FINITE SUMS THAT INVOLVE RECIPROCALS OF PRODUCTS OF GENERALIZED FIBONACCI NUMBERS

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Abstract

In this paper we find closed forms for certain finite sums. In each case the denominator of the summand consists of products of generalized Fibonacci numbers. Furthermore, we express each closed form in terms of *rational numbers*.

1. Introduction

The Fibonacci and Lucas numbers are defined, respectively, for all integers n, by

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1,$$

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \quad L_1 = 1.$$

Define, for all integers n, the sequences $\{U_n\}$ and $\{V_n\}$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0, \quad U_1 = 1, \tag{1}$$

$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p, \tag{2}$$

in which p is a positive integer. Then $\{U_n\}$ and $\{V_n\}$ are integer sequences that generalize the Fibonacci and Lucas numbers, respectively. Throughout this paper p is taken to be a positive integer. Let $\Delta = p^2 + 4$. Then with the use of standard difference techniques it can be shown that the closed forms (the Binet forms) for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n, \tag{3}$$

where $\alpha = \left(p + \sqrt{\Delta}\right)/2$, and $\beta = \left(p - \sqrt{\Delta}\right)/2$.

Next we define, for all integers n, the sequence $\{W_n\}$ by

$$W_n = pW_{n-1} + W_{n-2}, \quad W_0 = a, \quad W_1 = b, \tag{4}$$

[1] I dedicate this paper to my mother Maria. She continues to be a fountain of love and support.

where $a \geq 0$ and $b \geq 0$ are integers with $(a, b) \neq (0, 0)$. These conditions on a and b, together with the fact that p is assumed to be a positive integer, ensure that each of the reciprocal sums in this paper is well defined. It can be shown that

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{5}$$

where $A = b - a\beta$ and $B = b - a\alpha$. Associated with $\{W_n\}$ is the constant $e_W = AB = b^2 - pab - a^2$, which occurs in the sequel. Accordingly, $e_F = 1$ and $e_L = -5$.

An identity linking the Fibonacci and Lucas numbers is $L_n = F_{n-1} + F_{n+1}$, and a similar identity links the sequences $\{U_n\}$ and $\{V_n\}$. Motivated by this we define a companion sequence $\left\{\overline{W}_n\right\}$ of $\{W_n\}$ by $\overline{W}_n = W_{n-1} + W_{n+1}$. With the use of (5) we see that

$$\overline{W}_n = A\alpha^n + B\beta^n. \tag{6}$$

The sequences $\{W_n\}$ and $\left\{\overline{W}_n\right\}$ generalize the sequences $\{U_n\}$ and $\{V_n\}$, respectively. Furthermore $\overline{U}_n = V_n$, and $\overline{V}_n = \Delta U_n$, so that $\overline{F}_n = L_n$, and $\overline{L}_n = 5F_n$.

Let $k \geq 1$, $m \geq 0$, and $n \geq 2$ be integers. In this paper (in Section 3) we give a closed expression, in terms of *rational numbers*, for each of the following sums:

$$S(k, m, n) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}}{W_{ki+m}W_{k(i+1)+m}}, \tag{7}$$

$$T_1(k, m, n) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}U_{k(i+1)+m}}{W_{ki+m}W_{k(i+1)+m}W_{k(i+2)+m}}, \tag{8}$$

$$T_2(k, m, n) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}V_{k(i+1)+m}}{W_{ki+m}W_{k(i+1)+m}W_{k(i+2)+m}}, \tag{9}$$

$$T_3(k, m, n) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}\overline{W}_{k(i+1)+m}}{W_{ki+m}W_{k(i+1)+m}W_{k(i+2)+m}}, \tag{10}$$

$$X(k, m, n) = \sum_{i=1}^{n-1} \frac{1}{W_{ki+m}W_{k(i+1)+m}W_{k(i+2)+m}W_{k(i+3)+m}}. \tag{11}$$

In Section 2 we present some background and motivation for our study. In Section 3 we present our main results, and in Section 4 we present a detailed proof of one of our results. The method of proof that we outline can be used to prove all the results in this paper. It is reasonable to surmise that there are sums, analogous to (7)-(11), with longer products in their denominators, for which closed forms can be found. In Sections 5, 6, and 7 we select some of (7)-(11) and show that this is the case.

2. Background and Motivation

The broad question of summation of reciprocals that involve Fibonacci or generalized Fibonacci numbers has a long history. In this short paper it is not our intention to give this history. Instead, in the next few paragraphs, we give a brief commentary on work that we have cited on reciprocal sums that involve the summands in (7)-(11). After this, we indicate the motivation for the present paper.

André-Jeannin [1] considered the summand in (7) for the particular cases $W_n = U_n$ and $W_n = V_n$. Taking $m = 0$ and k an odd integer, he expressed the infinite sums in terms of the Lambert Series

$$L(x) = \sum_{i=1}^{\infty} \frac{x^i}{1-x^i}, \quad |x| < 1.$$

Inspired by the work of André-Jeannin [1], the authors in [5] considered the analogues of U_n and V_n for the recurrence $W_n = pW_{n-1} - W_{n-2}$, and obtained analogues of André-Jeannin's results for these sequences. The authors first obtained two finite sums in terms of the irrational roots of $x^2 - px + 1 = 0$. They then took the appropriate limits to obtain the corresponding infinite sums. Interestingly, these infinite sums did not involve the Lambert Series, but were expressed in terms of the irrational roots of $x^2 - px + 1 = 0$.

Filipponi [3] considered the summand in (7) for the particular cases $W_n = U_n$ and $W_n = V_n$. Taking $m = 0$ and k an even integer, he expressed the infinite sums in terms of α and β.

André-Jeannin [2] considered the more general sequence of integers defined by $W_n = pW_{n-1} - qW_{n-2}$, in which the initial values W_0 and W_1 are integers, and p and q are integers with $pq \neq 0$. For this sequence he studied the sums

$$\sum_{i=1}^{\infty} \frac{1}{W_{ki+m}W_{k(i+i_0)+m}} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{q^{ki+m}}{W_{ki+m}W_{k(i+i_0)+m}}, \tag{12}$$

for integers $k \geq 1$, $m \geq 0$, and $i_0 \geq 1$. In the course of his analysis, André-Jeannin expressed any finite sums in terms of one of the roots of $x^2 - px + q = 0$. Furthermore, in (12), he evaluated only the sum on the right in terms of rational numbers.

In [6] we studied the infinite sums

$$\sum_{i=1}^{\infty} \frac{\overline{W}_{k(i+m)}}{W_{ki}W_{k(i+m)}W_{k(i+2m)}}, \tag{13}$$

and

$$\sum_{i=1}^{\infty} \frac{(-1)^i}{W_{ki}W_{k(i+m)}W_{k(i+2m)}W_{k(i+3m)}}, \tag{14}$$

in which $k \geq 1$ and $m \geq 1$ are odd integers.

In [7] we studied the infinite sums

$$\sum_{i=1}^{\infty} \frac{\overline{W}_{k(i+m)}}{W_{ki}W_{k(i+m)}W_{k(i+2m)}}, \tag{15}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{W_{ki}W_{k(i+m)}W_{k(i+2m)}W_{k(i+3m)}}, \tag{16}$$

in which $k \geq 1$ and $m \geq 1$ are integers with k even. For (13)-(16) it was only in the case of (15) that we managed to express the infinite sum as a finite sum of rational numbers.

Reflecting upon the results in the foregoing paragraphs, we realized that sums (finite or infinite) involving summands that are similar to those in (7)-(11) are not usually evaluated in terms of rational numbers. It was this realization that prompted us to embark upon the investigation that led to the present paper. We list two finite sums (see [8]) that we recently evaluated in terms of rational numbers.

$$\sum_{i=1}^{n-1} \frac{1}{F_{ki+m}F_{k(i+1)+m}} = \frac{F_{k(n-1)}}{F_kF_{k+m}F_{kn+m}},\ n > 1, \tag{17}$$

where $k > 0$ is an even integer, and $m > 0$ is any integer.

$$\sum_{i=1}^{n-1} \frac{1}{L_{ki+m}L_{k(i+1)+m}} = \frac{F_{k(n-1)}}{F_kL_{k+m}L_{kn+m}},\ n > 1, \tag{18}$$

where $k \neq 0$ is an even integer, and m is any integer. In the present paper our evaluation of $S_1(k, m, n)$ generalizes both (17) and (18).

3. The Main Results

We now state our main results. As stated in Section 1, in (19)-(23) $k \geq 1$, $m \geq 0$, and $n \geq 2$ are assumed to be integers. We have

$$U_kW_{k+m}S(k, m, n) = \frac{(-1)^kU_{k(n-1)}}{W_{kn+m}}, \tag{19}$$

$$\begin{aligned} e_WU_{2k}W_{k+m}T_1(k, m, n) \quad &= \quad -\frac{1}{U_k}\left(\frac{W_{-k}U_{k(n-1)}}{W_{kn+m}} + \frac{(-1)^{k+1}W_kU_{kn}}{W_{k(n+1)+m}}\right) \\ &\quad + \frac{(-1)^{k+1}W_k}{W_{2k+m}}, \end{aligned} \tag{20}$$

$$\begin{aligned} e_W U_{2k} W_{k+m} T_2(k,m,n) &= \frac{1}{U_k}\left(\frac{\overline{W}_{-k} U_{k(n-1)}}{W_{kn+m}} + \frac{(-1)^{k+1}\overline{W}_k U_{kn}}{W_{k(n+1)+m}}\right) \\ &\quad + \frac{(-1)^k \overline{W}_k}{W_{2k+m}}, \end{aligned} \tag{21}$$

$$U_k W_{k+m} T_3(k,m,n) = \frac{(-1)^k}{W_{2k+m}} - \frac{(-1)^{kn} W_{k+m}}{W_{kn+m} W_{k(n+1)+m}}, \tag{22}$$

$$\begin{aligned} e_W V_k U_{3k} W_{k+m} X(k,m,n) &= \frac{(-1)^m \overline{W}_{4k+m}}{W_{2k+m} W_{3k+m}} + \frac{(-1)^{k+m}}{U_k^2}\left(\frac{U_{k(n-1)}}{W_{kn+m}}\right. \\ &\quad \left. + \frac{(-1)^{k+1} V_{2k} U_{kn}}{W_{k(n+1)+m}} + \frac{U_{k(n+1)}}{W_{k(n+2)+m}}\right). \end{aligned} \tag{23}$$

To illustrate, we present two examples of (23). Let $k = 1$ and $m = 0$. Then for $W_n = F_n$ (23) becomes

$$\sum_{i=1}^{n-1} \frac{1}{F_i F_{i+1} F_{i+2} F_{i+3}} = \frac{7}{4} - \frac{1}{2}\left(\frac{F_{n-1}}{F_n} + \frac{3F_n}{F_{n+1}} + \frac{F_{n+1}}{F_{n+2}}\right). \tag{24}$$

Let $k = 2$ and $m = 0$. Then for $W_n = L_n$ (23) becomes

$$\sum_{i=1}^{n-1} \frac{1}{L_{2i} L_{2(i+1)} L_{2(i+2)} L_{2(i+3)}} = -\frac{1}{432} - \frac{1}{360}\left(\frac{F_{2(n-1)}}{L_{2n}} - \frac{7F_{2n}}{L_{2(n+1)}} + \frac{F_{2(n+1)}}{L_{2(n+2)}}\right). \tag{25}$$

4. The Method of Proof

In this section, to illustrate our method of proof, we prove (23) in detail. We require the following four identities, which can be proved with the use of the closed forms:

$$\begin{aligned} U_{kn} W_{kn+m} - U_{k(n-1)} W_{k(n+1)+m} &= (-1)^{k(n+1)} U_k W_{k+m}, & (26) \\ U_{4k} W_{2k+m} - (-1)^k U_k W_{3k+m} - U_k^2 \overline{W}_{4k+m} &= (-1)^k U_{3k} W_{k+m}, & (27) \\ W_{2k+m} W_{3k+m} - W_{k+m} W_{4k+m} &= (-1)^{k+m} e_W U_k^2 V_k, & (28) \end{aligned}$$

and

$$\begin{aligned} & W_{kn+m} W_{k(n+1)+m} - V_{2k} W_{kn+m} W_{k(n+3)+m} \\ & + W_{k(n+2)+m} W_{k(n+3)+m} = (-1)^{kn+m} e_W U_k V_k U_{3k}. \end{aligned} \tag{29}$$

For $n \geq 2$ denote the right side of (23) by $R(k,m,n)$. Then

$$\begin{aligned} R(k,m,n+1) - R(k,m,n) &= \frac{(-1)^{k+m}}{U_k^2}\left(\frac{U_{kn}}{W_{k(n+1)+m}} - \frac{U_{k(n-1)}}{W_{kn+m}}\right) \\ &+ \frac{(-1)^{m+1}V_{2k}}{U_k^2}\left(\frac{U_{k(n+1)}}{W_{k(n+2)+m}} - \frac{U_{kn}}{W_{k(n+1)+m}}\right) \\ &+ \frac{(-1)^{k+m}}{U_k^2}\left(\frac{U_{k(n+2)}}{W_{k(n+3)+m}} - \frac{U_{k(n+1)}}{W_{k(n+2)+m}}\right). \end{aligned}$$

With the use of (26) the right side becomes

$$\frac{(-1)^{kn+m}W_{k+m}}{U_k}\left(\frac{1}{W_{kn+m}W_{k(n+1)+m}} - \frac{V_{2k}}{W_{k(n+1)+m}W_{k(n+2)+m}} + \frac{1}{W_{k(n+2)+m}W_{k(n+3)+m}}\right), \tag{30}$$

and with the use of (29) we see that (30) simplifies to

$$\frac{e_W V_k U_{3k} W_{k+m}}{W_{kn+m}W_{k(n+1)+m}W_{k(n+2)+m}W_{k(n+3)+m}}. \tag{31}$$

Thus

$$\begin{aligned} &R(k,m,n+1) - R(k,m,n) \\ &= e_W V_k U_{3k} W_{k+m}\left(X(k,m,n+1) - X(k,m,n)\right). \end{aligned} \tag{32}$$

Next,

$$\begin{aligned} R(k,m,2) &= \frac{(-1)^{k+m}}{U_k^2}\left(\frac{U_k}{W_{2k+m}} + \frac{(-1)^{k+1}U_{4k}}{W_{3k+m}} + \frac{U_{3k}}{W_{4k+m}}\right) \\ &+ \frac{(-1)^m\overline{W}_{4k+m}}{W_{2k+m}W_{3k+m}}. \end{aligned} \tag{33}$$

Expressing the right side of (33) as a fraction with denominator $U_k^2W_{2k+m}W_{3k+m}W_{4k+m}$, the numerator is

$$\begin{aligned} &(-1)^{m+1}W_{4k+m}\left(U_{4k}W_{2k+m} - (-1)^kU_kW_{3k+m} - U_k^2\overline{W}_{4k+m}\right) \\ &+ (-1)^{k+m}U_{3k}W_{2k+m}W_{3k+m}. \end{aligned} \tag{34}$$

Then, with the use of (27) and (28), we see that

$$\begin{aligned} R(k,m,2) &= \frac{(-1)^{k+m}U_{3k}\left(W_{2k+m}W_{3k+m} - W_{k+m}W_{4k+m}\right)}{U_k^2W_{2k+m}W_{3k+m}W_{4k+m}} \\ &= \frac{e_W U_k^2 V_k U_{3k}}{U_k^2W_{2k+m}W_{3k+m}W_{4k+m}} \\ &= \frac{e_W V_k U_{3k}}{W_{2k+m}W_{3k+m}W_{4k+m}} \\ &= e_W V_k U_{3k} W_{k+m} X(k,m,2). \end{aligned} \tag{35}$$

Taken together, (32) and (35) show that (23) is true.

We remark that all the results in this paper can be proved in a similar manner. Above we made use of identities (26)-(29) to assist in the proof. An alternative method is to simply use brute force. Specifically, to prove that two quantities are equal we substitute the closed forms of the sequences in question and expand with the use of a computer algebra system. All of the results in this paper can be proved with this method. With this method any occurrence of e_W is replaced by AB.

5. Sums that Belong to the Same Family as $T_3\left(k,m,n\right)$

Throughout the remainder of this paper $k \geq 1$, $m \geq 0$, and $n \geq 2$ are assumed to be integers. Let us write (22) as

$$U_k W_{k+m} W_{2k+m} T_3\left(k,m,n\right) = (-1)^k - \frac{(-1)^{kn} W_{k+m} W_{2k+m}}{W_{kn+m} W_{k(n+1)+m}}. \tag{36}$$

Now define the following sum, where the denominator of the summand consists of seven factors.

$$T_7\left(k,m,n\right) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} \overline{W}_{k(i+3)+m}}{W_{ki+m} W_{k(i+1)+m} \cdots W_{k(i+6)+m}}. \tag{37}$$

In (37) we have modified our notation to make it more suggestive. Specifically, the number 7 in the subscript of T_7 denotes seven factors in the denominator of the summand. We continue this convention in what follows.

Next, define the following sum, where the denominator of the summand consists of eleven factors.

$$T_{11}\left(k,m,n\right) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} \overline{W}_{k(i+5)+m}}{W_{ki+m} W_{k(i+1)+m} \cdots W_{k(i+10)+m}}. \tag{38}$$

Finally, define the following sum, where the denominator of the summand consists of fifteen factors.

$$T_{15}\left(k,m,n\right) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} \overline{W}_{k(i+7)+m}}{W_{ki+m} W_{k(i+1)+m} \cdots W_{k(i+14)+m}}. \tag{39}$$

In each of (37)-(39) the analogy with (10) is clear. Furthermore, the pattern in (36) can be extended to yield

$$U_{3k} W_{k+m} \cdots W_{6k+m} T_7\left(k,m,n\right) = (-1)^k - \frac{(-1)^{kn} W_{k+m} \cdots W_{6k+m}}{W_{kn+m} \cdots W_{k(n+5)+m}}, \tag{40}$$

$$U_{5k}W_{k+m}\cdots W_{10k+m}T_{11}\left(k,m,n\right)=(-1)^{k}-\frac{(-1)^{kn}W_{k+m}\cdots W_{10k+m}}{W_{kn+m}\cdots W_{k(n+9)+m}}, \quad (41)$$

and

$$U_{7k}W_{k+m}\cdots W_{14k+m}T_{15}\left(k,m,n\right)=(-1)^{k}-\frac{(-1)^{kn}W_{k+m}\cdots W_{14k+m}}{W_{kn+m}\cdots W_{k(n+13)+m}}. \quad (42)$$

The lists of formulas (37)-(39) and (40)-(42) have clearly defined patterns and can easily be extended by the reader.

Are there similar sums, for which closed forms exist, that have $5, 9, 13, \ldots$ factors in the denominator of the summand? We have discovered such sums. The first few representatives of these sums are

$$t_5\left(k,m,n\right) = \sum_{i=1}^{n-1}\frac{\overline{W}_{k(i+2)+m}}{W_{ki+m}W_{k(i+1)+m}\cdots W_{k(i+4)+m}}, \quad (43)$$

$$t_9\left(k,m,n\right) = \sum_{i=1}^{n-1}\frac{\overline{W}_{k(i+4)+m}}{W_{ki+m}W_{k(i+1)+m}\cdots W_{k(i+8)+m}}, \quad (44)$$

$$t_{13}\left(k,m,n\right) = \sum_{i=1}^{n-1}\frac{\overline{W}_{k(i+6)+m}}{W_{ki+m}W_{k(i+1)+m}\cdots W_{k(i+12)+m}}. \quad (45)$$

We have found that

$$U_{2k}W_{k+m}\cdots W_{4k+m}t_5\left(k,m,n\right) = 1-\frac{W_{k+m}\cdots W_{4k+m}}{W_{kn+m}\cdots W_{k(n+3)+m}}, \quad (46)$$

$$U_{4k}W_{k+m}\cdots W_{8k+m}t_9\left(k,m,n\right) = 1-\frac{W_{k+m}\cdots W_{8k+m}}{W_{kn+m}\cdots W_{k(n+7)+m}}, \quad (47)$$

$$U_{6k}W_{k+m}\cdots W_{12k+m}t_{13}\left(k,m,n\right) = 1-\frac{W_{k+m}\cdots W_{12k+m}}{W_{kn+m}\cdots W_{k(n+11)+m}}. \quad (48)$$

We have not been able to find closed forms for those counterparts to (43)-(45) where each summand is multiplied by $(-1)^{ki}$. The sums (43)-(45) together with their respective closed forms (46)-(48) have clearly defined patterns and can easily be extended by the reader.

Let $k=2$ and $m=0$. Then for $W_n=F_n$ (40) becomes

$$31933440\sum_{i=1}^{n-1}\frac{L_{2(i+3)}}{F_{2i}F_{2(i+1)}\cdots F_{2(i+6)}}=1-\frac{3991680}{F_{2n}F_{2(n+1)}\cdots F_{2(n+5)}}. \quad (49)$$

Let $k=1$ and $m=0$. Then for $W_n=L_n$ (40) becomes

$$33264\sum_{i=1}^{n-1}\frac{5(-1)^{i}F_{i+3}}{L_iL_{i+1}\cdots L_{i+6}}=-1-\frac{16632(-1)^{n}}{L_nL_{n+1}\cdots L_{n+5}}. \quad (50)$$

6. Sums that Belong to the Same Family as $S(k,m,n)$

Define the following sum, where the denominator of the summand consists of six factors.

$$S_6(k,m,n) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}}{W_{ki+m}W_{k(i+1)+m}\cdots W_{k(i+5)+m}}. \tag{51}$$

We have managed to find a closed form for (51). To present this closed form succinctly we define three quantities $c_i = c_i(k)$ as follows:

$$\begin{aligned} c_0 &= 1, \\ c_1 &= (-1)^{k+1}V_kV_{3k}, \\ c_2 &= \frac{(-1)^k\left(V_{6k}+V_{2k}+2(-1)^k\right)}{2}. \end{aligned}$$

Then

$$e_W^2U_k\cdots U_{5k}\left(S_6(k,m,n)-S_6(k,m,2)\right) = \tag{52}$$
$$U_{k(n-2)}\sum_{i=0}^{2} c_i\left(\frac{1}{W_{(2+i)k+m}W_{(n+i)k+m}} + \frac{1}{W_{(6-i)k+m}W_{(n+4-i)k+m}}\right).$$

Indeed, numerical evidence suggests that a similar closed form exists for the analogous sum $S_{10}(k,m,n)$ with ten factors in the denominator of the summand, and for the analogous sum $S_{14}(k,m,n)$ with fourteen factors in the denominator of the summand. Numerical evidence also suggests that similar closed forms exist for analogous sums that have $18, 22, 26, \ldots$ factors in the denominator of the summand.

Let $k=2$ and $m=0$. Then for $W_n = F_n$ (52) becomes

$$\begin{aligned} 27720\sum_{i=1}^{n-1}\frac{1}{F_{2i}F_{2(i+1)}\cdots F_{2(i+5)}} - \frac{1}{144} &= F_{2(n-2)}\left(\frac{1}{3F_{2n}} - \frac{27}{4F_{2(n+1)}}\right. \\ &\quad + \frac{331}{21F_{2(n+2)}} - \frac{54}{55F_{2(n+3)}} \\ &\quad \left. + \frac{1}{144F_{2(n+4)}}\right). \end{aligned} \tag{53}$$

Let $k=1$ and $m=0$. Then for $W_n = L_n$ (52) becomes

$$\begin{aligned} 750\sum_{i=1}^{n-1}\frac{(-1)^i}{L_iL_{i+1}\cdots L_{i+5}} + \frac{125}{2772} &= F_{n-2}\left(\frac{1}{3L_n} + \frac{1}{L_{n+1}} - \frac{19}{7L_{n+2}}\right. \\ &\quad \left. + \frac{4}{11L_{n+3}} + \frac{1}{18L_{n+4}}\right). \end{aligned} \tag{54}$$

7. Sums that Belong to the Same Family as $X(k,m,n)$

Define the following sum, where the denominator of the summand consists of eight factors.

$$X_8(k,m,n) = \sum_{i=1}^{n-1} \frac{1}{W_{ki+m}W_{k(i+1)+m}\cdots W_{k(i+7)+m}}. \tag{55}$$

We have managed to find a closed form for (55). In order to present this closed form we define four quantities $f_i = f_i(k)$ as follows:

$$\begin{aligned} f_0 &= 1, \\ f_1 &= -\frac{U_{3k}V_{4k}}{U_k}, \\ f_2 &= \frac{(-1)^k U_{3k}\left(V_{8k}+(-1)^k V_{2k}+1\right)}{U_k}, \\ f_3 &= \frac{(-1)^{k+1}V_{4k}\left(V_{8k}+V_{4k}+2(-1)^k V_{2k}+2\right)}{2}. \end{aligned}$$

Then

$$e_W^3 U_k \cdots U_{7k}\left(X_8(k,m,n) - X_8(k,m,2)\right) = \tag{56}$$
$$(-1)^m U_{k(n-2)} \sum_{i=0}^{3} f_i \left(\frac{1}{W_{(2+i)k+m}W_{(n+i)k+m}} + \frac{1}{W_{(8-i)k+m}W_{(n+6-i)k+m}}\right).$$

Numerical evidence suggests that similar closed forms exist for analogous sums that have $12, 16, 20, \ldots$ factors in the denominator of the summand.

Let $k=1$ and $m=0$. Then for $W_n = F_n$ (56) becomes

$$\begin{aligned} 3120\sum_{i=1}^{n-1}\frac{1}{F_iF_{i+1}\cdots F_{i+7}} - \frac{1}{21} &= F_{n-2}\left(\frac{1}{F_n} - \frac{7}{F_{n+1}} - \frac{30}{F_{n+2}} + \frac{70}{F_{n+3}}\right. \\ &\quad \left. - \frac{45}{4F_{n+4}} - \frac{14}{13F_{n+5}} + \frac{1}{21F_{n+6}}\right). \end{aligned} \tag{57}$$

8. Concluding Comments

We have discovered closed forms for variants of (8)-(10) that we do not present here. For instance, we have discovered a closed form for

$$\sum_{i=1}^{n-1} \frac{(-1)^{ki}U_{ki+m}}{W_{ki+m}W_{k(i+1)+m}W_{k(i+2)+m}}. \tag{58}$$

Furthermore, in the spirit of Sections 5-7, we have discovered lengthier analogues of these variants. The possibilities seem endless.

Acknowledgment of priority: Just prior to submitting this article I was made aware of Theorem 1 in [4]. In fact our result (19) is a consequence of Theorem 1 in [4], as is our main result in [8]. I thank Zhi-Wei Sun for alerting me to [4].

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