac{W_{x} W_{y}}{\mu \varepsilon Q_{x} Q_{y}} & \frac{Q_{x}^{\prime} Q_{z}^{\prime}}{Q_{x} Q_{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{y} Q_{z}^{\prime}}{\mu Q_{y} Q_{z}} & \frac{i W_{x} Q_{y}^{\prime}}{\mu Q_{x} Q_{y}} & 0 & 0 & -\frac{W_{y} W_{z}}{\mu \varepsilon Q_{y} Q_{z}} & \frac{Q_{x}^{\prime} Q_{y}^{\prime}}{Q_{x} Q_{y}}
\end{array}\right]
$$

By solving (5.50), twelve eigenvalues can be obtained, and the six eigenvalues of the $\Lambda$ can be written as follows

$$
\begin{equation*}
\lambda_{1}=\lambda_{2}=x_{1}+i y_{1}, \lambda_{3}=\lambda_{5}=x_{1}-i y_{1}, \lambda_{4}=\lambda_{6}=x_{2}+i y_{2} \tag{5.51}
\end{equation*}
$$

where

$$
y_{1}=2 j W_{x}^{2} W_{z}^{2} W_{y}^{4} Q_{x}^{2} Q_{x}^{\prime 2} Q_{y}{ }^{4}\binom{1+\frac{\mu^{2} \varepsilon^{2}}{2 Q_{x}^{\prime 2}} W_{x}^{2} Q_{x}^{2} Q_{z}^{\prime 2} Q_{z}^{4}-\frac{\mu^{2} \varepsilon^{2}}{W_{y}^{2} Q_{x}^{\prime 2}} W_{x}^{2} Q_{x}{ }^{2} Q_{z}^{\prime 2} Q_{y}^{\prime 2} Q_{z}^{4}-}{2\left(\frac{\mu \varepsilon}{Q_{x}^{\prime 2}} W_{x}^{2} Q_{x}{ }^{2} Q_{z}^{\prime 2} Q_{z}{ }^{4}\right)^{2}}
$$

$$
\begin{gathered}
x_{1}=\mu^{2} \varepsilon^{2} W_{y}^{4} W_{z}^{2} Q_{x}^{\prime 4} Q_{y}^{4} Q_{z}^{4} Q_{z}^{\prime 2} Q_{x}^{4} \\
x_{2}=\mu^{2} \varepsilon^{2} W_{y}^{4} W_{z}^{2} Q_{x}^{\prime 4} Q_{y}^{4} Q_{z}^{4} Q_{z}^{\prime 2} Q_{x}^{4}-2 \mu^{2} \varepsilon^{2}\left(\mu \varepsilon W_{y}^{4} W_{z}^{2} Q_{x}^{\prime 4} Q_{y}^{4} Q_{z}^{4} Q_{z}^{\prime 2} Q_{x}^{4}\right)^{2} \\
y_{2}=2 j W_{x}^{2} W_{z}^{2} W_{y}^{4} Q_{x}^{2} Q_{x}^{\prime 2} Q_{y}^{4}\binom{\frac{\mu \varepsilon}{2 Q_{x}^{\prime 2}} Q_{z}^{4} Q_{x}^{2} W_{x}^{2}-\frac{\mu^{2} \varepsilon^{2}}{Q_{x}^{\prime 2}} W_{x}^{2} Q_{x}{ }^{2} Q_{z}^{\prime 2} Q_{z}^{4}}{-\frac{\mu^{2} \varepsilon^{2}}{W_{y}^{2} Q_{x}^{\prime 2}} W_{x}^{2} Q_{x}^{2} Q_{z}^{\prime 2} Q_{y}^{\prime 2} Q_{z}^{4}+2\left(\frac{\mu \varepsilon}{Q_{x}^{\prime 2}} W_{x}^{2} Q_{x}^{2} Q_{z}^{\prime 2} Q_{z}^{4}\right)^{2}}
\end{gathered}
$$

From the equation (5.50), it can be observed that twelve eigenvalues can be obtained for the F-LOD-FDTD method but six eigenvalues of them are zero. So, rest of the six eigenvalues have been calculated. For the first six eigenvalues, only the first six terms of each eigenvalue have been considered; they can be written in compact form in (5.51). From (5.51), we can easily obtain the magnitude of the eigenvalues unity, i.e $\left|\lambda_{1}\right|=\left|\lambda_{2}\right|=\left|\lambda_{3}\right|=\left|\lambda_{4}\right|=\left|\lambda_{5}\right|=\left|\lambda_{6}\right|=1$. However, we have also computed the eigenvalues to check the unconditional stability. For instance, we compute the eigenvalues for the case of $k_{x}=0.2, k_{y}=0.2$ and $k_{z}=0.2$ and $\Delta x=\Delta y=\Delta z=1 \mathrm{~mm}$. The computed eigenvalues of the F-LOD-FDTD are tabulated in Table 5.3, from which we can observe that the magnitudes of the eigenvalues are never larger than unity. Hence, the F-LOD-FDTD method can be considered to be unconditionally stable.

Table 5.3
Computed eigenvalues of F-LOD-FDTD

| Eigen values | CFLN=2 | CFLN=4 | CFLN=6 |
| :---: | :---: | :---: | :---: |
| $\lambda_{1}$ | 1.00000 | 1.000000 | 1.000000 |
| $\lambda_{2}$ | 1.0000 | 1.0000 | 1.00000 |
| $\lambda_{3}$ | 0.996344 | 0.995244 | 0.994214 |
| $\lambda_{4}$ | 0.00574 | 0.00554 | 0.00443 |
| $\lambda_{5}$ | 0.000613 | 0.0005513 | 0.0004213 |
| $\lambda_{6}$ | 0.000488 | 0.005488 | 0.003588 |
| $\lambda_{7}-\lambda_{12}$ | 0.00000 | 0.00000 | 0.00000 |

### 5.5.2 Numerical Stability Analysis

To check the stability of the proposed F-LOD-FDTD method numerically, the problem of the electromagnetic scattering by an elongated thin PEC plate (as shown in Fig. 5.5) is considered [2]. The F-LOD-FDTD CPML model is discretised spatially with uniform cubic cells spanning $\Delta x=\Delta y=\Delta z=1 \mathrm{~mm}$. The F-LOD-FDTD lattice is terminated with a 10 cell CPML absorber. Excitation is provided by a z-directed electric dipole located 1 mm above one of the corners of the plate (as shown in Fig. 5.5). The time signature of the excitation is the differentiated Gaussian pulse with a half-width $\mathrm{t}_{\mathrm{w}}=53 \mathrm{ps}$ and a time delay $\mathrm{t}_{0}=4 \mathrm{t}_{\mathrm{w}}$.


Fig. 5.9 Computed $E_{y}$ field in the plane of the plate at its opposite corner, in a direction to the plate edge at a distance of (a) 1 mm (b) 2 mm from the edge for CFLN=5


Fig. 5.10 Computed $E_{y}$ field in the plane of the plate at its opposite corner, in a direction to the plate edge at a distance of 3 mm from the edge for (a) CFLN=10 (b) CFLN=12

The computed electric field in the plane of the plate at its opposite corner, in a direction normal to the plate edge at a distance of $1 \mathrm{~mm}(\mathrm{~A}), 2 \mathrm{~mm}(\mathrm{~B})$ from the edge is shown in Fig. 5.9 (a)-(b) respectively for $\mathrm{CFLN}=5$. The results demonstrate the numerical stability of the proposed F-LOD-FDTD CPML for a 3-D electromagnetic problem. From Fig. 5.10, it is also seen that F-LOD-FDTD CPML is stable for higher CFLN.

### 5.6 Near-Field to Far-Field Transformation for both the 3-D C-LOD-FDTD and F-LOD-FDTD Methods

The near-field to far-field transformation technique for the C-LOD-FDTD and F-LOD-FDTD methods is described in this section. Since the NF-FF transformation 173
procedure for the proposed approaches are the same as the NF-FF transformation procedure of the FDTD method, we briefly discuss the transformation technique for the C-LOD-FDTD and F-LOD-FDTD here. In many applications, such as the antennas and radar cross section (RCS) scatterer, it is necessary to find the radiation or scattered fields in the region that are far away from an antenna or scatterer. Therefore, the direct simulation of C-LOD-FDTD or F-LOD-FDTD for the far field requires a mesh extending many wavelengths from the object which leads to a huge increase in computational time, which is not practical in applications. Instead, the far zone electromagnetic fields are computed from the near field LOD-FDTD data through a near-field to far-field transformation technique. For the near-field to far-field transformation technique, an imaginary surface is first selected to enclose the electromagnetic object. Following the notation of [2], the radiation vectors $N$ and $L$ are defined as

$$
\begin{gather*}
\vec{N}=\int_{S^{\prime}} \vec{J}_{s} \exp \left(j k \vec{r}^{\prime} \cdot \hat{r}\right) d s^{\prime}  \tag{5.52a}\\
\vec{L}=\int_{S^{\prime}} \vec{M}_{s} \exp \left(j k \vec{r}^{\prime} \cdot \hat{r}\right) d s^{\prime} \tag{5.52b}
\end{gather*}
$$

where $j=\sqrt{-1}, \mathrm{k}$ the wavenumber, $\hat{r}$ the unit vector to the far zone field point, $r^{\prime}$ the vector to the source point of integration and $S^{\prime}$ the closed surface surrounding the scatterer. The electric and magnetic field components in the far field are expressed as:

$$
\begin{align*}
& E_{\theta}=-\frac{j k e^{-j k r}}{4 \pi r}\left(L_{\phi}+\eta_{0} N_{\theta}\right)  \tag{5.53a}\\
& E_{\phi}=+\frac{j k e^{-j k r}}{4 \pi r}\left(L_{\theta}-\eta_{0} N_{\phi}\right)  \tag{5.53b}\\
& H_{\theta}=+\frac{j k e^{-j k r}}{4 \pi r}\left(N_{\phi}-\frac{L_{\theta}}{\eta_{0}}\right)  \tag{5.53c}\\
& H_{\phi}=-\frac{j k e^{-j k r}}{4 \pi r}\left(N_{\theta}+\frac{L_{\phi}}{\eta_{0}}\right) \tag{5.53d}
\end{align*}
$$

where $N_{\theta}, N_{\phi}, L_{\theta}$ and $L_{\phi}$ can be expressed in terms of the following integrals:

$$
\begin{gather*}
N_{\theta}=\int_{S}\left(J_{x} \cos (\theta) \cos (\phi)+J_{y} \cos (\theta) \sin (\phi)-J_{z} \sin (\theta)\right) e^{-j k r^{\prime} \cos (\psi)} d S^{\prime}  \tag{5.54a}\\
N_{\phi}=\int_{S}\left(-J_{x} \sin \phi+J_{y} \cos \phi\right) e^{-j k r^{\prime} \cos (\psi)} d S^{\prime} \tag{5.54b}
\end{gather*}
$$

$$
\begin{gather*}
L_{\theta}=\int_{S}\left(M_{x} \cos (\theta) \cos (\phi)+M_{y} \cos (\theta) \sin (\phi)-M_{z} \sin (\theta)\right) e^{-j k r^{\prime} \cos (\psi)} d S^{\prime}  \tag{5.54c}\\
L_{\phi}=\int_{S}\left(-M_{x} \sin \phi+M_{y} \cos \phi\right) e^{-j k r^{\prime} \cos (\psi)} d S^{\prime} \tag{5.54d}
\end{gather*}
$$

Here, the currents $\vec{J}$ and $\vec{M}$ on the surface are determined by $E$ and $H$ fields which are computed using either C-LOD-FDTD or F-LOD-FDTD method inside the computational domain. These currents are transformed into the frequency domain while being captured. After completing for all the time steps, the far field terms $L_{\theta}, L_{\phi}$, $N_{\theta}$ and $N_{\phi}$ are calculated. These far field terms are calculated in the same way as given in [2] so will not be repeated here. Bistatic RCS can then be calculated using the following equation.

$$
\begin{align*}
R C S_{\theta} & =\frac{k^{2}}{8 \pi \eta_{0} P_{i n c}}\left|L_{\phi}+\eta_{0} N_{\theta}\right|^{2}  \tag{5.55a}\\
R C S_{\phi} & =\frac{k^{2}}{8 \pi \eta_{0} P_{i n c}}\left|L_{\theta}-\eta_{0} N_{\phi}\right|^{2} \tag{5.55b}
\end{align*}
$$

The $P_{\text {inc }}$ can be calculated as:

$$
\begin{equation*}
P_{i n c}=\frac{1}{2 \eta_{0}}\left|E_{i n c}(\omega)\right|^{2} \tag{5.56}
\end{equation*}
$$

where $E_{\text {inc }}(\omega)$ is the discrete Fourier transform (DFT) of the incident electric field waveform at the frequency for which RCS calculation is required. The near-field to farfield transformation techniques have been used for both C-LOD-FDTD and F-LODFDTD.

### 5.7 Pure Scattered Field Formulation for Both the 3-D C-LOD-FDTD and F-LOD-FDTD Methods

In this section, pure scattered field formulation for both 3-D C-LOD-FDTD and F-LOD-FDTD will be presented. For plane wave excitation in LOD-FDTD, the problem space is divided into two regions: the total field region and the scattered field region. The vectorial sum of incident and scattered fields present within a given space provides the total fields. The scattered field formulation for the 3-D C-LOD-FDTD in two substeps is given in the next sub-section.

### 5.7.1 Scattered Field Formulation for 3-D C-LOD-FDTD

Following the theory of scattered field formulation as described in Chapter 3 and the C -LOD-FDTD formulations as described in Section 5.2, the scattered field formulation for the C-LOD-FDTD method for the two-steps is given below.
Sub-step 1: The electric field updating equations of sub-step 1 are as follows:

$$
\begin{align*}
& \left.E_{\text {scat }, x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}=C_{\text {exe }} \times\left. E_{\text {scat }, x}\right|_{i+1 / 2, j, k} ^{n} \\
& +C_{e x h z} \times\left(\left.H_{\text {scat }, z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{\text {scat }, z}\right|_{i+1 / 2, j-1 / 2, k} ^{n+1 / 2}\right) \\
& +C_{e y h x} \times\left(\left.H_{s c a t, z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{\text {scat }, z}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right)  \tag{5.57a}\\
& +C_{\text {exeic }} \times\left. E_{\text {inc }, x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}+C_{\text {exeip }} \times\left. E_{\text {inc }, x}\right|_{i+1 / 2, j, k} ^{n} \\
& \left.E_{\text {scat }, y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}=C_{\text {eye }} \times\left. E_{\text {scat }, y}\right|_{i, j+1 / 2, k} ^{n} \\
& +C_{\text {eyhx }} \times\left(\left.H_{\text {scat }, x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{\text {scat }, x}\right|_{i, j+1 / 2, k-1 / 2} ^{n+1 / 2}\right) \\
& +C_{e y h x} \times\left(\left.H_{s c a t, x}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{s c a t, x}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right)  \tag{5.57b}\\
& +C_{\text {eyeic }} \times\left. E_{\text {inc }, y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}+C_{\text {eyeip }} \times\left. E_{i n c, y}\right|_{i, j+1 / 2, k} ^{n} \\
& \left.E_{s c a t, z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}=C_{e z e} \times\left. E_{s c a t, z}\right|_{i, j, k+1 / 2} ^{n} \\
& +C_{e z h y} \times\left(\left.H_{s c a t, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{s c a t, y}\right|_{i-1 / 2, j, k+1 / 2} ^{n+1 / 2}\right) \\
& +C_{e z h y} \times\left(\left.H_{s c a t, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{\text {scat }, y}\right|_{i-1 / 2, j, k+1 / 2} ^{n}\right)  \tag{5.57c}\\
& +C_{\text {ezeic }} \times\left. E_{\text {inc }, z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}+C_{\text {ezeip }} \times\left. E_{\text {inc }, z}\right|_{i, j, k+1 / 2} ^{n}
\end{align*}
$$

The magnetic field updating equations of sub-step 1 are as follows:

$$
\begin{align*}
& \left.H_{s c a t, x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}=C_{h x h} \times\left. H_{s c a t, x}\right|_{i, j+1 / 2, k+1 / 2} ^{n} \\
& \quad+C_{h x e y} \times\left(\left.E_{s c a t, y}\right|_{i, j+1 / 2, k+1} ^{n+1 / 2}-\left.E_{\text {scat }, y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right) \\
& \quad+C_{h x e y} \times\left(\left.E_{s c a t, y}\right|_{i, j+1 / 2, k+1} ^{n}-\left.E_{s c a t, y}\right|_{i, j+1 / 2, k} ^{n}\right)  \tag{5.57d}\\
& \quad+C_{h x h i c} \times\left. H_{i n c, x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}+C_{h x h i p} \times\left. H_{i n c, x}\right|_{i, j+1 / 2, k+1 / 2} ^{n} \\
& \left.H_{\text {scat }, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}=C_{h y h} \times\left. H_{s c a t, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n} \\
& \quad+C_{h y e z} \times\left(\left.E_{s c a t, z}\right|_{i+1, j, k+1 / 2} ^{n+1 / 2}-\left.E_{s c a t, z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right) \\
& \quad+C_{h y e z} \times\left(\left.E_{s c a t, z}\right|_{i+1, j, k+1 / 2} ^{n}-\left.E_{\text {scat }, z}\right|_{i, j, k+1 / 2} ^{n}\right)  \tag{5.57e}\\
& \quad+C_{h y h i c} \times\left. H_{i n c, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}+C_{h y h i p} \times\left. H_{i n c, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n}
\end{align*}
$$

$$
\begin{align*}
& \left.H_{\text {scat }, z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}=C_{h z h} \times\left. H_{\text {scat }, z}\right|_{i+1 / 2, j+1 / 2, k} ^{n} \\
& \quad+C_{h z e x} \times\left(\left.E_{\text {scat }, x}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2}-\left.E_{\text {scat }, y}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right) \\
& \quad+C_{h z e x} \times\left(\left.E_{\text {scat }, x}\right|_{i+1 / 2, j+1, k} ^{n}-\left.E_{s c a t, x}\right|_{i+1 / 2, j, k} ^{n}\right)  \tag{5.57f}\\
& \quad+C_{h z h i c} \times\left. H_{i n c, z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}+C_{h z h i p} \times\left. H_{i n c, z}\right|_{i+1 / 2, j+1 / 2, k} ^{n}
\end{align*}
$$

In a similar way, the updating equations for sub-step 2 can be derived as given below.
Sub-step 2:
The electric field updating equations of sub-step 2 are as follows:

$$
\begin{align*}
& \left.E_{\text {scat }, x}\right|_{i+1 / 2, j, k} ^{n+1}=C_{\text {exe }} \times\left. E_{\text {scat }, x}\right|_{i+1 / 2, j, k} ^{n+1 / 2} \\
& +C_{\text {exhy }} \times\left(\left.H_{\text {scat }, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{\text {scat }, y}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1}\right) \\
& +C_{e x h y} \times\left(\left.H_{s c a t, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{s c a t, y}\right|_{i+1 / 2, j, j, k-1 / 2} ^{n+1 / 2}\right)  \tag{5.58a}\\
& +C_{\text {exeic }} \times\left. E_{\text {inc }, x}\right|_{i+1 / 2, j, k} ^{n+1}+C_{\text {exeip }} \times\left. E_{\text {inc }, x}\right|_{i+1 / 2, j, k} ^{n+1 / 2} \\
& \left.E_{\text {scat }, y}\right|_{i+1 / 2, j, k} ^{n+1}=C_{\text {eye }} \times\left. E_{\text {scat }, y}\right|_{i+1 / 2, j, k} ^{n+1 / 2} \\
& +C_{e y h y} \times\left(\left.H_{s c a t, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{s c a t, y}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1}\right) \\
& +C_{\text {eyhy }} \times\left(\left.H_{\text {scat }, y}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{\text {scat }, y}\right|_{i, j+1 / 2, k-1 / 2} ^{n+1 / 2}\right)  \tag{5.58b}\\
& +C_{\text {eyeic }} \times\left. E_{\text {inc, },}\right|_{i, j+1 / 2, k} ^{n+1}+C_{\text {eyeip }} \times\left. E_{\text {inc }, y}\right|_{i, j+1 / 2, k} ^{n+1 / 2} \\
& \left.E_{\text {scat }, z}\right|_{i+1 / 2, j, k} ^{n+1}=C_{\text {eze }} \times\left. E_{\text {scat }, z}\right|_{i, j, k+1 / 2} ^{n+1 / 2} \\
& +C_{e z h x} \times\left(\left.H_{s c a t, x}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{s c a t, x}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1}\right) \\
& +C_{e z h x} \times\left(\left.H_{\text {scat }, x}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{\text {scat }, x}\right|_{i+1 / 2, j, j, k-1 / 2} ^{n+1 / 2}\right)  \tag{5.58c}\\
& +C_{\text {ezeic }} \times\left. E_{\text {inc }, z}\right|_{i+1 / 2, j, k} ^{n+1}+C_{\text {ezeip }} \times\left. E_{\text {inc }, z}\right|_{i+1 / 2, j, j} ^{n+1 / 2}
\end{align*}
$$

The magnetic field updating equations of sub-step 2 are as follows:

$$
\begin{align*}
& \left.H_{\text {scat }, x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}=C_{h x h} \times\left. H_{\text {scat }, x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2} \\
& \quad+C_{h x e z} \times\left(\left.E_{\text {scat }, z}\right|_{i, j+1, k+1 / 2} ^{n+1}-\left.E_{s c a t, z}\right|_{i, j, k+1 / 2} ^{n+1}\right) \\
& \quad+C_{h x e z} \times\left(\left.E_{\text {scat }, z}\right|_{i, j+1, k+1 / 2} ^{n+1 / 2}-\left.E_{\text {scat }, z}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right)  \tag{5.58d}\\
& \quad+C_{h x h i c} \times\left. H_{i n c, x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}+C_{h x h i p} \times\left. H_{i n c, x}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}
\end{align*}
$$

$$
\begin{align*}
& \left.H_{\text {scat }, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}=C_{h y h} \times\left. H_{\text {scat }, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2} \\
& \quad+C_{h y e x} \times\left(\left.E_{\text {scat }, x}\right|_{i+1 / 2, j, k+1} ^{n+1}-\left.E_{s c a t, x}\right|_{i+1 / 2, j, k} ^{n+1}\right) \\
& \quad+C_{h y e x} \times\left(\left.E_{\text {scat }, x}\right|_{i+1 / 2, j, k+1} ^{n+1 / 2}-\left.E_{\text {scat }, x}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right)  \tag{5.58e}\\
& \quad+C_{h y h i c} \times\left. H_{i n c, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}+C_{h y h i p} \times\left. H_{i n c, y}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2} \\
& \left.H_{\text {scat }, z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}=C_{h z h} \times\left. H_{s c a t, z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2} \\
& \quad+C_{h z e y} \times\left(\left.E_{s c a t, y}\right|_{i+1, j+1 / 2, k} ^{n+1}-\left.E_{\text {scat }, y}\right|_{i, j+1 / 2, k} ^{n+1}\right) \\
& \quad+C_{h z e y} \times\left(\left.E_{\text {scat }, y}\right|_{i+1, j+1 / 2, k} ^{n+1 / 2}-\left.E_{\text {scat }, y}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right)  \tag{5.58f}\\
& \quad+C_{h z h i c} \times\left. H_{i n c, z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}+C_{h z h i p} \times\left. H_{i n c, z}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}
\end{align*}
$$

where $C_{\text {exe }}, C_{\text {exxx }}, C_{\text {exhy }}, C_{\text {exkz }}$, and $C_{\text {hxh }}, C_{\text {hxex }}, C_{\text {hxey }}, C_{\text {hxez }}$ as well as other coefficients are the same as mentioned in Section 5.2, and

$$
\begin{gathered}
C_{\text {exeic }}=\left(4\left(\varepsilon_{0}-\varepsilon\right)-\sigma_{e} \Delta t\right) /\left(4 \varepsilon+\sigma_{e} \Delta t\right), C_{\text {exeip }}=-\left(4\left(\varepsilon_{0}-\varepsilon\right)+\sigma_{e} \Delta t\right) /\left(4 \varepsilon+\sigma_{e} \Delta t\right) \\
C_{\text {hxeic }}=\left(4\left(\mu_{0}-\mu\right)-\sigma_{m} \Delta t\right) /\left(4 \mu+\sigma_{m} \Delta t\right), C_{h x h i p}=-\left(4\left(\mu_{0}-\mu\right)+\sigma_{m} \Delta t\right) /\left(4 \mu+\sigma_{m} \Delta t\right)
\end{gathered}
$$

However, equations (5.57) and (5.58) of sub-step 1 and 2 cannot be solved directly; tridiagonal linear system can be formed from sub-steps 1and 2 by following the procedure which has been described in Section 5.2. Equations (5.57) and (5.58) have been used for the calculation of RCS from 3-D microwave structures in Section 5.8.

### 5.7.2 Scattered Field Formulation for 3-D F-LOD-FDTD

Similar to the scatter field formulation of the C-LOD-FDTD method, the scattered field formulation for the F-LOD-FDTD method can be derived. Following the scattered field formulation as described in Chapter 3 and the F-LOD-FDTD formulations as described in Section 5.4, the scattered field formulation for the F-LOD-FDTD method for the two-steps can be derived. For brevity, only sub-step 1 is given below.

Sub-step 1:
Auxiliary implicit updating for electric and magnetic fields are as follows:

$$
\begin{align*}
\left.e_{\text {scat }, x}\right|^{n+1 / 2}-\left.2 E_{s c a t, x}\right|^{n}= & \left.b \partial_{y} h_{\text {scat }, z}\right|^{n+1 / 2}  \tag{5.59a}\\
& +b\left(\left.e_{\text {inc }, x}\right|^{n+1 / 2}-\left.e_{i n c, x}\right|^{n}\right)
\end{align*}
$$

$$
\begin{align*}
\left.e_{s c a t, y}\right|^{n+1 / 2}-\left.2 E_{s c a t, y}\right|^{n}= & \left.b \partial_{z} h_{s c a t, x}\right|^{n+1 / 2} \\
& +b\left(\left.e_{i n c, y}\right|^{n+1 / 2}-\left.e_{i n c, y}\right|^{n}\right)  \tag{5.59b}\\
\left.e_{s c a t, z}\right|^{n+1 / 2}-\left.2 E_{s c a t, z}\right|^{n}= & \left.b \partial_{x} h_{s c a t, y}\right|^{n+1 / 2} \\
& +b\left(\left.e_{i n c, z}\right|^{n+1 / 2}-\left.e_{i n c, z}\right|^{n}\right)  \tag{5.59c}\\
\left.h_{s c a t, x}\right|^{n+1 / 2}-\left.2 H_{s c a t, x}\right|^{n}= & \left.d \partial_{z} e_{s c a t, y}\right|^{n+1 / 2} \\
& +d\left(\left.h_{i n c, x}\right|^{n+1 / 2}-\left.h_{i n c, x}\right|^{n}\right)  \tag{5.59d}\\
\left.h_{s c a t, y}\right|^{n+1 / 2}-\left.2 H_{s c a t, y}\right|^{n}= & \left.d \partial_{x} e_{s c a t, z}\right|^{n+1 / 2} \\
& +d\left(\left.h_{i n c, y}\right|^{n+1 / 2}-\left.h_{i n c, y}\right|^{n}\right)  \tag{5.59e}\\
\left.h_{s c a t, z}\right|^{n+1 / 2}-\left.2 H_{s c a t, z}\right|^{n}= & \left.d \partial_{y} e_{s c a t, x}\right|^{n+1 / 2} \\
& +d\left(\left.h_{i n c, z}\right|^{n+1 / 2}-\left.h_{i n c, z}\right|^{n}\right) \tag{5.59f}
\end{align*}
$$

and the explicit updating for the electric and magnetic fields are as follows:

$$
\begin{align*}
\left.E_{s c a t, \xi 1}\right|^{n+1 / 2} & =\left.e_{s c a t, \xi 1}\right|^{n+1 / 2}-\left.E_{\text {scat }, \xi 1}\right|^{n}  \tag{5.60}\\
\left.H_{\text {scat }, \xi 1}\right|^{n+1 / 2} & =\left.h_{\text {scat }, \xi 1}\right|^{n+1 / 2}-\left.H_{\text {scat }, \xi 1}\right|^{n} \tag{5.61}
\end{align*}
$$

In a similar way, the update scattered field equations for the sub-step 2 can be derived for the F-LOD-FDTD method. Therefore, for a given incident wave, the above equations can be used to calculate the scattered field. The total field can be obtained by adding the scattered field to the incident field. Note that incident fields are applied only in the internal region, and in the CPML region only scattered fields exist, which are absorbed by CPML. In the next section, the applications of fundamental scheme LODFDTD are discussed.

### 5.8 Computational Results on 3-D Microwave Structures Using C-LOD-FDTD and F-LOD-FDTD Methods

### 5.8.1 EM Scattering from Spheres

To validate the proposed F-LOD-FDTD CPML method for EM scattering problems, we first consider plane wave scattering from a perfectly electric conducting sphere of radius $k a=8.3$. The problem space is divided using uniform orthogonal meshes with
cell size $\Delta x=\Delta y=\Delta z=0.75 \mathrm{~cm}$. An eight-layer CPML is used for the PEC sphere which surrounds the entire computational domain with parameters $\sigma_{\mathrm{s}, \max }=0.7 \sigma_{\mathrm{opt}}, \sigma_{\mathrm{opt}}=11.21$ $(\mathrm{S} / \mathrm{m}), m=4, \kappa_{\max }=1$, and $\alpha_{\max }=0.2$. Fig 5.11 (a)-(b) shows the bistatic radar cross section of the PEC sphere obtained using the F-LOD-FDTD CPML for CFLN=2, 10 and compared with the results obtained by C-LOD-FDTD CPML as well as Mie series.


Fig. 5.11 $\operatorname{RCS}(\theta, 0)$ for a perfectly conducting sphere $k a=8.3$ compared with the results obtained using C-LOD-FDTD as well as Mie series solution for (a) CFLN=2 (b) CFLN=10

From Fig. 5.11 (a)-(b), it can be observed that the results on a scattered field at higher CFLN agree well with the results obtained from C-LOD-FDTD and with Mie series. As far as the computer resources are concerned, C-LOD-FDTD CPML requires around 28 MB memory and 6.5 mins of execution time, whereas F-LOD-FDTD CPML needs around 19.5 MB memory and 4.5 minutes of execution time.


Fig. 5.12 Transient scattering of PEC sphere of radius 0.5 m illuminated by a Gaussian plane wave

Our simulations were carried out on a six core Linux workstation with 3.4 GHz clock and 32 GB RAM using Matlab. The computational performance in terms of memory requirement and execution time show that F-LOD-FDTD CPML is computationally efficient compared to C-LOD-FDTD CPML. Fig. 5.12 shows the transient scattering from a PEC sphere of radius 0.5 m compared with the results obtained by C-LOD-FDTD as well as with published results obtained using TDIE [169]. It can be seen that the result obtained by F-LOD-FDTD method agrees reasonably well with the analytical result. From the Fig. 5.12, it can also be observed that the time domain solutions of the PEC sphere are stable in early time.

Now, we analyse the bistatic scattering from a lossless dielectric sphere ( $k a=3, \varepsilon_{r}=20$ ). Our numerical results will be compared with the results in the literature [170]. The problem space is divided using uniform orthogonal rectangular meshes with cell size $\Delta x=\Delta y=\Delta z=0.75 \mathrm{~cm}$. An eight-layer CPML is used for the dielectric sphere which surrounds the entire computational domain with parameter, $\sigma_{\mathrm{s}, \max }=0.7 \sigma_{\mathrm{opt}}, \sigma_{\mathrm{opt}}=11.21(\mathrm{~S} / \mathrm{m}), m=4, \kappa_{\max }=1$, and $\alpha_{\max }=0.2$. The computed bistatic RCS for the dielectric sphere obtained by F-LOD-FDTD is shown in Fig. 5.13 (a)-(b). From Fig. 5.13 (a)-(b), it is observed that the result obtained by the F-LOD-FDTD CPML with higher CFLN agrees reasonably well with the published results from [170]. The computational resources used for the simulation of the dielectric sphere are the same as those used for the PEC sphere. The transient scattering from a lossless dielectric sphere with radius 0.5 m and the relative permittivity $\varepsilon_{\mathrm{r}}=2$ has been computed next using F-LOD-FDTD.


Fig. 5.13 Bistatic RCS for a dielectric sphere $\left(k a=3, \varepsilon_{r}=20\right)$ compared with Mie series solution for (a) $\mathrm{CFLN}=5$ and (b) CFLN=10.


Fig. 5.14 Transient scattering from dielectric sphere of radius 0.5 m illuminated by a Gaussian plane wave

Fig. 5.14 shows the transient scattering compared with the results obtained by C-LODFDTD as well as with the published results obtained using TD-EFIE [171]. It can be observed that the result obtained by F-LOD-FDTD agrees reasonably well with the analytical result as well as with the result obtained using C-LOD-FDTD. From the transient response, it can be observed that the time domain solutions of the lossless dielectric sphere are stable in early time.

### 5.8.2 Modelling of Cylindrical Dielectric Resonator

To further validate the F-LOD-FDTD CPML and C-LOD-FDTD CPML, a cylindrical dielectric resonator is considered. Our numerical results on the resonant frequencies of the cylindrical dielectric resonator will be compared with the results obtained from published results in the literature [172]. The parameter used for the cylindrical dielectric resonator (as shown in Fig. 5.15) are $\varepsilon_{\mathrm{r}}=38$, $\mathrm{a}=5.25 \mathrm{~mm}, \mathrm{~h}=4.6 \mathrm{~mm}$. An eight-layer CPML is used for the cylindrical dielectric resonator which surrounds the entire computational domain with parameters $\sigma_{\mathrm{s}, \max }=0.7 \sigma_{\mathrm{opt}}, \sigma_{\mathrm{opt}}=11.21(\mathrm{~S} / \mathrm{m}), m=4, \kappa_{\max }=1$, and $\alpha_{\max }=0.2$. The resonant frequencies for the TE and HE modes of the resonator are computed using both the F-LOD-FDTD CPML and C-LOD-FDTD CPML and are tabulated in Table 5.4.


Fig. 5.15 The problem space for dielectric resonator with CPML boundary, $\varepsilon_{\mathrm{r}}=38, \mathrm{a}=5.25 \mathrm{~mm}, \mathrm{~h}=4.6$ mm

Table 5.4
Comparison of resonant frequencies for the cylindrical dielectric resonator [172]

| Mode | F-LOD-FDTD | C-LOD-FDTD | Glisson et al. [172] |
| :---: | :---: | :---: | :---: |
| $\mathrm{TE}_{01}$ | 4.831 | 4.811 | 4.82 |
| $\mathrm{HE}_{12}$ | 6.639 | 6.621 | 6.63 |

The results on resonant frequencies reveal that those obtained using F-LOD-FDTD CPML have closer agreement with the published data [172].

### 5.8.3 Arbitrary Shaped Thin Wire Antenna Modelling

To further validate the proposed approach, various dipole antennas have been analysed by F-LOD-FDTD CPML and C-LOD-FDTD CPML. First, a thin wire dipole antenna in free space has been analysed. A dipole antenna (as shown in Fig. 5.16) composed of two thin wires having a radius of 0.05 mm and length of 9.75 mm is considered. The computational domain is discretised with orthogonal meshes with cell sizes $\Delta x=0.001 \mathrm{~mm} \Delta y=0.001 \mathrm{~mm}$ and $\Delta z=0.025 \mathrm{~mm}$. For this, a CPML with a thickness of 8 cells and a 10 cell air gap on all sides is considered with parameters $\sigma_{\mathrm{s}, \max }=0.7 \sigma_{\mathrm{opt}}, \sigma_{\mathrm{opt}}=11.21(\mathrm{~S} / \mathrm{m}), m=4, \kappa_{\max }=1$, and $\alpha_{\max }=0.2$.


Fig. 5.16 A thin wire dipole antenna


Fig. 5.17 Transient current in the center of the thin wire dipole antenna

Note that the larger value $\alpha=0.2$ in the CPML region is considered to allow evanescent waves to be absorbed by the PML without reflection. Fig. 5.17 shows the induced transient current at the centre of the dipole antenna. The scattering parameters in the frequency domain as well as the radiation pattern are obtained using the proposed methods. Fig. 5.18 (a)-(b) shows the calculated $\mathrm{S}_{11}$ of the thin wire dipole antenna obtained using F-LOD-FDTD CPML and conventional LOD-FDTD CPML for CFLN=2 and 10. Far field radiation patterns in the xz and xy plane cut obtained by the F-LOD-FDTD and C-LOD-FDTD methods are shown in Fig. 5.19 and Fig. 5.20 at the frequency of 7 GHz . From Figs. 5.19-5.20, it can be observed that the calculated $\left|\mathrm{S}_{11}\right|$ and the radiation pattern by the proposed F-LOD-FDTD CPML agree reasonably well with the results obtained by C-LOD-FDTD.

(a)

(b)

Fig. 5.18 (a)-(b) Calculated $\left|\mathrm{S}_{11}\right|$ of the dipole antenna using F-LOD-FDTD, C-LOD-FDTD and FDTD


Fig. 5.19 Radiation pattern in the xz plane cut


Fig. 5.20 Radiation pattern in the xy plane cut

Next we present the computed transient current induced at the centre of a 1 m long thin wire dipole antenna having a radius of 5 cm operating at 300 MHz and compare this with the published results obtained using the time domain (TD) method of moment (MOM) [173] as shown in Fig. 5.21. For this, a CPML with a thickness of 8 cells and with a 10 cell air gap on all sides is considered with parameters $\sigma_{\mathrm{s}, \max }=0.7 \sigma_{\mathrm{opt}}$, $\sigma_{\mathrm{opt}}=11.21(\mathrm{~S} / \mathrm{m}), m=4, \kappa_{\max }=1$, and $\alpha_{\max }=0.2$.


Fig. 5.21 Calculated currents at the centre of the 1 m long dipole antenna

In terms of the computer resources for the calculation of the S-parameter of the dipole antenna, the C-LOD-FDTD CPML requires around 60.5 MB memory and 215.31 seconds of execution time, whereas F-LOD-FDTD CPML needs around 53 MB memory and 150.66 seconds of execution time. It can be observed that the F-LODFDTD CPML method requires less execution time as a result of the reduced number of arithmetic operations required for F-LOD-FDTD. Our simulations were carried out on a six core Linux workstation with 3.4 GHz clock and 32 GB RAM using Matlab.

To further demonstrate the accuracy and efficiency of the F-LOD-FDTD and C-LOD-FDTD approaches, they are applied to the analysis of thin wire bent and square loop antennas. First, the bent thin wire antenna (shown in Fig. 5.22) is analysed. A bent wire with $90^{\circ}$ bent angles with equal sides of length 9.75 mm and radius 0.05 mm are considered. The computational domain is discretised with orthogonal meshes with cell sizes $\Delta \mathrm{x}=0.001 \mathrm{~mm}, \Delta \mathrm{y}=0.001 \mathrm{~mm}$, and $\Delta \mathrm{z}=0.025 \mathrm{~mm}$. A CPML with a thickness of 8 cells and with a 10 cell air gap on all sides is considered with parameters $\sigma_{\mathrm{s}, \max }=0.7 \sigma_{\mathrm{opt}}$, $\sigma_{\mathrm{opt}}=11.21(\mathrm{~S} / \mathrm{m}), m=4, \kappa_{\max }=1$, and $\alpha_{\max }=0.2$.


Fig. 5.22 Bent thin wire antenna


Fig. 5.23 Induced current at the centre of the bent wire antenna


Fig. 5.24 Calculated $\left|\mathrm{S}_{11}\right|$ of the bent wire antenna using F-LOD-FDTD and C-LOD-FDTD


Fig. 5.25 Calculated phase of the bent wire antenna using F-LOD-FDTD CPML

Note that larger value $\alpha=0.2$ in the CPML region is considered to allow evanescent waves to penetrate into the PML without reflection. Fig. 5.23 shows the transient current induced at the centre of the bent wire antenna obtained by F-LOD-FDTD CPML. Fig. 5.24 shows the calculated $\mathrm{S}_{11}$ of the bent wire antenna by F-LOD-FDTD CPML compared with the result obtained by C-LOD-FDTD CPML for CFLN=5. Fig. 5.25 presents the computed phase of the bent thin wire antenna using F-LOD-FDTD compared with the result obtained by C-LOD-FDTD for CFLN=2. From Figs. 5.24 and 5.25 , it can be observed that the result obtained by F-LOD-FDTD CPML agrees reasonably well with the results obtained by conventional LOD-FDTD CPML. In respect of computer resources, the conventional LOD-FDTD CPML requires around 90 MB memory and 30 mins execution time whereas F-LOD-FDTD CPML needs a memory of 60 MB and execution time of 20 mins.

Next, we use F-LOD-FDTD CPML and C-LOD-FDTD CPML approaches to model one wavelength thin wire square loop antenna as shown in Fig. 5.26. Fig. 5.27 shows the induced transient current from the square loop wire antenna obtained by F-LODFDTD CPML. Figs. 5.28-5.29 show the calculated $\mathrm{S}_{11}$ and phase of the square loop antenna obtained using F-LOD-FDTD at CFLN=14 and 12. From Figs. 5.28 and 5.29, it can be observed that the computed results using the F-LOD-FDTD CPML and C-LODFDTD approaches provide a stable solution even at higher CFLN. We used the same computer resources as were used for the dipole antenna. The C-LOD-FDTD CPML requires 22 mins and 100 MB memory for the simulation whereas F-LOD-FDTD needs 18 mins and 89 MB memory.


Fig. 5.26 Geometry of square loop wire antenna


Fig. 5.27 Induced current from the square loop wire antenna


Fig. 5.28 Computed $\left|\mathrm{S}_{11}\right|$ of the square loop wire antenna using F-LOD-FDTD CPML and C-LOD-FDTD at $\mathrm{CFLN}=14$


Fig. 5.29 Computed phase of the square loop wire antenna using F-LOD-FDTD CPML and C-LODFDTD CPML at $\mathrm{CFLN}=12$

### 5.8.4 Rectangular Microstrip Patch Antenna Analysis

To further validate the proposed approach, the rectangular microstrip antenna is analysed. The parameters of the rectangular microstrip antenna (as shown in Fig. 5.30) are taken from [174]. The orthogonal meshes with the cell size $\Delta x=0.389 \mathrm{~mm}, \Delta \mathrm{y}=0.40$ mm , and $\Delta \mathrm{z}=0.265 \mathrm{~mm}$ have been used; thus the rectangular antenna is $32 \Delta \mathrm{x} \times 40 \Delta \mathrm{y}$.


Fig. 5.30 Rectangular microstrip antenna


Fig. 5.31 (a)-(b) Return loss of the rectangular antenna at CFLN=2 and 10

The length of the microstrip line from the source plane to the edge of the antenna is 50 $\Delta y$, and the reference plane for port 1 is $10 \Delta y$ from the edge of the patch. The microstrip line width is modelled as $6 \Delta x$. The Gaussian half width is $T=15 \mathrm{ps}$, and the time delay $\mathrm{t}_{0}$ is set at 3 T . The scattering coefficient results obtained by the F-LODFDTD CPML shown in Fig. 5.31(a)-(b) for CFLN=2, 10, show good agreement with the result obtained by C-LOD-FDTD CPML. The antenna resonates at 7.5 GHz which agrees with the published results in [174]. Additional resonances are also in good agreement, except for the highest resonance near 18 GHz , which is somewhat shifted.

### 5.8.5 Modelling of Microstrip Low Pass Filter

To further validate F-LOD-FDTD CPML, the microstrip low-pass filter [174] (shown in Fig. 5.32) is analysed and the results obtained by the F-LOD-FDTD method are compared with the results obtained by the C-LOD-FDTD method. The cell size $\Delta x, \Delta y$, and $\Delta \mathrm{z}$ for the low pass filter are carefully chosen to fit the dimensions of the circuit. The spatial cell sizes used are $\Delta x=0.4064 \mathrm{~mm}, \Delta \mathrm{y}=0.4233 \mathrm{~mm}$, and $\Delta \mathrm{z}=0.265 \mathrm{~mm}$, chosen to fit the dimension of the filter. The length of the rectangular patch becomes equal to $50 \Delta \mathrm{x} \times 6 \Delta \mathrm{y}$. The scattering coefficient $\mathrm{S}_{11}$ obtained by the F-LOD-FDTD is shown in Fig. 5.33 (a) for CFLN $=6$ and compared with the results obtained using the F -LOD-FDTD method. The scattering coefficient $\mathrm{S}_{21}$ results, shown in Fig. 5.33 (b), again show good agreement in the location of the nulls of frequency response. The phase of the s-parameters is shown in Fig. 5.34. The desired low-pass filter performance has sharp $\mathrm{S}_{21}$ with roll-off beginning at approximately 5 GHz . There is again some shift near the high end of the frequency range. However, from the figure, it can be observed that the results obtained by the C-LOD-FDTD method agree reasonably well with the results obtained by the C-LOD-FDTD method. As far as computer resources are concerned, the C-LOD-FDTD CPML requires around 549 MB memory and 136 sec execution time, whereas F-LOD-FDTD CPML needs around 417 MB and 110.66 sec . From the comparison, it is observed that the CPML with F-LOD-FDTD requires less execution time than conventional LOD-FDTD.


Fig. 5.32 Microstrip low pass filter

S11

(a)

(b)

Fig. 5.33 Return losses of microstrip low pass filter (a) $\mathrm{S}_{11}$ and (b) $\mathrm{S}_{21}$ for CFLN=6


Fig. 5.34 Phase of $\mathrm{S}_{11}$ for the microstrip low pass filter

### 5.8.6 Microstrip Transmission Line Model of VLSI Interconnect

Finally, we have analysed wave propagation on a lossy microstrip VLSI interconnect [175] as shown in Fig. 5.35 using the proposed F-LOD-FDTD CPML and C-LODFDTD CPML. It must be noted that the explicit-FDTD method requires excessive computational time to model such a small structure due to CFL constraint. The 5 mm long micro-strip transmission line having a width of $10 \mu m$ is printed on a lossy $\mathrm{SiO}_{2}$ ( $\varepsilon_{\mathrm{r}}=4.0, \sigma=0.5 \times 10^{-3} \mathrm{~S} / \mathrm{m}$ ) substrate which is $1 \mu m$ thick and is excited by a Gaussian pulse [175]. To analyse this structure using our proposed approach, we choose $\Delta x=20 \mu \mathrm{~m}, \Delta y=5 \mu \mathrm{~m}$ and $\Delta z=1 \mu \mathrm{~m}$. Figs. 5.36 and 5.37 show the electric fields at a distance of 1.5 and 2 mm from the load end of the microstrip interconnect computed using the proposed scheme and compared with the results published in [175].


Fig. 5.35 Lossy microstrip transmission line model of VLSI interconnect [175]


Fig. 5.36 Normalised $E_{z}$ field observed at 1.5 mm from the load end of the transmission line.


Fig. 5.37 Normalised $E_{z}$ field observed at 2 mm from the load end of the transmission line

The F-LOD-FDTD CPML requires 4536 sec of CPU time and 184.9 MB of memory on a 3.4 GHz Linux workstation with 32 GB RAM using Matlab, whereas C-LODFDTD requires 6804 sec of CPU time and 215.3 MB memory. This proves that F-LODFDTD CPML is computationally more efficient than C-LOD-FDTD. For all cases, F-LOD-FDTD CPML and C-LOD-FDTD CPML obtained numerically stable results.

### 5.9 Discussion

In this chapter, a modified two sub-step conventional 3-D LOD-FDTD was first developed. A two sub-step CPML was derived for C-LOD-FDTD which obtains less reflection error. The stability analysis of C-LOD-FDTD was provided to demonstrate the unconditional stability of the developed C-LOD-FDTD method. The performance of the proposed C-LOD-FDTD CPML was investigated and compared with standard explicit FDTD as well as the F-LOD-FDTD. The C-LOD-FDTD CPML was also validated using numerical results obtained from realistic 3-D microwave devices and antenna.

To reduce the number of arithmetic operations due to the matrix operator, F-LODFDTD has been developed with the inclusion of procedures to improve computational efficiency. CPML ABC for the F-LOD-FDTD method has also been presented. A comparison of F-LOD-FDTD with conventional LOD-FDTD has been tabulated which shows the improvement of the F-LOD-FDTD method. To the best of our knowledge, the stability analysis of the F-LOD-FDTD method is derived for the first time in this
thesis to demonstrate the stability of F-LOD-FDTD. Finally, both F-LOD-FDTD and C-LOD-FDTD have been validated using numerical results on various microwave devices and antenna. The performance comparison in terms of execution time and memory used by C-LOD-FDTD and F-LOD-FDTD for analysing various microwave devices and antenna have been provided, which also proves the usefulness of the LOD-FDTD methods. Comparing the CPU time required for both the conventional and fundamental approaches, the proposed F-LOD-FDTD CPML approach is characterised by lighter calculation burden and higher efficiency.

## Chapter 6

## 3-D LOD-NFDTD: LOD-FDTD Approaches Using Non-orthogonal Curvilinear Meshes

### 6.1 Introduction

In the previous chapter, we presented efficient C-LOD-FDTD and F-LOD-FDTD approaches using orthogonal rectangular meshes for analysing 3-D structures. In this chapter, we employ nonorthogonal curved meshes with 3-D LOD-FDTD for modelling 3-D electromagnetic structures. The conventional LOD-FDTD method that uses orthogonal meshes employs staircase approximation to model curved/arbitrarily shaped structured. Very fine meshes are required to model the curved structures using orthogonal mesh LOD-FDTD method which can reduce the discretisation error introduced as a result of staircase approximation. This staircase error can also be reduced if nonorthogonal, curved meshes are employed while modelling 3-D curved structures. With this aim, we develop C-LOD-NFDTD and F-LOD-NFDTD approaches in this chapter using nonorthogonal curvilinear meshes. We also present CPML absorbing boundary conditions for nonorthogonal meshes 3-D LOD-FDTDs. Theoretical stability analysis of C-LOD-NFDTD and F-LOD-NFDTD will also be provided to demonstrate their unconditional stability. Numerical analyses of the C-LOD-NFDTD and F-LOD-NFDTD methods are also presented. Using our approaches, three dimensional (3-D) meshes can be conformed to the curved discontinuities, thereby resulting in improved accuracies with shorter computation time. By using the fundamental scheme for non-orthogonal meshes with the LOD-FDTD method, the computational efficiency of the method is increased. Performance comparisons for both the C-LOD-NFDTD and F-LOD-NFDTD will also be presented.

### 6.2 C-LOD-FDTD Method Using Non-orthogonal Curvilinear Meshes

In Chapter 3, we presented the nonorthogonal mesh 2-D LOD-FDTD method for modeling curved 2-D structures. Compared to the orthogonal mesh LOD-FDTD scheme, fewer meshes are required to represent the curved or oblique boundary of electromagnetic structures when using nonorthogonal meshes, and computational efficiency can be improved using LOD-NFDTD method. Here, we derive 3-D C-LODNFDTD method by modifying the orthogonal mesh version described in Section 5.2 of Chapter 5. The theory related to nonorthogonal mesh FDTD (NFDTD) has been described in section 3.2 of Chapter 3, so will not be repeated here. However, the derivation of the 3-D C-LOD-NFDTD method is described next.

### 6.2.1 Derivation of the 3-D C-LOD-NFDTD Method

Following the theory of the nonorthogonal FDTD method as described in Section 3.2 of Chapter 3 and denoting the covariant electric and magnetic field components which represent the flow of field along the grid as $E_{m}, H_{m}(m=1,2,3)$, and the contra-variant electric and magnetic field components which represent the flow through facets of the grid as $E^{m}, H^{m}(m=1,2,3)$, Maxwell's equations in a lossy medium are written in the generalised curvilinear coordinate system (as shown in Fig. 6.1) as:

$$
\dot{\partial}_{e}\left[\begin{array}{c}
E^{1}  \tag{6.1a}\\
E^{2} \\
E^{3}
\end{array}\right]=\left[\begin{array}{l}
\Theta_{13} \frac{\partial H_{3}}{\partial u^{2}}-\Theta_{12} \frac{\partial H_{2}}{\partial u^{3}} \\
\Theta_{21} \frac{\partial H_{1}}{\partial u^{3}}-\Theta_{23} \frac{\partial H_{3}}{\partial u^{1}} \\
\Theta_{32} \frac{\partial H_{2}}{\partial u^{1}}-\Theta_{31} \frac{\partial H_{1}}{\partial u^{2}}
\end{array}\right]
$$

and

$$
\dot{\partial}_{h}\left[\begin{array}{l}
H^{1}  \tag{6.1b}\\
H^{2} \\
H^{3}
\end{array}\right]=\left[\begin{array}{l}
\Theta_{13} \frac{\partial E_{3}}{\partial u^{2}}-\Theta_{12} \frac{\partial E_{2}}{\partial u^{3}} \\
\Theta_{21} \frac{\partial E_{1}}{\partial u^{3}}-\Theta_{23} \frac{\partial E_{3}}{\partial u^{1}} \\
\Theta_{32} \frac{\partial E_{2}}{\partial u^{1}}-\Theta_{31} \frac{\partial E_{1}}{\partial u^{2}}
\end{array}\right]
$$



Fig. 6.1 Nonorthogonal curvilinear coordinate system $\left(\mathbf{u}^{1}, \mathbf{u}^{2}, \mathbf{u}^{3}\right)$; (a) unitary vectors $\left(\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}\right)$ (b) reciprocal unitary vectors ( $\left.\mathbf{a}^{1}, \mathbf{a}^{\mathbf{2}}, \mathbf{a}^{\mathbf{3}}\right)$
where $\Theta_{p q}=\sqrt{g^{p p} /\left(g \cdot g_{q q}\right)}(p, q=1,2,3), \dot{\partial}_{e}=\varepsilon \frac{\partial}{\partial t}+\sigma_{e}, \dot{\partial}_{h}=-\mu \frac{\partial}{\partial t}-\sigma_{e}$ and $g$ is the determinant of the metric with elements $g_{p q}$ can be calculated using (3.3) given in Chapter 3 . In a curvilinear coordinate system, the covariant components of the electric $(E)$ and magnetic $(H)$ fields are placed on the cells in the manner suggested by Holland [23]. In the explicit FDTD method, the contra-variant like $H^{p}$ and $E^{p}$ on the left hand sides of the equations (6.1a) and (6.1b) must be converted to a covariant such as $H_{p}$ and $E_{p}$. The transformation matrix $G$ is defined as

$$
G=\left[\begin{array}{lll}
G_{11} & G_{12} & G_{13}  \tag{6.2}\\
G_{21} & G_{22} & G_{23} \\
G_{31} & G_{32} & G_{33}
\end{array}\right]
$$

To implement the LOD-NFDTD method, each explicit method is changed into an implicit time step using a two sub-step procedure following the LOD principle. By applying the LOD principle to (6.1a) and (6.1b), the two sub-step equations for C-LODNFDTD can be written as follows:

Sub-step1:

$$
\begin{align*}
& {\left[\begin{array}{lll}
G_{11} & G_{12} & G_{13} \\
G_{21} & G_{22} & G_{23} \\
G_{31} & G_{32} & G_{33}
\end{array}\right] \cdot\left[\begin{array}{l}
\left.E_{1}\right|^{n+1 / 2}-\left.C_{e} E_{1}\right|^{n} \\
\left.E_{2}\right|^{n+1 / 2}-\left.C_{e} E_{2}\right|^{n} \\
\left.E_{3}\right|^{\mid n+1 / 2}-\left.C_{e} E_{3}\right|^{n}
\end{array}\right]=D_{e} \cdot\left[\begin{array}{l}
\left.\Theta_{13} \frac{\partial H_{3}}{\partial u^{2}}\right|^{n+1 / 2}+\left.\Theta_{13} \frac{\partial H_{3}}{\partial u^{2}}\right|^{n} \\
\left.\Theta_{21} \frac{\partial H_{1}}{\partial u^{3}}\right|^{n+1 / 2}+\left.\Theta_{21} \frac{\partial H_{1}}{\partial u^{3}}\right|^{n} \\
\left.\Theta_{32} \frac{\partial H_{2}}{\partial u^{1}}\right|^{n+1 / 2}+\left.\Theta_{32} \frac{\partial H_{2}}{\partial u^{1}}\right|^{n}
\end{array}\right]}  \tag{6.3a}\\
& {\left[\begin{array}{lll}
G_{11} & G_{12} & G_{13} \\
G_{21} & G_{22} & G_{23} \\
G_{31} & G_{32} & G_{33}
\end{array}\right] \cdot\left[\begin{array}{l}
\left.H_{1}\right|^{n+1 / 2}-\left.C_{h} H_{1}\right|^{n} \\
\left.H_{2}\right|^{n+1 / 2}-\left.C_{h} H_{2}\right|^{n} \\
\left.H_{3}\right|^{n+1 / 2}-\left.C_{h} H_{3}\right|^{n}
\end{array}\right]=D_{h} \cdot\left[\begin{array}{l}
\left.\Theta_{12} \frac{\partial E_{2}}{\partial u^{3}}\right|^{n+1 / 2}+\left.\Theta_{12} \frac{\partial E_{2}}{\partial u^{3}}\right|^{n} \\
\left.\Theta_{2 E_{3}} \frac{\partial u^{1}}{}\right|^{n+1 / 2}+\left.\Theta_{23} \frac{\partial E_{3}}{\partial u^{1}}\right|^{n} \\
\left.\Theta_{31} \frac{\partial E_{1}}{\partial u^{2}}\right|^{n+1 / 2}+\left.\Theta_{31} \frac{\partial E_{1}}{\partial u^{2}}\right|^{n}
\end{array}\right]} \tag{6.3b}
\end{align*}
$$

Sub-step2:

$$
\begin{align*}
& {\left[\begin{array}{lll}
G_{11} & G_{12} & G_{13} \\
G_{21} & G_{22} & G_{23} \\
G_{31} & G_{32} & G_{33}
\end{array}\right] \cdot\left[\begin{array}{l}
\left.E_{1}\right|^{n+1}-\left.C_{e} E_{1}\right|^{n+1 / 2} \\
\left.E_{2}\right|^{n+1}-\left.C_{e} E_{2}\right|^{n+1 / 2} \\
\left.E_{3}\right|^{n+1}-\left.C_{e} E_{3}\right|^{n+1 / 2}
\end{array}\right]=-D_{e} \cdot\left[\begin{array}{l}
\left.\Theta_{12} \frac{\partial H_{2}}{\partial u^{3}}\right|^{n+1}+\left.\Theta_{12} \frac{\partial H_{2}}{\partial u^{3}}\right|^{n+1 / 2} \\
\left.\Theta_{23} \frac{\partial H_{3}}{\partial u^{1}}\right|^{n+1}+\left.\Theta_{23} \frac{\partial H_{3}}{\partial u^{1}}\right|^{n+1 / 2} \\
\left.\Theta_{31} \frac{\partial H_{1}}{\partial u^{2}}\right|^{n+1}+\left.\Theta_{31} \frac{\partial H_{1}}{\partial u^{2}}\right|^{n+1 / 2}
\end{array}\right]}  \tag{6.4a}\\
& {\left[\begin{array}{lll}
G_{11} & G_{12} & G_{13} \\
G_{21} & G_{22} & G_{23} \\
G_{31} & G_{32} & G_{33}
\end{array}\right] \cdot\left[\begin{array}{l}
\left.H_{1}\right|^{n+1}-\left.C_{h} H_{1}\right|^{n+1 / 2} \\
\left.H_{2}\right|^{n+1}-\left.C_{h} H_{2}\right|^{n+1 / 2} \\
\left.H_{3}\right|^{n+1}-\left.C_{h} H_{3}\right|^{n+1 / 2}
\end{array}\right]=-D_{h} \cdot\left[\begin{array}{l}
\left.\Theta_{13} \frac{\partial E_{3}}{\partial u^{2}}\right|^{n+1}+\left.\Theta_{13} \frac{\partial E_{3}}{\partial u^{2}}\right|^{n+1 / 2} \\
\left.\Theta_{21} \frac{\partial E_{1}}{\partial u^{3}}\right|^{n+1}+\left.\Theta_{21} \frac{\partial E_{1}}{\partial u^{3}}\right|^{n+1 / 2} \\
\left.\Theta_{32} \frac{\partial E_{2}}{\partial u^{1}}\right|^{n+1}+\left.\Theta_{32} \frac{\partial E_{2}}{\partial u^{1}}\right|^{n+1 / 2}
\end{array}\right]} \tag{6.4b}
\end{align*}
$$

where $C_{e}=\left(4 \varepsilon-\sigma_{e} \Delta t\right) /\left(4 \varepsilon+\sigma_{e} \Delta t\right), D_{e}=2 \Delta t /\left(4 \varepsilon+\sigma_{e} \Delta t\right), D_{h}=2 \Delta t /\left(4 \mu+\sigma_{m} \Delta t\right)$ and $C_{h}=\left(4 \mu-\sigma_{m} \Delta t\right) /\left(4 \mu+\sigma_{m} \Delta t\right)$. The updating equations of the C-LOD-NFDTD method can be derived with indices from (6.3) and (6.4) as follows:
Sub-step 1:
The electric field updating equations of sub-step 1 are as follows:

$$
\begin{align*}
\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}=\left.C_{e} E_{1}\right|_{i+1 / 2, j, k} ^{n}+ & \frac{1}{G} D_{e} \Theta_{13}\left(1 / \Delta u^{2}\right)\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n+1 / 2}\right)  \tag{6.5a}\\
& +\frac{1}{G} D_{e} \Theta_{13} \times\left(1 / \Delta u^{2}\right)\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right) \\
\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}=\left.C_{e} E_{2}\right|_{i, j+1 / 2, k} ^{n} & +\frac{1}{G} D_{e} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{1}\right|_{i, j+1 / 2, k-1 / 2} ^{n+1 / 2}\right)  \tag{6.5b}\\
& +\frac{1}{G} D_{e} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{1}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right) \\
\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}=\left.C_{e} E_{3}\right|_{i, j, k+1 / 2} ^{n}+ & \frac{1}{G} D_{e} \Theta_{32} \times \frac{1}{\Delta u^{1}}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i-1 / 2, j, k+1 / 2} ^{n+1 / 2}\right) \\
& +\frac{1}{G} D_{e} \Theta_{32} \times \frac{1}{\Delta u^{1}}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{2}\right|_{i-1 / 2, j, k+1 / 2} ^{n}\right) \tag{6.5c}
\end{align*}
$$

The magnetic field updating equations of sub-step 1 are as follows:

$$
\begin{align*}
\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}=\left.C_{h} H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n}+ & \frac{1}{G} D_{h} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.E_{2}\right|_{i, j+1 / 2, k+1} ^{n+1 / 2}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right) \\
& +\frac{1}{G} D_{h} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.E_{2}\right|_{i, j+1 / 2, k+1} ^{n}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n}\right)  \tag{6.5d}\\
\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}=\left.C_{h} H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n} & +\frac{1}{G} D_{h} \Theta_{23} \times \frac{1}{\Delta u^{1}}\left(\left.E_{3}\right|_{i+1, j, k+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right)  \tag{6.5e}\\
& +\frac{1}{G} D_{h} \Theta_{23} \times \frac{1}{\Delta u^{1}}\left(\left.E_{3}\right|_{i+1, j, k+1 / 2} ^{n}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n}\right) \\
\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}=\left.C_{h} H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n} & +\frac{1}{G} D_{h} \Theta_{31} \times \frac{1}{\Delta u^{2}}\left(\left.E_{1}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right)  \tag{6.5f}\\
& +\frac{1}{G} D_{h} \Theta_{31} \times \frac{1}{\Delta u^{2}}\left(\left.E_{1}\right|_{i+1 / 2, j+1, k} ^{n}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n}\right)
\end{align*}
$$

In a similar way, the updating equations for sub-step 2 can be derived as follows.
Sub-step 2:
The electric field updating equations of sub-step 2 are as follows:

$$
\begin{align*}
\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1}=\left.C_{e} E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2} & -\frac{1}{G} D_{e} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{2}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1}\right) \\
& -\frac{1}{G} D_{e} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1 / 2}\right)  \tag{6.6a}\\
\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1}=\left.C_{e} E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2} & -\frac{1}{G} D_{e} \Theta_{23} \times \frac{1}{\Delta u^{1}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2, k} ^{n+1}\right)  \tag{6.6b}\\
& -\frac{1}{G} D_{e} \Theta_{23} \times \frac{1}{\Delta u^{1}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2, k} ^{n+1 / 2}\right)
\end{align*}
$$

$$
\begin{align*}
\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1}=\left.C_{e} E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2} & -\frac{1}{G} D_{e} \Theta_{31} \times \frac{1}{\Delta u^{2}}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}-\left.H_{1}\right|_{i, j-1 / 2, k+1 / 2} ^{n+1}\right)  \tag{6.6c}\\
& -\frac{1}{G} D_{e} \Theta_{31} \times \frac{1}{\Delta u^{2}}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{1}\right|_{i, j-1 / 2, k+1 / 2} ^{n+1 / 2}\right)
\end{align*}
$$

The magnetic field updating equations of sub-step 2 are as follows:

$$
\begin{align*}
\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}=\left.C_{h} H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2} & -\frac{1}{G} D_{h} \Theta_{13} \times \frac{1}{\Delta u^{2}}\left(\left.E_{3}\right|_{i, j+1, k+1 / 2} ^{n+1}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1}\right)  \tag{6.6d}\\
& -\frac{1}{G} D_{h} \Theta_{13} \times \frac{1}{\Delta u^{2}}\left(\left.E_{3}\right|_{i, j+1, k+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right) \\
\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}=\left.C_{h} H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2} & -\frac{1}{G} D_{h} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.E_{1}\right|_{i+1 / 2, j, k+1} ^{n+1}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1}\right)  \tag{6.6e}\\
& -\frac{1}{G} D_{h} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.E_{1}\right|_{i+1 / 2, j, k+1} ^{n+1 / 2}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right) \\
\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}=\left.C_{h} H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2} & -\frac{1}{G} D_{h} \Theta_{32} \times \frac{1}{\Delta u^{1}}\left(\left.E_{2}\right|_{i+1, j+1 / 2, k} ^{n+1}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1}\right) \\
& -\frac{1}{G} D_{h} \Theta_{32} \times \frac{1}{\Delta u^{1}}\left(\left.E_{2}\right|_{i+1, j+1 / 2, k} ^{n+1 / 2}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right) \tag{6.6f}
\end{align*}
$$

where $C_{e}, D_{e}, C_{h}$, and $D_{h}$ are the same as mentioned after (6.4). Note that the equations (6.5a)-(6.5f) and (6.6a)-(6.6f) of sub-steps 1 and 2 cannot be used directly for numerical calculation because the synchronous variables are included on both the left and right hand sides. Thus the modified equations must be derived. By placing (6.5f) into (6.5a), (6.5e) into (6.5c) and (6.5d) into (6.5b), and following the procedure given in [23], an approximation of the desired covariant and contra-variant field components are obtained which are second order accurate and they lead to simultaneous linear equations with tridiagonal form for $E_{1}, E_{2}$ and $E_{3}$ field components. Here, a simultaneous linear equation with tri-diagonal matrix form for the $E_{1}$ field component is provided below.

$$
\begin{align*}
& \begin{aligned}
&-\left.\alpha_{11} E_{1}\right|_{i+1 / 2, j-1, k} ^{n+1 / 2}+\left.\left(1+\alpha_{11}\right) E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}-\left.\alpha_{11} E_{1}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2} \\
&=\left.C_{e} E_{1}\right|_{i+1 / 2, j, k} ^{n}+ \frac{\Theta_{13}}{G} D_{e} \frac{1}{\Delta u^{2}} C_{h}\left(\left.H_{3}\right|_{i-1 / 2, j+1 / 2, k} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right) \\
&+\frac{\Theta_{13}}{G} D_{e} \frac{1}{\Delta u^{2}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right) \\
& \quad+\alpha_{11}\left(\left.E_{1}\right|_{i+1 / 2, j+1, k} ^{n}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n}+\left.E_{1}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right)
\end{aligned}
\end{align*}
$$

where $\alpha_{11}=\frac{\Theta_{13}}{G} D_{e} \frac{1}{\Delta u^{2}} \frac{\Theta_{31}}{G} D_{h} \frac{1}{\Delta u^{2}}$.

In a similar way, simultaneous linear equations with tri-diagonal matrix form for $E_{2}$ and $E_{3}$ field components can be derived. Similar to sub-step 1, simultaneous linear equations can be obtained for sub-step 2. By substituting (6.6f) into (6.6a), (6.6e) into (6.6c) and (6.6d) into (6.6b), and following the procedure given in [23], an approximation of the desired covariant and contra-variant field components are obtained which are second order accurate, and this leads to simultaneous linear equations with tri-diagonal form for $E_{1}, E_{2}$, and $E_{3}$ field components. In this case, only the simultaneous linear equation with tri-diagonal matrix form for $E_{1}$ is provided below.

$$
\begin{align*}
-\left.\beta_{11} E_{1}\right|_{i+1 / 2, j, k-1} ^{n+1}+ & \left.\left(1+2 \beta_{11}\right) E_{1}\right|_{i+1 / 2, j, k} ^{n+1}-\left.\beta_{11} E_{1}\right|_{i+1 / 2, j, k+1} ^{n+1}=\left.C_{e} E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2} \\
& -\frac{1}{G} D_{e} \Theta_{12} \times \frac{1}{\Delta u^{3}} C_{h}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1 / 2}\right)  \tag{6.8}\\
& +\beta_{11}\left(\left.E_{1}\right|_{i+1 / 2, j, j, k+1} ^{n+1}-\left.E_{1}\right|_{i+1 / 2, j, j} ^{n+1}+\left.E_{1}\right|_{i+1 / 2, j, j+1} ^{n+1 / 2}-\left.E_{1}^{n+1 / 2}\right|_{i+1 / 2, j, k-1} ^{n+1}\right) \\
& \quad-\frac{1}{G} D_{e} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i+1 / 2, j, k-k-1 / 2} ^{n+1 / 2}\right)
\end{align*}
$$

where $\beta_{11}=\frac{\Theta_{12}}{G} D_{e} \frac{1}{\Delta u^{3}} \frac{\Theta_{21}}{G} D_{h} \frac{1}{\Delta u^{3}}$. In a similar way, simultaneous linear equations with tri-diagonal matrix form for $E_{2}$ and $E_{3}$ field components can be derived. An important advantage of the present approach is that the above equations have the same form as the orthogonal mesh version C-LOD-FDTD method. The convolutional perfectly matched layer (CPML) ABC for the C-LOD-NFDTD method will be discussed next.

### 6.2.2 CPML ABC for the 3-D C-LOD-NFDTD Method

The theory of CPML ABC for non-orthogonal LOD-FDTD in 2-D has been described in Chapter 3, so will not be repeated here. By following its derivation, we can derive the CPML updating equations for sub-steps 1 and 2 for the 3-D LOD-NFDTD method as given below.

Sub-step 1:
The updating equations of electric field for sub-step 1 are as follows:

$$
\begin{align*}
\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}= & \left.C_{e} E_{1}\right|_{i+1 / 2, j, k} ^{n}+\frac{1}{G} D_{e} \Theta_{13} \times \frac{1}{\Delta u^{2}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n+1 / 2}\right)  \tag{6.9a}\\
& +\frac{1}{G} D_{e} \Theta_{13} \times \frac{1}{\Delta u^{2}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right)+\frac{2}{G} D_{e} \Theta_{13} \times\left.\psi_{E_{12}}\right|^{n} \\
\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}= & \left.C_{e} E_{2}\right|_{i, j+1 / 2, k} ^{n}+\frac{1}{G} D_{e} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{1}\right|_{i, j+1 / 2, k-1 / 2} ^{n+1 / 2}\right)  \tag{6.9b}\\
& +\frac{1}{G} D_{e} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{1}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right)+\frac{2}{G} D_{e} \Theta_{21} \times\left.\psi_{E_{23}}\right|^{n} \\
\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}= & \left.C_{e} E_{3}\right|_{i, j, k+1 / 2} ^{n}+\frac{1}{G} D_{e} \Theta_{32} \times \frac{1}{\Delta u^{1}}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i-1 / 2, j, j+k+1 / 2} ^{n+1 / 2}\right)  \tag{6.9c}\\
& +\frac{1}{G} D_{e} \Theta_{32} \times \frac{1}{\Delta u^{1}}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{2}\right|_{i-1 / 2, j, k+1 / 2} ^{n}\right)+\frac{2}{G} D_{e} \Theta_{32} \times\left.\psi_{E_{31}}\right|^{n}
\end{align*}
$$

The updating equations of magnetic field for sub-step 1 are as follows:

$$
\begin{align*}
\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}= & \left.C_{h} H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n}+\frac{1}{G} D_{h} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.E_{2}\right|_{i, j+1 / 2, k+1} ^{n+1 / 2}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right) \\
& +\frac{1}{G} D_{h} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.E_{2}\right|_{i, j+1 / 2, k+1} ^{n}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n}\right)+\frac{1}{G} D_{h} \Theta_{12} \times\left.\psi_{H_{13}}\right|^{n}  \tag{6.9d}\\
\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}= & \left.C_{h} H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n}+\frac{1}{G} D_{h} \Theta_{23} \times \frac{1}{\Delta u^{1}}\left(\left.E_{3}\right|_{i+1 / j, k+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right)  \tag{6.9e}\\
& +\frac{1}{G} D_{h} \Theta_{23} \times \frac{1}{\Delta u^{1}}\left(\left.E_{3}\right|_{i+1, j, k+1 / 2} ^{n}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n}\right)+\frac{1}{G} D_{h} \Theta_{23} \times\left.\psi_{H_{21}}\right|^{n} \\
\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}= & \left.C_{h} H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}+\frac{1}{G} D_{h} \Theta_{31} \times \frac{1}{\Delta u^{2}}\left(\left.E_{1}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right)  \tag{6.9f}\\
& +\frac{1}{G} D_{h} \Theta_{31} \times \frac{1}{\Delta u^{2}}\left(\left.E_{1}\right|_{i+1 / 2, j+1, k} ^{n}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n}\right)+\frac{1}{G} D_{h} \Theta_{31} \times\left.\psi_{H_{32}}\right|^{n}
\end{align*}
$$

In a similar way, the updating equations for sub-step 2 can be derived as follows.
Sub-step 2:
The updating equations of electric field for sub-step 2 are as follows:

$$
\begin{align*}
& \left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1}=\left.C_{e} E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}-\frac{1}{G} D_{e} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{2}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1}\right)  \tag{6.10a}\\
& -\frac{1}{G} D_{e} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1 / 2}\right)-\frac{1}{G} D_{e} \Theta_{12} \times\left.\psi_{E_{13}}\right|^{n+1 / 2} \\
& \left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1}=\left.C_{e} E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}-\frac{1}{G} D_{e} \Theta_{23} \times \frac{1}{\Delta u^{1}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2, k} ^{n+1}\right)  \tag{6.10b}\\
& -\frac{1}{G} D_{e} \Theta_{23} \times \frac{1}{\Delta u^{1}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2, k} ^{n+1 / 2}\right)-\frac{1}{G} D_{e} \Theta_{23} \times\left.\psi_{E_{21}}\right|^{n+1 / 2}
\end{align*}
$$

$$
\begin{align*}
\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1} & =\left.C_{e} E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}-\frac{1}{G} D_{e} \Theta_{31} \times \frac{1}{\Delta u^{2}}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}-\left.H_{1}\right|_{i, j-1 / 2, k+1 / 2} ^{n+1}\right)  \tag{6.10c}\\
& -\frac{1}{G} D_{e} \Theta_{31} \times \frac{1}{\Delta u^{2}}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{1}\right|_{i, j-1 / 2, k+1 / 2} ^{n+1 / 2}\right)-\frac{1}{G} D_{e} \Theta_{31} \times\left.\psi_{E_{32}}\right|^{n+1 / 2}
\end{align*}
$$

The updating equations of magnetic field for sub-step 2 are as follows:

$$
\begin{align*}
\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1} & =\left.C_{h} H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\frac{1}{G} D_{h} \Theta_{13} \times \frac{1}{\Delta u^{2}}\left(\left.E_{3}\right|_{i, j+1, k+1 / 2} ^{n+1}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1}\right)  \tag{6.10d}\\
& -\frac{1}{G} D_{h} \Theta_{13} \times \frac{1}{\Delta u^{2}}\left(\left.E_{3}\right|_{i, j+1, k+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right)-\frac{1}{G} D_{h} \Theta_{13} \times\left.\psi_{H_{12}}\right|^{n+1 / 2} \\
\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1} & =\left.C_{h} H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\frac{1}{G} D_{h} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.E_{1}\right|_{i+1 / 2, j, k+1} ^{n+1}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1}\right)  \tag{6.10e}\\
& -\frac{1}{G} D_{h} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.E_{1}\right|_{i+1 / 2, j, k+1} ^{n+1 / 2}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right)-\frac{1}{G} D_{h} \Theta_{21} \times\left.\psi_{H_{23}}\right|^{n+1 / 2} \\
\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1} & =\left.C_{h} H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\frac{1}{G} D_{h} \Theta_{32} \times \frac{1}{\Delta u^{1}}\left(\left.E_{2}\right|_{i+1, j+1 / 2, k} ^{n+1}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1}\right)  \tag{6.10f}\\
- & \frac{1}{G} D_{h} \Theta_{32} \times \frac{1}{\Delta u^{1}}\left(\left.E_{2}\right|_{i+1, j+1 / 2, k} ^{n+1 / 2}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right)-\frac{1}{G} D_{h} \Theta_{32} \times\left.\psi_{H_{31}}\right|^{n+1 / 2}
\end{align*}
$$

where $C_{e}=\left(4 \varepsilon-\sigma_{e} \Delta t\right) /\left(k(j)\left(4 \varepsilon+\sigma_{e} \Delta t\right)\right), D_{e}=2 \Delta t /\left(k(j)\left(4 \varepsilon+\sigma_{e} \Delta t\right)\right)$, $D_{h}=2 \Delta t /\left(k(j)\left(4 \mu+\sigma_{m} \Delta t\right)\right)$ and $C_{h}=\left(4 \mu-\sigma_{m} \Delta t\right) /\left(k(j)\left(4 \mu+\sigma_{m} \Delta t\right)\right)$. Note that equations (6.9a)-(6.9f) and (6.10a)-(6.10f) of sub-steps 1 and 2 cannot be solved directly because the synchronous variables are included on both the left and right hand sides. So, by placing (6.9f) into (6.9a), (6.9e) into (6.9c) and (6.9d) into (6.9b), and following the procedure given in [23], an approximation for the desired covariant and contra-variant field components is obtained which is second order accurate and leads to simultaneous linear equations with tri-diagonal form for $E_{1}, E_{2}$, and $E_{3}$ field components. Similar to sub-step 1, simultaneous linear equations can be formed for substep 2. By substituting (6.10f) into (6.10a), (6.10e) into (6.10c) and (6.10d) into (6.10b), and following the procedure given in [23], an approximation for the desired covariant and contra-variant field components is obtained which are second order accurate and leads to simultaneous linear equations with tri-diagonal form for $E_{1}, E_{2}$, and $E_{3}$ field components. However, the auxiliary variables $\psi_{E}$ and $\psi_{H}$ must satisfy the recursive relations [39]. The equation of auxiliary variable $\psi_{E_{12}}$ is provided below.

$$
\begin{gather*}
\psi_{E_{12}} l_{i+1 / 2, j, k}^{n+1 / 2}=\left.c_{u^{2}} \psi_{E_{12}}\right|_{i+1 / 2, j, k} ^{n}+\frac{d_{u_{j}^{2}}}{2 \Delta u^{2}}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right)  \tag{6.11a}\\
c_{s}=e^{-\left(\left(\sigma_{s} / \kappa_{s}\right)+\alpha_{s}\right)\left(\Delta t / \varepsilon_{0}\right)}  \tag{6.11b}\\
d_{s}=\frac{\sigma_{s}}{\kappa_{s}\left(\sigma_{s}+\kappa_{s} \sigma_{s}\right)} \times\left(c_{s}-1\right), \quad\left(s=u^{1}, u^{2}, \text { or } u^{3}\right)  \tag{6.11c}\\
\sigma_{s}(s)=\frac{\sigma_{s_{\max }}\left|s-s_{0}\right|^{m}}{\delta^{m}}  \tag{6.11d}\\
\kappa_{s}(s)=1+\left(\kappa_{\max }-1\right) \frac{\left|s-s_{0}\right|^{m}}{\delta^{m}} \tag{6.11e}
\end{gather*}
$$

where $\delta$ is the thickness of the PML absorber, $s_{0}$ is the PML interface, $m$ is the order of the polynomial. In a similar way, the updating equations of other auxiliary variables can be derived. Similar to LOD-FDTD CPML using orthogonal meshes, in the proposed C-LOD-NFDTD CPML approach only one variable $\psi$ is enough. The auxiliary term $\left.\psi_{E_{12}}\right|^{n+1 / 2}$ contains the same time index $(n+1 / 2)$ as that of the field term on the left side. This additional time variable at $(n+1 / 2)$ does not affect numerical results significantly except to contribute to additional complexities in the computation. Therefore, for the sake of simplification, both auxiliary variables will be considered at the same time index " $n$ ". The CPML formulations have been used with the C-LOD-NFDTD method for the numerical analysis of the 3-D microwave curved structures, as described in Section 6.8.

### 6.3 Stability Analysis of the 3-D C-LOD-NFDTD

As stated in Chapter 5, if all the eigenvalues of the amplification matrix are not larger than unity in magnitudes, the method is considered to be stable so in this section, we demonstrate the unconditional stability of the C-LOD-NFDTD method by evaluating the amplification matrix. Since the stability analysis of C-LOD-NFDTD requires detailed mathematical procedure, so only the key results of the analysis are summarised below. The field components in the spatial spectral domain are assumed to have the following form

$$
\begin{equation*}
E_{r}^{n}(i, j, k)=E_{r}^{n} e^{-j\left(k_{i} i \Delta u^{1}+k_{2} j \Delta u^{2}+k_{3} k \Delta u^{3}\right)} \tag{6.12a}
\end{equation*}
$$

$$
\begin{equation*}
H_{r}^{n}(i, j, k)=H_{r}^{n} e^{-j\left(k_{i} \Delta \Delta u^{1}+k_{2} j \Delta u^{2}+k_{3} k \Delta u^{3}\right)} \tag{6.12b}
\end{equation*}
$$

where $r=1,2,3$ which is equivalent to $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions respectively of the orthogonal coordinates while $k_{1}, k_{2}$ and $k_{3}$ are wave numbers along the $u^{1}, u^{2}, u^{3}$ directions respectively. By substituting (6.12a) and (6.12b) in (6.5)-(6.6) of sub-steps 1 and 2 and assuming lossless media, we derive the following equation,

Sub-step 1:

$$
\begin{equation*}
U^{n+1 / 2}=\Lambda_{1} U^{n} \text { where } \Lambda_{1} \text { is } 6 \times 6 \text { matrix } \tag{6.13}
\end{equation*}
$$

$$
\Lambda_{1}=\left[\begin{array}{cccccc}
\frac{G_{2}}{Q_{2}} & 0 & 0 & 0 & 0 & -\frac{i 2 A_{2}}{\varepsilon Q_{2}} \\
0 & \frac{G_{3}}{Q_{3}} & 0 & -\frac{i 2 A_{3}}{\varepsilon Q_{3}} & 0 & 0 \\
0 & 0 & \frac{G_{1}}{Q_{1}} & 0 & -\frac{i 2 A_{1}}{\varepsilon Q_{1}} & 0 \\
0 & -\frac{i 2 A_{3}}{\mu Q_{3}} & 0 & \frac{G_{3}}{Q_{3}} & 0 & 0 \\
0 & 0 & -\frac{i 2 A_{1}}{\mu Q_{1}} & 0 & \frac{G_{1}}{Q_{1}} & 0 \\
-\frac{i 2 A_{2}}{\mu Q_{2}} & 0 & 0 & 0 & 0 & \frac{G_{2}}{Q_{2}}
\end{array}\right]
$$

Sub-step 2:

$$
\begin{gather*}
U^{n+1}=\Lambda_{2} U^{n+1 / 2}  \tag{6.14}\\
\Lambda_{2}=\left[\begin{array}{cccccc}
\frac{G_{3}}{Q_{3}} & 0 & 0 & 0 & \frac{i 2 A_{3}}{\varepsilon Q_{3}} & 0 \\
0 & \frac{G_{1}}{Q_{1}} & 0 & 0 & 0 & \frac{i 2 A_{1}}{\varepsilon Q_{1}} \\
0 & 0 & \frac{G_{2}}{Q_{2}} & \frac{i 2 A_{2}}{\varepsilon Q_{2}} & 0 & 0 \\
0 & 0 & \frac{i 2 A_{2}}{\mu Q_{2}} & \frac{G_{2}}{Q_{2}} & 0 & 0 \\
\frac{i 2 A_{3}}{\mu Q_{3}} & 0 & 0 & 0 & \frac{G_{3}}{Q_{3}} & 0 \\
0 & \frac{i 2 A_{1}}{\mu Q_{1}} & 0 & 0 & 0 & \frac{G_{1}}{Q_{1}}
\end{array}\right]
\end{gather*}
$$

By combining (6.13) and (6.14), we obtain the following equation

$$
\begin{equation*}
U^{n+1}=\Lambda_{1} \Lambda_{2} U^{n}=\Lambda U^{n} \tag{6.15}
\end{equation*}
$$

where $\Lambda=\Lambda_{1} \Lambda_{2}$

$$
\Lambda=\left[\begin{array}{llcccc}
\frac{G_{y} G_{z}}{Q_{y} Q_{z}} & \frac{4 A_{1} A_{2}}{\mu \varepsilon Q_{1} Q_{2}} & 0 & 0 & \frac{i 2 A_{3} G_{2}}{\varepsilon Q_{2} Q_{3}} & -\frac{i 2 A_{2} G_{1}}{\varepsilon Q_{1} Q_{2}}  \tag{6.16}\\
0 & \frac{G_{1} G_{3}}{Q_{1} Q_{3}} & \frac{4 A_{2} A_{3}}{\mu \varepsilon Q_{2} Q_{3}} & -\frac{i 2 A_{3} G_{2}}{\varepsilon Q_{2} Q_{3}} & 0 & \frac{i 2 A_{1} G_{3}}{\varepsilon Q_{1} Q_{3}} \\
\frac{4 A_{x} A_{z}}{\mu \varepsilon Q_{x} Q_{z}} & 0 & \frac{G_{1} G_{2}}{Q_{1} Q_{2}} & \frac{i 2 A_{2} G_{1}}{\varepsilon Q_{1} Q_{2}} & -\frac{i 2 A_{1} G_{3}}{\varepsilon Q_{1} Q_{3}} & 0 \\
0 & -\frac{i 2 A_{3} G_{1}}{\mu Q_{1} Q_{3}} & \frac{i 2 A_{2} G_{3}}{\mu Q_{2} Q_{3}} & \frac{G_{2} G_{3}}{Q_{2} Q_{3}} & 0 & \frac{4 A_{1} A_{3}}{\mu \varepsilon Q_{1} Q_{3}} \\
\frac{i 2 A_{z} G_{x}}{\mu Q_{x} Q_{z}} & 0 & -\frac{i 2 A_{1} G_{2}}{\mu Q_{1} Q_{2}} & \frac{4 A_{1} A_{2}}{\mu \varepsilon Q_{1} Q_{2}} & \frac{G_{1} G_{3}}{Q_{1} Q_{3}} & 0 \\
-\frac{i 2 A_{y} G_{z}}{\mu Q_{y} Q_{z}} & \frac{i 2 A_{1} G_{2}}{\mu Q_{1} Q_{2}} & 0 & 0 & \frac{4 A_{2} A_{3}}{\mu \varepsilon Q_{2} Q_{3}} & \frac{G_{1} G_{2}}{Q_{1} Q_{2}}
\end{array}\right]
$$

Six eigenvalues are obtained from equation (6.16). Here, only the numerical computed eigenvalues are provided for the verification of the unconditional stability of the C -LOD-NFDTD method. By considering $k_{1}=0.25, k_{2}=0.25$ and $k_{z}=0.25$ and $\Delta u^{1}=\Delta u^{2}=\Delta u^{3}=1 \mathrm{~mm}$, six eigenvalues are computed. Table 6.1 presents the magnitude of the computed eigenvalues. From Table 6.1, it can be observed that the magnitudes of the eigenvalues are never larger than unity. Therefore, the 3-D C-LODNFDTD method can be considered to be unconditionally stable.

Table 6.1
Computed eigenvalues using two-steps procedure of C-LOD-NFDTD

| Eigen values | CFLN=2 | CFLN=4 | CFLN=6 |
| :---: | :---: | :---: | :---: |
| $\lambda_{1}$ | 1.00000 | 1.00000 | 1.00000 |
| $\lambda_{2}$ | 0.99976 | 0.99946 | 0.99936 |
| $\lambda_{3}$ | 0.99667 | 0.99347 | 0.99237 |
| $\lambda_{4}$ | 0.88598 | 0.82286 | 0.81238 |
| $\lambda_{5}$ | 0.73297 | 0.71287 | 0.69947 |
| $\lambda_{6}$ | 0.88598 | 0.82286 | 0.81238 |

### 6.4 3-D F-LOD-NFDTD: Nonorthogonal 3-D LOD-FDTD Using Fundamental Scheme Based on Curvilinear Meshes

The formulations of the 3-D non-orthogonal locally one dimensional finite difference time domain method based on fundamental scheme (F-LOD-NFDTD) is presented in this section. As we know, substantial arithmetic operations are required for the conventional LOD-FDTD even in the case of nonorthogonal meshes which may reduce the efficiency of the computational performance of the method. This can be improved by deriving the nonorthogonal F-LOD-FDTD method. This formulation will lead to matrix-operator-free forms on the right-hand sides of the resultant equations. The resultant F-LOD-NFDTD algorithm involves updating equations whose right-hand sides are much simpler and more concise than those in the conventional implementation. This achieves a substantial reduction in the number of arithmetic operations required for their computations. The F-LOD-NFDTD scheme will be derived following the formulation described in Section 5.5 of Chapter 5 for F-LOD-FDTD using orthogonal meshes.

### 6.4.1 Formulation of the 3-D F-LOD-NFDTD Method

The theory of nonorthogonal meshes curvilinear coordinates has been described in Chapter 3, so will not be repeated here. The theory of the fundamental scheme is also discussed in Section 5.4 of Chapter 5, and will also not be repeated here. However, by following the theories of nonorthogonal meshes and fundamental scheme, we can derive the 3-D fundamental scheme based nonorthogonal LOD-FDTD method viz. 3-D F-LOD-NFDTD method. Denoting the covariant electric and magnetic field components which represent the flow of field along the grid as $E_{m}, H_{m}, e_{m}, h_{m}(m=1,2,3)$, and the contra-variant electric and magnetic field components which represent the flow through facets of the grid as $E^{m}, H^{m}, e^{m}, h^{m}(m=1,2,3)$, the electric and magnetic field components of sub-steps 1 and 2 for the F-LOD-NFDTD method can be derived as follows:
Sub-step 1:
Auxiliary implicit updating for electric and magnetic fields are as follows:

$$
\begin{align*}
& \left.\frac{1}{2} e^{1}\right|^{n+1 / 2}-\left.\frac{b}{2 \sqrt{g}} \partial_{u^{2}} h_{3}\right|^{n+1 / 2}=\left.E^{1}\right|^{n}  \tag{6.17a}\\
& \left.\frac{1}{2} e^{2}\right|^{n+1 / 2}-\left.\frac{b}{\sqrt{g}} \frac{1}{2} \partial_{u^{3}} h_{1}\right|^{n+1 / 2}=\left.E^{2}\right|^{n}  \tag{6.17b}\\
& \left.\frac{1}{2} e^{3}\right|^{n+1 / 2}-\left.\frac{b}{\sqrt{g}} \frac{1}{2} \partial_{u^{1}} h_{2}\right|^{n+1 / 2}=\left.E^{3}\right|^{n}  \tag{6.17c}\\
& \left.\frac{1}{2} h^{1}\right|^{n+1 / 2}-\left.\frac{d}{\sqrt{g}} \frac{1}{2} \partial_{u^{3}} e_{2}\right|^{n+1 / 2}=\left.H^{1}\right|^{n}  \tag{6.17d}\\
& \left.\frac{1}{2} h^{2}\right|^{n+1 / 2}-\left.\frac{1}{\sqrt{g}} \frac{d}{2} \partial_{u^{1}} e_{3}\right|^{n+1 / 2}=\left.H^{2}\right|^{n}  \tag{6.17e}\\
& \left.\frac{1}{2} h^{3}\right|^{n+1 / 2}-\left.\frac{1}{\sqrt{g}} \frac{d}{2} \partial_{u^{2}} e_{1}\right|^{n+1 / 2}=\left.H^{3}\right|^{n} \tag{6.17f}
\end{align*}
$$

and the explicit updating for the electric and magnetic fields are as follows:

$$
\begin{align*}
& \left.E^{\xi}\right|^{n+1 / 2}=\left.e^{\xi}\right|^{n+1 / 2}-\left.E^{\xi}\right|^{n}  \tag{6.18}\\
& \left.H^{\xi}\right|^{n+1 / 2}=\left.h^{\xi}\right|^{n+1 / 2}-\left.H^{\xi}\right|^{n} \tag{6.19}
\end{align*}
$$

where $\xi$ is $1,2,3$ which is equivalent to $\mathrm{x}, \mathrm{y}$, and z of the orthogonal coordinate systems. Here, the covariant components of the $E$ and $H$ fields are placed on the cells in the manner suggested by Holland [23]. Similar to the NFDTD method, the contravariant like $e^{p}, h^{p}, E^{p}$, and $H^{p}$ on the left hand sides of (6.17a)-(6.17f) and both sides of (6.18) and (6.19) must be converted to the covariant like $e_{p}, h_{p}, E_{p}$, and $H_{p}$ field components as follows:

$$
\begin{align*}
& \left.e_{1}\right|^{n+1 / 2}-\left.2 E_{1}\right|^{n}=\left.\frac{b}{G \sqrt{g}} \partial_{u^{2}} h_{3}\right|^{n+1 / 2}  \tag{6.20a}\\
& \left.e_{2}\right|^{n+1 / 2}-\left.2 E_{2}\right|^{n}=\left.\frac{b}{G \sqrt{g}} \partial_{u^{3}} h_{1}\right|^{n+1 / 2}  \tag{6.20b}\\
& \left.e_{3}\right|^{n+1 / 2}-\left.2 E_{3}\right|^{n}=\left.\frac{b}{G \sqrt{g}} \partial_{u^{1}} h_{2}\right|^{n+1 / 2}  \tag{6.20c}\\
& \left.h_{1}\right|^{n+1 / 2}-\left.2 H_{1}\right|^{n}=\left.\frac{d}{G \sqrt{g}} \partial_{u^{3}} e_{2}\right|^{n+1 / 2} \tag{6.20d}
\end{align*}
$$

$$
\begin{gather*}
\left.h_{2}\right|^{n+1 / 2}-\left.2 H_{2}\right|^{n}=\left.\frac{d}{G \sqrt{g}} \partial_{u^{1}} e_{3}\right|^{n+1 / 2}  \tag{6.20e}\\
\left.h_{3}\right|^{n+1 / 2}-\left.2 H_{3}\right|^{n}=\left.\frac{d}{G \sqrt{g}} \partial_{u^{2}} e_{1}\right|^{n+1 / 2}  \tag{6.20f}\\
\left.E_{\xi}\right|^{n+1 / 2}=\left.e_{\xi}\right|^{n+1 / 2}-\left.E_{\xi}\right|^{n}  \tag{6.21}\\
\left.H_{\xi}\right|^{n+1 / 2}=\left.h_{\xi}\right|^{n+1 / 2}-\left.H_{\xi}\right|^{n} \tag{6.22}
\end{gather*}
$$

where the transformation matrix $G$ is defined in (6.2), and $g$ is the determinant of the metric with elements $g_{p q}$. From (6.20a)-(6.20f), it can be observed that these equations cannot be solved directly, so by placing (6.20f) into (6.20a), (6.20e) into (6.20c) and ( 6.20 d ) into ( 6.20 b ), a simultaneous linear system with tri-diagonal matrix which can be solved efficiently can be obtained as follows.

$$
\begin{align*}
& \left.e_{1}\right|^{n+1 / 2}-\left.b d\left(\frac{1}{G \sqrt{g}}\right)^{2} \partial_{u^{2}}^{2} e_{1}\right|^{n+1 / 2}=\left.2 E_{1}\right|^{n}+\left.\frac{2}{G \sqrt{g}} b \partial_{u^{2}} H_{3}\right|^{n}  \tag{6.23a}\\
& \left.e_{2}\right|^{n+1 / 2}-\left.b d\left(\frac{1}{G \sqrt{g}}\right)^{2} \partial_{u^{3}}^{2} e_{2}\right|^{n+1 / 2}=\left.2 E_{2}\right|^{n}+\left.\frac{2}{G \sqrt{g}} b \partial_{u^{3}} H_{1}\right|^{n}  \tag{6.23b}\\
& \left.e_{3}\right|^{n+1 / 2}-\left.b d\left(\frac{1}{G \sqrt{g}}\right)^{2} \partial_{u^{u}}^{2} e_{3}\right|^{n+1 / 2}=\left.2 E_{3}\right|^{n}+\left.\frac{2}{G \sqrt{g}} b \partial_{u^{1}} H_{2}\right|^{n} \tag{6.23c}
\end{align*}
$$

where $b=\frac{\Delta t}{2 \varepsilon}, d=\frac{\Delta t}{2 \mu}$, and the explicit updating equations for the electric and magnetic field components can be written from (6.21) and (6.22) using (6.23) and (6.20). Similar to sub-step 1, we can derive the following equations for sub-step 2.

Sub-step 2:
Auxiliary implicit updating for electric and magnetic field components as:

$$
\begin{gather*}
\left.e_{1}\right|^{n+1}=\left.2 E_{1}\right|^{n+1 / 2}-\left.\frac{1}{G \sqrt{g}} b \partial_{u^{3}} h_{2}\right|^{n+1}  \tag{6.24a}\\
\left.e_{2}\right|^{n+1}=\left.2 E_{2}\right|^{n+1 / 2}-\left.\frac{1}{G \sqrt{g}} b \partial_{u^{1}} h_{3}\right|^{n+1}  \tag{6.24b}\\
\left.e_{3}\right|^{n+1}=\left.2 E_{3}\right|^{n+1 / 2}-\left.\frac{1}{G \sqrt{g}} b \partial_{u^{2}} h_{1}\right|^{n+1}  \tag{6.24c}\\
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\end{gather*}
$$

$$
\begin{align*}
& \left.h_{1}\right|^{n+1}=\left.2 H_{1}\right|^{n+1 / 2}-\left.\frac{d}{G \sqrt{g}} \partial_{u^{2}} e_{3}\right|^{n+1}  \tag{6.24d}\\
& \left.h_{2}\right|^{n+1}=\left.2 H_{2}\right|^{n+1 / 2}-\left.\frac{d}{G \sqrt{g}} \partial_{u^{3}} e_{1}\right|^{n+1}  \tag{6.24e}\\
& \left.h_{3}\right|^{n+1}=\left.2 H_{3}\right|^{n+1 / 2}-\left.\frac{d}{G \sqrt{g}} \partial_{u^{1}} e_{2}\right|^{n+1} \tag{6.24f}
\end{align*}
$$

From (6.24a)-(6.24f), we can obtain the following simultaneous linear equations for the auxiliary electric field components as,

$$
\begin{align*}
& \left.e_{1}\right|^{n+1}-\left.b d\left(\frac{1}{G \sqrt{g}}\right)^{2} \partial_{u^{3}}^{2} e_{1}\right|^{n+1}=\left.2 E_{1}\right|^{n+1 / 2}-\left.\frac{2}{G \sqrt{g}} b \partial_{u^{3}} H_{2}\right|^{n+1 / 2}  \tag{6.25a}\\
& \left.e_{2}\right|^{n+1}-\left.b d\left(\frac{1}{G \sqrt{g}}\right)^{2} \partial_{u^{1}}^{2} e_{2}\right|^{n+1}=\left.2 E_{2}\right|^{n+1 / 2}-\left.\frac{2}{G \sqrt{g}} b \partial_{u^{1}} H_{3}\right|^{n+1 / 2}  \tag{6.25b}\\
& \left.e_{3}\right|^{n+1}-\left.b d\left(\frac{1}{G \sqrt{g}}\right)^{2} \partial_{u^{2}}^{2} e_{3}\right|^{n+1}=\left.2 E_{3}\right|^{n+1 / 2}-\left.\frac{2}{G \sqrt{g}} b \partial_{u^{2}} H_{1}\right|^{n+1 / 2} \tag{6.25c}
\end{align*}
$$

The explicit electric and magnetic field components can be obtained as follows:

$$
\begin{align*}
& \left.E_{\xi}\right|^{n+1}=\left.e_{\xi}\right|^{n+1}-\left.E_{\xi}\right|^{n+1 / 2}  \tag{6.26}\\
& \left.H_{\xi}\right|^{n+1}=\left.h_{\xi}\right|^{n+1}-\left.H_{\xi}\right|^{n+1 / 2} \tag{6.27}
\end{align*}
$$

The updating equations for sub-steps 1 and 2 with indices can be written as follows:
Sub-step 1:
Auxiliary implicit updating for electric field component:

$$
\begin{align*}
& \left.\left.\alpha_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j-1, k} ^{n+1 / 2}+\left.\left.\beta_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2} \\
& \quad+\left.\left.\gamma_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2}=\left.\left.C_{a 1}\right|_{i+1 / 2, j, k} E_{1}\right|_{i+1 / 2, j, k} ^{n}  \tag{6.28a}\\
& \quad+\left.C_{b 1}\right|_{i+1 / 2, j, k}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right) \\
& \left.\left.\alpha_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k-1} ^{n+1 / 2}+\left.\left.\beta_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2} \\
& \quad+\left.\left.\gamma_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k+1} ^{n+1 / 2}=\left.\left.C_{a 2}\right|_{i, j+1 / 2, k} E_{2}\right|_{i, j+1 / 2, k} ^{n}  \tag{6.28b}\\
& \quad+\left.C_{b 2}\right|_{i, j+1 / 2, k}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{1}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right)
\end{align*}
$$

$$
\begin{align*}
& \left.\left.\alpha_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i-1, j, k+1} ^{n+1 / 2}+\left.\left.\beta_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2} \\
& \quad+\left.\left.\gamma_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i+1, j, k+1 / 2} ^{n+1 / 2}=\left.\left.C_{a 3}\right|_{i, j, k+1 / 2} E_{3}\right|_{i, j, k+1 / 2} ^{n}  \tag{6.28c}\\
& \quad+\left.C_{b 3}\right|_{i, j, k+1 / 2}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{2}\right|_{i-1 / 2, j, k+1 / 2} ^{n}\right)
\end{align*}
$$

where $\left.\alpha_{1}\right|_{i+1 / 2, j, k}=-\frac{b d}{2}\left(\frac{1}{G \sqrt{g}}\right)^{2} \frac{1}{\left(\Delta u^{2}\right)^{2}},\left.\gamma_{1}\right|_{i+1 / 2, j, k}=-\frac{b d}{2}\left(\frac{1}{G \sqrt{g}}\right)^{2} \frac{1}{\left(\Delta u^{2}\right)^{2}}$

$$
\left.\beta_{1}\right|_{i+1 / 2, j, k}=1-\left.\alpha_{1}\right|_{i+1 / 2, j, k}-\left.\gamma_{1}\right|_{i+1 / 2, j, k},\left.C_{b 1}\right|_{i+1 / 2, j, k}=\frac{2}{G \sqrt{g}} b \frac{1}{\Delta u^{2}} .
$$

The explicit electric field components from (6.21) can be derived as follows:

$$
\begin{align*}
& \left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}=\left.e_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n}  \tag{6.29a}\\
& \left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}=\left.e_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n}  \tag{6.29b}\\
& \left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}=\left.e_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n} \tag{6.29c}
\end{align*}
$$

Sub-step 2: auxiliary updating for $\left.e_{1}\right|^{n+1}$ can be written as follows:

$$
\begin{align*}
& \left.\left.\alpha_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j-1, k} ^{n+1}+\left.\left.\beta_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j, k} ^{n+1} \\
& +\left.\left.\gamma_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j+1, k} ^{n+1}=\left.\left.C_{a 1}\right|_{i+1 / 2, j, k} E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}  \tag{6.30a}\\
& +\left.C_{b 1}\right|_{i+1 / 2, j, k}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n+1 / 2}\right) \\
& \left.\left.\alpha_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k-1} ^{n+1}+\left.\left.\beta_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k} ^{n+1} \\
& +\left.\left.\gamma_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k+1} ^{n+1}=\left.\left.C_{a 2}\right|_{i, j+1 / 2, k} \quad E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}  \tag{6.30b}\\
& +\left.C_{b 2}\right|_{i, j+1 / 2, k}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{1}\right|_{i, j+1 / 2, k-1 / 2} ^{n+1 / 2}\right) \\
& \left.\left.\alpha_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i-1, j, k+1} ^{n+1}+\left.\left.\beta_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i, j, k+1 / 2} ^{n+1} \\
& +\left.\left.\gamma_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i+1, j, k+1 / 2} ^{n+1}=\left.\left.C_{a 3}\right|_{i, j, k+1 / 2} E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}  \tag{6.30c}\\
& +\left.C_{b 3}\right|_{i, j, k+1 / 2}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i-1 / 2, j, k+1 / 2} ^{n+1 / 2}\right)
\end{align*}
$$

and the electric field components from (6.26) can be obtained as follows:

$$
\begin{gather*}
\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1}=\left.e_{1}\right|_{i+1 / 2, j, k} ^{n+1}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}  \tag{6.31a}\\
\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1}=\left.e_{2}\right|_{i, j+1 / 2, k} ^{n+1}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}  \tag{6.31b}\\
\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1}=\left.e_{3}\right|_{i, j, k+1 / 2} ^{n+1}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2} \tag{6.31c}
\end{gather*}
$$

The CPML absorbing boundary conditions for the F-LOD-NFDTD method will be discussed next.

### 6.4.2 CPML ABC for the 3-D F-LOD-NFDTD Method

Similar to the orthogonal mesh F-LOD-FDTD CPML, efficient ABCs are required to truncate the computational domains for modelling open region electromagnetic problems using the nonorthogonal mesh F-LOD-NFDTD method. In the literature, only the Mur's ABC [138] and PML ABC [126] have been developed for 3-D LOD-FDTD using fundamental scheme with orthogonal meshes. However, it is well known that Mur's ABC [138], and PML [126] are not efficient at absorbing low frequency and evanescent waves while at the same time preserving the unconditional stability, even in nonorthogonal mesh methods. In this section, we therefore develop the CPML absorbing boundary condition for 3-D LOD-NFDTD using fundamental scheme, which is highly effective in absorbing low frequency contents and evanescent waves. The theory presented in this section will later be used to analyse 3-D structures using nonorthogonal meshes. The theory of CPML ABC for nonorthogonal 2-D LOD-FDTD was presented in Chapter 3, so will not be repeated here. However, by following the F-LOD-NFDTD of Section 6.4.1 and CPML for nonorthogonal coordinates of Chapter 3, the CPML equations of sub-steps 1 and 2 for F-LOD-NFDTD method can be derived. Similar to the formulations of CPML ABC for orthogonal mesh F-LOD-FDTD given in Section 5.4.2, the nonorthogonal mesh CPML formulation for sub-steps 1 and 2 of the F-LOD-NFDTD method can be derived as follows:

Sub-step 1: auxiliary implicit updating for electric and magnetic fields are as follows:

$$
\begin{align*}
& \left.e_{1}\right|^{n+1 / 2}-\left.2 E_{1}\right|^{n}=\left.\frac{b}{G \sqrt{g}} \frac{1}{\kappa_{2_{j}}} \partial_{u^{2}} h_{3}\right|^{n+1 / 2}  \tag{6.32a}\\
& \left.e_{2}\right|^{n+1 / 2}-\left.2 E_{2}\right|^{n}=\left.\frac{b}{G \sqrt{g}} \frac{1}{\kappa_{3_{k}}} \partial_{u^{3}} h_{1}\right|^{n+1 / 2}  \tag{6.32b}\\
& \left.e_{3}\right|^{n+1 / 2}-\left.2 E_{3}\right|^{n}=\left.\frac{b}{G \sqrt{g}} \frac{1}{\kappa_{1_{i}}} \partial_{u^{1}} h_{2}\right|^{n+1 / 2}  \tag{6.32c}\\
& \left.h_{1}\right|^{n+1 / 2}-\left.2 H_{1}\right|^{n}=\left.\frac{d}{G \sqrt{g}} \frac{1}{\kappa_{3_{k+1 / 2}}} \partial_{u^{3}} e_{2}\right|^{n+1 / 2}  \tag{6.32d}\\
& \left.h_{2}\right|^{n+1 / 2}-\left.2 H_{2}\right|^{n}=\left.\frac{d}{G \sqrt{g}} \frac{1}{\kappa_{1_{i+1 / 2}}} \partial_{u^{1}} e_{3}\right|^{n+1 / 2} \tag{6.32e}
\end{align*}
$$

$$
\begin{equation*}
\left.h_{3}\right|^{n+1 / 2}-\left.2 H_{3}\right|^{n}=\left.\frac{d}{G \sqrt{g}} \frac{1}{\kappa_{2_{j+1 / 2}}} \partial_{u^{2}} e_{1}\right|^{n+1 / 2} \tag{6.32f}
\end{equation*}
$$

and explicit updating for the electric and magnetic fields are as follows:

$$
\begin{align*}
& \left.E_{\xi 1}\right|^{n+1 / 2}=\left.e_{\xi 1}\right|^{n+1 / 2}-\left.E_{\xi 1}\right|^{n}+2 a_{p} \psi_{E_{\xi 1 \xi 2}}^{n}  \tag{6.33}\\
& \left.H_{\xi 1}\right|^{n+1 / 2}=\left.h_{\xi 1}\right|^{n+1 / 2}-\left.H_{\xi 1}\right|^{n}+2 b_{p} \psi_{H_{\xi 1 \xi 2}}^{n} \tag{6.34}
\end{align*}
$$

Equations (6.32a)-(6.32f) cannot be solved directly, so by placing (6.32f) into (6.32a), (6.32e) into (6.32c) and (6.32d) into (6.32b), a simultaneous linear system of equations can be derived, as given below, which can be solved efficiently by forming tri-diagonal matrix formulation and solution methodologies.

$$
\begin{align*}
&\left.e_{1}\right|^{n+1 / 2}-\left.b d(1 / G \sqrt{g})^{2} \cdot\left(1 / \kappa_{2_{j+1 / 2}} \kappa_{2_{j}}\right) \partial_{u^{2}}^{2} e_{1}\right|^{n+1 / 2} \\
&=\left.2 E_{1}\right|^{n}+2 /\left.\left(G \sqrt{g} \cdot \kappa_{2_{j}}\right) b \partial_{u^{2}} H_{3}\right|^{n}  \tag{6.35a}\\
&\left.e_{2}\right|^{n+1 / 2}-b d\left(\frac{1}{G \sqrt{g}}\right)^{2} \frac{1}{\kappa_{3_{k+1 / 2}}}\left.\frac{1}{\kappa_{3_{k}}} \partial_{u^{3}}^{2} e_{2}\right|^{n+1 / 2}  \tag{6.35b}\\
&=\left.2 E_{2}\right|^{n}+\left.\frac{2}{G \sqrt{g}} \frac{1}{\kappa_{3_{k}}} b \partial_{u^{3}} H_{1}\right|^{n} \\
&\left.e_{3}\right|^{n+1 / 2}-\left.b d\left(\frac{1}{G \sqrt{g}}\right)^{2} \frac{1}{\kappa_{1_{i}}} \frac{1}{\kappa_{1_{i+1 / 2}}} \partial_{u^{1}}^{2} e_{3}\right|^{n+1 / 2} \\
&=\left.2 E_{3}\right|^{n}+\left.\frac{2}{G \sqrt{g}} \frac{1}{\kappa_{1_{i}}} b \partial_{u^{1}} H_{2}\right|^{n} \tag{6.35c}
\end{align*}
$$

where $b=\frac{\Delta t}{2 \varepsilon}, d=\frac{\Delta t}{2 \mu}$, and the explicit updating equations for the electric and magnetic field components can be derived from (6.33) and (6.34) using (6.35) and (6.32). Similar to sub-step 1, we can write the following equations for sub-step 2 as: Sub-step 2:

$$
\begin{align*}
& \left.e_{1}\right|^{n+1}=\left.2 E_{1}\right|^{n+1 / 2}-\left.\frac{1}{G \sqrt{g}} \frac{1}{\kappa_{3_{k+1 / 2}}} b \partial_{u^{3}} h_{2}\right|^{n+1}  \tag{6.36a}\\
& \left.e_{2}\right|^{n+1}=\left.2 E_{2}\right|^{n+1 / 2}-\left.\frac{1}{G \sqrt{g}} \frac{1}{\kappa_{1_{i+1 / 2}}} b \partial_{u^{1}} h_{3}\right|^{n+1} \tag{6.36b}
\end{align*}
$$

$$
\begin{align*}
& \left.e_{3}\right|^{n+1}=\left.2 E_{3}\right|^{n+1 / 2}-\left.\frac{1}{G \sqrt{g}} \frac{1}{\kappa_{2_{j+1 / 2}}} b \partial_{u^{2}} h_{1}\right|^{n+1}  \tag{6.36c}\\
& \left.h_{1}\right|^{n+1}=\left.2 H_{1}\right|^{n+1 / 2}-\left.\frac{1}{G \sqrt{g}} \frac{1}{\kappa_{2_{j+1}}} d \partial_{u^{2}} e_{3}\right|^{n+1}  \tag{6.36d}\\
& \left.h_{2}\right|^{n+1}=\left.2 H_{2}\right|^{n+1 / 2}-\left.\frac{1}{G \sqrt{g}} \frac{1}{\kappa_{3_{k+1}}} d \partial_{u^{3}} e_{1}\right|^{n+1}  \tag{6.36e}\\
& \left.h_{3}\right|^{n+1}=\left.2 H_{3}\right|^{n+1 / 2}-\left.\frac{1}{G \sqrt{g}} \frac{1}{\kappa_{1_{i+1}}} d \partial_{u^{1}} e_{2}\right|^{n+1} \tag{6.36f}
\end{align*}
$$

and the explicit electric and magnetic field components can be written as follows:

$$
\begin{align*}
& \left.E_{\xi}\right|^{n+1}=\left.e_{\xi}\right|^{n+1}-\left.E_{\xi}\right|^{n+1 / 2}+2 a_{p} \psi_{e \xi 1 \xi 2}^{n+1 / 2}  \tag{6.37}\\
& \left.H_{\xi}\right|^{n+1}=\left.h_{\xi}\right|^{n+1}-\left.H_{\xi}\right|^{n+1 / 2}+2 b_{p} \psi^{n+1 / 2} h 1 \xi 2 \tag{6.38}
\end{align*}
$$

Similar to sub-step 1, we obtain the simultaneous linear equations for electric field components from the above equations (6.36a)-(6.36f). The updating equations for substeps 1 and 2 with indices $(i, j, k)$ can be written as follows.
Sub-step 1: auxiliary updating for $\left.e_{1}\right|^{n+1 / 2}$ can be written as follows:

$$
\left.\begin{array}{l}
\left.\left.\alpha_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j-1, k} ^{n+1 / 2}+\left.\left.\beta_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2} \\
\quad+\left.\left.\gamma_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2}=\left.\left.C_{a 1}\right|_{i+1 / 2, j, k} E_{1}\right|_{i+1 / 2, j, k} ^{n} \\
\quad+\left.C_{b 1}\right|_{i+1 / 2, j, k}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right)
\end{array}\right\} \begin{aligned}
& \left.\left.\alpha_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k-1} ^{n+1 / 2}+\left.\left.\beta_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2} \\
& \quad+\left.\left.\gamma_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k+1} ^{n+1 / 2}=\left.\left.C_{a 2}\right|_{i, j+1 / 2, k} E_{2}\right|_{i, j+1 / 2, k} ^{n} \\
& \quad+\left.C_{b 2}\right|_{i, j+1 / 2, k}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{1}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right)
\end{aligned} \quad \begin{aligned}
& \left.\left.\alpha_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i-1, j, k+1} ^{n+1 / 2}+\left.\left.\beta_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2} \\
& \quad+\left.\left.\gamma_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i+1, j, k+1 / 2} ^{n+1 / 2}=\left.\left.C_{a 3}\right|_{i, j, k+1 / 2} E_{3}\right|_{i, j, k+1 / 2} ^{n} \\
& \quad+\left.C_{b 3}\right|_{i, j, k+1 / 2}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{2}\right|_{i-1 / 2, j, k+1 / 2} ^{n}\right)
\end{aligned}
$$

where
$\left.\alpha_{1}\right|_{i+1 / 2, j, k}=-\frac{b d}{2}\left(\frac{1}{G \sqrt{g}}\right)^{2} \frac{1}{\kappa_{y_{j+1 / 2}}} \frac{1}{\kappa_{y_{j}}} \frac{1}{\left(\Delta u^{2}\right)^{2}},\left.\gamma_{1}\right|_{i+1 / 2, j, k}=-\frac{b d}{2}\left(\frac{1}{G \sqrt{g}}\right)^{2} \frac{1}{\kappa_{y_{j+1 / 2}}} \frac{1}{\kappa_{y_{j}}} \frac{1}{\left(\Delta u^{2}\right)^{2}}$

$$
\left.\beta_{1}\right|_{i+1 / 2, j, k}=1-\left.\alpha_{1}\right|_{i+1 / 2, j, k}-\left.\gamma_{1}\right|_{i+1 / 2, j, k},\left.C_{b 1}\right|_{i+1 / 2, j, k}=\frac{2}{G \sqrt{g}} b \frac{1}{\kappa_{y_{j}}} \frac{1}{\Delta u^{2}} .
$$

The explicit electric field components from (6.33) can be written as follows:

$$
\begin{array}{r}
\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}=\left.e_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n}+2 a_{1} \psi_{\left.E_{12}\right|_{i+1 / 2, j, k}}^{n} \\
\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}=\left.e_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n}+\left.2 a_{2} \psi_{E_{23}}^{n}\right|_{i, j+1 / 2, k} \\
\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}=\left.e_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n}+2 a_{3} \psi_{\left.E_{31}\right|_{i, j, k+1 / 2}}^{n} \tag{6.40c}
\end{array}
$$

The magnetic field components from (6.34) can be written as follows:

$$
\begin{align*}
& \left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}=\left.h_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n}+2 b_{1} \psi_{H_{12} l_{i, j+1 / 2, k+1 / 2}^{n}}^{n}  \tag{6.41a}\\
& \left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}=\left.h_{2}\right|_{i+1 / 2, j, j+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n}+\left.2 b_{2} \psi_{H_{23}}^{n}\right|_{i+1 / 2, j, k+1 / 2}  \tag{6.41b}\\
& \left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}=\left.h_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}-\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}+2 b_{3} \psi_{H_{31}}^{n} l_{i+1 / 2, j+1 / 2, k} \tag{6.41c}
\end{align*}
$$

Sub-step 2: auxiliary updating for $\left.e_{\varepsilon}\right|^{n+1}$ can be written as follows:

$$
\begin{align*}
& \left.\left.\alpha_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j-1, k} ^{n+1}+\left.\left.\beta_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j, k} ^{n+1} \\
& +\left.\left.\gamma_{1}\right|_{i+1 / 2, j, k} e_{1}\right|_{i+1 / 2, j+1, k} ^{n+1}=\left.\left.C_{a 1}\right|_{i+1 / 2, j, k} E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}  \tag{6.42a}\\
& +\left.C_{b 1}\right|_{i+1 / 2, j, k}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2, k} ^{n+1 / 2}\right) \\
& \left.\left.\alpha_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k-1} ^{n+1}+\left.\left.\beta_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k} ^{n+1} \\
& +\left.\left.\gamma_{2}\right|_{i, j+1 / 2, k} e_{2}\right|_{i, j+1 / 2, k+1} ^{n+1}=\left.\left.C_{a 2}\right|_{i, j+1 / 2, k} \quad E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}  \tag{6.42b}\\
& +\left.C_{b 2}\right|_{i, j+1 / 2, k}\left(\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{1}\right|_{i, j+1 / 2, k-1 / 2} ^{n+1 / 2}\right) \\
& \left.\left.\alpha_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i-1, j, k+1} ^{n+1}+\left.\left.\beta_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i, j, k+1 / 2} ^{n+1} \\
& +\left.\left.\gamma_{3}\right|_{i, j, k+1 / 2} e_{3}\right|_{i+1, j, k+1 / 2} ^{n+1}=\left.\left.C_{a 3}\right|_{i, j, k+1 / 2} E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}  \tag{6.42c}\\
& +\left.C_{b 3}\right|_{i, j, k+1 / 2}\left(\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i-1 / 2, j, k+1 / 2} ^{n+1 / 2}\right)
\end{align*}
$$

and the electric field and magnetic field components from (6.37) and (6.38) can be written as follows:

$$
\begin{align*}
& \left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1}=\left.e_{1}\right|_{i+1 / 2, j, k} ^{n+1}-\left.E_{1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}+2 a_{1} \psi_{\left.E_{12}\right|_{i+1 / 2, j, k}}^{n+1 / 2}  \tag{6.43a}\\
& \left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1}=\left.e_{2}\right|_{i, j+1 / 2, k} ^{n+1}-\left.E_{2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}+2 a_{2} \psi_{E_{23}}^{n+1 / 2}, i_{i, j+1 / 2, k}  \tag{6.43b}\\
& \left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1}=\left.e_{3}\right|_{i, j, k+1 / 2} ^{n+1}-\left.E_{3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}+\left.2 a_{3} \psi_{E_{31}}^{n+1 / 2}\right|_{i, j, k+1 / 2}  \tag{6.43c}\\
& \left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}=\left.h_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}-\left.H_{1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}+2 b_{1} \psi_{H_{12} \mid l_{i, j+1 / 2, k+1 / 2}^{n+1 / 2}}^{n+1} \tag{6.44a}
\end{align*}
$$

$$
\begin{align*}
& \left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}=\left.h_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}+2 b_{2} \psi_{H_{23}|l|+1 /, j k+1 / 2}^{n+1 / 2}  \tag{6.44b}\\
& \left.H_{3}\right|_{i+1 / 2, j+j+1 / 2, k} ^{n+1}=\left.h_{3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}-H_{3}{ }_{i+1 / 2, j, j+1 / 2, k}^{n+1}+2 b_{3} \psi_{H_{31}+i / 2 / 2, j+1 / 2, k}^{n+1} \tag{6.44c}
\end{align*}
$$

where 1,2 , and 3 are equivalent to $\mathrm{x}, \mathrm{y}, \mathrm{z}$ of the orthogonal coordinate system respectively, and $b_{3}=\Delta t / 2 \mu, a_{1}=\Delta t / 2 \varepsilon$. The auxiliary variables $\psi_{E_{\xi!52}}$ can be written as follows.

$$
\begin{align*}
& c_{s}=e^{-\left(\left(\sigma_{s} / \kappa_{s}\right)+\alpha_{s}\right)\left(\Delta t / \varepsilon_{0}\right)}, \quad \xi 3 \text { is } \mathrm{z}, x \text { or } y  \tag{6.46a}\\
& d_{s}=\sigma_{s} /\left(\kappa_{s}\left(\sigma_{s}+\kappa_{s} \sigma_{s}\right)\right) \times\left(c_{s}-1\right), \quad(s=x, y, \text { or } z)  \tag{6.46b}\\
& \sigma_{s}(s)=\left(\sigma_{s_{\max }}\left|s-s_{0}\right|^{m}\right) / \delta^{m}  \tag{6.46c}\\
& \kappa_{s}(s)=1+\left(\kappa_{\max }-1\right)\left(\left|s-s_{0}\right|^{m}\right) / \delta^{m} \tag{6.46d}
\end{align*}
$$

where $\delta$ is the thickness of the PML absorber, $s_{0}$ is the PML interface, and $m$ is the order of the polynomial. Other auxiliary variables $\psi_{H_{\xi!\xi_{2}}}$ can be written in a similar way. Compared to CPML for 3-D C-LOD-NFDTD, in the proposed F-LOD-NFDTD approach $\psi$ is required only to explicitly update the electric and magnetic field components given in (6.40a)-(6.40c) and (6.41a)-(6.41c) of sub-step 1 but not during the implicit updating process for auxiliary electric fields (6.39a)-(6.39c). From (6.40a), it can also be seen that one variable $\psi$ is enough for the proposed approach, because the auxiliary term $\psi_{E_{12}}^{n+1 / 2}$ contains the same time index $(n+1 / 2)$ as that of the field on the left side. For the sake of simplification, both auxiliary variables will be considered at the same time index ' $n$ '. These features obtain improved efficiency and ease of programming for the proposed F-LOD-NFDTD CPML.

To demonstrate the benefits in terms of efficiency and accuracy, a comparison of the number of required arithmetic operations between C-LOD-NFDTD CPML and the F-LOD-NFDTD CPML over one complete time step is made and shown in Table 6.2. It can be observed from Table 6.2 that the total flop count for the right-hand side of the resultant equations for the fundamental scheme is 100 , which is considerably less than 196 for the conventional scheme, thus offering an efficiency gain of 1.96 . Table 6.2 also includes the arithmetic operations needed to invert the tri-diagonal matrices as there
is a cost involved in solving them, but the proposed fundamental scheme still offers superior performance in reducing arithmetic operations, thus achieving an overall efficiency gain of 1.76. In addition, a modified procedure is introduced for the fundamental scheme in which implicit and explicit equations are alternatively calculated; thus, it is necessary to retain only 7 field arrays, as against 12 field arrays required by the C-LOD-NFDTD. In this way, the F-LOD-NFDTD method can obtain improved memory savings.

Table 6.2 Comparison of the number of arithmetic operations per grid between C-LOD-
NFDTD-CPML and F-LOD-NFDTD-CPML

| Arithmetic Operations |  | C- LOD-NFDTD- <br> CPML | F-LOD-NFDTD- <br> CPML |
| :---: | :---: | :---: | :---: |
| Implicit, RHS | $\mathrm{M} / \mathrm{D}$ | 64 | 30 |
|  | $\mathrm{~A} / \mathrm{S}$ | 70 | 18 |
|  | $\mathrm{M} / \mathrm{D}$ | 32 | 20 |
|  | $\mathrm{~A} / \mathrm{S}$ | 30 | 32 |
| Tridiag. Matrices | $\mathrm{M} / \mathrm{D}$ | 18 | 50 |
|  | $\mathrm{~A} / \mathrm{S}$ | 12 | 50 |
|  | $\mathrm{M} / \mathrm{D}+\mathrm{A} / \mathrm{S}$ | 226 | 18 |
| Efficiency gain | RHS | 100 | 12 |
|  | overall | 1 | 128 |

The stability analysis of the F-LOD-NFDTD method will be discussed next.

### 6.5 Stability Analysis of 3-D F-LOD-NFDTD Method

Since the stability analysis of the F-LOD-NFDTD requires large mathematical derivation, only the key results of this analysis are summarised below. The field components in the spatial spectral domain are considered from (6.12a) and (6.12b). By

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substituting (6.12a) and (6.12b) into (6.20)-(6.23) and (6.24)-(6.27), we obtain the follow equations for the two sub-steps:
Sub-Step 1:

$$
\begin{equation*}
U^{n+1 / 2}=\Lambda_{1} U^{n} \tag{6.47}
\end{equation*}
$$

where $\Lambda_{1}$ is a $12 \times 12$ matrix as follows:

$$
\Lambda_{1}=\left[\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{2}} & 0 & 0 & 0 & 0 & \frac{i W_{2}}{\varepsilon Q_{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{3}} & 0 & \frac{i W_{3}}{\varepsilon Q_{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{1}} & 0 & \frac{i W_{1}}{\varepsilon Q_{1}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{3}}{\mu Q_{3}} & 0 & \frac{1}{Q_{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{1}}{\mu Q_{1}} & 0 & \frac{1}{Q_{1}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{2}}{\mu Q_{2}} & 0 & 0 & 0 & 0 & \frac{1}{Q_{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{2}^{\prime}}{Q_{2}} & 0 & 0 & 0 & 0 & \frac{i W_{2}}{\varepsilon Q_{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{3}^{\prime}}{Q_{3}} & 0 & \frac{i W_{3}}{\varepsilon Q_{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{1}^{\prime}}{Q_{1}} & 0 & \frac{i W_{1}}{\varepsilon Q_{1}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{3}}{\mu Q_{3}} & 0 & \frac{Q_{3}^{\prime}}{Q_{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{1}}{\mu Q_{1}} & 0 & \frac{Q_{1}^{\prime}}{Q_{1}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{2}}{\mu Q_{2}} & 0 & 0 & 0 & 0 & \frac{Q_{2}^{\prime}}{Q_{2}}
\end{array}\right]
$$

Sub-step 2:

$$
\begin{equation*}
U^{n+1}=\Lambda_{2} U^{n+1 / 2} \tag{6.48}
\end{equation*}
$$

where $\Lambda_{2}$ is $12 \times 12$ matrix as follows:

$$
\Lambda_{2}=\left[\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{3}} & 0 & & 0 & 0 & \frac{i W_{3}}{\varepsilon Q_{3}}
\end{array}\right] \quad 0 \quad\left[\begin{array}{ccccccccccccccc}
0 & 0 & \frac{1}{Q_{1}} & 0 & 0 & 0 & \frac{i W_{1}}{\varepsilon Q_{1}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{Q_{2}} & \frac{i W_{2}}{\varepsilon Q_{2}} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{2}}{\mu Q_{2}} & \frac{1}{Q_{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{3}}{\mu Q_{3}} & 0 & 0 & 0 & \frac{1}{Q_{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{1}}{\mu Q_{1}} & 0 & 0 & 0 & \frac{1}{Q_{1}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{3}^{\prime}}{Q_{3}} & 0 & 0 & 0 & \frac{i W_{3}}{\varepsilon Q_{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{1}^{\prime}}{Q_{1}} & 0 & 0 & 0 & \frac{i W_{1}}{\varepsilon Q_{1}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{2}^{\prime}}{Q_{2}} & \frac{i W_{2}}{\varepsilon Q_{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{2}}{\mu Q_{2}} & \frac{Q_{2}^{\prime}}{Q_{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{3}}{\mu Q_{3}} & 0 & 0 & \frac{Q_{3}^{\prime}}{Q_{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{1}}{\mu Q_{1}} & 0 & 0 & 0 & \frac{Q_{1}^{\prime}}{Q_{1}}
\end{array}\right]
$$

where $W_{1}=\frac{1}{G \sqrt{g}} \frac{\Delta t}{\Delta u^{1}} \sin \left(\frac{k_{1} \Delta u^{2}}{2}\right), W_{2}=\frac{1}{G \sqrt{g}} \frac{\Delta t}{\Delta u^{2}} \sin \left(\frac{k_{2} \Delta u^{2}}{2}\right)$,
$W_{3}=\frac{1}{G \sqrt{g}} \frac{\Delta t}{\Delta u^{3}} \sin \left(\frac{k_{3} \Delta u^{3}}{2}\right)$
$Q_{1}^{\prime}=1-\frac{\left(W_{1}\right)^{2}}{\mu \varepsilon}, Q_{2}^{\prime}=1-\frac{\left(W_{2}\right)^{2}}{\mu \varepsilon}, Q_{3}^{\prime}=1-\frac{\left(W_{3}\right)^{2}}{\mu \varepsilon}$, and $Q_{1}=1+\frac{\left(W_{1}\right)^{2}}{\mu \varepsilon}, Q_{2}=1+\frac{\left(W_{2}\right)^{2}}{\mu \varepsilon}$,
$Q_{3}=1+\frac{\left(W_{3}\right)^{2}}{\mu \varepsilon}$
By combining the above equation for both the steps we obtain,

$$
\begin{equation*}
U^{n+1}=\Lambda_{1} \Lambda_{2} U^{n}=\Lambda U^{n} \tag{6.49}
\end{equation*}
$$

where $\Lambda=\Lambda_{1} \Lambda_{2}$

$$
\Lambda=\left[\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{3}^{\prime}}{Q_{2} Q_{3}} & -\frac{W_{1} W_{2}}{\mu \varepsilon Q_{1} Q_{2}} & 0 & 0 & \frac{i W_{3}}{\varepsilon Q_{2} Q_{3}} & \frac{i W_{2} Q_{1}^{\prime}}{\varepsilon Q_{1} Q_{2}}  \tag{6.50}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{1}^{\prime}}{Q_{1} Q_{3}} & -\frac{W_{2} W_{3}}{\mu \varepsilon Q_{2} Q_{3}} & \frac{i W_{3} Q_{2}^{\prime}}{\varepsilon Q_{2} Q_{3}} & 0 & \frac{i W_{1}}{\varepsilon Q_{1} Q_{3}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{W_{1} W_{3}}{\mu \varepsilon Q_{1} Q_{3}} & \frac{Q_{2}^{\prime}}{Q_{1} Q_{2}} & \frac{i W_{2}}{\varepsilon Q_{1} Q_{2}} & \frac{i W_{1} Q_{3}^{\prime}}{\varepsilon Q_{1} Q_{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{3} Q_{1}^{\prime}}{\mu Q_{1} Q_{3}} & \frac{i W_{2}}{\mu Q_{2} Q_{3}} & \frac{Q_{2}^{\prime}}{Q_{2} Q_{3}} & 0 & -\frac{W_{1} W_{3}}{\mu \varepsilon Q_{1} Q_{3}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{3}}{\mu Q_{1} Q_{3}} & \frac{i W_{1} Q_{2}^{\prime}}{\mu Q_{1} Q_{2}} & -\frac{W_{1} W_{2}}{\mu \varepsilon Q_{1} Q_{2}} & \frac{Q_{3}^{\prime}}{Q_{1} Q_{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{2} Q_{3}^{\prime}}{\mu Q_{2} Q_{3}} & \frac{i W_{x}}{\mu Q_{x} Q_{y}} & 0 & 0 & -\frac{W_{2} W_{3}}{\mu \varepsilon Q_{2} Q_{3}} & \frac{Q_{1}^{\prime}}{Q_{1} Q_{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{2}^{\prime} Q_{3}^{\prime}}{Q_{2} Q_{3}} & -\frac{W_{2} W_{3}}{\mu \varepsilon Q_{2} Q_{2}} & 0 & 0 & \frac{i W_{3} Q_{2}^{\prime}}{\varepsilon Q_{2} Q_{3}} & \frac{i W_{2} Q_{1}^{\prime}}{\varepsilon Q_{1} Q_{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q_{1}^{\prime} Q_{3}^{\prime}}{Q_{1} Q_{3}} & -\frac{W_{2} W_{3}}{\mu \varepsilon Q_{2} Q_{3}} & \frac{i W_{3} Q_{2}^{\prime}}{\varepsilon Q_{2} Q_{3}} & 0 & \frac{i W_{1} Q_{3}^{\prime}}{\varepsilon Q_{1} Q_{3}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{W_{1} W_{3}}{\mu \varepsilon Q_{1} Q_{3}} & \frac{Q_{1}^{\prime Q_{2}^{\prime}}}{Q_{1} Q_{2}} & \frac{i W_{2} Q_{1}^{\prime}}{\varepsilon Q_{1} Q_{2}} & \frac{i W_{1} Q_{3}^{\prime}}{\varepsilon Q_{1} Q_{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{3} Q_{1}^{\prime}}{\mu Q_{1} Q_{3}} & \frac{i W_{2} Q_{3}^{\prime}}{\mu Q_{2} Q_{3}} & \frac{Q_{2}^{\prime} Q_{3}^{\prime}}{Q_{2} Q_{3}} & 0 & -\frac{W_{1} W_{3}}{\mu \varepsilon Q_{1} Q_{3}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i W_{3} Q_{1}^{\prime}}{\mu Q_{1} Q_{3}} & \frac{i W_{1} Q_{2}^{\prime}}{\mu Q_{1} Q_{2}} & -\frac{W_{1} W_{2}}{\mu \varepsilon Q_{1} Q_{2}} & \frac{Q_{1}^{\prime Q_{3}^{\prime}}}{Q_{1} Q_{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i W_{2} Q_{3}^{\prime}}{\mu Q_{1} Q_{2}} \frac{i W_{1} Q_{2}^{\prime}}{\mu Q_{2}} & 0 & 0 & -\frac{W_{2} W_{3}}{\mu \varepsilon Q_{2} Q_{3}} & \frac{Q_{1}^{\prime Q_{2}^{\prime}}}{Q_{1} Q_{2}}
\end{array}\right]
$$

The eigenvalues of (6.50) can be obtained as:

$$
\begin{equation*}
\lambda_{1}=\lambda_{2}=x_{1}+i y_{1}, \lambda_{3}=\lambda_{5}=x_{1}-i y_{1}, \lambda_{4}=\lambda_{6}=x_{2}+i y_{2} \tag{6.51}
\end{equation*}
$$

where

$$
\begin{gathered}
x_{1}=\mu^{2} \varepsilon^{2} W_{2}^{4} W_{3}^{2} Q_{1}^{\prime 4} Q_{2}^{4} Q_{3}^{4} Q_{3}^{\prime 2} Q_{1}^{4} \\
y_{1}=2 j W_{1}^{2} W_{3}^{2} W_{2}^{4} Q_{1}^{2} Q_{1}^{\prime 2} Q_{2}^{4}\binom{1+\frac{\mu^{2} \varepsilon^{2}}{2 Q_{1}^{\prime 2}} W_{1}^{2} Q_{1}^{2} Q_{3}^{\prime 2} Q_{3}^{4}-\frac{\mu^{2} \varepsilon^{2}}{Q_{1}^{\prime 2} W_{2}^{2}} W_{1}^{2} Q_{1}^{2} Q_{3}^{\prime 2} Q_{2}^{\prime 2} Q_{3}^{4}-}{2\left(\frac{\mu \varepsilon}{Q_{1}^{\prime 2}} W_{1}^{2} Q_{1}^{2} Q_{3}^{\prime 2} Q_{3}^{4}\right)^{2}}
\end{gathered}
$$

$$
\begin{array}{r}
x_{2}=\mu^{2} \varepsilon^{2} W_{2}^{4} W_{3}^{2} Q_{1}^{\prime 4} Q_{2}^{4} Q_{3}^{4} Q_{3}^{\prime 2} Q_{1}^{4}-2 \mu^{2} \varepsilon^{2}\left(\mu \varepsilon W_{2}^{4} W_{3}^{2} Q_{1}^{\prime 4} Q_{2}^{4} Q_{3}^{4} Q_{3}^{\prime 2} Q_{1}^{4}\right)^{2} \\
y_{2}=2 j W_{1}^{2} W_{3}^{2} W_{2}^{4} Q_{1}^{2} Q_{1}^{\prime 2} Q_{2}{ }^{4}\binom{\frac{\mu \varepsilon}{2 Q_{1}^{\prime 2}} Q_{3}^{4} Q_{1}^{2} W_{1}^{2}-\frac{\mu^{2} \varepsilon^{2}}{Q_{1}^{\prime 2}} W_{1}^{2} Q_{1}^{2} Q_{3}^{\prime 2} Q_{3}^{4}}{-\frac{\mu^{2} \varepsilon^{2}}{W_{2}^{2} Q_{1}^{\prime 2}} W_{1}^{2} Q_{1}^{2} Q_{3}^{\prime 2} Q_{2}^{\prime 2} Q_{3}^{4}+2\left(\frac{\mu \varepsilon}{Q_{1}^{\prime 2}} W_{1}^{2} Q_{1}^{2} Q_{3}^{\prime 2} Q_{3}^{4}\right)^{2}}
\end{array}
$$

For the first six eigenvalues, only the first six terms have been considered which can be written in compact form in (6.51). From (6.51), we can easily obtain the magnitude of the eigenvalues unity, i.e $\left|\lambda_{1}\right|=\left|\lambda_{2}\right|=\left|\lambda_{3}\right|=\left|\lambda_{4}\right|=\left|\lambda_{5}\right|=\left|\lambda_{6}\right|=1$. However, we have also computed the eigenvalues to check the unconditional stability. For instance, we compute the eigenvalues for the case of $k_{1}=0.2, k_{2}=0.2$ and $k_{3}=0.2$ and $\Delta u^{1}=\Delta u^{2}=\Delta u^{3}=1 \mathrm{~mm}$. The computed eigenvalues for the F-LOD-NFDTD method are tabulated in Table 6.3. From Table 6.3, it can be observed that the magnitudes of the eigenvalues are never larger than unity. Therefore, the 3-D F-LOD-NFDTD method is unconditionally stable.

Table 6.3
Computed eigenvalues of F-LOD-NFDTD

| Eigen values | CFLN=2 | CFLN=4 | CFLN=6 |
| :---: | :---: | :---: | :---: |
| $\lambda_{1}$ | 1.0000 | 1.00000 | 1.00000 |
| $\lambda_{2}$ | 1.0000 | 1.00000 | 1.00000 |
| $\lambda_{3}$ | 0.996544 | 0.995544 | 0.994514 |
| $\lambda_{4}$ | 0.00584 | 0.00584 | 0.00483 |
| $\lambda_{5}$ | 0.000623 | 0.0005523 | 0.0004223 |
| $\lambda_{6}$ | 0.000498 | 0.005498 | 0.003598 |
| $\lambda_{7}-\lambda_{12}$ | 0.00000 | 0.00000 | 0.00000 |

The near-field to far-field transformation technique for both the C-LOD-NFDTD and F-LOD-NFDTD methods will be discussed next.

### 6.6 Near-Field to Far-Field Transformation for both the 3-D C-LOD-NFDTD and F-LOD-NFDTD Methods

In many applications such as antennas and radar cross section (RCS) scatterer, it is necessary to find the radiation or scattered fields in a region that is far away from an antenna or scatterer. However, direct LOD-NFDTD simulation for the far field requires a mesh extending many wavelengths from the object which leads to a huge increase in computational time, which is not practical in application. Instead the far zone electromagnetic fields are computed from the near field LOD-NFDTD data through a near-field to far-field transformation technique. For the near-field to far-field transformation technique, an imaginary surface is first selected to enclose the electromagnetic object. Following the notation of [2], the radiation vectors $N$ and $L$ are defined as

$$
\begin{gather*}
\vec{N}=\int_{S^{\prime}} \vec{J}_{s} \exp \left(j k \vec{r}^{\prime} \cdot \hat{r}\right) d s^{\prime}  \tag{6.52a}\\
\vec{L}=\int_{S^{\prime}} \vec{M}_{s} \exp \left(j k \vec{r}^{\prime} \cdot \hat{r}\right) d s^{\prime} \tag{6.52b}
\end{gather*}
$$

where $j=\sqrt{-1}, \mathbf{k}$ the wavenumber, $\hat{r}$ the unit vector to the far zone field point, $r^{\prime}$ the vector to the source point of integration and $S^{\prime}$ the closed surface surrounding the scatterer. The electric and magnetic field components in the far field are expressed as:

$$
\begin{align*}
& E_{\theta}=-\frac{j k e^{-j k r}}{4 \pi r}\left(L_{\phi}+\eta_{0} N_{\theta}\right)  \tag{6.53a}\\
& E_{\phi}=+\frac{j k e^{-j k r}}{4 \pi r}\left(L_{\theta}-\eta_{0} N_{\phi}\right)  \tag{6.53b}\\
& H_{\theta}=+\frac{j k e^{-j k r}}{4 \pi r}\left(N_{\phi}-\frac{L_{\theta}}{\eta_{0}}\right)  \tag{6.53c}\\
& H_{\phi}=-\frac{j k e^{-j k r}}{4 \pi r}\left(N_{\theta}+\frac{L_{\phi}}{\eta_{0}}\right) \tag{6.53d}
\end{align*}
$$

where $N_{\theta}, N_{\phi}, L_{\theta}$ and $L_{\phi}$ can be expressed in terms of the following integrals:

$$
\begin{gather*}
N_{\theta}=\int_{S}\left(J_{1} \cos (\theta) \cos (\phi)+J_{2} \cos (\theta) \sin (\phi)-J_{3} \sin (\theta)\right) e^{-j k r^{\prime} \cos (\psi)} d S^{\prime}  \tag{6.54a}\\
N_{\phi}=\int_{S}\left(-J_{1} \sin \phi+J_{2} \cos \phi\right) e^{-j k r^{\prime} \cos (\psi)} d S^{\prime}  \tag{6.54b}\\
L_{\theta}=\int_{S}\left(M_{1} \cos (\theta) \cos (\phi)+M_{2} \cos (\theta) \sin (\phi)-M_{3} \sin (\theta)\right) e^{-j k r^{\prime} \cos (\psi)} d S^{\prime} \tag{6.54c}
\end{gather*}
$$

$$
\begin{equation*}
L_{\phi}=\int_{S}\left(-M_{1} \sin \phi+M_{2} \cos \phi\right) e^{-j k r^{\prime} \cos (\psi)} d S^{\prime} \tag{6.54d}
\end{equation*}
$$

However, $J_{1}, J_{2}$, and $J_{3}$ are equivalent to $J_{x}, J_{y}$, and $J_{z}$ that are used for orthogonal mesh LOD-FDTD. Here, the currents $\vec{J}$ and $\vec{M}$ on the surface are determined by $E$ and $H$ fields which are computed using either C-LOD-NFDTD or F-LOD-NFDTD method inside the computational domain. These currents are transformed into the frequency domain while being captured. After completing for all time steps, the far field terms $L_{\theta}, L_{\phi}, N_{\theta}$ and $N_{\phi}$ are calculated. These far field terms are calculated in the same way as given in [2] so will not be repeated here. Bistatic RCS can then be calculated using the following equation.

$$
\begin{gather*}
R C S_{\theta}=\frac{k^{2}}{8 \pi \eta_{0} P_{\text {inc }}}\left|L_{\phi}+\eta_{0} N_{\theta}\right|^{2}  \tag{6.55a}\\
R C S_{\phi}=\frac{k^{2}}{8 \pi \eta_{0} P_{\text {inc }}}\left|L_{\theta}-\eta_{0} N_{\phi}\right|^{2} \tag{6.55b}
\end{gather*}
$$

where $P_{\text {inc }}$ can be calculated as: $\quad P_{\text {inc }}=\frac{1}{2 \eta_{0}}\left|E_{\text {inc }}(\omega)\right|^{2}$
where $E_{\text {inc }}(\omega)$ is the discrete Fourier transform (DFT) of the incident electric field waveform at the frequency for which RCS calculation is required. The near-field to farfield transformation technique has been used for both C-LOD-NFDTD and F-LODNFDTD to analyse microwave curved 3-D structures.

### 6.7 Pure Scattered Field Formulation for both the C-LODNFDTD and F-LOD-NFDTD Methods

In this section, pure scattered field formulation for both 3-D C-LOD-NFDTD and F-LOD-NFDTD methods will be presented. For the plane wave excitation in LODNFDTD, the problem space is divided into two regions; the total field region and the scattered field region. The vectorial sum of incident and scattered fields present within a given space provides the total fields. Following the theory of scattered field formulation given in Chapter 3, we can write the scattered field formulation for both 3-D C-LODNFDTD and F-LOD-NFDTD as follows.

### 6.7.1 Scattered Field Formulation for 3-D C-LOD-NFDTD

Following the theory of scattered field formulation as described in Chapter 3 and the C-LOD-NFDTD formulations described in Section 6.2, the scattered field formulation for C-LOD-NFDTD for the two-steps can be derived as follows.

Sub-step 1:
The updating equations of the electric field for sub-step 1 are as follows:

$$
\begin{align*}
& E_{\text {scat, },\left.\right|_{i+1 / 2, j, k} ^{n+1 / 2}}=\left.C_{e} E_{\text {scat } 1}\right|_{i+1 / 2, j, k} ^{n}+\frac{1}{G} D_{e} \Theta_{13} \times \frac{1}{\Delta u^{2}}\left(\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j-1 / 2, k} ^{n+1 / 2}\right) \\
& +\frac{1}{G} D_{e} \Theta_{13} \times \frac{1}{\Delta u^{2}}\left(\left.H_{s c a t, 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}-\left.H_{s c a t, 3}\right|_{i+1 / 2, j-1 / 2, k} ^{n}\right)  \tag{6.56a}\\
& +\frac{1}{G} D_{e} \Theta_{13} \times\left. E_{i n c, 1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}-\frac{1}{G} D_{e} \Theta_{13} \times\left. E_{i n c, 1}\right|_{i+1 / 2, j, k} ^{n} \\
& \left.E_{\text {scat }, 2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}=\left.C_{e} E_{\text {scat }, 2}\right|_{i, j+1 / 2, k} ^{n}+\frac{1}{G} D_{e} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.H_{\text {scat },}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{\text {scat }, 1}\right|_{i, j+1 / 2, k-1 / 2} ^{n+1 / 2}\right) \\
& +\frac{1}{G} D_{e} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.H_{\text {scat }, 1}\right|_{i, j+1 / 2, k+1 / 2} ^{n}-\left.H_{\text {scat, } 1}\right|_{i, j+1 / 2, k-1 / 2} ^{n}\right)  \tag{6.56b}\\
& +\frac{1}{G} D_{e} \Theta_{21} \times E_{\text {inc }, 2} 2_{i, j+1 / 2, k}^{n+1 / 2}-\frac{1}{G} D_{e} \Theta_{21} \times\left. E_{\text {inc }, 2}\right|_{i, j+1 / 2, k} ^{n} \\
& \left.E_{\text {scat }, 3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}=\left.C_{e} E_{s c a t, 3}\right|_{i, j, k+1 / 2} ^{n}+\frac{D_{e}}{G} \Theta_{32} \frac{1}{\Delta u^{1}} \times\left(\left.H_{s c a t, 2}\right|_{i+1 / 2, j, j+k+1 / 2} ^{n+1 / 2}-\left.H_{\text {scat } 2}\right|_{i-1 / 2, j, k+1 / 2} ^{n+1 / 2}\right) \\
& +\frac{1}{G} D_{e} \Theta_{32} \times \frac{1}{\Delta u^{1}}\left(\left.H_{s c a t, 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n}-\left.H_{s c a t, 2}\right|_{i-1 / 2, j, k+1 / 2} ^{n}\right)  \tag{6.56c}\\
& +\frac{1}{G} D_{e} \Theta_{32} \times E_{i n c, 3}{ }_{i, j, j, k+1 / 2}^{n+12}-\frac{1}{G} D_{e} \Theta_{32} \times\left. E_{i n c, 3}\right|_{i, j, k+1 / 2} ^{n}
\end{align*}
$$

The updating equations of the magnetic field for sub-step 1 are as follows:

$$
\begin{align*}
& \left.H_{\text {scat }, 1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}=C_{h} H_{\text {scat },\left.\right|_{i, j+1 / 2, k+1 / 2} ^{n}}^{n}+\frac{D_{h} \Theta_{12}}{G} \frac{1}{\Delta u^{3}}\left(\left.E_{s c a t, 2}\right|_{i, j+1 / 2, k+1} ^{n+1 / 2}-\left.E_{\text {scat }, 2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right) \\
& +\frac{1}{G} D_{h} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.E_{\text {scat } 2}\right|_{i, j+1 / 2, k+1} ^{n}-\left.E_{\text {scat }, 2}\right|_{i, j+1 / 2, k} ^{n}\right)  \tag{6.56d}\\
& +\frac{1}{G} D_{h} \Theta_{12} \times\left. H_{i n c, 1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\frac{1}{G} D_{h} \Theta_{12} \times\left. H_{i n c, 1}\right|_{i, j+1 / 2, k+1 / 2} ^{n} \\
& \left.H_{\text {scat }, 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}=\left.C_{h} H_{\text {scat } 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n}+\frac{D_{h} \Theta_{23}}{G} \times \frac{1}{\Delta u^{1}}\left(\left.E_{\text {scat }, 3}\right|_{i+1, j, k+1 / 2} ^{n+1 / 2}-\left.E_{\text {scat }, 3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right) \\
& +\frac{1}{G} D_{h} \Theta_{23} \times \frac{1}{\Delta u^{1}}\left(\left.E_{\text {scat } 3}\right|_{i+1, j, k+1 / 2} ^{n}-\left.E_{\text {scat }, 3}\right|_{i, j, k+1 / 2} ^{n}\right)  \tag{6.56e}\\
& +\frac{1}{G} D_{h} \Theta_{23} \times\left. H_{i n c, 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\frac{1}{G} D_{h} \Theta_{23} \times\left. H_{i n c, 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n}
\end{align*}
$$

$$
\begin{array}{r}
\left.H_{\text {scat }, 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}=\left.C_{h} H_{s c a t, 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}+\frac{D_{h} \Theta_{31}}{G} \times \frac{1}{\Delta u^{2}}\left(\left.E_{s c a t, 1}\right|_{i+1 / 2, j+1, k} ^{n+1 / 2}-\left.E_{s c a t, 1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right) \\
+\frac{1}{G} D_{h} \Theta_{31} \times \frac{1}{\Delta u^{2}}\left(\left.E_{s c a t, 1}\right|_{i+1 / 2, j+1, k} ^{n}-\left.E_{s c a t, 1}\right|_{i+1 / 2, j, k} ^{n}\right)  \tag{6.56f}\\
+\frac{1}{G} D_{h} \Theta_{31} \times\left. H_{i n c, 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}-\frac{1}{G} D_{h} \Theta_{31} \times\left. H_{i n c, 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n}
\end{array}
$$

Sub-step 2: The updating equations of the electric field for sub-step 2 are as follows:

$$
\begin{align*}
& E_{\text {scat }, 1_{i+1 / 2, j, k}^{n+1}}=\left.C_{e} E_{\text {scat, } 1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}-\frac{D_{e} \Theta_{12}}{G} \times \frac{1}{\Delta u^{3}}\left(\left.H_{\text {scat }, 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}-\left.H_{\text {scat }, 2}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1}\right) \\
& -\frac{1}{G} D_{e} \Theta_{12} \times \frac{1}{\Delta u^{3}}\left(\left.H_{\text {scat }, 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\left.H_{\text {scat } 2}\right|_{i+1 / 2, j, k-1 / 2} ^{n+1 / 2}\right)  \tag{6.57a}\\
& -\frac{1}{G} D_{e} \Theta_{12} \times E_{\text {inc, }, 1} 1_{i+1 / 2, j, k}^{n+1}+\frac{1}{G} D_{e} \Theta_{12} \times E_{\text {inc, }} 1_{l_{i+1 / 2, j, k}^{n+1 / 2}} \\
& \left.E_{s c a t, 2}\right|_{i, j+1 / 2, k} ^{n+1}=\left.C_{e} E_{s c a t, 2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}-\frac{D_{e} \Theta_{23}}{G} \times \frac{1}{\Delta u^{1}}\left(\left.H_{s c a t, 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}-\left.H_{s c a t, 3}\right|_{i-1 / 2, j+1 / 2, k} ^{n+1}\right) \\
& -\frac{1}{G} D_{e} \Theta_{23} \times \frac{1}{\Delta u^{1}}\left(H_{\text {scat } 3} 3_{i+1 / 2, j+1 / 2, k}^{n+1 / 2}-\left.H_{\text {scat }, 3}\right|_{i-1 / 2, j+1 / 2, k} ^{n+1 / 2}\right)  \tag{6.57b}\\
& -\frac{D_{e}}{G} \Theta_{23} \times\left. E_{\text {inc }, 2}\right|_{i, j+1 / 2, k} ^{n+1}+\frac{D_{e}}{G} \Theta_{23} \times\left. E_{\text {inc }, 2}\right|_{i, j+1 / 2, k} ^{n+1 / 2} \\
& \left.E_{s c a t, 3}\right|_{i, j, k+1 / 2} ^{n+1}=\left.C_{e} E_{s c a t, 3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}-\frac{D_{e} \Theta_{31}}{G} \times \frac{1}{\Delta u^{2}}\left(\left.H_{s c a t, 1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1}-\left.H_{s c a t, 1}\right|_{i, j-1 / 2, k+1 / 2} ^{n+1}\right) \\
& -\frac{1}{G} D_{e} \Theta_{31} \times \frac{1}{\Delta u^{2}}\left(\left.H_{\text {scat }, 1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\left.H_{\text {scat }, 1}\right|_{i, j-1 / 2, k+1 / 2} ^{n+1 / 2}\right)  \tag{6.57c}\\
& -\frac{1}{G} D_{e} \Theta_{31} \times E_{\text {inc, } 3} 3_{i+1 / 2, j, k}^{n+1}+\frac{1}{G} D_{e} \Theta_{31} \times\left. E_{\text {inc }, 3}\right|_{l_{i+1 / 2, j, k}^{n+1 / 2}}
\end{align*}
$$

The updating equations of the magnetic field for sub-step 2 are as follows:

$$
\begin{align*}
& H_{\text {scat }, 1_{i, j+1 / 2, k+1 / 2}^{n+1}}=\left.C_{h} H_{\text {scat }, 1}\right|_{i, j+1 / 2, k+1 / 2} ^{n+1 / 2}-\frac{D_{h} \Theta_{13}}{G} \times \frac{1}{\Delta u^{2}}\left(\left.E_{\text {scat }, 3}\right|_{i, j+1, k+1 / 2} ^{n+1}-\left.E_{\text {scat } 3}\right|_{i, j, k+1 / 2} ^{n+1}\right) \\
& -\frac{1}{G} D_{h} \Theta_{13} \times \frac{1}{\Delta u^{2}}\left(\left.E_{\text {scat } 3}\right|_{i, j+1, k+1 / 2} ^{n+1 / 2}-\left.E_{\text {scat, } 3}\right|_{i, j, k+1 / 2} ^{n+1 / 2}\right)  \tag{6.57d}\\
& -\frac{1}{G} D_{h} \Theta_{13} \times H_{i n c, 1} l_{i, j+1 / 2, k+1 / 2}^{n+1}+\frac{1}{G} D_{h} \Theta_{13} \times H_{i n c, 1} l_{i, j+1 / 2, k+1 / 2}^{n+1 / 2} \\
& \left.H_{\text {scat }, 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}=\left.C_{h} H_{\text {scat }, 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}-\frac{D_{h} \Theta_{21}}{G} \times \frac{1}{\Delta u^{3}}\left(\left.E_{\text {scat } 1}\right|_{i+1 / 2, j, k+1} ^{n+1}-\left.E_{\text {scat }, 1}\right|_{i+1 / 2, j, j, k} ^{n+1}\right) \\
& -\frac{1}{G} D_{h} \Theta_{21} \times \frac{1}{\Delta u^{3}}\left(\left.E_{\text {scat }, 1}\right|_{i+1 / 2, j, k+1} ^{n+1 / 2}-\left.E_{\text {scat }, 1}\right|_{i+1 / 2, j, k} ^{n+1 / 2}\right)  \tag{6.57e}\\
& -\frac{1}{G} D_{h} \Theta_{21} \times\left. H_{i n c, 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1}+\frac{1}{G} D_{h} \Theta_{21} \times\left. H_{i n c, 2}\right|_{i+1 / 2, j, k+1 / 2} ^{n+1 / 2}
\end{align*}
$$

$$
\begin{align*}
&\left.H_{\text {scat } 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}=\left.C_{h} H_{s c a t, 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}-\frac{D_{h} \Theta_{32}}{G} \times \frac{1}{\Delta u^{1}}\left(\left.E_{\text {scat }, 2}\right|_{i+1, j+1 / 2, k} ^{n+1}-\left.E_{\text {scat }, 2}\right|_{i, j+1 / 2, k} ^{n+1}\right) \\
&-\frac{1}{G} D_{h} \Theta_{32} \times \frac{1}{\Delta u^{1}}\left(\left.E_{s c a t, 2}\right|_{i+1, j+1 / 2, k} ^{n+1 / 2}-\left.E_{s c a t, 2}\right|_{i, j+1 / 2, k} ^{n+1 / 2}\right)  \tag{6.57f}\\
&-\frac{1}{G} D_{h} \Theta_{32} \times\left. H_{i n c, 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1}+\frac{1}{G} D_{h} \Theta_{32} \times\left. H_{i n c, 3}\right|_{i+1 / 2, j+1 / 2, k} ^{n+1 / 2}
\end{align*}
$$

Equations (6.56) and (6.57) of sub-steps 1 and 2 cannot be solved directly, but a tridiagonal linear system can be formed from sub-steps 1 and 2 by following the procedure described in section 6.2.

### 6.7.2 Scattered Field Formulation for 3-D F-LOD-NFDTD Method

Similar to the scatter field formulation of the C-LOD-NFDTD method, the scattered field formulation for the F-LOD-NFDTD method can be derived. Following the theory of scattered field formulation as described in Chapter 3 and the F-LOD-NFDTD formulations as described in Section 6.4, the scattered field formulation for the F-LODNFDTD method for the two-steps can be derived. For brevity only sub-step 1 is given below.

Sub-step 1:
Auxiliary implicit updating for electric and magnetic fields are as follows:

$$
\begin{align*}
&\left.e_{\text {scat }, 1}\right|^{n+1 / 2}-\left.2 E_{\text {scat }, 1}\right|^{n}=\left.(b / G \sqrt{g}) \partial_{u^{2}} h_{\text {scat }, 3}\right|^{n+1 / 2} \\
&+(b / G \sqrt{g})\left(\left.e_{i n c, 1}\right|^{n+1 / 2}-\left.e_{i n c, 1}\right|^{n}\right)  \tag{6.58a}\\
&\left.e_{\text {scat }, 2}\right|^{n+1 / 2}-\left.2 E_{\text {scat }, 2}\right|^{n}=\left.(b / G \sqrt{g}) \partial_{u^{3}} h_{\text {scat }, 1}\right|^{n+1 / 2} \\
&+(b / G \sqrt{g})\left(\left.e_{i n c, 2}\right|^{n+1 / 2}-\left.e_{i n c, 2}\right|^{n}\right)  \tag{6.58b}\\
&\left.e_{\text {scat }, 3}\right|^{n+1 / 2}-\left.2 E_{s c a t, 3}\right|^{n}=\left.(b / G \sqrt{g}) \partial_{u^{1}} h_{s c a t, 2}\right|^{n+1 / 2} \\
&+(b / G \sqrt{g})\left(\left.e_{i n c, 3}\right|^{n+1 / 2}-\left.e_{i n c, 3}\right|^{n}\right)  \tag{6.58c}\\
& h_{\text {scat }, 1}| |^{n+1 / 2}-\left.2 H_{s c a t, 1}\right|^{n}=\left.(d / G \sqrt{g}) \partial_{u^{3}} e_{\text {scat }, 2}\right|^{n+1 / 2} \\
&+(d / G \sqrt{g})\left(\left.h_{i n c, 1}\right|^{n+1 / 2}-\left.h_{i n c, 1}\right|^{n}\right)  \tag{6.58d}\\
&\left.h_{\text {scat }, 2}\right|^{n+1 / 2}-\left.2 H_{s c a t, 2}\right|^{n}=\left.(d / G \sqrt{g}) \partial_{u^{1}} e_{s c a t, 3}\right|^{n+1 / 2} \\
&+(d / G \sqrt{g})\left(\left.h_{i n c, 2}\right|^{n+1 / 2}-\left.h_{i n c, 2}\right|^{n}\right) \tag{6.58e}
\end{align*}
$$

$$
\begin{align*}
\left.h_{\text {scat }, 3}\right|^{n+1 / 2}-\left.2 H_{\text {scat }, 3}\right|^{n}= & \left.(d / G \sqrt{g}) \partial_{u^{2}} e_{\text {scat }, 1}\right|^{n+1 / 2} \\
& +(d / G \sqrt{g})\left(\left.h_{\text {inc }, 3}\right|^{n+1 / 2}-\left.h_{\text {inc }, 3}\right|^{n}\right) \tag{6.58f}
\end{align*}
$$

and the explicit updating for the electric and magnetic fields are as follows:

$$
\begin{gather*}
\left.E_{s c a t, \xi}\right|^{n+1 / 2}=\left.e_{s c a t, \xi}\right|^{n+1 / 2}-\left.E_{s c a t, \xi}\right|^{n}  \tag{6.59}\\
\left.H_{s c a t, \xi}\right|^{n+1 / 2}=\left.h_{s c a t, \xi}\right|^{n+1 / 2}-\left.H_{s c a t, \xi}\right|^{n} \tag{6.60}
\end{gather*}
$$

In a similar way, the updated scattered field equations for sub-step 2 can be derived for the F-LOD-NFDTD method. Therefore, for a given incident wave, the above equations can be used to calculate the scattered field. The total field can be obtained by adding the scattered field to the incident field. Incident fields are applied only in internal region, only the scattered fields exist in the CPML region, which are absorbed by CPML ABC.

### 6.8 Numerical Computations Using Both 3-D C-LODNFDTD and F-LOD-NFDTD Methods

### 6.8.1 EM Scattering from Conducting and Dielectric Spheres

In this section, computational results obtained using the proposed methods are provided in both the frequency domain and the time domain. To test the validity and efficiency of the proposed C-LOD-NFDTD and F-LOD-NFDTD algorithms, we first consider the scattering by a perfectly electric conducting sphere of radius $k a=8.3$, as shown in Fig 6.2 (a). Fig. 6.2 (b) presents the nonorthogonal meshes of the conducting sphere which was generated following the nonorthogonal mesh generation technique outlined in Chapter 3. The structured nonorthogonal gridding techniques discussed in Chapter 3 are used here to generate the meshes for 3-D structures. Fig. 6.2 (c)-(d) illustrates the validation of the numerical results obtained by both F-LOD-NFDTD and C-LOD-NFDTD for CFLN= 4 and 10 with the results obtained using the Mie series solution. The result obtained by F-LOD-NFDTD and C-LOD-NFDTD agrees well with the Mie series solution, as shown in Fig. 6.2 (c) for CFLN=4. Fig. 6.2 (d) shows the RCS of the conducting sphere obtained by both C-LOD-NFDTD and F-LOD-NFDTD at $\mathrm{CFLN}=10$.

(a)

(b)

(c)

(d)

Fig. 6.2 (a) Conducting sphere (b) Nonorthogonal meshes of the conducting sphere, (c)-(d) $\operatorname{RCS}(\theta, 0)$ for a perfectly conducting sphere $k a=8.3$ compared with Mie series solution for CFLN=4 and 10

It can be observed that the results agree with Mie series even for higher CFLN. An eight-layer CPML is used for the conducting sphere which surrounds the entire computational domain with parameters $\sigma_{\mathrm{s}, \max }=0.7 \sigma_{\mathrm{opt}}, \sigma_{\mathrm{opt}}=11.21(\mathrm{~S} / \mathrm{m}), m=4, \kappa_{\max }=1$, and $\alpha_{\max }=0.2$. In terms of computational resources, the C-LOD-NFDTD CPML requires around 30 MB and 6 mins whereas F-LOD-NFDTD needs around 20 MB and 4.4 mins. Our simulations were carried out on a six core Linux workstation with 3.4 GHz clock and 32 GB RAM using Matlab. From the computational performance in terms of memory requirement, it is seen that F-LOD-NFDTD CPML can be computationally efficient compared to C-LOD-NFDTD CPML.

Fig. 6.3 (a)-(c) shows the transient scattering obtained using F-LOD-NFDTD from a PEC sphere of radius 0.5 m and compares it with the results obtained by C-LODNFDTD for $\mathrm{CFLN}=2,10$ and 15 as well as with published results obtained using TDIE [169].

(a)

(b)


Fig. 6.3 (a)-(c) Transient scattering of PEC sphere of radius 0.5 m illuminated by a Gaussian plane wave

It can be observed that the result obtained by the F-LOD-NFDTD method agrees reasonably well with the results obtained using C-LOD-NFDTD as well as with the analytical result. From the transient response, it can also be observed that the time domain solutions of the PEC sphere are stable in early time. Now we analyse the bistatic scattering from the lossless dielectric sphere (as shown in Fig. 6.4 (a)) ( $k a=3, \varepsilon_{r}=20$ ). Our numerical results will be compared with the results available in the literature [170]. Fig 6.4 (b) presents the nonorthogonal meshes of the dielectric sphere and Fig. 6.4 (c) shows the cut view of one quarter of the sphere. An eight-layer CPML is used for the dielectric sphere which surrounds the entire computational domain with parameters $\sigma_{\mathrm{s}, \max }=0.7 \sigma_{\mathrm{opt}}, \sigma_{\mathrm{opt}}=11.21(\mathrm{~S} / \mathrm{m}), m=4, \kappa_{\max }=1$, and $\alpha_{\max }=0.2$.

(a)

(b)

(c)

Fig. 6.4 (a) Lossless dielectric sphere (b) Nonorthogonal meshes for the dielectric sphere (c) Nonorthogonal meshes of one quarter of the dielectric sphere


Fig. 6.5 Bistatic RCS for a dielectric sphere $\left(k a=3, \varepsilon_{r}=20\right)$ compared Mie series solution


Fig. 6.6 (a)-(b) Bistatic RCS for a dielectric sphere ( $k a=3, \varepsilon_{r}=20$ ) compared Mie series as well as with the results obtained C-LOD-NFDTD for $\mathrm{CFLN}=5$ and 10

The computed bistatic RCS for the lossless dielectric sphere obtained by F-LODNFDTD for CFLN=5, 12 are shown in Fig. 6.5 (a)-(b) compared with the Mie series. From the figures, it is observed that the result obtained by the F-LOD-NFDTD with higher CFLN agrees reasonably well with the Mie series [170]. Next the computed RCS using F-LOD-NFDTD from the lossless dielectric sphere has been compared with the results obtained using C-LOD-NFDTD as well as with the Mie series as shown in Fig. 6.6 (a)-(b). The RCS obtained using F-LOD-NFDTD also agrees reasonably well with the result obtained using C-LOD-NFDTD. The computational resources for the simulation of the dielectric sphere are the same as those for the conducting sphere. Next the transient scattering from the lossless dielectric sphere with radius 0.5 m and the relative permittivity $\varepsilon_{\mathrm{r}}=2$ is computed using F-LOD-NFDTD and C-LOD-NFDTD.


Fig. 6.7 (a)-(c) Transient scattering from dielectric sphere of radius 0.5 m illuminated by a Gaussian plane wave for $\mathrm{CFLN}=5,10,20$

Fig. 6.7 (a)-(c) shows the transient scattering compared with the results obtained by C-LOD-NFDTD for CFLN $=5,15$ and 20, as well as with published results obtained using TD-EFIE [171]. It can be seen that the result obtained by C-LOD-NFDTD and F-LODNFDTD methods agree reasonably well with the analytical result. From the transient response, it can be observed that the time domain solutions of the lossless dielectric sphere are stable in early time.

### 6.8.2 EM Scattering from Dielectric Coated Conducting and Two

## Layer Dielectric Spheres

The proposed C-LOD-NFDTD and F-LOD-NFDTD methods are next applied to compute the RCS from the dielectric coated conducting sphere (as shown in Fig. 6.8 (a)) which is illuminated by the following modulated Gaussian plane pulse.

$$
\begin{equation*}
\mathbf{E}^{i n c}(\mathbf{r}, t)=\mathbf{E}_{0} \frac{4}{T \sqrt{\pi}} \exp \left(-\left(\frac{\tau-t_{0}}{\sqrt{2} \sigma}\right)^{2}\right) \cos \left(2 \pi f_{0} \tau\right) \tag{6.70}
\end{equation*}
$$

where $E_{0}=120 \pi, f_{0}$ is the centre carrier frequency, $t_{0}=8 \sigma$ and $\sigma=6 /\left(2 \pi f_{b w}\right)$ with $f_{b w}$ being the nominal bandwidth. The conducting sphere has a diameter of 0.4 m coated by a dielectric shell with thickness 0.05 m and relative permittivity $\varepsilon_{r}=2.0$. The nonorthogonal meshes of the dielectric coated conducting sphere are shown in Fig. 6.8 (b).

(a)

(b)

(c)

Fig. 6.8 (a) Dielectric coated conducting sphere (b) Nonorthogonal meshes for the dielectric coated conducting sphere (c) Nonorthogonal meshes of three-quarters of the outer dielectric sphere with conducting sphere

Fig. 6.8 (c) shows the nonorthogonal meshes of three-quarters of the outer dielectric sphere with inner conducting sphere. For this example, $f_{0}=500 \mathrm{MHz}, f_{b w}=1.0 \mathrm{GHz}$, $\Delta t=0.125 \mathrm{~ns}$ or $c \Delta t=0.0375 \mathrm{LM}$ (LM = light meter: the time that light takes to travel 1 m in vacuum) are used.


Fig. 6.9 (a)-(b) RCS for a dielectric coated conducting sphere obtained using the proposed methods and comparison of the results obtained using TDIE [176] available in the literature


Fig. 6.10 (a)-(b) Transient responses of the coated conducting sphere using the proposed method compared with the results using TDIE [176] available in the literature

The wideband RCS from zero frequency to 1.0 GHz obtained using the C-LODNFDTD, F-LOD-NFDTD and compared with the results available in the literature are shown in Fig. 6.9 (a)-(b) for CFLN=2, 10. Fig. 6.10 (a)-(b) shows the transient response obtained using F-LOD-NFDTD and C-LOD-NFDTD for CFLN=2 and 12 compared with the published results extracted from [176]. From Figs. 6.9 and 6.10, it can be observed that the results obtained using the proposed approaches agree reasonably well with the published results available in the literature.
Now, the scattering cross-sections of the two layered small lossless dielectric sphere are computed using the proposed methods. It must be noted that the explicit-NFDTD method requires excessive computational time to model such a small structure due to the CFL constraint.


Fig. 6.11 (a) Two layered small dielectric sphere (b) Nonorthogonal meshes for the two layered dielectric sphere (c) Nonorthogonal meshes of one quarter of the outer dielectric sphere with inner dielectric sphere

(a)

(b)

Fig. 6.12 (a)-(b) Scattering cross-section for the two layered spheres for CFLN=2 and 10

Fig. 6.11 (a) shows the two layered spheres where the outer sphere has the radius $b=600$ nm and the refractive indices are $\mathrm{n}_{1}=1.44$ and $\mathrm{n}_{2}=2.7$, while the inner sphere has the radius $\mathrm{a}=400 \mathrm{~nm}$. The nonorthogonal meshes for the two layered spheres are shown in Fig. 6.11 (b), and Fig. 6.11 (c) shows the nonorthogonal meshes of one quarter of the outer dielectric sphere with inner dielectric sphere. Fig. 6.12 (a)-(b) shows the scattering cross-section obtained using the proposed methods compared with the results obtained from [177] for CFLN=2 and 10. From Fig. 6.12, it can be observed that the results obtained using the proposed approaches agree reasonably well even in very small structures.

### 6.8.3 Modelling of Cylindrical Dielectric Resonators and Filters

To further validate the proposed methods, a cylindrical dielectric resonator (as shown in Fig. 6.13 (a)) is considered. Our numerical results on the cylindrical dielectric resonator will be compared with the existing results in the literature that were obtained by using analytical methods [172]. The parameters of the cylindrical dielectric resonator are $\varepsilon_{\mathrm{r}}=38, \mathrm{a}=5.25 \mathrm{~mm}, \mathrm{~h}=4.6 \mathrm{~mm}$. An eight-layer CPML is used for the cylindrical

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dielectric resonator which surrounds the entire computational domain with parameters $\sigma_{\mathrm{s}, \max }=0.7 \sigma_{\mathrm{opt}}, \sigma_{\mathrm{opt}}=11.21(\mathrm{~S} / \mathrm{m}), m=4, \kappa_{\max }=1$, and $\alpha_{\max }=0.2$. The nonorthogonal meshes of the cylindrical dielectric resonator are shown in Fig. 6.13 (b). The computed resonant frequencies of TE and HE modes of a cylindrical dielectric resonator [172] using both the F-LOD-NFDTD CPML and C-LOD-NFDTD CPML are tabulated in Table 6.4. The results on resonant frequencies reveal that those obtained using C-LODNFDTD and F-LOD-NFDTD CPML have closer agreement with the published data extracted from [172].

(a)

(b)

Fig. 6.13 (a) Cylindrical dielectric resonator, (b) Nonorthogonal meshes for the cylindrical dielectric resonator

Table 6.4
Comparison of resonant frequencies for the cylindrical dielectric resonator [172] with $\varepsilon_{\mathrm{r}}=38, \mathrm{a}=5.25 \mathrm{~mm}, \mathrm{~h}=4.6 \mathrm{~mm}$

| Mode | F-LOD-NFDTD | LOD-NFDTD | Glisson et al. [172] |
| :---: | :---: | :--- | :---: |
| $\mathrm{TE}_{01}$ | 4.823 | 4.801 | 4.82 |
| $\mathrm{HE}_{12}$ | 6.634 | 6.611 | 6.63 |

The proposed methods are next applied for analysing a single cavity $\mathrm{TE}_{01 \delta}$-mode DR filter [178], as shown in Fig. 6.14. The parameters of the dielectric resonator are dielectric constant, $\varepsilon_{\mathrm{r}}=38$, diameter of the resonator $=27.8 \mathrm{~mm}$, disk_D=25.4 mm, DR_H=14.5 mm and disk_H=3mm. The length and width of the metal cavity are both 46 mm , and the height is 41 mm . The nonorthogonal meshes for the single cavity DR filter are shown in Fig. 6.15 (a). Fig. 6.15 (b) shows the open ended nonorthogonal meshes of the cavity. The transmission zero computed using the proposed methods is shown in Fig. 6.16 (a)-(b) and compared with the results extracted from [178].


Fig. 6.14 Single cavity DR filter with two probes on opposite sides of axis


Fig. 6.15 (a) Nonorthogonal meshes for single cavity dielectric resonator (b) Nonorthogonal meshes for single cavity dielectric resonator with open end

The computed transmission zero using the proposed methods agrees reasonably well with the results obtained from the published results.

## Curvilinear Meshes



Fig. 6.16 (a)-(b) Computed transmission zero using the proposed methods for CFLN=2, and 10

Using the analytical solution for the calculation of the conducting sphere as reference [156], the maximum errors of the C-LOD-FDTD and F-LOD-FDTD using orthogonal and non-orthogonal meshes will be provided in Chapter 7 to demonstrate the superiority of the 3-D LOD-FDTD with non-orthogonal meshes.

### 6.9 Discussion

In this chapter, nonorthogonal 3-D LOD-FDTD using the curvilinear coordinate system has been developed for the first time. The formulations of the C-LOD-NFDTD and F-LOD-NFDTD methods have been described. Both formulations can be applied to the analysis of curved microwave structures. The CPML ABC for both 3-D C-LOD-

## Chapter 6. 3-D LOD-NFDTD: LOD-FDTD Approaches Using Non-orthogonal

 Curvilinear MeshesNFDTD and F-LOD-NFDTD have also been developed along with the stability analysis. A theoretical study of both C-LOD-NFDTD and F-LOD-NFDTD shows that nonorthogonal meshes are more advantageous for modelling objects with curved features and can reduce computational burden compared to orthogonal meshes. New pure scattered field formulations of the 3-D C-LOD-NFDTD and F-LOD-NFDTD methods have also been presented.

The proposed C-LOD-NFDTD and F-LOD-NFDTD have been applied to analyse various microwave curved structures. From the analysis, it can be observed that the proposed method is unconditionally stable and the numerical results agree closely with the results in the literature. A performance comparison in terms of the execution time and memory used by C-LOD-NFDTD and F-LOD-NFDTD for analysing various microwave devices has been provided and proves the usefulness of the nonorthogonal 3-D LOD-FDTD methods. Comparing the CPU time required for both the conventional and fundamental approaches, the proposed F-LOD-NFDTD CPML approach is characterised to be computationally more efficient.

## Chapter 7

## Conclusions and Future work

### 7.1 Overall Contributions

In this thesis, efficient implicit LOD-FDTD using orthogonal and non-orthogonal meshes has been investigated for solving various electromagnetics problems. Based on the investigations and the results obtained in this thesis, this chapter highlights the outcomes and contributions that have been achieved. The contributions made in this thesis can be grouped into five main areas:
(1) Development and implementation of a segmented LOD-FDTD method for analysing electrically large symmetric structures.
(2) Development and implementation of nonorthogonal LOD-NFDTD for solving 2D curved structures.
(3) Development and implementation of rotationally symmetric LOD-FDTD with dispersion control parameters.
(4) Development and implementation of efficient two sub-step conventional and fundamental scheme based CPML for LOD-FDTD using orthogonal meshes for analysing 3-D structures.
(5) Development and implementation of efficient C-LOD-NFDTD and F-LODNFDTD along with CPML employing nonorthogonal meshes for analysing 3-D curved structures

The specific contributions made in each of these areas will now be listed. The following contributions have been made in area (1):

- Derived the formulation of 2-D LOD-FDTD with CPML for the TE and TM cases (Chapter 2).
- Developed a new segmented locally one dimensional FDTD (S-LOD-FDTD) technique for analysing electrically large symmetric structures (Chapter 2).
- Developed S-ADI-FDTD to compare with the S-LOD-FDTD method
- Numerical analysis of various electrically large symmetric tunnel structures has been performed using the S-LOD-FDTD method (Chapter 2).
- Provided performance comparison and error analysis of the S-LOD-FDTD method (Chapter 2).

The contributions in area (2) are:

- Development of the formulation of 2-D LOD-FDTD using nonorthogonal meshes based on curvilinear coordinates (Chapter 3).
- The CPML formulation of 2-D LOD-NFDTD has also been developed (Chapter 3).
- Stability analysis and numerical dispersion analysis of the 2-D LOD-NFDTD (Chapter 3).
- Demonstration of the developed 2-D LOD-NFDTD for analysing various 2-D curved structures (Chapter 3).
- Computational performance analysis of the 2-D LOD-NFDTD method (Chapter 3).

In area (3) the original outcomes are:

- Derivation of RS-LOD-FDTD with dispersion control parameters (Chapter 4).
- Development of CPML ABC for D-RS-LOD-FDTD (Chapter 4).
- Applied D-RS-LOD-FDTD algorithm with CPML ABC to analyse rotationally symmetric structures (Chapter 4).

The original outcomes which have been made in area (4)

- Derivation of CPML ABC for the two sub-step conventional LOD-FDTD method with orthogonal meshes (Chapter 5).
- Development of CPML ABC for the fundamental scheme based LOD-FDTD (F-LOD-FDTD) method (Chapter 5).
- Stability analysis of the F-LOD-FDTD method (Chapter 5).
- Applied the conventional LOD-FDTD with two sub-steps and F-LOD-FDTD with CPML ABC for analysing various 3-D microwave structures (Chapter 5).

The original contributions in area (5) are:

- Development of 3-D conventional LOD-FDTD based on curvilinear coordinates (Chapter 6).
- Development of CPML ABC for 3-D LOD-NFDTD (Chapter 6).
- Development of 3-D nonorthogonal LOD-FDTD using the fundamental scheme (Chapter 6).
- Derivation of CPML ABC for the F-LOD-NFDTD methods (Chapter 6)
- Stability analysis of the 3-D conventional LOD-NFDTD and F-LOD-NFDTD method (Chapter 6).
- Application of the LOD-NFDTD and F-LOD-NFDTD methods in analysis of 3D microwave structures (Chapter 6).

These contributions, and accompanying background material, have been addressed within this thesis in the following order. Chapter 1 contains the literature review, outlines of the thesis and list of the refereed journal and conference publications which have resulted from the research undertaken in this thesis. Chapter 2 introduces the relevant background theory of LOD-FDTD and the theory of segmented LOD-FDTD for analysing propagation along electrically large symmetric structures. The nonorthogonal LOD-FDTD for analysing 2-D curved structures is presented in Chapter 3. Chapter 4 proposed the computationally efficient D-RS-LOD-FDTD to control dispersion for the analysis of rotationally symmetric structures. The two sub-step C-LOD-FDTD and the F-LOD-FDTD have been presented for analyzing 3-D structures in Chapter 5. Chapter 6 proposes a nonorthogonal scheme for 3-D C-LOD-NFDTD and F-LOD-NFDTD for analysing 3-D curved structures.

### 7.2 Conclusions Based on Individual Chapters

### 7.2.1 2-D LOD-FDTD for EM Propagation Modelling in Electrically Large Symmetric Structures

In this chapter, the theory of the LOD-FDTD method has been introduced. More specifically LOD-FDTD in 2-D using orthogonal meshes has been described. The CPML formulation of the 2-D LOD-FDTD has also been discussed and the developed segmented LOD-FDTD technique has been presented. Note that the segmented LODFDTD method has been applied to modified 2-D large tunnel structures. The application of the S-LOD-FDTD method to the analysis of electrically large symmetric structures
has been described. Straight, curved and branched tunnels have been analysed as electrically large symmetric structures. The results obtained by S-LOD-FDTD for the large tunnel structures have been compared with the results obtained by S-ADI-FDTD as well as with the published results in the literature. The results indicate higher signal attenuation for the junction/transition regions compared to regions away from such abrupt transitions. The predictions on path loss agree reasonably well with published measured data. Here, the averaged path loss for Roux tunnel obtained using S-LODFDTD is shown in Fig. 7.1 and compared with the results obtained by S-ADI-FDTD as well as with the measured data extracted from the published results in [153]. The averaged path loss over 50 m length from the published results is considered to compare with the averaged simulated data.


Fig. 7.1 Comparison of averaged (over 50 m ) path loss for Roux tunnel

The comparison shows that the result obtained by S-LOD-FDTD agrees reasonably well with the results obtained by other methods. The results also reveal that the proposed segmentation approach can help to reduce computational resources and hence can be extended for electromagnetic modeling of any long symmetric path propagation problems. Analysing the electrically large symmetric structures using the developed S-LOD-FDTD method can provide a large savings in computer resources and offer good accuracies. Table 7.1 summarises the computational performance in terms of the execution time and memory of the proposed S-LOD-FDTD and S-ADI-FDTD methods. The tabulated results indicate that the developed S-LOD-FDTD is more effective than S-ADI-FDTD because it requires less time and memory.

Table 7.1
Required computational time and memory using S-LOD-FDTD method for Roux tunnel
(CFLN=2)

| Tunnel Structures | S-LOD-FDTD |  | S-ADI-FDTD |  |
| :---: | :---: | :---: | :---: | :---: |
|  | CPU Times <br> (hrs) | Memory <br> $(\mathrm{MB})$ | CPU Times <br> $(\mathrm{hrs})$ | Memory <br> $(\mathrm{MB})$ |
| Roux Tunnel <br> $[153]$ | 3.5 hrs | 1010 | 4.32 hrs | 1090 |
| Branch Tunnel <br> $[154]$ | 2.86 hrs | 900 | 3.45 hrs | 1010 |

The relative error of the S-LOD-FDTD has been compared with the S-ADI-FDTD method. The results and error analysis confirm that the proposed S-LOD-FDTD method is computationally more efficient compared than S-ADI-FDTD and provides accurate results for propagation predictions inside large symmetric tunnels. One journal paper has been published based on this in IEEE Transactions on Magnetics and a conference paper published in 2011 APSAEM conference (ISBN 978-4-931455-16-0), (see publications list in Chapter 1).

### 7.2.2 Non-orthogonal LOD-FDTD Method for EM Scattering from Two Dimensional (2-D) Structures

Chapter 3 of this thesis is concerned with the investigation of the nonorthogonal LOD-NFDTD method for solving 2-D electromagnetic scattering from curved structures. The technique can overcome the limitation of using orthogonal meshes for modeling curved bodies using 2-D LOD-FDTD. For the first time, the LOD-NFDTD method in 2-D has been proposed in this chapter based on a curvilinear coordinate system. A theoretical study of LOD-NFDTD has been performed for both TE and TM cases. CPML absorbing boundary condition for the LOD-NFDTD method has also been developed for the first time to improve the absorbing efficiency. A new pure scattered field formulation of the LOD-NFDTD has also been presented along with stability and dispersion error analysis. The new LOD-NFDTD has been applied for analysing EM scattering from circular conducting, dielectric, coated conducting and layered elliptic
cylinders as well as overfilled dielectric and bent PEC cavity structures to validate the method. For the numerical analysis, it is seen that the proposed method is unconditionally stable and the numerical results agree reasonably well with the results in the literature, as well as with the ADI-NFDTD results. In modelling bodies having curvatures for EM scattering, it is found that the proposed nonorthogonal LOD-NFDTD technique is more advantageous than employing orthogonal meshes and reduces computational burden.

A performance comparison of the new LOD-NFDTD method has also been conducted. Compared with ADI-NFDTD, the proposed method provides savings in computational resources. The performance comparisons of LOD-NFDTD with ADINFDTD in terms of arithmetic operations, execution time and memory, and maximum error are shown in Tables 7.2 to 7.4. It is worth mentioning that only two equations are solved in each sub-step for the LOD-NFDTD method. This leads to a reduction in the number of arithmetic operations for the updating equations as compared to the ADINFDTD. Table 7.3 shows the comparison of the required execution time and CPU memory for the LOD-NFDTD and ADI-NFDTD methods for calculating the RCS of the PEC cylinder [156].

Table 7.2
Number of arithmetic operations

|  | Implicit |  | Explicit |  | Total |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | M/D | $\mathrm{A} / \mathrm{S}$ | $\mathrm{M} / \mathrm{D}$ | $\mathrm{A} / \mathrm{S}$ | $\mathrm{M} / \mathrm{D}$ | $\mathrm{A} / \mathrm{S}$ |
|  | $40+40$ | $20+20$ | $11+11$ | $6+6$ | 102 | 52 |
| LOD-NFDTD | $29+29$ | $9+9$ | $6+6$ | $4+4$ | 70 | 20 |

Table 7.3
Comparison of computer resources for calculation of RCS of a PEC cylinder of diameter $=10 \lambda$

|  | $\Delta \mathrm{t}=\mathrm{CFLN} \times \Delta \mathrm{t}_{\text {CFL }}$ <br> $(\mathrm{ps})$ | CPU Time <br> $(\mathrm{s})$ | Memory <br> $(\mathrm{KB})$ |
| :--- | :---: | :---: | :---: |
| LOD-NFDTD | $(8 \times 1.5)=12$ | 43 | 925 |
| ADI-NFDTD | $(8 \times 1.5)=12$ | 58 | 970 |

Table 7.4
Comparison of the maximum errors for calculation of RCS of overfilled cavity using LOD-NFDTD and ADI-NFDTD for $\mathrm{TM}_{\mathrm{z}}$ wave

| CFLN | \% Maximum error of LOD- <br> NFDTD Method | \% Maximum error of ADI- <br> NFDTD Method |
| :--- | :---: | :---: |
| 2 | 0.002 | 0.008 |
| 4 | 0.017 | 0.025 |
| 6 | 0.104 | 0.151 |
| 8 | 0.345 | 0.515 |
| 10 | 0.423 | 0.672 |

Table 7.3 shows the comparison of the required execution time and CPU memory for the LOD-NFDTD and ADI-NFDTD methods to calculate the RCS of the PEC cylinder. It can be observed from Table 7.3 that given the same grids and higher CFLN, the LOD-NFDTD method requires less computational time than the ADI-NFDTD method when used to calculate the RCS of the PEC cylinder. Using the analytical solution provided in [164] for an overfilled cavity for the TM case as reference, the maximum errors of the LOD-NFDTD and ADI-NFDTD method for different CFLN are calculated as shown in Table 7.4. From the comparison of the above three cases, it is seen that the proposed LOD-NFDTD method is superior to the conventional ADINFDTD method. From the contributions in Chapter 3, two papers have been published. One journal paper has been published in IEEE Transactions on Electromagnetic Compatibility and a conference paper published in 2011 IEEE AP-S/URSI conference (available in IEEE xplore).

### 7.2.3 Rotationally Symmetric LOD-FDTD with Dispersion Control Parameters

In Chapter 4, the CPML ABC for RS-LOD-FDTD for both TE and TM cases has been developed. Next, to improve the computational efficiency and reduced dispersion, a novel D-RS-LOD-FDTD method along with CPML ABC has been presented. Various rotationally symmetric structures such as resonators, open tip monopole (OTM) antenna
and expanded tip wire (ETW) antennas, have been analysed by the proposed method to demonstrate the validation. From the S-parameters and SAR calculations, it can be observed that the result obtained by D-RS-LOD-FDTD CPML has lower dispersion as well as minimise the computational resources.

To compare performances between the D-RS-LOD-FDTD and RS-LOD-FDTD methods, we first present the relative error calculation. Fig. 7.2 shows the relative errors of D-RS-LOD-FDTD and conventional RS-LOD-FDTD with respect to CFLN (CFL number). From Fig. 7.2 it is clear that D-RS-LOD-FDTD has less relative error than RS-LOD-FDTD. Additionally, a comparison between D-RS-LOD-FDTD and RS-LODFDTD methods in terms of execution time for calculating SAR of the ETW antenna is shown in Fig. 7.3 from which it is seen that the computational time required for D-RS-LOD-FDTD is less than the RS-LOD-FDTD method.


Fig. 7.2 Relative error with respect to CFLN


Fig. 7.3 Computational time for D-RS-LOD-FDTD and RS-LOD-FDTD with different CFLN

As a result, the simple use of dispersion control parameters enables us to use larger times steps, and reduce error and less computational time is required to efficiently model rotationally symmetric microwave structures and devices. One paper based on the contributions from Chapter 4 has been presented in ISAP 2012 conference (available in IEEE xplore)

### 7.2.4 Efficient LOD-FDTD Approaches for 3-D Bodies Using Orthogonal Meshes

To analyse realistic 3-D structures using the LOD-FDTD method, we have developed a two sub-step C-LOD-FDTD along with two sub-step CPML. Low numerical reflections have been demonstrated for the CPML scheme. Analytical stability analysis of the C-LOD-FDTD is presented to demonstrate the unconditional stability of the proposed approach. The performance of the proposed C-LOD-FDTD CPML is investigated and compared with standard explicit FDTD.

Because simultaneous linear systems with tri-diagonal matrix are involved in C -LOD-FDTD, a large number of arithmetic operations are required, which can be computationally expensive for modelling many realistic 3-D structures. To overcome this cost, a fundamental scheme based LOD-FDTD has been presented. A modified calculation procedure is also proposed which further improves the computational efficiency. CPML ABC for the F-LOD-FDTD method is also derived for the first time. A comparison of the F-LOD-FDTD with the C-LOD-FDTD has been tabulated which shows the improvement of the F-LOD-FDTD method. Also, stability analysis of the F-LOD-FDTD method is provided that demonstrates the unconditional stability of the proposed F-LOD-FDTD method. Finally, the fundamental LOD-FDTD (F-LOD-FDTD) has been validated using numerical results obtained on resonant frequencies of a dielectric resonator as well as current distribution and input reflection coefficients of many antennas and microwave 3-D structures. The results obtained by both the LODFDTD methods agree reasonably well with the results in the literature. Fig. 7.4 shows the computed transient current using F-LOD-FDTD induced at the centre of a 1 m long thin wire dipole antenna having a radius of 5 cm operating at 300 MHz and the results are compared with the published results obtained using MOM [173].


Fig.7.4 Calculated currents at the center of the 1 m long dipole antenna


Fig.7.5 Absolute relative error computed of the proposed approach with reference to C-LOD-FDTD

Fig. 7.5 presents the absolute computational error of the proposed method with reference to the C-LOD-FDTD CPML approach which shows that the results obtained by F-LOD-FDTD CPML offer lower errors. Fig. 7.6 shows the electric field at a distance of 2 mm from the load end of the transmission line computed using the proposed scheme and compared with the results published in [175].

It can be observed from the Fig. 7.6 that the result obtained by the proposed F-LOD-FDTD-CPML method agrees reasonably well with the published result [175]. Fig. 7.7 presents the absolute computational error of the proposed method with reference to the C-LOD-FDTD CPML approach which shows that the results obtained by F-LODFDTD CPML offer lower errors. The performance comparison in terms of execution time and memory used by the conventional LOD-FDTD and F-LOD-FDTD for analysing each problem have been provided as shown in Table 7.5 which also proves the usefulness of the LOD-FDTD methods.


Fig. 7.6 Normalised $E_{z}$ field observed at 2 mm from the load end of the microstrip interconnect


Fig. 7.7 Absolute error calculated of the proposed approach with C-LOD-FDTD as reference method

Table 7.5
Comparison of CPU time and memory for various 3-D structures

| Scheme |  | Dipole | VLSI Interconnect |
| :---: | :---: | :---: | :---: |
| F-LOD- | Execution time (s) | 150.66 | 4536 |
| FDTD | Memory (MB) | 53 | 184.9 |
| C-LOD- | Execution time (s) | 215.31 | 6804 |
| FDTD | Memory (MB) | 60.5 | 215.3 |

Comparing the CPU time and memory of the conventional and fundamental approaches, large savings in computer memory and CPU time are obtained for the F-LOD-FDTD approach. The main contribution of Chapter 5 of this thesis is the development of the efficient two sub-step CPML ABC for C-LOD-FDTD for analysing 3-D microwave structures. The developed C-LOD-FDTD CPML approach provides better performance
than the standard explicit FDTD method. One paper from this contribution has been accepted in the 2013 IEEE $\boldsymbol{W}$ WEM conference (to be published in IEEE xplore). Another contribution of this chapter is the development of F-LOD-FDTD CPML approach. The developed approach is more computationally efficient compared to C-LOD-FDTD. The numerical results on 3-D microwave structures are provided to demonstrate the superiority of the F-LOD-FDTD CPML approach over C-LOD-FDTD CPML approach. One journal paper on this has been accepted for publication in IEEE Microwave and Wireless Components Letters (see publications list in Chapter 1).

### 7.2.5 3-D LOD-NFDTD: LOD-FDTD Approaches Using Nonorthogonal Curvilinear Meshes

This thesis also made the contributions on both conventional LOD-FDTD (C-LODFDTD) and F-LOD-FDTD based on a curvilinear coordinate system for analysing 3-D curved structures. The formulation of C-LOD-NFDTD has been described. The C-LOD-NFDTD formulation is easy to formulate and can be applied easily for analysing curved structures. The CPML ABC for 3-D C-LOD-NFDTD has also been developed and the stability analysis of C-LOD-NFDTD has been presented. Numerical verification of the stability analysis is also presented to demonstrate the unconditional stability of the C-LOD-NFDTD method. An efficient C-LOD-NFDTD has been combined with CPML ABC to analyse the 3-D curved structures. Several 3-D curved structures have been analysed. Good agreement has been obtained between C-LOD-NFDTD and published results from the literature. Large savings in CPU time and computer memory have been achieved in the analysis of 3-D structures compared to other methods.

To further improve the computational efficiency of analysing 3-D curved microwave structures, nonorthogonal LOD-FDTD with fundamental scheme (F-LOD-NFDTD) has been developed to minimise the number of matrix operations and field variables. The CPML ABC for the F-LOD-NFDTD method has also been developed. The computational performance in terms of arithmetic operations has been provided which exhibits the merits of the F-LOD-NFDTD method over C-LOD-NFDTD method. The stability analysis of the F-LOD-NFDTD method is derived and then the numerical verification of the stability analysis proves the unconditional stability of the F-LOD-

NFDTD method. The 3-D curved microwave structures have also been analysed with the proposed F-LOD-NFDTD CPML. Good agreement between F-LOD-NFDTD CPML and C-LOD-NFDTD CPML has been obtained. Computational performance in terms of CPU time and memory requirements (as shown in Table 7.6) has been provided to illustrate the significance of the nonorthogonal 3-D LOD-FDTD approaches. The main contribution of Chapter 6 of this thesis is the development of the 3-D C-LOD-NFDTD CPML, and F-LOD-NFDTD CPML. Numerical results on 3-D microwave curved structures obtained by C-LOD-NFDTD and F-LOD-NFDTD methods are presented to illustrate the usefulness of the proposed approaches. Finally, using the analytical solution for the calculation of the conducting sphere as reference [156], the maximum errors of the C-LOD-FDTD and F-LOD-FDTD using orthogonal and non-orthogonal meshes are shown in Table 7.7. From the comparison of maximum errors, it can be concluded that the LOD-FDTD with non-orthogonal mesh has improved performance over LOD-FDTD with orthogonal meshes.

Table 7.6
Comparison of CPU time and memory for various 3-D curved structures

| Scheme |  | Conducting Sphere | Dielectric Sphere |
| :---: | :---: | :---: | :---: |
| F-LOD- | Execution time (min) | 4.5 | 11.5 |
| NFDTD | Memory (MB) | 18 | 34.5 |
| C-LOD- | Execution time (min) | 6.5 | 19.8 |
| NFDTD | Memory (MB) | 28 | 36.2 |

Table 7.7
Comparison of the maximum errors in RCS calculation of conducting sphere using C-LOD-FDTD and F-LOD-FDTD using orthogonal and non-orthogonal meshes

| CFLN | \% Maximum error of C-LOD- <br> FDTD Method |  | \% Maximum error of F-LOD- <br> FDTD Method |  |
| :--- | :---: | :---: | :---: | :---: |
|  | Orthogonal <br> meshes | Non-orthogonal <br> meshes | Orthogonal <br> meshes | Non-orthogonal <br> meshes |
| 8 | 0.345 | 0.0845 | 0.215 | 0.0615 |
| 10 | 0.423 | 0.0923 | 0.272 | 0.082 |
| 258 |  |  |  |  |

Papers based on these contributions will be submitted to IEEE Transactions soon.

### 7.3 Suggestions for Future Work

The work from this thesis could be extended in the future. One is the application of a fundamental scheme with D-RS-LOD-FDTD to analyse the electrically large rotationally symmetric antennas and scatterers. The developed 3-D LOD-FDTD CPML could be used to analyse more complex problems. Higher order schemes can also be investigated for the F-LOD-FDTD to reduce the dispersion error. More complex 3-D curved structures could be analysed using the developed F-LOD-NFDTD method. Numerical dispersion analysis for the C-LOD-NFDTD and F-LOD-NFDTD could also be studied. Higher order scheme or dispersion control parameters scheme can also be developed for reducing the dispersion of the 3-D nonorthogonal LOD-FDTD approaches.

## Appendix A

## A1. Numerical Formulation of the 2-D ADI-FDTD Method Using

## Orthogonal Meshes:

For a 2-D TE Wave:
The numerical formulation of the ADI-FDTD method for the 2D TE and TM wave are presented below which have been used in each segment of the S-ADI-FDTD method in Chapter 2. These formulations are available for an inhomogeneous lossy medium and when uniform cells are used. Two procedures are used to calculate one discrete timestep. The updating equations for the TE wave are presented first.

Sub-step 1:

$$
\left.\left.\begin{array}{rl}
\left.E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}= & \left.C(i+1 / 2, j) E_{x}\right|_{i+1 / 2, j} ^{n} \\
& \quad+D_{b}(i+1 / 2, j)\left\{\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right\}
\end{array}\right\} \begin{array}{rl}
\left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}= & \left.C(i, j+1 / 2) E_{y}\right|_{i, j+1 / 2} ^{n} \\
& \quad-D_{a}(i, j+1 / 2)\left\{\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right\}
\end{array}\right\} \begin{aligned}
\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}= & \left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& +B_{b}(i+1 / 2, j+1 / 2)\left\{\left.E_{x}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}\right\} \\
& \quad-B_{a}(i+1 / 2, j+1 / 2)\left\{\left.E_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}\right\}
\end{aligned}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E_{x}\right|_{i+1 / 2, j} ^{n+1}=\left.C(i+1 / 2, j) E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2} \\
& +D_{b}(i+1 / 2, j)\left\{\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right\}  \tag{A.2a}\\
& \left.E_{y}\right|_{i, j+1 / 2} ^{n+1}=\left.C(i, j+1 / 2) E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2} \\
& -D_{a}(i, j+1 / 2)\left\{\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right\}  \tag{A.2b}\\
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \\
& +B_{b}(i+1 / 2, j+1 / 2)\left\{\left.E_{x}\right|_{i+1 / 2, j+1} ^{n+1}-\left.E_{x}\right|_{i+1 / 2, j} ^{n+1}\right\}  \tag{A.2c}\\
& -B_{a}(i+1 / 2, j+1 / 2)\left\{\left.E_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}\right\}
\end{align*}
$$

where

$$
\begin{aligned}
& B_{a}(i, j)=\frac{\Delta t}{\mu(i, j)} \frac{1}{\Delta x(i)}, B_{b}(i, j)=\frac{\Delta t}{\mu(i, j)} \frac{1}{\Delta y(j)}, \\
& C(i, j)=\frac{\varepsilon(i, j)}{\varepsilon(i, j)+\sigma(i, j) \Delta t}, D_{a}(i, j)=\frac{\Delta t}{\varepsilon(i, j)+\sigma(i, j) \Delta t} \frac{1}{\Delta x(i)} \\
& D_{b}(i, j)=\frac{\Delta t}{\varepsilon(i, j)+\sigma(i, j) \Delta t} \frac{1}{\Delta y(j)}
\end{aligned}
$$

In sub-step 1, (A.1b) and (A.1c) cannot be used for direct numerical calculation, thus, (A.3) is derived from (A.1b) and (A.1c) by eliminating the $\left.H_{z}\right|^{n+1 / 2}$ components as follows:

$$
\begin{align*}
& \left.\alpha E_{y}\right|_{i-1, j+1 / 2} ^{n+1 / 2}+\left.\beta E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}+\left.\gamma E_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2} \\
& =\left.C(i, j+1 / 2) E_{y}\right|_{i, j+1 / 2} ^{n}+D_{a}(i, j+1 / 2)\left[\left\{\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}\right\}\right. \\
& \quad-B_{b}(i+1 / 2, j+1 / 2)\left\{\left.E_{x}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}\right\}  \tag{A.3}\\
& \\
& \left.\quad+B_{b}(i-1 / 2, j+1 / 2) .\left\{\left.E_{x}\right|_{i-1 / 2, j+1} ^{n}-\left.E_{x}\right|_{i-1 / 2, j} ^{n}\right\}\right]
\end{align*}
$$

where $\alpha=-B_{a}(i-1 / 2, j+1 / 2) D_{a}(i, j+1 / 2), \gamma=-B_{a}(i+1 / 2, j+1 / 2) D_{a}(i, j+1 / 2)$
$\beta=1-\alpha-\gamma$
In the sub-step 2, (A.2a) and (A.2c) cannot be used for direct numerical calculation, thus, (A.4) is derived from (A.2a) and (A.2c) by eliminating the $\left.H_{z}\right|^{n+1 / 2}$ components as follows:

$$
\begin{align*}
& \left.\alpha E_{x}\right|_{i+1 / 2, j-1} ^{n+1}+\left.\beta E_{x}\right|_{i+1 / 2, j} ^{n+1}+\left.\gamma E_{x}\right|_{i+1 / 2, j+1} ^{n+1} \\
& =\left.C(i+1 / 2, j) E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}+D_{b}(i+1 / 2, j)\left[\left\{\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}\right\}\right. \\
& \quad-B_{a}(i+1 / 2, j+1 / 2)\left\{\left.E_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}\right\}  \tag{A.4}\\
& \\
& \left.\quad+B_{a}(i+1 / 2, j-1 / 2)\left\{\left.E_{y}\right|_{i+1, j-1 / 2} ^{n+1 / 2}-\left.E_{y}\right|_{i, j-1 / 2} ^{n+1 / 2}\right\}\right]
\end{align*}
$$

where $\alpha=-B_{b}(i+1 / 2, j-1 / 2) D_{b}(i+1 / 2, j), \gamma=-B_{b}(i+1 / 2, j+1 / 2) D_{b}(i+1 / 2, j)$
$\beta=1-\alpha-\gamma$

## For a 2-D TM Wave:

The numerical formulation of the ADI-FDTD method for a 2D TM wave is presented below. The calculation is made in the same way as in the case of the TE wave as follows:

Sub-step 1:

$$
\begin{gather*}
\left.H_{x}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.H_{x}\right|_{i, j+1 / 2} ^{n}-B_{b}(i, j+1 / 2)\left\{\left.E_{z}\right|_{i, j+1} ^{n}-\left.E_{z}\right|_{i, j} ^{n}\right\}  \tag{A.5a}\\
\left.H_{y}\right|_{i+1 / 2, j} ^{n+1 / 2}=\left.H_{y}\right|_{i+1 / 2, j} ^{n}+B_{a}(i+1 / 2, j)\left\{\left.E_{z}\right|_{i+1, j} ^{n+1 / 2}-\left.E_{z}\right|_{i, j} ^{n+1 / 2}\right\}  \tag{A.5b}\\
\left.E_{z}\right|_{i, j} ^{n+1 / 2}=\left.C(i, j) \cdot E_{y}\right|_{i, j} ^{n} \\
\quad+D_{a}(i, j)\left\{\left.H_{y}\right|_{i+1 / 2, j} ^{n+1 / 2}-\left.H_{y}\right|_{i-1 / 2, j} ^{n+1 / 2}\right\}  \tag{A.5c}\\
\quad-D_{b}(i, j)\left\{\left.H_{x}\right|_{i, j+1 / 2} ^{n}-\left.H_{x}\right|_{i, j-1 / 2} ^{n}\right\}
\end{gather*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.H_{x}\right|_{i, j+1 / 2} ^{n+1}=\left.H_{x}\right|_{i, j+1 / 2} ^{n+1 / 2}-B_{b}(i, j+1 / 2)\left\{\left.E_{z}\right|_{i, j+1} ^{n+1}-\left.E_{z}\right|_{i, j} ^{n+1}\right\}  \tag{A.6a}\\
& \left.H_{y}\right|_{i+1 / 2, j} ^{n+1}=\left.H_{y}\right|_{i+1 / 2, j} ^{n+1 / 2}+B_{a}(i+1 / 2, j)\left\{\left.E_{z}\right|_{i+1, j} ^{n+1 / 2}-\left.E_{z}\right|_{i, j} ^{n+1 / 2}\right\}  \tag{A.6b}\\
& \left.E_{z}\right|_{i, j} ^{n+1}=\left.C(i, j) \cdot E_{y}\right|_{i, j} ^{n+1 / 2} \\
& \quad+D_{a}(i, j)\left\{\left.H_{y}\right|_{i+1 / 2, j} ^{n+1}-\left.H_{y}\right|_{i-1 / 2, j} ^{n+1}\right\}  \tag{A.6c}\\
& \quad-D_{b}(i, j)\left\{\left.H_{x}\right|_{i, j+1 / 2} ^{n+1}-\left.H_{x}\right|_{i, j-1 / 2} ^{n+1}\right\}
\end{align*}
$$

Unlike (A.1b) and (A.1c), the equations of sub-step 1 and 2 of the $\mathrm{TM}_{\mathrm{z}}$ case cannot be used for direct numerical calculation. Placing (A.5b) into (A.5c) of sub-step 1 and (A.6a) into (A.6c) of sub-step 2 yields the simultaneous linear equations $\left.E_{z}\right|_{i, j} ^{n+1 / 2}$ and $\left.E_{z}\right|_{i, j} ^{n+1}$ that result in the tri-diagonal matrix form which can be solved by using Approach A) that has been described in Section 2.3.2.

## A2. Numerical Formulation of the 2-D ADI-FDTD Method with CPML Absorbing Boundary Conditions

The CPML equations for the 2-D ADI-FDTD method for the $\mathrm{TE}_{\mathrm{z}}$ case are formulated into two sub-steps which are shown below.

Sub-step 1:

$$
\begin{align*}
\left.E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}=C(i+1 / & 2, j)\left.E_{x}\right|_{i+1 / 2, j} ^{n}+1 / \kappa_{y_{j}} D_{b}(i+1 / 2, j) \\
& \left\{\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right\}+\left.C_{\psi_{e x}} \psi_{e_{x y}}\right|_{i+1 / 2, j} ^{n} \tag{A.7a}
\end{align*}
$$

$$
\begin{align*}
& \left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.C(i, j+1 / 2) E_{y}\right|_{i, j+1 / 2} ^{n}-1 / \kappa_{x_{i}} D_{a}(i, j+1 / 2) . \\
& \qquad\left\{\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right\}+\left.C_{\psi_{e y}} \psi_{e_{y x}}\right|_{i, j+1 / 2} ^{n}  \tag{A.7b}\\
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& +1 / \kappa_{y_{j+1 / 2}} B_{b}(i+1 / 2, j+1 / 2)\left\{\left.E_{x}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{x}\right|_{i+1 / 2, j} ^{n}\right\} \\
& \quad-1 / \kappa_{x_{i+1 / 2}} B_{a}(i+1 / 2, j+1 / 2)\left\{\left.E_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}\right\}  \tag{A.7c}\\
& \quad-\left.C_{\psi_{h z}} \psi_{h_{z x}}\right|_{i+1 / 2, j+1 / 2} ^{n}+\left.C_{\psi_{h z}} \psi_{h_{z y}}\right|_{i+1 / 2, j+1 / 2} ^{n}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E_{x}\right|_{i+1 / 2, j} ^{n+1}=\left.C(i+1 / 2, j) E_{x}\right|_{i+1 / 2, j} ^{n+1 / 2}+1 / \kappa_{y_{j}} D_{b}(i+1 / 2, j) . \\
& \quad\left\{\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right\}+\left.C_{\psi_{e x}} \psi_{e_{x y}}\right|_{i+1 / 2, j} ^{n+1 / 2}  \tag{A.8a}\\
& \left.E_{y}\right|_{i, j+1 / 2} ^{n+1}=\left.C(i, j+1 / 2) E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}-1 / \kappa_{x_{i}} D_{a}(i, j+1 / 2) . \\
& \left\{\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{z}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right\}+\left.C_{\psi_{e y}} \psi_{e_{y x}}\right|_{i, j+1 / 2} ^{n+1 / 2}  \tag{A.8b}\\
& \left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \\
& +1 / \kappa_{y_{j+1 / 2}} B_{b}(i+1 / 2, j+1 / 2)\left\{\left.E_{x}\right|_{i+1 / 2, j+1} ^{n+1}-\left.E_{x}\right|_{i+1 / 2, j} ^{n+1}\right\} \\
& -1 / \kappa_{x_{i+1 / 2}} B_{a}(i+1 / 2, j+1 / 2)\left\{\left.E_{y}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{y}\right|_{i, j+1 / 2} ^{n+1 / 2}\right\}  \tag{A.8c}\\
& \quad-\left.C_{\psi_{h z}} \psi_{h_{z x}}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}+\left.C_{\psi_{h z}} \psi_{h_{z y}}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \tag{A.9a}
\end{align*}
$$

where $\left.\quad \psi_{e_{x y}}\right|_{i+1 / 2, j} ^{n}=\left.b_{r} \psi_{e_{x y}}\right|_{i+1 / 2, j} ^{n-1 / 2}+a_{r}\left(\left.H_{z}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{z}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)$

$$
\begin{gather*}
\left.\psi_{h_{z y}}\right|_{i+1 / 2, j+1 / 2} ^{n}=b_{r} \psi_{h_{z y}} n_{i+1 / 2, j+1 / 2}^{n-1 / 2}+a_{r}\left(\left.E_{z}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{z}\right|_{i+1 / 2, j} ^{n}\right)  \tag{A.9b}\\
b_{r}=e^{-\left(\left(\sigma_{r} / \kappa_{r}\right)+\alpha_{r}\right)\left(\Delta t / \varepsilon_{0}\right)}  \tag{A.9c}\\
a_{r}=\frac{\sigma_{r}}{\kappa_{r}\left(\sigma_{r}+\kappa_{r} \sigma_{r}\right)} \times\left(b_{r}-1\right), \quad(\mathrm{r}=x, y) \tag{A.9d}
\end{gather*}
$$

Here, subscripts $e$ and $h$ indicate the coefficients for the electric and magnetic fields. $\psi_{h x y}, \psi_{h y x}, \psi_{e z x}$ and $\psi_{e z y}$ are discrete variables that have non-zero values only in some CPML regions and are necessary for the implementation of the absorbing boundary. In a similar way the updating equations of the 2-D ADI-FDTD CPML for the TM case can be derived.

## A3. Numerical Formulation of the 2-D ADI-FDTD Method Using Nonorthogonal Meshes:

For a 2-D TE Wave:
To simplify the problem, it is assumed that the wave propagates is in an isotropic lossy medium. In the 2-D ADI-NFDTD formulation, each explicit time step is changed to implicit time step using two procedures as follows:

Sub-step 1:

$$
\begin{align*}
& \left.E^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}=\left.a E^{1}\right|_{i+1 / 2, j} ^{n} \\
& +\frac{b}{\Delta u^{2}}\left\{\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right\}  \tag{A.7a}\\
& \left.E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.a E^{2}\right|_{i, j+1 / 2} ^{n} \\
& -\frac{b}{\Delta u^{1}}\left\{\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right\}  \tag{A.7b}\\
& \left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.c H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& +\frac{d}{\Delta u^{2}}\left\{\left.E_{1}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{1}\right|_{i+1 / 2, j} ^{n}\right\}  \tag{A.7c}\\
& -\frac{d}{\Delta u^{1}}\left\{\left.E_{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{2}\right|_{i, j+1 / 2} ^{n+1 / 2}\right\}
\end{align*}
$$

Sub-step 2:

$$
\left.\begin{align*}
& \left.E^{1}\right|_{i+1 / 2, j} ^{n+1}=\left.a E^{1}\right|_{i+1 / 2, j} ^{n+1 / 2} \\
&  \tag{A.8a}\\
& \quad+\frac{b}{\Delta u^{2}}\left\{\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right\}
\end{aligned} \begin{aligned}
&\left.E^{2}\right|_{i, j+1 / 2} ^{n+1}=\left.a E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2} \\
&-\frac{b}{\Delta u^{1}}\left\{\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right\} \tag{A.8b}
\end{align*} H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.c H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} .
$$

$$
\begin{gathered}
\text { where, } a=\left(4 \varepsilon-\sigma_{e} \Delta t\right) /\left(4 \varepsilon+\sigma_{e} \Delta t\right), b=2 \Delta t / \sqrt{g}\left(4 \varepsilon+\sigma_{e} \Delta t\right), \\
c=\left(4 \mu-\sigma_{m} \Delta t\right) /\left(4 \mu+\sigma_{m} \Delta t\right), d=2 \Delta t / \sqrt{g}(4 \mu+\sigma \Delta t)
\end{gathered}
$$

The covariant $E_{m}, H_{m}$ and contravariant $E^{m}, H^{m}(m=1,2,3)$, together with $g$, are all defined in [23] and this can also be calculated using (3.4). The relationship between covariant fields $H_{m}$, and contra-variant fields $H^{m}(m=1,2,3)$ are given by $H_{m}=g_{m 1} H^{1}+g_{m 2} H^{2}+g_{m 3} H^{3}$ and $H^{m}=g^{m 1} H_{1}+g^{m 2} H_{2}+g^{m 3} H_{3}$, where $g_{m l}$ and $g^{m l}$ ( $m, l=1,2,3$ ), are tensors defined in (3.4). A similar relation holds for $E_{m}$ and $E^{m}$. Here $H^{m}$ and $E^{m}(m=1,2,3)$, are true fields components. For 2-D TEz case, $H_{3}=H^{3}$, the $\left.E^{2}\right|^{n+1 / 2},\left.H_{3}\right|^{n+1 / 2}$ in (A.7ba) and $\left.H^{3}\right|^{n+1 / 2},\left.E_{2}\right|^{n+1 / 2}$ in (A.7c) are defined as synchronous variables. Since (A.7b) cannot be calculated directly, simultaneous linear equations have to be formed from (A.7b) and (A.7c) by eliminating the synchronous variables $H_{3}{ }_{i+1 / 2, j+1 / 2}^{n+1 / 2},\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}$. Since $H_{3}=H^{3}$ for the 2-D TEz case, we can obtain the expression of the $\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1 / 2}$, from (A.7c). (A.7c) is placed in (A.7b), then, according to [23], the desired covariant field components are averaged by known contra-variant fields to give a second order accurate approximation

## For a 2-D TM Wave:

The numerical formulation of the ADI-NFDTD method for the 2D TM wave is presented below. The calculation is made in the same way as in the case of the TE wave as follows:

Sub-step 1:

$$
\begin{gather*}
\left.H^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}=\left.a H^{2}\right|_{i+1 / 2, j} ^{n}-\frac{b}{\Delta u^{2}}\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.E_{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)  \tag{A.9a}\\
\left.H^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.a H^{2}\right|_{i, j+1 / 2} ^{n}+\frac{b}{\Delta u^{1}}\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)  \tag{A.9b}\\
\left.E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.c E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
\quad+\frac{d}{\Delta u^{1}}\left(\left.H_{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i, j+1 / 2} ^{n+1 / 2}\right)-\frac{d}{\Delta u^{2}}\left(\left.H_{1}\right|_{i+1 / 2, j+1} ^{n}-\left.H_{1}\right|_{i+1 / 2, j} ^{n}\right) \tag{A.9c}
\end{gather*}
$$

Sub-step 2:

$$
\begin{gather*}
\left.H^{1}\right|_{i+1 / 2, j} ^{n+1}=\left.a H^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}-\frac{b}{\Delta u^{2}}\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.E_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right)  \tag{A.10a}\\
\left.H^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.a H^{2}\right|_{i, j+1 / 2} ^{n}+\frac{b}{\Delta u^{1}}\left(\left.E_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.E_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right)  \tag{A.10b}\\
\left.E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.c E^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \\
\quad-\frac{d}{\Delta u^{2}}\left(\left.H_{1}\right|_{i+1 / 2, j+1} ^{n+1}-\left.H_{1}\right|_{i+1 / 2, j} ^{n+1}\right)+\frac{d}{\Delta u^{1}}\left(\left.H_{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.H_{2}\right|_{i, j+1 / 2} ^{n+1 / 2}\right) \tag{A.10c}
\end{gather*}
$$

where $a, b, c, d$ are same as those mentioned previously. Similar to the TE case, the covariant and contra-variant together with $g$, are taken to be the same as those given in [23] and this can be calculated using (3.4). For $2-\mathrm{D} \mathrm{TM}_{\mathrm{z}}$ case, $E_{3}=E^{3}$, the $E_{3}, H^{2}$ in equation (A.9b) and $H_{2}$ and $E^{3}$ in equation (A.9c) are defined as synchronous variables. Since the equation (A.9c) is not directly solved, by placing (A.9c) in (A.9b), and as per [23], the desired covariant field components are averaged by known contra-variant fields to give a second order accurate approximation, leading to tri-diagonal matrix equation.

## A4. Numerical Formulation of the 2-D ADI-NFDTD Method with

## CPML Absorbing Boundary Conditions

The CPML equations for the 2-D ADI-FDTD method for the $\mathrm{TE}_{z}$ case are formulated into two sub-steps which are shown below.

Sub-step 1:

$$
\begin{align*}
& \left.E^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}=\left.a E^{1}\right|_{i+1 / 2, j} ^{n}+1 / \kappa_{u_{j}^{2}} \frac{b}{\Delta u^{2}}  \tag{A.11a}\\
& \qquad\left\{\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right\}+\left.C_{\psi_{e^{1}}} \psi_{e_{12}}\right|_{i+1 / 2, j} ^{n} \\
& \left.E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}=\left.a E^{2}\right|_{i, j+1 / 2} ^{n}-1 / \kappa_{u_{i}^{1}} \frac{b}{\Delta u^{1}}  \tag{A.11b}\\
& \qquad\left\{\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right\}+\left.C_{\psi_{e^{2}}} \psi_{e_{21}}\right|_{i, j+1 / 2} ^{n}
\end{align*}
$$

$$
\begin{align*}
& \left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}=\left.c H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n} \\
& +1 / \kappa_{u^{2}{ }_{j+1 / 2}} \frac{d}{\Delta u^{2}}\left\{\left.E_{1}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{1}\right|_{i+1 / 2, j} ^{n}\right\} \\
& \quad-1 / \kappa_{u_{i+1 / 2}^{1}} \frac{d}{\Delta u^{1}}\left\{\left.E_{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{2}\right|_{i, j+1 / 2} ^{n+1 / 2}\right\}  \tag{A.11c}\\
& \quad-\left.C_{\psi_{h^{3}}} \psi_{h_{31}}\right|_{i+1 / 2, j+1 / 2} ^{n}+\left.C_{\psi_{h^{3}}} \psi_{h_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n}
\end{align*}
$$

Sub-step 2:

$$
\begin{align*}
& \left.E^{1}\right|_{i+1 / 2, j} ^{n+1}=\left.a E^{1}\right|_{i+1 / 2, j} ^{n+1 / 2}+1 / \kappa_{u^{2}{ }_{j}} \frac{b}{\Delta u^{2}} \\
& \left\{\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n+1}\right\}+\left.C_{\psi_{e^{\prime}}} \psi_{e_{12}}\right|_{i+1 / 2, j} ^{n+1 / 2}  \tag{A.12a}\\
& \left.E^{2}\right|_{i, j+1 / 2} ^{n+1}=\left.a E^{2}\right|_{i, j+1 / 2} ^{n+1 / 2}-1 / \kappa_{u_{i}^{1}} \frac{b}{\Delta u^{1}}  \tag{A.12b}\\
& \left\{\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}-\left.H_{3}\right|_{i-1 / 2, j+1 / 2} ^{n+1 / 2}\right\}+\left.C_{\psi_{e^{2}}} \psi_{e_{21}}\right|_{i, j+1 / 2} ^{n+1 / 2} \\
& \left.H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1}=\left.c H^{3}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2} \\
& +1 / \kappa_{u^{2}{ }_{j+1 / 2}} \frac{d}{\Delta u^{2}}\left\{\left.E_{1}\right|_{i+1 / 2, j+1} ^{n+1}-\left.E_{1}\right|_{i+1 / 2, j} ^{n+1}\right\} \\
& -1 / \kappa_{u_{i+1 / 2}^{1}} \frac{d}{\Delta u^{1}}\left\{\left.E_{2}\right|_{i+1, j+1 / 2} ^{n+1 / 2}-\left.E_{2}\right|_{i, j+1 / 2} ^{n+1 / 2}\right\}  \tag{A.12c}\\
& -\left.C_{\psi_{h^{3}}} \psi_{h_{31}}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}+\left.C_{\psi_{h^{3}}} \psi_{h_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n+1 / 2}
\end{align*}
$$

where $b=2 \Delta t / k(j) \sqrt{g}\left(4 \varepsilon+\sigma_{e} \Delta t\right), C_{\psi_{e^{\prime}}}=\Delta t / k(j) \sqrt{g}\left(4 \varepsilon+\sigma_{e} \Delta t\right) d u^{2}$

$$
\begin{align*}
& d=2 \Delta t / k(j) \sqrt{g}\left(4 \mu+\sigma_{m} \Delta t\right), C_{\psi_{h^{3}}}=\Delta t / k(j) \sqrt{g}\left(4 \mu+\sigma_{m} \Delta t\right) d u^{1} \\
& \left.\psi_{e_{12}}\right|_{i+1 / 2, j} ^{n}=\left.b_{r} \psi_{e_{1},}\right|_{i+1 / 2, j} ^{n-1 / 2}+a_{r}\left(\left.H_{3}\right|_{i+1 / 2, j+1 / 2} ^{n}-\left.H_{3}\right|_{i+1 / 2, j-1 / 2} ^{n}\right)  \tag{A.13a}\\
& \left.\psi_{h_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n}=\left.b_{r} \psi_{h_{32}}\right|_{i+1 / 2, j+1 / 2} ^{n-1 / 2}+a_{r}\left(\left.E_{1}\right|_{i+1 / 2, j+1} ^{n}-\left.E_{1}\right|_{i+1 / 2, j} ^{n}\right) \tag{A.13b}
\end{align*}
$$

Unlike (A.7b) and (A.7c), (A.11b) and (A.11c) as well as (A.12a) and (A.12c) cannot be used for direct numerical calculation. Placing (A.11c) in (A.11b) and (A.12c) in (A.12a) yields the simultaneous linear equations that result in the tri-diagonal matrix.
In a similar way the updating equations of the 2-D ADI-NFDTD CPML for the TM case can be derived.

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