Fractal Algorithm for Finding Global Optimal Solution

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Abstract

For solving global optimization problems, a new algorithm, which is called Fractal Algorithm, is presented. Feasible region is partitioned by fractal combining with golden section. Bad region is deleted gradually and finally optimal solution is remained. The full advantage of the local fine structure of fractal and the quick convergence of golden section method were taken. Hence it is high efficient and high speedy. The algorithm has following character: strong adaptability, adapting to a class of complex function. It needs only that the object function has one order derivative. The minimum can be found at any precision at which a computer can work. On the other hand, this method needs so little memory that it almost can be implemented on any personal computer and its efficiency is not almost influenced. The proof of convergence of the algorithm is given. The illustrations show the algorithm is effective.

1. Introduction

As a leading edge of branch of mathematics, fractal has become a hot studying topic. It has showed strong vitality in many fields such as physics, image graphics, biology etc[1]. Fractal also brings a great change for mathematics itself. It extends greatly the studying and application range of mathematics.

On the other hand, nonlinear problems have been a hot and difficult point of studying all the while. Because it is necessary for solving many engineering problems and control problems, it has been paid close attention to. Lots of algorithms for solving constraint and unrestricted optimal problems have been explored. Such as conjugate direction method, conjugate gradient method, variable metric algorithm and Powell direction speedup method etc[2] . These classical algorithms have played an important role in many practical problems and showed its particular advantages. However, because of the multiformity of the optimal problems, most of these methods have more or less shortage. For example, some of them give only local optimal solution, some require the object function has one or more order derivative and some is not effective. Recent years, there are lots of new algorithms appearing, for example, chaos genetic algorithm[3] , hybrid chaos algorithm [4], ant colony algorithm [5], interval-slope algorithm[6][7] , filled function method[8] and so on. Although they all are effective, they still have the shortage of calculating slow and requiring too much to object function. Otherwise, all they are too complex.

On studying fractal, it is found that fractal generally set up on simple mathematical foundation and shows complex geometrical character. It has unique advantage for solving nonlinear problem. Literature [9] discuss the two dimensional global optimization with fractal. But it deals with too large number of data. Here arising from the one-dimensional search method golden section and fractal, a new method for solving three-dimensional global optimization is presented, that is called as Fractal Algorithm (hereinafter it is shortened as FA), in which the feasible region is divided be fractal. Then the method of picking out the bad point and remaining the good point is taken. Resulting from this the final global solution is found. Numeric experiments show this is a quiet simple and effective method. Comparison with literature[9], not only the dimension is heightened but also the efficiency got increase. The following is the main characters. (1) It has strong adaptability. It need just the object function is one order derivative. (2) It has high precision. It can reach as high the precision as the computer can reach. (3) It needs little computer internal memories. It can be implemented in any personal computer. Finally, the proof of the convergence is given.

2. Basic Principle of FA and Algorithm

The Golden Section Method is a classical algorithm of optimization. It is well known for its simple and effective. It is the foundation of many other algorithms. But it only holds true for unimodal function in one-dimensional interval $[a, b]$. The basic idea is: according to the principle

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”take out bad and remain good”, the principle of symmetry and the principle of geometric proportion shrinking to reduce the search interval step by step. That is say, take $x_1 = a + 0.382(b - a)$, $x_2 = a + 0.618(b - a)$ in the interval $[a, b]$, if $f(x_1) > f(x_2)$, let $a = x_1$, then repeat this step. Or else, let $b = x_2$, then repeat this step. So every time the interval is reduce 0.382 or 0.618 time, until a point. This is a one-dimensional search method that has quite fast rate of convergence. Resulting from this method, here a three-dimensional search method is developed.

2.1 Basic Principle of FA

Because this method is an expending of Golden Section Method, it also has the same character: simple and quick. Besides this, FA improves the fore algorithm from finding only the local optimal solution of unimodal function to finding the global optimal solution of multimodal. The basic principle is: take arbitrary point in a feasible of cuboid:

$$\Omega = \{(x, y, z)|x_1 < x < x_2, y_1 < y < y_2, z_1 < z < z_2\} \tag{1}$$

as an initial optimal point (x^*, y^*, z^*) , and the value of the object function at the point as the initial optimal value f^* Then divide respectively the feasible region Ω in three directions of length, width and height at the position 0.382 and 0.618. As Fig.1 shows, so Ω is divided into 27 little cu-

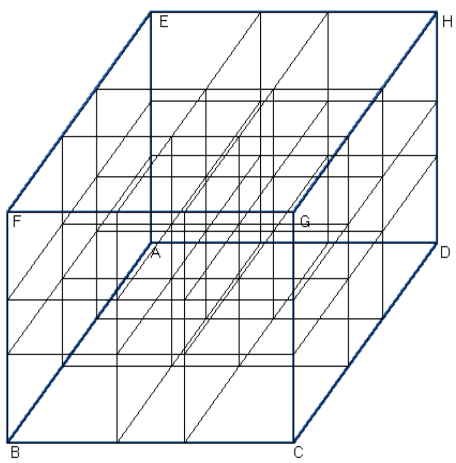

Figure 1. the division of feasible region

biods. Calculate the function value of the object function at the centroid of the little cuboid. If it is less than f^*, replace f^* with it and replace (x^*, y^*, z^*) with the centroid point. Then repeat the former step with the new little cuboid as new initial cuboid. Or else, make a simple estimate to the object function in the little cuboid, if the lower bound is greater than f^*,so we call the little cuboid is ”bad” one, remove the little cuboid. Else take the little cuboid (so it is called ”good” little cuboid.) as new initial cuboid and repeat again. Do this until the diameter of the little cuboid is less than a given precision. Finally, the remained point (x^*, y^*, z^*) will be the global optimal point, and the f^* will be the global optimal value.

2.2 Algorithm

Given the object function: $f(x, y, z) \in C^1$ $(x, y, z) \in \Omega \subset R^3$,$f(x, y, z)$ is lower bounded in Ω , the optimal problem is:

$$(x^*, y^*, z^*) = \arg \min_{(x,y,z)\in\subset\Omega} f(x, y, z) \tag{2}$$

Find the optimal solution of function $u = f(x, y, z)$ in region of cuboid $\Omega = \{a, b; c, d; e, f\}$. The following algorithm is taken:

step0: Given $\varepsilon > 0$, points a, b, c, d, e, f, g, h , from below to up arrange counterclockwise. e is above a. Take arbitrary point as initial optimal point (x^*, y^*, z^*) . Calculate $f^* = f(x^*, y^*, z^*)$;

Step1: If the diameter of cuboid is less than ε , stop. Else divide the cuboid into 27 little cuboids in three directions of length, width and height at position 0.382 and 0.618. Calculate the position of every node. (as show in Fig.1);

Step2: Calculate the function value of centroid of every little cuboid. If it less than f^* , go to step3. Else go to step4.

Step3: Assign the eight vertex of the new cuboid to a, b, c, d, e, f, g, h . Go to step1.

Step4: Estimate the value of object function in the little cuboid. If its lower bound is greater than f^* , remove it. Or else, assign the eight vertex of the cuboid to a, b, c, d, e, f, g, h . Go to step1.

Step5: print f^* and (x^*, y^*, z^*), .

3 The Proof of Convergence

Suppose the optimal problem is given in equation (2).

Set up the initial cuboid is $\Omega = \{a_1, b_1; c_1, d_1; e_1, f_1\}$. Where $a_1 < x < b_1$ is the range of x coordinate of the cuboid, $c_1 < y < d_1$ is the range of y coordinate of the cuboid, $e_1 < z < f_1$ is the range of z coordinate of the cuboid. The length of the cuboid is $b_1 - a_1$, the width is $d_1 - c_1$ and the height is $f_1 - e_1$. Set up the diameter of the cuboid is Φ_0 . So:

$$\Phi_0 = \sqrt{(b_1 - a_1)^2 + (d_1 - c_1)^2 + (f_1 - e_1)^2} \tag{3}$$

For the first time dividing the feasible region, the cuboid is divided into 27 little cuboids. According to dividing method mentioned above, every little cuboid has different diameter, but all of them are less than Φ_0 . I.e.:$\varphi_i^1 \leq \Phi_1 = 0.618\Phi_0 \quad (i = 1, 2, \ldots, 27)$ Then estimate the value of object function in every cuboid according to the basic principle mentioned above. Remove the bad little cuboids, remain the good cuboids. For the second time dividing, every good cuboids coming from last 27 cuboids is divided in to more little cuboids by the same way. The total number is less than 27^2. As well as the first time, all diameters of them φ_i^2 are less than the diameters of their parent cuboid 0.618 times. I.e.:$\varphi_i^2 = \Phi_2 \leq 0.618\Phi_1 \leq 0.618^2\Phi_0 \quad (i = 1, 2, \ldots, 27^2)$ Suppose after n times dividing, no more than 27^n cuboids are obtained. They satisfy $\varphi_i^n = \Phi_n \leq 0.618^n\Phi_0 \quad (i = 1, 2, \ldots, 27^n)$. After $n+1$ times dividing, no more than 27^{n+1} cuboids are obtained. Their diameters are $\varphi_i^{n+1} \quad (i = 1, 2, \ldots, 27^{n+1})$ Of cause all the diameters of the cuboids are less than that of the 0.618 times of their parent cuboids. That is: $\varphi_i^{n+1} \leq \Phi_{n+1} = 0.618\Phi_n \leq 0.618^{n+1}\Phi_0 \quad (i = 1, 2, \ldots, 27^{n+1})$. According to mathematical induction, for any n it is satisfied: $0 < \varphi_i^n \leq 0.618^n\Phi_0$. Sequentially, $\lim_{n\to\infty} \varphi_i^n = 0 \quad (i = 1, 2, \ldots)$.

On the other hand, at beginning arbitrary point in feasible region is taken as initial optimal point (x_0^*, y_0^*, z_0^*) . The value of the object function at the point is taken as initial optimal value f_0^* . Then, once a little cuboid is constructed, the value of the object function at the centriod is compared with f_0^*. If it is less than f_0^* , replace f_0^* with it and replace (x_0^*, y_0^*, z_0^*) with the centriod. After n times replacement, they are denoted as $(x_n^*, y_n^*, z_n^*)$and f_n^* . Where f_n^* satisfies $f_n^* > f_{n+1}^*$. Namely, f_n^* is monotone decreasing. Moreover according to the supposition,$f(x)$ is lower bounded in Ω . So f_n^* is convergence.

Finally, we prove f^* , the limit value of f_n^* , $\lim_{n\to\infty} f_n^* = f^*$, is the global optimal value. In fact, if there is a point (x_1, y_1, z_1) , at which the value of the object function is less than f^* , according to the continuity of the object function, there must exist a neighbourhood $U((x_1, y_1, z_1), \delta)$ in which the object function value of all point are less than f^* . Taking note of that diameter of little cuboid is go to 0, so there must exist a little cuboid which whole fall in the neighbourhood $U((x_1, y_1, z_1), \delta)$. So the value of the object function at the centroid of it is less than f^* . This is conflict with the fact that f^* is the minimum of the value of object function of the centroid of all little cuboids. Therefore f^* is the global optimal value.

4 Numerical Example

In the following examples resulting from the FA are given. They test the effective of FA.

Example 1 $F_1 = (x_1 - 0.3)^2 + (x_2 - 0.5)^2 + (x_3 - 0.7)^2$ $(-2 < x_i < 2)$ $i = 1, 2, 3$. x^*=0.299979; y^* =0.499658; z^*=0.700127. $f(x^*, y^*, z^*)$=1.33450E-7. The exact value is $f(0.3, 0.5, 0.7) = 0$. The search process of example 1 is shown in Fig.2. Example 2 $F_2 = (x_1 - x_2^2)^2 + (x_1 - 1)^2 +$

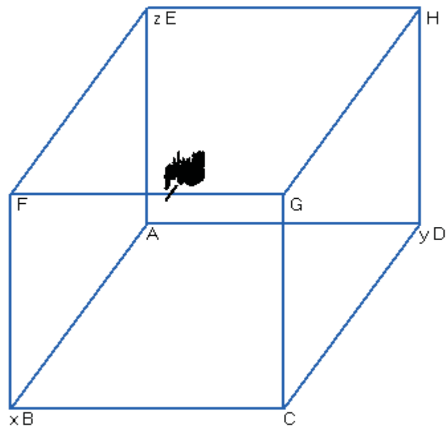

Figure 2. the search process of example 1

$(x_2 - x_3^2)^2 (-2 < x_i < 2)$ $i = 1, 2, 3$ x^*=0.999529; y^* =0. 999529; z^*=0. 999529. $f(x^*, y^*, z^*)$=6.6430578E-7. The exact value is $f(1, 1, 1) = 0$. The search process of example 2 is shown in Fig.3. Example 3

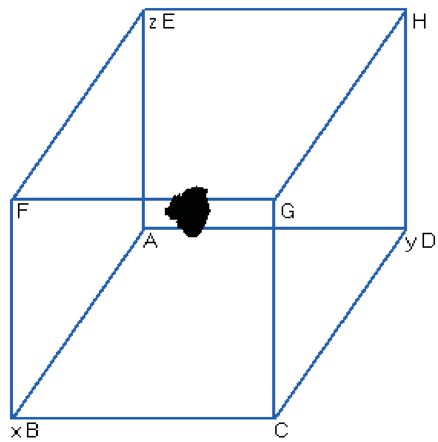

Figure 3. the search process of example 2

$F_3 = \frac{\sin^2\sqrt{x_1^2+x_2^2+x_3^2} - 0.5}{\sqrt{1+0.001(x_1^2+x_2^2+x_3^2)}} - 0.5 \qquad (-2.48 < x_i <$

2.48) $i = 1, 2, 3$ x^*=9.328297E-04; y^* =9.328297E-04; z^*=9.328297E-04. $f(x^*, y^*, z^*)$=.9999973. The exact value is $f(0,0,0) = -1$. The search process of example 3 is shown in Fig.4.

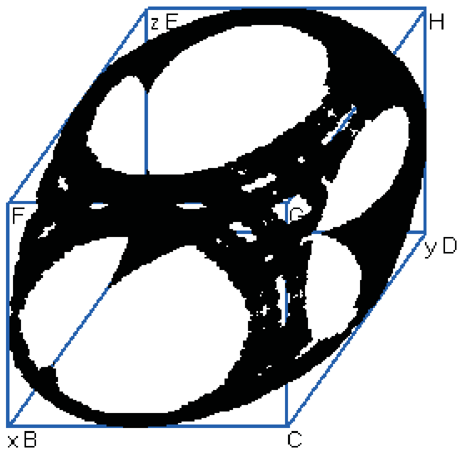

Figure 4. the search process of example 3

From the examples above it can be seen that FA is effective. The calculating precision can reach a high degree. And from diagrams of the search course it can be seen that FA searched only very little part of the whole feasible region. It is specially so for convex function. For instance example 1 and example 2 is so. For other object function whose value vibrate more greatly, the searched region will be larger than that for a convex function. For instant example 3 is so. Because it has periodicity in some meaning, so the feasible that was searched region is relatively larger. Moreover, here when remove a bad little cuboid, the one order Taylor formula is used. If more order Taylor formula is used, the feasible removed will be more than that now. That is we need search further little for getting the final result. But then the calculating amount will increase times. So the efficiency will not increases necessarily

5 Conclusion

The calculating result above show FA can find quiet precision optimal solution in three-dimensional space. It just needs that object function has one-order derivative. In fact, as long as the object function can be estimated in a little region it works. Moreover, FA needs very little computer internal memory. So FA can be applied broad, hence it is an algorithm that has great practicality. Further work is to extend this method to higher dimensional space.

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