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A Chebyshev interval method for nonlinear dynamic systems under uncertainty

by

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Abstract

This paper proposes a new interval analysis method for the dynamic response of nonlinear systems with uncertain-but-bounded parameters using Chebyshev polynomial series. Interval model can be used to describe nonlinear dynamic systems under uncertainty with low-order Taylor series expansions. However, the Taylor series-based interval method can only suit problems with small uncertain levels. To account for larger uncertain levels, this study introduces Chebyshev series expansions into interval model to develop a new uncertain method for dynamic nonlinear systems. In contrast to the Taylor series, the Chebyshev series can offer a higher numerical accuracy in the approximation of solutions. The Chebyshev inclusion function is developed to control the overestimation in interval computations, based on the truncated Chevbyshev series expansion. The Mehler integral is used to calculate the coefficients of Chebyshev polynomials. With the proposed Chebyshev approximation, the set of ordinary differential equations (ODEs) with interval parameters can be transformed to a new set of ODEs with deterministic parameters, to which many numerical solvers for ODEs can be directly applied. Two numerical examples are applied to demonstrate the effectiveness of the proposed method, in particular its ability to effectively control the overestimation as a non-intrusive method.

Key words: Interval model; Chebyshev polynomial series; Dynamic response of nonlinear systems; Ordinary differential equations (ODEs).

1. Introduction

In engineering, a number of dynamic systems governed by Ordinary Differential Equations (ODEs) are in the presence of uncertainty. In practical mechanical and structural systems, a variety of uncertainties are inherent in loads, parameters, material properties, fraction tolerance, boundary conditions and geometric dimensions, due to the complexity of real-world problems. Parameter uncertainties may lead to obvious changes of system dynamic responses, especially for nonlinear systems. Many methods can be applied to account for various uncertainties in mechanical dynamic systems [1, 2], including the reliability-based optimization (RBO) [3] and robust design optimization (RDO) approaches [4]. It is noted [5-7] that smaller parameter uncertainties might be propagated in the design and thus result in relatively larger uncertainties in the dynamic response of nonlinear systems involving uncertain parameters.

Probabilistic methods [8-12] have been widely applied to a range of uncertain problems, in which the uncertain parameters are usually expressed as stochastic variables with precise probability distributions, under the assumption of knowing complete information. However, it is not always available to get the complete statistical information to define probability distribution functions in engineering problems. Furthermore, Ben-Haim and Elishakoff [2] denoted that even small variations deviating from the real values may cause relatively large errors of the probability distributions in the feasible region of the design space. As a result, the non-probabilistic uncertain method has emerged as the beneficial supplement to the conventional probabilistic method [13].

Interval method based on the set theory belongs to one of the typical non-probabilistic methods, in which interval variables are used to represent upper and lower bounds of the uncertain-but-bounded parameters. Interval model makes it possible to measure uncertainties for bounded parameters without knowing the complete information of the system. Recently, the interval method has drawn much attention and is experiencing popularity in the areas of uncertain analysis and design. The determination of lower and upper bounds for an uncertain parameter is much easier than the identification of a precise probability distribution. Based on the interval arithmetic, the interval method calculates the upper and lower bounds of the true solution. However, one of the major shortcomings is the relatively large overestimation caused by the so-called "wrapping effect", intrinsic in interval computations. As a result, how to effectively control the overestimation is the key in the interval arithmetic.

Several interval methods have been established to solve a number of static engineering problems [14, 15]. In practical applications, there are also a great number of dynamic problems governed by ODEs. If the ODEs with parameter uncertainty are solved using the conventional interval method, the overestimation will be accumulated in the process of numerical iterations. Hence, some particular interval algorithms tailored for special problems have been proposed to solve the ODE-based dynamic uncertain problems. These methods can be roughly classified into two categories, the first of which is rigorous enclosure methods, and the second of which is approximation methods. Interval Taylor series [16-18] and Taylor model [19] methods are the two typical rigorous enclosure methods. Taylor model method uses the high-order Taylor series expansion to approximate system responses by adding a remainder interval term to enclose the true solution, which can reduce the wrapping effect caused by the dependency of interval variables. Lin [20] proposed a VSPODE method which combined the two methods to solve ODEs with interval parameters, for sharper interval results. In the above methods, the overestimation cannot be ignored, because the remainder interval terms are included in a number of iterative computations. Furthermore, even if higher-order Taylor series expansions are employed, the overestimation may still exist because the wrapping effect is intrinsic in conventional interval computations.

The approximation interval method can produce a solution, which is not the precise solution of the problem but close to the exact solution. Qiu et al. [21, 22] developed non-probabilistic interval analysis method for the dynamic response analysis of structures using the finite element and parameter perturbation methods. The interval analysis method and convex model were employed in the analysis of structure dynamics response under uncertain conditions [23, 24]. Wu et al. [25] employed the first-order Taylor expansion to analysis the dynamic response of linear structural systems with interval parameters. Zhang et al. [26] used the matrix perturbation theory and interval arithmetic to estimate upper and lower bounds of dynamic responses of closed-loop systems with interval parameters. With the first-order Taylor expansion, Han et al. [27] approximated transient responses of composite-laminated plates via a linear function of uncertain parameters. Qiu et al. [28] studied the dynamic response of nonlinear vibration systems with uncertainties, where the ranges of nonlinear dynamic responses were estimated through the interval mathematics based on the second-order Taylor series expansion.

Most of the aforementioned methods can be regarded as a type of simplified Taylor series methods, without considering the remainder interval terms. Most of these methods adopted the low-order Taylor series to approximate dynamic responses, and so they were limited to problems with lower-level of uncertainties. For problems exhibiting high-level uncertainties, the high-order Taylor series expansion has to be employed in order to reduce the approximation error, as a result, which leads to large overestimation for dynamic nonlinear systems.

To obtain the dynamic response ranges of nonlinear systems with large uncertainty, this study employs the Chebyshev series to approximate the true solution, through which higher numerical accuracy of the approximation can be achieved. The Chebyshev inclusion function is proposed to calculate the bounds of interval functions, due to its capability to effectively control the overestimation compared to the Taylor inclusion function. The Mehler integral method is applied to find coefficients of the Chebyshev inclusion function. One ODE with uncertain parameters will be transformed to several ODEs with deterministic parameters based on the Chebyshev inclusion function, to which many standard numerical methods for solving ODEs can be directly applied.

2. Problem description

In most cases, the dynamic response of nonlinear systems is governed by a set of ODEs, especially by the second-order ODEs. Since the second-order ODEs can be generally transformed to the first-order ODEs in numerical implementation, this study is focused on the first-order ODEs only for the sake of simplicity but without losing any generality. The ODEs of a l-dimensional problem can be described as follows:

$$\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}, \mathbf{x}),\ \mathbf{y}(t_0) = \tilde{\mathbf{y}} \tag{1}$$

where t denotes time, $\mathbf{f}=[f_1, f_2, \ldots, f_l]^{\mathrm{T}}$ is a l-dimensional vector comprising nonlinear functions, $\mathbf{x}=[x_1, x_2, \ldots, x_k]^{\mathrm{T}}$ is a k-dimensional vector consisting of uncertain parameters, and $\tilde{\mathbf{y}}$ is the l-dimensional initial vector. The notation in bold denotes vector, while the notation in italic denotes scalar.

Assuming the complete statistical information of parameter uncertainty is unknown and only the bounds of uncertain parameters are known, the uncertain parameter vector can be expressed as follows:

$$\underline{\mathbf{x}} \le \mathbf{x} \le \overline{\mathbf{x}} \tag{2}$$

where $\underline{\mathbf{x}} = [\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k]^{\mathrm{T}}$ and $\overline{\mathbf{x}} = [\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k]^{\mathrm{T}}$ denote the vectors including the lower and upper bounds of uncertain parameters, respectively.

Introducing the interval notation, the k-dimensional interval vector **[x]** can be defined as

$$[\mathbf{x}] = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{x_i : \underline{x}_i \le x_i \le \overline{x}_i, i = 1, 2, \ldots, k\} \tag{3}$$

The solution of Eq. (1) subject to Eq. (2) can be expressed as an interval vector [**y**]

$$[\mathbf{y}] = \{\mathbf{y} : \mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}, \mathbf{x}), \mathbf{y}(t_0) = \tilde{\mathbf{y}}, \mathbf{x} \in [\mathbf{x}]\} \tag{4}$$

The lower bound and upper bound to be obtained can be given by

$$\underline{\mathbf{y}} = \min_{\mathbf{x} \in [\mathbf{x}]} \{\mathbf{y} : \mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}, \mathbf{x}), \mathbf{y}(t_0) = \tilde{\mathbf{y}}, \mathbf{x} \in [\mathbf{x}]\} \tag{5}$$

$$\overline{\mathbf{y}} = \max_{\mathbf{x} \in [\mathbf{x}]} \{\mathbf{y} : \mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}, \mathbf{x}), \mathbf{y}(t_0) = \tilde{\mathbf{y}}, \mathbf{x} \in [\mathbf{x}]\} \tag{6}$$

In general, the precise bounds of the solution cannot be easily obtained through Eqs. (5) and (6), as the numerical solver have to rely on global searching algorithms which will expense the computational cost. However, the interval arithmetic can be used to estimate the range of the solution, to reduce the computational cost of the global search. Interval arithmetic operations are defined on the real set R, such that the interval results close to all possible real results. Given the two real intervals [x] and [y], the four arithmetic operations are defined by

$$[x] * [y] = \{x * y : x \in [x], y \in [y]\} \quad \text{for} * \in \{+, -, \times, \div\} \tag{7}$$

In general, most continuous functions can be transformed to the above four arithmetic operations through the Taylor series expansion, which can mathematically keep the equivalence of the transformation. For the arithmetic operations, more detailed information can be found in the relevant references [29]. It is easy to use interval arithmetic operations to generate ranges of the solution, as aforementioned, but the interval arithmetic will lead to large overestimation if no particular algorithm is incorporated. In the following sections, the algorithm based on the Chebyshev series is proposed to reduce the overestimation.

3. Chebyshev inclusion function for interval functions

3.1 Taylor inclusion function

Consider a function f from R^n to R^m, the interval function $[f]$ from IR^n to IR^m can be defined as an inclusion function for the function f if it satisfies

$$\forall [x] \in IR^n, f([x]) \subset [f]([x]) \tag{8}$$

The direct calculation of an enclosure for a function using the interval arithmetic will often lead to large overestimation. As an alternative, the high-order Taylor series expansion can be used to make the result sharper. If the function f is n+1 times differentiable on the interval [x], the nth-order Taylor inclusion function [30] can be expressed as follows:

$$\left[f_{T_n}\right]([x]) = f(x_c) + f'(x_c)[\Delta x] + \dots + \frac{1}{n!} f^{(n)}(x_c)[\Delta x]^n + \frac{1}{(n+1)!}\left[f^{(n+1)}([x])\right][\Delta x]^{n+1} \tag{9}$$

where x_c denotes the midpoint of [x]

$$x_c = mid([x]) = \frac{1}{2}(\underline{x} + \overline{x}) \tag{10}$$

Here $[\Delta x]$ is a symmetry interval of [x], which is expressed by

$$[\Delta x] = \left[\frac{\underline{x} - \overline{x}}{2}, \frac{\overline{x} - \underline{x}}{2}\right] \tag{11}$$

In the above, Eq. (9) calculates the rigorous enclosure for the function $f(x)$. The last term in the right hand side of Eq. (9) is usually neglected to obtain the approximation of the enclosure of $f(x)$ in engineering.

3.2 Interval trigonometric function

Trigonometric functions, one of the basic theories to support the proposed method, cannot be evaluated reasonably by using the interval arithmetic, because the overestimation is involved in the Taylor inclusion function. However, trigonometric functions can be calculated by a special algorithm without leading to overestimation [30]. For an interval variable [x], the cosine function cos([x]) can be calculated as follows:

$$\begin{aligned}
&if \quad \exists k \in Z \,|\, 2k\pi - \pi \in [x] \quad then \quad \inf([\cos([x])]) = -1;\\
&else \quad \inf([\cos([x])]) = \min(\cos\underline{x},\ \cos\overline{x});\\
&if \quad \exists k \in Z \,|\, 2k\pi \in [x] \quad then \quad \sup([\cos([x])]) = 1;\\
&else \quad \sup([\cos([x])]) = \max(\cos\underline{x},\ \cos\overline{x}).
\end{aligned} \tag{12}$$

In this equation, Z is the integer set. For example, cos([1, 5.5])=[-1, 0.7087], which is shown in Fig. 1.

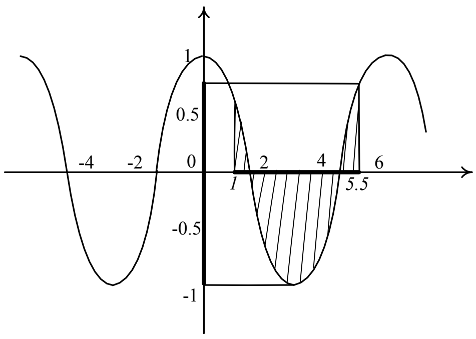


Figure. 1 Computation of cos([x])

3.3 Chebyshev inclusion function

If the function $f(x)$ is included in $C[a, b]$, which means $f(x)$ is continuous over $[a, b]$, there exists a polynomials $p(x)$ which converges to the function $f(x)$ on $[a, b]$ [31], that is

$$\left\| f(x) - p(x) \right\|_\infty < \varepsilon, \quad x \in [a,b] \tag{13}$$

This expression is validity for any $\varepsilon > 0$. Let $P_n(x)$ denote the set of polynomials of degree not bigger than n, for every nonnegative integer n, there exists a unique polynomial $p_n^*(x)$ in $P_n(x)$, such that

$$\left\| f(x) - p(x) \right\|_\infty \geq \left\| f(x) - p_n^*(x) \right\|_\infty = E_n(f), \text{ where } x \in [a,b] \tag{14}$$

For all $p(x) \in P_n(x)$ other than $p_n^*(x)$, $E_n(f)$, the infimum of maximum error, is defined as

$$E_n(f) = \inf_{p_n \in P_n} \left\| f(x) - p_n(x) \right\|_\infty, \text{ where } x \in [a,b] \tag{15}$$

Here $p_n^*(x)$ is the best uniform approximation of degree n to $f(x)$ on $[a, b]$. However, it's difficult to obtain $p_n^*(x)$ when the degree of polynomials $n>2$. The truncate Chebyshev series are very close to the best uniform approximation polynomials, which are employed to approximate the original function.

The Chebyshev polynomial for $x \in [a,b]$ of degree n is denoted by C_n and defined by [32]

$$C_n(x) = \cos n\theta, \text{ where } \theta = \arccos\left(\frac{2x - (b+a)}{b-a} \right) \in [0, \pi] \tag{16}$$

where n denotes the nonnegative integer. The Chebyshev series can also be expressed as the polynomial of $x \in [-1,1]$, and the recurrence relations are defined by

$$\begin{cases} C_0(x) = 1, \quad C_1(x) = x, \\ C_{n+1}(x) = 2xC_n(x) - C_{n-1}(x) \end{cases}. \tag{17}$$

For the multi-dimensional problem, the polynomials are the tensor product of each one-dimensional polynomial, e.g. the k-dimensional Chebyshev polynomials of $x_i \in [-1,1]$ ($i = 1, 2, \ldots, k$) is defined as:

$$C_{n_1, n_2, \ldots, n_k}(x_1, x_2, \ldots, x_k) = \cos(n_1\theta_1)\cos(n_2\theta_2)\ldots\cos(n_k\theta_k) \tag{18}$$

where $\theta_i = \arccos(x_i)$. It is noted that the one-dimensional problem with $x \in [-1,1]$ is considered here only for the consideration of simplicity without losing any generality.

The Chebyshev series are orthogonal, such that

$$\int_{-1}^{1}\frac{C_n(x)C_m(x)}{\sqrt{1-x^2}}dx=\int_0^{\pi}(\cos n\theta\cos m\theta)\,d\theta=\begin{cases}\pi, & m=n=0\\ \pi/2, & m=n\neq 0\\ 0, & m\neq n\end{cases} \tag{19}$$

where $1/\sqrt{1-x^2}$ is the weighting function. The function f(x) included in C[a, b] can be approximated as the truncate Chebyshev series of degree n:

$$f(x)\approx p_n(x)=\frac{1}{2}f_0+\sum_{i=1}^{n}f_iC_i(x) \tag{20}$$

where f_i are the constant coefficients. The error between the truncated Chebyshev series $p_n(x)$ and original function $f(x)$ is shown as follow [33]:

$$e_n(f)=\left|f(x)-p_n(x)\right|\leq\frac{2^{-n}}{(n+1)!}\left\|f^{(n+1)}\right\|_{\infty} \tag{21}$$

In the above equation, if n is large enough, $e_n(f)$ can be neglected. The truncated Chebyshev series can approximate the original function better than the truncated Taylor series, as shown by Example 1 as below. Similar to the Taylor inclusion function, the Chebyshev inclusion function is defined by

$$\left[f_{C_n}\right]([x])=\frac{1}{2}f_0+\sum_{i=1}^{n}f_iC_i([x])=\frac{1}{2}f_0+\sum_{i=1}^{n}f_i\cos i[\theta] \tag{22}$$

where $[\theta]=[0,\pi]$. It should be noted that Eq. (22) is not a rigorous inclusion function, because the error term $e_n(f)$ is not incorporated in the function $[f_{C_n}]([x])$. Fortunately, in most cases the error can be neglected, as the error is submerged in the overestimation induced by the lower terms. Since $[\theta]=[0,\pi]$, through the algorithm of interval trigonometric functions shown in Section 3.2, Eq. (22) can be calculated easily. The Chebyshev inclusion function usually produces sharper result than the Taylor inclusion function, which can be displayed by Example 1.

Considering Eqs. (19) and (20), the constant coefficients f_i can then be calculated by

$$f_i=\frac{2}{\pi}\int_{-1}^{1}\frac{f(x)C_i(x)}{\sqrt{1-x^2}}dx=\frac{2}{\pi}\int_0^{\pi}\big(f(\cos\theta)\cos i\theta\big)d\theta\text{ , where }i=0,1,2,\ldots,n \tag{23}$$

Eq. (23) will be calculated through numerical integral methods. Similarly, the multi-dimensional coefficients can be calculated as follows:

$$f_{i_1,i_2,\ldots,i_k}=\left(\frac{2}{\pi}\right)^k\int_{-1}^{1}\ldots\int_{-1}^{1}\frac{f(\mathbf{x})C_{i_1,\ldots,i_k}(\mathbf{x})}{\sqrt{1-x_1^2}\ldots\sqrt{1-x_k^2}}dx_1\ldots dx_k$$

$$=\left(\frac{2}{\pi}\right)^k\int_0^\pi\ldots\int_0^\pi\left(f\left(\cos\theta_1,\ldots,\cos\theta_k\right)\cos i_1\theta_1\ldots\cos i_k\theta_k\right)d\theta_1\ldots d\theta_k \tag{24}$$

where k denotes the number of dimensions, and the subscripts are $i_1,\ldots,i_k=0,1,2,\ldots,n$. The Mehler integral, also called Gaussian-Chebyshev integration formula [34] is employed to calculate Eqs. (23) and (24), which is a type of interpolation integral formula expressed as follows:

$$\int_a^b \rho(x) f(x)\,dx \approx \sum_{j=1}^m A_j f\left(x_j\right) \tag{25}$$

where A_j are the integral coefficients.

If the interpolation points x_1, $x_2 \ldots x_m$ are the zeros of the orthogonal polynomials of degree m with the weight function $\rho(x)$, the algebraic precision order of this integral formula is $2m$-1. For Eq. (23), if the weight function is chosen as $\rho(x)=1/\sqrt{1-x^2}$, the corresponding orthogonal polynomials will be the Chebyshev polynomials. The interpolation points are the zeros of the Chebyshev polynomials of degree m

$$x_j=\cos\theta_j,\ \text{where}\ \theta_j=\frac{2j-1}{m}\frac{\pi}{2},\ j=1,2,\ldots,m \tag{26}$$

In this case, all the integral coefficients $A_j=\pi/m$ [33]. Substitute A_j into Eq. (25), we get the Mehler integral formula as follows:

$$\int_{-1}^1\frac{1}{\sqrt{1-x^2}}f(x)\,dx\approx\frac{\pi}{m}\sum_{j=1}^m f\left(x_j\right)=\frac{\pi}{m}\sum_{j=1}^m f\left(\cos\theta_j\right) \tag{27}$$

With the Mehler integral method, the coefficients of Chebyshev polynomials are calculated as

$$f_i=\frac{2}{\pi}\int_{-1}^1\frac{f(x)C_i(x)}{\sqrt{1-x^2}}dx\approx\frac{2}{\pi}\frac{\pi}{m}\sum_{j=1}^m f\left(x_j\right)C_i\left(x_j\right)=\frac{2}{m}\sum_{j=1}^m f\left(\cos\theta_j\right)\cos i\theta_j \tag{28}$$

$$f_{i_1,\ldots,i_k}=\left(\frac{2}{\pi}\right)^k\int_{-1}^1\ldots\int_{-1}^1\frac{f(\mathbf{x})C_{i_1,\ldots,i_k}(\mathbf{x})}{\sqrt{1-x_1^2}\ldots\sqrt{1-x_k^2}}dx_1\ldots dx_k$$

$$\approx\left(\frac{2}{m}\right)^k\sum_{j_1=1}^m\ldots\sum_{j_k=1}^m f\left(\cos\theta_{j_1},\ldots,\cos\theta_{j_k}\right)\cos i_1\theta_{j_1}\ldots\cos i_k\theta_{j_k} \tag{29}$$

Eqs. (28) and (29) express the Chebyshev coefficients of a function as a linear combination of values of the function. To minimize the integral error, the parameter m is usually not less than $n+1$.

Example 1 Considering $f(x)=\arctan x$, where $x\in[-1,1]$, the approximation of degree 5 of truncated Chebyshev polynomials is given by

$$p_5(x)=0.8284\cos\theta-0.0474\cos3\theta+0.0049\cos5\theta \tag{30}$$

where $\theta=\arccos x\in[0,\pi]$. The truncated Taylor series approximation of degree 5 is expressed by

$$T_5(x)=x-x^3/3+x^5/5 \tag{31}$$

The errors for Eqs. (30) and (31) related to the original function are displayed in Fig. 2. The results show that the error of the Chebyshev polynomials is much smaller than the conventional Taylor polynomials.

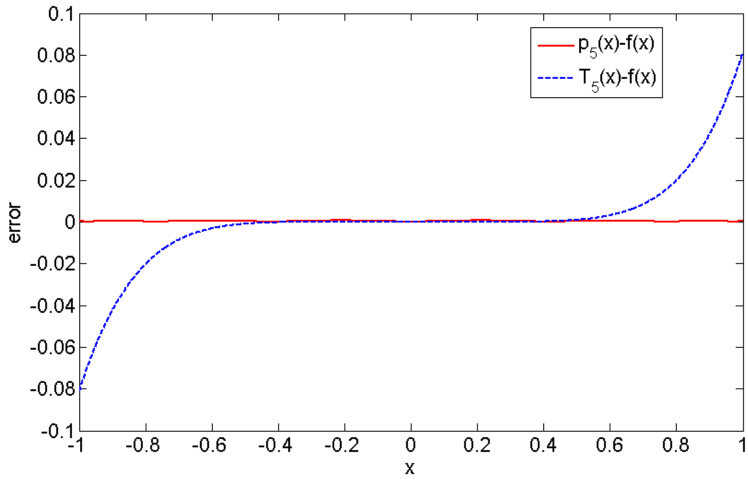


Figure. 2 Errors of $\arctan x$ for Chebyshev polynomials and Taylor polynomials

Based on the Chebyshev and Taylor inclusion function, the interval range can be given as follows:

$$\left[f_{C_5}\right]([-1,1])=[-0.8807,0.8807]\text{, and }\left[f_{T_5}\right]([-1,1])=[-1.5333,1.5333] \tag{32}$$

The actual interval range of $f(x)$ is given by

$$f([-1,1])=\arctan[-1,1]=\left[-\frac{\pi}{4},\frac{\pi}{4}\right]=[-0.7854,0.7854] \tag{33}$$

Here Eqs. (32) and (33) show that the Chebyshev inclusion function can result in less overestimation than the Taylor inclusion function.

4. Chebyshev method for solving ODEs with interval parameters

The solution of Eq. (1) can be regarded as a vector including continuous functions, with respect to time t and uncertain parameter vector $\mathbf{x}$

$$\mathbf{y}(t,\mathbf{x})=\left\{\mathbf{y}:\mathbf{y}'(t)=\mathbf{f}(t,\mathbf{y},\mathbf{x}),\mathbf{y}(t_0)=\tilde{\mathbf{y}}\right\} \tag{34}$$

Expanding Eq. (20) to a k-dimensional expression, the approximation of the vector $\mathbf{y}$ with respect to the uncertain parameter vector $\mathbf{x}$ can be given as

$$\mathbf{y}(t,\mathbf{x}) \approx \sum_{i_1=0}^{n} \cdots \sum_{i_k=0}^{n} \left(\frac{1}{2}\right)^{p} \mathbf{y}_{i_1,\ldots,i_k} C_{i_1,\ldots,i_k}(\mathbf{x}) \tag{35}$$

where p denotes the total number of *zero*(s) to be occurred in the subscripts $i_1,\ldots,i_k$, $C_{i_1,\ldots,i_k}(\mathbf{x})$ is the k-dimensional Chebyshev polynomials given in Eq. (18), and $\mathbf{y}_{i_1,\ldots,i_k}$ denotes the vector including the coefficients of Chebyshev polynomials. The Chebyshev polynomials are only the function of $\mathbf{x}$, so the coefficients vector $\mathbf{y}_{i_1,\ldots,i_k}$ is the function of time t.

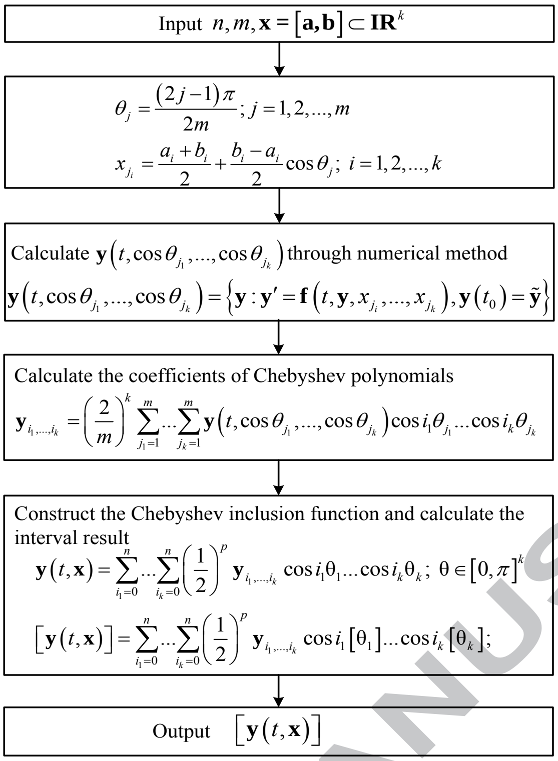
Consider Eq. (29), the coefficients are calculated by

$$\mathbf{y}_{i_1,\ldots,i_k} \approx \left(\frac{2}{m}\right)^{k} \sum_{j_1=1}^{m} \cdots \sum_{j_k=1}^{m} \mathbf{y}\left(t, \cos\theta_{j_1}, \ldots, \cos\theta_{j_k}\right) \cos i_1\theta_{j_1} \ldots \cos i_k\theta_{j_k} \tag{36}$$

where θ_j denotes the zeros of Chebyshev polynomials with degree m, as shown in Eq. (26). With Eq. (34), if the uncertain parameter vector is chosen as $\mathbf{x} = [\cos\theta_{j_1}, \ldots, \cos\theta_{j_k}]^{\mathrm{T}}$, $\mathbf{y}(t, \cos\theta_{j_1}, \ldots, \cos\theta_{j_k})$ in the above equation will be the solution of Eq. (1). As a result, any conventional numerical methods can be applied to solve the differential equations. Substituting Eq. (36) into Eq. (35), and using the concept of interval arithmetic, the interval solution of Eq. (1) can then be obtained.

In a briefly description, the proposed method can transform the original each differential equation with interval parameters into several differential equations of deterministic without any interval parameters. The detailed algorithm of the Chebyshev method can be described as **Algorithm 1**, which involves four main steps: (1) to produce the interpolation points through Eq. (26), (2) to solve the differential equations at the interpolation points to calculate $\mathbf{y}$, (3) to calculate the coefficients of Chebyshev polynomials by Eq. (29), and (4) to construct the Chebyshev inclusion function and obtain the corresponding interval $[\mathbf{y}]$.

From **Algorithm 1**, we find that the algorithm solving the uncertain problem is similar to a type of sampling method, but the sampling points (interpolation points) have been defined in advance and the post-processing are particular. Thus, the proposed method may be used to solve other uncertain dynamic problems. For example, solving the ODEs system in the third step may be replaced by solving the algebraic system, DAEs system, or even PDEs system, and then this theory would solve uncertain multibody systems [35] and uncertain structure systems, and so on. However, the accuracy and efficiency should be researched further.



Algorithm 1

To compare the Chebyshev method with the conventional Taylor method, a second-order Taylor series approximation is re-derived in this Section according to the work of [28]. Expanding **f** in Eq. (1) with respect to the vector **x** including uncertain parameters with the second-order Taylor series, we have

$$\mathbf{f} \approx \mathbf{f}(t, \mathbf{y}_c, \mathbf{x}_c) + \sum_{j=1}^{k} \left(\sum_{i=1}^{l} \frac{\partial \mathbf{f}}{\partial y_i} \frac{\partial y_i}{\partial x_j} + \frac{\partial \mathbf{f}}{\partial x_j} \right)\Bigg|_{(\mathbf{y}_c, \mathbf{x}_c)} \Delta x_j + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \left(\sum_{p=1}^{l} \sum_{q=1}^{l} \frac{\partial^2 \mathbf{f}}{\partial y_p \partial y_q} \frac{\partial y_p}{\partial x_i} \frac{\partial y_q}{\partial x_j} + 2 \sum_{p=1}^{l} \frac{\partial^2 \mathbf{f}}{\partial y_p \partial x_i} \frac{\partial y_p}{\partial x_j} + \sum_{p=1}^{l} \frac{\partial \mathbf{f}}{\partial y_p} \frac{\partial^2 y_p}{\partial x_i \partial x_j} + \frac{\partial^2 \mathbf{f}}{\partial x_i \partial x_j} \right)\Bigg|_{(\mathbf{y}_c, \mathbf{x}_c)} \Delta x_i \Delta x_j \quad (37)$$

Similarly, expanding **y** in Eq. (1) with respect to **x** by the second-order Taylor series will lead to

$$\mathbf{y} \approx \mathbf{y}_c + \sum_{j=1}^{k} \frac{\partial \mathbf{y}}{\partial x_j}\bigg|_{(\mathbf{y}_c, \mathbf{x}_c)} \Delta x_j + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{\partial^2 \mathbf{y}}{\partial x_i \partial x_j}\bigg|_{(\mathbf{y}_c, \mathbf{x}_c)} \Delta x_i \Delta x_j \quad (38)$$

Differentiating Eq. (38) with respect to t on both sides, the derivative of **y** can be expressed as

$$. \quad \mathbf{y}' \approx \mathbf{y}_c' + \sum_{j=1}^{k} \left(\frac{\partial \mathbf{y}}{\partial x_j}\bigg|_{(\mathbf{y}_c, \mathbf{x}_c)} \right)' \Delta x_j + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \left(\frac{\partial^2 \mathbf{y}}{\partial x_i \partial x_j}\bigg|_{(\mathbf{y}_c, \mathbf{x}_c)} \right)' \Delta x_i \Delta x_j \quad (39)$$

Substituting Eqs. (37) and (39) into Eq. (1), we can obtain the following ODEs via the perturbation theory

$$\mathbf{y}_c' = \mathbf{f}(t, \mathbf{y}_c, \mathbf{x}_c) \quad (40)$$

$$\left(\left. \frac{\partial \mathbf{y}}{\partial x_j} \right|_{(\mathbf{y}_c, \mathbf{x}_c)} \right)' = \left. \left(\sum_{i=1}^{l} \frac{\partial \mathbf{f}}{\partial y_i} \frac{\partial y_i}{\partial x_j} + \frac{\partial \mathbf{f}}{\partial x_j} \right) \right|_{(\mathbf{y}_c, \mathbf{x}_c)} \tag{41}$$

$$\left(\left. \frac{\partial^2 \mathbf{y}}{\partial x_i \partial x_j} \right|_{(\mathbf{y}_c, \mathbf{x}_c)} \right)' = \left. \left(\sum_{p=1}^{l} \sum_{q=1}^{l} \frac{\partial^2 \mathbf{f}}{\partial y_p \partial y_q} \frac{\partial y_p}{\partial x_i} \frac{\partial y_q}{\partial x_j} + 2 \sum_{p=1}^{l} \frac{\partial^2 \mathbf{f}}{\partial y_p \partial x_i} \frac{\partial y_p}{\partial x_j} + \sum_{p=1}^{l} \frac{\partial \mathbf{f}}{\partial y_p} \frac{\partial^2 y_p}{\partial x_i \partial x_j} + \frac{\partial^2 \mathbf{f}}{\partial x_i \partial x_j} \right) \right|_{(\mathbf{y}_c, \mathbf{x}_c)} \tag{42}$$

where $i, j = 1, \ldots, k$. Solving the ODEs given from Eq. (40) to Eq. (42) and then substituting the solutions into Eq. (38), the interval solution of **y** can be obtained.

Both the Chebyshev and Taylor methods transform the ODEs with uncertain parameters into new ODEs with deterministic parameters. However, the expression of ODEs obtained with the Taylor method is different from the original ODEs, while the Chebyshev method keeps the expression of the original ODEs unchanged. Thus, the Chebyshev method is more convenient in the numerical implementation.

5. Numerical examples

The algorithms for the Chebyshev and Taylor methods are programmed using MATLAB tool: INTLAB [36]. This paper is mainly focused on the presentation of a new interval analysis methodology to solve uncertain problems of nonlinear dynamic systems. Two numerical examples have been carefully chosen to demonstrate the effectiveness of the proposed method. Both examples include nonlinear elements, in particular the pendulum mechanism model which is highly nonlinear. The results obtained by the Chebyshev and Taylor methods are compared with the results of the scanning method [37].

5.3.1 Vehicle ride analysis

A two-degree-of-freedom quarter-car model [38] is shown in Fig. 3.

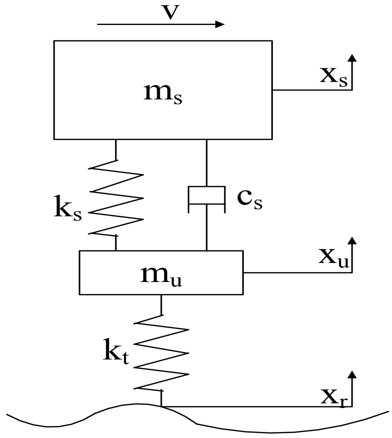


Figure. 3 Two-degree-of-freedom quarter car model

The cubic nonlinear characteristic of the spring stiffness can be expressed as follows:

$$\begin{cases} F_s = k_s\left(x_s - x_u\right) + K_s\left(x_s - x_u\right)^3 \\ F_t = k_t\left(x_u - x_r\right) + K_t\left(x_u - x_r\right)^3 \end{cases} \quad (43)$$

where F_s and F_t denote the suspension force and tyre force, k_s and k_t are the linear stiffness characteristics of the suspension and tyre, and K_s and K_t represent the cubic stiffness characteristics of the suspension and tyre, respectively. The differential equations of the system motion model are expressed as follows:

$$\begin{cases} m_s\ddot{x}_s + c_s\left(\dot{x}_s - \dot{x}_u\right) + k_s\left(x_s - x_u\right) + K_s\left(x_s - x_u\right)^3 = 0 \\ m_u\ddot{x}_u - c_s\left(\dot{x}_s - \dot{x}_u\right) - k_s\left(x_s - x_u\right) - K_s\left(x_s - x_u\right)^3 + k_t\left(x_u - x_r\right) + K_t\left(x_u - x_r\right)^3 = 0 \end{cases} \quad (44)$$

The initial condition for the above problem are given as: $x_s(t_0) = 0$, $x_u(t_0) = 0$, $\dot{x}_s(t_0) = 0$ and $\dot{x}_u(t_0) = 0$. Here, m_s, m_u, and c_s are the sprung mass, unsprung mass, and suspension damping rate, respectively. To test the vehicle ride characteristics, it is supposed that the automotive drives through a triangle road-block with a speed of u=10 *m/s*. The road model is shown in Fig. 4, and the road inputs are described as Eq (45):

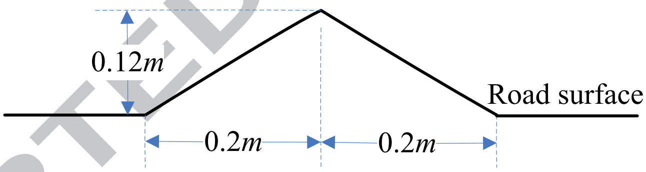


Figure. 4 Road-block model

$$x_r = \begin{cases} 0.12ut/0.2 = 6t & 0 \le t < 0.02 \\ 0.12 - 0.12u\left(t - 0.01\right)/0.2 = 0.24 - 6t & 0.02 \le t < 0.04 \\ 0 & t \ge 0.04 \end{cases} \quad (45)$$

The parameters c_s, k_s, and k_t are considered as interval parameters, and their values are shown in Table 1.

Table 1 Parameters for a quarter-car model

	m_s (*kg*)	m_u (*kg*)	c_s (*Ns/m*)	k_s (*N/m*)	k_t (*N/m*)	K_s (*N/m*^{3})	K_t (*N/m*^{3})
Mean value	375	60	1000	15000	200000	1.5×10^6	2×10^7
Range	-	-	[950,1050]	$[145,155]\times10^2$	$[19,21]\times10^4$	-	-

To solve the second-order ODEs, Eq. (44) is transformed into the following first-order ODEs

$$\begin{cases} \dot{x}_s = v_s \\ \dot{x}_u = v_u \\ \dot{v}_s = -\dfrac{1}{m_s}\left(c_s\left(v_s - v_u\right) + k_s\left(x_s - x_u\right) + K_s\left(x_s - x_u\right)^3\right) \\ \dot{v}_u = \dfrac{1}{m_u}\left(c_s\left(v_s - v_u\right) + k_s\left(x_s - x_u\right) + K_s\left(x_s - x_u\right)^3 + k_t\left(x_r - x_u\right) + K_t\left(x_r - x_u\right)^3\right) \end{cases} \tag{46}$$

Table 2 Vehicle displacement

Time (*s*)	**Lower bound**			**Upper bound**		
	Scanning	Chebyshev	Taylor	Scanning	Chebyshev	Taylor
0.1	0.0096	0.0096	0.0096	0.0100	0.0100	0.0100
0.2	0.0122	0.0121	0.0120	0.0125	0.0125	0.0127
0.3	0.0086	0.0085	0.0080	0.0092	0.0092	0.0097
0.4	0.0019	0.0019	0.0010	0.0033	0.0033	0.0042
0.5	-0.0043	-0.0043	-0.0056	-0.0026	-0.0025	-0.0013
1	0.0005	0.0004	-0.0027	0.0020	0.0020	0.0051
1.5	-0.0009	-0.0009	-0.0059	0.0002	0.0003	0.0053
2	-0.0004	-0.0005	-0.0074	0.0004	0.0004	0.0073
2.5	-0.0001	-0.0001	-0.0090	0.0004	0.0005	0.0093
3	-0.0003	-0.0004	-0.0111	0.0000	0.0001	0.0108

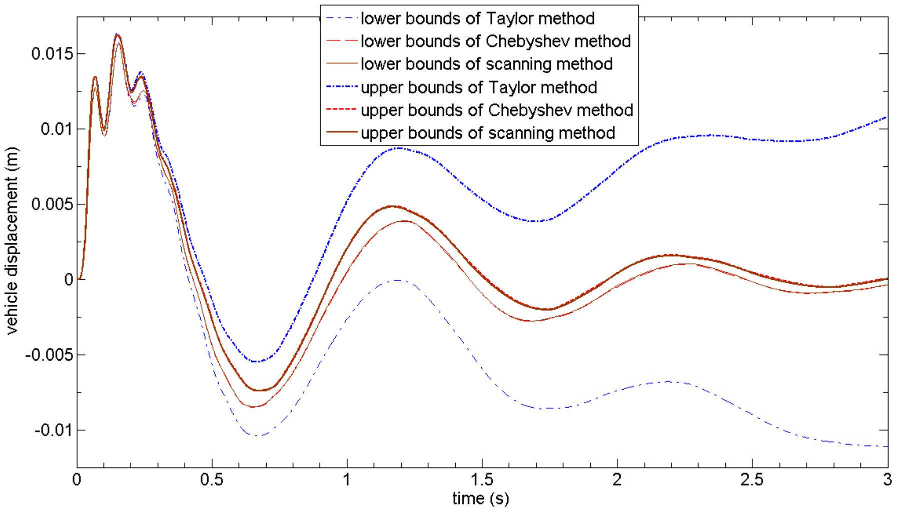


Figure. 5 Lower bounds and upper bounds of the vehicle displacement

The third-order Chebyshev method (n=3 and m=4 in **Algorithm 1**) is used to solve the ODEs within the period of 0 to 3 second. At the same time, the Taylor method is also used to solve this problem only for the comparison purpose. To calculate actual bounds of the result, the scanning method with symmetrical 10 sampling points is applied to each uncertain parameter. The iterative numbers for the Chebyshev method and scanning method are m^3=64 and 10^3=1000, respectively, which shows that the Chebyshev method spends less computational time than the scanning method. The vehicle displacement is shown in Fig. 5, and the other details are listed in Table 2. From Fig. 5 and Table 2, it can be found that the Taylor method will gradually enlarge the interval range in the numerical process, while the Chebyshev method can enclose the actual result tight in the entire numerical procedure.

5.3.2 Double pendulum problem

The schematic of a double pendulum is shown in Fig. 6, where m_1 and m_2 represent the masses of the two pendulums, respectively, and l_1 and l_2 denote the lengths of the two pendulum rods, respectively.

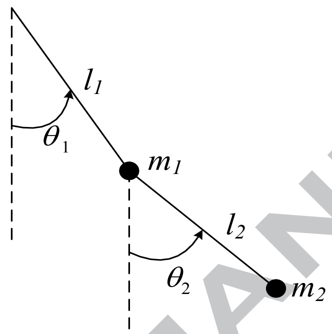


Figure. 6 Schematic of a double pendulum

According to [20], the ODEs of this system can be expressed as follows:

$$\begin{cases} \dot{\theta}_1 = \omega_1 \\ \dot{\theta}_2 = \omega_2 \\ \dot{\omega}_1 = \dfrac{-g\left(2m_1+m_2\right)\sin\theta_1 - m_2 g\sin\left(\theta_1-2\theta_2\right) - 2m_2\sin\left(\theta_1-\theta_2\right)\left(\omega_2^2 l_2 - \omega_1^2 l_1\cos\left(\theta_1-\theta_2\right)\right)}{l_1\left(2m_1+m_2-m_2\cos\left(2\theta_1-2\theta_2\right)\right)} \\ \dot{\omega}_2 = \dfrac{2\sin\left(\theta_1-\theta_2\right)\left(\omega_1^2 l_1\left(m_1+m_2\right)+g\left(m_1+m_2\right)\cos\theta_1+\omega_2^2 l_2 m_2\cos\left(\theta_1-\theta_2\right)\right)}{l_2\left(2m_1+m_2-m_2\cos\left(2\theta_1-2\theta_2\right)\right)} \end{cases} \tag{47}$$

Here θ_1 and θ_2 denote the angles of the pendulum rods, ω_1 and ω_2 are the angle velocities of the two rods, and g is the gravity acceleration. It is assumed that two interval parameters are included in the ODEs. The lengths of the two rods are considered as the interval parameters that are expressed as l_1=[0.18, 0.22]m and l_2=[0.27, 0.33]m, and the gravity acceleration is set to the point value as g=9.8m/s^2. The initial conditions are $[\theta_1,\theta_2,\omega_1,\omega_2]=[\pi/3,\pi/2,0,0]$. The Chebyshev method, Taylor method, and scanning method are applied to solve the ODEs within the period of 0s to 10s, respectively. For the Chebyshev method, we choose n=4 and m=5 in **Algorithm 1**. With respect to the scanning method, the symmetrical 10 sampling points are used for each uncertain parameter, and the numbers of iterations are m^2=25 and 10^2=100, respectively. The results of the two pendulum rod angles are shown in Fig. 7 and Fig. 8, and more detailed results are shown in Table 3 and Table 4, respectively.

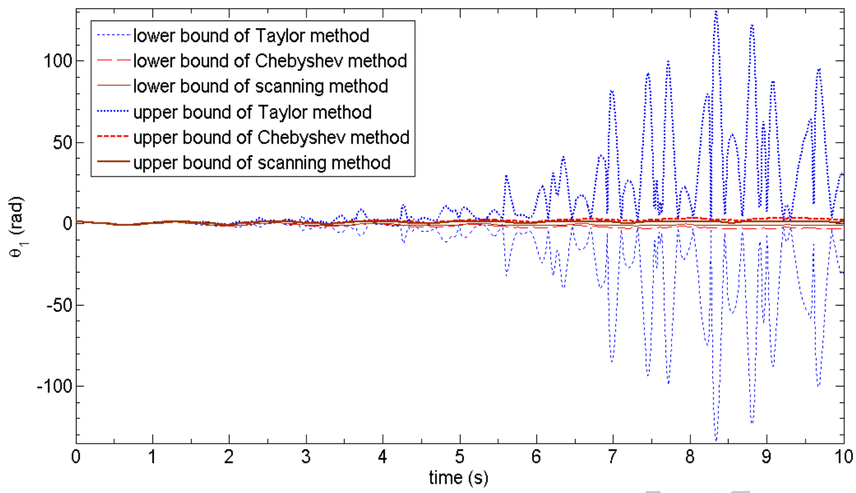


Figure. 7(a) The angle of top pendulum

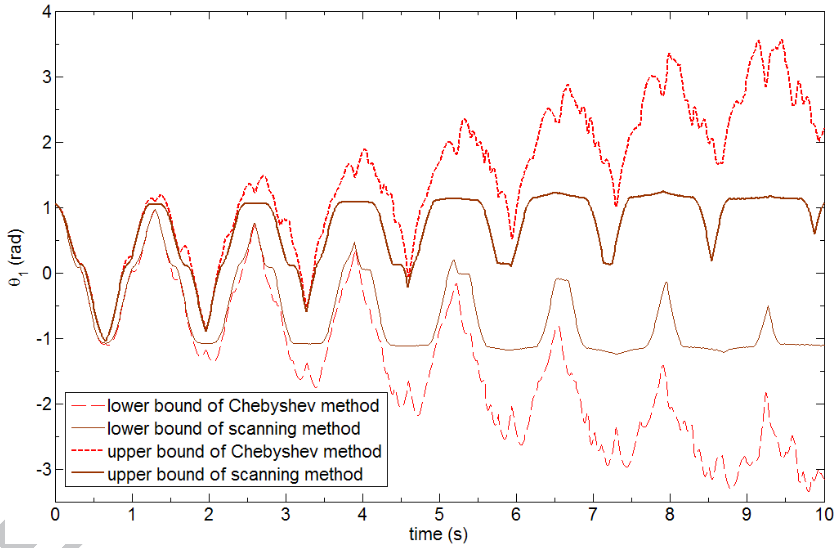


Figure. 7(b) The angle of top pendulum removing the results of Taylor method

Table 3 The angle of top pendulum

Time	**Lower bound**			**Upper bound**		
(s)	Scanning	Chebyshev	Taylor	Scanning	Chebyshev	Taylor
1	0.0405	0.0109	-0.0936	0.2229	0.2518	0.3551
2	-1.0818	-1.2651	-1.2526	-0.7598	-0.7575	-0.7693
3	-0.7636	-0.9644	-1.5272	0.2360	0.8083	1.1623
4	0.0622	-0.1984	-0.1822	1.0783	1.8313	1.7667
5	-0.6910	-0.9107	-1.2058	1.1097	1.6752	2.3863
6	-1.1744	-2.3797	-12.8006	0.3423	0.8866	11.4587
7	-1.1646	-2.5892	-84.5239	0.9261	2.2411	81.8147
8	-0.3745	-2.0719	-14.1746	1.2055	3.3500	14.7794
9	-1.1335	-2.886	-26.7229	1.1368	2.9360	28.2597

10	-1.1061	-3.147	-31.1329	1.0554	2.1891	31.0808

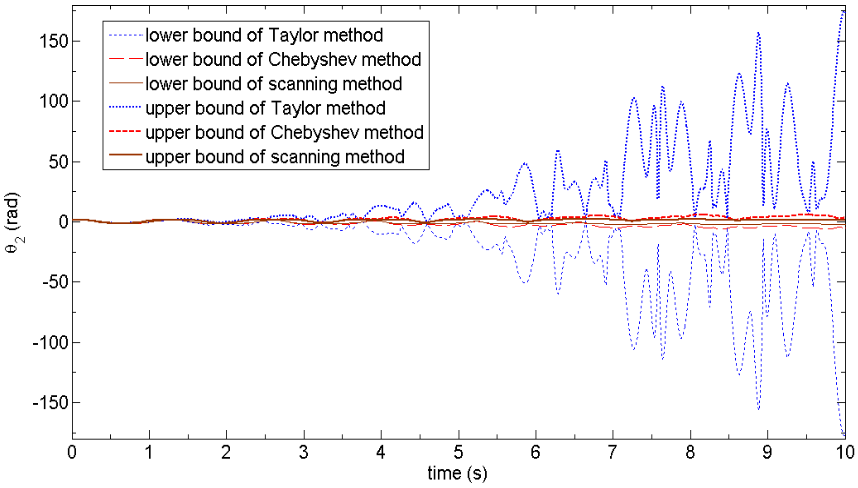


Figure. 8(a) The angle of bottom pendulum

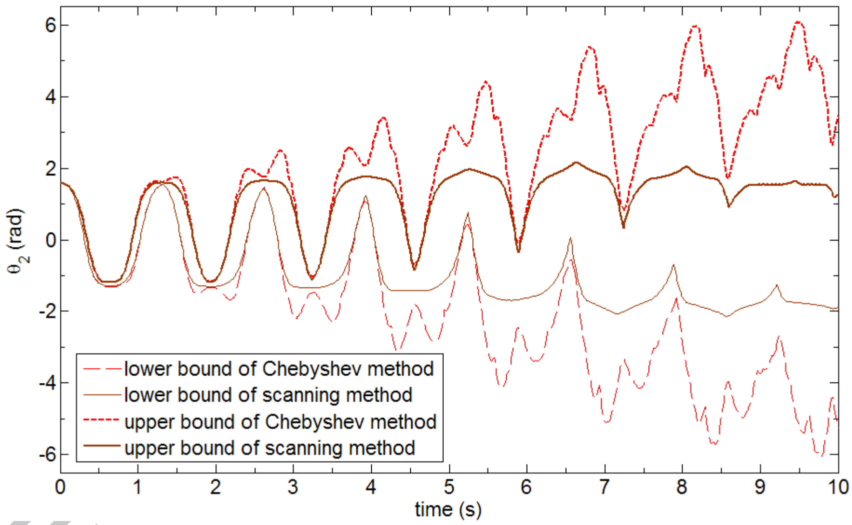


Figure. 8(b) The angle of bottom pendulum removing the results of Taylor method

Table 4 The angle of bottom pendulum

Time	**Lower bound**			**Upper bound**		
(s)	Scanning	Chebyshev	Taylor	Scanning	Chebyshev	Taylor
1	-0.3433	-0.4205	-0.5104	0.6814	0.7078	0.7991
2	-1.3028	-1.3512	-2.2099	-1.1352	-1.1227	-0.2664
3	-1.2997	-2.1218	-2.7629	1.0027	1.3839	2.1181
4	0.6821	0.6724	-10.0460	1.7480	2.3941	13.1759
5	-1.0245	-1.8788	-2.6981	1.7200	3.0183	3.4816
6	-1.6262	-3.0095	-23.1644	0.6188	0.7689	20.7134
7	-1.9158	-5.0298	-12.0554	1.7778	4.2641	12.3499
8	-1.5435	-2.6869	-51.2302	1.9604	4.8478	53.7140
9	-1.7605	-3.7274	-78.7272	1.5362	3.8724	77.2593
10	-1.8968	-5.1708	-177.2278	1.2417	3.3829	174.8497

From Figs. 7(a) and 8(a) and Tables 3 and 4, we can find that the Taylor method enlarges the range of results along with the process of the iteration, while Chebyshev method can enclosure the actual results in the entire numerical period without large overestimation. However, it is noted that there are still some overestimation for the proposed Chebyshev method from Figs. 7(b) and 8(b), which cannot be totally avoided when the interval arithmetic is applied to systems with high nonlinearity, except that additional design optimization algorithms are incorporated, which will cost the computational effort.

6. Conclusions

A new interval method with the Chebyshev series has been proposed for the analysis of the dynamic response of nonlinear systems with uncertain-but-bounded parameters. The truncated Chebyshev series expansion is used to approximate the solution in a general way with higher accuracy than the truncated Taylor series expansion, especially for non-monotonic solutions. The Chebyshev inclusion function, based on the Chebyshev polynomials, has better ability to control the overestimation compared with the conventional Taylor inclusion function. The Mehler numerical integral method is employed to calculate the coefficients of Chebyshev polynomials. With the Chebyshv polynomials, the set of ODEs with interval parameters are transformed to a new set of ODEs with deterministic parameters. Then, many numerical methods for solving ODEs can be applied to obtain the solution. The second-order Taylor method is also derived by using the perturbation theory, in order to compare with the Chebyshev method.

Two typical numerical examples are used to demonstrate the effectiveness of the proposed methodology. The numerical results show that the Chebyshev method can achieve a tighter enclosure of the results than the Taylor method, with respect to the scanning method. The proposed Chebyshev method, therefore, produces less overestimation when applied to solve nonlinear dynamic problems of ODEs. The proposed interval method can be used to evaluate the dynamic response of nonlinear systems with relatively large uncertainties. Another advantage of the proposed Chebyshev method is non-intrusive, which can be easily processed in the numerical implementation, while the Taylor method is intrusive due to the demand of additional efforts. Our ongoing research is to combine the proposed Chebyshev interval analysis method with optimization algorithms to achieve more accurate results.

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