



STUDY OF SOME FUNCTIONALS OF STANDARD AND
FRACTIONAL BROWNIAN MOTIONS WITH
APPLICATIONS IN QUANTITATIVE FINANCE AND
STATISTICS

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Abstract

This thesis contains results on two important problems arising in quantitative finance and statistics.

The first problem is about option pricing with a volume weighted average price (VWAP) as an underlying process. The VWAP is a very important quantity in finance; it appears in Australian taxation law to specify the price of share-buybacks for publicly-listed companies and it is a standard benchmark price for market participants. Pricing options on VWAP is a challenging problem from a mathematical point of view because it involves two sources of randomness: the price of the asset and its traded volume. To solve the problem we have applied the moment-matching approach to a range of "stock and volume" models and as a result obtained an accurate approximation for prices of "call" options. All results have been verified by intensive Monte Carlo simulation.

The second problem is concerned with study of analytical properties and simulation algorithms for a fractional Brownian motion (fBM) which is considered a good alternative to modelling stochastic processes with long range dependence in modern Mathematical finance. In particular, we have reviewed the known simulation algorithms and have implemented the fastest of them ("circulant embedding") on a modern multicore computer. We applied the algorithm to two longstanding open problems in statistics, namely, to study distributions of exponential functionals of fBM and the maximum of fBM. The results of our simulations exhibit new and striking properties of these distributions.

Contents

Chapter 1	Introduction	2
Chapter 2	Mathematical background	3
2.1	Basic tools from probability theory	3
2.1.1	Conditional expectation	3
2.1.2	Multivariate normal distribution	8
2.2	Stochastic processes	13
2.2.1	Brownian motion	14
2.3	Stochastic Integration	14
2.4	Stochastic differential equations	29
2.4.1	Geometric Brownian motion	30
2.4.2	The Ornstein-Uhlenbeck process	32

I Options with volume-weighted average price (VWAP) as underlying **35**

Chapter 3	Introduction	36
3.1	Overview	36
3.2	Existing literature	36
3.2.1	Volume weighted average price (VWAP)	36
3.2.2	Asian options	37
3.2.3	Options on VWAP	39
3.3	Summary of the contributions of part I	39

Chapter 4	The VWAP model and calculation of its moments	40
4.1	The volume process	41
4.2	Calculation of VWAP moments	43
4.3	Derivation of the Laplace transforms	45
4.4	Integrability of VWAP moments	51
Chapter 5	Pricing options with VWAP as underlying	55
5.1	Pricing by the moment-matching method	55
Chapter 6	Numerical results	59
6.1	Implementation of the moment-matching pricing scheme	59
6.1.1	Symbolic algebra with Mathematica	59
6.2	Verification by Monte Carlo simulation	60
6.2.1	Basics of Monte Carlo simulation	60
6.2.2	Simulating stochastic processes	64
6.3	Results	64
II	The distribution of maximum of fractional Brownian motion and asymptotic variance of optimal Bayesian estimators	68
Chapter 7	Introduction	69
7.1	Overview	69
7.2	The problem settings and existing literature	70
7.2.1	Distribution of the maximum of fBm	70
7.2.2	Variance of the limiting distribution of certain Pitman estimators	70
7.3	Summary of the contributions of part II	73
Chapter 8	Fractional Brownian motion	74
8.1	Basic properties	74
8.2	Simulation of fBm	75
8.2.1	Exact simulation methods	75

8.2.2	Approximate simulation methods	80
Chapter 9	Parallel Computation	82
9.1	Basic Computer Architecture	82
9.1.1	The “Power Wall”	83
9.1.2	The Memory System	84
9.2	Parallel architectures	86
9.2.1	Flynn’s taxonomy	86
9.2.2	Shared and distributed memory MIMD architectures	87
9.3	Concepts of parallel programming	88
9.3.1	Load balancing	88
9.3.2	Synchronisation	88
9.4	Common approaches to parallel programming	89
9.5	Amdahl’s law	90
Chapter 10	Parallel implementation of Monte Carlo schemes	92
10.1	Simulation of realisations for problems 1 and 2	92
10.2	Implementation details	94
10.2.1	Third-party libraries used	94
10.2.2	Thread safety	94
10.2.3	Overview of the main classes	95
Chapter 11	Empirical results for certain functionals of fractional Brownian motion	97
11.1	Results for problem 1	97
11.2	Results for problem 2	98
11.3	Conjectures	99
III	Appendices	101
Appendix A	VWAP calculations	102
A.1	Calculations for model 1	102

Appendix B Extra mathematical results	105
B.1 Some results from mathematical analysis	105
B.2 Some more results from measure theory	105
B.3 Some more results from probability theory	105
Appendix C Code listings	111
C.1 VWAP First Moment	111
C.2 VWAP Monte Carlo C++ code	113
C.3 Fractional Brownian Motion Monte Carlo C++ code	119
C.3.1 Code to generate fractional Brownian motion trajectories	119
C.3.2 Code to estimate $\text{var}(\text{zeta})$	124
C.3.3 Code to estimate probMax	126
C.3.4 WorkDistributor code	129
C.3.5 FBMSimulationData code	130
References	133

List of Tables

4.1	Market impact example	40
6.1	Numerical comparisons for first VWAP moment.	65
6.2	Numerical comparisons for second VWAP moment.	65
11.1	Standard errors for $T^{1-H}\mathbb{P}\left(\max_{0\leq t\leq T} W_t^{(H)} < 1\right)$	97

List of Figures

6.1 Comparison of VWAP call option prices	66
6.2 Relative error of VWAP call option prices for $K = 100$	66
6.3 Relative error of VWAP call option prices for $K = 110$	67
9.1 Memory Hierarchy	85
9.2 An example of a simple cache.	85
9.3 Shared memory microprocessor with two processors	87
10.1 Parallel implementation of crude Monte Carlo simulation	93
11.1 $T^{1-H}\mathbb{P}\left(\max_{0\leq t\leq T} W_t^{(H)} < 1\right)$ for $H = 0.40$	98
11.2 $T^{1-H}\mathbb{P}\left(\max_{0\leq t\leq T} W_t^{(H)} < 1\right)$ for $H = 0.75$	98
11.3 $T^{1-H}\mathbb{P}\left(\max_{0\leq t\leq T} W_t^{(H)} < 1\right)$ for $H = 0.95$	99
11.4 $\text{var}(\zeta_H)$ for a range of H values. Region of integration was truncated to [-10000,10000].	99

CERTIFICATE OF ORIGINAL AUTHORSHIP

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Signature of Student:

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