

**BEAMFORMING AND TIME REVERSAL IMAGING FOR
NEAR-FIELD ELECTROMAGNETIC LOCALISATION
USING PLANAR ANTENNA ARRAYS**

MOHAMMED JAINUL ABEDIN

FACULTY OF ENGINEERING AND INFORMATION TECHNOLOGY
UNIVERSITY OF TECHNOLOGY, SYDNEY (UTS)

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Monammed Jainui Abedin

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Abstract

The localisation of radiating sources of electromagnetic waves in the near-field of a receiver antenna array are of use in a vast range of applications, such as in microwave imaging, wireless communications, RFID, real time localisation systems and remote sensing etc. Localisation of targets embedded in a background dielectric medium, which is usually the case in Radar, UWB imaging and remote sensing, can be done using the scattered response received at the antennas. In this thesis, we investigate methods for localisation of both near-field radiating as well as scattering sources of electromagnetic waves.

For localisation of near-field radiating sources, planar antenna arrays such as concentric circular ring array (CCRA), uniform rectangular array (URA), uniform circular array (UCA) and elliptic array are employed. The thesis employs beamforming and parameter estimation methods for localisation and proposes novel algorithms that are based on standard Capon beamformer (SCB), subspace based superresolution algorithms (MUSIC and ESPRIT) and maximum likelihood (ML) methods. Complex array geometries can suffer from severe mutual coupling and are susceptible to array modelling errors. These errors impair the performance of algorithms that are used for beamforming and parameter estimation for localisation. To overcome the limitations of standard Capon beamformer (SCB), a modified capon beamforming method is proposed to make SCB robust against both array modelling error and mutual coupling effects. The proposed method is applied with planar antenna arrays for localisation of near-field sources. Planar arrays are also used with MUSIC and ESPRIT superresolution algorithms for performance investigation in a near-field source localisation. Here, to reduce the computational burden of standard MUSIC and ESPRIT algorithms, a novel method to estimate the range using the time-delay is proposed. Lastly, to overcome the performance limitations of superresolution algorithms with planar arrays, the ML estimation is investigated for the localisation of near-field sources using planar arrays. Since ML method cannot automatically detect the number of sources, a novel method is proposed here for detecting the number of sources. Finally, performance comparisons of all the methods under investigation have been presented using computer simulations.

In order to localise targets embedded either in homogeneous or in heterogeneous background medium, we employ time reversal (TR) techniques that localise based on the received scattering responses from the embedded targets. We propose a novel beamspace-TR technique that can achieve efficient focusing on targets embedded in both a homogeneous and heterogeneous dielectric background media. It is shown that prior to back propagation, applying beamspace processing to the TR operation in the receiving mode helps achieve a reduced dimensional computation and achieves selective focusing. We have also proposed beamspace-TR-MUSIC algorithm for improving the resolution of standard TR-MUSIC algorithm. Performance of these techniques is investigated for localising the target embedded in a clutter rich dielectric background where the dielectric contrast between the target and the background medium is very low. We also propose to extend the maximum likelihood based TR (TR-ML) to improve the focusing ability and to help to localise dielectric targets embedded in a highly cluttered dielectric medium. To prove the ability of the proposed algorithms, they are applied to the problem of UWB radar imaging for the detection of early stage breast cancer. Computer simulations are used for the investigation of the imaging performance of TR, beamspace-TR, TR-MUSIC, beamspace-TR-MUSIC and TR-ML methods on a two-dimensional electromagnetic heterogeneous dielectric scattering model of the breast.

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List of Abbreviations

ULA	Uniform linear array
UCA	Uniform circular array
CCRA	Concentric circular ring array
URA	Uniform rectangular array
SCB	Standard Capon beamformer
M-SCB	Modified standard Capon beamformer
MVDR	Minimum variance distortionless response
RCB	Robust Capon beamformer
DL	Diagonal loading
SOI	Signal of interest
MUSIC	Multiple signal classification
ESPRIT	Estimation of signal parameter via rotational invariance
ML	Maximum likelihood
MDL	Minimum description length
TR	Time reversal
TR-MUSIC	Time reversal multiple signal classification
TR-ML	Maximum likelihood based Time reversal
CRLB	Cramer-Rao lower bound
EVD	Eigenvalue decomposition
SVD	Singular value decomposition
DWBA	Distorted wave Born approximation

Symbols and Notations

$(.)^*$	Complex conjugate of a matrix
$(.)^T$	Transpose of a matrix
$(.)^H$	Conjugate transpose of a matrix
$Tr(.)$	Trace of matrix
\Re	The real part of matrix
\Im	Imaginary part of a matrix
\oplus	Direct sum
$\text{Min}\{\dots\}$	Minimum of the argument list
$\text{Max}\{\dots\}$	Maximum of the argument list
\mathbf{I}_m	$(m \times m)$ identity matrix
$\text{Diag}(x)$	Diagonal matrix built with the component of the vector x
$E[.]$	Statistical expectation operator
$\ .\ _F$	Frobenious norm
$J_n(.)$	Bessel function of the first kind of order n
$H_n(.)$	Hankel function of the first kind of order n
$J'_n(.)$	Derivative of Bessel function of the first kind of order n
$H'_n(.)$	Derivative of Hankel function of the first kind of order n
$h_n(.)$	Spherical Hankel function of the first kind of order n
$Y(.)$	Legendre polynomial