A complete classification of spatial relations using the Voronoi-based 9-intersection model*

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Abstract

In this paper we show that the Voronoi-based 9-intersection model (V9I) proposed by Chen *et al.* (2001) is more expressive than what has been believed before. Given any two spatial entities A, B, the V9I relation between A and B is represented as a 3×3 Boolean matrix. For each pair of types of spatial entities *points*, *lines*, and *regions*, we first show that most Boolean matrices do not represent a V9I relation by using topological constraints and the definition of Voronoi regions, and then find illustrations for all the remaining matrices. This guarantees that our method is sound and complete. In particular, we show that there are 18 V9I relations between two areas with connected interior, while there are only nine 4-intersection relations. Our investigations also show that, unlike many other spatial relation models, V9I relations are *context-* or *shape-sensitive*. That is, the existence of other entities or the shape of the entities may affect the validity of certain relations.

Keywords: Voronoi region; 9-intersection; topological relation; qualitative spatial reasoning

1 Introduction

Construction of Voronoi diagrams of a finite system of points is very useful in computer graphics, geographical information systems, and many other research areas (Aurenhammer and Klein, 1996). Okabe *et al.* (2009) give a comprehensive introduction to various theories and applications of Voronoi diagrams. Since it is highly related to distance, Voronoi diagram has been used extensively in solving distance problems. For instance, Geng *et al.* (2011) and De Rezende and Westrupp (1999) discuss the Voronoi-based nearest neighbor problem, and Sud *et al.* (2006) study the fast

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Table 1: Numbers of V9I relations

Type	4-Int. Relations	V9I Relations	V9I Relations identified in (Chen *et al.*, 2001)
$(area, area)$	9	18	13
$(line, line)$	16	17	8
$(line, area)$	11	16	13
$(point, point)$	2	3	3
$(point, line)$	3	4	4
$(point, area)$	3	6	5

proximity computation. Voronoi diagram has also been used as a better representation of space between obstacles for path planning (Takahashi and Schilling, 1989; Bhattacharya and Gavrilova, 2008; Foskey *et al.*, 2001). Moreover, the distance under which Voronoi diagram is defined has been generalized in various ways. For example, the 'boat-sail' distance is proposed and generalized in (Sugihara, 1990, 2011), which apply Voronoi diagram to an anisotropic and dynamic problem. Another example is the A-distance in ambulance facility location problem studied by Hsia *et al.* (2009). Nowadays, Voronoi diagram has been used to help make energy efficient network architecture (Nojeong and Varshney, 2005) and to compute and improve the coverage of sensors (Meguerdichian *et al.*, 2001).

Chen *et al.* (2001) propose a Voronoi-based 9-intersection (V9I) model for describing spatial relations between entities in the real plane.[1] The idea is similar to the influential 9-Intersection Model (9IM) of Egenhofer and Herring (1991), but the exteriors of the two spatial entities are replaced with their Voronoi regions. Three types of spatial entities are considered in their paper, viz. (simple) points, (simple) lines, and regions with connected interior (i.e. simple regions with holes). Chen *et al.* (2001) argue that the replacement of exterior with Voronoi region can circumvent two imperfections of 9IM, viz. the *linear dependency* between a region's interior, boundary, and exterior, and the *inconsistency* of a line's topological property in $\mathbb{R}$ and in $\mathbb{R}^2$.

The authors further give a classification for the relations between area/line/point and area/line/point, where an area is represented by a simple region with holes, which has a connected interior, and lines and points are all simple entities. The idea of replacing exterior with Voronoi region in 9IM is novel and brilliant, but the classification made in Chen *et al.* (2001) is informal, sometimes imprecise, and far from complete.

In this paper, we revisit the V9I model proposed by Chen *et al.* (2001) and provide a formal and complete classification of spatial relations using the V9I model. Our results show (see Table 1) that there are more V9I relations than what have been identified by Chen *et al.* (2001). For each pair of types of spatial entities *points*, *lines*, and *regions*, we first show that most Boolean matrices do not represent a V9I relation by using topological constraints and the definition of Voronoi regions, and then provide illustrations for all the remaining matrices. This guarantees that our method is sound and complete.

Both 9IM and the V9I refine the 4-Intersection Model (4IM) (Egenhofer and Franzosa, 1991). The 9IM splits one 4IM relation into several different topological relations, but the V9I splits a

[1] According to Google Scholar, this work has been cited more than 100 times before December 20, 2012.

4IM relation in a different way. Very often, two different V9I relations that are contained in the same 4IM relation are topologically equivalent. This implies that these distinctions made by V9I cannot be obtained by 9IM. A closer examination shows that, unlike many other spatial relation models, V9I relations are *context-* or *shape-sensitive*. That is, the existence of other entities or the shape of the entities may affect the validity of certain relations.

In the remainder of this paper, we first introduce basic notions that will be used later in Section 2. These include basic topological terminologies, definition of Voronoi regions, and the definition of 9IM relations and the V9I relations. In Section 3, we examine for each pair of types of spatial entities how many V9I relations are there and provide illustrations (i.e. realizations) for each realizable matrix. We also explain here how V9I splits 4IM relations. In Section 4 we analyse how different definitions of Voronoi regions affect the V9I relations between spatial entities. The last section concludes the paper.

2 Basic Notions

In this paper, we are concerned with the two dimensional Euclidean plane $\mathbb{R}^2$, with the usual distance and topology. For two points p, q, we write $d(p,q)$ for the distance between p and q. The distance of p and a subset A of $\mathbb{R}^2$ is defined as

$$d(p,A) = \inf\{d(p,q) \mid q \in A\}. \tag{1}$$

For a point p and a positive real number ε, we write $B(p,\varepsilon)$ for the closed disk centred at p with radius ε, and write $B^\circ(p,\varepsilon)$ for the corresponding open disk. A set U in $\mathbb{R}^2$ is said *open* if for any point p in U, there exists $\varepsilon > 0$ such that $B^\circ(p,\varepsilon) \subset U$. A set A in $\mathbb{R}^2$ is said *closed* if its set complement in $\mathbb{R}^2$ is open. For a set X in $\mathbb{R}^2$, the *interior* of X, denoted as X°, is defined as the largest open set contained in X; the *closure* of X, denoted as $\overline{X}$, is defined as the smallest closed set that contains X; the *boundary* of X, denoted as ∂X, is defined as the set difference of $\overline{X}$ and X°. The *exterior* of X, denoted as X^e, is defined as the interior of the set complement of X. Note that if X is a closed set, then X^e is exactly the set complement of X.

In real-world applications, spatial entities are often represented as ‘points’, ‘lines’, or ‘areas’. A *point* is interpreted, as usual, as an element of $\mathbb{R}^2$. A *line* is a simple curve, i.e. a set in the plane that is topological equivalent to the closed interval [0,1]. A *simple area* is a set in the plane that is topologically equivalent to a closed disk. A simple area is also known as a *simple region* in e.g. (Egenhofer and Herring, 1991). In this paper, following Chen *et al.* (2001), we use a more general notion of area. Let X be a set in $\mathbb{R}^2$. Then X is an *area* if

(i) X is bounded, i.e. it is contained in some closed disk; and

(ii) X is regular closed, i.e. $X = \overline{X^\circ}$ is the closure of its interior; and

(iii) X° is connected, i.e. for any two disjoint open sets $U, V \neq \varnothing$, $U \cup V \neq X^\circ$.

Therefore, an area is actually a *simple region with holes* in (Egenhofer and Herring, 1991).

We next recall the 9-intersection model (9IM) of Egenhofer and Herring. Given two spatial entities A and B in the plane, the topological relation between A and B, written as $\mathbf{R}_9(A,B)$, is characterized by examining whether the nine intersections between the interiors, the boundaries, and the exteriors of A and B are empty or not. If an intersection is empty, then we put a '0' in the corresponding entry in the matrix, and a '1' otherwise. In this way, we represent the topological relation between A and B as a 3×3 Boolean matrix.

$$\mathbf{R}_9(A,B) = \left(\begin{array}{ccc} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^e \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^e \\ A^e \cap B^\circ & A^e \cap \partial B & A^e \cap B^e \end{array} \right). \tag{2}$$

The 9IM relations can be defined for any pair of points/lines/areas. For points and areas, the topological notions of interior, boundary, and exterior are interpreted as in classical topology, but for a line L, we regard it as an embedding in the one-dimensional $\mathbb{R}$. Therefore, each simple line L has two boundary points, viz. the two end points of L, and each non-boundary point of L is an interior point. Because $A = A^\circ \cup \partial A \neq \varnothing$, $B = B^\circ \cup \partial B \neq \varnothing$, and $A^\circ \cup \partial A \cup A^e = B^\circ \cup \partial B \cup B^e = \mathbb{R}^2$, it is clear that not all 3×3 Boolean matrices are valid 9IM relations. We refer the reader to (Egenhofer *et al.*, 1993) and (Li, 2006) for detailed discussion.

A simplified version of 9IM is obtained by considering the top left 2×2 sub-matrix of the matrix $\mathbf{R}_9(A,B)$ in (2). This is known as the 4-intersection model (Egenhofer and Franzosa, 1991), and such a relation will be addressed as a 4IM relation henceforth.

Let A be an area. Note that the exterior of A is the same as its set complement. This linear dependency among the interior, the boundary, and the exterior of an area is regarded as one imperfection of the 9IM in (Chen *et al.*, 2001). The use of the embedding topology of a line in $\mathbb{R}$ is regarded as another imperfection of the 9IM, as the interior and exterior of a line in the plane are connected to each other. Having observed these imperfections, Chen *et al.* (2001) suggest to replace the exterior with the corresponding Voronoi region of an entity. In this way, they show that the resulting V9I model is able to make more distinctions of spatial relations between spatial entities and the above two imperfections can be circumvented.

The classification made in (Chen *et al.*, 2001) is, however, informal and incomplete. It is the aim of this paper to examine the Voronoi-based 9-intersection (V9I) model proposed in (Chen *et al.*, 2001) and make a complete classification of spatial relations using the V9I model.

The notion of Voronoi regions is usually defined for points in a finite system of points in the plane. Intuitively, a Voronoi region is a 'region of influence' of a particular point, and all these Voronoi regions together form a partition of the plane into convex polygons. This definition has been extended to entities other than points. We next present a formal definition of Voronoi regions.

Definition 1. (Chen *et al.*, 2001) Let Γ be a finite set of spatial entities (points/lines/areas) and $E \in \Gamma$ be an arbitrary entity. The Voronoi region of E w.r.t. Γ, written as E^v, is defined as

$$E^v = \{p \mid (\forall E' \in \Gamma) d^*(p,E) \leq d^*(p,E')\}, \tag{3}$$

where $d^*(p,X) = d(p,X)$ if X is a point or a line, and $d^*(p,X) = d(p,\partial X)$ if X is an area.

Remark 1. Γ was not explicitly mentioned in (Chen *et al.*, 2001). We think it is necessary to include Γ in the definition, because Voronoi regions may change if entities are added to or deleted from Γ. We note that there are other interpretations of Voronoi regions. In particular, several authors define the Voronoi region of an entity E as the set of points which are closer to E than to any other entities E' in Γ. Moreover, when E is an area, sometimes it is more desirable to define E^v as the set of points that is close to E itself instead of to its boundary. In this paper, to facilitate the comparison, we adopt the interpretation used in (Chen *et al.*, 2001). A detailed analysis will be given in Section 4.

Having defined Voronoi region for each spatial entity in a finite system of spatial entities Γ, we are now ready to define the Voronoi-based 9-intersection relation of two entities in Γ. Let A, B be two entities in Γ. We define

$$\mathbf{R}_{v9}(A,B) = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^v \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^v \\ A^v \cap B^\circ & A^v \cap \partial B & A^v \cap B^v \end{pmatrix}. \tag{4}$$

As in 9IM, we represent the V9I relation between two spatial entities A, B by a 3×3 Boolean matrix $\mathbf{R}_{v9}(A,B)$, where an entry is 0 if the corresponding intersection is empty and 1 otherwise.

Given a 3×3 Boolean matrix $\mathbf{M}$, a natural question arises whether $\mathbf{M}$ represents a V9I relation between two spatial entities. In the following section, we will give a complete analysis to this question. The results show that there are more V9I relations than identified in (Chen *et al.*, 2001). For ease of discussion, we assign a unique decimal integer d between 0 to 511 to each 3*3 Boolean matrix $\mathbf{M}$ as follows[2]

$$d = \mathbf{M}(1,1) * 2^8 + \mathbf{M}(1,2) * 2^7 + ... + \mathbf{M}(3,2) * 2 + \mathbf{M}(3,3),$$

and call d the *matrix identity* of $\mathbf{M}$. For example, matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

has identity $371 = 1 \times 2^8 + 0 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$. There are 512 different Boolean matrices in total, but very few of them are realizable. Table 2 gives all matrixes that represent a realizable V9I relation in each case respectively.

3 V9I relations between spatial entities

In this section, we give a complete classification of V9I relations between two spatial entities. According to whether these spatial entities are points, lines, or areas, there are nine cases to consider. Since the case $(line, point)$ is the converse of the case $(point, line)$, the case $(area, point)$ is the converse of the case $(point, area)$, and the case $(area, line)$ is the converse of the case $(line, area)$, we need only consider six cases.

In the following, we will show, for each pair of types (*Type*-1,*Type*-2) of spatial entities and each 3×3 Boolean matrix $\mathbf{M}$, whether there exist a finite system Γ of spatial entities and a *Type*-1 entity $A \in \Gamma$ and a *Type*-2 entity $B \in \Gamma$ such that $\mathbf{M} = \mathbf{R}_{v9}(A,B)$.

[2]We believe this or similar encoding has been used elsewhere before.

Table 2: Realizable Relations in V9I

Type	Corresponding Matrix Identities of Realizable Relations in V9I
$(area, area)$	$\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_{27}, \mathbf{M}_{31}, \mathbf{M}_{91}, \mathbf{M}_{95}, \mathbf{M}_{351}, \mathbf{M}_{292}, \mathbf{M}_{293}, \mathbf{M}_{319}, \mathbf{M}_{383}, \mathbf{M}_{448}, \mathbf{M}_{449}, \mathbf{M}_{475}, \mathbf{M}_{479}, \mathbf{M}_{484}, \mathbf{M}_{485}, \mathbf{M}_{511}$
$(line, line)$	$\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_{27}, \mathbf{M}_{45}, \mathbf{M}_{63}, \mathbf{M}_{195}, \mathbf{M}_{219}, \mathbf{M}_{239}, \mathbf{M}_{255}, \mathbf{M}_{325}, \mathbf{M}_{351}, \mathbf{M}_{365}, \mathbf{M}_{383}, \mathbf{M}_{455}, \mathbf{M}_{479}, \mathbf{M}_{495}, \mathbf{M}_{511}$
$(line, area)$	$\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_{27}, \mathbf{M}_{31}, \mathbf{M}_{195}, \mathbf{M}_{199}, \mathbf{M}_{219}, \mathbf{M}_{223}, \mathbf{M}_{287}, \mathbf{M}_{455}, \mathbf{M}_{479}, \mathbf{M}_{292}, \mathbf{M}_{293}, \mathbf{M}_{319}, \mathbf{M}_{487}, \mathbf{M}_{511}$
$(point, point)$	$\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_{27}$
$(point, line)$	$\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_{45}, \mathbf{M}_{27}$
$(point, area)$	$\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_{27}, \mathbf{M}_{31}, \mathbf{M}_{36}, \mathbf{M}_{37}$

3.1 The V9I relation between two points

Suppose $\mathbf{M}$ is a 3×3 Boolean matrix. We now determine when $\mathbf{M}$ represents the V9I relation between two points p, q. Note that for two points p, q, we have $\{p\}^\circ = \{q\}^\circ = \varnothing$, and $\partial\{p\} = \{p\}$, $\partial\{q\} = \{q\}$. Moreover, we have $p \in \{p\}^v$ and $q \in \{q\}^v$. Therefore, if $\mathbf{M}$ represents the V9I relation between p, q, then the first row and the first column of $\mathbf{M}$ are both $(0, 0, 0)$.

As a consequence, the V9I relation between two points p, q is represented by the following matrix:

$$\mathbf{R}_{v9}(p, q) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \{p\} \cap \{q\} & \{p\} \cap \{q\}^v \\ 0 & \{p\}^v \cap \{q\} & \{p\}^v \cap \{q\}^v \end{pmatrix}.$$

We will need the following result, which shows that, in particular, p is in $\{q\}^v$ iff $p = q$.

Proposition 1. *Let Γ be a finite set of spatial entities and p be a point in Γ. If X is a point or a line in Γ, then $p \in X^v$ iff $p \in X$; if X is an area in Γ, then $p \in X^v$ iff $p \in \partial X$.*

Proof. Take the case when X is a line as an example. If $p \in X$, then $d(p, X) = d(p, p) = 0$, and hence $p \in X^v$. On the other hand, if $p \notin X$, then $d(p, p) = 0 < d(p, X)$ because X is a closed set. Hence $p \notin X^v$. $\square$

Note that $\{p\} \cap \{q\} \neq \varnothing$ iff $p = q$. By $p \in \{p\}^v$ and $q \in \{q\}^v$, we know if $\{p\} \cap \{q\}$ is nonempty, then the other three intersections are all nonempty. On the other hand, if $\{p\} \cap \{q\} = \varnothing$, then by Proposition 1 we know $p \notin \{q\}^v$ and $q \notin \{p\}^v$. So, in this case, $\{p\}^v \cap \{q\}^v$ is the only undetermined intersection. This suggests that there are at most three possible matrices. These are all realizable and their realizations are shown in Figure 1. We note that the first two relations are *context-sensitive* in the sense that existence of other entities in the finite system Γ will affect the validity of these relations.

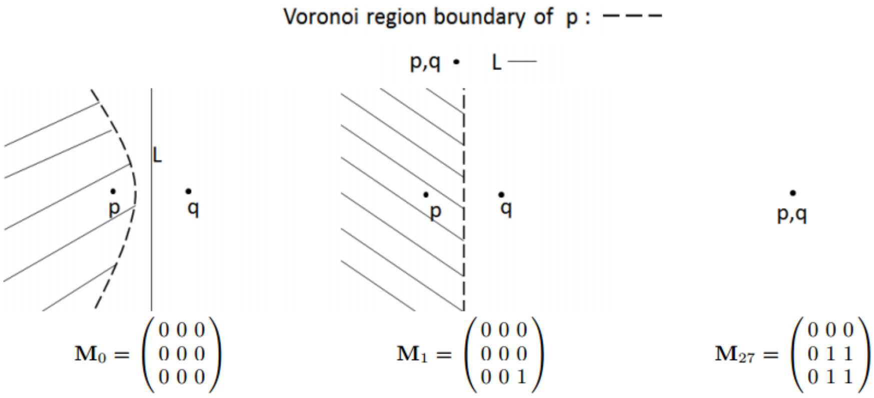


Figure 1: V9I $(point, point)$ relations between two points p and q, where L is an auxiliary line.

Definition 2. We say a V9I relation α is *context-sensitive* if there exists an instance (A, B) of α such that the V9I relation between A, B changes if we add (or delete) a third entity C to Γ without changing all the other entities.

Remark 2. Let A, B be two spatial entities that are related by a context-sensitive V9I relation α. Suppose the relation between A, B changes to another V9I relation α' if we add or delete a third region C from Γ. Because A, B have not changed, it is clear that the topological relation between them does not change. Therefore, the V9I relations α and α' are contained in the same 4IM relation. Moreover, these relations cannot be distinguished by 9IM.

3.2 The V9I relation between a point and a line

Suppose $\mathbf{M}$ is a 3×3 Boolean matrix. We now determine when $\mathbf{M}$ represents the V9I relation between a point p and a line L. Note that $\{p\}^\circ = \varnothing$ and $\partial\{p\} = \{p\} \subset \{p\}^v$. Therefore, the V9I relation between p and L is represented by the following matrix:

$$\mathbf{R}_{v9}(p, L) = \begin{pmatrix} 0 & 0 & 0 \\ \{p\} \cap L^\circ & \{p\} \cap \partial L & \{p\} \cap L^v \\ \{p\}^v \cap L^\circ & \{p\}^v \cap \partial L & \{p\}^v \cap L^v \end{pmatrix}.$$

There are two subcases according to whether p is in L.

First, suppose p is not in L. Then by Proposition 1 we know $p \notin L^v$, i.e. $\{p\} \cap L^v = \varnothing$. Similarly, for any $q \in L$, we have $d(q, L) = 0 < d(q, p)$, and hence $q \notin \{p\}^v$. We have $\{p\}^v \cap L = \varnothing$. So there are only two possible matrices in this subcase:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Second, suppose p is a point of L. Then it is straightforward to show that $\{p\}^v \cap L = \{p\}^v \cap L^v = \{p\}$. If $p \in L^\circ$, then we have $\{p\} \cap \partial L = \varnothing$ and $\{p\}^v \cap \partial L = \varnothing$. If $p \in \partial L$, then we have $\{p\} \cap L^\circ = \varnothing$ and $\{p\}^v \cap L^\circ = \varnothing$. Because $\{p\} \subset \{p\}^v$ and $L \subset L^v$, there are only two possible matrices in this subcase:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

These four matrices are all realizable, and illustrations are given in Figure 2. We note that the first two relations are context-sensitive.

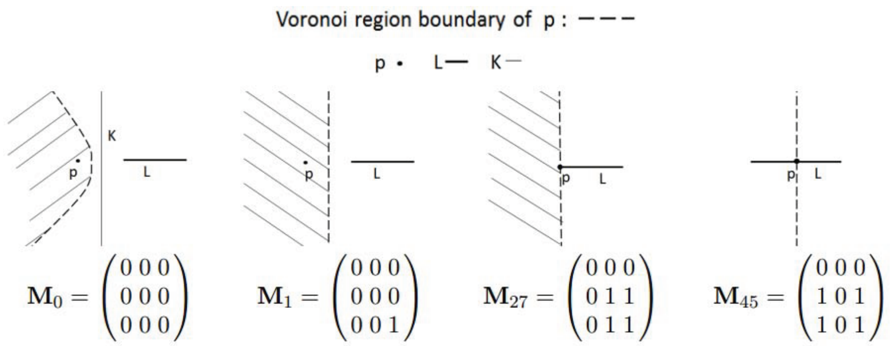


Figure 2: V9I $(point, line)$ relations between a point p and a line L, where K is an auxiliary line.

3.3 The V9I relation between a point and an area

Suppose $\mathbf{M}$ is a 3×3 Boolean matrix. We now determine when $\mathbf{M}$ represents the V9I relation between a point p and an area A. Note that $\{p\}^\circ = \varnothing$, $\partial\{p\} = \{p\} \subseteq \{p\}^v$ and $\partial A \subseteq A^v$. The V9I relation between p and a is represented by the following matrix:

$$\mathbf{R}_{v9}(p, A) = \begin{pmatrix} 0 & 0 & 0 \\ \{p\} \cap A^\circ & \{p\} \cap \partial A & \{p\} \cap A^v \\ \{p\}^v \cap A^\circ & \{p\}^v \cap \partial A & \{p\}^v \cap A^v \end{pmatrix}.$$

There are three subcases according to the topological relation between p and a. First, if $p \in A^\circ$, then $\{p\} \cap A^\circ \neq \varnothing$, $\{p\}^v \cap A^\circ \neq \varnothing$, and $\{p\} \cap \partial A = \{p\}^v \cap \partial A = \{p\} \cap A^v = \varnothing$. Thus the only undetermined intersection is $\{p\}^v \cap A^v$, and there are only two possible matrices:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Second, if $p \in \partial A$, then $\{p\} \cap A^\circ = \varnothing$ and $\{p\} \cap \partial A = \{p\}^v \cap \partial A = \{p\} \cap A^v = \{p\}^v \cap A^v \neq \varnothing$. The only undetermined intersection is $\{p\}^v \cap A^\circ$. Thus there are also two possible matrices in this subcase:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Third, when $p \notin A$, then $\{p\}^v \cap A = \varnothing$ and $\{p\} \cap A^v = \varnothing$. The only undetermined intersection is $\{p\}^v \cap A^v$. Thus we also get two possible matrices for this subcase:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These six matrices are all realizable, and illustrations are given in Figure 3. Note that relations $\mathbf{M}_0$ and $\mathbf{M}_1$ in Figure 3 are context-sensitive and split the 4IM $(point, area)$ relation *disjoint*. They cannot be distinguished by 9IM. Similarly, the relations $\mathbf{M}_{36}$ and $\mathbf{M}_{37}$ are context-sensitive and split the 4IM $(point, area)$ relation *contained_by*. They cannot be distinguished by 9IM either. In addition, the V9I relations $\mathbf{M}_{27}$ and $\mathbf{M}_{31}$[3] split the 4IM $(point, area)$ relation *touch*, and they could neither be distinguished by 9IM. These relations are, however, not context-sensitive.

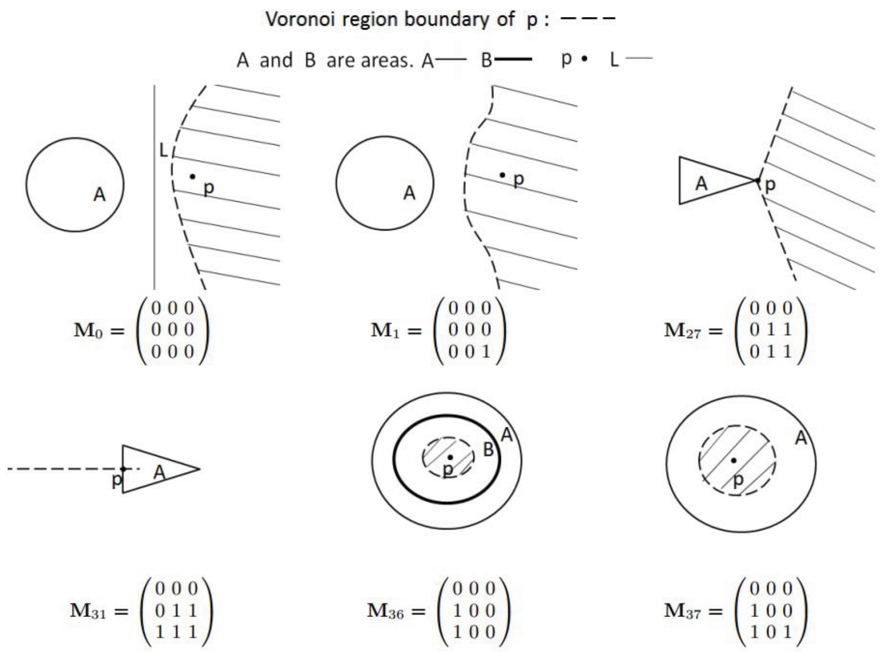


Figure 3: V9I $(point, area)$ relations between a point p and an area A, where L is an auxiliary line and B is an auxiliary area.

[3]This relation was not considered in (Chen *et al.*, 2001).

Definition 3. We say a V9I $(point, region)$ relation α is *shape-sensitive* if there exists an instance (p, A) of α such that the V9I relation between p and A changes if we replace a tiny part of the boundary of A around p with either a straight line segment or an angle centred at p.

Remark 3. The V9I relations $\mathbf{M}_{27}$ and $\mathbf{M}_{31}$ are shape-sensitive (see Figure 3). In fact, $\mathbf{M}_{27}$ occurs when the boundary of A around p is "angled", while $\mathbf{M}_{31}$ occurs when the boundary of A around p is "straight".

3.4 The V9I relation between two lines

Suppose $\mathbf{M}$ is a 3×3 Boolean matrix. We now determine when $\mathbf{M}$ represents the V9I relation between two lines L and K, i.e. when $M = \mathbf{R}_{v9}(L, K)$.

$$\mathbf{R}_{v9}(L,K) = \begin{pmatrix} L^\circ \cap K^\circ & L^\circ \cap \partial K & L^\circ \cap K^v \\ \partial L \cap K^\circ & \partial L \cap \partial K & \partial L \cap K^v \\ L^v \cap K^\circ & L^v \cap \partial K & L^v \cap K^v \end{pmatrix}.$$

Recall that $L \subset L^v$ and $K \subset K^v$. We have the following proposition.

Proposition 2. *Let Γ be a finite set of spatial entities and L, K be two lines in Γ. Suppose $X \subseteq L$ and $Y \subseteq K$. Then we have*

(i) If $X \cap Y \neq \varnothing$, then $L^v \cap Y \neq \varnothing$, $X \cap K^v \neq \varnothing$, and $L^v \cap K^v \neq \varnothing$.

(ii) If $X \cap K = \varnothing$ then $X \cap K^v = \varnothing$.

(iii) If $L \cap Y = \varnothing$ then $L^v \cap Y = \varnothing$.

Proof. Item (i) follows directly from the fact that $L \subset L^v$ and $K \subset K^v$. As for (ii), suppose $X \cap K = \varnothing$. Then for any point p in X, we have $d(p, L) = 0 < d(p, K)$, i.e. p is not in K^v. Therefore $X \cap K^v = \varnothing$. Item (iii) is completely similar to (ii). □

We first consider the case when $L^\circ \cap K^\circ \neq \varnothing$. By Proposition 2(i) we know $L^\circ \cap K^v \neq \varnothing$, $L^v \cap K^\circ \neq \varnothing$ and $L^v \cap K^v \neq \varnothing$. Therefore, the representation matrix becomes:

$$\mathbf{R}_{v9}(L,K) = \begin{pmatrix} 1 & L^\circ \cap \partial K & 1 \\ \partial L \cap K^\circ & \partial L \cap \partial K & \partial L \cap K^v \\ 1 & L^v \cap \partial K & 1 \end{pmatrix}.$$

Consider the intersection between L° and ∂K. By Proposition 2 we know $L^v \cap \partial K \neq \varnothing$ if $L^\circ \cap \partial K \neq \varnothing$, and $\partial L \cap K^v \neq \varnothing$ if $\partial L \cap K^\circ \neq \varnothing$, and $\partial L \cap K = \varnothing$ iff $\partial L \cap K^v = \varnothing$, and $L \cap \partial K = \varnothing$ iff $L^v \cap \partial K = \varnothing$. In addition, if $\partial L \cap \partial K \neq \varnothing$, then we have $\partial L \cap K^v \neq \varnothing$ and $L^v \cap \partial K \neq \varnothing$. Summing these up, we know that there are at most eight possible matrices with nonempty interior-interior intersection:

$$\begin{pmatrix} 1&0&1\\0&0&0\\1&0&1 \end{pmatrix} \begin{pmatrix} 1&1&1\\0&0&0\\1&1&1 \end{pmatrix} \begin{pmatrix} 1&0&1\\1&0&1\\1&0&1 \end{pmatrix} \begin{pmatrix} 1&0&1\\0&1&1\\1&1&1 \end{pmatrix} \begin{pmatrix} 1&1&1\\0&1&1\\1&1&1 \end{pmatrix} \begin{pmatrix} 1&0&1\\1&1&1\\1&1&1 \end{pmatrix} \begin{pmatrix} 1&1&1\\1&1&1\\1&1&1 \end{pmatrix} \begin{pmatrix} 1&1&1\\1&0&1\\1&1&1 \end{pmatrix}.$$

We next consider the case when $L^\circ \cap K^\circ = \varnothing$. By Proposition 2, we know $L^\circ \cap K = \varnothing$ iff $L^\circ \cap K^v = \varnothing$, $L \cap K^\circ = \varnothing$ iff $L^v \cap K^\circ = \varnothing$, $\partial L \cap K = \varnothing$ iff $\partial L \cap K^v = \varnothing$, and $L \cap \partial K = \varnothing$ iff $L^v \cap \partial K = \varnothing$. Combing with the fact that $K \subset K^v$ and $L \subset L^v$, we know there are only nine possible matrices when $L^\circ \cap K^\circ = \varnothing$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Thus altogether we have 17 possible matrices. These matrices are all realizable and illustrations are given in Figure 4.

Remark 4. The first pair of relations in Figure 4 are context-sensitive and split the 4IM relation *disjoint*. Again, these context-sensitive relations cannot be distinguished by 9IM. As there are sixteen $(line, line)$ 4IM relations (Hadzilacos and Tryfona, 1992), the V9I model leaves all 4IM relations other than *disjoint* unchanged. We also note that only eight V9I $(line, line)$ relations have been found in (Chen *et al.*, 2001).

3.5 The V9I relation between a line and an area

Suppose $\mathbf{M}$ is a 3×3 Boolean matrix. We now determine when $\mathbf{M}$ represents the V9I relation between a line L and an area A. The corresponding Voronoi-based 9-intersection model is:

$$\mathbf{R}_{v9}(L, A) = \begin{pmatrix} L^\circ \cap A^\circ & L^\circ \cap \partial A & L^\circ \cap A^v \\ \partial L \cap A^\circ & \partial L \cap \partial A & \partial L \cap A^v \\ L^v \cap A^\circ & L^v \cap \partial A & L^v \cap A^v \end{pmatrix}.$$

Recall that $L \subset L^v$ and $\partial A \subset A^v$. We have the following proposition.

Proposition 3. *Let Γ be a finite set of spatial entities and L be a line, and A be an area, both in Γ. Suppose X is a set. Then we have*

(i) If $L \cap X \neq \varnothing$, then $L^v \cap X \neq \varnothing$; if $X \cap \partial A \neq \varnothing$, then $X \cap A^v \neq \varnothing$.

(ii) If $\partial L \cap \partial A = \varnothing$ then $\partial L \cap A^v = \varnothing$.

(iii) If $L \cap \partial A = \varnothing$ then $L^v \cap \partial A = \varnothing$.

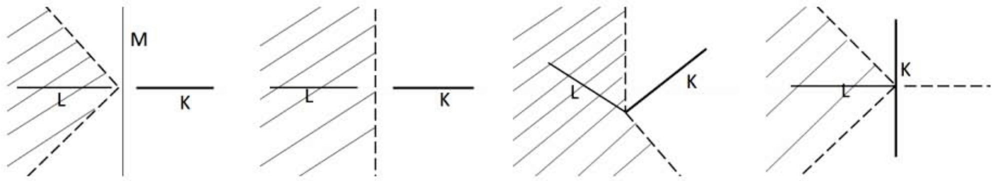
Proof. Items (i) follows directly from the fact that $L \subset L^v$ and $\partial A \subset A^v$. As for (ii), suppose $\partial L \cap \partial A$ is empty. Then for any point p in ∂L, we know $d(p, \partial L) = 0 < d(p, \partial A)$, hence p is in not in A^v. For (iii), suppose $L \cap \partial A$ is empty. Then for any point p in ∂A we have $d(p, \partial A) = 0 < d(p, L)$, hence p is not in L^v. □

<u>Case 1.</u> We begin by consider the case when $L^\circ \cap A^\circ$ is nonempty. By Proposition 3(i) we know $L^v \cap A^\circ$ is also nonempty.
<u>Subcase 1.1.</u> If $L^\circ \cap \partial A \neq \varnothing$, then by Proposition 3(i) we have $L^v \cap \partial A$, $L^\circ \cap A^v$, and $L^v \cap A^v$ are all nonempty. Therefore the representation matrix becomes:

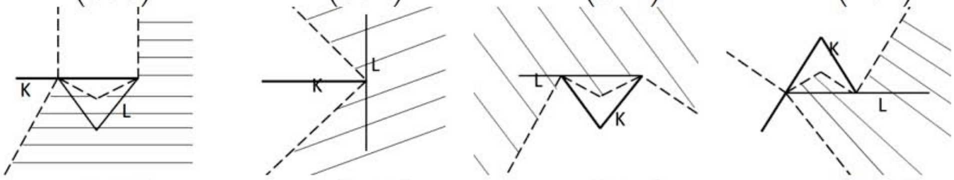
$$\mathbf{R}_{v9}(L, A) = \begin{pmatrix} 1 & 1 & 1 \\ \partial L \cap A^\circ & \partial L \cap \partial A & \partial L \cap A^v \\ 1 & 1 & 1 \end{pmatrix}.$$

Voronoi region boundary of L: – – –

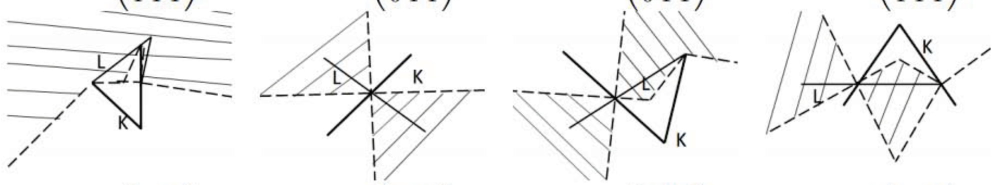
L — K — M —



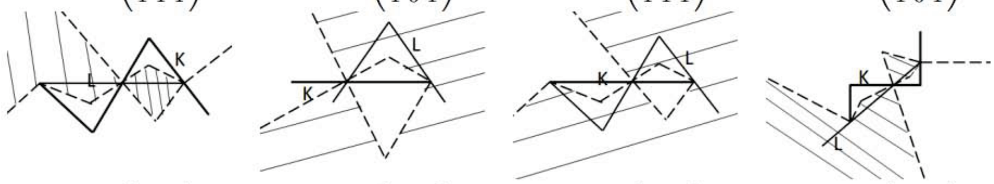
$\mathbf{M}_0 = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$ $\mathbf{M}_1 = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{pmatrix}$ $\mathbf{M}_{27} = \begin{pmatrix} 0&0&0 \\ 0&1&1 \\ 0&1&1 \end{pmatrix}$ $\mathbf{M}_{45} = \begin{pmatrix} 0&0&0 \\ 1&0&1 \\ 1&0&1 \end{pmatrix}$



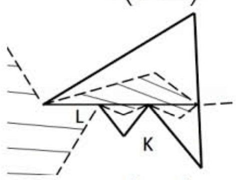
$\mathbf{M}_{63} = \begin{pmatrix} 0&0&0 \\ 1&1&1 \\ 1&1&1 \end{pmatrix}$ $\mathbf{M}_{195} = \begin{pmatrix} 0&1&1 \\ 0&0&0 \\ 0&1&1 \end{pmatrix}$ $\mathbf{M}_{219} = \begin{pmatrix} 0&1&1 \\ 0&1&1 \\ 0&1&1 \end{pmatrix}$ $\mathbf{M}_{239} = \begin{pmatrix} 0&1&1 \\ 1&0&1 \\ 1&1&1 \end{pmatrix}$



$\mathbf{M}_{255} = \begin{pmatrix} 0&1&1 \\ 1&1&1 \\ 1&1&1 \end{pmatrix}$ $\mathbf{M}_{325} = \begin{pmatrix} 1&0&1 \\ 0&0&0 \\ 1&0&1 \end{pmatrix}$ $\mathbf{M}_{351} = \begin{pmatrix} 1&0&1 \\ 0&1&1 \\ 1&1&1 \end{pmatrix}$ $\mathbf{M}_{365} = \begin{pmatrix} 1&0&1 \\ 1&0&1 \\ 1&0&1 \end{pmatrix}$



$\mathbf{M}_{383} = \begin{pmatrix} 1&0&1 \\ 1&1&1 \\ 1&1&1 \end{pmatrix}$ $\mathbf{M}_{455} = \begin{pmatrix} 1&1&1 \\ 0&0&0 \\ 1&1&1 \end{pmatrix}$ $\mathbf{M}_{479} = \begin{pmatrix} 1&1&1 \\ 0&1&1 \\ 1&1&1 \end{pmatrix}$ $\mathbf{M}_{495} = \begin{pmatrix} 1&1&1 \\ 1&0&1 \\ 1&1&1 \end{pmatrix}$



$\mathbf{M}_{511} = \begin{pmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{pmatrix}$

Figure 4: V9I $(line, line)$ relations between two lines L and K, where M is an auxiliary line.

Moreover, by Proposition 3(i) and (ii) we know $\partial L \cap \partial A = \varnothing$ iff $\partial L \cap A^v = \varnothing$. Summing these up, we get four possible matrices for this subcase:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

<u>Subcase 1.2.</u> If $L^\circ \cap \partial A = \varnothing$, then by Proposition 3(i) we know $L^\circ \cap A^v = \varnothing$. The representation matrix then becomes:

$$\mathbf{R}_{v9}(L, A) = \begin{pmatrix} 1 & 0 & 0 \\ \partial L \cap A^\circ & \partial L \cap \partial A & \partial L \cap A^v \\ 1 & L^v \cap \partial A & L^v \cap A^v \end{pmatrix}.$$

In addition, if the boundaries of L and A intersect, then by Proposition 3(i) we know $\partial L \cap A^v \neq \varnothing$, $L^v \cap \partial A \neq \varnothing$ and $L^v \cap A^v \neq \varnothing$.

If, on the other hand, the boundaries of L and A do not intersect, then by Proposition 3(ii) we know $\partial L \cap A^v = \varnothing$. Since $L^\circ \cap A^\circ \neq \varnothing$, we must have $\partial L \cap A^\circ \neq \varnothing$ in this case. This is because, otherwise, boundary points of L are outside A but some interior points of L are interior points of A, and hence L should cross the boundary of A (i.e. $L^\circ \cap \partial A \neq \varnothing$). A contradiction. Furthermore, by Proposition 3(iii) we have $L^v \cap \partial A = \varnothing$ since $L \cap \partial A = \varnothing$. Summing these results up, we get four possible matrices for this case:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

<u>Case 2.</u> The case when $L^\circ \cap A^\circ = \varnothing$ is much like the previous case. In this case, we note that when L and A are separated (i.e. $L \cap A = \varnothing$), then $L^v \cap A^\circ = \varnothing$. This is because, for any interior point p of A, we have $d(p, \partial A) < d(p, L)$. In this case, we have eight possible matrices:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Therefore we have in total 16 possible matrices for a line and an area. These matrices are all realizable and illustrations are shown in Figure 5.

Definition 4. We say a V9I $(line, region)$ relation α is *shape-sensitive* if there exists an instance (L, A) of α such that the V9I relation between L and A changes if we replace a tiny part of the boundary of A and/or the line L around a point p with either a straight line segment or an angle centred at p, where p is a touching point of A and L, i.e. a 0-dimensional connected component of $\partial A \cap L$.

Remark 5. Egenhofer *et al.* (1993) have shown that there are eleven 4IM relations between a simple line and an area. The V9I splits a 4IM $(line, area)$ relation when the line is *disjoint from*, or is *contained_by*, or *touch* the area. Note that there are three different $(line, area)$ touch 4IM relations according to whether the line touches the region at an interior point or at a boundary

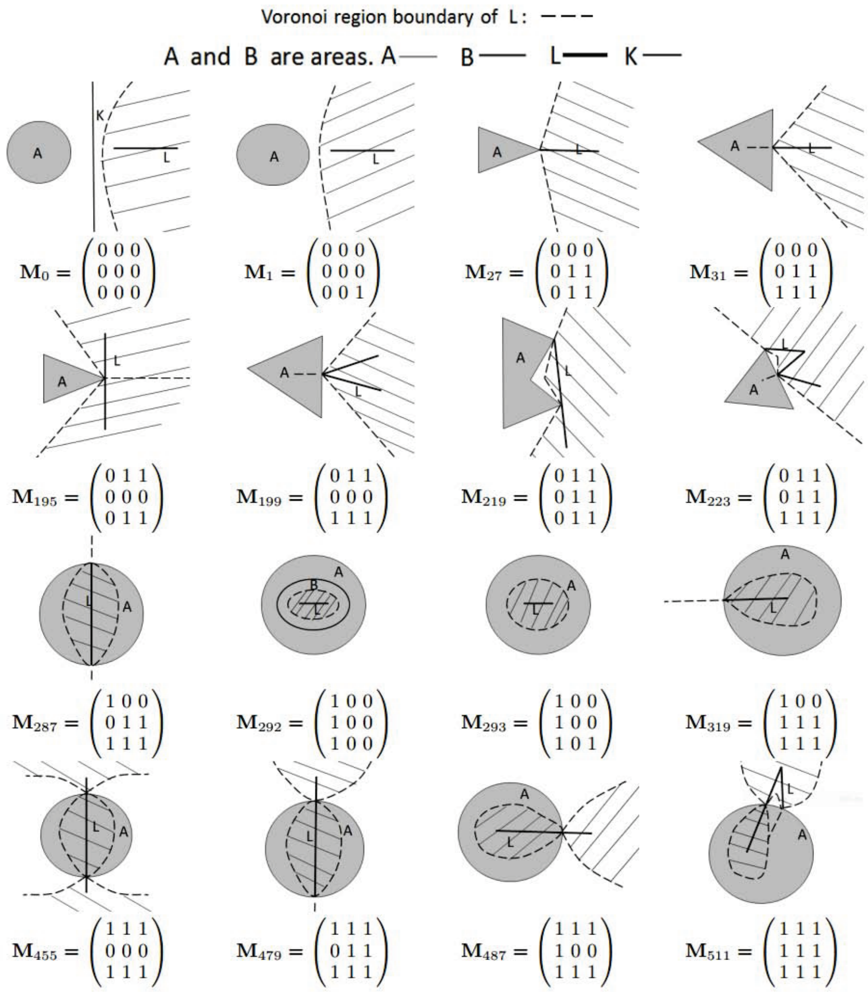


Figure 5: V9I $(line, area)$ relations, where L is a line, A is an area, K is an auxiliary line, and B is an auxiliary area.

Table 3: 4-intersection $(area, area)$ relations and their matrix representations.

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	A and B are disjoint	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	A and B touch	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	A equals B
$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$	A contains B	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	A covers B	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	A is contained by B
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	A is covered by B	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	A and B overlap with disjoint boundaries	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	A and B overlap with intersecting boundaries

point of the line, or both, viz. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ($touch\_boundary$), $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ($touch\_interior$), and $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ ($touch\_both$). The *disjoint* relation is split into $\mathbf{M}_0$ and $\mathbf{M}_1$, which are context-sensitive; and the *contained_by* relation is split into $\mathbf{M}_{292}$ and $\mathbf{M}_{293}$, which are also context-sensitive. The $touch\_boundary$ relation is split into $\mathbf{M}_{27}$ and $\mathbf{M}_{31}$, the $touch\_interior$ relation is split into $\mathbf{M}_{195}$ and $\mathbf{M}_{199}$, and the $touch\_both$ relation is split into $\mathbf{M}_{219}$ and $\mathbf{M}_{223}$. These six $touch$ V9I relations are shape-sensitive. As before, these context- and shape-sensitive relations are not distinguishable by 9IM.

3.6 The V9I relation between two areas

Suppose $\mathbf{M}$ is a 3×3 Boolean matrix. We now determine when $\mathbf{M}$ represents the V9I relation between two areas A and B. The corresponding Voronoi-based 9-intersection model is:

$$\mathbf{R}_{v9}(A,B) = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^v \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^v \\ A^v \cap B^\circ & A^v \cap \partial B & A^v \cap B^v \end{pmatrix}.$$

Egenhofer and Franzosa (1991) have shown that there are nine topological relations (see Table 3) between two areas A and B, using the 4-intersection model.

By the definition of Voronoi region, we have $\partial X \subset X^v$ for any area X. It is easy to conclude the following proposition:

Proposition 4. *Let Γ be a finite set of spatial entities and X, Y be two areas in Γ. Then we have*

(i) If $\partial X \cap Y^v \neq \varnothing$ then $X^v \cap Y^v \neq \varnothing$.

(ii) If $\partial X \cap Y^\circ \neq \varnothing$ then $X^v \cap Y^\circ \neq \varnothing$.

(iii) $\partial X \cap \partial Y \neq \varnothing$ iff $X^v \cap \partial Y \neq \varnothing$.

(iv) If $X \cap \partial Y = \varnothing$ then $X^\circ \cap Y^v = \varnothing$.

Proof. Items (i) and (ii) follow directly from the fact that ∂X is a subset of X^v. So does the 'only-if' part of (iii). As for the 'if' part of (iii), suppose $\partial X \cap \partial Y$ is empty. Then for any point

p in ∂Y, we know $d(p, \partial Y) = 0 < d(p, \partial X)$, hence p is in not in X^v. For (iv), suppose $X \cap \partial Y$ is empty. Then for any point p in X° we have $d(p, \partial X) < d(p, \partial Y)$, because any path from p to any boundary point of Y crosses the boundary of X first. That is, no interior point of X is in the Voronoi region of Y. $\square$

We next consider the V9I relation between A and B case by case.

If A and B are disjoint, then by Proposition 4(iii), we know both $\partial A \cap B^v$ and $A^v \cap \partial B$ are empty. Similarly, by Proposition 4(iv), we know both $A^\circ \cap B^v$ and $A^v \cap B^\circ$ are empty. Hence the representation matrix becomes:

$$\mathbf{R}_{v9}(A, B) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A^v \cap B^v \end{pmatrix},$$

and there are only two possible matrices in this case:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If A and B touch, then by Proposition 4(iii) and (i), we know $\partial A \cap B^v$, $A^v \cap \partial B$, and $A^v \cap B^v$ are all nonempty. The representation matrix becomes:

$$\mathbf{R}_{v9}(A, B) = \begin{pmatrix} 0 & 0 & A^\circ \cap B^v \\ 0 & 1 & 1 \\ A^v \cap B^\circ & 1 & 1 \end{pmatrix},$$

and there are only four possible matrices for this case:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

If A equals B, the only relation is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

If A contains B, then $A^\circ \cap B^v \neq \varnothing$ by Proposition 4(ii), and both $\partial A \cap B^v$ and $A^v \cap \partial B$ are empty by Proposition 4(iii), and $A^v \cap B^\circ = \varnothing$ by Proposition 4(iv) and $\partial A \cap B = \varnothing$. Therefore, the representation matrix becomes:

$$\mathbf{R}_{v9}(A, B) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & A^v \cap B^v \end{pmatrix},$$

and there are only two matrices for this case:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Symmetrically, for the case that B contains A, the only two matrices are

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

If A covers B, then we have $A^\circ \cap B^v \neq \varnothing$ by Proposition 4(ii), and $\partial A \cap B^v$, $A^v \cap \partial B$, and $A^v \cap B^v$ are all nonempty by Proposition 4(iii) and (i). Therefore, the representation matrix becomes:

$$\mathbf{R}_{v9}(A, B) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ A^v \cap B^\circ & 1 & 1 \end{pmatrix}.$$

Thus the only two relations for this case are:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Symmetrically, for the case that B covers A, the only two matrices are

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

If A and B overlap with disjoint boundaries, then we have $A^\circ \cap B^v$ and $A^v \cap B^\circ$ are both nonempty by Proposition 4(ii), and both $A^v \cap \partial B$ and $\partial A \cap B^v$ are empty by Proposition 4(iii). Therefore the representation matrix becomes:

$$\mathbf{R}_{v9}(A, B) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & A^v \cap B^v \end{pmatrix},$$

and there are only two matrices in this case:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

If A and B overlap with intersecting boundaries, then we have $A^\circ \cap B^v$ and $A^v \cap B^\circ$ are both nonempty by Proposition 4(ii), and $A^v \cap \partial B$, $\partial A \cap B^v$, and $A^v \cap B^v$ are all nonempty by Proposition 4(iii) and (i). Thus there is only one matrix in this case, i.e.:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Summing these up, we get 18 possible matrices for $(area, area)$ relations. These matrices are all realizable and illustrations are given in Figure 6.

Definition 5. We say a V9I $(region, region)$ relation α is *shape-sensitive* if there exists an instance (A, B) of α such that the V9I relation between A and B changes if we replace a tiny part of the boundary of A and/or the boundary of B around a point p with either a straight line segment or an angle centred at p, where p is a touching point of A and B, i.e. a 0-dimensional connected component of $\partial A \cap \partial B$.

Remark 6. The V9I splits each 4IM $(area, area)$ relation except *equal* and *overlap with_intersecting_boundaries*. On one hand, *disjoint* is split into $\mathbf{M}_0$ and $\mathbf{M}_1$, *contained_by* is split into $\mathbf{M}_{292}$ and $\mathbf{M}_{293}$, *contain* is split into $\mathbf{M}_{448}$ and $\mathbf{M}_{449}$, and $overlap\_with\_disjoint\_boundaries$ is split into $\mathbf{M}_{484}$ and $\mathbf{M}_{485}$. These V9I relations are context-sensitive. On the other hand, *touch* is split into $\mathbf{M}_{27}$, $\mathbf{M}_{31}$, $\mathbf{M}_{91}$, and $\mathbf{M}_{95}$, *covered_by* is split into $\mathbf{M}_{319}$ and $\mathbf{M}_{383}$, and *covers* is split into $\mathbf{M}_{475}$ and $\mathbf{M}_{479}$. These V9I relations are shape-sensitive. Finally, we note that all split V9I relations are not distinguishable by 9IM.

Remark 7. Context-sensitive V9I relations are useful in, for instance, identifying 'safe' evacuation route for people trapped in bushfire. To be specific, let area A and area B be two disjoint fire zones which are still spreading radially. Suppose there is a person P who is trapped at a point C near A and B and trying to go to a safe area D. Could P run from C to D before the fire from A and B blocks the way? Questions like this can be answered by using the V9I model. Assume that the running speed of P is the same as the spreading speed of the fire. If C and D are 'blocked' by A and B in the V9I model (i.e. the Voronoi regions of C and D are disconnected because of the existence of A and B), then it is highly possible that P will be engulfed by fire when running to D. Note that the V9I model can also be generalised to take the spreading speed of fire and the running speed of P as weights of the distance. In this way, we can get more reliable suggestions that are critical in bushfire (or flood) evacuation route planning.

Due to shape-sensitivity, V9I is able to distinguish four kinds of *touch* relation according to whether the boundaries around a touching point are 'angled' or 'straight'. For example, in Figure 6, $\mathbf{M}_{27}$ is an 'angled_angled' touch, $\mathbf{M}_{31}$ is an 'angled_straight' touch, $\mathbf{M}_{91}$ is an 'straight_angled' touch, and $\mathbf{M}_{95}$ is a combination of the latter two. In the same sense, V9I is also distinguishes between 'angled_angled' $covered\_by$ ($\mathbf{M}_{319}$) and 'angled_straight' $covered\_by$ ($\mathbf{M}_{383}$).

These distinctions may enhance the recognition of objects or events in image processing. Take loading cubic boxes (e.g. containers) into a truck as an example. Using the V9I, we could differentiate between stable and unstable states. It is easy to see that 'straight_straight' touch represents a stable state, while the other three touch relations, i.e. 'angled_angled', 'angled_straight', and 'straight_angled' touch, all represent unstable states. When an unstable state detected, the monitor system can sound an alarm to avoid potential danger of overturning.

The above distinctions of touch relations is also useful in monitoring changes of complex objects in GIS. For example, in (Jiang and Worboys, 2009, page 58), the authors remarked that their model cannot distinguish the two different splits of regions shown in Figure 11 of their paper (first proposed by Galton (1997)). This problem can be fixed by using the V9I model. Actually, the two different splits of regions involve, respectively, 'angled_angled' touch and

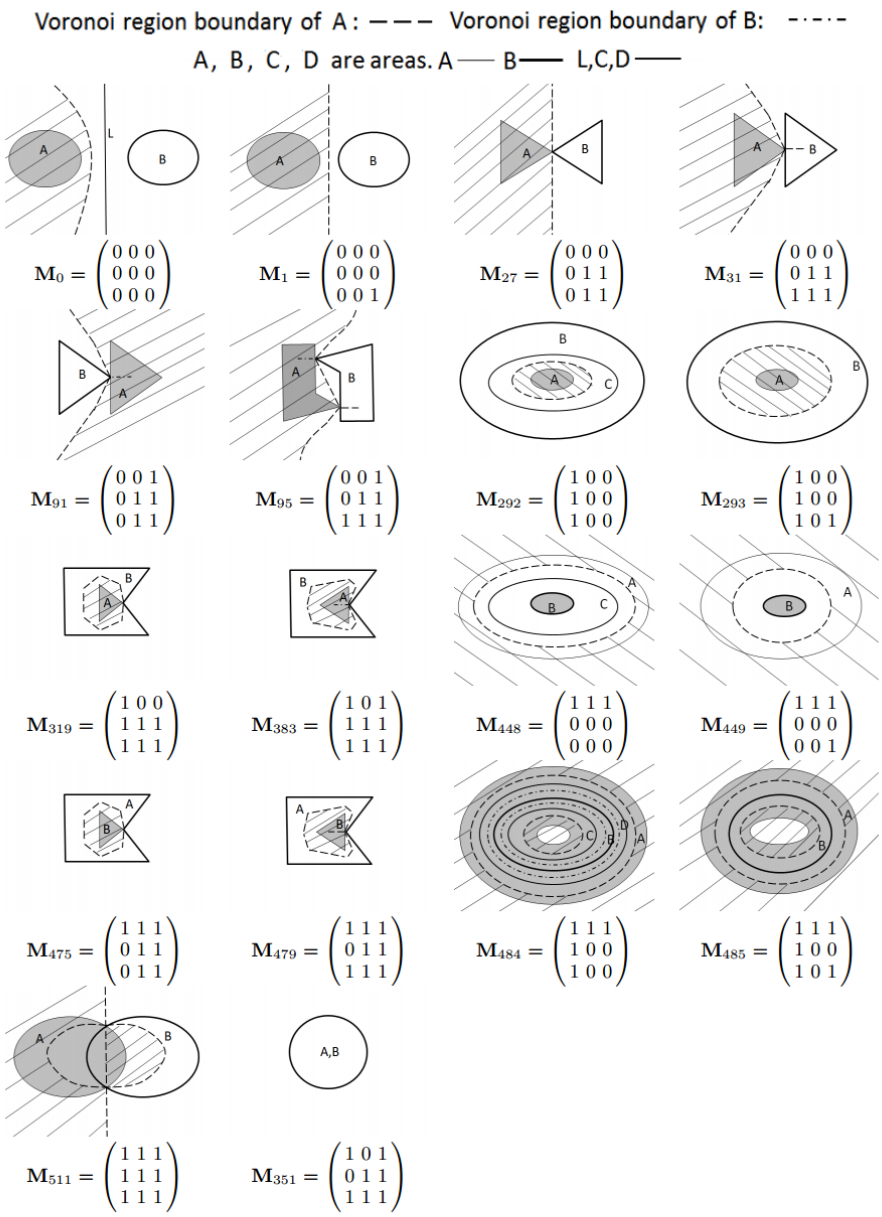


Figure 6: V9I $(area, area)$ relations, where A and B are two areas, L is an auxiliary line, C and D are two auxiliary simple regions.

'straight_straight' touch. In general, this distinguishing power of V9I is useful to the analysis of dynamic changes involving different touch relations, which are likely to result in different states or processes in real world.

4 Further discussions

It is clear that the analysis in Section 3 heavily relies on the definition of Voronoi regions. There are two variant definitions of Voronoi regions.

Let Γ be a finite system of spatial entities. The first variant defines the Voronoi region of a point or a line as in Definition 1, but defines the Voronoi region of an area A as

$$A^v = \{p : (\forall X \in \Gamma) d(p, A) \leq d(p, X)\}. \tag{5}$$

The second variant defines the Voronoi region of an entity E as

$$E^v = \{p : (\forall Y \in \Gamma) d^*(p, E) < d^*(p, Y)\}. \tag{6}$$

Compared with Definition 1, these two variants are not desirable in modelling spatial relations. Take $(area, area)$ relations as an example. If we adopt (5), then there are ten different V9I $(area, area)$ relations. This is because, by (5), we have $A \subset A^v$ and $B \subset B^v$, and $A^v \cap B^v$ is the only (possibly) undetermined entry in the V9I representation matrix if the 4-Intersection matrix is given. More specifically, only the disjoint relation splits because the existence of a third entity may block the connection between those two areas.

On the other hand, if we adopt (6), then there are only nine different V9I $(area, area)$ relations, i.e. the same as the nine 4-Intersection relations. This is because by (6) we have $\partial A \cap B^v = A^v \cap \partial B = A^v \cap B^v$, and $A^\circ \cap B^v \neq \varnothing$ iff $A^\circ \cap \partial B \neq \varnothing$, and, symmetrically, $A^v \cap B^\circ \neq \varnothing$ iff $A^\partial \cap B^\circ \neq \varnothing$. In other words, every V9I matrix entry is uniquely determined by if 4-Intersection matrix is given.

This shows that the definition adopted in Definition 1 of this paper and in Chen *et al.* (2001) is the best choice if we want to make more distinctions.

5 Conclusion

In this paper we have given a formal and complete classification for Voronoi-based nine intersection (V9I) relations between various types of spatial entities. Our results have shown that the V9I model is in fact more expressive than what has been believed before. A very interesting phenomenon we found in this paper is that some V9I relations are context- or shape-sensitive. This means that the existence of other entities or the shape of the entities may affect the validity of certain relations. These context- or shape-sensitive relations cannot be obtained by 9IM. Our future work will develop efficient algorithms for computing the Voronoi regions of a system of areas or lines, and develop algorithms for computing and/or detecting these sensitive relations.

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