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TIME-DELAY SYSTEMS: STABILITY, SLIDING MODE CONTROL AND STATE ESTIMATION

by

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Certificate of Authorship/Originality

I certify that the work in this thesis has not been previously submitted for a degree nor has it been submitted as a part of the requirements for other degree except as fully acknowledged within the text.

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That Dinh Nguyen

ABSTRACT

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Time delays and external disturbances are unavoidable in many practical control systems such as robotic manipulators, aircraft, manufacturing and process control systems and it is often a source of instability or oscillation. This thesis is concerned with the stability, sliding mode control and state estimation problems of time-delay systems. Throughout the thesis, the Lyapunov-Krasovskii (L-K) method, in conjunction with the Linear Matrix Inequality (LMI) techniques is mainly used for analysis and design.

Firstly, a brief survey on recent developments of the L-K method for stability analysis, discrete-time sliding mode control design and linear functional observer design of time-delay systems, is presented. Then, the problem of exponential stability is addressed for a class of linear discrete-time systems with interval time-varying delay. Some improved delay-dependent stability conditions of linear discrete-time systems with interval time-varying delay are derived in terms of linear matrix inequalities.

Secondly, the problem of reachable set bounding, essential information for the control design, is tackled for linear systems with time-varying delay and bounded disturbances. Indeed, minimisation of the reachable set bound can generally result in a controller with a larger gain to achieve better performance for the uncertain dynamical system under control. Based on the L-K method, combined with the delay decomposition approach, sufficient conditions for the existence of ellipsoid-based bounds of reachable sets of a class of linear systems with interval time-varying delay

and bounded disturbances, are derived in terms of matrix inequalities. To obtain a smaller bound, a new idea is proposed to minimise the projection distances of the ellipsoids on axes, with respect to various convergence rates, instead of minimising its radius with a single exponential rate. Therefore, the smallest possible bound can be obtained from the intersection of these ellipsoids.

This study also addresses the problem of robust sliding mode control for a class of linear discrete-time systems with time-varying delay and unmatched external disturbances. By using the L-K method, in combination with the delay decomposition technique and the reciprocally convex approach, new LMI-based conditions for the existence of a stable sliding surface are derived. These conditions can deal with the effects of time-varying delay and unmatched external disturbances while guaranteeing that all the state trajectories of the reduced-order system are exponentially convergent to a ball with a minimised radius. Robust discrete-time quasi-sliding mode control scheme is then proposed to drive the state trajectories of the closed-loop system towards the prescribed sliding surface in a finite time and maintain it there after subsequent time.

Finally, the state estimation problem is studied for the challenging case when both the system's output and input are subject to time delays. By using the information of the multiple delayed output and delayed input, a new minimal order observer is first proposed to estimate a linear state functional of the system. The existence conditions for such an observer are given to guarantee that the estimated state converges exponentially within an ϵ-bound of the original state. Based on the L-K method, sufficient conditions for ϵ-convergence of the observer error, are derived in terms of matrix inequalities. Design algorithms are introduced to illustrate the merit of the proposed approach.

From theoretical as well as practical perspectives, the obtained results in this thesis are beneficial to a broad range of applications in robotic manipulators, airport navigation, manufacturing, process control and in networked systems.

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Nomenclature and Notation

Throughout the thesis, capital letters denote matrices and lower-case alphabet and Greek letters denote column vectors and scalars, respectively and the following nomenclatures and notations are used:

- TDS: Time-delay system
- FWM: Free weighting matrix
- SMC: Sliding mode control
- VSS: Variable structure system
- CTSMC: Continuous-time sliding mode control
- DTSMC: Discrete-time sliding mode control
- QSM: Quasi-sliding mode
- QSMB: Quasi-sliding mode band
- L-K: Lyapunov-Krasovskii
- LQR: Linear quadratic regulator
- LMI: Linear matrix inequality
- LKF: Lyapunov-Krasovskii functional
- MAB: Maximum allowable bound
- PAM: Piecewise analysis method
- IQC: Integral quadratic constraint
- SISO Single input single output
- MIMO: Multi input multi output
- $\mathbb{R}$: Field of real numbers
- $\mathbb{R}^+$: Set of non-negative reals
- $\mathbb{Z}$: Set of all integer numbers
- $\mathbb{Z}^+$: Set of non-negative integer numbers
- $\mathbb{Z}[a, b] = \mathbb{Z} \cap [a, b]$

- $\mathbb{N}$: Set of all natural numbers
- $\mathbb{R}^n$: n-dimensional space
- $\mathbb{R}^{n\times m}$: Space of all matrices of $(n \times m)$-dimension
- $\mathcal{C}$: Continuous function
- A^T: Transpose of matrix A
- A^{-1}: Inverse of matrix A
- $A > B$: Inequality between real vectors or matrices are understood componentwise
- I_n: Identity matrix of dimension $n \times n$
- 0_n: Zero matrix of dimension $n \times n$
- $(*)$: in a matrix means the symmetric term
- $\lambda(A)$: Set of all eigenvalues of matrix A
- $\lambda_m(A)$: Smallest eigenvalue of matrix A
- $\lambda_M(A)$: Largest eigenvalue of matrix A
- diag$[A, B, C]$: Block diagonal matrix with diagonal entries A, B, C
- $\circ$: the Hadamard product, i.e., $(A \circ B)_{i,j} = A_{i,j}.B_{i,j}$
- $||.||$: Euclidean norm of a vector or spectral norm of a matrix

Chapter 1

Introduction

1.1 Background

It is well-known that uncertainties including parametric variations, time delay and external disturbances are usually unavoidable in practical control systems due to data transfer, signal transformation, modelling inaccuracies, linearization approximations, unknown inputs and measurement errors. and they can quite often be a source of instability or oscillations and poor control performance in control systems. Therefore, the design of any control schemes for dynamic systems subject to time delays and disturbances, must take into account these influences on the closed-loop performance, see, e.g., [99, 119, 69] and the references therein.

Stability of time-delay systems (TDS) is a basic issue in control theory and there have been considerable efforts devoted to the problem of stability analysis [40, 69]. First of all, frequency-domain methods based on the distribution of the roots of its characteristic equation or the solutions of a complex Lyapunov matrix function equation, were used to determine the stability of time-delay systems [93, 6]. However, the disadvantage of this method is that it is only suitable for systems with constant delays, since the solutions of systems with time-varying delay are much more complicated, due to its transcendental characteristic equations. Therefore, the application of frequency-domain methods for stability analysis of TDS has serious limitations [142]. Then, time-domain methods, in which the most common approaches to the stability analysis of TDS are the LKF and Razumikhin function methods, were introduced. These methods were established in the 1950s by the Russian mathematicians Krasovskii and Razumikhin, respectively. The main idea of these methods is to construct a set of appropriate LKFs or an appropriate Lyapunov function to obtain the stability criteria of TDS in the form of existence conditions. At that time, this idea was theoretically very important, however, it was impossible to derive a general solution. However, this problem was solved when Riccati equations, linear matrix inequalities (LMIs) [5], and Matlab toolboxes came into use. Consequently, time-domain methods have been widely used to address the problem of stability analysis of TDS.

Along with the concerns of stability analysis, the problem of state bounding including reachable set bounding and state convergence for TDS has received considerable attention, see, e.g., [30] and the references therein. The reachable set estimation plays an important role in designing controllers for TDS with bounded disturbances. The fact is that the smaller bound of the reachable sets is, the larger control gain is achieved, which results in the better performance of the system [70].

For TDS with disturbances and uncertainties, sliding mode control (SMC) has been recognized as a control methodology, belonging to the variable structure systems which are characterised by their robustness with respect to parameter variations and external disturbances [139]. The basic idea of the sliding mode is to drive the system trajectories into a predetermined hyperplane or surface in finite time, and maintain the trajectory on it for all subsequent time. During the ideal sliding motion, the system is completely insensitive to uncertainties or external disturbances. The dynamics and performance of the system then depend on the selection of the sliding surface. In sliding mode control, a sliding surface is first constructed to meet existence conditions of the sliding mode. Then, a discontinuous control law is synthesised to drive the system state towards the sliding surface in a finite time and maintain it thereafter on that surface. With the widespread use of digital controllers, many researchers have focused on discrete-time sliding mode control. For discrete-time sliding mode control, the state trajectory of the system will reach the switching surface and cross it in a finite time. When the trajectory has crossed the switching surface the first time, it will cross the surface again in every successive sampling period which leads to a zigzag motion about the switching surface. The size of each successive zigzagging step is stable and the trajectory stays within a specified band which is called a quasi-sliding mode band (QSMB) [36, 4]. For discrete-time sliding mode control, a reaching law is first constructed. Then, a control law is often synthesised from the reaching law, in conjunction with a specified QSMB.

For the problem of control design, the assumption of state available for feedback is often made. However, in practical control systems, due to some reasons such as mechanical structure constraint and working conditions, the state variables are not fully available for feedback. In that case, a state observer is used to estimate the system state which can not be measured directly from the system output. A full-order state observer is a model of the actual system plus a corrective term the so-called the estimation error between the model output and the actual system output. When the model output is the same as the system output, the estimation error term vanishes and the observer will be a duplicate of the system itself. On the other hand, in state feedback control applications, only a linear combination of the state variables, i.e., $Kx(t)$ is required, rather than complete knowledge of the entire state vector $x(t)$. The question therefore arises as to whether a less complex observer can be constructed to estimate a linear combination of some of the unmeasurable state variables [138]. For TDS, there remain challenging problems as to how to construct a linear functional observer for systems with delay in the state, control input and measured output.

1.2 Thesis Objectives

The main objectives of the thesis are described as follows:

First of all, a brief survey on: i)the recent development of the L-K method, in combination with the bounding, delay decomposition, free weighting matrix techniques and the descriptor system approach, in deriving the delay-dependent stability conditions; ii) the sliding mode control design for discrete-time systems; iii) linear functional observer design for TDS is presented, respectively. Then, the stability analysis for a class of discrete-time systems with interval time-varying delay is investigated. Here, a new set of LKFs, containing an augmented vector and some triple summation terms, is proposed to obtain improved delay-dependent exponential stability conditions. The conservatism of the obtained stability conditions is further reduced by utilizing some advanced techniques, including the delay decomposition and the reciprocally convex approach. In order to prove the effectiveness of the proposed approach, the obtained results will be compared with the outcomes in existing literature.

The second objective of the thesis is to derive new new sufficient conditions for the existence of the smallest possible reachable set bounding of linear systems with time-varying delay and bounded disturbances in terms of linear matrix inequalities. To achieve this objective, the L-K method, combined with the delay decomposition and free weighting matrix approach is first used to obtain the ellipsoidal bound of the reachable set of the system. Then, by utilizing the idea of minimising the projection distance of the ellipsoids on each axis with different exponential rate, the smallest possible bound is obtained from the intersection of a family of ellipsoids.

The third objective is to investigate the problem of discrete-time quasi-sliding mode control design for linear discrete-time systems with time-varying delay and unmatched disturbances. Based on the L-K method, combined with the reciprocally convex approach, new sufficient conditions for the existence of a stable sliding surface are derived in terms of LMIs. These conditions also guarantee that the effects of time-varying delay and unmatched external disturbances can be suppressed when the system is in the sliding mode and all the state trajectories are exponentially convergent to a ball whose radius can be minimised. Then, a robust discrete-time quasi-sliding mode controller is designed to drive the state trajectories of the closed-loop system towards the prescribed sliding surface in finite time and maintain it there afterwards.

Finally, the problem of the state observer design for linear systems with delays in the state, input and measured output is addressed. By using the information of the delayed output and delayed input, a novel linear functional observer is first proposed. Existence conditions of such an observer are provided to guarantee that the estimate converges robustly within an ϵ-bound of the true state. Sufficient conditions for ϵ-convergence of the observer error are derived in terms of matrix inequalities.

1.3 Thesis organization

This thesis is organised as follows:

- *Chapter 2*: In this chapter, a brief survey on the recent development of the L-K method, sliding mode control design for discrete-time systems and linear functional observer design for time-delay systems is presented, respectively.

- *Chapter 3*: This chapter addresses the problem of exponential stability of linear systems with interval time-varying delay. For the sake of relaxing the conservatism of the stability conditions, a new set of LKFs, containing an augmented vector and some triple summation terms, are introduced. Based on the combination of the reciprocally convex approach and the delay-decomposition technique, improved delay-dependent conditions for exponential stability of the system are derived in terms of LMIs.

- *Chapter 4*: The problem of reachable set bounding for a class of linear discrete-time systems that are subject to state delay and bounded disturbances is considered. Based on the L-K method, in conjunction with the delay-decomposition and free weighting techniques, sufficient conditions for the existence of ellipsoid-based bounds of reachable sets of a linear uncertain discrete system, are derived in terms of matrix inequalities. Here, a new idea is to minimise the projection distances of the ellipsoids on each axis with different exponential convergence rates, instead of minimisation of their radius with a single exponential rate. A much smaller bound can thus be obtained from the intersection of these ellipsoids.

- *Chapter 5*: This chapter considers the problem of robust discrete-time quasi-sliding mode control design for a class of discrete-time systems with time-varying delay and unmatched disturbances. Based on the L-K method, combined with the reciprocally convex approach, novel LMI-based conditions for the existence of a stable sliding surface, are first derived. These conditions also guarantee that the effects of time-varying delay and unmatched disturbances are mitigated when the system is in the sliding mode and the sliding dynamics are exponentially convergent within a ball whose radius can be minimised. Then, a robust discrete-time quasi-sliding mode controller is proposed to drive the system state trajectories towards the prescribed sliding surface in finite time and maintain it there after subsequent time.

- *Chapter 6*: This chapter deals with the problem of partial state observer design for linear systems that are subject to time delays in the control input as well as the measured output. By using the information of both the multiple delayed output and input, a novel minimal-order observer is proposed to guarantee that the estimated state converges exponentially to the original state. The conditions for the existence of such an observer are first given and then, by choosing a set of appropriate augmented LKFs, containing a triple-integral term, sufficient conditions for ϵ-convergence of the estimation error are derived in terms of matrix inequalities. Constructive design algorithms are provided. Numerical examples are given to illustrate the design procedure, practicality and effectiveness of the proposed observer.

- *Chapter 7:-Conclusion:* A brief summary of the thesis contents and its contributions are given in the final chapter. Details of the originality, novelty, contribution and innovations of the thesis have been reviewed in this chapter. Possible future works and discussions are given.

1.4 List of Publications

Journal Articles

1. **Nguyen D. That** (2014), *Discrete-time quasi-sliding mode control for a class of underactuated mechanical systems with bounded disturbances*, Journal of Computer Science and Cybernetics, vol. 30(2), pp. 93-105.

2. Q.P. Ha, **Nguyen D. That**, Phan T. Nam and H. Trinh (2014), *Partial state estimation for linear systems with output and input time delays*, ISA Transactions, vol. 53(2), pp. 327-334.

3. **Nguyen D. That**, Phan T. Nam and Q.P. Ha (2013), *Reachable set bounding for linear discrete-time systems with delays and bounded disturbances*, Journal of Optimization Theory and Applications, vol. 57(1), pp. 96-107.

4. **Nguyen D. That**, Phan T. Nam and Q.P. Ha (2012), *On sliding dynamics bounding for discrete-time systems with state delay and disturbances*, Special Issue

on Variable Structure Systems in Australian Journal of Electrical Electronics Engineering, vol. 9 (3), pp. 255-262.

5. **Nguyen D. That** and Q.P. Ha (2014), *Further results on exponential stability of linear discrete-time systems with interval time-varying delay*, **submitted for journal publication**.

6. **Nguyen D. That** and Q.P. Ha (2014), *Discrete-time sliding mode control design for linear discrete-time system with time-varying delay and unmatched disturbances*, **submitted for journal publication**.

Conference Papers

7. **Nguyen D. That**, Nguyen Khanh Quang, Raja M. T. Raja Ismail, Phan T. Nam and Q.P. Ha (2012), *Improved reachable set bounding for linear systems with discrete and distributed delays*, The 1st International Conference on Control, Automation and Information Sciences, Ho Chi Minh, Vietnam, pp. 137-141.

8. **Nguyen D. That**, Nguyen Khanh Quang, Pham Tam Thanh and Q.P. Ha (2013), *Robust exponential stabilization of underactuated mechanical systems in the presence of bounded disturbances using sliding mode control*, The 2nd International Conference on Control, Automation and Information Sciences, Nha Trang, Vietnam, pp. 206-211.

9. **Nguyen D. That** (2013), *Robust discrete-time quasi-sliding mode control design of the Pendubot in the presence of bounded disturbances*, The 2nd Vietnam Conference on Control and Automation, Da Nang, Vietnam, pp. 314-320.

Relevant Conferences

10. Raja M. T. Raja Ismail, **Nguyen D. That**, and Q.P. Ha (2012), *Observer-based trajectory tracking for a class of underactuated Lagrangian systems using higher-*

order sliding modes, The 8th IEEE International Conference on Automation Science and Engineering, Seoul, Korea, pp.1200-1205.

11. Tri Tran, **Nguyen D. That**, and Q.P. Ha (2012), *Decentralised Predictive Controllers with Parameterised Quadratic Constraints for Nonlinear Interconnected Systems*, The 1st International Conference on Control, Automation and Information Sciences, Ho Chi Minh, Vietnam, pp. 48-53- **Best Paper Award**.

12. R.M.T. Raja Ismail, **Nguyen D. That**, and Q.P. Ha (2013), *Adaptive fuzzy sliding mode control for uncertain nonlinear underactuated mechanical systems*, The 2nd International Conference on Control, Automation and Information Sciences, Nha Trang, Vietnam, pp. 212-217.

14. Nguyen Khanh Quang, Doan Quang Vinh, **Nguyen D. That** and Q. P. Ha (2013), *Observer-based integral sliding mode control for sensorless PMSM drive using FPGA*, The 2nd International Conference on Control, Automation and Information Sciences, Nha Trang, Vietnam, pp. 218-223.

15. Tri Tran, **Nguyen D. That**, and Q.P. Ha (2013), *APRC-based decentralised model predictive control for parallel splitting systems with a matrix annihilation*, The 2nd International Conference on Control, Automation and Information Sciences, Nha Trang, Vietnam, pp. 184-189.

16. Nguyen Khanh Quang, **Nguyen D. That**, Nguyen Hong Quang and Q.P. Ha (2012), *FPGA-based fuzzy sliding mode control for sensorless PMSM drive*, The 8th IEEE International Conference on Automation Science and Engineering, Seoul, Korea, pp. 172-177.

17. Pham Tam Thanh and **Nguyen D. That** (2014), *Nonlinear controller based on flatness for Permanent Magnet-Excited Synchronous Motor*, The 31st International Symposium on Automation and Robotics in Construction and Mining, Sydney, Aus-

tralia, pp. 120-125.

18. R.M.T. Raja Ismail, **Nguyen D. That**, and Q.P. Ha (2014), *Offshore container crane systems with robust optimal sliding mode control*, The 31st International Symposium on Automation and Robotics in Construction and Mining, Sydney, Australia, pp. 149-156.

Chapter 2

Literature survey

In this chapter, a brief survey on stability analysis, discrete-time sliding mode control and state estimation of time-delay systems will be presented.

2.1 Stability analysis of time-delay systems

2.1.1 Introduction

Time delay is encountered in many dynamic systems such as chemical or process control systems and networked control systems. It has been shown that time delay is the main source of the generation of oscillation which often results in poor performance and can lead to instability, see, e.g., [69, 81] and references therein. Due to the crucial importance in both theoretical and practical perspectives, during the past decades, there has been considerable effort devoted to the problem of stability of TDS. The main concern of the stability analysis in these systems, particularly for the time-varying delay case, has been how to enlarge the feasible stability region, or how to increase the maximum allowable bound (MAB) for the time delay such that the system stability is still guaranteed, or how to develop stability conditions by using the fewest decision variables, while keeping the same maximal allowable delay. Stability conditions for TDS can be divided into two types, namely delay-dependent and delay-independent. In general, delay-dependent stability conditions, which include information of the size of delays, are less conservative than the delay-independent ones.

Among the methods of stability analysis for TDS, the L-K approach has been considered to be one of the most popular methods to obtain the delay-dependent stability criteria of time-delay systems. The main philosophy of this method is first to select a positive-definite functional, then compute its time derivative (continuous-time systems) and difference (discrete-time systems) along the solutions of the system. Finally, some negativity conditions for the derivative are proposed to obtain the stability criteria in terms of linear matrix inequalities (LMIs). These LMIs can cast into a convex optimisation problem which can be solved efficiently by using the numerical algorithms [5]. Moreover, to reduce the conservatism of the delay-dependent stability conditions, some techniques such as bounding techniques, the delay decomposition, the free weighting matrix and the descriptor model transformation, were introduced to combine with the L-K method.

2.1.2 Basic stability theorems

The stability of a system, in general, refers to the ability to return to its equilibrium point in the absence of external disturbances. The stability of a control system is the primary condition for its normal operation.

Definition 1 *[142] Consider the following continuous-time system:*

$$\begin{aligned} \dot{x}(t) =& f(t, x(t)), \forall t \geq t_0, \\ x(t_0) =& x_0, \end{aligned} \tag{2.1}$$

where $x(t)$ is the state vector; $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and t is a continuous time variable. A point $x_e \in \mathbb{R}^n$ is called an equilibrium point of system (2.1) if $f(t, x_e) = 0, \forall t \geq t_0$. Without loss of generality, assume $f(t, 0) = 0$, i.e., $x_e = 0$.

- *For any $t_0 \geq 0$ and a positive scalar $\epsilon > 0$, if there exists a positive scalar $\delta_1 = \delta(t_0, \epsilon) > 0$ such that*

$$\|x(t_0)\| < \delta(t_0, \epsilon) \Rightarrow \|x(t)\| < \epsilon, \forall t \geq t_0, \tag{2.2}$$

then the system is stable in the Lyapunov sense at the equilibrium point, $x_e = 0$.

- *For any* $t_0 \geq 0$, *if there exists a positive scalar* $\delta_2 = \delta(t_0) > 0$ *such that*

$$\|x(t_0)\| < \delta_2 \Rightarrow \lim_{t \to \infty} x(t) = 0, \forall t \geq t_0, \tag{2.3}$$

  the system is asymptotically stable at the equilibrium point, $x_e = 0$.

- *For given positive scalars* $\alpha > 0$ *and* $\beta > 0$, *if there exists a positive scalar* δ_3 *such that*

$$\|x(t_0)\| < \delta_3 \Rightarrow \|x(t)\| \leq \beta \|x(t_0)\| e^{-\alpha(t-t_0)}, \forall t \geq t_0, \tag{2.4}$$

  then the system is exponentially stable at the equilibrium point, $x_e = 0$.

- *If* δ_1 *in (2.2) or* δ_2 *in (2.3) can be chosen independently of* t_0, *then the system is uniformly stable (or uniformly asymptotically stable) at the equilibrium point,* $x_e = 0$.

- *If* δ_2 *in (2.3) or* δ_3 *in (2.4) can be an arbitrarily large, finite number, then the system is globally asymptotically stable (or globally exponentially stable) at the equilibrium point,* $x_e = 0$.

Theorem 1 *(Lyapunov Stability Theorem for Continuous-Time Systems) [142] Consider system (2.1) with the equilibrium point* $x_e = 0$ *(i.e.,* $f(t, 0) = 0, \forall t$*).*

- *If there exists a positive definite function* $V(t, x(t))$ *such that*

$$\dot{V}(t, x(t)) = \frac{d}{dt} V(t, x(t)) \leq 0 \tag{2.5}$$

  then the system is stable at the equilibrium point, $x_e = 0$.

- *If there exists a positive definite function* $V(t, x(t))$ *such that*

$$\dot{V}(t, x(t)) = \frac{d}{dt} V(t, x(t)) < 0 \tag{2.6}$$

  then the system is asymptotically stable at the equilibrium point, $x_e = 0$.

- *If system (2.1) is asymptotically stable at the equilibrium point,* $x_e = 0$ *such that*

$$V(t, x(t)) \to \infty \text{ as } \|x\| \to \infty, \tag{2.7}$$

then the system is globally asymptotically stable at the equilibrium point.

Definition 2 *[142]Consider the following discrete-time system*

$$\begin{aligned} x(k+1) =& f(k, x(k)), k \geq k_0 \\ x(k_0) =& x_0, \end{aligned} \tag{2.8}$$

where $x(k) \in \mathbb{R}^n$ *is the state vector;* $f : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^n$*. If* $f(k, x_e) = 0$ *for all* $k \geq k_0$*, then* x_e *is called an equilibrium point of system (2.8). Again, without loss of generality, assume* $f(k, 0) = 0, \forall k \geq k_0$*, i.e.,* $x_e = 0$*.*

- *For any given positive integer* $k_0 \geq 0$ *and a scalar* $\epsilon > 0$*, if there exists a positive scalar* $\delta_1 = \delta(k_0, \epsilon) > 0$ *such that*

$$\|x(k_0)\| < \delta(k_0, \epsilon) \Rightarrow \|x(k)\| < \epsilon, \forall k \geq k_0, \tag{2.9}$$

then the system is stable in the Lyapunov sense at the equilibrium point, $x_e = 0$*.*

- *If the system is stable at the equilibrium point,* $x_e = 0$ *and there exists a positive scalar* $\delta_2 = \delta(k_0, \epsilon) > 0$ *such that*

$$\|x(k_0)\| < \delta(k_0, \epsilon) \Rightarrow \lim_{k \to \infty} x(k) = 0, \tag{2.10}$$

then the system is asymptotically stable at the equilibrium point, $x_e = 0$*.*

- *If there exist positive constants* $\delta_3 > 0$*,* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\|x(k_0)\| < \delta_3 \Rightarrow \|x(k\| < \beta \|x(k_0)\| e^{-\alpha(k-k_0)}, \tag{2.11}$$

then the system is exponentially stable at the equilibrium point, $x_e = 0$*.*

- *If scalars* δ_1 *or* δ_2 *can be chosen independently of* k_0*, then the system is uniformly stable or uniformly asymptotically stable, respectively, at the equilibrium point,* $x_e = 0$.

- *If scalars* δ_2 *or* δ_3 *can be an arbitrarily large, finite number, then the system is globally asymptotically stable or globally exponentially stable, respectively, at the equilibrium point,* $x_e = 0$.

Theorem 2 *(Lyapunov Stability Theorem for Discrete-Time Systems) Consider system (2.8) with the equilibrium point* x_e.

- *If there exists a positive definite function* $V(k, x(k))$ *such that*

$$\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k)) \leq 0, \forall k, \forall x \neq 0, \tag{2.12}$$

then the system is stable at the equilibrium point.

- *If there exists a positive definite function* $V(k, x(k))$ *such that*

$$\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k)) < 0, \forall k, \forall x \neq 0, \tag{2.13}$$

then the system is asymptotically stable at the equilibrium point.

- *If system (2.1) is asymptotically stable at the equilibrium point,* $x_e = 0$ *such that*

$$V(k, x(k)) \to \infty \text{ as } \|x(k)\| \to \infty, \tag{2.14}$$

then the system is globally asymptotically stable at the equilibrium point.

2.1.3 Time-delay Systems

The systems whose future evolution depends on not only the current values of the state variables but also their history, are called time-delay systems or retarded systems. This means that the future evolution of the state variable x at time t not only depends on t and $x(t)$, but also on the values of x before time t. Consequently, the Lyapunov function is a functional, $V(t, x(t+s))$, that depends on t and $x(t+s)$. This type of functional is called a LKF. The LKF methods are of crucial importance in the study of stability of time-delay systems.

Let $\mathcal{C}_n = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ be the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence, where τ is a given positive scalar representing the maximum delay. Denote $\|\phi\|_c = \sup_{-\tau \leq s \leq 0} \|\phi(s)\|$ where $\|\phi(s)\|$ represents the Euclidean norm of $\phi(s) \in \mathbb{R}^n$. Consider a class of TDS as follows

$$\begin{aligned} &\dot{x}(t) = f(t, x_t), \forall t \geq t_0, \\ &x_{t0}(\theta) = \phi(\theta), \quad -\tau \leq \theta \leq 0, \end{aligned} \tag{2.15}$$

where $x(t) \in \mathbb{R}^n$ is the state vector with $x_t = x(t+\theta)$; t_0 and $x_{t0} \in \mathbb{R}^n$ are, respectively, the initial time instant and initial state; $f \in \mathbb{R}^+ \times \mathcal{C}_n \to \mathbb{R}^n$ is assumed to be a continuous function and $f(t, 0) = 0$ for all $t \in \mathbb{R}$. System (2.15) is also assumed to have a unique solution.

Theorem 3 *(Lyapunov-Krasovskii Stability Theorem)[44, 142]*
Suppose that the function f takes bounded sets of $\mathcal{C}_n$ into bounded sets of $\mathbb{R}^n$, and $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, non-decreasing functions satisfying $u(0) = v(0) = 0$ and $u(s), v(s) > 0$ for $s > 0$.

- *If there exists a continuous function $V : \mathbb{R} \times \mathcal{C}_n \to \mathbb{R}^+$ such that*

  (a) $u(\|x(t)\|) \leq V(t, x_t) \leq v(\|x_t\|_c)$.

  (b) *The time derivative of $V(t, x_t)$ along the solutions of system (2.15), defined as $\dot{V}(t, x_t) = \lim_{s \to 0^+} \sup \frac{1}{s}[V(t+s, x_{t+s}) - V(t, x_t)]$, satisfies*

$$\dot{V}(t, x_t) \leq -w(\|x(t)\|), \tag{2.16}$$

  then the trivial solution of (2.15) is uniformly stable.

- *If the trivial solution of (2.15) is uniformly stable and $w(s) > 0$ for $s > 0$, then the trivial solution of (2.15) is uniformly asymptotically stable.*
- *If the trivial solution of (2.15) is uniformly asymptotically stable and if $\lim_{s\to\infty} u(s) = \infty$, then the trivial solution of (2.15) is globally uniformly asymptotically stable.*

For the L-K stability theorem, the information of the state variable $x(t)$ in the interval $[t-\tau,\ t]$ is required to necessitate the manipulation of functionals. Consequently, this may make the L-K theorem difficult to apply. This difficulty can sometimes be circumvented by using the Razumikhin theorem which involves only functions, but no functionals [142]. The main idea of the Razumikhin theorem is the use of a function, $V(x)$, to represent the size of $x(t)$

$$\overline{V}(x_t) = \max_{\theta\in[-\tau,0]} V(x_t). \tag{2.17}$$

It can be seen clearly that the function $\overline{V}(x_t)$ indicates the size of x_t. If $V(x(t)) < \overline{V}(x_t)$, then $\overline{V}(x_t)$ does not grow when $V(x(t)) > 0$. In fact, if the function $\overline{V}(x_t)$ does not grow, it is only necessary that the time-derivative $\dot{V}(x(t))$ is not positive whenever $V(x(t)) = \overline{V}(x_t)$.

Theorem 4 *(Razumikhin Stability Theorem)[44, 142] Suppose that the function f takes bounded sets of $\mathcal{C}_n$ into bounded sets of $\mathbb{R}^n$, and $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, non-decreasing functions and $u(s), v(s) > 0$ are positive for $s > 0, u(0) = v(0) = 0$. Let $p : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous non-decreasing function satisfying $p(s) > s$ for $s > 0$.*

- *If there exists a continuous function $V : \mathbb{R} \times \mathcal{C}_n \to \mathbb{R}^+$ such that*

  (a) $u(\|x\|) \leq V(t,x) \leq v(\|x\|), \quad t \in \mathbb{R}, x \in \mathbb{R}^n.$

(b) *The time derivative of* $V(t, x_t)$ *along the solutions of system (2.15), defined as* $\dot{V}(t, x_t) = \lim_{s\to 0^+} \sup \frac{1}{s}[V(t+s, x_{t+s}) - V(t, x_t)]$, *satisfies*

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|), \tag{2.18}$$

whenever $V(t+\theta, x(t+\theta)) \leq p(V(t, x(t))), \forall \theta \in [-\tau, 0]$, *then the trivial solution of (2.15) is uniformly stable.*

- *If the trivial solution of (2.15) is uniformly stable and* $w(s) > 0$ *for* $s > 0$, *then the trivial solution of (2.15) is uniformly asymptotically stable.*
- *If the trivial solution of (2.15) is uniformly asymptotically stable and if* $\lim_{s\to\infty} u(s) = \infty$, *then the trivial solution of (2.15) is globally uniformly asymptotically stable.*

2.1.4 Lyapunov-Krasovskii approach

Recently, the L-K approach has been recognised as a strong tool for stability analysis and stabilization of TDS. The first stage of the L-K approach is to construct a set of appropriate LKFs, then compute its time derivative or difference along the solutions of the system. By letting the derivative or difference be some negative conditions, the delay-dependent conditions for stability of the system are derived in terms of LMIs. Therefore, the selection of the LKF plays a crucial role in deriving the stability criteria. On the other hand, to further reduce the conservatism of the obtained stability conditions, some techniques such as the bounding technique, delay decomposition, free weighting matrix and model transformation are used. In this section, we will review the development of the L-K method, in combination with other techniques to derive the stability criteria of TDS. Here, for the sake of simplicity on representation, we consider two classes of continuous TDS as follows

$$\Sigma_1 : \begin{aligned} \dot{x}(t) =& Ax(t) + A_1 x(t-\tau), t \geq 0, \\ x(t) =& \phi(t), \forall t \in [-\tau, 0], \end{aligned} \tag{2.19}$$

and

$$\Sigma_2 : \begin{aligned} \dot{x}(t) =& Ax(t) + A_1 x(t - \tau(t)), t \geq 0, \\ x(t) =& \phi(t), \forall t \in [-\overline{\tau}, 0], \end{aligned} \tag{2.20}$$

where $x(t) \in \mathbb{R}^n$ is the system state; A, A_1 are known constant matrices with appropriate dimensions; $\phi(t)$ is the continuous initial condition; τ and $\tau(t)$ are, respectively, the constant time delay and time-varying delay in the state for system Σ_1 and Σ_2. The time-varying delay is assumed to be continuous in the whole process and satisfying,

$$0 < \tau(t) \leq \overline{\tau}, \tag{2.21}$$

where $\overline{\tau}$ is a positive constant representing the maximum delay, which is also used in this thesis as the upper bound of the constant delay τ of system $\sum_1$, i.e., $\tau \leq \overline{\tau}$.

2.1.5 Bounding technique

It is known that finding a better bound on some weighted cross-products arising in the stability analysis may lead to less conservative delay-dependent stability conditions. In [78, 7], some delay-dependent stability conditions for linear uncertain systems with time delay were obtained by expanding the obvious relation $|Xa + b|^2 X^{-1} \geq 0$, to the well-known inequality on the upper bound for the inner product of two vectors

$$-2ab \leq a^T X a + b^T X^{-1} b, \tag{2.22}$$

where $a \in \mathbb{R}^n, b \in \mathbb{R}^n$ and $X > 0, X \in \mathbb{R}^{n \times n}$. Later, to reduce the conservatism of stability conditions obtained in [78, 79, 7, 8], by using the Schur complement, an improved inequality was proposed in [105] as follows:

Park's Inequality [104]: Assume that $a(s) \in \mathbb{R}^{n_a}$ and $b(s) \in \mathbb{R}^{n_b}$ are given for $s \in \Omega$, where Ω is a given interval. Then, for any $X > 0, X \in \mathbb{R}^{n_a \times n_a}$ any matrix

$M \in \mathbb{R}^{n_b \times n_b}$, we have

$$-2\int_{\Omega} a(s)^T b(s)ds \leq \int_{\Omega} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} X & XM \\ \star & (M^T X + I)X^{-1}(M^T X + I)^T \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds. \tag{2.23}$$

By using this inequality and constructing the following L-K functional

$$V(t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)Qxs)ds + \int_{-\tau}^{0}\int_{t+u}^{t} \dot{x}^T(s)A_1^T X A_1 \dot{x}(s)dsdu, \tag{2.24}$$

an improved stability condition for time-delay system Σ_1 was obtained in [104] which is restated as follows:

Theorem 5 *[104] For given positive constants* $0 \leq \tau \leq \overline{\tau}$, *if there exist symmetric positive definite matrices* P, Q, V *and* W *such that the following linear matrix inequality holds*

$$\begin{bmatrix} \Delta & -W^T A_1 & A^T A_1 V & \overline{\tau}(W^T + P) \\ \star & -Q & A_1^T A_1^T V & 0 \\ \star & \star & -V & 0 \\ \star & \star & \star & -V \end{bmatrix} < 0, \tag{2.25}$$

where $\Delta = (A + A_1)^T P + P(A + A_1) + W^T A_1 + A_1^T W + Q$,

then system Σ_1 *is asymptotically stable.*

In [39], another inequality was introduced to further reduce the conservatism of the existing stability conditions in terms of the number of decision variables. This inequality directly relaxes the integral term of quadratic quantities of the LKFs into the quadratic term of the integral quantities, resulting in a linear combination of positive functions weighted by the inverses of convex parameters.

Jessen's Inequality [39]: For any constant matrix $M > 0, M \in \mathbb{R}^{m\times n}$, scalars $b > a$, vector function $y : [a, b] \to \mathbb{R}^m$ such that the following integrations are well defined, then

$$(b - a)\int_a^b y(s)^T M y(s)ds \geq \left[\int_a^b y(s)ds\right]^T M \left[\int_a^b y(s)ds\right]. \tag{2.26}$$

Based on this inequality and the following L-K functional

$$V(t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)Qx(s)ds + \tau \int_{-\tau}^{0} \int_{t+u}^{t} \dot{x}^T(s)R\dot{x}(s)dsdu, \qquad (2.27)$$

a delay-dependent stability condition for time-delay system Σ_1 was obtained in [38], which can be restated in the following theorem.

Theorem 6 *[38] For a given positive constant $\overline{\tau}$ such that $0 \leq \tau \leq \overline{\tau}$, if there exist symmetric positive definite matrices P, Q and R such that the following linear matrix inequality holds*

$$\begin{bmatrix} PA + A^T P + Q - R & PA_1 + R & \overline{\tau} A^T R \\ \star & -Q - R & \overline{\tau} A_1^T R \\ \star & \star & -Z \end{bmatrix} < 0, \qquad (2.28)$$

then system Σ_1 is asymptotically stable.

By constructing a new set of appropriate LKFs, combined with Jessen's inequality, some improved stability criteria were derived for different kinds of time-delay systems, see e.g., [40, 45, 46, 82, 103, 72, 11] and [47].

Then, by using the Schur complement, the Park's inequality was further developed in [92] to obtain an improved delay dependent stability condition for time-delay systems.

Moon's Inequality [92]: Assume that $a(s) \in \mathbb{R}^{n_a}$ and $b(s) \in \mathbb{R}^{n_b}$ and $N(s) \in \mathbb{R}^{n_a \times n_b}$ are given for $s \in \Omega$. Then, for any $X \in \mathbb{R}^{n_a \times n_a}$, $Y \in \mathbb{R}^{n_a \times n_b}$ and $Z \in \mathbb{R}^{n_b \times n_b}$, we have

$$-2 \int_{\Omega} a(s)^T N(s) b(s) ds \leq \int_{\Omega} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} X & Y - N(s) \\ \star & Z \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds, \qquad (2.29)$$

where

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0.$$

By using this inequality and choosing the L-K functional as

$$V(t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)Qx(s)ds + \int_{-\tau}^{0}\int_{t+u}^{t} \dot{x}^T(s)Z\dot{x}(s)dsdu, \qquad (2.30)$$

improved delay-dependent stability conditions were obtained in [92] which can be expressed in the following theorem.

Theorem 7 *[92] For given positive constants* $0 \leq \tau \leq \overline{\tau}$*, if there exist symmetric positive definite matrices* P, Q, X, Y *and* Z *such that the following linear matrix inequality holds*

$$\begin{bmatrix} PA + A^TP + \overline{\tau}X + Y + Y^T + Q & PA_1 - Y & \overline{\tau}A^TZ \\ \star & -Q & \overline{\tau}A_1^TZ \\ \star & \star & -\overline{\tau}Z \end{bmatrix} < 0, \qquad (2.31)$$

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0, \qquad (2.32)$$

then system Σ_1 *is asymptotically stable.*

It should be noted that Moon's inequality is a generalization of inequalities (2.22) and (2.23). Consequently, Moon's inequality has been extensively used to deal with a variety of issues of time-delay systems, see e.g., [102, 34, 10, 41, 83, 13, 11] and the references therein.

Recently, by a much improved inequality, that so-called Wirtinger-based integral inequality was introduced in [121].

Wirtinger-based integral inequality[121] For a given $n \times n$-matrix $S > 0$, any differentiable function $\varphi : [a, b] \to \mathbb{R}^n$, then the following inequality holds

$$\int_a^b \dot{\varphi}(u)S\dot{\varphi}(u)du \geq \frac{1}{b-a}(\varphi(b) - \varphi(a))^T S(\varphi(b) - \varphi(a)) + \frac{12}{b-a}\Omega^T S\Omega, \qquad (2.33)$$

where

$$\Omega = \frac{\varphi(b) + \varphi(a)}{2} - \frac{1}{b-a}\int_a^b \varphi(u)du.$$

By using the Wirtinger-based integral inequality and choosing the following LKF

$$V(x(t),\dot{x}(t)) = \begin{bmatrix} x(t) \\ \int_{t-\tau}^{t} x(s)ds \end{bmatrix}^T \begin{bmatrix} P & Q \\ \star & Z \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-\tau}^{t} x(s)ds \end{bmatrix} + \int_{t-\tau}^{t} x(s)Sx(s)ds + \int_{t-\tau}^{t} (\tau - t + s)\dot{x}(s)R\dot{x}(s)ds, \tag{2.34}$$

the following stability condition of system Σ_1 subject to an interval delay was obtained as

Theorem 8 *[121] For a given positive constant delay τ satisfying $0 < \tau_m \leq \tau \leq \tau_M$, if there exist symmetric positive definite matrices P, Q and R such that the following conditions hold*

$$\begin{bmatrix} P & Q \\ \star & Z + \tau^{-1}S \end{bmatrix} > 0, \tag{2.35}$$

$$\Sigma(\tau) - \frac{1}{\tau}\Pi(R) < 0, \tag{2.36}$$

where

$$\Sigma(\tau) = \begin{bmatrix} \Delta & PA_1 - Q & \tau A^T Q + \tau Z \\ \star & -S & \tau A_1^T Q - \tau Z \\ \star & \star & 0 \end{bmatrix} + \tau \begin{bmatrix} A^T \\ A_1^T \\ 0 \end{bmatrix} R \begin{bmatrix} A^T \\ A_1^T \\ 0 \end{bmatrix},$$

$$\Pi(R) = \begin{bmatrix} R & R & 0 \\ \star & R & 0 \\ \star & \star & 0 \end{bmatrix} + \frac{\pi^2}{4} \begin{bmatrix} R & R & -2R \\ \star & R & -2R \\ \star & \star & 4R \end{bmatrix},$$

$$\Delta = PA + A^T P + S + Q + Q^T,$$

then system Σ_1 is asymptotically stable.

Recently, by selecting some new sets of appropriate LKFs, combined with the Wirtinger-based integral inequality and the reciprocally convex approach, some improved delay-dependent conditions were obtained in [122, 96, 55].

2.1.6 Delay decomposition technique

It is known that by introducing the half delay parameter into the TDS, we will get more information on the system. Therefore, the conservatism of the obtained stability conditions may be reduced. More generality, the main idea of the delay decomposition technique is to divide the delay interval into N segments (N-a positive integer), then an appropriate LKF is chosen with different weighted matrices corresponding to different segments in the LKF.

To illustrate the idea of this method, the following TDS is considered

$$\dot{x}(t) = Ax(t) + A_1 x(t - \sum_{i=1}^{N} \tau_i(t)), x(t) \quad = \Phi(t), \quad \forall t \in [-\overline{\tau}, 0], \tag{2.37}$$

where $0 < \tau_i(t) \leq \overline{\tau}_i < \infty, \dot{\tau}(t) \leq \lambda_i, i = 1, \cdots, N$ are positive scalars, representing the time-varying delay components in the state and

$$\tau = \sum_{i=1}^{N} \tau_i(t) \leq \overline{\tau},$$

with $\overline{\tau}$ is the upper bound on the sum of the time delays. To proceed further, define

$$\alpha_j(t) = \sum_{i=1}^{j} \tau_i(t), \overline{\alpha}_j = \sum_{i=1}^{j} \overline{\tau}_i,$$

where $\alpha_0(t) = 0, \overline{\tau}_0 = 0$ in the boundary expression of the summation. It should be noted that $\tau_i(t)$ and $\overline{\tau}_i$ now represent, respectively, a partition of the lumped time-varying delay $\alpha_N(t)$ and $\overline{\alpha}_N$.

By constructing the following LKF

$$\begin{aligned} V(t) =& x^T(t)Px(t) + \int_{t-\overline{\alpha}_N}^{t} x^T(s)Qx(s)ds + \int_{-\overline{\alpha}_N}^{0}\int_{t+u}^{t} \dot{x}^T(s)R\dot{x}(s)dsdu \\ &+ \sum_{i=1}^{N} \int_{t-\alpha_i}^{t-\alpha_{i-1}} x^T(s)S_i x(s)ds, \end{aligned} \tag{2.38}$$

the delay-dependent stability conditions were derived in [21] as follows:

Theorem 9 *[21] For system (2.37), if there exist symmetric positive definite matrices P, Q, R and $S_i \geq S_{i+1}, i = 1, \ldots, N$ such that the following condition holds*

$$B^{\perp T} \begin{bmatrix} \Theta_1 + \Theta_2 & 0 \\ \star & \Theta_3 \end{bmatrix} B^{\perp} < 0, \tag{2.39}$$

where $B^{\perp} \in \mathbb{R}^{(2N+3)n \times (N+2)n}$ is the orthogonal complement of

$$B_{(N+1)n \times (2N+3)n} = \left[\begin{array}{ccccc|cccc} I_n & -I_n & 0 & \ldots & 0 & -I_n & 0 & \ldots & 0 \\ 0 & I_n & -I_n & \ldots & 0 & 0 & -I_n & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_n & -I_n & 0 & 0 & \ldots & -I_n \end{array}\right]$$

and

$$\Theta_1 = \begin{bmatrix} \triangle_{11} & 0 & \ldots & 0 & PA_1 \\ \star & \triangle_{22} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & \ldots & -(1-\sum_{i=1}^{N-1}\lambda_i)(Q_{N-1}-Q_N) & 0 \\ \star & \star & \ldots & \star & -(1-\sum_{i=1}^{N}\lambda_i)Q_N \end{bmatrix},$$

$$\begin{aligned} \triangle &= PA + A^T P + S_1 + Q, \triangle_{22} = -(1-\lambda_1)(S_1 - S_2), \\ \Theta_2 &= \overline{\alpha}_N [A \ \ 0 \ \ \ldots \ \ 0 \ \ A_1]^T R [A \ \ 0 \ \ \ldots \ \ 0 \ \ A_1], \\ \Theta_3 &= diag\{-Q, -\overline{\tau}_1^{-1} R, \ldots, -\overline{\tau}_N^{-1} R, -(\overline{\alpha}_N)^{-1} R\}, \end{aligned}$$

then system (2.37) is asymptotically stable.

Recently, by constructing some new sets of LKFs, combined with the delay decomposition approach, some improved delay-dependent stability conditions were reported in [85, 132, 134, 94, 55, 48, 94, 96, 95].

2.1.7 Free weighting matrix

The objective of this method is to introduce some free weighting matrices (FWMs) to express the relationships among the terms of the state equation of the system in the derivation of delay-dependent stability criteria. There are two ways to do this. The first way is to replace term $\dot{x}(t)$ with the system equation in the conventional way. The second is to retain term $\dot{x}(t)$ and use FWMs to express the relationships among the terms of the state equation of the system. Generally speaking, the results obtained by the two methods are equivalent. Here, we focus on the first treatment. For this, the LKF is first constructed. Then, some free weighting matrices which indicate the relationship between the terms in the Leibniz-Newton formula, are added to the derivative of the Lyapunov functional.

In [144], the following LKF was considered

$$V(t) = x^T(t)Px(t) - \int_{t-\tau}^{t} x^T(\alpha)Qx(\alpha)d\alpha + \tau \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^T(\alpha)R\dot{x}(\alpha)d\alpha d\theta. \quad (2.40)$$

By using the Newton-Leibniz formula

$$x(t-\tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(\alpha)d\alpha, \quad (2.41)$$

we have

$$x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(\alpha)d\alpha = 0. \quad (2.42)$$

For any matrices M, N with appropriate dimensions, we have the following equations,

$$x^T(t)M\Big[x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(\alpha)d\alpha\Big] = 0. \quad (2.43)$$

$$x^T(t-\tau)M\Big[x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(\alpha)d\alpha\Big] = 0. \quad (2.44)$$

Moreover, for any matrix Z, the following equation is obtained

$$\tau\eta^T(t)Z\eta(t) - \int_{t-\tau}^{t} \eta^T(t)Z\eta(t)ds = 0, \quad (2.45)$$

where

$$\eta(t) = [x(t) \quad x(t-\tau)], Z = \begin{bmatrix} Z_{11} & Z_{12} \\ \star & Z_{22} \end{bmatrix}.$$

By adding the terms on the left hand side of equations (2.34) (2.35) and (2.36) to the time derivative of the Lyapunov functional (2.31) along the solutions of system

Σ_1, the following delay-dependent stability conditions were derived as stated in the following theorem.

Theorem 10 *[144] For given positive constants* $0 \leq \tau \leq \overline{\tau}$, *if there exist symmetric positive definite matrices* P, Q, R *and matrices* $M, N, Z_{11}, Z_{12}, Z_{22}$ *such that the following linear matrix inequality holds*

$$\begin{bmatrix} PA + A^T P + M + M^T + Q + \overline{\tau} Z_{11} & PA_1 - M + N^T + \overline{\tau} Z_{12} & \overline{\tau} A^T R \\ \star & -Q - N - N^T + \overline{\tau} Z_{22} & \overline{\tau} A_1^T R \\ \star & \star & -\overline{\tau} R \end{bmatrix} < 0, \tag{2.46}$$

$$\begin{bmatrix} Z_{11} & Z_{12} & M \\ \star & Z_{22} & N \\ \star & \star & R \end{bmatrix} \geq 0, \tag{2.47}$$

then system Σ_1 *is asymptotically stable.*

By using the same LKF (2.31) and retaining the term $\dot{x}(t)$, without the use of inequality (2.36), another stability condition was reported in [150] as follows.

Theorem 11 *[150] For given positive constants* $0 \leq \tau \leq \overline{\tau}$, *if there exist symmetric positive definite matrices* P, Q, R *and matrices* M, N *such that the following linear matrix inequality holds*

$$\begin{bmatrix} PA + A^T P + M + M^T + Q & PA_1 - M + N^T & -\overline{\tau} M & \overline{\tau} A^T R & \\ \star & -Q - N - N^T & -\overline{\tau} N & \overline{\tau} A_1^T R & \\ \star & \star & -\overline{\tau} R & 0 & \\ \star & \star & \star & -\overline{\tau} R & 0 \end{bmatrix} < 0, \tag{2.48}$$

then system Σ_1 *is asymptotically stable.*

For time-varying delay system Σ_2, by choosing the LKF (2.31) and using inequalities (2.31) and (2.31), the following result was obtained.

Theorem 12 *[151] For given positive constants* $0 \leq \tau(t) \leq \overline{\tau}$*, if there exist symmetric positive definite matrices* P, Q *and matrices* M, N *such that the following linear matrix inequality holds*

$$\begin{bmatrix} PA + A^TP + M + M^T & PA_1 - M + N^T & -\overline{\tau}M & \overline{\tau}A^TQ \\ \star & -N - N^T & -\overline{\tau}N & \overline{\tau}A_1^TQ \\ \star & \star & -\overline{\tau}Q & 0 \\ \star & \star & \star & -\overline{\tau}Q & 0 \end{bmatrix} < 0, \qquad (2.49)$$

then system Σ_2 *is asymptotically stable.*

The FWM approach has been recognised as an effective tool in stability analysis in terms of reducing the conservatism of the stability criteria. Therefore, it has been extensively used in the derivation of delay-dependent stability conditions for time-delay systems, see, e.g., [52, 53, 143, 54, 34, 113, 110] and references therein.

2.1.8 Descriptor system approach

Motivated by the fact that the delay-dependent stability conditions of a time-delay system can be analysed via another transformed model which is equivalent to the original model, the descriptor system approach has been developed in [28]. The main idea of this method is first to transform the original TDS into a descriptor system model, then to derive the stability conditions for the transformed model.

From system Σ_1, we have

$$\begin{aligned} \dot{x}(t) &= y(t), t \geq 0, \\ 0 &= -y(t) + (A + A_1)x(t) - A_1 \int_{t-\tau}^{t} y(\alpha)d\alpha. \end{aligned} \qquad (2.50)$$

By rearrangement, system (2.50) can be rewritten in the form of

$$E\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) - \overline{A}_1 \int_{t-\tau}^{t} y(\alpha)d\alpha, \qquad (2.51)$$

where

$$\overline{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \overline{A} = \begin{bmatrix} 0 & I \\ A + A_1 & -I \end{bmatrix}, \overline{A}_1 = \begin{bmatrix} 0 \\ A_1 \end{bmatrix}.$$

It should be noted that the descriptor time-delay system (2.51) is equivalent to system Σ_1. Therefore, instead of considering the stability of system Σ_1, we consider the problem of stability analysis of system (2.51). Based on Moon's inequality and by selecting the following LKF

$$V(t) = \overline{x}^T(t)EP\overline{x}(t) + \int_{t-\tau}^{t} x^T(\alpha)Sx(\alpha)d\alpha + \int_{-\tau}^{0}\int_{t+\theta}^{t} y^T(\alpha)Ry(\alpha)d\alpha, \tag{2.52}$$

delay-dependent stability conditions for TDS Σ_1 was derived in the following theorem.

Theorem 13 *[28] For a given positive constant $\overline{\tau}$ such that $0 \leq \tau \leq \overline{\tau}$, if there exist symmetric positive definite matrices P_1, S, R and matrices P_2, P_3, X, Y such that the following linear matrix inequality holds*

$$\begin{bmatrix} \Omega + \overline{\tau}Y & P^T\overline{A}_1 - X^T \\ \star & -S \end{bmatrix} < 0, \tag{2.53}$$

$$\begin{bmatrix} R & X \\ \star & Y \end{bmatrix} \geq 0, \tag{2.54}$$

where

$$\Omega = P^T \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix}^T P + \begin{bmatrix} S & 0 \\ 0 & \overline{\tau}R \end{bmatrix} + \begin{bmatrix} X \\ 0 \end{bmatrix} + \begin{bmatrix} X \\ 0 \end{bmatrix}^T, P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix},$$

then system Σ_1 is asymptotically stable.

Similarly, for time-varying system Σ_2, the descriptor system is obtained in the form

$$E\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) - \overline{A}_1 \int_{t-\tau(t)}^{t} y(\alpha)d\alpha. \tag{2.55}$$

The following delay-dependent stability of the descriptor time-delay system (2.22) was obtained in [29] as

Theorem 14 *[29] For a given upper bound of the time-varying delay, $\overline{\tau}$ such that $0 \leq \tau(t) \leq \overline{\tau}$, if there exist symmetric positive definite matrices P_1, S, R and matrices P_2, P_3, X, Y such that the following linear matrix inequality holds*

$$\Omega + \overline{\tau} Y < 0, \tag{2.56}$$

$$\begin{bmatrix} R & \overline{A}_1 P \\ \star & -Y \end{bmatrix} \geq 0, \tag{2.57}$$

where

$$\Omega = P^T \begin{bmatrix} 0 & I \\ A + A_1 & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A + A_1 & -I \end{bmatrix}^T P + \begin{bmatrix} 0 & 0 \\ 0 & \overline{\tau} R \end{bmatrix},$$

then system Σ_2 is asymptotically stable.

Later, by using the descriptor system approach and choosing the following LKF

$$V(t) = x^T(t) P x(t) + \int_0^{\tau} (\tau - \theta) \dot{x}^T(t - \theta) Q_{33} \dot{x}(t - \theta) + \int_0^{t} \int_{\theta - \tau}^{\theta} \rho^T R \rho ds d\theta, \tag{2.58}$$

where

$$\rho = \begin{bmatrix} x(\theta) \\ x(\theta - \tau) \\ \dot{x}(s) \end{bmatrix}, Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ \star & Q_{22} & Q_{23} \\ \star & \star & Q_{33} \end{bmatrix},$$

an improved stability condition was derived in [64] as follows.

Theorem 15 *[64] For a given positive constant $\overline{\tau}$ satisfying $0 \leq \tau(t) \leq \overline{\tau}$, if there exist symmetric positive definite matrices P_1, S, R and matrices P_2, P_3, X, Y such that the following linear matrix inequality holds*

$$\Omega + \overline{\tau} Y < 0, \tag{2.59}$$

$$\begin{bmatrix} R & \overline{A}_1 P \\ \star & -Y \end{bmatrix} \geq 0, \tag{2.60}$$

where

$$\Omega = P^T \begin{bmatrix} 0 & I \\ A + A_1 & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A + A_1 & -I \end{bmatrix}^T P + \begin{bmatrix} 0 & 0 \\ 0 & \overline{\tau} R \end{bmatrix},$$

then system Σ_2 is asymptotically stable.

Recently, by constructing improved sets of appropriate LKFs combined with the descriptor system approach, some less conservative stability criteria for TDS was derived, see, e.g., [29, 152, 33, 46, 47, 10, 26, 45] and [56].

2.2 Discrete-time sliding mode control

2.2.1 Introduction

Over the past decades, there has been a growing interest in the control design for variable structure systems (VSS) whose structure varies under certain conditions [139]. Sliding mode control is a well-known control methodology belonging to the variable structure control that is characterised by their robustness with respect to parameter variations and external disturbances. This property is extremely important in practical control where most systems are heavily affected by time delay, parametric variations and external disturbances. The basic idea of the sliding mode control is to drive the system trajectories into a predetermined hyperplane or surface, and maintain the trajectory on it for all subsequent time. During the ideal sliding motion, the system is completely insensitive to uncertainties or external disturbances. The dynamics and performance of the systems then depend on the selection of the sliding surface. The design procedure of a discrete-time sliding mode controller consists of two stages as follows:

- Stage 1: Design a stable sliding surface with desired performance characteristics.
- Stage 2: Design a discontinuous control law to drive the state trajectory towards the sliding surface and maintain it on this surface over time.

The closed-loop dynamical behavior under a variable structure control law comprises two distinct types of motion. The reaching mode, occurring whilst the system states are being driven towards the sliding surface, is in general affected by the presence of any matched and unmatched disturbances. With the sliding mode, the system state trajectories reach the sliding surface and remain on it. In the second phase, the sliding motion takes place and the system becomes insensitive to all matched disturbances or other source of uncertainties. Therefore, it is necessary to design a sliding mode controller such that the initial reaching phase is as short as possible [23].

2.2.2 Sliding surface design

The problem of the switching surface design is significantly important because the performance of closed-loop dynamics is governed by the parameters of the sliding surface. Over the past decades, several design schemes of sliding surface have been proposed such as the pole placement method [2, 42, 58], linear quadratic regulator (LQR) method [139, 23, 118, 62] and LMI approach [14, 133, 131, 130, 154, 145, 49].

In the following, a brief review on the sliding surface design methods, including the pole placement method, LQR method and LMI approach will be presented.

Let us consider the discrete-time system as follows

$$x(k+1) = Ax(k) + Bu(k) + D\omega(k), k \in \mathbb{Z}^+, \tag{2.61}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are, respectively, the system state vector and the control input. Matrices A, B and D are constant, with appropriate dimensions and $\omega(k) \in \mathbb{R}^p$ is the external disturbances. Here, in the general case, we consider the system with unmatched disturbances. It should be noted that in the sliding mode, the system is insensitive to parameter variations and matched disturbances. However, the system is still affected by the effects of unmatched disturbances and time delay in the sliding mode. This is characterised by the fact that the system state is repeatedly crossing the sliding surface rather than remaining on the surface and this motion is highly undesirable in practice. Therefore, instead of designing a sliding mode controller to drive the states to lie on the sliding surface, the controller will force the states to remain within a small boundary layer about the surface.

It is well known that for the controllable system (2.61), there exists a transformation matrix T which can always be chosen such that $TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$, where $B_2 \in \mathbb{R}^{m\times m}$ is a non-singular matrix. With $z(k) = Tx(k)$, system (2.61) can be transformed into the following regular form:

$$z(k+1) = \overline{A}z(k) + \overline{B}u(k) + \overline{D}\omega(k)), k \in \mathbb{Z}^+, \tag{2.62}$$

where

$$\overline{A} = TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \overline{B} = TB, \overline{D} = TD = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

Equation (2.62) can be partitioned into the following form

$$\begin{aligned} z_1(k+1) &= A_{11}z_1(k) + A_{12}z_2(k) + D_1\omega(k), \\ z_2(k+1) &= A_{21}z_1(k) + A_{22}z_2(k) + B_2u(k) + D_2\omega(k)). \end{aligned} \tag{2.63}$$

It can be seen that the first equation of (2.63) does not depend on the control, while the dimension of the second equation is equal to that of the control. Therefore, $z_2(k)$ is handled as a control in the first equation and designed as a linear function of $z_1(k)$. The sliding surface is then proposed as

$$s(k) = Cz(k) = C_1z_1(k) + C_2z_2(k), \tag{2.64}$$

where C_1 and C_2 are the design parameters which define the sliding surface and they should be chosen such that in the sliding mode, all remaining dynamics are stable. During the sliding motion, we have $s(k) = 0$ so that

$$z_2(k) = -C_2^{-1}C_1z_1(k) = -Fz_1(k), \tag{2.65}$$

where $F = C_2^{-1}C_1$. The reduced-order sliding motion can thus be obtained as

$$z_1(k+1) = [A_{11} - A_{12}F]z_1(k) + D_1\omega(k). \tag{2.66}$$

It can be seen that equation (2.66) describes all dynamics of the closed-loop system in sliding mode. Therefore, stability of the system in the sliding mode, is ensured when all eigenvalues of the matrix $A_{11} - A_{12}F$ lie within the unit circle. The problem of finding the design matrix F is, in fact, a classical stabilization problem. It can be found that if the pair (A, B) is controllable, then the pair (A_{11}, A_{12}) is controllable as well.

Pole placement technique

It can be seen that the eigenvalues of the sliding mode system (2.66) can be placed arbitrarily in the complex plane by selecting an appropriate matrix F. Therefore, the problem of designing a suitable sliding surface with desired performance and stable dynamics for the closed-loop system now depends on the assignment problem. The remaining degrees of freedom available in the assignment problem will be used to modally shape the system response using a judicious selection of eigenvector forms during the sliding mode.

During the sliding motion, the eigenvalues of the system will consist of the set of $(n - m)$ stable eigenvalues assigned to the spectrum of $(A_{11} - A_{12}F)$ from equation (2.66), plus the value zero repeated m times. For systems with unmatched disturbances, the objective is to make a nonzero sliding mode eigenvalues insensitive to unmatched disturbances using robust eigenstructure assignment. This will minimise the effects of unmatched disturbances $D_1\omega(k)$ in the sliding mode.

The methodology for determining F is the same as a full-state feedback matrix which was reported in [2]. It can be found by assigning an arbitrary self-conjugate set of eigenvalues to the controllable system.

To show how the vector F may be found in an explicit form without the sliding motion equation using pole placement technique, we consider the following linear system as

$$z_1(k+1) = A_{11}z_1(k) + A_{12}u(k). \tag{2.67}$$

where $u(k)$ is the state feedback controller chosen as

$$u(k) = -Fz_1(k). \tag{2.68}$$

By using Ackermann's formula, matrix F can be determined as follows [2]:

$$F = e^T P(\lambda), \tag{2.69}$$

where

$$e^T = (0, \ldots, 0, 1)(A_{12}, A_{11}A_{12}, \ldots, A_{11}^{n-m-1}A_{12})$$

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_{n-m-1})(\lambda - \lambda_{n-m}),$$

with $\lambda_1, \lambda_2, \ldots, \lambda_{n-m}$ are the desired eigenvalues of the system, assumed to be distinct.

One of the desirable properties of any closed-loop system is that the eigenvalues are rendered insensitive to perturbations and disturbances in the coefficient matrices of the system equations. The problem of designing a state feedback controller using the pole placement technique is generally underdetermined with many degrees of freedom. Therefore, the eigenstructure assignment which restricts the degrees of freedom should be taken into account.

LQR-based approach

The objective of this method is to design the sliding surface to minimise the following quadratic performance index

$$J = \sum_{k=0}^{\infty} z^T(k)Qz(k), \tag{2.70}$$

where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

is a symmetric positive definite weighing matrix. This approach was first proposed in [140]. In general, it can be particularly useful for the system model- following variable structure controller design where variable structure control systems are required for a model-following error system. The advantage of this method is that it will enable desirable weighting values to be placed upon particular elements; for instance, it is necessary to design a sliding surface whose deflections of the real system follow those of the model closely.

The quadratic performance index (2.70) can be expressed in the form of

$$J = \sum_{k=0}^{\infty} \Big[z_1^T(k)Q_{11}z_1(k) + 2z_1^T(k)Q_{12}z_2(k) + z_2^T(k)Q_{22}z_2(k) \Big]. \tag{2.71}$$

To proceed further, we will express equation (2.70) in the form of the standard LQR problem where $z_1(k)$, which determines the system dynamics in the ideal sliding mode, has the role of the state and the effective control input is the function of $z_2(k)$.

By noting that

$$\begin{aligned} &2z_1^T(k)Q_{12}z_2(k) + z_2^T(k)Q_{22}z_2(k) \\ &= (z_2(k) + Q_{22}^{-1}Q_{21}z_1(k))^T Q_{22}(z_2^T(k) + Q_{22}^{-1}Q_{21}z_1(k)) \\ &\quad - z_1^T(k)Q_{21}^T Q_{22}^{-1} Q_{21} z_1(k), \end{aligned}$$

equation (2.71) can be rewritten in the form of

$$J = \sum_{k=0}^{\infty} \Big[z_1^T(k)\overline{Q}z_1(k) + \upsilon^T(k)Q_{22}\upsilon(k) \Big], \tag{2.72}$$

where

$$\begin{aligned} \overline{Q} &= Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}, \\ \upsilon(k) &= z_2(k) + Q_{22}^{-1}Q_{21}z_1(k). \end{aligned}$$

From (2.63), the nominal system with respect to $z_1(k)$ can be obtained as

$$z_1(k+1) = A_{11}z_1(k) + A_{12}z_2(k). \tag{2.73}$$

Consequently, we obtain

$$z_1(k+1) = \mathcal{A}z_1(k) + A_{12}\upsilon(k), \tag{2.74}$$

where $\mathcal{A} = A_{11} - A_{12}Q_{22}^{-1}Q_{21}$. It should be noted that the problem of minimising the cost performance index (2.70) is now equivalent to that of minimising the functional (2.72) subject to dynamic constraint (2.74). Since Q is a symmetric positive definite matrix, it is easy to see that $\overline{Q} > 0$. In addition, with the assumption of controllability of the original system (A, B), the system (A_{11}, A_{12}) and $(\mathcal{A}, A_{12})$ are also controllable.

Based on the LQR minimisation of J in association with the nominal system in (2.74), we obtain

$$\upsilon(k) = -Q_{22}^{-1}A_{12}^T P z_1(k), \tag{2.75}$$

where P is a positive definite matrix which satisfies the following equation

$$\mathcal{A}^T P + P\mathcal{A} - PA_{12}Q_{22}^{-1}A_{12}^T P + \overline{Q} = 0. \tag{2.76}$$

Thus, we obtain

$$z_2(k) = -Q_{22}^{-1}(A_{12}^T P + Q_{21})z_1(k). \tag{2.77}$$

By comparing equation (2.77) with equation (2.65), the design matrix F of the sliding function is obtained as

$$F = -Q_{22}^{-1}(A_{12}^T P + Q_{21}). \tag{2.78}$$

LMI approach

Recently, the LMI approach is extensively used to design the switching function. The main advantage of this method is that it can deal with the system with uncertainties directly and the switching gain can be obtained explicitly from the solution of LMI conditions. This method starts with the selection of a positive-definite functional subject to the reduced-order system, then computes its forward difference along the solutions of the system. Finally, some LMI conditions are obtained from the stability constraints in the sense of Lyapunov. In the following, the LMI approach to design the sliding surface (2.64) will be considered.

Consider the following Lyapunov functional

$$V = z_1^T(k)Pz_1(k), \tag{2.79}$$

where P is a symmetric positive definite matrix. Then, by taking the forward difference of functional (2.79) along the solution of system (2.66), we obtain

$$\begin{aligned} \Delta V(k) &= z_1^T(k+1)Pz_1(k+1) - z_1^T(k)Pz_1(k) \\ &= [(A_{11} - A_{12}F)z_1(k) + D_1\omega(k)]^T P[(A_{11} - A_{12}F)z_1(k) + D_1\omega(k)] \\ &\quad - z_1(k)Pz_1(k). \end{aligned} \tag{2.80}$$

From above, we obtain

$$\Delta V(k) \leq \xi^T(k)\left[\Phi + \Psi P \Psi^T\right]\xi(k), \tag{2.81}$$

where $\xi^T(k) = [z_1^T(k) \quad v^T(k)]$ and

$$\Phi = \begin{bmatrix} -P & \star \\ 0 & 0 \end{bmatrix}, \Psi = \begin{bmatrix} (A_{11} - A_{12}F)^T \\ D_1^T \end{bmatrix}.$$

Thus

$$\Delta V(k) \leq 0, \tag{2.82}$$

if the following inequality holds

$$\Phi + \Psi P \Psi^T < 0. \tag{2.83}$$

Now by using the Schur inequality, (2.83) can be brought to the form,

$$\begin{bmatrix} -P & \star & \star \\ 0 & 0 & \star \\ (A_{11} - A_{12}F) & D_1 & -P^{-1} \end{bmatrix} < 0. \tag{2.84}$$

By multiplying equation (2.84) from the right and the left by $diag(P^{-1}, I, I)$ and its transpose, respectively, we obtain

$$\begin{bmatrix} -P^{-1} & \star & \star \\ 0 & 0 & \star \\ (A_{11} - A_{12}F)P^{-1} & D_1 & -P^{-1} \end{bmatrix} < 0. \tag{2.85}$$

By defining $Q = P^{-1}$ and $M = FQ = FP^{-1}$, the following linear matrix inequality is obtained

$$\begin{bmatrix} -Q & \star & \star \\ 0 & 0 & \star \\ A_{11}Q - A_{12}M & D_1 & Q \end{bmatrix} < 0. \tag{2.86}$$

Inequality (2.85) can be solved by using the Matlab's LMI toolbox. Therefore, the switching gain matrix F can be obtained as

$$F = MP = MQ^{-1}. \tag{2.87}$$

2.2.3 Reaching law

As mentioned above, the reaching law plays an important role in designing a discrete-time sliding mode controller. In the following, we will review some reaching laws in the literature.

Sarpturk's reaching law

Motivated by the fact that the control law for continuous-time sliding mode control can be obtained from the Lyapunov function and the reaching law is constructed from which the control law automatically follows. A reaching condition for single input single output system (SISO) was proposed by Sarpturk as follows

$$|s(k+1)| < |s(k)|, \tag{2.88}$$

where $s(k)$ is the sliding surface. Equation (2.88) can be formulated in the form of

$$[s(k+1) - s(k)]\text{sign}(s(k)) < 0, \tag{2.89a}$$

$$[s(k+1) + s(k)]\text{sign}(s(k)) > 0. \tag{2.89b}$$

Equation (2.89a) implies that the closed-loop system should be moving towards the sliding surface, whereas the second condition implies that the closed-loop system is not allowed to go too far in that direction. By using this reaching law, the following control law for system (2.61) was given in [120] as

$$u(k) = K(x, s)x(k), \tag{2.90}$$

where $K(\cdot,\cdot) \in \mathbb{R}^{m \times n}$ represents the switching feedback gains. The elements of $K(\cdot,\cdot)$ denoted by $K_{ij}(x, s); i = 1 \ldots n, j = 1 \ldots n$ can be selected as follows

$$K_{ij}(x, s) = \begin{cases} K_{ij}^{+} \text{ if } s_i(x)x_j(k) > 0 \\ K_{ij}^{-} \text{ if } s_i(x)x_j(k) < 0. \end{cases} \tag{2.91}$$

The coefficients K_{ij}^{+} and K_{ij}^{-} can be determined by evaluating conditions (2.90) and (2.91) resulting in an upper and a lower bound for each K_{ij}^{+} and K_{ij}^{-}. However, it is pointed out that the condition (2.89b) is only a sufficient condition for the existence of the discrete-time sliding mode [124].

Gao's reaching law

As with the reaching law proposed in [120] for SISO systems, to deal with the multi input multi output (MIMO) systems, an improved reaching law was introduced in [36] as follows

$$s(k+1) - s(k) = -qTs(k) - \varepsilon T\text{sgn}(s(k)), \tag{2.92}$$

where T is the sampling period, $q > 0$ and $\varepsilon > 0$ are positive scalars, chosen such that $1 - qT > 0$.

With this reaching law, the closed-loop system should possess the following properties:

- The system trajectory, starting from any initial state, will move monotonically towards the switching plane and cross it in a finite time.
- Once the trajectory has crossed the switching plane the first time, it will cross the plane again in every successive sampling period, resulting in a zigzag motion along the sliding surface.
- The size of each successive zigzagging step is nonincreasing and the trajectory stays within a specified band, the so called quasi-sliding mode band (QSMB). Despite the fact that the above conditions were stated for a single input system, they can be applied to a multiple input system by applying the three rules to the m entries of the switching function $s(k)$ independently.

Based on this reaching law, the following control law for system (2.66) was derived in [36] as

$$u(k) = -(CB)^{-1}\Big[CAz(k) - Cz(k) + qTs(k) - \varepsilon T\mathrm{sgn}(s(k))\Big]. \tag{2.93}$$

Moreover, the quasi-sliding mode band is determined as

$$\varDelta = \frac{\varepsilon T_s}{1 - qT_s}. \tag{2.94}$$

Bartoszewicz's reaching law

Consider the following linear discrete-time uncertain system as

$$x(k+1) = (A + \varDelta A)x(k) + Bu(k) + D\omega(k), \tag{2.95}$$

where the matrix $\varDelta A$ represents parameter uncertainties. Disturbances and parameter uncertainties are assumed to be bounded and

$$d_m \leq d(k) = C^T \varDelta A x(k) + D\omega(k) \leq d_M, \tag{2.96}$$

where d_m and d_M are, respectively, known constants. Let us define

$$d_0 = \frac{d_m + d_M}{2}, \delta_d = \frac{d_M - d_m}{2}.$$

Motivated by the fact that the system is no longer required to cross the sliding hyperplane in each successive control step, but only to remain in a small band around it, the control strategy can be linear and the undesirable chattering is avoided. In [3], another reaching law was proposed for system (2.62) as follows:

$$s(k+1) = d(k) - d_0 + s_d(k+1), \tag{2.97}$$

where $s_d(k+1)$ is an a *priori* known function such that the following conditions are satisfied:

- if $s(0) > 2\delta_d$, then

$$\begin{cases} s_d(0) = s(0), \\ s_d(k)s(0) \geq 0, \text{ for all } k \geq 0, \\ s_d(k) = 0, \text{ for all } k \geq k*, \\ |s_d(k+1)| < |s(k)| - 2\delta_d \text{ for any } k < k*. \end{cases}$$

- otherwise $s_d(k) = 0$ for any $k \geq 0$.

The constant k^* is a positive integer chosen by the designer in order to achieve good tradeoff between the fast convergence rate of the system and the magnitude of the control input required to achieve this convergence rate. Based on this reaching law, Bartoszewicz (1998), proposed the following control law for system (2.62) as

$$u(k) = -(CB)^{-1}\Big[CAz(k) + d_0 - s_d(k+1)\Big]. \tag{2.98}$$

This control law guarantees that, for any $k \geq k*$, the system state satisfies the following inequality:

$$|s(k)| = |d(k-1) - d_0| \leq \delta_d. \tag{2.99}$$

Recently, inspired by Gao and Bartoszewicz's reaching law, some improved reaching laws have been proposed for complicated systems with time-varying delay and matched/unmatched disturbances, in order to design some robust discrete-time quasi-sliding mode controllers to achieve better performance; see, e.g., [60, 61, 154, 145, 146, 100, 117] and references therein.

2.3 Linear functional observer design of time-delay systems

2.3.1 Introduction

In many practical control systems, the physical states of the system are usually not fully available for feedback. Therefore, the problem of constructing observers to estimate these states is of practical importance in control engineering. Generally speaking, a state observer is a system that allows the reconstruction of the entire state vector from a minimum set of measurements for which the system is completely observable. A state observer is a model of the actual system plus a corrective term which is the error between the model output and the actual system output.

When the model output is the same as the system output, the error term vanishes and the observer will be a duplicate of the system itself. These observers are called the full-order state observer as its order is equal to the order of the original system. Full-order observers are designed to reconstruct all of the state variables. However, in practice, some of the state variables may be accurately measured. For the sake of practical implementation, in general, the state feedback control law only requires a linear combination of the state variables, i.e., $Fx(t)$ rather than the complete knowledge of the entire state vector $x(t)$ [138]. In this context, to reduce the order and complexity of the designed observers, the linear functional observers are used to estimate linear functions of the state vector of the system without needing to estimate all the individual states.

In [65], an optimum state estimator-the Kalman filter, was first introduced. Later, some state observers were developed to estimate states of linear systems [87]. The first idea of the functional observer design was first presented in the context of designing low-order observers to estimate a desired linear function or functions of the system states [88]. Generally speaking, a functional observer is a general form of the state observer as the linear functions are constructed from the individual states of the system. The problem of functional observation therefore reduces to the problem of state observation, see, e.g., [88, 135, 17, 137] and references therein..

Time delays often appear in many control systems either in the state, the control input or the output. Some typical systems such as large-scale or networked control systems, where computation units are located far from the plant, measurement and control data have to be transmitted through a communication channel, time delay is therefore unavoidable and, in general, may affect system stability. One technique to stabilize delay systems is based on state estimation where an observer is designed to guarantee the asymptotic stability of the estimation error dynamics [9]. Thus, during the past decades, there has been a considerable amount of research devoted to the topics of observer design for control systems with time delay, see, e.g., [138, 119, 69] and references therein. Several design schemes, including the spectrum assignment [89, 106] and the linear matrix inequality approach [138, 137, 17, 96, 43] have been proposed to design asymptotic observers. For the finite spectrum assignment, the observer structure contains an integro-differential equation form and the derivatives of the observer state vector often depends on the time-delayed observer state vector. Consequently, the order of the proposed observers is normally high as it depends on the number of eigenvalues lying to the right-half s- plane. Therefore, these observers require memory units for internal delays and integration of past values in practical implementation [138].

Consider the linear system as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), t \geq 0, \\ y(t) &= Cx(t) \end{aligned} \tag{2.100}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^l$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$ and $C \in \mathbb{R}^{p\times n}$ are known and constant.

Without loss of generality, we assume that the pair (A, B) is controllable, and the pair (C, A) is observable, and matrix C has full-row rank (i.e., $\text{rank}(C) = p$).

The objective of the linear functional observer design is to reconstruct a partial set of the state vector or a linear combination of the state vector of system (2.100), with a reduced-order structure such that the estimate converges to the original state.

Let $z(t) \in \mathbb{R}^m$ be a vector that is required to be estimated with the following form

$$z(t) = Lx(t), \tag{2.101}$$

where $L \in \mathbb{R}^{m\times n}$ is a known and constant matrix. We assume that $\text{rank}(L) = m$ and $\text{rank}\begin{bmatrix} C \\ L \end{bmatrix} = (p + m)$. To reconstruct the linear state function, $z(t)$, the following m-order observer $(m \leq n - p)$ was proposed in [138, 16]:

$$\begin{aligned} \hat{z}(t) &= w(t) + Ey(t), t \geq 0, \\ \dot{w}(t) &= Nw(t) + Jy(t) + Hu(t) \end{aligned} \tag{2.102}$$

where $\hat{z}(t) \in \mathbb{R}^m$ denotes the estimate of $z(t)$ and E, N, J and H are constant matrices of appropriate dimensions to be determined such that the estimate $\hat{z}(t)$ approaches to $z(t)$ or the estimation error converges asymptotically to zero as $t \to \infty$.

It should be noted that matrix L can always be chosen by the designer to represent any desired partial set of the state vector to be estimated. For example, if matrix L is chosen as a controller gain of a state feedback controller which stabilises the closed-loop system matrix $(A + BL)$, then the linear functional observer proposed in (2.102) would provide an estimate of the corresponding control signal to be directly feedback into the system. Moreover, it was shown in [16] that for m linear functions to be estimated, one may propose an observer with order as low as m. It is clear that the order of the linear functional observer will be increased when the number of functions needing be estimated increases. Therefore, when

$m = n - p$, the design procedure of a linear functional observer is the same as that of a reduced-order state observer.

The conditions for the existence of the linear functional observer (2.102) were provided in [16] which is restated in the following theorem.

Theorem 16 *For linear system (2.100), the estimate $\hat{z}(t)$ converges asymptotically to $z(t)$ for any $x(0), \hat{z}(0)$ and $u(t)$, if and only if the following conditions hold*

$$\begin{aligned} &1) \quad N \quad \textit{is Hurwitz}, \\ &2) \quad PA - NP - JC = 0, \\ &3) \quad H = PB, \\ &4) \quad P = L - EC. \end{aligned}$$

It is known that the estimate $\hat{z}(t)$ approaches to $z(t)$ as $t \to \infty$, if

$$\lim_{t\to\infty} [\hat{z}(t) - z(t)] = 0.$$

Let us denote the estimation error $e(t) = \hat{z}(t) - z(t)$. Then, the dynamics of this estimation error are determined by

$$\dot{e}(t) = Ne(t) + (PA - NP - JC)x(t) + (PB - H)u(t). \tag{2.103}$$

Necessary and sufficient conditions for the existence of the observer were derived in [16] as:

$$\text{rank} \begin{bmatrix} LA \\ CA \\ C \\ L \end{bmatrix} = \text{rank} \begin{bmatrix} CA \\ C \\ L \end{bmatrix}. \tag{2.104}$$

2.3.2 Linear functional observers for systems with delay in the state

Consider the linear time-delay system as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + Bu(t), t \geq 0, \\ y(t) &= Cx(t) \\ z(t) &= Lx(t), \end{aligned} \tag{2.105}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^l$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively. The time delay τ is a known positive scalar and vector $z(t) \in \mathbb{R}^m$ is the partial state to be estimated. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{p\times n}$ and $L \in \mathbb{R}^{m\times n}$ are known and constant.

Without loss of generality, we assume that the pair (A, B) is controllable, and the pair (C, A) is observable, rank$(C) = p$, rank$(L) = m$ and rank $\begin{bmatrix} C \\ L \end{bmatrix} = (p+m)$.

The objective is to design an mth-order observer to estimate the linear state functional $z(t)$ of system (2.105) with time delay in the state such that the estimate $\hat{z}(t)$ approaches to $z(t)$ or the estimation error converges asymptotically to zero.

For time-delay system (2.105), a linear functional observer was proposed in [17] as follows:

$$\begin{aligned} \hat{z}(t) &= w(t) + Fy(t), \\ \dot{w}(t) &= Nw(t) + N_d w(t-\tau) + Dy(t) + D_d y(t-\tau) + Eu(t) \end{aligned} \tag{2.106}$$

where $\hat{z}(t) \in \mathbb{R}^m$ denotes the estimate of $z(t)$ with an initial condition $\zeta(\theta) = \phi(\theta)$, $\forall\theta \in [-\tau, 0]$, and $\phi \in \mathcal{C}([-\tau, 0])$ and N, N_d, D, D_d and E are constant matrices of appropriate dimensions to be determined.

By defining the estimation error as follows

$$e(t) = z(t) - \hat{z}(t) = Lx(t) - \hat{z}(t), \tag{2.107}$$

and $\Psi = L - FC$.

Some conditions for the existence of mth-order observer (2.106) are stated in the following theorem [17].

Theorem 17 *For time-delay system (2.105) and* m*th-order observer (2.106), with a given positive scalar τ, the estimate $\hat{z}(t)$ approaches to $z(t)$ if and only if the following conditions hold*

$$
\begin{aligned}
&1)\quad \dot{e}(t) = Ne(t) + N_d e(t-\tau) \qquad \textit{is asymptotically stable},\\
&2)\quad \Psi A - N\Psi - DC = 0,\\
&3)\quad \Psi A_d - N_d\Psi - D_d C = 0,\\
&4)\quad E = \Psi B.
\end{aligned}
$$

The following necessary and sufficient conditions for the existence of the observer (2.106) such that the estimation error converges asymptotically to zero when $t \to \infty$, must be satisfied :

$$
\operatorname{rank}\begin{bmatrix} 0 & C \\ CA & CA_d \\ C & 0 \\ LA & LA_d \\ 0 & L \\ L & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & C \\ CA & CA_d \\ C & 0 \\ 0 & L \\ L & 0 \end{bmatrix}. \tag{2.108}
$$

For mth-order observer (2.106), if the existence condition (2.108) is not satisfied, one can increase the order of the observer by inserting additional rows into the matrix L until the condition (2.108) holds. It is shown that when the order of the observer reaches $(n-p)$, the condition (2.108) is automatically satisfied as both its sides reach the maximum value of $2n$. Consequently, this reduces the advantages of the design of a reduced-order state observer for the time-delay system as the structure of the observer becomes more complex [138].

By using the multiple delayed information of both input and output, another observer was proposed in [138] as follows:

$$\hat{z}(t) = w(t) + Fy(t) + F_d y(t-\tau),$$
$$\dot{w}(t) = Nw(t) + N_d w(t-\tau) + Dy(t) + D_{1d} y(t-\tau) + D_{2d} y(t-2\tau) \quad (2.109)$$
$$+ Eu(t) + E_d u(t-\tau),$$

where N, N_d, D, D_{1d}, D_{2d}, E and E_d are constant matrices of appropriate dimensions to be designed such that the estimate $\hat{z}(t)$ approaches to $z(t)$.

Some less conservative conditions for existence of mth-order observer (2.106) were derived in [138] which are re-stated as follows:

Theorem 18 *For time-delay system (2.105) and* m*th-order observer (2.106), with a given positive scalar τ, the estimate $\hat{z}(t)$ converges asymptotically to $z(t)$ for any initial conditions if and only if the following conditions hold*

1) $\dot{e}(t) = Ne(t) + N_d e(t-\tau)$ *is asymptotically stable,*
2) $LA - NL + NFC - DC - FCA = 0,$
3) $LA_d + NF_dC - N_dL + N_dFC - D_{1d}C - FCA_d - F_{1d}CA = 0,$
4) $N_dF_{1d}C - D_{2d}C - F_{1d}CA_d = 0,$
5) $LB - FCB - E = 0,$
6) $E_d + F_{1d}CB = 0.$

The problem of the design of mth-order order linear functional observer for the time-delay system (2.105) is now equivalent to the problem of solving the set of the above matrix equations for all the unknown matrices N, N_d, D, D_{1d}, D_{2d}, E and E_d. The necessary and sufficient conditions for the existence of the observer (2.106) were stated as follows

$$\text{rank}\begin{bmatrix} 0 & C & 0 \\ CA & CA_d & 0 \\ C & 0 & 0 \\ 0 & L & 0 \\ L & 0 & 0 \\ LA & LA_d & 0 \\ 0 & CA & CA_d \\ 0 & 0 & C \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & C & 0 \\ CA & CA_d & 0 \\ C & 0 & 0 \\ 0 & L & 0 \\ L & 0 & 0 \\ 0 & CA & CA_d \\ 0 & 0 & C \end{bmatrix}. \tag{2.110}$$

By employing the idea on reachable set bounding, an observer in the form of (2.106) was proposed to deal with the design of linear functional observers for a class of time-delay systems which are subjected to unknown bounded disturbances [96]. Sufficient conditions were derived to ensure that the estimation error converges exponentially within a ball whose radius can be minimised.

2.3.3 Linear functional observers for systems with delay in the output

In practical control systems, the information of the output measurements is quite often available for processing only after a certain time delay, for example, in a large scale or networked control system where computation units are located far from the plant, and measurement data is transmitted via a communication channel with a limited bandwidth and thus time delay is unavoidable. As a result, the assumption about the availability of the instantaneous information of the output no longer holds. Therefore, it is extremely important to take the systems delayed output into account in the state observation problem.

Consider system (2.100) when the output information of $y(t)$ is available only after a time delay τ, an m-order observer was proposed in [138] as

$$\begin{aligned}\hat{z}(t) &= w(t) + Ey(t-\tau),\\ \dot{w}(t) &= Nw(t) + N_d w(t-\tau) + D_{1d}y(t-\tau) + D_{2d}y(t-2\tau)\\ &\quad + Hu(t) + H_d u(t-\tau)\end{aligned} \tag{2.111}$$

where $w(t) \in \mathbb{R}^m$ and τ is a positive scalar representing the time delay associated with the output information being received at the observer's terminal. Matrices N, N_d, D_{1d}, D_{2d}, E and E_d are of appropriate dimensions to be determined such that the estimation error (i.e., $e(t) = z(t) - \hat{z(t)})$ converges asymptotically to zero. It is easy to see that instead of using the information of $y(t)$ which is not available, the delayed output information $w(t-\tau)$, $y(t-\tau)$, $y(t-2\tau)$, $u(t-\tau)$ are used to reconstruct the estimate of $z(t)$.

The conditions for the existence of the observer (2.111) were derived in [138] which are restated in the following theorem.

Theorem 19 *For* m*th-order observer (2.111), with a given positive scalar τ, the estimate $\hat{z}(t)$ converges asymptotically to $z(t)$ for any initial conditions $x(0), w(0)$ and $u(t)$ if and only if the following conditions hold*

1) $\dot{e}(t) = Ne(t) + N_d e(t-\tau)$ *is asymptotically stable,*
2) $LA - NL = 0,$
3) $NEC - D_{1d}C - N_d L - ECA = 0,$
4) $N_d EC - D_{2d}C = 0,$
5) $LB - H = 0,$
5) $ECB + H_d = 0.$

Other approaches for linear systems with constant output delay were reported in [125, 68, 37]. Recently, another linear functional observer for a class of linear systems, where the input $u(t)$ and output $y(t)$ are both subject to known and constant time delays τ_1 and τ_2, respectively was proposed in [43]. By constructing a set of appropriate augmented LyapunovKrasovskii functionals with a triple-integral term and using the information of both the delayed output and input, some conditions for the existence of a minimal-order observer were derived. These conditions also guarantee that the estimation error is ϵ-convergent with an exponential rate. The necessary and sufficient conditions for the existence of this observer were derived in terms of LMIs.

2.4 Summary

This chapter has presented a brief review on on the recent development of the L-K method for stability analysis, discrete-time sliding mode control and linear functional observer design of time-delay systems.

For stability analysis of time-delay systems, it is worth pointing out that for the sake of conservatism reduction of the obtained delay-dependent stability criteria, the L-K approach, in combination with the bounding technique, delay decomposition approach, descriptor system approach and free weighting technique were used to get more information on the system or to estimate the upper bound of the derivative of the Lyapunov functional without ignoring some useful terms. However, to do this, the obtained results will be dependent on the partitioning part or the number of decision variables will be increased. Consequently, the computational burden may significantly increase. Therefore, the remaining interesting question is how to develop new methods to further reduce the conservatism of the existing stability criteria, while maintaining a reasonably low computational burden.

For DTSMC, the problem of sliding surface design and reaching law are extremely important in order to achieve the desired system performance. In DTSMC, the ideal sliding mode can no longer be attained and the closed-loop system is only driven into a quasi sliding mode. Moreover, it is known that a smaller QSMB tends to yield better control performance. Thus, it is interesting to obtain a smallest possible QSMB, and from that information, one can design a controller so that the induced sliding dynamics is prescribed in that band. Therefore, how to design the sliding surface and reaching law which may obtain the smallest QSMB is still an open problem.

For the problem of linear functional observer design for linear systems with time delay, associated in both measurement output and input. This means that the assumption about the availability of the instantaneous information of the output/input is no longer hold. As a consequence, the design of functional observers is more complicated and designers are left with no other choice but to use the delayed information to reconstruct the state or partial state vector. Consequently, this imposes additional constraints on the structure of the designed observers. Therefore, it is necessary to to develop linear functional observers with new architectures to deal with delayed information. In addition, the problem of linear functional observer design will be more interesting when the time- delay systems with time-varying delay in both measurement output and input.

Chapter 3

Exponential stability of time-delay systems

3.1 Introduction

Time delays are frequently encountered in various areas of science and engineering, including physical and chemical processes, economics, engineering, communication networks and biological systems. The existence of time delays is often a main cause of oscillations, instability and poor performance of the system. During the past decades, the stability analysis of TDS has received considerable attention from researchers, see, e.g. [99, 40, 142, 84, 20, 109, 111] and the references therein. On the other hand, in many practical control systems, the system response is required to be as fast as possible. As a result, the state trajectories of the system are often expected to converge sufficiently fast. Therefore, it is important for designers to be able to estimate the convergence rate of the system. For continuous-time systems with time-varying delay, several stability analysis schemes have been proposed for deriving the exponential stability conditions; see., e.g, [86, 112, 73, 73, 56, 114, 98, 149] and the references therein.

Along with many advantages of digital control, including cost-effectiveness and high flexibility of embedded systems, the problem of stability analysis for discrete-time systems with delay has received considerable attention; see., e.g, [119] and reference therein.

In [34], a delay-dependent stability condition for discrete-time systems with time-varying delay was derived by using Moon's inequality, which depends on the minimum and maximum delay bounds. Then, further results were later reported in

[35], where a set of augmented LKFs was constructed to use in conjunction with a bounding technique. Another delay-dependent stability criteria of linear continuous/discrete systems with time-varying delay were developed in [157] by using a piecewise analysis method (PAM). Based on the combination of a Lyapunov functional and the delay-partitioning approach, some stability conditions were proposed in [90], where the results were compared with those obtained by using output feedback stabilization in [51]. Improved delay-dependent stabilization criteria were reported in [107] by using a piecewise LKF and a finite sum inequality. By using the model transformation approach, another stability criteria was proposed in terms of linear matrix inequalities [80]. Based on the integral quadratic constraint (IQC) and assumption of bounded interval time-varying delay, a set of new stability criteria was presented in [66]. Recently, some improved results were reported in [80, 24, 74].

However, it should be noted that not much attention has been paid to discrete-time systems. Moreover, the conservatism of stability conditions for linear systems with time-varying delay can be relaxed with the combination of existing LKFs, which include quadratic and double summation terms, with some free-weighting matrices. However, the use of FWMs may increase the computational complexity due to an increase in the number of decision variables. As a consequence, it is worth finding a more effective method to ultimately improve stability criteria of these systems that can be obtained in a computationally-effective manner. These together have been the motivation in the current work.

In this chapter, we consider the problem of exponential stability of discrete-time systems with interval time-varying delay. Here, without using any FWMs, we introduce a new set of LKFs containing an augmented vector and some triple summation terms. To enhance the feasible region of stability conditions, the reciprocally convex approach is used to evaluate the double summation terms in the derivative of the proposed LKFs. As a result, improved results on exponential stability are obtained, in comparison with existing stability conditions in the literature. Numerical examples are provided to illustrate the effectiveness of the proposed approach.

3.2 Problem statement and preliminaries

Consider the following linear discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-\tau(k)), \quad k \in \mathbb{Z}^+, \\ x(k) &= \phi(k), \quad k \in \mathbb{Z}[-\tau_M, 0], \end{aligned} \tag{3.1}$$

where $x(k) \in \mathbb{R}^n$ is the system state and $A, A_d \in \mathbb{R}^{n\times n}$ are constant matrices. The time-varying delay $\tau(k)$ is assumed to belong to a given interval

$$0 < \tau_m \leq \tau(k) \leq \tau_M, \ \forall k \in \mathbb{Z}^+, \tag{3.2}$$

where $\tau_m < \tau_M$ are positive constants representing the minimum and maximum delays respectively and $\phi(k), k \in \mathbb{Z}[-\tau_M, 0]$, is the initial string for system (3.1).

It should be noted that system (3.1) is very popular in the literature and it is extensively studied as the time-varying delay $\tau(k)$ is frequently encountered in many engineering systems such as networked control systems, chemical process and long transmission lines in pneumatic systems. A typical system containing time delays is the networked control system where the delays induced by the network transmission (either from sensor to controller or from controller to actuator) are actually time-varying.

The aim here is to derive new delay-dependent conditions such that system (3.1) is exponential stable with the maximum allowable bound for the time delay. The following definition and lemmas are first introduced.

Definition 3 *System* (3.1) *is said to be exponentially stable if there exist positive constants* $\alpha > 1$ *and* $N > 1$ *such that all solutions* $x(k, \phi)$ *of system* (3.1) *satisfy*

$$\|x(k,\phi)\| \leq N\|\phi\|\alpha^{-k}, \ \forall k \in \mathbb{Z}^+,$$

where α *is the exponential decay rate of system* (3.1)*, and* $\|\phi\| = \max\{\|\phi(k)\| : k \in \mathbb{Z}[-\tau_M, 0]\}$.

For the Lyapunov-Krasovskii method, the construction of LKFs plays a crucial role in deriving the less conservative delay-dependent stability conditions. However, the estimation of the double summation terms in the different LKFs is always a challenging problem. Thus, the following lemma is used frequently.

Lemma 1 *[76] Let* P *be a symmetric positive-definite matrix and* $\tau_1, \tau_2 \in \mathbb{Z}$, $0 < \tau_1 < \tau_2$. *Then for any* $r > 1$ *and a vector function* $x(k), k \in \mathbb{Z}$, *the following inequality holds*

$$a)\ \sum_{s=k-\tau_2}^{k-\tau_1} r^{s-k} x^T(s)Px(s) \geq r_a \left[\sum_{s=k-\tau_2}^{k-\tau_1} x(s)\right]^T P \left[\sum_{s=k-\tau_2}^{k-\tau_1} x(s)\right], \tag{3.3a}$$

$$b)\ \sum_{s=-\tau_2}^{-\tau_1-1} \sum_{u=s}^{-1} r^u x^T(k+u)Px(k+u) \geq r_b \left[\sum_{s=-\tau_2}^{-\tau_1-1} \sum_{u=s}^{-1} x(k+u)\right]^T P \left[\sum_{s=-\tau_2}^{-\tau_1-1} \sum_{u=s}^{-1} x(k+u)\right], \tag{3.3b}$$

where

$$r_a = \frac{1-r}{r^{\tau_1} - r^{\tau_2+1}}, r_b = \frac{(1-r)^2}{r\left[r^{1+\tau_2} - r^{1+\tau_1} + (1-r)(\tau_2 - \tau_1)\right]}.$$

Proof. By using the Schur complement, we have the following inequality for any $r > 1$ and $s \in \mathbb{Z}$

$$\begin{bmatrix} r^{s-k}x^T(s)Px(s) & x^T(s) \\ \star & r^{k-s}P^{-1} \end{bmatrix} \geq 0. \tag{3.4}$$

For $k - \tau_2 \leq s \leq k - \tau_1$, by summing the above inequality from $k - \tau_2$ to $k - \tau_1$, we obtain

$$\begin{bmatrix} \sum_{k-\tau_2}^{k-\tau_1} r^{s-k}x^T(s)Px(s) & \sum_{k-\tau_2}^{k-\tau_1} x^T(s) \\ \star & \sum_{k-\tau_2}^{k-\tau_1} r^{k-s}P^{-1} \end{bmatrix} \geq 0. \tag{3.5}$$

By using Schur complement again, it follows that

$$\sum_{k-\tau_2}^{k-\tau_1} r^{-s}x^T(s)Px(s) \geq \Big[\sum_{k-\tau_2}^{k-\tau_1} x(s)\Big]^T \Big(\sum_{k-\tau_2}^{k-\tau_1} r^{k-s}P^{-1}\Big)^{-1} \Big[\sum_{k-\tau_2}^{k-\tau_1} x(s)\Big], \tag{3.6}$$

with

$$\Big(\sum_{k-\tau_2}^{k-\tau_1} r^{k-s} P^{-1}\Big)^{-1} = \frac{1-r}{r^{\tau_2} - r^{\tau_1+1}} P = r_a P.$$

It can be seen that the inequality (3.6) is equivalent to the inequality (3.3a). Similarly, for $s \leq u \leq -1$ and $-\tau_2 \leq s \leq -\tau_1 - 1$, by changing the variable s into u in (3.5), we obtain the following inequality

$$\sum_{s=-\tau_2}^{-\tau_1-1} \begin{bmatrix} \sum_s^{-1} r^u x^T(k+u) P x(k+u) & \sum_s^{-1} x^T(k+u) \\ \star & \sum_s^{-1} r^{-u} P^{-1} \end{bmatrix} \geq 0. \tag{3.7}$$

Therefore, the following inequality can be obtained as

$$\begin{aligned} \sum_{s=-\tau_2}^{-\tau_1-1} \sum_s^{-1} r^u x^T(k+u) P x(k+u) \geq \Big[\sum_{s=-\tau_2}^{-\tau_1-1} \sum_s^{-1} x(k+u) \Big]^T \Big[\sum_{s=-\tau_2}^{-\tau_1-1} \sum_s^{-1} r^{-u} P^{-1} \Big]^{-1} \\ \times \Big[\sum_{s=-\tau_2}^{-\tau_1-1} \sum_s^{-1} x(k+u) \Big], \end{aligned} \tag{3.8}$$

with

$$\begin{aligned} \Big[\sum_{s=-\tau_2}^{-\tau_1-1} \sum_s^{-1} r^{-u} P^{-1} \Big]^{-1} &= \Big[\sum_{s=-\tau_2}^{-\tau_1-1} \sum_s^{-1} r^{-u} \Big]^{-1} P \\ &= r_b P. \end{aligned}$$

The proof is completed. □

The reciprocally convex combination lemma provided in [105] is used in this chapter. This inequality is reformulated as follows:

Lemma 2 *[105] For a given scalar $\beta \in (0,1)$, an $n \times n$-matrix $R > 0$ and two vectors $\eta_1, \eta_2 \in \mathbb{R}^n$, define function $\Theta(\beta, R)$ as*

$$\Theta(\beta, R) = \frac{1}{\beta} \eta_1^T R \eta_1 + \frac{1}{1-\beta} \eta_2^T R \eta_2.$$

If there is a matrix $X \in \mathbb{R}^{n \times n}$ such that $\begin{bmatrix} R & X \\ * & R \end{bmatrix} \geq 0$, then the following inequality holds

$$\min_{\beta \in (0,1)} \Theta(\beta, R) \geq \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}.$$

Lemma 3 *Let $V(k)$ be a Lyapunov functional, if there exist scalars $\gamma_1 > 0$, $\gamma_2 > 0$ and $r > 1$ such that*

$$\gamma_1\|x(k)\|^2 \leq V(k) \leq \gamma_2\|x(k)\|^2, \tag{3.9a}$$

$$\Delta V(k) + (1 - r^{-1})V(k) \leq 0, \quad k \in \mathbb{Z}^+, \tag{3.9b}$$

then every solution $x(k, \phi)$ of system (3.1) *satisfies the following estimation*

$$\|x(k, \phi)\| \leq \sqrt{\frac{\gamma_2}{\gamma_1}}\|\phi\|\alpha^{-k}, \quad k \in \mathbb{Z}^+. \tag{3.10}$$

where the Lyapunov factor and exponential decay rate are respectively determined as $N = \sqrt{\frac{\gamma_2}{\gamma_1}}$ and $\alpha = \sqrt{r}$.

Proof. From (3.9b), we have

$$\begin{aligned} V(k+1) &\leq r^{-1}V(k) \\ &\leq \dots \\ &\leq r^{-k-1}V(0). \end{aligned}$$

Thus,

$$V(k) \leq \gamma_2 r^{-k}\|\phi\|^2, \quad k \in \mathbb{Z}^+.$$

By taking (3.9a) into account, we obtain

$$\|x(k, \phi)\| \leq \sqrt{\frac{\gamma_2}{\gamma_1}}\|\phi\| r^{-\frac{1}{2}k} = N\|\phi\|\alpha^{-k}, \quad k \in \mathbb{Z}^+,$$

where $\alpha = \sqrt{r}$. This completes the proof. □

3.3 Main Results

The following notations are specifically used in this development. For given integers τ_m, τ_M satisfying $0 < \tau_m < \tau_M$ and any integer number $\delta \in (0, \tau_M - \tau_m)$, matrices X, G and symmetric positive definite matrices $P, Q_j, R_j, j = 1, 2, 3, S_1, S_2$ of appropriate dimensions, let us denote $\tau = \tau_m + \delta$. We also denote $e_i = [0_{n\times(i-1)n}\ I_n\ 0_{n\times(12-i)n}]$,

$i = 1, 2, \ldots, 12$, as entry matrices, and the following constants

$$\tau_a = \frac{\tau_m(\tau_m+1)}{2}, \tau_b = \frac{(\tau_M-\tau_m)(\tau_M+\tau_m+1)}{2},$$
$$r_1 = \frac{1-r}{r-r^{\tau_m+1}}, r_2 = \frac{1-r}{r^{\tau+1}-r^{\tau_M+1}}, r_3 = \frac{1-r}{r^{\tau_m+1}-r^{\tau+1}},$$
$$r_4 = \frac{(1-r)^2}{r\left[r^{1+\tau_m}-(1+\tau_m)r+\tau_m\right]},$$
$$r_5 = \frac{(1-r)^2}{r\left[r^{1+\tau_M}-r^{1+\tau_m}+(1-r)(\tau_M-\tau_m)\right]},$$

vectors

$$\xi(k) = \Big[x^T(k) \;\; x^T(k-\tau_m) \;\; x^T(k-\tau(k)) \;\; x(k-\tau) \;\; x^T(k-\tau_M)$$
$$\sum_{s=k-\tau_m}^{k-1} x^T(s) \; \sum_{s=k-\tau}^{k-\tau(k)-1} x^T(s) \; \sum_{s=k-\tau(k)}^{k-\tau_m-1} x^T(s) \; \sum_{s=k-\tau_M}^{k-\tau-1} x^T(s)$$
$$\sum_{s=k-\tau(k)}^{k-\tau-1} x^T(s) \; \sum_{s=k-\tau}^{k-\tau_m-1} x^T(s) \; \sum_{s=k-\tau_M}^{k-\tau(k)-1} x^T(s)\Big]^T,$$
$$\zeta(k) = \left[x^T(k) \quad \sum_{s=k-\tau_m}^{k-1} x^T(s) \quad \sum_{s=k-\tau}^{k-\tau_m-1} x^T(s) \quad \sum_{s=k-\tau_M}^{k-\tau-1} x^T(s)\right]^T,$$
$$\rho(k) = \left[x^T(k) \; y^T(k)\right]^T,$$

and matrices

$$R_c = \tau_m R_1 + (\tau_M-\tau)R_2 + (\tau-\tau_m)R_3,$$
$$\Pi_1 = [(Ae_1 + A_d e_3)^T \;\; (e_1-e_2+e_6)^T \;\; (e_2-e_4+e_{11})^T \;\; (e_4-e_5+e_9)^T]^T,$$
$$\Pi_2 = [e_1^T \;\; e_6^T \;\; e_{11}^T \;\; e_9^T]^T, \Pi_3 = [e_1^T \; ((A-I)e_1 + A_d e_3)^T]^T,$$
$$\Pi_4 = [e_6^T \;\; (e_1-e_2)^T]^T, \Pi_5 = [e_9^T \;\; (e_4-e_5)^T]^T,$$
$$\Pi_6 = [e_7^T \;\; (e_3-e_4)^T \;\; e_8^T \;\; (e_2-e_3)^T]^T, \Pi_7 = (A-I)e_1 + A_d e_3,$$

$$\Pi_8 = \tau_m e_1 - e_6, \Pi_9 = (\tau_M - \tau_m)e_1 - e_8 - e_{12},$$

$$\Pi_{10} = [e_{11}^T \quad (e_2 - e_4)^T]^T, \Pi_{11} = [e_{12}^T \quad (e_3 - e_5)^T \quad e_{10}^T \quad (e_4 - e_3)^T]^T,$$

$$\Xi_1 = \begin{bmatrix} R_3 & X_1 \\ \star & R_3 \end{bmatrix}; \Xi_2 = \begin{bmatrix} R_2 & X_2 \\ \star & R_2 \end{bmatrix},$$

$$\begin{aligned} \Omega_0(r) = {} & \Pi_1^T P \Pi_1 - r^{-1} \Pi_2^T P \Pi_2 + e_1^T Q_1 e_1 + r^{-\tau_m} e_2^T (Q_2 - Q_1) e_2 \\ & + r^{-\tau} e_4^T (Q_3 - Q_2) e_4 - r^{-\tau_M} e_5^T Q_3 e_5 + \Pi_3^T R_c \Pi_3 - r_1 \Pi_4^T R_1 \Pi_4 \\ & + \Pi_7^T (\tau_a S_1 + \tau_b S_2) \Pi_7 - r_4 \Pi_8^T S_1 \Pi_8 - r_5 \Pi_9^T S_2 \Pi_9, \end{aligned}$$

$$\Omega_1(r) = r_2 \Pi_5^T R_2 \Pi_5 + r_3 \Pi_6^T \Xi_1 \Pi_6,$$

$$\Omega_2(r) = r_3 \Pi_{10}^T R_3 \Pi_{10} + r_2 \Pi_{11}^T \Xi_2 \Pi_{11},$$

Now, we are ready to present the main result that gives sufficient conditions for exponentially stable of system (3.1) as follows:

Theorem 20 *For given integers $0 < \tau_m < \tau_M$, if there exist a scalar $r > 1$, an integer $\delta \in (0, \tau_M - \tau_m)$, symmetric positive definite matrices $P, Q_j, R_j, j = 1, 2, 3, S_1, S_2$ and a matrix X_1, X_2 such that the following inequalities hold*

$$\Omega_0(r) - \Omega_1(r) < 0, \tag{3.11a}$$

$$\Omega_0(r) - \Omega_2(r) < 0, \tag{3.11b}$$

$$\Xi_1 \geq 0, \Xi_2 \geq 0, \tag{3.11c}$$

Then system (3.1) is exponentially stable with the exponential decay rate $\alpha = \sqrt{r}$. Moreover, every solutions of system (3.1) satisfies

$$\|x(k, \phi)\| \leq \sqrt{\frac{\gamma_2}{\gamma_1}} \|\phi\| \alpha^{-k}, \quad k \in \mathbb{Z}^+ \tag{3.12}$$

where

$$
\begin{aligned}
\gamma_1 =& \lambda_m(P), \\
\gamma_2 =& (1+\tau_M)\lambda_M(P) \\
&+ \frac{1-r^{-\tau_m}}{1-r^{-1}}\lambda_M(Q_1) + \frac{r^{-\tau_m}-r^{-\tau}}{1-r^{-1}}\lambda_M(Q_2) + \frac{r^{-\tau}-r^{-\tau_M}}{1-r^{-1}}\lambda_M(Q_3) \\
&+ 9\frac{(1-r^{-1})\,\tau_m + r^{-\tau_m-1} - r^{-1}}{(1-r^{-1})^2}\lambda_M(R_1) \\
&+ 9\frac{(1-r^{-1})\,(\tau_M-\tau) + r^{-\tau_M-1} - r^{-\tau-1}}{(1-r^{-1})^2}\lambda_M(R_2) \\
&+ 9\frac{(1-r^{-1})\,(\tau-\tau_m) + r^{-\tau-1} - r^{-\tau_m-1}}{(1-r^{-1})^2}\lambda_M(R_3) \\
&+ 4\frac{\tau_a\,(1-r^{-1})^2 + (1+\tau_m)r^{-2} - \tau_m r^{-1} - r^{-\tau_m-2}}{(1-r^{-1})^3}\lambda_M(S_1) \\
&+ 4\frac{\lambda_M(S_2)}{(1-r^{-1})^3}\left[\tau_b\left(1-r^{-1}\right)^2 + r^{-\tau-2} - (\tau_M-\tau_m)\left(r^{-1}-r^{-2}\right) - r^{-\tau_M-2}\right].
\end{aligned}
$$

Proof. Define $y(k) = x(k+1) - x(k) = (A-I)x(k) + A_d x(k-\tau(k))$. Now, for the sake of getting more information on system (3.1), the interval time-varying delay $[\tau_m \;\; \tau_M]$ is divided into two nonuniform subintervals. Note that, for any $k \in \mathbb{Z}$, we have either $\tau(k) \in [\tau_m, \tau]$ or $\tau(k) \in (\tau, \tau_M]$. Let us define two sets

$$
\mathit{\Gamma}_1 = \{k \in \mathbb{Z} | \tau(k) \in [\tau_m, \tau]\}
$$

$$
\mathit{\Gamma}_2 = \{k \in \mathbb{Z} | \tau(k) \in (\tau, \tau_M]\}.
$$

Consider the following Lyapunov-Krasovskii functional

$$
V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{3.13}
$$

where

$$
\begin{aligned}
V_1(k) &= \zeta^T(k) P \zeta(k), \\
V_2(k) &= \sum_{s=k-\tau_m}^{k-1} r^{s-k+1} x^T(s) Q_1 x(s) + \sum_{s=k-\tau}^{k-\tau_m-1} r^{s-k+1} x^T(s) Q_2 x(s) \\
&\quad + \sum_{s=k-\tau_M}^{k-\tau-1} r^{s-k+1} x^T(s) Q_3 x(s),
\end{aligned}
$$

$$V_3(k) = \sum_{s=-\tau_m}^{-1} \sum_{v=k+s}^{k-1} r^{v-k+1} \rho^T(v) R_1 \rho(v) + \sum_{s=-\tau_M}^{-\tau-1} \sum_{v=k+s}^{k-1} r^{v-k+1} \rho^T(v) R_2 \rho(v),$$
$$+ \sum_{s=-\tau}^{-\tau_m-1} \sum_{v=k+s}^{k-1} r^{v-k+1} \rho^T(v) R_3 \rho(v),$$
$$V_4(k) = \sum_{s=-\tau_m}^{-1} \sum_{u=s}^{-1} \sum_{v=k+u}^{k-1} r^{v-k+1} y^T(v) S_1 y(v) + \sum_{s=-\tau_M}^{-\tau_m-1} \sum_{u=s}^{-1} \sum_{v=k+u}^{k-1} r^{v-k+1} y^T(v) S_2 y(v).$$

By taking the forward difference of $V_1(k)$ along the solutions of system (3.1), we have

$$\Delta V_1(k) = \zeta^T(k+1) P \zeta(k+1) - r^{-1} \zeta^T(k) P \zeta(k) + (r^{-1} - 1) V_1(k),$$

where

$$\zeta(k+1) = \begin{bmatrix} x(k+1) \\ \sum_{s=k+1-\tau_m}^{k} x(s) \\ \sum_{s=k+1-\tau}^{k-\tau_m} x(s) \\ \sum_{s=k+1-\tau_M}^{k-\tau} x(s) \end{bmatrix} = \begin{bmatrix} Ax(k) + A_d x(k-\tau(k)) \\ x(k) - x(k-\tau_m) + \sum_{s=k-\tau_m}^{k-1} x(s) \\ x(k-\tau_m) - x(k-\tau) + \sum_{s=k-\tau}^{k-\tau_m-1} x(s) \\ x(k-\tau) - x(k-\tau_M) + \sum_{s=k-\tau_M}^{k-\tau-1} x(s) \end{bmatrix}.$$

Therefore, $\Delta V_1(k)$ can be obtained of the form

$$\Delta V_1(k) = \xi^T(k)(\Pi_1^T P \Pi_1 - r^{-1} \Pi_2^T P \Pi_2) \xi(k) + (r^{-1} - 1) V_1(k). \tag{3.14}$$

The forward differences of $V_2(k)$ and $V_3(k)$ are obtained as

$$\begin{aligned} \Delta V_2(k) = {} & x^T(k) Q_1 x(k) + r^{-\tau_m} x^T(k-\tau_m) Q_2 x(k-\tau_m) + r^{-\tau} x^T(k-\tau) Q_3 x(k-\tau) \\ & - r^{-\tau_m} x^T(k-\tau_m) Q_1 x(k-\tau_m) - r^{-\tau} x^T(k-\tau) Q_2 x(k-\tau) \\ & - r^{-\tau_M} x^T(k-\tau_M) Q_3 x(k-\tau_M) + (r^{-1} - 1) V_2(k), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \Delta V_3(k) = {} & \rho^T(k) R_c \rho(k) - \sum_{s=k-\tau_m}^{k-1} r^{s-k} \rho^T(s) R_1 \rho(s) - \sum_{s=k-\tau_M}^{k-\tau-1} r^{s-k} \rho^T(s) R_2 \rho(s) \\ & - \sum_{s=k-\tau}^{k-\tau_m-1} r^{s-k} \rho^T(s) R_3 \rho(s) + (r^{-1} - 1) V_3(k). \end{aligned} \tag{3.16}$$

Similarly, the difference of $V_4(k)$ along solutions of system (3.1) is calculated as follows.

$$\begin{aligned} \mathit{\Delta} V_4 =& y^T(k)\left(\tau_a S_1 + \tau_b S_2\right) y(k) - \sum_{s=-\tau_m}^{-1} \sum_{v=s}^{-1} r^v y^T(k+v) S_1 y(k+v) \\ &- \sum_{s=-\tau_M}^{-\tau_m-1} \sum_{v=s}^{-1} r^v y^T(k+v) S_2 y(k+v) + (r^{-1}-1) V_4(k). \end{aligned} \tag{3.17}$$

We distinguish two sub-intervals of the time delay in the following cases.

Case I: For $k \in \mathit{\Gamma}_1$, i.e. the time delay $\tau(k) \in [\tau_m, \tau]$, from Lemma 1, the following estimations are obtained as

$$\begin{aligned} -\sum_{s=k-\tau_m}^{k-1} r^{s-k} \rho^T(s) R_1 \rho(s) \leq& -r_1 \begin{bmatrix} \sum_{s=k-\tau_m}^{k-1} x(s) \\ x(k) - x(k-\tau_m) \end{bmatrix}^T R_1 \begin{bmatrix} \sum_{s=k-\tau_m}^{k-1} x(s) \\ x(k) - x(k-\tau_m) \end{bmatrix} \\ \leq& -r_1 \xi^T(k) \mathit{\Pi}_4^T R_1 \mathit{\Pi}_4 \xi(k). \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} -\sum_{s=k-\tau_M}^{k-\tau-1} r^{s-k} \rho^T(s) R_2 \rho(s) \leq& -r_2 \begin{bmatrix} \sum_{s=k-\tau_M}^{k-\tau-1} x(s) \\ x(k-\tau) - x(k-\tau_M) \end{bmatrix}^T R_2 \times \\ &\times \begin{bmatrix} \sum_{s=k-\tau_M}^{k-\tau-1} x(s) \\ x(k-\tau) - x(k-\tau_M) \end{bmatrix} \\ \leq& -r_2 \xi^T(k) \mathit{\Pi}_5^T R_2 \mathit{\Pi}_5 \xi(k). \end{aligned} \tag{3.19}$$

Moreover, we also have

$$\begin{aligned} -\sum_{s=k-\tau}^{k-\tau_m-1} r^{s-k} \rho^T(s) R_3 \rho(s) =& -\sum_{s=k-\tau}^{k-\tau(k)-1} r^{s-k} \rho^T(s) R_3 \rho(s) \\ &- \sum_{s=k-\tau(k)}^{k-\tau_m-1} r^{s-k} \rho^T(s) R_3 \rho(s). \end{aligned} \tag{3.20}$$

By using Lemma 1, the following inequality is obtained as

$$
\begin{aligned}
&-\sum_{s=k-\tau}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_3\rho(s)-\sum_{s=k-\tau(k)}^{k-\tau_m-1} r^{s-k}\rho^T(s)R_3\rho(s)\\
&\le -\frac{1-r}{r^{\tau(k)+1}-r^{\tau+1}}\Big[\sum_{s=k-\tau}^{k-\tau(k)-1}\rho(s)\Big]^T R_3\Big[\sum_{s=k-\tau}^{k-\tau(k)-1}\rho(s)\Big]\\
&\quad -\frac{1-r}{r^{\tau_m+1}-r^{\tau(k)+1}}\Big[\sum_{s=k-\tau(k)}^{k-\tau_m-1}\rho(s)\Big]^T R_3\Big[\sum_{s=k-\tau(k)}^{k-\tau_m-1}\rho(s)\Big]
\end{aligned}
\tag{3.21}
$$

From Lemma 2, we have

$$
\begin{aligned}
&-\sum_{s=k-\tau}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_3\rho(s)-\sum_{s=k-\tau(k)}^{k-\tau_m-1} r^{s-k}\rho^T(s)R_3\rho(s)\\
&\le -r_3\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}^T\begin{bmatrix}R_3 & X_1\\ \star & R_3\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}\\
&\le -r_3\xi^T(k)\Pi_6^T\Xi_1\Pi_6\xi(k),
\end{aligned}
\tag{3.22}
$$

where $\beta_1=\dfrac{r^{\tau(k)}-r^{\tau}}{r^{\tau_m}-r^{\tau}}$ and

$$
\eta_1=\begin{bmatrix}\sum_{s=k-\tau}^{k-\tau(k)-1}x(s)\\ x(k-\tau(k))-x(k-\tau)\end{bmatrix},\eta_2=\begin{bmatrix}\sum_{s=k-\tau(k)}^{k-\tau_m-1}x(s)\\ x(k-\tau_m)-x(k-\tau(k))\end{bmatrix}.
$$

Similarly, for the different $\varDelta V_4(k)$, by using Lemma 1, the following inequalities are obtained as

$$
\begin{aligned}
&-\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1} r^v y^T(k+v)S_1y(k+v)\\
&\le -r_4\left(\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1}y(k+v)\right)^T S_1\left(\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1}y(k+v)\right)\\
&\le -r_4\left(\tau_m x(k)-\sum_{s=k-\tau_m}^{k-1}x(s)\right)^T S_1\left(\tau_m x(k)-\sum_{s=k-\tau_m}^{k-1}x(s)\right)\\
&\le -r_4\xi^T(k)\Pi_8^T S_1\Pi_8\xi(k).
\end{aligned}
\tag{3.23}
$$

and

$$
\begin{aligned}
&-\sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1} r^v y^T(k+v)S_2 y(k+v) \\
&\leq -r_5\left(\sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1} y(k+v)\right)^T S_2 \left(\sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1} y(k+v)\right) \\
&\leq -r_5\left[(\tau_M-\tau_m)x(k)-\sum_{s=k-\tau}^{k-\tau_m-1} x(s)\right]^T S_2\left[(\tau_M-\tau_m)x(k)-\sum_{s=k-\tau_M}^{k-\tau_m-1} x(s)\right] \\
&\leq -r_5\left[(\tau_M-\tau_m)x(k)-\sum_{s=k-\tau_M}^{k-\tau(k)-1} x(s)-\sum_{s=k-\tau(k)}^{k-\tau_m-1} x(s)\right]^T S_2 \\
&\quad\times\left[(\tau_M-\tau_m)x(k)-\sum_{s=k-\tau}^{k-\tau(k)-1} x(s)-\sum_{s=k-\tau(k)}^{k-\tau_m-1} x(s)\right] \\
&\leq -r_5\xi^T(k)\mathit{\Pi}_9^T S_2\mathit{\Pi}_9\xi(k).
\end{aligned}
\tag{3.24}
$$

From (3.13)-(3.24), we now obtain

$$
\mathit{\Delta}V(k)+(1-r^{-1})V(k)\leq \xi^T(k)(\mathit{\Omega}_0-\mathit{\Omega}_1)\xi(k), \quad \forall k\in\mathbb{Z}^+. \tag{3.25}
$$

Case II: For $k\in\mathit{\Gamma}_2$, i.e. the time delay $\tau(k)\in(\tau,\tau_M]$, by Lemma 1, we obtain

$$
\begin{aligned}
&-\sum_{s=k-\tau}^{k-\tau_m-1} r^{s-k}\rho^T(s)R_3\rho(s) \\
&\leq -r_3\begin{bmatrix}\sum_{s=k-\tau}^{k-\tau_m-1} x(s) \\ x(k-\tau_m)-x(k-\tau)\end{bmatrix}^T R_3 \begin{bmatrix}\sum_{s=k-\tau}^{k-\tau_m-1} x(s) \\ x(k-\tau_m)-x(k-\tau)\end{bmatrix} \\
&\leq -r_3\xi^T(k)\mathit{\Pi}_{10}^T R_3\mathit{\Pi}_{10}\xi(k).
\end{aligned}
\tag{3.26}
$$

In addition, we can have

$$
\begin{aligned}
-\sum_{s=k-\tau_M}^{k-\tau-1} r^{s-k}\rho^T(s)R_2\rho(s) = &-\sum_{s=k-\tau_M}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_2\rho(s) \\
&-\sum_{s=k-\tau(k)}^{k-\tau-1} r^{s-k}\rho^T(s)R_2\rho(s).
\end{aligned}
\tag{3.27}
$$

Similarly, we have the following estimation

$$
\begin{aligned}
&-\sum_{s=k-\tau_M}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_2\rho(s)-\sum_{s=k-\tau(k)}^{k-\tau-1} r^{s-k}\rho^T(s)R_2\rho(s)\\
&\leq -\frac{1-r}{r^{\tau(k)+1}-r^{\tau_M+1}}\Big[\sum_{s=k-\tau_M}^{k-\tau(k)-1}\rho(s)\Big]^T R_2\Big[\sum_{s=k-\tau_M}^{k-\tau(k)-1}\rho(s)\Big] \qquad (3.28)\\
&-\frac{1-r}{r^{\tau+1}-r^{\tau(k)+1}}\Big[\sum_{s=k-\tau(k)}^{k-\tau-1}\rho(s)\Big]^T R_2\Big[\sum_{s=k-\tau(k)}^{k-\tau-1}\rho(s)\Big],
\end{aligned}
$$

which yields

$$
\begin{aligned}
&-\sum_{s=k-\tau_M}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_2\rho(s)-\sum_{s=k-\tau(k)}^{k-\tau-1} r^{s-k}\rho^T(s)R_2\rho(s)\\
&\leq -r_2\begin{bmatrix}\eta_3\\ \eta_4\end{bmatrix}^T\begin{bmatrix}R_2 & X_2\\ \star & R_2\end{bmatrix}\begin{bmatrix}\eta_3\\ \eta_4\end{bmatrix} \qquad (3.29)\\
&\leq -r_2\xi^T(k)\Pi_{11}^T\Xi_2\Pi_{11}\xi(k),
\end{aligned}
$$

where $\beta_2=\dfrac{r^{\tau(k)}-r^{\tau_M}}{r^{\tau}-r^{\tau_M}}$ and

$$
\eta_3=\begin{bmatrix}\sum_{s=k-\tau_M}^{k-\tau(k)-1}x(s)\\ x(k-\tau(k))-x(k-\tau_M)\end{bmatrix}, \eta_4=\begin{bmatrix}\sum_{s=k-\tau(k)}^{k-\tau-1}x(s)\\ x(k-\tau)-x(k-\tau(k))\end{bmatrix}.
$$

Similarly, we obtain the following inequality

$$
\Delta V(k)+(1-r^{-1})V(k)\leq \xi^T(k)(\Omega_0-\Omega_2)\xi(k),\quad \forall k\in\mathbb{Z}^+. \qquad (3.30)
$$

It follows from Lemma 3, (3.25) and (3.35) that

$$
\Delta V(k)+(1-r^{-1})V(k)\leq 0,\quad \forall k\in\mathbb{Z}^+. \qquad (3.31)
$$

On the other hand, it can be verified from (3.13) that

$$
\gamma_1\|x(k)\|^2\leq V(k)\leq \gamma_2\|x_k\|^2, \qquad (3.32)
$$

From Lemma 3, (3.31) and (3.32), we have

$$
\|x(k,\phi)\|\leq N\|\phi\|\alpha^{-k},\quad k\in\mathbb{Z}^+,
$$

where $N = \sqrt{\frac{\gamma_2}{\gamma_1}}$. Thus, from Definition 3, system (3.1) is exponentially stable with the exponential decay rate $\alpha = \sqrt{r}$. The proof is completed. □

Remark 1 By comparison to other LKFs existing in the literature, the new LKFs proposed in this paper contain a new augmented vector $\zeta(k)$ in $V_1(k)$ and two triple summation terms in $V_4(k)$. As a result, the information about the current values of the state variables $x(k)$ and their history are exploited, leading to less conservative stability conditions.

Remark 2 It should be noted that the stability conditions, established in Theorem 20, contain the tuning parameters α and δ so there remains the interesting question as how to find the optimal combination of these parameters. A direct method to solve that problem is to choose a cost function t_{min} that is obtained while solving the feasibility problem using Matlab's LMI toolbox. When t_{min} is positive, the combination of the tuning parameters α and δ does not allow a feasible solution to the set of LMIs [32]. By using a numerical optimisation algorithm, such as the program *fminsearch* in the optimisation toolbox of Matlab, we can find the solution of the cost function t_{min}. The optimal combination of two parameters is obtained when the minimum value of the cost function is negative [30].

Remark 3 The convergence rate of the system can be directly chosen from this proposed approach. Moreover, the exponential stability conditions given in Theorem 20 are derived in terms of LMIs without introducing any free-weighting matrices. Therefore, the obtained stability conditions may involve fewer decision variables, and hence reduce the computational complexity.

3.4 Numerical examples

3.4.1 Example 3.1

For illustration of the effectiveness of the proposed approach in relaxing the conservatism of the stability conditions, let us consider the system, given in [34, 51, 66, 90] as follows

$$x(k+1) = \begin{pmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{pmatrix} x(k) + \begin{pmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{pmatrix} x(k-\tau(k)). \tag{3.33}$$

For this system, the asymptotic stability conditions proposed in the aforementioned studies gave the allowable upper bounds τ_M with various values of τ_m as listed in Table 3.1 below. Here, from Theorem 20 and Remark 2, we find the optimal value for r as $r = 1.001$. As a result, the exponential convergence rate is calculated as $\alpha = 1.0005$ and the upper bounds τ_M are also given correspondingly in Table 3.1. It can be seen that all the upper bounds τ_M obtained by using Theorem 20 are larger than those obtained in the papers mentioned above. As explained in Remark 1, this also confirms the improvement of this approach on stability conditions as compared to existing results.

Table 3.1. MABs of τ_M for different values of τ_m

τ_m	1	3	5	7	11	13	15
Gao et al. (2007)	12	13	13	14	16	17	18
Zhang et al. (2008)	12	13	14	15	17	19	20
He et al. (2008)	17	17	18	18	20	22	23
Meng et al. (2010)	17	17	18	18	20	22	23
Li and Gao (2011)	-	18	19	21	25	25	26
Kao (2012)	17	18	19	21	25	25	-
This paper	19	19	20	21	22	23	24

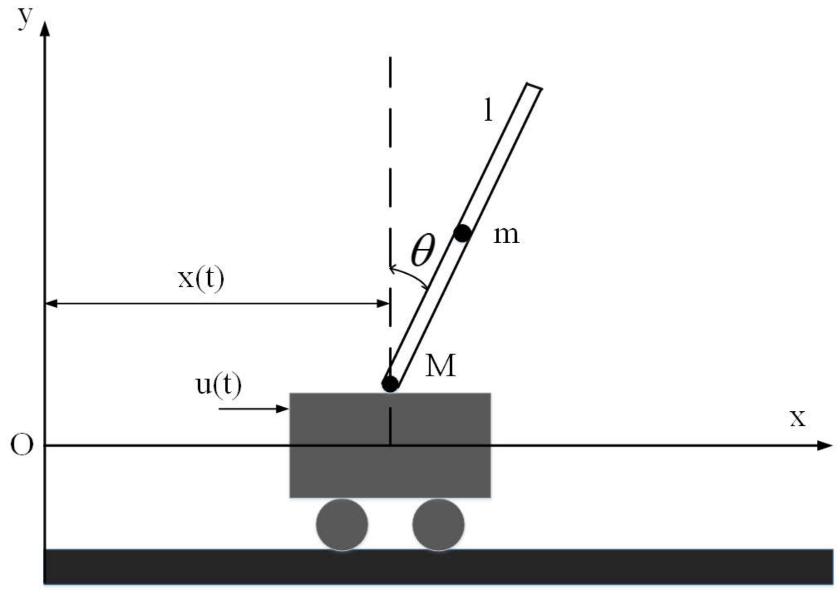

Figure 3.1 : Inverted Pendulum.

3.4.2 Example 3.2

Consider the following system given in [27, 90] as

$$x(k+1) = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.97 \end{pmatrix} x(k) + \begin{pmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{pmatrix} x(k-\tau(k)). \tag{3.34}$$

From Theorem 20 and Remark 2, we find $r = 1.01$ and the exponential stability of the system is still guaranteed for all $0 < \tau(k) \leq 18$, while the obtained feasible regions in [27, 90] are respectively $0 \leq \tau(k) \leq 13$ and $0 \leq \tau(k) \leq 15$.

3.4.3 Example 3.3

Consider the inverted Pendulum system as shown in Figure 3.1. The motion equation of the inverted Pendulum is obtained in the form of [35, 158]

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \frac{3(M+m)g}{l(4M+m)} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -\frac{3}{l(4M+m)} \end{bmatrix} u(t), \tag{3.35}$$

where $u(t)$ is the control force applied to the cart in the x direction, with the purpose of keeping the pendulum balanced upright, x is the displacement of the center of mass of the cart from the origin O; θ is the angle of the pendulum from the top vertical; M and m are the masses of the cart and the pendulum, respectively; l is the half length of the pendulum (i.e., the distance from the pivot to the center of mass of the pendulum). By selecting $M = 8$ kg, $m = 2.0$ kg, $l = 0.5$ m, $g = 9.8$ m/s^2 and sampling time $T_s = 30$ ms, the discrete-time model of the the inverted pendulum is obtained as follows

$$x(k+1) = \begin{pmatrix} 1.0078 & 0.0301 \\ 0.5202 & 1.0078 \end{pmatrix} x(k) + \begin{pmatrix} -0.0001 \\ -0.0053 \end{pmatrix} u(k). \tag{3.36}$$

By using the same state-feedback delayed control law in [35, 158] as

$$u(k) = [102.9100 \quad 80.7916] x(k - \tau(k)),$$

we obtained the closed-loop system as

$$x(k+1) = \begin{pmatrix} 1.0078 & 0.0301 \\ 0.5202 & 1.0078 \end{pmatrix} x(k) + \begin{pmatrix} -0.0111 & -0.0035 \\ -0.5866 & -0.1839 \end{pmatrix} x(k - \tau(k)). \tag{3.37}$$

Here, the objective is to find the maximum allowable upper bound τ_M such that the exponential stability of the closed-loop system (3.37) is still guaranteed. From Theorem 20 and Remark 2, by choosing $r = 1.01$, we obtain the maximum allowable upper bound $\tau_M = 7$ while the stability conditions obtained in [35, 158] give $\tau_M = 6$ and $\tau_M = 5$ respectively. The following simulation has been carried out with the initial condition $x(0) = [0.2 \quad 0.5]^T$ and time-varying delay $1 \leq \tau(k) \leq 7$. As clearly shown in Figure 3.2, the states $x_1(k)$ and $x_2(k)$ of system (3.37) are exponentially converge to zero.

3.5 Conclusion

This chapter has addressed the problem of exponential stability for a class of discrete-time systems with interval time-varying delay. A new set of Lyapunov-Krasvoskii functionals containing an augmented vector and some triple summations

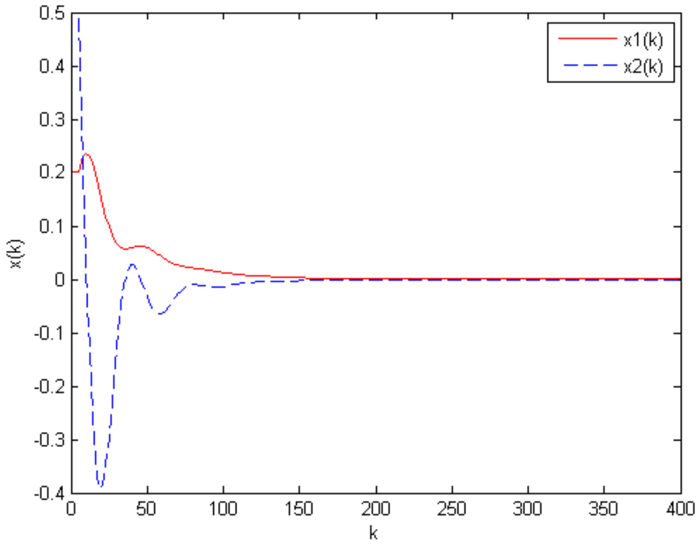

Figure 3.2 : The state trajectories of $x_1(k)$ and $x_2(k)$.

has been proposed. By combining the reciprocally convex approach with the delay-decomposition technique, improved exponential stability conditions are derived in terms of LMIs. In comparison with existing results in the literature, the obtained conditions are less conservative, judging by a larger maximum allowable bound. The effectiveness of the proposed approach is illustrated through numerical examples.

Chapter 4

Reachable set bounding

4.1 Introduction

Reachable set bounding was first investigated in the late 1960s in the field of state estimation and it has subsequently received considerable attention due to its extensive applications in peak-to-peak gain minimization [1], control systems with actuator saturation [129, 57], parameter estimation [22] and other areas. A reachable set is defined as a set of all the states that can be reached from the origin, in a finite time, by inputs with bounded peak value and subject to uncertainties.

Bounding reachable sets is of practical importance in the design of a suitable controller for these systems. Indeed, minimization of the reachable set bound can generally result in a controller with a larger gain to achieve better performance for the uncertain dynamical system under control [15]. When designing robust controllers for systems with time delays, which are known to cause system instability or performance degradation [108], information of the reachable set bound is of particular interest [70].

In [30], by applying the Lyapunov-Razumikhin approach, a delay-dependent condition for the existence of an ellipsoidal bound of reachable sets of a linear system with time-varying delay and bounded peak input, was derived in terms of LMIs. To solve this condition, five positive scalars have to be treated as tuning parameters to find the possible smallest ellipsoid. Based on the modified LKF, an improved condition for the reachable set bounding was proposed in [70] with one nonconvex scalar. However, this condition also involves seven matrices and one scalar to be determined. Therefore, if the value of the derivative of time delay is large, this method will yield a larger ellipsoid bounding the reachable set than that in [30]. On the other hand, the time-varying delay was assumed to be differentiable and the value of the derivative of time delay is less than one. Consequently, the application of the proposed method is limited. Later, a maximal LKF approach was used to derive conditions for reachable set bounding of linear systems with polytopic uncertainties and non-differential time-varying delays [160]. A further result for uncertain polytopic systems with interval time-varying delay was reported in [94]. For linear neutral systems, a recent result was reported in [123] to determine the ellipsoidal bound. By using convex-hull properties and the Lyapunov method, sufficient conditions on reachable set bounding for uncertain dynamic systems with time-varying delays and bounded peak disturbances were established in terms of LMIs [75].

It is worth pointing out that all the aforementioned works have been devoted to continuous-time systems and their discrete-time counterparts seem to receive less attention. In addition, the problem of finding the smallest possible ellipsoidal bound of reachable sets is formulated into that of minimizing the volume of an ellipsoid. In the control design of practical systems, it is interesting to estimate the projection distances of the ellipsoid on all axes, since they reflect the physical magnitude of each coordinate of the system state trajectories.

With the rapid development of digital technology, computer-based controllers have been used intensively in practical control systems. As a result, the direct use of discrete-time models are more suitable for the control design. These facts motivated the present study.

This chapter addresses the problem of reachable set bounding for linear discrete-time systems that are subject to state delay and bounded disturbances. Based on the Lyapunov method, combined with the delay decomposition approach, sufficient conditions for the existence of ellipsoid-based bounds of reachable sets of a linear uncertain discrete system are derived in terms of matrix inequalities. Here, a new idea is proposed to minimize the projection distances of the ellipsoids on each axis with different exponential convergence rates, instead of minimization of their radius with a single exponential rate. A smallest possible bound can thus be obtained from the intersection of these ellipsoids. A numerical example is given to illustrate the effectiveness of the proposed approach.

4.2 Problem statement and preliminaries

Consider the following linear discrete-time system:

$$\begin{aligned} x(k+1) &= A_0 x(k) + A_d x(k-\tau(k)) + D\omega(k), \quad k \in \mathbb{Z}^+, \\ x(k) &\equiv 0, \quad k \in \mathbb{Z}[-\tau_M, 0], \end{aligned} \tag{4.1}$$

where $x(k) \in \mathbb{R}^n$ is the system state; A_0, A_d, D are constant matrices with appropriate dimensions; $\tau(k)$ is the time-varying delay in the state; $\omega(k)$ is the disturbance. The following assumptions are made. The delay $\tau(k)$ is interval time-varying delay in the whole process and satisfies

$$0 \leq \tau_m \leq \tau(k) \leq \tau_M, \tag{4.2}$$

where τ_m and τ_M are known positive integers representing respectively the minimum and maximum delay bounds. The disturbance $\omega(k) \in \mathbb{R}^p$ is assumed to be bounded:

$$\omega^T(k)\omega(k) \leq \omega_p^2, \quad \forall k \geq 0, \tag{4.3}$$

where ω_p is a positive scalar.

Before presenting the main objective in this chapter, the following definitions are first introduced.

Definition 4 *A reachable set for delay system (4.1) subject to bounded disturbance (4.3) is defined as*

$$\mathcal{R}_x = \{x(k) \in \mathbb{R}^n \Big| x(k), \omega(k) \ \textit{satisfy} \ (4.1) \ \textit{and} \ (4.3), k \geq 0\}. \tag{4.4}$$

Definition 5 *For a positive-definite symmetric matrix* $P > 0$, *we define an ellipsoid* $\varepsilon(P, 1)$ *bounding the reachable set (4.4) as follows*

$$\varepsilon(P, 1) = \{x \in \mathbb{R}^n \Big| x^T P x \leq 1\}, \tag{4.5}$$

whose projection distance on the h-th axis ($h = 1, 2, \ldots, n$*) is determined as*

$$d_h(P) = \sup\{2|x_h| \Big| x = [x_1, x_2 \ldots, x_n]^T \in \varepsilon(P, 1)\}. \tag{4.6}$$

The objective is to derive delay-dependent conditions for the existence of the possible smallest bound of reachable sets of system (4.1) subject to time-varying delays (4.2) and bounded disturbances (4.3), in terms of matrix inequalities.

4.3 Main results

The following lemma is often used to estimate the reachable set bounding of systems with interval time-varying delay and bounded input disturbances.

Lemma 4 *For a given positive scalar* ω_p, *if there exist a scalar* $r > 1$ *and a positive definite function* $V(k)$ *which satisfies*

$$\Delta V(k) + (1 - r^{-1})V(k) - (1 - r^{-1})\omega^T(k)\omega(k) \leq 0, \tag{4.7}$$

then the following inequality holds

$$V(k) \leq \omega_p^2 + V(0)e^{-\gamma k}, \forall k \geq 0, \tag{4.8}$$

where $\gamma = ln(r) > 0$. *Therefore, we obtain the following inequality*

$$\lim_{k \to \infty} supV(k) \leq \omega_p^2. \tag{4.9}$$

Proof. From (4.3) and (4.7), we have

$$\begin{aligned} V(k+1) &\leq r^{-1}V(k) + (1 - r^{-1})\overline{\omega}_p^2 \\ &\leq r^{-2}V(k-1) + (1 - r^{-2})\overline{\omega}_p^2 \\ &\quad \ldots \\ &\leq r^{-(k+1)}V(0) + (1 - r^{-(k+1)})\overline{\omega}_p^2. \end{aligned}$$

Since $r > 1$, thus

$$V(k) \leq \omega_p^2 + V(0)e^{-\gamma k}.$$

Consequently, we have

$$\lim_{k\to\infty} \sup V(k) \leq \omega_p^2. \tag{4.10}$$

This completes the proof. □

We are now ready to state the main results for obtaining a sufficient condition for the existence of a smaller bound for the reachable sets of linear discrete-time systems with interval time-varying delays and bounded disturbances.

The following notations are specifically used in this development. For an axis index $h = 1, 2, \cdots, n$, we denote an integer number $\lambda_h \in (0, \tau_M - \tau_m]$; a positive scalar $r_h > 1$; symmetric positive definite matrices P_h, Q_{1h}, Q_{2h}, Q_{3h}, R_{1h}, R_{2h}, $R_{3h} \in \mathbb{R}^{n\times n}$; $(2n \times 2n)$-matrices

$$X_h = \begin{bmatrix} X_{11h} & X_{12h} \\ \star & X_{22h} \end{bmatrix}, Y_h = \begin{bmatrix} Y_{11h} & Y_{12h} \\ \star & Y_{22h} \end{bmatrix}, Z_h = \begin{bmatrix} Z_{11h} & Z_{12h} \\ \star & Z_{22h} \end{bmatrix},$$

$$U_h = \begin{bmatrix} U_{11h} & U_{12h} \\ \star & U_{22h} \end{bmatrix}, V_h = \begin{bmatrix} V_{11h} & V_{12h} \\ \star & V_{22h} \end{bmatrix}, S_h = \begin{bmatrix} S_{11h} & S_{12h} \\ \star & S_{22h} \end{bmatrix};$$

and also $(n \times 2n)$-matrices

$$W_h^T = [W_{1h}^T \quad W_{2h}^T], K_h^T = [K_{1h}^T \quad K_{2h}^T], L_h^T = [L_{1h}^T \quad L_{2h}^T], M_h^T = [M_{1h}^T \quad M_{2h}^T],$$

$$N_h^T = [N_{1h}^T \quad N_{2h}^T], T_h^T = [T_{1h}^T \quad T_{2h}^T], H_h^T = [H_{1h}^T \quad H_{2h}^T], E_h^T = [E_{1h}^T \quad E_{2h}^T].$$

We also define $\tau_h = \tau_m + \lambda_h$; $\tau_{Mh} = \tau_M - \tau_h; \tau_{hm} = \tau_h - \tau_m$; $G_h = [G_{ij}]$, where

$$G_{ij} = \begin{cases} 1, & \text{if } i = j = h, \\ 0, & \text{otherwise,} \end{cases}$$

and matrices

$$\Omega_h = \begin{bmatrix} \Omega_{11h} & \Omega_{12h} & L_{1h} - M_{1h} & K_{1h} - L_{1h} & -W_{1h} & \Omega_{16h} \\ \star & \Omega_{22h} & L_{2h} - M_{2h} & K_{2h} - L_{2h} & -W_{2h} & \Omega_{26h} \\ \star & \star & -r_h^{-\tau_m} Q_{3h} & 0 & 0 & 0 \\ \star & \star & \star & -r_h^{-\tau_h} Q_{2h} & 0 & 0 \\ \star & \star & \star & \star & -r_h^{-\tau_M} Q_{1h} & 0 \\ \star & \star & \star & \star & \star & \Omega_{66h} \end{bmatrix}, \tag{4.11}$$

$$\Sigma_h = \begin{bmatrix} \Sigma_{11h} & \Sigma_{12h} & N_{1h} - T_{1h} & H_{1h} - E_{1h} & -H_{1h} & \Sigma_{16h} \\ \star & \Sigma_{22h} & N_{2h} - T_{2h} & H_{2h} - E_{2h} & -H_{2h} & \Sigma_{26h} \\ \star & \star & -r_h^{-\tau_m} Q_{3h} & 0 & 0 & 0 \\ \star & \star & \star & -r_h^{-\tau_h} Q_{2h} & 0 & 0 \\ \star & \star & \star & \star & -r_h^{-\tau_M} Q_{1h} & 0 \\ \star & \star & \star & \star & \star & \Sigma_{66h} \end{bmatrix}, \tag{4.12}$$

where

$$A = A_0 - I_n, O_{1h} = \tau_{Mh} R_{1h} + \tau_{hm} R_{2h} + \tau_m R_{3h},$$

$$\begin{aligned}
\Omega_{11h} &= (1 - r_h^{-1}) P_h + A^T O_{1h} A + Q_{1h} + Q_{2h} + Q_{3h} + M_{1h} + M_{1h}^T + \tau_{Mh} X_{11h} \\
&\quad + \tau_{hm} Y_{11h} + \tau_m Z_{11h} + P_h A + A^T P_h^T + A^T P_h A, \\
\Omega_{12h} &= P_h A_d + A^T O_{1h} A_d + A^T P_h A_d + W_{1h} - K_{1h} + M_{2h}^T \\
&\quad + \tau_{Mh} X_{12h} + \tau_{hm} Y_{12h} + \tau_m Z_{12h}, \\
\Omega_{16h} &= P_h D + A^T O_{1h} D + A^T P_h D, \\
\Omega_{22h} &= A_d^T P_h A_d + A_d^T O_{1h} A_d + W_{2h} + W_{2h}^T - K_{2h} - K_{2h}^T \\
&\quad + \tau_{Mh} X_{22h} + \tau_{hm} Y_{22h} + \tau_m Z_{22h}, \\
\Omega_{26h} &= A_d^T O_{1h} D + A_d^T P_h D, \Omega_{66h} = D^T P_h D + D^T O_{1h} D - (1 - r_h^{-1}) I_p,
\end{aligned}$$

$$\Sigma_{11h} = (1 - r_h^{-1})P_h + A^T O_{1h} A + Q_{1h} + Q_{2h} + Q_{3h} + T_{1h} + T_{1h}^T + \tau_{Mh} U_{11h}$$
$$+ \tau_{hm} V_{11h} + \tau_m S_{11h} + P_h A + A^T P_h^T + A^T P_h A,$$
$$\Sigma_{12h} = P_h A_d + A^T O_{1h} A_d + A^T P_h A_d + E_{1h} - N_{1h} + T_{2h}^T$$
$$+ \tau_{Mh} U_{12h} + \tau_{hm} V_{12h} + \tau_m S_{12h},$$
$$\Sigma_{16h} = P_h D + A^T O_{1h} D + A^T P_h D,$$
$$\Sigma_{22h} = A_d^T P_h A_d + A_d^T O_{1h} A_d + E_{2h} + E_{2h}^T - N_{2h} - N_{2h}^T$$
$$+ \tau_{Mh} U_{22h} + \tau_{hm} V_{22h} + \tau_m S_{22h},$$
$$\Sigma_{26h} = A_d^T O_{1h} D + A_d^T P_h D, \Sigma_{66h} = D^T P_h D + D^T O_{1h} D - (1 - r_h^{-1}) I_p,$$
$$\Psi_{1h} = \begin{bmatrix} X_h & W_h \\ \star & r_h^{-\tau_M} R_{1h} \end{bmatrix}, \Psi_{2h} = \begin{bmatrix} X_h & K_h \\ \star & r_h^{-\tau_M} R_{1h} \end{bmatrix}, \Psi_{3h} = \begin{bmatrix} Y_h & L_h \\ \star & r_h^{-\tau_h} R_{2h} \end{bmatrix},$$
$$\Psi_{4h} = \begin{bmatrix} Z_h & M_h \\ \star & r_h^{-\tau_m} R_{3h} \end{bmatrix}, \Psi_{5h} = \begin{bmatrix} U_h & H_h \\ \star & r_h^{-\tau_M} R_{1h} \end{bmatrix}, \Psi_{6h} = \begin{bmatrix} V_h & E_h \\ \star & r_h^{-\tau_h} R_{2h} \end{bmatrix},$$
$$\Psi_{7h} = \begin{bmatrix} V_h & N_h \\ \star & r_h^{-\tau_h} R_{2h} \end{bmatrix}, \Psi_{8h} = \begin{bmatrix} S_h & T_h \\ \star & r_h^{-\tau_m} R_{3h} \end{bmatrix}.$$

Theorem 21 *For each $h \in \{1, 2, \cdots, n\}$, with given positive integers τ_m, τ_M and a positive scalar ω_p, if there exist an integer number $\lambda_h \in (0, \tau_M - \tau_m]$, scalars $r_h > 1, \delta_h > 0$, symmetric positive definite matrices P_h, Q_{1h}, Q_{2h}, Q_{3h}, R_{1h}, R_{2h}, R_{3h}, X_h, Y_h, Z_h, U_h, V_h, S_h, and matrices W_h, K_h, L_h, M_h, H_h, E_h, N_h, T_h, such that the following conditions hold:*

$$P_h > \delta_h G_h, \tag{4.13}$$

$$\Omega_h < 0, \quad \Sigma_h < 0, \tag{4.14}$$

$$\Psi_{jh} \geq 0 \quad \forall j \in \{1, \cdots, 8\}; \tag{4.15}$$

then the reachable sets of system (4.1) are bounded by

$$\bigcap_{h=1}^{n} \varepsilon(P_h, 1). \tag{4.16}$$

Moreover, for each $h = 1, 2, \cdots, n$, the projection distance of the ellipsoid $\varepsilon(P_h, 1)$ on the h-th axis is

$$d_h(P_h) = \frac{2\omega_p}{\sqrt{\lambda_{min}(P_h)}}. \tag{4.17}$$

Proof. From (4.1), by defining $y(k) = x(k+1) - x(k)$, we have

$$y(k) = (A_0 - I_n)x(k) + A_d x(k - \tau(k)) + D\omega(k).$$

Apart from the idea of getting more information on the system when constructing the LKFs, the conservatism of the delay-dependent conditions may be reduced. The given interval time-varying delay τ_m τ_M is divided into two nonuniform subintervals. For an axis index $h \in \{1, 2, \cdots, n\}$, note that, for any $k \in \mathbb{N}$, we have either $\tau(k) \in [\tau_m, \tau_h]$ or $\tau(k) \in (\tau_h, \tau_M]$. Let us define two sets

$$\Pi_{1h} = \{k \in \mathbb{N} | \tau(k) \in (\tau_h, \tau_M]\},$$

$$\Pi_{2h} = \{k \in \mathbb{N} | \tau(k) \in [\tau_m, \tau_h]\}.$$

Consider the following Lyapunov-Krasovskii functional

$$V_h(k) = V_{1h}(k) + V_{2h}(k) + V_{3h}(k), \tag{4.18}$$

where

$$\begin{aligned}
V_{1h}(k) &= x^T(k) P_h x(k),\\
V_{2h}(k) &= \sum_{s=k-\tau_M}^{k-1} r_h^{s-k} x^T(s) Q_{1h} x(s) + \sum_{s=k-\tau_h}^{k-1} r_h^{s-k} x^T(s) Q_{2h} x(s)\\
&\quad + \sum_{s=k-\tau_m}^{k-1} r_h^{s-k} x^T(s) Q_{3h} x(s),\\
V_{3h}(k) &= \sum_{s=-\tau_M}^{-\tau_h-1} \sum_{v=k+s}^{k-1} r_h^{s-k} y^T(v) R_{1h} y(v) + \sum_{s=-\tau_h}^{-\tau_m-1} \sum_{v=k+s}^{k-1} r_h^{s-k} y^T(v) R_{2h} y(v)\\
&\quad + \sum_{s=-\tau_m}^{-1} \sum_{v=k+s}^{k-1} r_h^{s-k} y^T(v) R_{3h} y(v).
\end{aligned}$$

By taking the forward differences of $V_{1h}(k)$ and $V_{2h}(k)$ along the solution of system (4.1), we obtain

$$\begin{aligned}
\Delta V_{1h}(k) &= x^T(k+1)P_h x(k+1) - r_h^{-1}x^T(k)P_h x(k) - (1-r_h^{-1})V_{1h} \\
&= [y(k)+x(k)]^T P_h[y(k)+x(k)] - r_h^{-1}x^T(k)Px(k) - (1-r_h^{-1})V_{1h} \\
&= x^T(k)(1-r_h^{-1})P_h x(k) + 2x^T(k)P_h y(k) + y^T(k)P_h y(k) - (1-r_h^{-1})V_{1h}(k),
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
\Delta V_{2h}(k) = &\sum_{s=k+1-\tau_M}^{k} r^{s-k-1}x^T(s)Q_{1h}x(s) - \sum_{s=k-\tau_M}^{k-1} r^{s-k-1}x^T(s)Q_{1h}x(s) \\
&+ \sum_{s=k+1-\tau_h}^{k} r^{s-k-1}x^T(s)Q_{2h}x(s) - \sum_{s=k-\tau_h}^{k-1} r^{s-k-1}x^T(s)Q_{2h}x(s) \\
&+ \sum_{s=k+1-\tau_m}^{k} r^{s-k-1}x^T(s)Q_{3h}x(s) - \sum_{s=k-\tau_m}^{k-1} r^{s-k-1}x^T(s)Q_{3h}x(s) \\
&- (1-r_h^{-1})V_{2h} \\
= &\, x^T(k)(Q_{1h}+Q_{2h}+Q_{3h})x(k) - r_h^{-\tau_M}x^T(k-\tau_M)Q_{1h}x(k-\tau_M) \\
&- r_h^{-\tau_h}x^T(k-\tau_h)Q_{2h}x(k-\tau_h) - r_h^{-\tau_m}x^T(k-\tau_m)Q_{3h}x(k-\tau_m) \\
&- (1-r_h^{-1})V_{2h}(k),
\end{aligned} \tag{4.20}$$

Next, the forward difference of $V_{3h}(k)$ along the solution of system (4.1) is obtained as follows

$$\begin{aligned}
\Delta V_{3h}(k) = &\sum_{s=-\tau_M}^{-\tau_h-1}\Big[\sum_{v=k+s+1}^{k} r_h^{s-k-1}y^T(v)R_{1h}y(v) - \sum_{v=k+s}^{k-1} r_h^{s-k-1}y^T(v)R_{1h}y(v)\Big] \\
&+ \sum_{s=-\tau_h}^{-\tau_m-1}\Big[\sum_{v=k+s+1}^{k} r_h^{s-k-1}y^T(v)R_{2h}y(v) - \sum_{v=k+s}^{k-1} r_h^{s-k-1}y^T(v)R_{2h}y(v)\Big] \\
&+ \sum_{s=-\tau_m}^{-1}\Big[\sum_{v=k+s+1}^{k} r_h^{s-k-1}y^T(v)R_{3h}y(v) - \sum_{v=k+s}^{k-1} r_h^{s-k-1}y^T(v)R_{3h}y(v)\Big] \\
&- (1-r_h^{-1})V_{3h}(k)
\end{aligned}$$

Finally, we have

$$\begin{aligned}\varDelta V_{3h}(k) = y^T(k)\Big[(\tau_M-\tau_h)R_{1h}+(\tau_h-\tau_m)R_{2h}+\tau_m R_{3h}\Big]y(k)\\ -\sum_{s=k-\tau_M}^{k-\tau_h-1} r_h^{s-k}y^T(s)R_{1h}y(s)-\sum_{s=k-\tau_h}^{k-\tau_m-1} r_h^{s-k}y^T(s)R_{2h}y(s)\\ -\sum_{s=k-\tau_m}^{k-1} r_h^{s-k}y^T(s)R_{3h}y(s)-(1-r_h^{-1})V_{3h}(k).\end{aligned} \tag{4.21}$$

By noting that $r_h^s > 1, \forall s > 0$, the following estimation is obtained as

$$\begin{aligned}\varDelta V_h(k) \leq & x^T(k)(1-r_h^{-1})P_h x(k)+2x^T(k)P_h y(k)+y^T(k)P_h y(k)\\ &+x^T(k)(Q_{1h}+Q_{2h}+Q_{3h})x(k)-r_h^{-\tau_M}x^T(k-\tau_M)Q_{1h}x(k-\tau_M)\\ &-r_h^{-\tau_h}x^T(k-\tau_h)Q_{2h}x(k-\tau_h)-r_h^{-\tau_m}x^T(k-\tau_m)Q_{3h}x(k-\tau_m)\\ &+y^T(k)\Big[(\tau_M-\tau_h)R_{1h}+(\tau_h-\tau_m)R_{2h}+\tau_m R_{3h}\Big]y(k)\\ &-\sum_{s=k-\tau_M}^{k-\tau_h-1} r_h^{-\tau_M}y^T(s)R_{1h}y(s)-\sum_{s=k-\tau_h}^{k-\tau_m-1} r_h^{-\tau}y^T(s)R_{2h}y(s)\\ &-\sum_{s=k-\tau_m}^{k-1} r_h^{-\tau_m}y^T(s)R_{3h}y(s)-(1-r_h^{-1})V_h(k).\end{aligned} \tag{4.22}$$

We distinguish two sub-intervals of the time delay in the following cases.

Case I: For $k \in \varPi_{1h}$, i.e. the time delay $\tau(k) \in (\tau_h, \tau_M]$, we have

$$-\sum_{s=k-\tau_M}^{k-\tau_h-1} y^T(s)R_{1h}y(s) = -\sum_{s=k-\tau_M}^{k-\tau(k)-1} y^T(s)R_{1h}y(s)-\sum_{s=k-\tau(k)}^{k-\tau_h-1} y^T(s)R_{1h}y(s). \tag{4.23}$$

The following notations $\xi^T(k) = [x^T(k) \quad x^T(k-\tau(k))]$,$\eta^T(k) = [x^T(k) \quad x^T(k-\tau(k)) \quad x^T(k-\tau_m) \quad x^T(k-\tau_h) \quad x^T(k-\tau_M) \quad \omega^T(k)]$, and $\zeta^T(k,s) = [\xi^T(k) \quad y^T(s)]$ are further introduced. For any matrices W_h, K_h, L_h and M_h, the following equations always hold:

$$2\xi^T(k)W_h\Big[x(k-\tau(k))-x(k-\tau_M)-\sum_{s=k-\tau_M}^{k-\tau(k)-1} y(s)\Big]=0, \tag{4.24}$$

$$2\xi^T(k)K_h\Big[x(k-\tau_h)-x(k-\tau(k))-\sum_{s=k-\tau(k)}^{k-\tau_h-1} y(s)\Big]=0, \tag{4.25}$$

$$2\xi^T(k)L_h\Big[x(k-\tau_m)-x(k-\tau_h)-\sum_{s=k-\tau_h}^{k-\tau_m-1}y(s)\Big]=0, \tag{4.26}$$

$$2\xi^T(k)M_h\Big[x(k)-x(k-\tau_m)-\sum_{s=k-\tau_m}^{k-1}y(s)\Big]=0. \tag{4.27}$$

We also have

$$(\tau_M-\tau_h)\xi^T(k)X_h\xi(k)-\sum_{s=k-\tau_M}^{k-\tau(k)-1}\xi^T(k)X_h\xi(k)-\sum_{s=k-\tau(k)}^{k-\tau_h-1}\xi^T(k)X_h\xi(k)=0, \tag{4.28}$$

$$(\tau_h-\tau_m)\xi^T(k)Y_h\xi(k)-\sum_{s=k-\tau_h}^{k-\tau_m-1}\xi^T(k)Y_h\xi(k)=0, \tag{4.29}$$

$$\tau_m\xi_h^T(k)Z_h\xi(k)-\sum_{s=k-\tau_m}^{k-1}\xi_h^T(k)Z_h\xi(k)=0. \tag{4.30}$$

Thus, from (4.18)-(4.30), we obtain

$$\begin{aligned}\Delta V_h(k)+(1-r_h^{-1})V_h(k)-(1-r_h^{-1})\omega^T(k)\omega(k) &\le \eta^T(k)\Omega_h\eta(k)\\ &-\sum_{s=k-\tau_M}^{k-\tau(k)-1}\zeta^T(k,s)\Psi_{1h}\zeta(k,s)ds\\ &-\sum_{s=k-\tau(k)}^{k-\tau_h-1}\zeta^T(k,s)\Psi_{2h}\zeta(k,s)ds\\ &-\sum_{s=k-\tau_h}^{k-\tau_m-1}\zeta^T(k,s)\Psi_{3h}\zeta(k,s)ds\\ &-\sum_{s=k-\tau_m}^{k-1}\zeta^T(k,s)\Psi_{4h}\zeta(k,s)ds\\ &<0.\end{aligned} \tag{4.31}$$

Case II: For $k\in\Pi_{2h}$, i.e. the time delay $\tau(k)\in[\tau_m,\tau_h]$, we have

$$-\sum_{s=k-\tau_h}^{k-\tau_m-1}y^T(s)R_{2h}y(s)=-\sum_{s=k-\tau_h}^{k-\tau(k)-1}y^T(s)R_{2h}y(s)-\sum_{s=k-\tau(k)}^{k-\tau_m-1}y^T(s)R_{2h}y(s).$$

Similarly, we also obtain

$$\begin{aligned}\Delta V_h(k) + (1 - r_h^{-1})V_h(k) - (1 - r_h^{-1})\omega^T(k)\omega(k) &\leq \eta^T(k)\Sigma_h\eta(k) \\ &- \sum_{s=k-\tau_M}^{k-\tau(k)-1} \zeta^T(k,s)\Psi_{5h}\zeta(k,s)ds \\ &- \sum_{s=k-\tau(k)}^{k-\tau_h-1} \zeta^T(k,s)\Psi_{6h}\zeta(k,s)ds \\ &- \sum_{s=k-\tau_h}^{k-\tau_m-1} \zeta^T(k,s)\Psi_{7h}\zeta(k,s)ds \\ &- \sum_{s=k-\tau_m}^{k-1} \zeta^T(k,s)\Psi_{8h}\zeta(k,s)ds \\ &< 0.\end{aligned} \tag{4.32}$$

Therefore, it follows from conditions (4.14) and (4.15) of Theorem 21 that

$$\Delta V_h(k) + (1 - r_h^{-1})V_h(k) - (1 - r_h^{-1})\omega^T(k)\omega(k) \leq 0,$$

which yields

$$V(k) \leq \omega_p^2 + V(0)e^{-\gamma k}, \forall k \geq 0.$$

By using Lemma 4, we have

$$\lim_{k\to\infty} \sup V(k) \leq \omega_p^2.$$

Therefore, the following inequality can be obtained as

$$x^T(k)P_h x(k) < \omega_p^2, \forall k \geq 0.$$

By using the spectral properties of symmetric positive-definite matrix, we have

$$\lambda_{min}(P_h)\|x(k)\|^2 \leq \omega_p^2. \tag{4.33}$$

Consequently, we obtain

$$\|x(k)\| \leq \frac{\omega_p}{\sqrt{\lambda_{min}(P_h)}}.$$

Thus, the reachable sets of system (4.1) for $k \geq 0$ is bounded by ellipsoid $\varepsilon(P_h, 1)$, according to definition (4.5). Moreover, a smaller bound of the reachable sets of the system (4.1) can be obtained from the intersection of the ellipsoids given in (4.13). From condition (4.6), the projection distances can then be obtained as (4.16). The proof is completed. □

Remark 4 It is worth mentioning that Lemma 4 is useful to deal with the problem of reachable set bounding for linear discrete-time systems with time-varying delay and bounded disturbances. However, it can also be used for the case of without disturbance (i.e., $\omega(k) = 0$). In that case, the considered problem reduces to the γ-exponential stability analysis for linear discrete-time systems with interval time-varying delay.

Remark 5 It should be noted that conditions established in Theorem 21 are not LMIs to be computationally solved with the use of Matlab's LMI toolbox. One may find the solution to this problem by using the method, mentioned in Remark 2

Remark 6 In the case of a single convergence rate $r_h = r$, $\delta_h = \delta$, $G_h = I_n$, $P_h = P$, and $Q_{jh} = Qj, R_{jh} = R_j$ for $j = 1, 2, 3; h = 1, 2, \cdots, n$, this approach is similar to the ones proposed in [30, 70, 160, 123, 94, 75]. The reachable set (4.4) of system (4.1) is then bounded by ellipsoid $\varepsilon(P, 1)$, whose volume, as suggested therein, can be minimised by solving the following optimisation problem for a scalar $\delta > 0$:

$$\min(\frac{1}{\delta}) \text{ s.t } \begin{cases} (a) \quad P \geq \delta I \\ (b) \quad (4.14) \text{ and } (4.15). \end{cases}$$

Remark 7 It can be seen from Theorem 21 that the constraints $P_h \geq \delta_h G_h$ are less conservative than $P \geq \delta I$. Thus, the projection distances of the ellipsoid $\varepsilon(P_h, 1)$, obtained with respect to the h-th axis, is always smaller than the diameter of ellipsoid $\varepsilon(P, 1)$. Furthermore, by taking the intersection of the ellipsoids, we can even further improve the bound for the reachable set as the intersection of the ellipsoids $\varepsilon = \left(\bigcap_{h=1}^{n} \varepsilon(P_h, 1)\right) \bigcap \varepsilon(P, 1)$.

Remark 8 It is worth pointing out that the conservatism of the conditions obtained in Theorem 21 can be further reduced by using the set of the LKFs and the reciprocally convex approach proposed in Chapter 3.

4.4 Numerical examples

In this section, numerical examples are provided to illustrate the effectiveness of our proposed approach.

Consider system (4.1) with the following matrices:

$$A_0 = \begin{bmatrix} 0.8 & -0.01 \\ -0.5 & 0.09 \end{bmatrix}, A_d = \begin{bmatrix} -0.02 & 0 \\ -0.1 & -0.01 \end{bmatrix}, D = \begin{bmatrix} 0.01 \\ 0.15 \end{bmatrix}.$$

The time-varying delay is assumed to belong to an interval $\tau(k) \in [0, 15]$ and the external disturbance is bounded as $\omega(k) = sin(7k)$.
From Theorem 21, and Remark 5, we obtain the following matrices

$$P = \begin{bmatrix} 741.3221 & -34.4791 \\ -34.4791 & 12.7422 \end{bmatrix}, P_1 = 10^3 \times \begin{bmatrix} 1.2159 & -0.0407 \\ -0.0407 & 0.0077 \end{bmatrix},$$
$$P_2 = 10^3 \times \begin{bmatrix} 1.0440 & -0.0608 \\ -0.0608 & 0.0202 \end{bmatrix}$$

and convergence rates $r = 1.118, r_1 = 1.150$, and $r_2 = 1.155$.

The ellipsoidal bound by $\varepsilon(P, 1)$ as obtained from [30, 70, 160, 123, 94, 75], $\varepsilon(P_1, 1)$ and $\varepsilon(P_2, 1)$ for system (4.1) proposed in this paper are respectively depicted in Fig. 4.1 and Fig. 4.2. By using Theorem 21 and Remark 6, a smaller bound is obtained from the intersection of these ellipsoids as $\varepsilon = \left(\bigcap_{h=1}^{n} \varepsilon(P_h, 1)\right) \bigcap \varepsilon(P, 1)$ for system (4.1) as depicted in Fig. 4.3, whereby it can be seen that the system trajectory

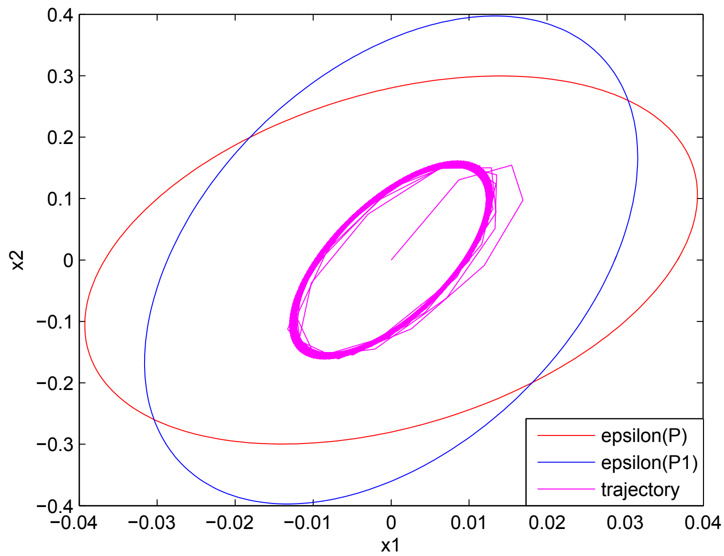

Figure 4.1 : Ellipsoidal bounds of the reachable set by $\varepsilon(P, 1)$ and $\varepsilon(P_1, 1)$

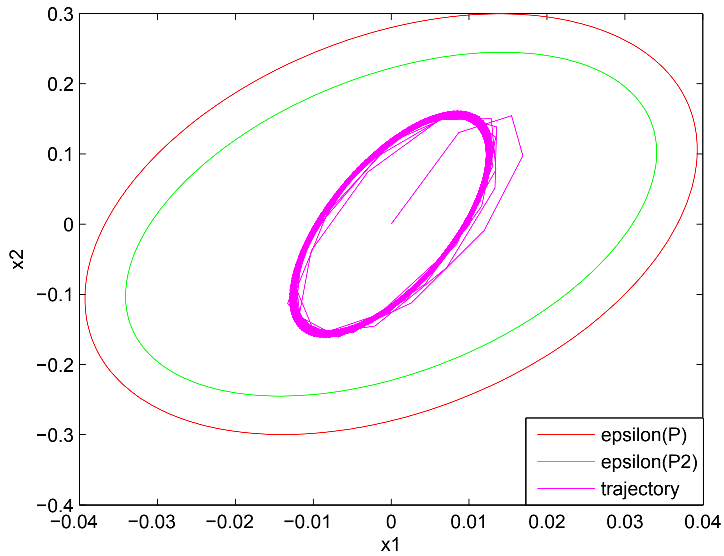

Figure 4.2 : Ellipsoidal bounds of the reachable set by $\varepsilon(P, 1)$ and $\varepsilon(P_2, 1)$

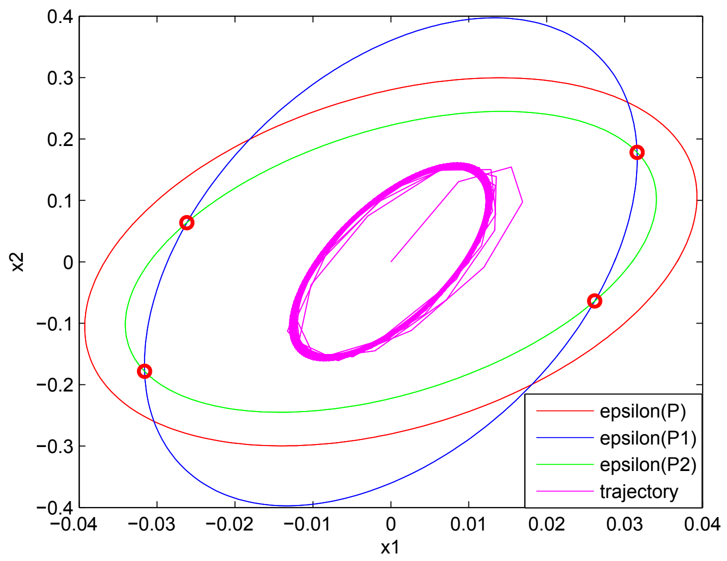

Figure 4.3 : An improved bound of the reachable set by $\varepsilon = \left(\bigcap_{h=1}^{n} \varepsilon(P_h, 1)\right) \bigcap \varepsilon(P, 1)$.

is prescribed in a smaller bound. Moreover, in order to design a robust controller, the information that the coordinates x_1 and x_2 of the trajectory are respectively bounded by $d_1 = 0.06$ and $d_2 = 0.48$, obtained from the proposed approach, may be more favorable to the control design than by $d = 0.6$ as obtained from $\varepsilon(P, 1)$.

4.5 Conclusion

This chapter has addressed the problem of reachable set bounding for a class of linear discrete-time systems with time-varying delay in the state and bounded disturbances. Based on the L-K approach, sufficient conditions for the existence of ellipsoidal bounds of the reachable sets are derived in terms of matrix inequalities. By minimising the projection distances of the ellipsoids on each coordinate axis with different convergence rates, a much smaller bound of the reachable sets is obtained from the intersection of these ellipsoids. a numerical example is given to illustrate the effectiveness of the proposed approach.

Chapter 5

Quasi-sliding mode control

5.1 Introduction

Time delays and external disturbances are unavoidable in many practical control applications, e.g., in robotics, manufacturing, and process control and it is often a source of instability or oscillations, see, e.g., [119, 69] and the references therein. Therefore, the design of control and observation schemes has been an interesting problem for dynamical systems to compensate for time delays [97] and to estimate external disturbances [136]. To enhance robustness, the sliding mode control methodology has been recognised as an effective strategy for uncertain systems, see, e.g., [139, 67, 156] and references therein. In this context, there have been considerable efforts devoted to the problem of sliding mode control design for uncertain systems with matched disturbances, see, e.g., [42, 148, 50] and references therein. However, when the matching conditions for disturbances are not satisfied, their effects can be only partially rejected in the sliding mode. Therefore, the control design for this case remains a challenging problem.

For a class of linear systems with time-varying delay and unmatched disturbances, a sliding-mode control strategy was developed in [49] and sufficient conditions were derived in terms of linear matrix inequalities (LMIs) to guarantee that the state trajectories of the system converge towards a ball with a pre-specified convergence rate. By using the invariant ellipsoid method, another sliding mode control design algorithm was proposed for a class of linear quasi-Lipschitz disturbed system to minimise the effects of unmatched disturbances to system motions in the sliding mode [116]. Later, by combining the predictor-based sliding mode control with the invariant ellipsoid method, an improved result was reported to take into account also time delay in the control input [115]. Recently, a disturbance observer-based sliding mode control was presented in [155], where mismatched uncertainties were considered.

Owing to advantages of digital technology, there has been increasing attention paid to the discrete-time sliding mode control. In [36], the quasi-sliding mode control and the associated quasi-sliding mode band (QSMB) and reaching law were introduced for single input discrete systems. Another quasi-sliding mode control design algorithm was reported in [4], adopting a different reaching law. A discrete-time sliding mode controller was synthesised to drive the system state trajectories into a small bounded region for a class of linear multi-input systems with matching perturbations [12]. A robust quasi-sliding mode control strategy was proposed in [61] for uncertain systems using multirate output feedback. In [146], a predictor-based sliding mode control law was used to deal with discrete-time uncertain systems subject also to an input delay. In [154], a sufficient condition for the existence of stable sliding surfaces, depending on the lower and upper delay bounds, was derived in terms of LMIs. Recently, some improved results for this problem have been reported in [147, 100, 145, 59, 117, 132].

In the framework of discrete-time sliding mode control, the problem of compensation for time-varying delay and rejection of the unmatched disturbance effects has not received much attention and so it will be addressed in this chapter. Here, by using the L-K method, in combination with the reciprocally convex approach, sufficient conditions for the existence of a stable sliding surface are derived in terms of LMIs. Moreover, these conditions guarantee that the effects of interval time-varying delay and unmatched disturbances are mitigated and the induced sliding dynamics are exponentially convergent within a ball with a radius to be minimised. A robust discrete-time quasi-sliding mode is then synthesised to drive the state trajectories of the closed-loop system towards the prescribed sliding surface and remain in this ball after a finite time.

The chapter is organized as follows. After the introduction, Section 2 presents the system definition and some preliminaries. The main results are included in Section 3. The effectiveness of the proposed control approach is illustrated in Section 4 through numerical examples. Finally, Section 5 concludes the chapter.

5.2 Problem Statement and Preliminaries

Consider a class of linear discrete-time uncertain systems described in the following form

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-\tau(k)) + Bu(k) + D\omega(k), \quad k \in \mathbb{Z}^+, \\ x(k) &\equiv \phi(k), \quad k \in \mathbb{Z}[-\tau_M, 0], \end{aligned} \tag{5.1}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are, respectively, the system state vector and the control input. Matrices A, A_d, B and D are constant, with appropriate dimensions, where $rank(B) = m \leq n$. The initial function of system (5.1), $\phi(k), k \in \mathbb{Z}[-\tau_M, 0]$, has its norm given by

$$\|\phi\|_\tau = \max\{\|\phi(k)\| : k \in \mathbb{Z}[-\tau_M, 0]\}. \tag{5.2}$$

The delay $\tau(k)$ is time-varying delay in the whole process and satisfying

$$0 \leq \tau_m \leq \tau(k) \leq \tau_M, \tag{5.3}$$

where τ_m and τ_M satisfying $\tau_m < \tau_M$, are known positive integers representing, respectively, the minimum and maximum delay bounds. The unmatched external disturbance $\omega(k) \in \mathbb{R}^p$ is assumed to be bounded, i.e., for any $k \geq 0$,

$$\omega^T(k)\omega(k) \leq \omega_p^2, \quad \forall k \geq 0, \tag{5.4}$$

where ω_p is a positive scalar.

It can be shown that if B is a full-column rank matrix, i.e., $rank(B) = m$, there exists a nonsingular transformation matrix T which can always be chosen such that $TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$, where $B_2 \in \mathbb{R}^{m\times m}$ is a non-singular matrix [71]. With $z(k) = Tx(k)$, system (5.1) can be transformed into the following regular form:

$$z(k+1) = \overline{A}z(k) + \overline{A}_d z(k-\tau(k)) + \overline{B}u(k) + \overline{D}\omega(k)), \tag{5.5}$$

where

$$\begin{aligned} \overline{A} =& TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \overline{A}_d = TA_dT^{-1} \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}, \\ \overline{B} =& TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \overline{D} = TD = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}. \end{aligned}$$

Now, by partitioning $z(k) = [z_1(k) \ z_2(k)]^T$, where $z_1(k) \in \mathbb{R}^{n-m}$ and $z_2(k) \in \mathbb{R}^m$, the dynamics of system (5.1) can be described by

$$\begin{aligned} z_1(k+1) &= \sum_{i=1}^{2}(A_{1i}z_i(k) + A_{d1i}z_i(t-\tau(k))) + D_1\omega(k), \\ z_2(k+1) &= \sum_{i=1}^{2}(A_{2i}z_i(k) + A_{d2i}z_i(t-\tau(k))) + B_2u(k) + D_2\omega(k)). \end{aligned} \tag{5.6}$$

The main purpose is first to derive sufficient conditions for the existence of a stable sliding surface such that in the induced sliding dynamics, the effects of time-varying delay and unmatched disturbances can be mitigated. These conditions also guarantee that all the state trajectories are exponentially convergent to a ball whose radius can be minimised. Finally, a discrete-time quasi-sliding mode controller is proposed to drive the system state trajectories to the quasi-sliding mode.

5.3 Robust quasi-sliding mode control design

5.3.1 Sliding function design

The sliding function for system (5.1) is proposed as follows,

$$s(k) = \overline{C}z(k) = [-C \quad I]z(k) = -Cz_1(k) + z_2(k), \tag{5.7}$$

where $C \in \mathbb{R}^{m\times(n-m)}$ is a constant matrix to be designed. In the induced sliding mode, we have $s(k) = 0$ so that $z_2(k) = Cz_1(k)$. The reduced-order sliding motion can thus be obtained as

$$z_1(k+1) = [A_{11} + A_{12}C]z_1(k) + [A_{d11} + A_{d12}C]z_1(k-\tau(k)) + D_1\omega(k). \tag{5.8}$$

Note that the sliding surface design is now equivalent to the stabilisation problem for system $(A_{11}, A_{12}, A_{d11}, A_{d12})$ where (A_{11}, A_{12}) and (A_{d11}, A_{d12}) are assumed to be controllable. Reduced-order system (5.8) will be stabilised by choosing an appropriate matrix C. Due to the presence of the unmatched disturbances $\omega(k)$, in general, the asymptotic convergence of state trajectories of system (5.8) can not be achieved. In that case, instead of investigating asymptotic stability of the system, we consider the system state convergence within the neighborhood of the equilibrium point. However, the shape of such a neighborhood is, in general, very complex and hard to determine exactly. Hence, the estimation of outer or inner bounding simple convex shapes as balls or ellipsoids or boxes will be considered. This is formalised in term of the existence problem, which must be solved to determine the switching surface.

In the following, for the sake of simplicity, we denote $e_1 = [I_{n-m}\ 0_{(n-m)\times 8(n-m)}]$, $e_i = [0_{(n-m)\times(i-1)(n-m)}\ I_{n-m}\ 0_{(n-m)\times(9-i)(n-m)}]$, $i = 2, 3, \ldots, 8$, $e_9 = [0_{(n-m)\times 8(n-m)}\ I_{n-m}]$ as entry matrices. The following notations are specifically used in our development. For given integers τ_m, τ_M satisfying $0 < \tau_m < \tau_M$, any scalar λ, nonsingular matrix $K \in \mathbb{R}^{(n-m)\times(n-m)}, F = K^{-1}$, matrices X, G, and symmetric positive definite matrices $P, Q_j, R_j, S_j, j = 1,2$ of appropriate dimensions, we denote the following vectors

$$
\begin{aligned}
&y(k) = z_1(k+1) - z_1(k), \rho(k) = \left[z_1^T(k)F^T \; y^T(k)F^T\right]^T,\\
&\xi(k) = \Bigg[z_1^T(k)F^T \; z_1^T(k-\tau_m)F^T \; z_1^T(k-\tau(k))F^T\\
&\qquad z_1^T(k-\tau_M)F^T \sum_{s=k-\tau_m}^{k-1} z_1^T(s)F^T \sum_{s=k-\tau(k)}^{k-\tau_m-1} z_1^T(s)F^T\\
&\qquad \sum_{s=k-\tau_M}^{k-\tau(k)-1} z_1^T(s)F^T \; y^T(k)F^T \; \omega^T(k)\Bigg]^T,\\
&\zeta(k) = \Bigg[z_1^T(k)F^T \sum_{s=k-\tau_m}^{k-1} z_1^T(s)F^T \sum_{s=k-\tau_M}^{k-\tau_m-1} z_1^T(s)F^T\Bigg]^T,
\end{aligned}
$$

constants

$$
\begin{aligned}
\tau_a &= \frac{\tau_m(\tau_m+1)}{2}, \tau_b = \frac{(\tau_M-\tau_m)(\tau_M+\tau_m+1)}{2},\\
\alpha_1 &= \frac{1-\alpha}{\alpha-\alpha^{\tau_m+1}}, \alpha_2 = \frac{1-\alpha}{\alpha^{\tau_m+1}-\alpha^{\tau_M+1}},\\
\alpha_3 &= \frac{(1-\alpha)^2}{\alpha\left[\alpha^{1+\tau_m}-(1+\tau_m)\alpha+\tau_m\right]},\\
\alpha_4 &= \frac{(1-\alpha)^2}{\alpha\left[\alpha^{1+\tau_M}-\alpha^{1+\tau_m}+(1-\alpha)(\tau_M-\tau_m)\right]},
\end{aligned}
$$

and matrices

$$
\begin{aligned}
&G = CK, \mathit{\Gamma} = e_1^T + \lambda e_8^T, R_c = \tau_m R_1 + (\tau_M-\tau_m)R_2,\\
&\mathcal{A}_c = [A_{11}K + A_{12}G - K \;\; 0_{n-m} \;\; A_{d11}K + A_{d12}G \;\; 0_{(n-m)\times 4(n-m)} \;\; -K \;\; D_1],\\
&\mathit{\Pi}_1 = e_1 + e_8, \mathit{\Pi}_2 = \begin{bmatrix} e_1^T & e_8^T \end{bmatrix}^T, \mathit{\Pi}_3 = \begin{bmatrix} e_5^T & (e_1-e_2)^T \end{bmatrix}^T,\\
&\mathit{\Pi}_4 = \begin{bmatrix} e_6^T & (e_2-e_3)^T & e_7^T & (e_3-e_4)^T \end{bmatrix}^T,\\
&\mathit{\Pi}_5 = \tau_m e_1 - e_5, \mathit{\Pi}_6 = (\tau_M-\tau_m)e_1 - e_6 - e_7.
\end{aligned}
$$

Now, we are ready to present the first theorem that gives sufficient conditions for the existence of a stable sliding surface as follows.

Theorem 22 *For system (5.8), with given positive integers τ_m and τ_M for the delay, where $0 < \tau_m < \tau_M$, and disturbance bound $\overline{\omega}_p > 0$, if there exist scalars λ and α, where $\alpha > 1$, a nonsingular matrix $K = F^{-1}$, matrices X, G and symmetric positive-definite matrices $P, Q_j, R_j, S_j, j = 1, 2,$ of appropriate dimensions such that the following inequalities hold*

$$\Omega(\alpha) < 0, \tag{5.9a}$$

$$\begin{bmatrix} R_2 & X \\ \star & R_2 \end{bmatrix} \geq 0, \tag{5.9b}$$

where

$$\begin{aligned} \Omega(\alpha) =& \Pi_1^T P \Pi_1 - \alpha^{-1} e_1^T P e_1 + e_1^T Q_1 e_1 + \alpha^{-\tau_m} e_2^T (Q_2 - Q_1) e_2 - \alpha^{-\tau_M} e_4^T Q_2 e_4 \\ &+ \Pi_2^T R_c \Pi_2 - \alpha_1 \Pi_3^T R_1 \Pi_3 - \alpha_2 \Pi_4^T \begin{bmatrix} R_2 & X \\ \star & R_2 \end{bmatrix} \Pi_4 + e_8^T (\tau_a S_1 + \tau_b S_2) e_8 \\ &- \alpha_3 \Pi_5^T S_1 \Pi_5 - \alpha_4 \Pi_6^T S_2 \Pi_6 + \Gamma \mathcal{A}_c + \mathcal{A}_c^T \Gamma^T - (1 - \alpha^{-1}) e_9^T e_9, \end{aligned}$$

then the state trajectories of the sliding dynamics (5.8) are exponentially convergent within a ball $\mathcal{B}(0, r)$ with radius

$$r = \frac{\overline{\omega}_p}{\sqrt{\lambda_{min}(F^T P F)}}. \tag{5.10}$$

Moreover, the design matrix C in (5.7) can be obtained explicitly as

$$C = GF. \tag{5.11}$$

Proof. Let us recall $y(k) = z_1(k+1) - z_1(k) = [A_{11} + A_{12}C - I] z_1(k) + [A_{d11} + A_{d12}C] z_1(k - \tau(k)) + D_1 \omega(k)$. Consider the following Lyapunov-Krasovskii functional

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{5.12}$$

where

$$
\begin{aligned}
V_1(k) &= z_1^T(k)F^T PF z_1(k),\\
V_2(k) &= \sum_{s=k-\tau_m}^{k-1} \alpha^{s-k+1} z_1^T(s)F^T Q_1 F z_1(s) + \sum_{s=k-\tau_M}^{k-\tau_m-1} \alpha^{s-k+1} z_1^T(s)F^T Q_2 F z_1(s),\\
V_3(k) &= \sum_{s=-\tau_m}^{-1} \sum_{v=k+s}^{k-1} \alpha^{v-k+1} \rho^T(v) R_1 \rho(v) + \sum_{s=-\tau_M}^{-\tau_m-1} \sum_{v=k+s}^{k-1} \alpha^{v-k+1} \rho^T(v) R_2 \rho(v),\\
V_4(k) &= \sum_{s=-\tau_m}^{-1} \sum_{u=s}^{-1} \sum_{v=k+u}^{k-1} \alpha^{v-k+1} y^T(v) F^T S_1 F y(v)\\
&\quad + \sum_{s=-\tau_M}^{-\tau_m-1} \sum_{u=s}^{-1} \sum_{v=k+u}^{k-1} \alpha^{v-k+1} y^T(v) F^T S_2 F y(v).
\end{aligned}
$$

By taking the forward difference of $V_1(k)$ along the solutions of system (5.8), we have

$$
\Delta V_1(k) = z_1^T(k+1)F^T PF z_1(k+1) - \alpha^{-1} z_1^T(k) F^T PF z_1(k) + (\alpha^{-1} - 1)V_1,
$$

where $z_1(k+1) = y(k) + z_1(k)$. Therefore, $\Delta V_1(k)$ can be obtained of the form

$$
\Delta V_1(k) = \xi^T(k)\Big[\Pi_1^T P \Pi_1 - \alpha^{-1} e_1^T P e_1\Big]\xi(k) + (\alpha^{-1} - 1)V_1. \tag{5.13}
$$

The forward differences of $V_2(k)$ and $V_3(k)$ along the solutions of system (5.8) are obtained as

$$
\begin{aligned}
\Delta V_2(k) &= z_1^T(k)F^T Q_1 F z_1(k) + \alpha^{-\tau_m} z_1^T(k-\tau_m) F^T (Q_2 - Q_1) F z_1(k-\tau_m)\\
&\quad + \alpha^{-\tau_M} z_1^T(k-\tau_M) F^T Q_2 F z_1(k-\tau_M) + (\alpha^{-1} - 1)V_2\\
&= \xi^T(k)\Big[e_1^T Q_1 e_1 + \alpha^{-\tau_m} e_2^T (Q_2 - Q_1) e_2 + \alpha^{-\tau_M} e_4^T Q_2 e_4\Big]\xi(k) + (\alpha^{-1} - 1)V_2,
\end{aligned} \tag{5.14}
$$

and

$$
\begin{aligned}
\Delta V_3(k) = \rho^T(k) R_c \rho(k) - \sum_{s=k-\tau_m}^{k-1} \alpha^{s-k} \rho^T(s) R_1\\
\rho(s) - \sum_{s=k-\tau_M}^{k-\tau_m-1} \alpha^{s-k} \rho^T(s) R_2 \rho(s) + (\alpha^{-1} - 1)V_3.
\end{aligned} \tag{5.15}
$$

For $\tau_m \leq \tau(k) \leq \tau_M, k \in \mathbb{Z}^+$ and $\tau_m < \tau_M$, we have

$$\begin{aligned}\varDelta V_3(k) =&\rho^T(k)R_c\rho(k) - \sum_{s=k-\tau_m}^{k-1} \alpha^{s-k}\rho^T(s)R_1\rho(s) + (\alpha^{-1}-1)V_3 \\ &- \sum_{s=k-\tau_M}^{k-\tau(k)-1} \alpha^{s-k}\rho^T(s)R_2\rho(s) - \sum_{s=k-\tau(k)}^{k-\tau_m-1} \alpha^{s-k}\rho^T(s)R_2\rho(s).\end{aligned} \tag{5.16}$$

By using Lemma 1, the following estimation can be obtained as

$$\begin{aligned}- \sum_{s=k-\tau_m}^{k-1} \alpha^{s-k}\rho^T(s)R_1\rho(s) \leq& -\alpha_1 \begin{bmatrix} \sum_{s=k-\tau_m}^{k-1} Fz_1(s) \\ Fz_1(k) - Fz_1(k-\tau_m) \end{bmatrix}^T R_1 \times \\ &\times \begin{bmatrix} \sum_{s=k-\tau_m}^{k-1} Fz_1(s) \\ Fz_1(k) - Fz_1(k-\tau_m) \end{bmatrix} \\ &\leq -\alpha_1 \xi^T(k)\varPi_3^T R_1 \varPi_3 \xi(k).\end{aligned} \tag{5.17}$$

Using the same argument, we also have

$$\begin{aligned}&- \sum_{s=k-\tau(k)}^{k-\tau_m-1} \alpha^{s-k}\rho^T(s)R_2\rho(s) - \sum_{s=k-\tau_M}^{k-\tau(k)-1} \alpha^{s-k}\rho^T(s)R_2\rho(s) \\ &\leq -\frac{1-\alpha}{\alpha^{\tau_m+1}-\alpha^{\tau(k)+1}} \Big[\sum_{s=k-\tau}^{k-\tau(k)-1} \rho(s)\Big]^T R_2 \Big[\sum_{s=k-\tau}^{k-\tau(k)-1} \rho(s)\Big] \\ &\quad -\frac{1-\alpha}{\alpha^{\tau(k)+1}-\alpha^{\tau_M+1}} \Big[\sum_{s=k-\tau(k)}^{k-\tau_m-1} \rho(s)\Big]^T R_2 \Big[\sum_{s=k-\tau(k)}^{k-\tau_m-1} \rho(s)\Big] \\ &\leq -\frac{1-\alpha}{\alpha^{\tau_m+1}-\alpha^{\tau_M+1}} \Big[\frac{1}{\beta_1}\eta_1^T R_2 \eta_1 + \frac{1}{1-\beta_1}\eta_2^T R_2 \eta_2\Big],\end{aligned} \tag{5.18}$$

where $\beta = \dfrac{\alpha^{\tau_m} - \alpha^{\tau(k)}}{\alpha^{\tau_m} - \alpha^{\tau_M}}$, and

$$\eta_1 = \begin{bmatrix} \sum_{s=k-\tau(k)}^{k-\tau_m-1} Fz_1(s) \\ Fz_1(k-\tau_m) - Fz_1(k-\tau(k)) \end{bmatrix}, \eta_2 = \begin{bmatrix} \sum_{s=k-\tau_M}^{k-\tau(k)-1} Fz_1(s) \\ Fz_1(k-\tau(k)) - Fz_1(k-\tau_M) \end{bmatrix}.$$

Thus, from Lemma 2, we have the following estimation

$$\begin{aligned}
&-\sum_{s=k-\tau(k)}^{k-\tau_m-1}\alpha^{s-k}\rho^T(s)R_2\rho(s)-\sum_{s=k-\tau_M}^{k-\tau(k)-1}\alpha^{s-k}\rho^T(s)R_2\rho(s)\\
&\leq -\alpha_2\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}^T\begin{bmatrix}R_2 & X\\ \star & R_2\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}\\
&\leq -\alpha_2\xi^T(k)\mathit{\Pi}_4^T\begin{bmatrix}R_2 & X\\ \star & R_2\end{bmatrix}\mathit{\Pi}_4\xi(k).
\end{aligned}\tag{5.19}$$

Next, the difference of $V_4(k)$ along the solutions of system (5.8) is calculated as follows.

$$\begin{aligned}
\mathit{\Delta}V_4 =&y^T(k)F^T\left(\tau_aS_1+\tau_bS_2\right)Fy(k)+(\alpha^{-1}-1)V_4\\
&-\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1}\alpha^v y^T(k+v)F^TS_1Fy(k+v)\\
&-\sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1}\alpha^v y^T(k+v)F^TS_2Fy(k+v).
\end{aligned}$$

By using Lemma 1, we have

$$\begin{aligned}
&-\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1}\alpha^v y^T(k+v)F^TS_1Fy(k+v)\\
&\leq -\alpha_3\left(\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1}y(k+v)\right)^T F^TS_1F\left(\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1}y(k+v)\right)\\
&\leq -\alpha_3\left(\tau_mz_1(k)-\sum_{s=k-\tau_m}^{k-1}z_1(s)\right)^T F^TS_1F\left(\tau_mz_1(k)-\sum_{s=k-\tau_m}^{k-1}z_1(s)\right)\\
&\leq -\alpha_3\xi^T(k)\mathit{\Pi}_5^TS_1\mathit{\Pi}_5\xi(k).
\end{aligned}$$

Similarly, we also obtain

$$
\begin{aligned}
&-\sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1}\alpha^v y^T(k+v)F^T S_2 F y(k+v)\\
&\leq -\alpha_4\left[(\tau_M-\tau_m)z_1(k)-\sum_{s=k-\tau_M}^{k-\tau_m-1}z_1(s)\right]^T F^T S_2 F\times\\
&\qquad\times\left[(\tau_M-\tau_m)z_1(k)-\sum_{s=k-\tau_M}^{k-\tau_m-1}z_1(s)\right]\\
&\leq -\alpha_4\left[(\tau_M-\tau_m)z_1(k)-\sum_{s=k-\tau(k)}^{k-\tau_m-1}z_1(s)-\sum_{s=k-\tau_M}^{k-\tau(k)-1}z_1(s)\right]^T F^T S_2 F\times\\
&\qquad\times\left[(\tau_M-\tau_m)z_1(k)-\sum_{s=k-\tau(k)}^{k-\tau_m-1}z_1(s)-\sum_{s=k-\tau_M}^{k-\tau(k)-1}z_1(s)\right]\\
&\leq -\alpha_4\xi^T(k)\mathit{\Pi}_6^T S_2\mathit{\Pi}_6\xi(k).
\end{aligned}
$$

Moreover, for any scalar λ and a nonsingular matrix F, by using the descriptor method, we always have the following equation

$$
\begin{aligned}
2(z_1^T(k)F^T+\lambda y^T(k)F^T)\Big[&(A_{11}+A_{12}C-I)z_1(k)\\
&+(A_{d11}+A_{d12}C)z_1(k-\tau(k))+D_1\omega(k)-y(k)\Big]=0.
\end{aligned}
\tag{5.20}
$$

Note that from the above notations with some simple computations, equation (5.20) can be rewritten in the form of

$$
\xi(k)^T(\mathit{\Gamma}\mathcal{A}_c+\mathcal{A}_c^T\mathit{\Gamma}^T)\xi(k)=0. \tag{5.21}
$$

Finally, from (5.12)-(5.21), we obtain

$$
\mathit{\Delta}V(k)+(1-\alpha^{-1})V(k)-(1-\alpha^{-1})\omega^T(k)\omega(k)\leq\xi^T(k)\mathit{\Omega}(\alpha)\xi(k),\forall k\in\mathbb{Z}^+. \tag{5.22}
$$

Therefore, it follows from conditions (5.9a) and (5.9b) of Theorem 22 that

$$
\mathit{\Delta}V(k)+(1-\alpha^{-1})V(k)-(1-\alpha^{-1})\omega^T(k)\omega(k)\leq 0
$$

which yields

$$
V(k)\leq\overline{\omega}_p^2+V(0)e^{-\gamma k}\forall k\in\mathbb{Z}^+.
$$

From Lemma 3, we have

$$\lim_{k\to\infty} \sup V(k) \le \varpi_p^2, \quad k \in \mathbb{Z}^+. \tag{5.23}$$

Thus, by using the spectral properties of symmetric positive-definite matrix, we obtain,

$$\begin{aligned}\lambda_{min}(F^T PF) \lim_{k\to\infty} \sup||z_1(k)||^2 &\le \lim_{k\to\infty} \sup V(k) \\ &\le \varpi_p^2.\end{aligned}$$

This means that $\lim_{k\to\infty} \sup||z_1(k)|| \le r$. Thus, the induced sliding dynamics are bounded within a ball with radius r defined in (5.10). The proof is completed.

Remark 9 Note that the obtained conditions in Theorem 22 are also not LMIs and the solution to this problem can be found by using the method, presented in Remark 2.

Remark 10 As the radius of the ball $B(0, r)$ in equation (5.10) is determined by $r = \frac{\varpi_p}{\sqrt{\delta}}$ where $\delta = \lambda_{min}(F^T PF)$, to find the possible smallest radius r, one may proceed with a simple optimisation process as suggested in [30, 134] to maximise δ subject to $\delta I \le F^T PF$, i.e., to formulate the following optimisation problem:

$$\text{minimise}(\frac{\varpi_p}{\sqrt{\delta}}) \text{ subject to } \begin{cases} (a) \quad F^T PF \ge \delta I \\ (b) \quad (5.9a) \text{ and } (5.9b). \end{cases}$$

Note that inequality $F^T PF \ge \delta I$ is equivalent to $(K^{-1})^T P(K^{-1}) \ge \delta I$. Pre- and post- multiplying this inequality by K^T and its transpose, respectively, we obtain

$$-P + \delta K^T K \le 0. \tag{5.24}$$

By using the Schur complement, we have

$$\begin{bmatrix} -P & K^T \\ \star & -\frac{1}{\delta} I \end{bmatrix} \le 0. \tag{5.25}$$

5.3.2 Robust quasi-sliding mode controller design

In discrete-time quasi-sliding mode control, under the appropriate controller, the system trajectory, starting from any initial state, will be driven towards the sliding surface in finite time. After reaching the sliding surface, the state trajectories cross the sliding surface for the first time, and repeat that again in successive sampling periods, resulting in a zigzag motion along the sliding surface. This motion will be bounded inside a specified region, the so-called quasi-sliding mode band (QSMB)[36]. In the previous section, under appropriate conditions, in the sliding mode, the state trajectories of the system are convergent within a ball whose radius can be minimised. In the following, the objective is to design a robust discrete-time quasi-sliding mode controller to drive the system dynamics towards the above ball in finite time and maintain it there afterwards.

First, it is noted from (5.3) that the external disturbance $\omega(k)$ is bounded, and so is the uncertain term $d(k) = \overline{C}\ \overline{D}\omega(k)$. Without loss of generality, we have, componentwise:

$$d_m \leq d(k) \leq d_M. \tag{5.26}$$

From a physical perspective, by assuming the boundedness of $z(k-\tau(k))$, and hence of vector $a(k) = \overline{C}\ \overline{A}_d z(k-\tau(k))$. We then have, similarly:

$$a_m \leq a(k) \leq a_M. \tag{5.27}$$

Let us define

$$\begin{aligned} a_0 &= \frac{a_m + a_M}{2},\ d_0 = \frac{d_m + d_M}{2}, \\ a_1 &= \frac{a_M - a_m}{2},\ d_1 = \frac{d_M - d_m}{2}. \end{aligned} \tag{5.28}$$

From sliding function (5.7), in which the design matrix $C = [C_1 \quad C_2 \quad \ldots C_m]^T$ is obtained from (5.11), we have $s(k) = [s_1(k)\ s_2(k)\ \ldots\ s_m(k)]^T$, where $s_i(k) = -C_i z_1(k) + z_{2i}(k)$ and C_i is a row vector in $\mathbb{R}^{1\times(n-m)}$.

Theorem 23 *For given positive integers τ_m and τ_M of the delay, satisfying $0 < \tau_m < \tau_M$, and a bound $\overline{\omega}_p > 0$ of the external disturbance, if there exist scalars λ and α, where $\alpha > 1$, a feasible solution of X, K, G, P and Q_j, R_j, S_j, $j = 1, 2$, for matrix inequalities (5.9a), (5.9b) and (5.25), and with the sliding function chosen as in (5.7) for sliding motion (5.8), the state trajectories of system (5.5) are driven towards the sliding surface in a finite time under the following control law:*

$$u(k) = -(\overline{C}\,\overline{B})^{-1}\Big[\overline{C}\,\overline{A}z(k) - (I - qT_s)s(k) + a_0 + d_0 + (\varepsilon T_s + a_1 + d_1) \circ sgn(s(k))\Big], \tag{5.29}$$

where $sgn(s(k)) = [sgn(s_1(k)), sgn(s_2(k)), ..., sgn(s_m(k))]^T$, T_s is the sampling period, $q = diag(q_1,\ q_2, ...,\ q_m)$ and $\varepsilon = [\varepsilon_1, \varepsilon_2, ..., \varepsilon_m]^T$, in which positive scalars q_i and ε_i, $i = 1, 2, \ldots, m$, are chosen such that $1 - T_s q_i > 0$ for quasi-sliding mode bands $\Delta_i(k)$ given by

$$\Delta_i = \frac{\varepsilon_i T_s}{1 - T_s q_i}. \tag{5.30}$$

Proof. From the designed sliding function $s(k) = \overline{C}z(k)$, we have

$$\begin{aligned} \Delta s(k) &= \overline{C}z(k+1) - \overline{C}z(k) \\ &= \overline{C}\Big[(\overline{A} - I)z(k) + \overline{A}_d z(k - \tau(k)) + \overline{B}u(k) + \overline{D}\omega(k)\Big]. \end{aligned} \tag{5.31}$$

By substituting the control law (5.29) into equation (5.31), we obtain

$$\begin{aligned} \Delta s(k) =& - qT_s s(k) - \varepsilon T_s \circ \mathrm{sgn}(s(k)) \\ &+ [a(k) - a_0 - a_1 \circ \mathrm{sgn}(s(k))] \\ &+ [d(k) - d_0 - d_1 \circ \mathrm{sgn}(s(k))]. \end{aligned} \tag{5.32}$$

Now, we have to show that the proposed control scheme satisfies the reaching condition and the existence of the quasi-sliding mode is guaranteed. This requires that the sign of the incremental change $\Delta s(k) = s(k+1) - s(k)$ should be opposite to the sign of $s(k)$, componentwise.

It is easy to see that when $s(k) > 0$, we have

$$a(k) \leq a_0 + a_1,$$

$$d(k) \leq d_0 + d_1,$$

and when $s(k) < 0$,

$$a(k) \geq a_0 - a_1,$$

$$d(k) \geq d_0 - d_1.$$

Thus, by judging the sign of the four terms constituting $\Delta s(k)$ in (5.32), we can see that the sign of the increment $\Delta s(k)$ of (5.31) is always opposite to the sign of $s(k)$, componentwise. Thus, if design parameters $q_i > 0$ and $\varepsilon_i > 0$ are chosen with $1 - q_i T_s > 0$, $i = 1, 2, \ldots, m$, then a quasi-sliding mode exists with quasi-sliding mode bands $\Delta_i = \frac{\varepsilon_i T_s}{1 - q_i T_s}$ [36]. This completes the proof. □

Remark 11 It is worth mentioning that in this chapter, the quasi-sliding mode control law (5.29) is obtained from the sliding function (5.7), whereby the design matrix C can be computed directly from (5.11) after the solution of conditions (5.9a), (5.9b) and the optimisation process mentioned in Remark 2. This gives designers a certain liberty in selecting the controller parameters in (5.29) for a desired QSMB as compared to existing methods in the literature, where the QSMB is determined subsequently from the design of a quasi-sliding mode control law.

5.4 Examples

In this section, some numerical examples are given to illustrate the effectiveness of the proposed approach.

5.4.1 Example 5.1

Consider a truck-trailer system for the case of unmatched external disturbance which was given in [153, 154] as follows

$$x(k+1) = \begin{bmatrix} 1.3461 & 0 & 0 \\ 0.3461 & 1 & 0 \\ 0.0086 & -0.05 & 1 \end{bmatrix} x(k) + \begin{bmatrix} -0.0384 & 0 & 0 \\ 0.0384 & 0 & 0 \\ 0.0001 & 0 & 0 \end{bmatrix} x(k-\tau(k))$$
$$+ \begin{bmatrix} -0.5747 \\ 0 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ -0.01 \\ 0.01 \end{bmatrix} \omega(k),$$

where $x(k) = [x_1(k)\ x_2(k)\ x_3(k)]^T$ is the system state vector of the angle difference between the trailer and the truck, the angle of the trailer, and the vertical position of the rear end of the trailer, respectively. The control input signal $u(k)$ is the steering angle. Here, the truck-trailer system is assumed to be subject to an external disturbance $\omega(k)$ with an upper bound $\overline{\omega}_p = 0.3$. The control objective is to minimise the effects of time-varying delay and unmatched disturbances, while backing the trailer-truck along the horizontal line $x_3(k) = 0$ in a safe and robust manner. Note that the proposed approaches in [153, 154] are available for linear discrete-time systems with time-varying delay and matched disturbances. Therefore, it can not apply for this case. For this, the sampling period is chosen as $T_s = 0.1$ sec. Matrix T of the transformation $z(k) = Tx(k)$ can be obtained from a singular value decomposition of matrix B as:

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

From Theorem 22 and Remarks 9 and 10, by choosing $\alpha = 1.15, \lambda = 0.6$ and solving matrix inequalities (5.9a) and (5.9b), we obtain the following matrices

$$G = 1.0e - 003 * [-0.6807 \ \ 0.0009], K = 1.0e - 003 * \begin{bmatrix} 0.9842 & 0.2487 \\ 0.2701 & 0.1449 \end{bmatrix}.$$

Thus, the switching gain is calculated as $C = [-1.3105 \ \ 2.2547]$. As a result, the sliding function is obtained as

$$s(k) = [-1.3105 \ \ 2.2547 \ \ 1]z(k).$$

Moreover, the possible smallest radius of the ball which bounds the state trajectories of the reduced-order system can be obtained as $r = 0.01$. With the assumption of boundedness of the external disturbance $\overline{\omega}_p = 0.3$, the average and variation magnitude of the disturbance-related uncertainty $d(k)$ can be found as

$$d_0 = 0, d_1 = 0.0107.$$

Similarly, we have for the delay-related uncertain term $a(k)$ $a_0 = 0$ and $a_1 = 0.0012$. By choosing $q = 2$ and $\varepsilon = 0.016$ for a QSMB of $\mathit{\Delta} = 0.002$, from Theorem 23, the robust discrete-time quasi-sliding mode controller is obtained in the form

$$u(k) = 1.74\Big([1.4232 \ \ -2.2547 \ 1.8190]z(k) - 0.8s(k) + 0.0119\text{sgn}(s(k))\Big). \quad (5.33)$$

With an initial condition of the system of $x(k) = [0.2 \ 0 \ \ -0.85]^T$, and the time-varying delay $\tau(k)$ is a random integer belonging to the interval [1 5], the state responses of the reduced-order system via $z(k)$ and closed-loop system are shown in Fig. 5.1 and Fig. 5.2, respectively. It can be seen in the inset of Fig. 5.1 that after reaching the sliding surface, the states trajectories of the reduced-order system exponentially converges within a ball with radius $r \leq 0.01$ in spite of time-varying delay and unmatched external disturbances. The responses of the control input signal and the sliding surface are depicted respectively in Fig. 5.3 and Fig. 5.4 to illustrate the steering process of the truck-trailer system. These indicate that

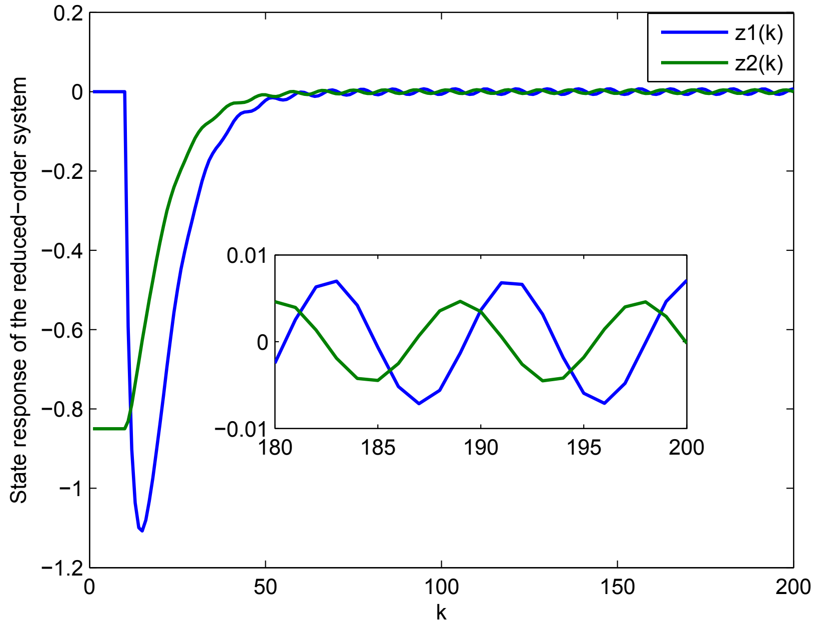

Figure 5.1 : State responses of the reduced-order system with unmatched disturbances

the effects of time-varying delay and unmatched bounded disturbances have been successfully suppressed by using the proposed controller.

5.4.2 Example 5.2

Now, consider the truck-trailer system in the case of without external disturbances (i.e., $\omega(k) = 0$). The control objective is still the same. From Theorem 22, by choosing $\alpha = 1.15, \lambda = 1.4$ and solving matrix inequalities (5.9a) and (5.9b), we obtain the following matrices

$$G = 1.0e - 003 * [-0.3920 \quad -0.0152], K = 1.0e - 003 * \begin{bmatrix} 0.8726 & 0.3033 \\ 0.3193 & 0.2134 \end{bmatrix}.$$

Thus, the design matrix C is calculated as $C = [-0.8817 \;\; 1.1819]$. As a result, the sliding surface is obtained as

$$s(k) = [-0.8817 \;\; 1.1819 \;\; 1]z(k).$$

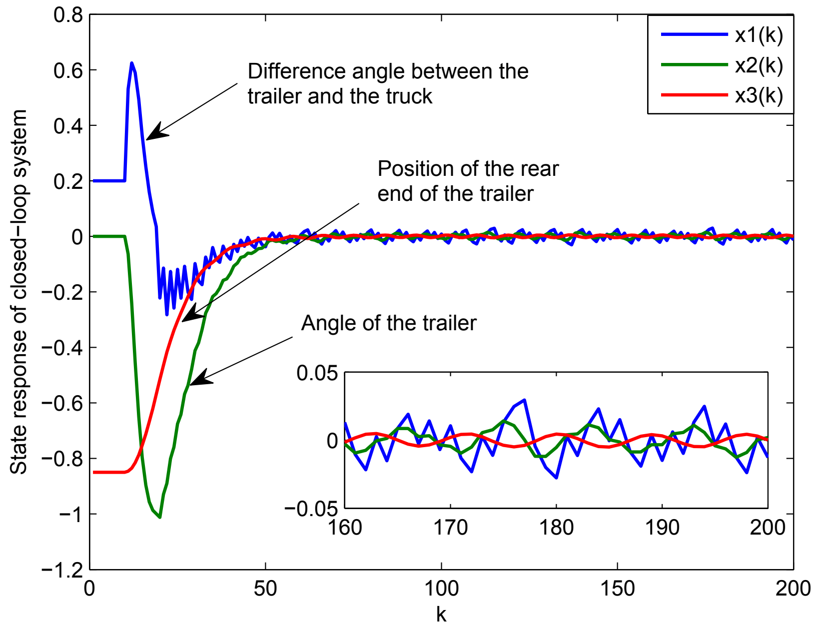

Figure 5.2 : State responses of the closed-loop system with unmatched disturbances

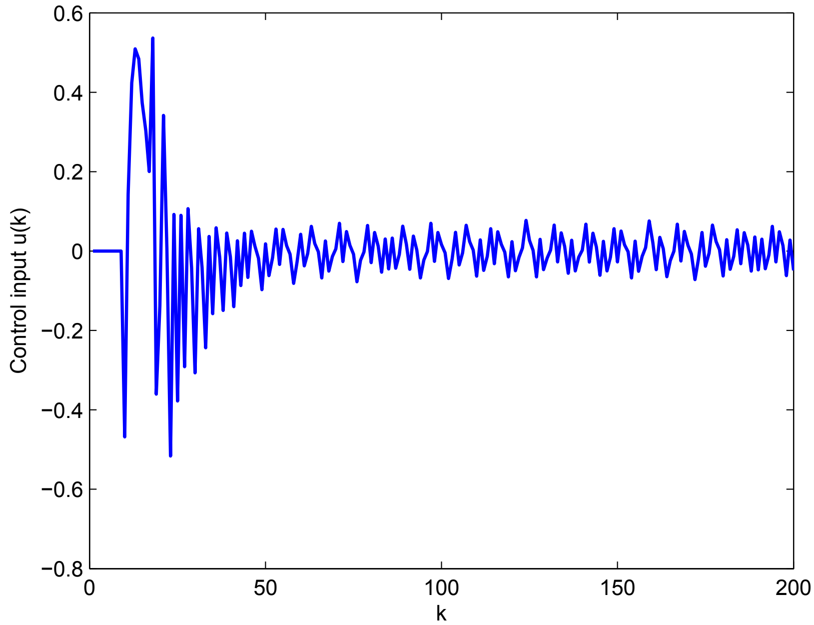

Figure 5.3 : Steer angle $u(k)$ of the truck-trailer system with unmatched disturbances

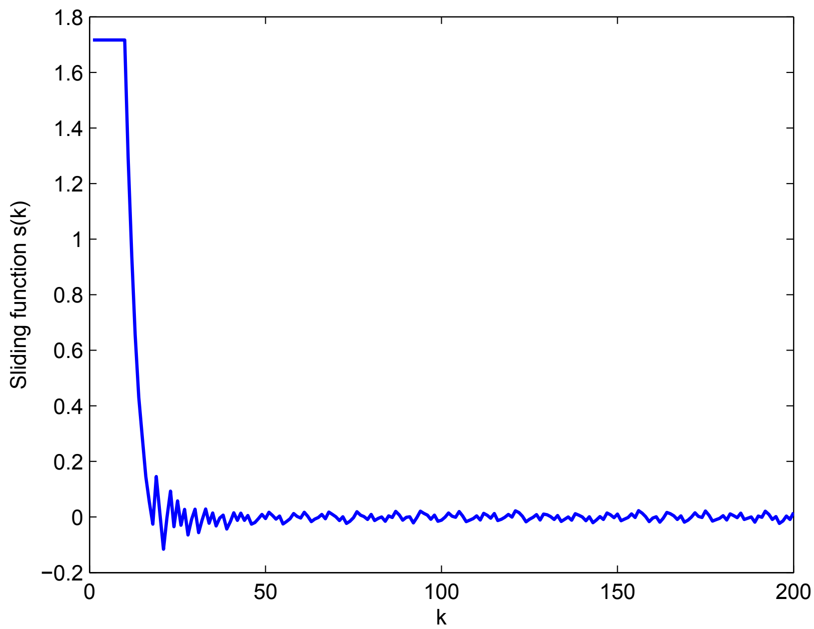

Figure 5.4 : Sliding function $s(k)$ of the truck-trailer system with unmatched disturbances

Similarly, the bound of uncertainty $a(k)$ is determined as $a_1 = 7.9694\text{x}10^{-5}$. By using the same controller parameters as in Example 5.1, from Theorem 23, the robust discrete-time quasi-sliding mode controller is obtained of the form

$$u(k) = 1.74\Big([0.9408 \ -1.1819 \ 1.6614]z(k) - 0.8s(k) + 7.9694e - 005\text{sgn}(s(k))\Big). \tag{5.34}$$

For the sake of demonstration the effectiveness of the proposed control schemes in terms of robustness to time-varying delay, the initial conditions $x(k) = [0.1 \ 0 \ -0.1]^T$ will be used. The obtained conditions are still feasible with an interval time-varying delay $[\tau_m \quad \tau_M]$, where $\tau_M \leq 16$. As shown in Fig. 5.5, the state responses of the closed-loop system exponentially converge to the origin. The control input signal and sliding surface are depicted in Fig. 5.6 and Fig. 5.7. It can be seen clearly that the close-loop control system is robustly stable with a large range of time-varying delay. It is worth pointing out that by comparing simulation results with Example 5.1, obviously, the proposed control scheme is more effective for the truck-trailer

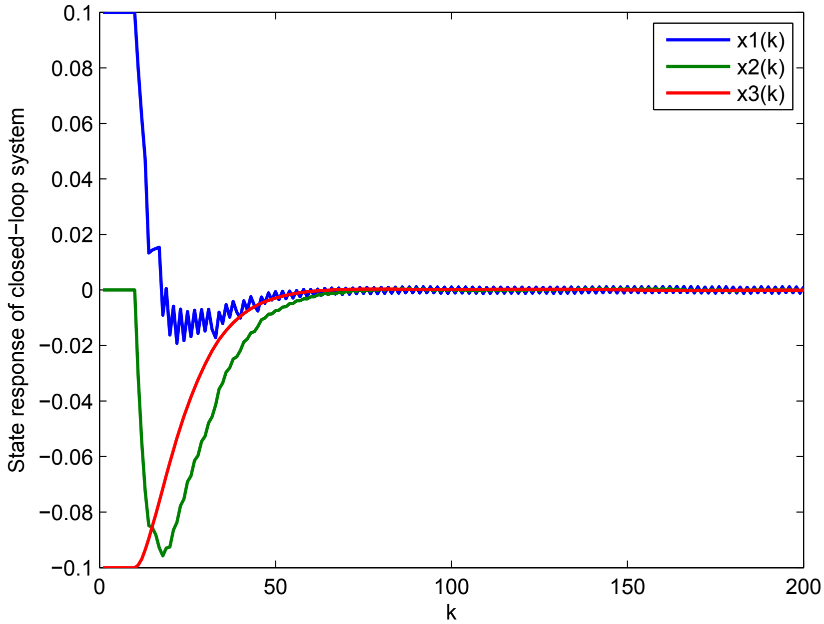

Figure 5.5 : State responses of the closed-loop system without external disturbances

system in the absence of external disturbances.

5.5 Conclusion

In this chapter, the problem of robust discrete-time quasi-sliding mode control design for a class of linear discrete-time systems with time-varying delay and unmatched disturbances has been addressed. Based on the Lyapunov-Krasovskii method, combined with the reciprocally convex approach, sufficient conditions for the existence of a stable sliding surface are derived in terms of matrix inequalities. These conditions also guarantee that the effects of time-varying delay and unmatched disturbances are mitigated when the system is in the sliding mode. Finally, a discrete-time quasi-sliding mode controller is proposed to satisfy the reaching condition. Numerical examples are provided to illustrate the feasibility of the proposed approach.

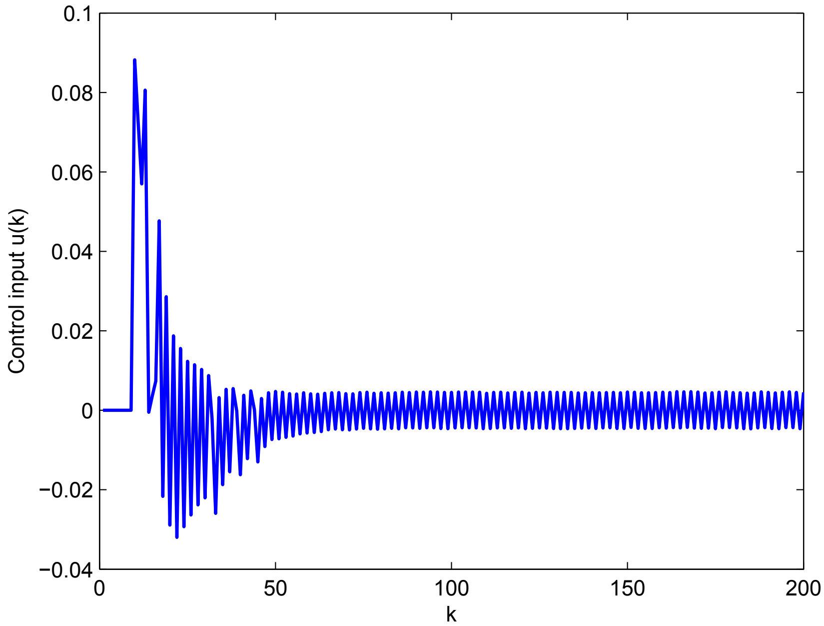

Figure 5.6 : Steer angle $u(k)$ of the truck-trailer system without external disturbances

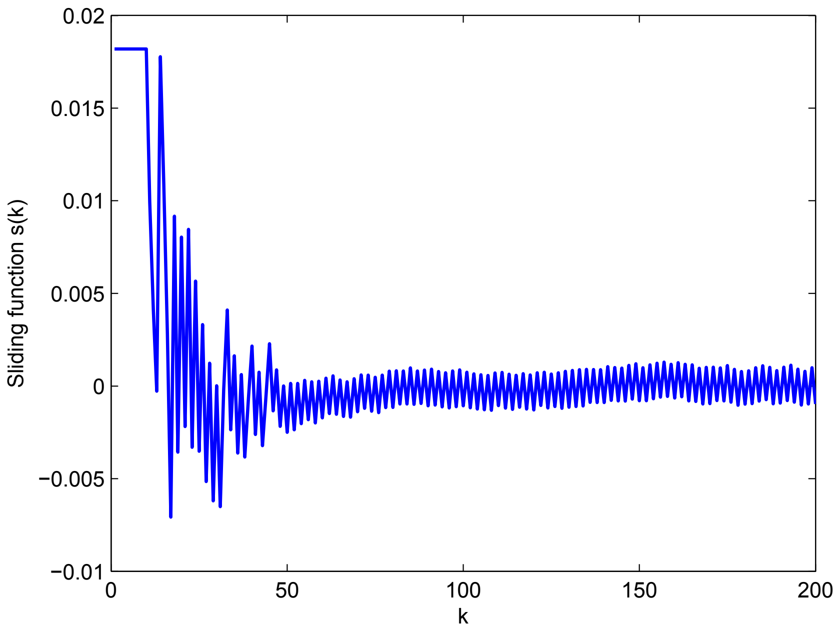

Figure 5.7 : Sliding function $s(k)$ of the truck-trailer system without external disturbances

Chapter 6

Functional observer design of time-delay systems

6.1 Introduction

The observer design problem for control systems with time delay has been an interesting research topic over the years, see, e.g., [138, 119, 69] and references therein. For a class of dynamic systems that are subject to time delays in the state and unknown inputs, there has been significant research devoted to the design of observers for simultaneous estimation of both state and unknown inputs [43, 136, 159]. However, in practice, information of the output measurements in applications such as power grids, rolling mills, biochemical reactors, automotive engineering, aeronautical systems or robotic networks is quite often available for processing only after a certain time delay. It is thus important to take time delay in the system output into account in the state observation problem. This has been tackled in [37], by using a chain structure for the observation algorithm with globally Lipschitz continuous invertible observability maps. A delay-free transformation approach is proposed for observer-based fault diagnosis and isolation via estimates of the state vector [128].

In these papers, the control input was assumed to be available without a time delay. However, due to a limited bandwidth of the communication channel or sensor and actuator speed constraints, or in general dual-rate systems with different updating and sampling periods [18, 19], a time delay in the system input should also be considered. This is important in networked control systems, where network-induced time delay appearing in both the sensor to controller and controller to actuator channels in addition to packet dropouts may cause data loss problems [63], and observer-based approaches may offer promising solutions [77, 141].

In practical implementation, control processes, on the other hand, usually require the availability only of a linear function of the system states rather than a full state vector to perform fault diagnosis and/or to realize a stabilizing feedback control law in the presence of disturbances and failures. Due to the retard problem in measurement and control data of control systems in robotic telehandlers, master-slave manipulators or applications as mentioned above, output and input time delay should be considered when reconstructing the linear feedback signals from using the delayed information. Although the design of linear functional observers has been well-addressed in the literature (see, e.g., [17, 25, 138] and references therein), it appears that very little attention has been paid to the problem of estimation of a state functional for linear systems that are subject to time delays in both the measurement output as well as the control input.

In this chapter, a new problem of estimating linear functions of the state vector in the case when the output information and control signal are both available not instantaneously but only after certain time delays is investigated. By using the information of the delayed output and delayed input, the proposed observer can estimate system state functionals in the challenging case of different time delays present in both the output and input of the control system.

6.2 Problem Statement and Preliminaries

Consider a linear time-invariant system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), t \geq 0, \\ y(t) &= Cx(t) \\ z(t) &= Lx(t), \end{aligned} \tag{6.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^l$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively. Vector $z(t) \in \mathbb{R}^m$ is the partial state to be estimated. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{p\times n}$ and $L \in \mathbb{R}^{m\times n}$ are known and constant.

The main purpose in this part of the research is to design an mth-order observer to estimate the partial state $z(t)$ of system (6.1), where the input $u(t)$ and output $y(t)$ are both subject to known and constant time delays τ_1 and τ_2, respectively. For this, let us first introduce some assumptions.

Assumption 1 The pair (A, B) is controllable, and the pair (C, A) is observable, $\text{rank}(C) = p$, $\text{rank}(L) = m$ and $\text{rank}\begin{bmatrix} C \\ L \end{bmatrix} = (p + m)$.

Assumption 2 The control input satisfies a generally-met condition for slowly time-varying processes in practical applications, i.e., for all $t \geq 0$, there exists $\delta > 0, h > 0$ such that $||u(t) - u(t - t_1)|| \leq \delta, \forall t_1 \in [0 \;\; h]$.

The following definition is introduced for a general dynamic system subject to an external disturbance $\omega(t)$,

$$\begin{cases} \dot{x}(t) = f(t, x_t, \omega(t)), \; t \geq 0 \\ x(\theta) = \varphi(\theta), \; \theta \in [-\tau, 0], \end{cases} \tag{6.2}$$

where $\omega(t)$ is assumed to be bounded by $\bar{\omega} > 0$, i.e., $||\omega(t)|| \leq \bar{\omega}, \; \forall t \geq 0$ and $x(\theta) = \varphi(\theta)$ is the state initial condition with $\varphi \in \mathcal{C}([-\tau, 0])$, the set of continuous functions on $[-\tau, 0]$:

Definition 6 *Let $\epsilon > 0$ be a given positive scalar and denote $\mathcal{B}(0, \bar{\omega}) = \{\omega(t) : ||\omega(t)|| \leq \bar{\omega}\}$. The system (6.2) is said to be ϵ-convergent if for all $\omega(t) \in \mathcal{B}(0, \bar{\omega})$ and for any initial condition $\varphi(\theta)$, we have*

$$\lim_{t\to\infty} ||x(t, \varphi(\theta), \omega(t))|| \leq \epsilon.$$

Moreover, if there exist a positive scalar α and a positive functional $\xi(.)$ such that

$$||x(t, \varphi(\theta), \omega(t))|| \leq \epsilon + \xi(\varphi(\theta))e^{-\alpha t}, \forall t \geq 0$$

then system (6.2) is said to be ϵ-convergent with an exponential rate α.

Remark 12 Since the input and output time delays τ_1 and τ_2 are known, information of input and output is both available after a time delay that is greater than or equal τ, which is defined by

$$\tau = \max\{\tau_1, \tau_2\}.$$

From Remark 12, we propose to use information of $y(t-\tau)$, $y(t-2\tau)$ and $u(t-\tau)$ in the following mth-order observer to obtain the partial state estimate $\hat{z}(t)$ for system (6.1) with time delays in both output and input:

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + N_d\zeta(t-\tau) + D_1y(t-\tau) + D_2y(t-2\tau) + Eu(t-\tau), \\ \hat{z}(t) &= \zeta(t) + Fy(t-\tau), \end{aligned} \tag{6.3}$$

where $\zeta(t) \in \mathbb{R}^m$ is the observer state with an initial condition $\zeta(\theta) = \phi(\theta)$, $\forall\theta \in [-\tau, 0]$, and $\phi \in \mathcal{C}([-\tau, 0])$. The design matrices N, N_d, D_1, D_2, E and F are of appropriate dimensions to be determined such that the estimate $\hat{z}(t)$ approaches to $z(t)$ or at least remains as closely to $z(t)$ as possible when $t \to \infty$. Let us first define the observer error $e(t)$ between the system partial state $z(t)$ given in (6.1) and its estimate $\hat{z}(t)$ obtained from the proposed observer (6.3) as,

$$e(t) = z(t) - \hat{z}(t). \tag{6.4}$$

Under the condition of no instantaneous information of the system output and input being available for estimation, an asymptotical convergence of the observer error $e(t)$ would be hardly achievable. The problem is thus to determine the observer design matrices such that the observer error $e(t)$ is bounded by a given positive scalar $\epsilon > 0$, for sufficiently large t, i.e., $\lim_{t\to\infty}||e(t)|| \leq \epsilon$, or in other words, the estimate $\hat{z}(t)$ converges within an ϵ-bound of $z(t)$.

To recall some lemmas in the following, we first define x_t, $x_t := \{x(t+s), s \in [-\tau, 0]\}$ as the segment of the trajectory $x(t)$ in the time interval $[t-\tau, t]$, with its norm $\|x_t\| = \sup_{s\in[-\tau,0]} \|x(t+s)\|$.

Lemma 5 *[27, 101] Let $V(t, x_t)$ be a Lyapunov functional for system (6.2) and α be a positive scalar. If $\dot{V} + 2\alpha V - \frac{2\alpha}{\bar{\omega}^2}\omega^T(t)\omega(t) \leq 0,\ \forall t \geq 0$, then there exists a positive functional $\xi(.)$ such that $V(t, x_t) \leq 1 + \xi(\varphi(\theta))e^{-2\alpha t}, \forall t \geq 0$.*

Lemma 6 *[39, 126] For any constant positive-definite matrix $P = P^T > 0$ and a scalar $\gamma > 0$ such that the following integrations are well-defined, then*
a) $-\int_{t-\gamma}^{t} x^T(s)Px(s)ds \leq -\frac{1}{\gamma}\Big(\int_{t-\gamma}^{t} x(s)ds\Big)^T P\Big(\int_{t-\gamma}^{t} x(s)ds\Big)$,
b) $-\int_{-\gamma}^{0}\int_{t+\theta}^{t} x^T(s)Px(s)dsd\theta \leq -\frac{2}{\gamma^2}\Big(\int_{-\gamma}^{0}\int_{t+\theta}^{t} x(s)dsd\theta\Big)^T P\Big(\int_{-\gamma}^{0}\int_{t+\theta}^{t} x(s)dsd\theta\Big)$.

The first lemma is needed for showing convergence with an exponential rate. The second lemma is useful in the delay-decomposition technique combined with the Lyapunov-Krasovskii method. In the case of no disturbance in system (6.2), i.e. $\bar{\omega} = 0$, the following lemma states a condition for its exponential stability.

Lemma 7 *[91] In association with any solution $x(t, \varphi)$ of system (6.2) where $\bar{\omega} = 0$, let $V(t, x_t)$ be a Lyapunov-Krasovskii functional satisfying*

a) $m_1||x(t)||^2 \leq V(t, x_t) \leq m_2||x_t||^2$,

b) $\frac{d}{dt}V(t, x_t) + 2\alpha V(t, x_t) \leq 0$.

Then the following exponential estimate holds:

$$||x(t, \varphi)|| \leq \sqrt{\frac{m_2}{m_1}}||\varphi||e^{-\alpha t}, \forall t \geq 0.$$

6.3 Main Results

The proposed observer (6.3) involves several constant matrices that need to be determined. For computing these design matrices, we first introduce our lemma as follows.

Lemma 8 *Given matrices A, C and L of system (6.1), the following matrix equations*

$$\begin{aligned} NL - LA &= 0, \\ N_d L + UC + FCA &= 0, \end{aligned} \tag{6.5}$$

for unknown matrices $N \in \mathbb{R}^{m\times m}$, $N_d \in \mathbb{R}^{m\times m}$, $U \in \mathbb{R}^{m\times p}$ and $F \in \mathbb{R}^{m\times p}$ are solvable if and only if the following condition holds

$$\text{rank}\begin{bmatrix} L \\ LA \end{bmatrix} = \text{rank}(L). \tag{6.6}$$

Proof. The matrix equation (6.5) can be expressed as

$$\begin{bmatrix} N & N_d & U & F \end{bmatrix} \Psi = \Phi, \tag{6.7}$$

where $\Psi \in \mathbb{R}^{(2m+2p)\times 2n}$ and $\Phi \in \mathbb{R}^{m\times 2n}$ are known matrices defined by

$$\Psi = \begin{bmatrix} L & 0 \\ 0 & L \\ 0 & C \\ 0 & CA \end{bmatrix}, \; \Phi = \begin{bmatrix} LA & 0 \end{bmatrix}. \tag{6.8}$$

In (6.7), the unknown matrix $\begin{bmatrix} N & N_d & U & F \end{bmatrix}$ exists if and only if the following condition holds

$$\text{rank}\begin{bmatrix} \Psi \\ \Phi \end{bmatrix} = \text{rank}(\Psi), \tag{6.9}$$

which is equivalent to condition (6.6) of Lemma 4.

Furthermore, the unknown matrix $\begin{bmatrix} N & N_d & U & F \end{bmatrix}$ is given as

$$\begin{bmatrix} N & N_d & U & F \end{bmatrix} = \Phi\Psi^{+} + Z(I - \Psi\Psi^{+}), \tag{6.10}$$

where $\Psi^{+} \in \mathbb{R}^{2n\times(2m+2p)}$ is a generalized inverse of Ψ and $Z \in \mathbb{R}^{m\times(2m+2p)}$ is an arbitrary matrix. Matrices N, N_d, U and F can now be extracted from (6.10) and are expressed as follows

$$N = N_1 + ZN_2, \;\; N_d = N_{d1} + ZN_{d2}, \tag{6.11}$$

$$U = U_1 + ZU_2, \;\; F = F_1 + ZF_2, \tag{6.12}$$

where N_1, N_2, N_{d1}, N_{d2}, U_1, U_2, F_1 and F_2 are known matrices. For clarity, we show how N_i and N_{di} are obtained

$$N_1 = \Phi\Psi^+ \begin{bmatrix} I_m \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; N_2 = (I - \Psi\Psi^+) \begin{bmatrix} I_m \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{6.13}$$

$$N_{d1} = \Phi\Psi^+ \begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix}, \; N_{d2} = (I - \Psi\Psi^+) \begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix}, \tag{6.14}$$

where I, I_m are identity matrices with dimension $(2m+2p)\times(2m+2p)$ and $m \times m$, respectively. This completes the proof of Lemma 8. □

6.3.1 Systems with both output and input delays

Theorem 24 *For system (6.1) subject to delays in output and input, given a positive scalar $\epsilon > 0$, the estimate $\hat{z}(t)$ obtained from observer (6.3) converges within an ϵ-bound of $z(t)$ for any initial conditions $x(0)$, $\phi(\theta)$ and control $u(t)$ if the following conditions hold*

1) $\dot{e}(t) = Ne(t) + N_d e(t-\tau) + LB(u(t) - u(t-\tau))$ *is ϵ-convergent,*

2) $LA - NL = 0,$

3) $(NF - D_1)C - N_d L - FCA = 0,$

4) $(N_d F - D_2)C = 0,$

5) $FCB - LB + E = 0.$

Proof. From (6.1), (6.2) and (6.3), we have

$$\zeta(t) = Lx(t) - e(t) - Fy(t-\tau).$$

Taking the time derivative of $e(t)$ along the solutions of system (6.1), we have

$$\begin{aligned}\dot{e}(t) &= L\dot{x}(t) - \dot{\hat{z}}(t) = L\dot{x}(t) - \dot{\zeta}(t) - F\dot{y}(t-\tau) \\ &= L\left(Ax(t) + Bu(t)\right) + N\left(e(t) - Lx(t) + Fy(t-\tau)\right) \\ &\quad + N_d\left(e(t-\tau) - Lx(t-\tau) + Fy(t-2\tau)\right) \\ &\quad - D_1y(t-\tau) - D_2y(t-2\tau) - Eu(t-\tau) \\ &\quad - FC\left(Ax(t-\tau) + Bu(t-\tau)\right).\end{aligned}$$

By substituting $y(t-\tau) = Cx(t-\tau)$,$y(t-2\tau) = Cx(t-2\tau)$ and collecting like terms, the following error dynamics equation is obtained

$$\begin{aligned}\dot{e}(t) &= Ne(t) + N_de(t-\tau) + LB(u(t) - u(t-\tau)) \\ &\quad + (NFC - D_1C - N_dL - FCA)x(t-\tau) \\ &\quad + (LA - NL)x(t) + (N_dFC - D_2C)x(t-2\tau) \\ &\quad - (FCB - LB + E)u(t-\tau).\end{aligned}$$

Therefore if the five conditions of Theorem 24 are met, then $\lim_{t\to\infty}||e(t)|| \leq \epsilon$. It means that the estimate $\hat{z}(t)$ converges within an ϵ-bound of $z(t)$.

To complete the design of the proposed observer (6.2), we need to determine the unknown matrices N, N_d, D_1, D_2, E and F such that all five conditions of Theorem 24 are satisfied. From the condition 5) of Theorem 24, matrix E can be obtained as

$$E = LB - FCB. \tag{6.15}$$

Consider the matrix equation 4) of Theorem 24, since C has full-row rank, condition 4) of Theorem 24 holds if and only if

$$(N_dF - D_2) = 0, \tag{6.16}$$

which implies that matrix D_2 can be obtained as $D_2 = N_dF$.

By denoting a matrix $U = -(NF - D_1)$, matrix D_1 can be obtained as $D_1 = U + NF$. Now if condition (6.6) of Lemma 8 holds then a solution of the two matrix equations 2) and 3) of Theorem 24 can be obtained as given in (6.10)-(6.11). □

After having shown the existence of the proposed observer and how to obtain its design matrices, the remaining task is now to establish conditions for the ϵ-convergence of its error $e(t)$, i.e., $\lim_{t\to\infty}||e(t)|| \leq \epsilon$, where $e(t)$ is the solution of the following observer error system

$$\begin{aligned}\dot{e}(t) =&(N_1 + ZN_2)e(t) + (N_{d1} + ZN_{d2})e(t-\tau)\\&+ LB(u(t) - u(t-\tau)),\end{aligned} \tag{6.17}$$

obtained by substituting (6.11) into condition 1) of Theorem 24. Note that from Assumption 2, we also have $||u(t) - u(t-\tau)|| \leq \bar{\omega}$, where $\bar{\omega} := \left\lceil \frac{\tau}{h} \right\rceil \delta$, and $\left\lceil \frac{\tau}{h} \right\rceil$ is the smallest integer not less than $\frac{\tau}{h}$, the so-called the ceiling function of $\frac{\tau}{h}$.

Before proceeding further, let us introduce additional notation. For a scalar λ, matrices $M, G, H, K \in \mathbb{R}^{m\times m}$, $J \in \mathbb{R}^{m\times(2m+2p)}$, positive-definite symmetric matrices

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ \star & P_{22} & P_{23} \\ \star & \star & P_{33} \end{bmatrix}, Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ \star & Q_{22} & Q_{23} \\ \star & \star & Q_{33} \end{bmatrix} \in \mathbb{R}^{3m\times 3m},$$

$$R = \begin{bmatrix} R_{11} & R_{12} \\ \star & R_{22} \end{bmatrix} \in \mathbb{R}^{2m\times 2m}, \text{ and } S \in \mathbb{R}^{m\times m},$$

a matrix $\varSigma$ is formed as

$$\varSigma = \begin{bmatrix} \varSigma_{11} & \varSigma_{12} \\ \star & -\frac{2\alpha}{\bar{\omega}^2} I \end{bmatrix}, \tag{6.18}$$

where

$$\Sigma_{11} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & P_{12} & \Omega_{16} & \Omega_{17} \\ \star & \Omega_{22} & -e^{-\alpha\tau}Q_{12} & \Omega_{24} & \Omega_{25} & 2\alpha P_{23} & \Omega_{27} \\ \star & \star & -e^{-\alpha\tau}Q_{22} & \Omega_{34} & -e^{-\alpha\tau}Q_{23} & 0 & 0 \\ \star & \star & \star & \Omega_{44} & 0 & P_{13} & 0 \\ \star & \star & \star & \star & -e^{-\alpha\tau}Q_{33} & P_{23} & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} & \Omega_{67} \\ \star & \star & \star & \star & \star & \star & \Omega_{77} \end{bmatrix}, \Sigma_{12} = \begin{bmatrix} MLB \\ 0 \\ 0 \\ \lambda MLB \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

in which

$$\Omega_{11} = Q_{11} + G + G^T + MN_1 + JN_2 + N_1^T M^T + N_2^T J^T + \frac{\tau^2}{4}R_{11} - 2e^{-\alpha\tau}S + 2\alpha P_{11},$$

$$\Omega_{12} = Q_{12} - G + H^T + 2\alpha P_{12}, \Omega_{13} = MN_{d1} + JN_{d2},$$

$$\Omega_{14} = Q_{13} + P_{11} + \lambda N_1^T M^T + \lambda N_2^T J^T - M + \frac{\tau^2}{4}R_{12},$$

$$\Omega_{16} = \frac{4}{\tau}e^{-\alpha\tau}S + 2\alpha P_{13}, \Omega_{17} = P_{13} + K^T - G,$$

$$\Omega_{22} = Q_{22} - e^{-\alpha\tau}Q_{11} - H - H^T + 2\alpha P_{22},$$

$$\Omega_{24} = Q_{23} + P_{12}^T, \Omega_{25} = -e^{-\alpha\tau}Q_{13} + P_{22},$$

$$\Omega_{27} = P_{23} - K^T - H, \Omega_{34} = \lambda N_{d1}^T M^T + \lambda N_{d2}^T J^T,$$

$$\Omega_{44} = Q_{33} + \frac{\tau^2}{4}R_{22} + \frac{\tau^2}{8}S - \lambda M - \lambda M^T,$$

$$\Omega_{66} = -e^{-\alpha\tau}R_{11} - e^{-\alpha\tau}\frac{8}{\tau^2}S + 2\alpha P_{33},$$

$$\Omega_{67} = P_{33} - e^{-\alpha\tau}R_{12}, \Omega_{77} = -e^{-\alpha\tau}R_{22} - K - K^T.$$

Now, we are ready to formulate our new ϵ-convergence condition for (6.17) in the following theorem.

Theorem 25 *For a given positive scalar $\epsilon > 0$, system* (6.17) *is ϵ-convergent if there exist a scalar λ, a positive scalar α, a matrix $J \in \mathbb{R}^{m\times(2m+2p)}$, a non-singular matrix $M \in \mathbb{R}^{m\times m}$, matrices $G, H, K \in \mathbb{R}^{m\times m}$, and positive-definite symmetric matrices $P \in \mathbb{R}^{3m\times 3m}$, $Q \in \mathbb{R}^{3m\times 3m}$, $R \in \mathbb{R}^{2m\times 2m}$ and $S \in \mathbb{R}^{m\times m}$, such that the* *following conditions hold*

$$P \geq \begin{bmatrix} \frac{1}{\epsilon^2}I & 0 & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix}, \tag{6.19}$$

$$\Sigma \leq 0. \tag{6.20}$$

Moreover, matrix Z is obtained as $Z = M^{-1}J$.

Proof. Let us define

$$\begin{aligned} a^T(t) &= [e^T(t) \quad e^T(t-\frac{\tau}{2}) \quad \int_{t-\tau/2}^{t} e^T(s)ds], \\ b^T(s) &= [e^T(s) \quad e^T(s-\frac{\tau}{2}) \quad \dot{e}^T(s)], \\ c^T(s) &= [e^T(s) \quad \dot{e}^T(s)]. \end{aligned}$$

Consider the following Lyapunov-Krasovskii functional

$$V = V_1 + V_2 + V_3 + V_4, \tag{6.21}$$

where

$$\begin{aligned} V_1 &= a^T(t)Pa(t), \\ V_2 &= \int_{t-\frac{\tau}{2}}^{t} b^T(s)e^{2\alpha(s-t)}Qb(s)ds, \\ V_3 &= \frac{\tau}{2}\int_{-\frac{\tau}{2}}^{0}\int_{\beta}^{0} c^T(t+s)e^{2\alpha(s-t)}Rc(t+s)dsd\beta, \\ V_4 &= \int_{-\frac{\tau}{2}}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{e}^T(s)e^{2\alpha(s-t)}S\dot{e}(s)dsd\lambda d\theta. \end{aligned}$$

By taking the time derivative of $V(t)$ along the solution of system (6.17), we can obtain

$$\begin{aligned} \dot{V}_1 &= 2a^T(t)P\dot{a}(t) + 2\alpha a^T(t)Pa(t) - 2\alpha V_1, \\ \dot{V}_2 &= b^T(t)Qb(t) - b^T(t-\frac{\tau}{2})e^{-\alpha\tau}Qb(t-\frac{\tau}{2}) - 2\alpha V_2, \\ \dot{V}_3 &= \frac{\tau^2}{4}c^T(t)Rc(t) - \frac{\tau}{2}\int_{t-\frac{\tau}{2}}^{t} c^T(s)e^{-\alpha\tau}Rc(s)ds - 2\alpha V_3, \\ \dot{V}_4 &= \frac{1}{2}\frac{\tau^2}{4}\dot{e}^T(t)S\dot{e}(t) - \int_{-\frac{\tau}{2}}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)e^{-\alpha\tau}S\dot{e}(s)dsd\theta - 2\alpha V_4. \end{aligned}$$

Now by using Lemma 6, we have

$$\begin{aligned}\dot{V}+2\alpha V \leq\ & 2a^T(t)P\dot{a}(t)+2\alpha a^T(t)Pa(t)+b^T(t)Qb(t)\\ & -b^T(t-\frac{\tau}{2})e^{-\alpha\tau}Qb(t-\frac{\tau}{2})+\frac{\tau^2}{4}c^T(t)Rc(t)\\ & -\Big(\int_{t-\frac{\tau}{2}}^{t}c(s)ds\Big)^Te^{-\alpha\tau}R\Big(\int_{t-\frac{\tau}{2}}^{t}c(s)ds\Big)\\ & +\frac{1}{2}\frac{\tau^2}{4}\dot{e}^T(t)S\dot{e}(t)-2e^T(t)e^{-\alpha\tau}Se(t)\\ & +\frac{8}{\tau}e^T(t)e^{-\alpha\tau}S\int_{t-\frac{\tau}{2}}^{t}e(s)ds\\ & -\frac{8}{\tau^2}\Big(\int_{t-\frac{\tau}{2}}^{t}e(s)ds\Big)^Te^{-\alpha\tau}S\Big(\int_{t-\frac{\tau}{2}}^{t}e(s)ds\Big).\end{aligned}\tag{6.22}$$

On one hand, from the Newton-Leibnitz formula, we have

$$0=2\Big(e^T(t)G+e^T(t-\frac{\tau}{2})H+\int_{t-\frac{\tau}{2}}^{t}\dot{e}^T(s)dsK\Big)\Big(e(t)-e(t-\frac{\tau}{2})-\int_{t-\frac{\tau}{2}}^{t}\dot{e}(s)ds\Big).\tag{6.23}$$

On the other hand, by using the descriptor method, we have

$$\begin{aligned}0=\ & 2\Big(e^T(t)M+\dot{e}^T(t)\lambda M\Big)\Big(-\dot{e}(t)+(N_1+ZN_2)e(t)\\ & +(N_{d1}+ZN_{d2})e(t-\tau)+LB(u(t)-u(t-\tau))\Big).\end{aligned}\tag{6.24}$$

Thus, by denoting $J=MZ$, $\omega(t)=u(t)-u(t-\tau)$,

$$\begin{aligned}d^T(t)=[&e^T(t)\ \ e^T(t-\frac{\tau}{2})\quad e^T(t-\tau)\quad \dot{e}^T(t)\quad \dot{e}^T(t-\frac{\tau}{2})\\ & (\int_{t-\frac{\tau}{2}}^{t}e(s)ds)^T\ (\int_{t-\frac{\tau}{2}}^{t}\dot{e}(s)ds)^T\ \omega^T(t)]\end{aligned}$$

and incorporating (6.23) and (6.24) into (6.22), we can now obtain

$$\dot{V}+2\alpha V-\frac{2\alpha}{\bar{\omega}^2}\omega^T(t)\omega(t)\leq d^T(t)\Sigma d(t)\leq 0.\tag{6.25}$$

By Lemma 5, there exists a positive functional $\xi(.)$ such that

$$V\leq 1+\xi(\varphi(\theta))e^{-2\alpha t},\forall t\geq 0.\tag{6.26}$$

From condition (6.19), we have

$$V\geq\frac{1}{\epsilon^2}||e(t)||^2.\tag{6.27}$$

From (6.25), (6.26) and by using the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, $\forall a, b \geq 0$, we finally obtain

$$||e(t)|| \leq \epsilon + \epsilon\sqrt{\xi(\varphi(\theta))}e^{-\alpha t}, \forall t \geq 0, \tag{6.28}$$

which implies that $e(t)$ is ϵ-convergent with an exponential rate α. The proof is completed. □

Remark 13 It should be noted that conditions (6.19) and (6.20) are not LMIs because of product terms αP_{11}, αP_{12} and so on. However, the feasible solution of these conditions can be found by using the method, proposed in Remark 2.

Based on the above development, we propose a computational algorithm as stated in the following.

Design Algorithm 1.

Step 1: Given a matrix L, check if condition (6.6) holds. If it does, proceed to step 2.

Step 2: From (6.12)-(6.13), obtain matrices N_1, N_2, N_{d1} and N_{d2}.

Step 3: Obtain matrices M, G, H, K, J, P, Q, R, and S by solving matrix inequalities (6.19) and (6.20), as suggested in Remark 6.2.

Step 4: Obtain $Z = M^{-1}J$ and matrices N, N_d, U and F from (6.10), (6.11). Then D_1, D_2 and E are derived as $D_1 = U + NF$, $D_2 = N_dF$ and $E = LB - FCB$. The observer design is completed.

6.3.2 Systems with output delay and instantaneous input

In the case when the control input is available without delay ($\tau_1 = 0$), its instantaneous information is useful for improving the estimation accuracy. Indeed, by augmenting (6.2) with $u(t)$ via a gain matrix E_1, we can derive conditions for exponential convergence of the observer error. For this, the proposed observer (6.2) for estimating the partial state $z(t)$, in the case $\tau = \tau_2, \tau_1 = 0$, now becomes:

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + N_d\zeta(t-\tau) + D_1 y(t-\tau) + D_2 y(t-2\tau) + Eu(t-\tau) + E_1 u(t) \\ \hat{z}(t) &= \zeta(t) + Fy(t-\tau), \end{aligned} \tag{6.29}$$

where $\zeta(t) \in \mathbb{R}^m$ is again the observer state with a continuous initial condition $\zeta(\theta) = \phi(\theta)$, $\forall \theta \in [-\tau, 0]$.

Theorem 26 *For system (6.1) subject to output delay only, the estimate $\hat{z}(t)$ obtained by observer* (6.29) *converges exponentially to $z(t)$ for any initial conditions $x(0)$, $\phi(\theta)$ and control $u(t)$ if the following conditions hold*

$$\begin{aligned} &1') \quad \dot{e}(t) = Ne(t) + N_d e(t-\tau) \quad \textit{is exponentially stable,} \\ &2') \quad LA - NL = 0, \\ &3') \quad (NF - D_1)C - N_d L - FCA = 0, \\ &4') \quad (N_d F - D_2)C = 0, \\ &5') \quad FCB + E = 0, \\ &6') \quad LB - E_1 = 0. \end{aligned}$$

Proof. The proof for Theorem 26 can be obtained similarly as in the proof of Theorem 24 by noting that the time derivative of $e(t)$ in this case is

$$\begin{aligned} \dot{e}(t) &= L\dot{x}(t) - \dot{\hat{z}}(t) = L\dot{x}(t) - \dot{\zeta}(t) - F\dot{y}(t-\tau) \\ &= L\left(Ax(t) + Bu(t)\right) + N\left(e(t) - Lx(t) + Fy(t-\tau)\right) \\ &\quad + N_d\left(e(t-\tau) - Lx(t-\tau) + Fy(t-2\tau)\right) \\ &\quad - D_1 y(t-\tau) - D_2 y(t-2\tau) - Eu(t-\tau) - E_1 u(t) \\ &\quad - FC\left(Ax(t-\tau) + Bu(t-\tau)\right). \end{aligned}$$

Similarly, by substituting $y(t-\tau) = Cx(t-\tau)$,$y(t-2\tau) = Cx(t-2\tau)$ and collecting like terms, the following error dynamics equation is obtained

$$\begin{aligned}\dot{e}(t) = {} & Ne(t) + N_d e(t-\tau) + (LA - NL)x(t) \\ & + (NFC - D_1C - N_dL - FCA)x(t-\tau) \\ & + (N_dFC - D_2C)x(t-2\tau) \\ & - (FCB + E)u(t-\tau) + (LB - E_1)u(t).\end{aligned}$$

Again, to complete the design, unknown matrices in observer (6.28) need to be determined. The solution to the matrix equations 2')-4') of Theorem 26 can be obtained in the same way as the solution to the matrix equations 2)-4) of Theorem 24, and hence not detailed here. Note that from conditions 5') and 6') of Theorem 26, matrices E and E_1 can be obtained respectively as

$$E = -FCB, \quad E_1 = LB. \tag{6.30}$$

□

The remaining design task is now to derive conditions for exponential stability of the observer error system defined in condition 1') of Theorem 6.3. By substituting (6.10) into condition 1'), we obtain the observer error system for observer (6.29) as

$$\dot{e}(t) = (N_1 + ZN_2)e(t) + (N_{d1} + ZN_{d2})e(t-\tau). \tag{6.31}$$

We now establish the new sufficient conditions for exponential stability of the estimation error dynamics (6.30).

Theorem 27 *The observer error system* (6.31) *is exponentially stable with a given exponential rate* α *if there exist a scalar* λ*, matrices* G, H, K, J*, a non-singular matrix* M*, and positive-definite symmetric matrices* S*,* P*,* Q *and* R *such that the following condition holds*

$$\Sigma_{11} < 0, \tag{6.32}$$

where Σ_{11} *is defined in (6.18). Moreover, matrix* Z *is obtained as* $Z = M^{-1}J$*.*

Proof. By using the same Lyapunov-Krasovskii functional as given in the proof of Theorem 2 within the proposed technique, and taking its time derivative, we obtain:

$$\dot{V} + 2\alpha V \leq g^T(t)\Sigma_{11}g(t) \leq 0, \tag{6.33}$$

where matrix Σ_{11} is defined in (6.18) and where
$g(t) = [e^T(t) \;\; e^T(t-\frac{\tau}{2}) \quad e^T(t-\tau) \quad \dot{e}^T(t) \quad \dot{e}^T(t-\frac{\tau}{2}) \quad (\int_{t-\frac{\tau}{2}}^{t} e(s)ds)^T \; (\int_{t-\frac{\tau}{2}}^{t} \dot{e}(s)ds)^T]^T.$

By using Lemma 7, condition (6.33) implies that system (6.31) is exponentially stable with exponential rate α. This completes the proof of Theorem 27. □

Design Algorithm 2.

Step 1: Given a matrix L, obtain $E_1 = LB$ and check if condition (6.6) holds. If it does, proceed to step 2.

Step 2: From (6.12)-(6.13), obtain matrices N_1, N_2, N_{d1} and N_{d2}.

Step 3: Obtain matrices $M, G, H, K, J, P, Q, R,$ and S by solving the matrix inequality (6.32), as suggested in Remark 2.

Step 4: Obtain $Z = M^{-1}J$ and matrices N, N_d, U and F from (6.10)-(6.11). Then D_1, D_2 and E are obtained as $D_1 = U + NF$, $D_2 = N_dF$, and $E = -FCB$. The observer design is completed.

6.4 Numerical Examples

In this section, numerical examples are given to illustrate the application of the obtained results and the merits of the proposed approach.

6.4.1 Example 6.1

We first illustrate the design procedure of the approach in this example. Consider the following system with different delays in the input and output:

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 \\ 0 & -1.3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$
$$z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t),$$

where the input and output are subject to delays respectively $\tau_1 = 1$ and $\tau_2 = 0.9$, and the difference $u(t) - u(t - t_1)$ is bounded by 0.1, for all $t_1 \in [0 \;\; 0.1]$, $\forall t \geq 0$. By using Design Algorithm 1, we first check the matrix rank condition in (6) with $L = [0 \;\; 1]$ (Step 1). After obtaining Ψ, Φ from (8) and N_1, N_2, N_{d1} and N_{d2} are obtained from (13),(14) (Step 2). Following Remark 13 and using the LMI solution in Step 3, we obtain, for $\epsilon = 0.075$, $\lambda = 0.6$ and $\alpha = 1.05$, matrices $N = -1.3$, $N_d = -0.114$, $U = 0.228$ and $F = 0.114$ can be computed from (11), (12). Step 4 is then completed with the calculation of the observer parameters as $D_1 = 0.08$, $D_2 = -0.013$ and $E = 0.886$. Thus, the following observer will estimate $z(t)$ to an error bound of $\epsilon = 0.075$ with an exponential rate $\alpha = 1.05$:

$$\dot{\zeta}(t) = -1.3\zeta(t) - 0.114\zeta(t-1) + 0.08y(t-1) - 0.013y(t-2) + 0.886u(t-1),$$
$$\hat{z}(t) = \zeta(t) + 0.114y(t-1).$$

For the sake of numerical simulation, the control input is chosen as $u(t) = 0.1\sin(t) - 1, t \geq 0$, which satisfies assumption $||u(t) - u(t - t_1)|| \leq 0.1$, $0 \leq t_1 \leq 0.1$, $t \geq 0$. As shown in Fig. 6.1 and Fig. 6.2, the estimate $\hat{z}(t)$ exponentially converges to the state $z(t) = x_2(t)$ within the bound $\epsilon = 0.075$ and the estimation error of the observer also exponentially converges to the ϵ- bound.

Now we illustrate the case when only the output is subject to a time delay, i.e., $\tau_2 = 1$ and $\tau_1 = 0$. By using Design Algorithm 2, we first check again condition (6)

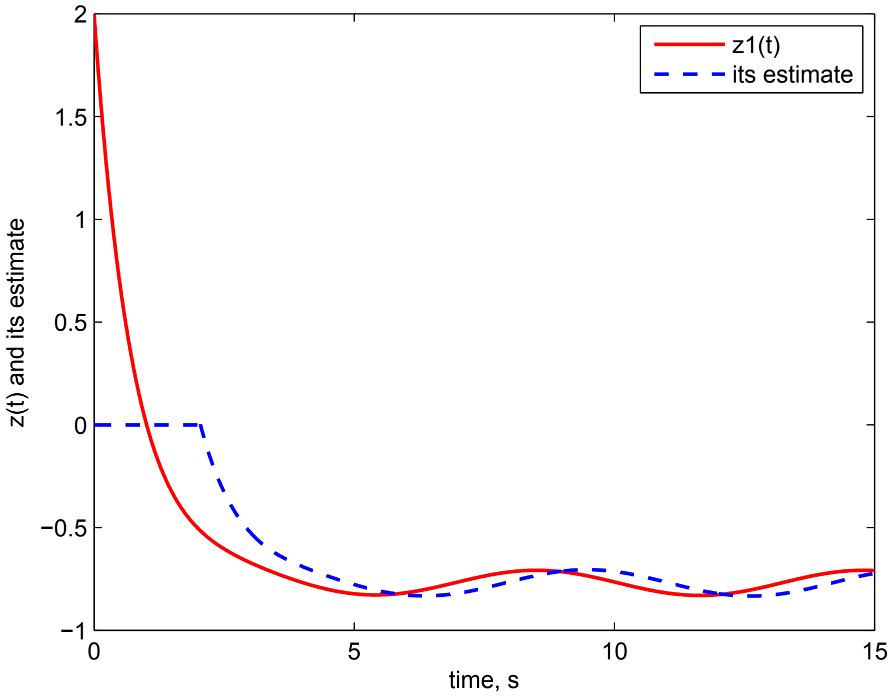

Figure 6.1 : System responses with $\tau_1 = 1$ and $\tau_2 = 0.9$.

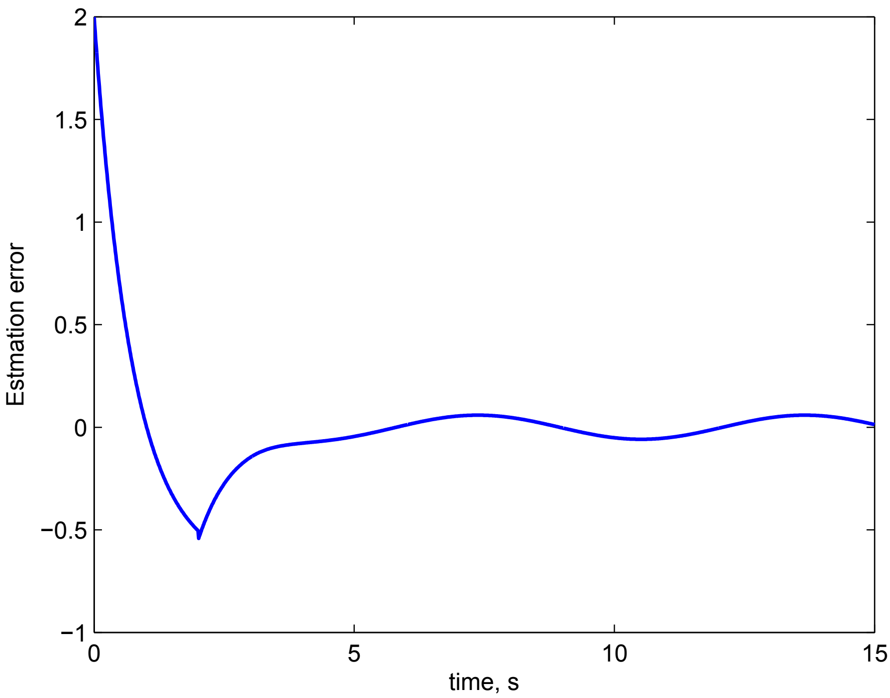

Figure 6.2 : Estimation error for the case $\tau_1 = 1$ and $\tau_2 = 0.9$.

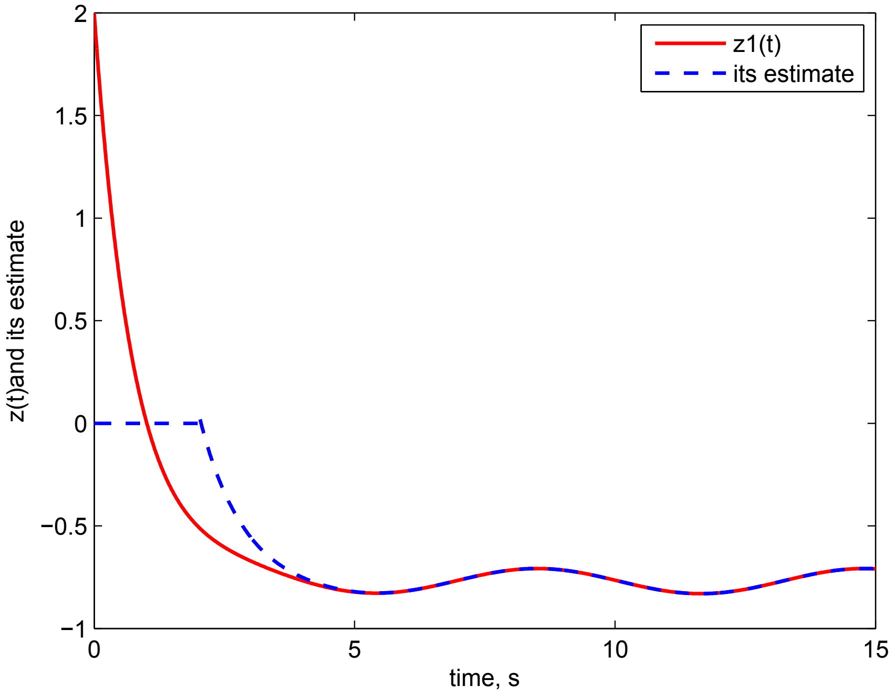

Figure 6.3 : System responses with $\tau_1 = 0$ and $\tau_2 = 1$.

and obtain $E_1 = 1$ (Step 1). We then proceed similarly with Steps 2-4 to obtain $N = -1.3$, $N_d = -0.1115$, $U = 0.223$ and $F = 0.1115$ for $\lambda = 0.2$ and $\alpha = 1.4$. To complete the procedure, the observer parameters are then calculated as $D_1 = 0.0781$, $D_2 = -0.0124$ and $E = -0.1115$. Thus, the following observer, of the form (6.29), can estimate exponentially $z(t)$ with an exponential rate $\alpha = 1.4$:

$$\begin{aligned}\dot{\zeta}(t) =& -1.3\zeta(t) - 0.1115\zeta(t-1) + 0.0781y(t-1)\\ & -0.0124y(t-2) - 0.1115u(t-1) + u(t)\\ \hat{z}(t) =& \zeta(t) + 0.1115y(t-1).\end{aligned}$$

With the same input $u(t) = 0.1\sin(t) - 1, t \geq 0$, the time responses of $z(t)$ and its estimate $\hat{z}(t)$ and the estimation error are shown in Fig. 6.3 and Fig. 6.4. It can be seen clearly that the estimate $\hat{z}(t)$ approaches to the state $z(t) = x_2(t)$ after 5 sec while the estimation error exponentially converges to zero.

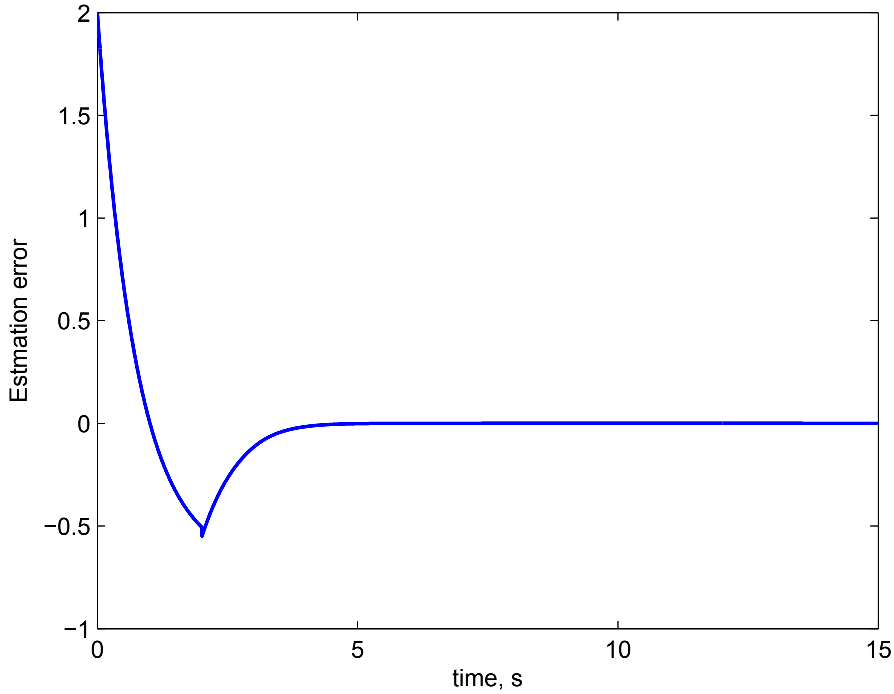

Figure 6.4 : Estimation error for the case $\tau_1 = 0$ and $\tau_2 = 1$.

6.4.2 Example 6.2

This example considers a practical master-slave tele-operated system by a two-finger robot hand [127], where the fingers are driven by DC servomotors and a belt-pulley mechanism. By ignoring the disturbance effects come from the spring, the mathematical model of the two-finger robot hand is given as follows:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -26 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 4.54 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), x(t) = [x_1(t) \quad x_2(t)]^T \\ z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t). \end{aligned}$$

Due to the transmission delay between the master and slave in the forward path and in the backward path, the control input at the slave side is subject to a delay $\tau_1 = 1$ sec. and the output displacement is available at the master side also with a delay

$\tau_2 = 1$ sec. The DC servomotor rotation angle is measured by a rotary encoder, and the finger displacement $x_1(t)$ can be obtained by multiplying the pulley radius by the rotation angle. However, in teleoperation of the control system, we need to have also information of the slave finger velocity. Thus, a partial state observer is designed to estimate the speed $x_2 = \dot{x}_1$ of the finger. Different with the approach given in [127], where τ_1, τ_2 are rendered to state delays in an augmented system, here we construct an observer using delayed information of both the input and output. By using Design Algorithm 1, we obtain $\epsilon = 0.01$, $\lambda = 0.5$, $\alpha = 1.0001$, $N = -26$, $N_d = -0.0377$, $D_1 = -0.98025$, $D_2 = -0.00142$, $E = 4.5$, $F = 0.0377$. Thus, with $\tau = \max\{\tau_1, \tau_2\} = 1$, an observer of the form (3) is obtained as:

$$\begin{aligned}\dot{\zeta}(t) =& -26\zeta(t) - 0.0377\zeta(t-1) - 0.98025y(t-1) - 0.00142y(t-2) + 4.5u(t-1),\\ \hat{z}(t) =& \zeta(t) + 0.0377y(t-1).\end{aligned}$$

After a delay of $\tau_1 + \tau_2 = 2$ sec., this observer can estimate the speed of the finger $z(t) = x_2(t) = \dot{x}_1(t)$ to an error bound of $\epsilon = 0.01$ with an exponential rate $\alpha = 1.0001$. Since the motion of the finger is restricted by the physical mechanism of the two-finger robot hand structure, the control input is therefore required to be changed slowly. Here, for the sake of illustration of the proposed approach, the control input chosen as $u(t) = 0.45 + 0.01\sin(t), t \geq 0$, which satisfies assumption $||u(t) - u(t - t_1)|| \leq 0.01$, $0 \leq t_1 \leq 0.1$, $t \geq 0$. As shown in Fig. 6.5 and Fig. 6.6, the estimated speed of the finger $\hat{z}(t)$ exponentially converges to the actual speed $z(t) = \dot{x}_1(t)$ within the given bound $\epsilon = 0.01$. To judge the state estimation performance, it can be seen that the estimation error practically converges to ϵ-bound after some initial conditions of the system.

6.4.3 Example 6.3

In this example, we compare effectiveness of the observer design approach by considering the same system given in [31] and [126] with the observer error dynamics

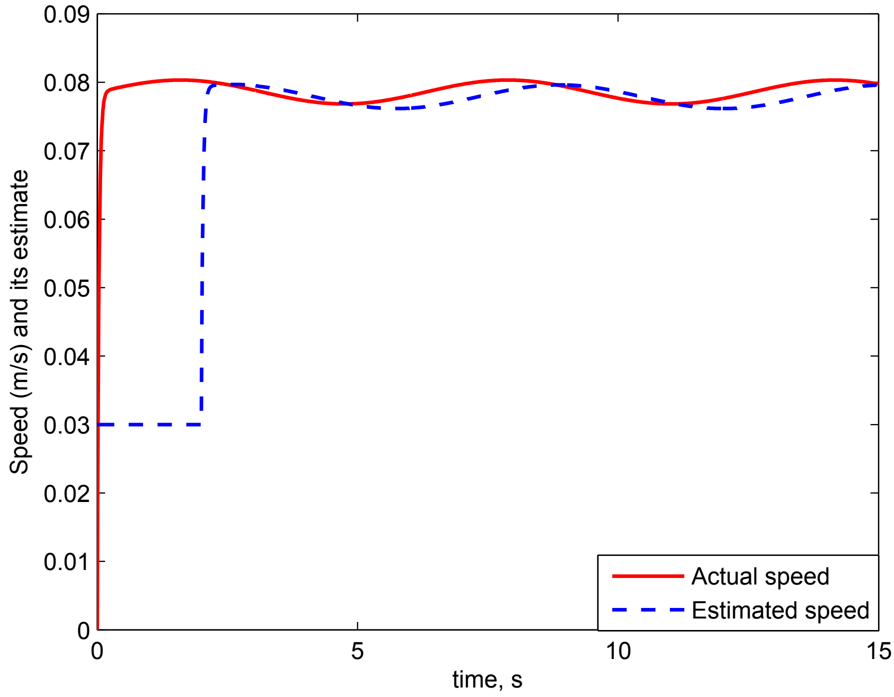

Figure 6.5 : System responses of a master-slave tele-operated two-hand robot.

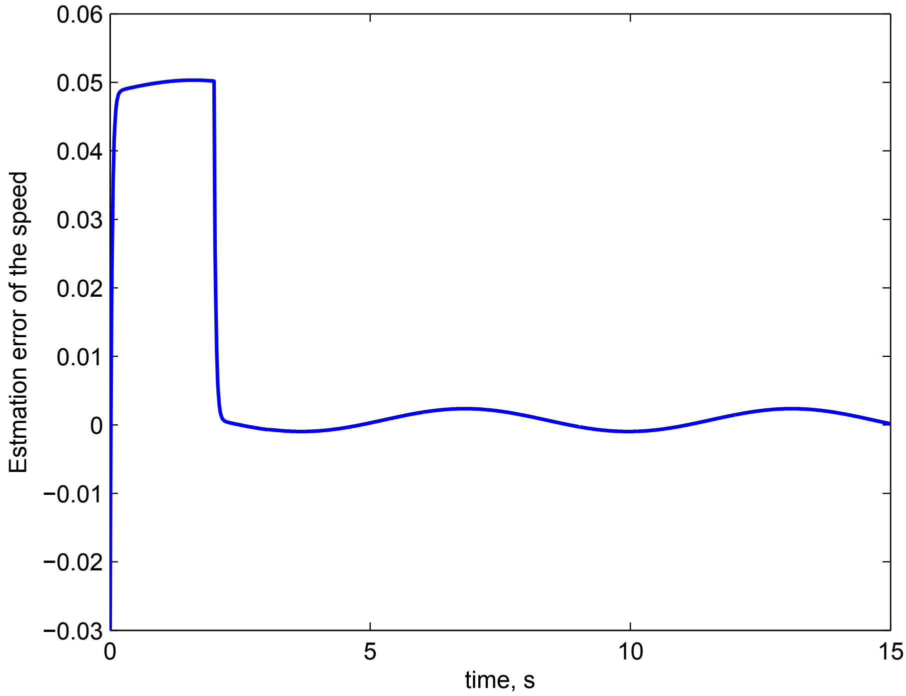

Figure 6.6 : Estimation error of the finger speed.

(6.31), where

$$N = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \ N_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \ N_2 = N_{d2} = 0.$$

The maximum value of time delay τ to ensure its asymptotic stability by this approach can be found as $\tau = 5.863$; by the approaches proposed in [31] and [126] are respectively $\tau = 5.717$ and $\tau = 5.30$. Thus, compared with the results obtained by stability criteria given therein, this approach appears to be less conservative. This is accounted for by the proposed combination of the delay-decomposition technique and the augmentation of a triple-integral term in the selected Lyapunov-Krasovskii functional.

6.5 Conclusion

In this chapter, we have presented a novel approach to design partial state observers with minimum order for linear systems that are subject to time delays in both the measured output and the control input. By coupling the chosen Lyapunov-Krasovskii functional containing a triple-integral term with the delay-decomposition technique, improved conditions are derived to guarantee that the estimation error dynamics converges within a given bound ϵ. In the case when the control input is instantaneously available without delay, new conditions are also derived for the existence of a minimal-order observer that can guarantee exponentially asymptotic convergence of the estimation error. Numerical examples are provided to illustrate the design procedure, practicality and effectiveness of the proposed approach.

Chapter 7

Conclusion

7.1 Summary

Throughout the thesis, the L-K method has been used to address some issues of stability analysis, reachable set bounding, discrete-time quasi-sliding mode control and state estimation for a class of time-delay systems. For stability analysis, some novel LMI based-conditions for exponential stability for a class of linear discrete-time systems with time-varying delay have been derived. Then, the obtained results were extended to address the problem of reachable set bounding for linear discrete-time systems with time-varying delay and bounded disturbances. For this, a new idea was proposed to obtain the possible smallest reachable set bound. Apart from the LKFs in Chapter 3, the problem of robust discrete-time quasi-sliding mode control of linear systems with time-varying delay and unmatched disturbances has been addressed. A new control design methodology for sliding surface design was proposed in which the switching gain can be explicitly obtained from the solution of the LMIs. Finally, we consider the new problem of partial state observer design for linear systems that are subject to time delays in both the system measured output and the control input. Here, a new observer was proposed to estimate the linear state functional and the existence conditions for such an observer were derived to guarantee that the estimate exponentially converges to the original state in a finite time. Necessary and sufficient conditions for ϵ- convergence of the estimation error were also provided in terms of LMIs.

7.2 Thesis contributions

The contributions of the thesis are summarized as follows:

- By constructing a suitable set of Lyapunov-Krasvoskii functionals, combined with the use of the delay decomposition and the reciprocally convex approach, new exponential stability conditions for a class of linear discrete-time systems with interval time-varying delay were derived in terms of LMIs.

- A new approach for the reachable set estimation problem of linear discrete-time systems with time-varying delay and bounded disturbances was developed in Chapter 4. By using the Lyapunov-Krasvoskii method and the delay decomposition approach, sufficient conditions for the existence of ellipsoid-based bounds of reachable sets were derived in terms of matrix inequalities. The novel feature of this approach was to propose an idea to minimise the projection distances of the ellipsoids on the h-th axis with various convergence rates instead of minimising their radius with a single exponential rate; a much smaller reachable set bounding can thus be obtained from the intersection of these ellipsoids.

- Some developments of the sliding mode control design for a class of linear systems with time-varying delay and bounded disturbances are introduced in Chapter 5. By using the Lyapunov-Krasvoskii method, combined with the reciprocally convex approach, new sufficient conditions for the existence of a stable sliding surface were derived in terms of LMIs. These conditions also guaranteed that in the induced sliding dynamics, all the state trajectories were exponentially convergent within a ball whose radius can be minimised. A robust quasi-sliding mode controller was then proposed to drive the system state trajectories to the sliding surface in a finite time and maintain it there after subsequent time.

- A new approach to design the linear functional state observer for linear systems with time delays in the measured output as well as the control input was presented in Chapter 6. The information of the delayed output and delayed input was used to design a minimal order observer to estimate system state functionals in which the estimation error dynamics was ϵ-convergence.

7.3 Future works

The obtained results of this thesis can be applied to some practical problems in the chemical industry, fault detection systems, wireless communications, mechanical systems, power systems and systems biology, where time delay and external disturbances need to be taken into account. Based on these results, some potential works will be considered for implementation in the future to real physical systems. Some future works may be considered as follows:

- For stability analysis and reachable set bounding, the delay-dependent stability conditions may be further reduced by using an improved set of appropriate LKFs, combined with Wirtinger based inequality and the reciprocally convex approach.

- The existing works on the problem of sliding mode control for linear systems subject to time-varying delay and bounded disturbances, often assume that the system states are available for sliding surface design and accessible to the control law. However, in practice, all of these states are not physically available for design. As a consequence, the full-state feedback sliding mode control law cannot be implemented. The future work could focus on the problem of dynamic output feedback sliding mode control for a class of linear systems with time-varying delay and unmatched bounded disturbances where the sliding function is considered as a linear state functional to be estimated.

- The observer, proposed in this thesis, deals with the problem of designing functional observers when there are constant time delays associated in both measurement output and input. However, in many practical engineering systems, the delay is time-varying delay and belonging to an interval. Therefore, it is more interesting to extend the current work to the problem of linear functional observer for systems with different time-varying delay in the input and output. Moreover, the proposed structure is still complicated to design so further research is needed to develop reduced-order decentralized linear functional observers with new architectures to cope with delayed information and any type of decentralized information structures.

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