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# The Application of Discrete Sliding Mode Control in Parabolic PDE Dynamics

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**Abstract**—In this paper, the problem of applying Discrete Sliding Mode Control (DSMC) on spatially finite-dimensional systems arising from discretization of bi-variate Partial Differential Equations (PDEs) describing spatio-temporal systems is studied. To this end, heat transfer PDE is discretized to create 2D discrete dynamics and eventually this 2D spatio-temporal discrete form is represented in 1D vectorial form. In order to study the effect of discrepancy between original PDE dynamics and their discrete schemes, an uncertainty term is also considered for the obtained discrete dynamics. According to the notion of strong stability and, in addition, using scaling matrices (similarity transformation), a new method for considering the stability of discrete-time systems in the presence of general uncertainty term (matched and unmatched) is developed. It is also shown that the proposed method in this paper can be used for the case with spatial constraints on the actuation. Consequently, as special cases, the problem of spatially piecewise constant, sparse and also boundary control input are studied.

## I. INTRODUCTION

Although PDEs problem are usually much harder to address than ODEs, discretizing method which converts PDEs to linear 2D systems can be used as an alternative simple solution [1]. In this paper, we will use the discrete finite-dimensional approximation (numerical approximation) of infinite-dimensional parabolic PDEs to deploy the Discrete Sliding Mode Control (DSMC) method to the dynamics defined by PDEs.

There are a number of studies on the dynamics described by PDEs. For instance, in [2] - [3], controlling an unstable linear system with a wave PDE in the actuation path is studied. Moreover, Iterative Learning Control (ILC) is used in [4] for a distributed parameter system which is governed by a second order hyperbolic PDE. In [5], the ILC method is deployed to distributed parameter systems governed by parabolic PDEs. In [6], an irrigation application in dry-land farming governed by a set of PDEs which include three spatial variables and one temporal independent variable is explained. In [7], ILC method is applied to parabolic classes of spatio-temporal systems based on the stability theory of 2D discrete systems. Basically, Sliding Mode Control (SMC) is a nonlinear control method and a special kind of Variable Structure Control (VSC) which can change its structure automatically in order to improve the dynamic performance. The main idea of SMC is to force the states into the sliding surface and remain

on the sliding surface after reaching. The most significant characteristics of SMC is its robustness, as it provides dynamics with an invariance property to matched uncertainties [8]. Therefore, SMC can address the modeling uncertainties and especially it is useful when the exact model of the dynamics is not available. However, there is a significant drawback for Continuous SMC (which mainly involves a discontinuous control term) which is known as chattering. Mainly, the method of softening the discontinuous control part by a continuous approximation is used for reducing the chattering [26]. On the other hand, for DSMC, elimination of the discontinuous control part from the control law leads to the eliminating the chattering issue [17], [20], [18]. The obtained control law is called *linear control law*.

Obviously, one approach to applying SMC to PDEs is to work directly with the determining equations; for instance, continuous SMC is used as a boundary control method for stabilizing infinite-dimensional parabolic PDE systems [9], and also with application to Arc Welding [10]. As an alternative method, one can discretize PDEs by an implicit discretization scheme which is appropriate for dynamics over spatially finite problems. In this paper, using DSMC a new method for dealing with the problem of stabilizing parabolic PDE dynamics via discretizing the heat PDEs by an implicit discretization scheme is proposed. To this end, after discretizing the PDE equation and making it a 2D system, it is converted to a 1D form with vector states, and then the 1D sliding mode control is used to stabilize the obtained dynamics.

On the other hand, physical constraints in many distributed parameter systems are arising from the fact that only boundary control can be used for controlling this kind of dynamics. However, distributed sensors and also actuators have been recently developed in a number of fields. Hence, distributed controllers have had effective application in infinite-dimensional dynamics [7]. This can also justify our innovation to use DSMC for stabilizing the dynamics arising from heat PDEs which covers general actuation constraints. The problem is then extended to spatially constrained actuators, for more generality.

Besides, the discrepancy between the original PDE model and its discrete approximation which mainly depends on the space and time step sizes, makes it inevitable to consider an uncertainty term in the obtained discrete state space. With a glimpse into the literature, it can be found that DSMC is mainly used for the systems which contain matched uncertainty and/or disturbance only [17], [18]. In this paper, a more general case of uncertainty (matched and unmatched)

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boundary controllers are the only choice. In this case, we assume that in (7), boundary condition on one side is the control inputs of the system and as a result it is not constant value. To this end, system (7) is changed to the following form

$$H(k+1) = \Lambda H(k) + Bu_b(k) + \hat{v}(k), \quad (10)$$

where

$$\hat{v}(k) = [\kappa \hat{\alpha}_k \ 0 \ \dots \ 0]^T, B = [0 \ 0 \ \dots \ \mu]^T, \quad (11)$$

and  $\hat{v} \in R^{n-1}$ ,  $B \in R^{(n-1) \times 1}$ . Here,  $u_b(k)$  is a scalar which represents the control input on the boundary. Generally, based on the physical position of actuators, control matrix  $B$  can have any different form, including the mentioned forms. In all these cases, it is assumed that matrix  $B$  has full rank,  $\text{rank}(B) = m$ .

### III. DESIGNING DSMC

Discrete Sliding Mode Control (DSMC) was first introduced in the mid 80s [15]. This idea was followed by a large number of publications [16] - [19]. Similar to Continuous Sliding Mode Control (CSMC), the design procedure of the DSMC for stabilizing problems is split into two steps:

- 1) Design a sliding surface which should have stable internal dynamics.
- 2) Create a control law which drives the closed-loop system into the sliding surface and forces the system trajectories to stay on or at least as close as possible to the surface.

Here, the objective is to design a DSMC to stabilize the states of the dynamics in (7). Generally, it is assumed that  $B \in R^{(n-1) \times m}$  and  $m \leq n-1$ . Obviously, the dynamics in (7) is not in regular form with control matrices (9). Besides,  $\text{rank}(B) = m$  (matrix  $B$  has full rank) and it is assumed that the pair  $(\Lambda, B)$  is controllable [19]. Since  $\text{rank}(B) = m$ , there exists an orthogonal matrix  $T_r \in R^{(n-1) \times (n-1)}$  such that

$$T_r B = \begin{bmatrix} 0_{[(n-1)-m] \times m} \\ \bar{B}_2 \end{bmatrix}, \quad (12)$$

where the matrix  $\bar{B}_2 \in R^{m \times m}$  is nonsingular [20] (The orthogonal matrix  $T_r$  can be computed using  $QR$  decomposition [20]). After the coordinate transformation, the dynamics in (7) is converted to

$$\begin{bmatrix} \bar{h}_1(k+1) \\ \bar{h}_2(k+1) \end{bmatrix} = \begin{bmatrix} \bar{\Lambda}_{11} & \bar{\Lambda}_{12} \\ \bar{\Lambda}_{21} & \bar{\Lambda}_{22} \end{bmatrix} \begin{bmatrix} \bar{h}_1(k) \\ \bar{h}_2(k) \end{bmatrix} + \begin{bmatrix} 0_{[(n-1)-m] \times m} \\ \bar{B}_2 \end{bmatrix} U(k) + T_r v(k), \quad (13)$$

where

$$\bar{H}(k) = \begin{bmatrix} \bar{h}_1(k) \\ \bar{h}_2(k) \end{bmatrix} = T_r H(k),$$

$$\begin{aligned} \bar{h}_1(k) &\in R^{(n-1)-m} \text{ and } \bar{h}_2(k) \in R^m \\ \bar{\Lambda}_{11} &\in R^{[(n-1)-m] \times [(n-1)-m]}, \bar{\Lambda}_{12} \in R^{[(n-1)-m] \times m} \\ \bar{\Lambda}_{21} &\in R^{m \times [(n-1)-m]}, \bar{\Lambda}_{22} \in R^{m \times m}. \end{aligned} \quad (14)$$

To design the sliding surface, since the boundary conditions do not influence the stability situation, we can temporarily ignore the term  $T_r v(k)$  arising from boundary conditions. Now, sliding surface is introduced as

$$\sigma_h(k) = \bar{S}\bar{H}(k) = \bar{S}_1 \bar{h}_1(k) + \bar{S}_2 \bar{h}_2(k), \quad (15)$$

where  $\bar{S}_1 \in R^{m \times [(n-1)-m]}$  and  $\bar{S}_2 \in R^{m \times m}$  are the design parameters which determine the sliding surface and should be chosen such that, in the case that  $\sigma_h(k) = 0$ , all remaining dynamics are stable. During ideal sliding on the surface,  $\sigma_h(k) = 0$  for all  $k \geq k_s$ , where  $k_s$  is the time when sliding starts, therefore

$$\bar{h}_2(k) = -\bar{S}_2^{-1} \bar{S}_1 \bar{h}_1(k). \quad (16)$$

Substituting the equation (16) into the equation (13) leads to

$$\bar{h}_1(k+1) = (\bar{\Lambda}_{11} - \bar{\Lambda}_{12} \bar{S}_2^{-1} \bar{S}_1) \bar{h}_1(k). \quad (17)$$

Hence, stability in the sliding mode is satisfied when all eigenvalues of the matrix  $(\bar{\Lambda}_{11} - \bar{\Lambda}_{12} \bar{S}_2^{-1} \bar{S}_1)$  are located inside the unit circle. On the other hand, in [26], it is presented that if the pair  $(\Lambda, B)$  is controllable, the pair  $(\bar{\Lambda}_{11}, \bar{\Lambda}_{12})$  is controllable as well. Therefore, the problem of finding the term  $\bar{S}_2^{-1} \bar{S}_1$  can be addressed as a classical state feedback problem which can be solved by pole placement, LQR-design or LMI method. It is also assumed that the matrix  $\bar{S}_2$  plays the role of a scaling parameter which can be selected arbitrarily but invertible. Here, we design  $\bar{S}_1, \bar{S}_2$  by solving a certain discrete Riccati equation with proper choices of weighting matrices. Eventually, the sliding surface in the original coordinate can be found by

$$S = [\bar{S}_1 \ \bar{S}_2] T_r. \quad (18)$$

In this stage, we want to design a controller which ensures the sliding mode of system. Firstly, a transformation matrix  $T_s$  is defined as

$$T_s = \begin{bmatrix} I_{(n-1)-m} & 0_{[(n-1)-m] \times m} \\ \bar{S}_1 & \bar{S}_2 \end{bmatrix}. \quad (19)$$

Using the above transformation matrix the dynamics in (13) is converted to

$$\begin{aligned} \tilde{H}(k+1) &= \begin{bmatrix} \bar{h}_1(k+1) \\ \sigma_h(k+1) \end{bmatrix} = \begin{bmatrix} \bar{\Lambda}_{11} & \bar{\Lambda}_{12} \\ \bar{\Lambda}_{21} & \bar{\Lambda}_{22} \end{bmatrix} \begin{bmatrix} \bar{h}_1(k) \\ \sigma_h(k) \end{bmatrix} + \\ &\begin{bmatrix} 0_{[(n-1)-m] \times m} \\ \bar{S}_2 \bar{B}_2 \end{bmatrix} U(k) + \begin{bmatrix} \tilde{v}_1(k) \\ \tilde{v}_2(k) \end{bmatrix}, \end{aligned} \quad (20)$$

where  $[\tilde{v}_1^T(k) \ \tilde{v}_2^T(k)]^T = T_s T_r v(k)$  and  $\tilde{v}_1(k) \in R^{(n-1)-m}$ ,  $\tilde{v}_2(k) \in R^m$ . In order to design a controller which forces the closed-loop system into the sliding mode we use the linear reaching law presented in [21] and [18], which is given by

$$\sigma_h(k+1) = \Phi \sigma_h(k), \quad (21)$$

where  $\Phi \in R^{m \times m}$  has been chosen to be a diagonal matrix with all its diagonal elements  $\phi_r$ ,  $r = 1, \dots, m$ , satisfying  $0 \leq \phi_r < 1$ .

*Theorem 1:* The system (20) is stable under the control input  $U$  as

$$U(k) = (\bar{S}_2 \bar{B}_2)^{-1} ([\Phi - \tilde{\Lambda}_{22}] \sigma_h(k) - \tilde{\Lambda}_{21} \bar{h}_1(k) - \tilde{v}_2(k)). \quad (22)$$

*Proof:* Applying the above control law to the system (20) results in the following closed-loop system

$$\begin{bmatrix} \bar{h}_1(k+1) \\ \sigma_h(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{\Lambda}_{11} & \tilde{\Lambda}_{12} \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \bar{h}_1(k) \\ \sigma_h(k) \end{bmatrix} + \begin{bmatrix} \tilde{v}_1(k) \\ 0 \end{bmatrix}. \quad (23)$$

Obviously, the eigenvalues of  $\Phi$  are designed to be stable. In addition, it can be easily proved that  $\tilde{\Lambda}_{11} = \tilde{\Lambda}_{11} - \tilde{\Lambda}_{12} \bar{S}_2^{-1} \bar{S}_1$  which is designed to be a stable matrix by (17). Consequently, the system (20) is stable under the control law (22). ■

*Remark 2:* As seen from (21) and (23), the states  $\sigma_h(k)$  in sliding surface are not influenced by the boundary conditions. However, broadly speaking, the boundary conditions have an obvious effect on the dynamics of the total states and, in addition, the final value that the states approach to. The bound of states will be found in Section V when the system (7) is assumed to have boundary conditions and an additional uncertainty term.

It should be mentioned that instead of control law (22), another direct method is also possible to obtain the sliding control law [18], [20]. Assuming that matrices  $\bar{S}_1$  and  $\bar{S}_2$  have been designed (by for instance LQR design) such that the reduced order dynamics (17) is stable. Now, from (21), (15) and (7) we have

$$\Phi \sigma_h(k) = S[\Lambda H(k) + BU(k) + v(k)], \quad (24)$$

where  $S$  is defined in (18). Thus, the control law can be determined by

$$U(k) = (SB)^{-1} [\Phi SH(k) - S\Lambda H(k) - Sv(k)]. \quad (25)$$

This control law is called direct control law which can be obtained directly after computing sliding matrix  $S$ .

#### IV. STRONG STABILITY FOR DISCRETE-TIME SYSTEMS

The stability studied in Theorem 1 is based on the notion of asymptotic stability. To analysis the effect of uncertainty, in this section, another notion of stability referred to as *strong asymptotic stability* or briefly *strong stability* is studied.

*Definition 1:* Consider the following LTI discrete-time system

$$x(k+1) = Ax(k), \quad k \geq 0, \quad x \in R^n. \quad (26)$$

The system (26) is strong asymptotically stable if and only if  $\|x(k+1)\| < \|x(k)\|$ , for all  $k > 0 : x(k) \neq 0$ .

*Lemma 1 ( [23]):* The system in (26) is strong asymptotically stable if and only if

$$\|A\| < 1. \quad (27)$$

Although it is clear that all the systems are not strong stable with respect to Euclidean norm, in the following it will be shown that each asymptotically stable system is strong asymptotically stable with respect to a specific norm.

*Lemma 2 ( [23]):* For each asymptotically stable LTI discrete-time system (26) there exists a similarity transformation matrix  $T$ , such that  $\|TAT^{-1}\| < 1$ .

*Lemma 3 ( [22]):* If  $T$  is a nonsingular similarity transformation matrix then

$$\|A\|_T \triangleq \|TAT^{-1}\|, \quad (28)$$

is a matrix norm.

Thus, it should be noted that  $\|TAT^{-1}\|$  can be assumed as a special norm ( $T$ -norm) of the matrix  $A$ . Lemmas 2 and 3 ensure that for every asymptotically stable system there exists a specific norm such that, under that norm, the closed-loop system is strong asymptotically stable [23]. The issue of strong asymptotic stability for uncertain systems will be studied in the next section. More importantly, a specific similarity transformation  $T$  such that  $\|TAT^{-1}\| < 1$  which is known as the balancing transformation is introduced in [24].

*Lemma 4:* Define a vector norm  $\|x\|_T$  as

$$\|x\|_T \triangleq \|Tx\|. \quad (29)$$

Then, the matrix norm in (28) is an induced norm for the vector norm in (29).

*Proof:* It can be shown that for every  $A \in R^{n \times n}$ ,

$$\begin{aligned} \max_{\|x\|_T=1} \|Ax\|_T &= \max_{\|x\|_T=1} \|T Ax\| = \\ &= \max_{\|Tx\|=1} \|TAT^{-1}Tx\| = \|TAT^{-1}\| = \|A\|_T. \end{aligned} \quad (30)$$

*Remark 3:* Moreover, for every vector  $x \in R^n$  we have

$$\|T\|^{-1} \|x\|_T \leq \|x\| \leq \|T\| \|x\|_T. \quad (31)$$

Besides, for every matrix  $A \in R^{n \times n}$ ,

$$\text{cond}(T)^{-1} \|A\|_T \leq \|A\| \leq \text{cond}(T) \|A\|_T, \quad (32)$$

where  $\text{cond}(T) = \|T\| \|T^{-1}\|$  is the *condition number* of the transformation matrix  $T$ .

#### V. DSMC FOR UNCERTAIN FORM

Generally, the discrepancy between the original PDE model (1) and its discrete 1D vectorial approximation (7) depends on the step sizes  $\Delta x$  and  $\Delta t$ . This matter can be assumed as uncertainty in the state space (7). It means that we have

$$H(k+1) = \Lambda H(k) + BU(k) + v(k) + f(k), \quad (33)$$

where the uncertain term  $f(k) \in R^{(n-1)}$ . We consider the following general uncertainty

$$f(k) = \Delta H(k) + d_k, \quad (34)$$

where  $\Delta$  shows the unknown uncertainty with the bound  $\|\Delta\| < \alpha_0$  ( $\|\cdot\|$  is the induced Euclidean or induced spectral norm). Moreover, the term  $d_k \in R^{(n-1)}$ , indicates the external disturbance and it is assumed that  $\|d_k\| < \beta_0$ , where  $\beta_0$  is a known positive constant. As a result, we can write

$$\|f(k)\| < \alpha_0 \|H(k)\| + \beta_0. \quad (35)$$

Thus, with the control law (22) the closed-loop system is

$$\begin{bmatrix} \tilde{h}_1(k+1) \\ \sigma_h(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{\Lambda}_{11} & \tilde{\Lambda}_{12} \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \tilde{h}_1(k) \\ \sigma_h(k) \end{bmatrix} + \begin{bmatrix} \tilde{v}_1(k) \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{f}_1(k) \\ \tilde{f}_2(k) \end{bmatrix}, \quad (36)$$

where  $\tilde{f}(k) = T_s T_r f(k)$  and also it is obvious that  $\tilde{f}_2(k) = S f(k)$ ; here  $S$  is defined in (18). Now, back to the original coordinate  $H(k)$  from (36), it is obtained that

$$H(k+1) = \Lambda_{eq} H(k) + v_{eq}(k) + f(k), \quad (37)$$

where

$$\Lambda_{eq} = T_r^{-1} T_s^{-1} \begin{bmatrix} \tilde{\Lambda}_{11} & \tilde{\Lambda}_{12} \\ 0 & \Phi \end{bmatrix} T_s T_r, v_{eq}(k) = T_r^{-1} T_s^{-1} \begin{bmatrix} \tilde{v}_1(k) \\ 0 \end{bmatrix}. \quad (38)$$

Here, a new theoretical justification of the robustness of the closed-loop system against uncertainty based on the strong stability notion represented in Section IV, is given. According to Lemmas 2 and 3, it is known that a special induced norm ( $T$ -norm) of the closed loop matrix ( $\Lambda_{eq}$ ) is necessarily smaller than 1. Hence, from (37) and (38), one can write that

$$\|H(k+1)\|_T \leq \|\Lambda_{eq}\|_T \|H(k)\|_T + \|v_{eq}(k)\|_T + \|f(k)\|_T. \quad (39)$$

Here, the induced norm  $\|\Lambda_{eq}\|_T$  is used as proved in Lemma 4. Besides, the bound of uncertainty in this new coordinate can be shown as

$$\begin{aligned} \|f(k)\|_T &\leq \|T \Delta T^{-1} T H(k)\| + \|T d_k\| \leq \\ &\|T\| \|\Delta\| \|T^{-1}\| \|T H(k)\| + \|T\| \|d_k\| \leq \\ &\alpha_0 \|T\| \|T^{-1}\| \|H(k)\|_T + \beta_0 \|T\|. \end{aligned} \quad (40)$$

Introducing  $\alpha = \text{cond}(T) \alpha_0$  and  $\beta = \|T\| \beta_0$ , the following theorem is represented to consider the bound on system states under  $T$ -norm and in the presence of uncertainty of the form (34).

*Theorem 2:* Consider the uncertain system (37) with uncertainty defined in (34). If we have  $\Psi + \alpha < 1$ , where  $\Psi = \|\Lambda_{eq}\|_T$ , then for any  $\varepsilon > 0$ , the states of the closed-loop system are bounded under  $T$ -norm by  $\rho_\varepsilon = \frac{\beta_1}{1-\Psi-\alpha} + \varepsilon$ , when  $k \rightarrow \infty$ , where  $\beta_1 = \beta + \max_k (\|v_{eq}(k)\|_T)$ . Moreover, the distance from sliding surface is bounded by  $S_T \rho_\varepsilon$ , where  $S_T = \|S\|_T$ .

*Proof:* From (39) and (40), we derive

$$\|H(k+1)\|_T < (\Psi + \alpha) \|H(k)\|_T + \beta_1. \quad (41)$$

Thus, it can be shown that

$$\|H(k)\|_T < (\Psi + \alpha)^k \|H(0)\|_T + \sum_{i=0}^{k-1} (\Psi + \alpha)^i \beta_1. \quad (42)$$

As a result, when  $\Psi + \alpha < 1$ ,

$$\|H(k)\|_T < (\Psi + \alpha)^k \|H(0)\|_T + \frac{\beta_1}{1-\Psi-\alpha}. \quad (43)$$

Hence, one can derive that

$$\forall \varepsilon > 0, \exists K > 0, s.t. \quad (44)$$

$$\|H(k)\|_T < \rho_\varepsilon = \varepsilon + \frac{\beta_1}{1-\Psi-\alpha}, \forall k > K.$$

Besides, from (15) it can be concluded that

$$\|\sigma_h(k)\|_T \leq S_T \|H(k)\|_T \quad (45)$$

Consequently, the bound on the distance from sliding surface is  $S_T \rho_\varepsilon$ . ■

*Remark 4 (disturbance only):* When there is no model-based uncertainty and the system includes only disturbance  $d_k$ , the parameter  $\alpha$  is zero in (35). As a consequence, the states of the closed loop system will be bounded by  $\rho_\varepsilon = \frac{\beta_1}{1-\Psi} + \varepsilon$ .

*Remark 5:* To find the bound in (44) in terms of Euclidean norm ( $\hat{\rho}_\varepsilon$ ), one can simply obtain from (44) and (31) that

$$\|H(k)\| < \hat{\rho}_\varepsilon = \varepsilon + \frac{\text{cond}(T) [\beta_0 + \max_k (\|v_{eq}(k)\|)]}{1-\Psi-\alpha}. \quad (46)$$

## VI. SIMULATION RESULTS

We consider the heat flow equation (1) over  $0 \leq t \leq 40(\text{sec})$  and  $0 \leq x \leq 42(\text{m})$  with  $\Upsilon = 1(\frac{m}{\text{sec}})$ . Using the discretization equations (5) and (6), with  $\Delta t = 1(\text{sec})$  and  $\Delta x = 2(\text{m})$ , we have  $\kappa = 0.25$ . Since  $0 \leq \kappa \leq 0.5$  the obtained discrete scheme will be stable. In addition, it is supposed that the state initial conditions are equal to 1, and also initial control inputs are equal to 0. Moreover, it is supposed that the state boundary conditions (2) and boundary control inputs are set to zero. Also, it is assumed that the control input is in spatially piece-wise constant form with the control input matrix of the form  $B_1$  in (9) with  $\mu = 1$ . As a result, control is defined only at the space points 1, 3, 5,  $\dots$ , 19 on the grid. It is clear that the pair  $(\Lambda, B)$  is controllable. To compute orthogonal matrix  $T_r$  we have used Matlab  $QR$  command. The matrix  $\tilde{S}_2 \in R^{10 \times 10}$  is chosen as  $\tilde{B}_2^{-1}$  and in addition matrix  $\tilde{S}_1 \in R^{10 \times 10}$  is determined by LQR design with state weighting of  $10I_{10}$  and control weighting of  $10^2 I_{10}$ . Eventually, matrix  $S$  is achieved from equation (18). Furthermore, it is assumed that  $f(k) = 0.2H(k) + 0.2 \sin(\frac{H(k)}{3\pi}) \cos(\frac{H(k)}{2\pi})$ . Here,  $\Phi = 0.3I_{10}$  is used in control law (22) and the results of applying SMC for discretized state space (7) are given in Figures 1-3. These figures show the system states, 2D control effort and also 2-norm of control law and states, respectively. Since the 2D figures are not good indexes for studying the performance of controller, 2-norm of vectors  $U(k)$  and  $H(k)$  are given in Figure 3 as an index to study the performance of DSMC.

## VII. CONCLUSION

In this paper, we have developed new results on the application of DSMC to a kind of dynamics described by PDEs. Firstly, the underlying PDE is discretized by the use of Taylor's expansion and then the attained 2D discrete system is converted to 1D vector form. As a result of the proposed method of this paper, space variable is hidden in the final 1D process structure. Secondly, to stabilize an unstable discretized 1D vectorial form, DSMC is applied to our problem successfully. Next, the discretizing error which is modelled in uncertainty term is studied. To this end, using notion of

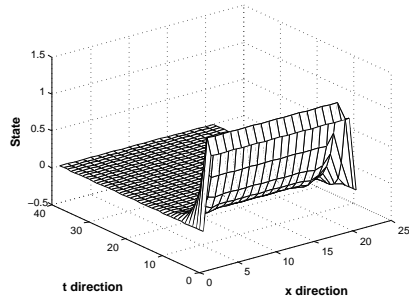


Fig. 1. States for spatially piece-wise constant controller, with  $\Phi = 0.3I_{10}$

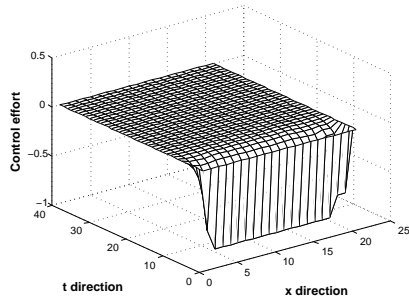


Fig. 2. Spatially piece-wise constant control effort, with  $\Phi = 0.3I_{10}$

strong stability and similarity transformation matrices a new framework to analyze the state bound in the presence of general uncertainty is proposed. Furthermore, since there are space constraints on the location of actuators (spatially piece-wise constant, sparse form or boundary control), a general strategy which can handle all these schemes is proposed in this paper. The simulation results confirm that the proposed DSMC is efficient and effective for the underlying PDEs.

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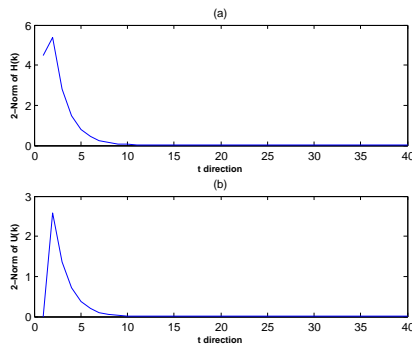


Fig. 3. 2-norm of the (a) state vector and (b) control effort with  $\Phi = 0.3I_{10}$

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