Analytical solutions for the ray perturbation in depth-varying media

Beverley J. Moore

School of Mathematical Sciences, University of Technology, Sydney, PO Box 123, Broadway, NSW 2007, Australia

Accepted 1992 December 21. Received 1992 December 10; in original form 1992 July 20

SUMMARY
In this paper, we present a simple closed-form solution to the problem of ray propagation through media which have weak lateral inhomogeneities superimposed on an unperturbed velocity field which varies with depth only. We use a Lagrangian formulation of ray-perturbation theory which incorporates corrections for differences in the arclength parameter in perturbed and unperturbed media at every instant along the ray path. We show how it is possible to reduce the first-order solution for the two-point boundary-value problem to the solution of a single initial-value problem. In this way, the total Green's function for two-point boundary-value problems can be related to the propagator for initial-value problems. Thus the analytical expressions derived for the propagators in this paper may be used to determine analytical expressions for the Green's function of the corresponding two-point boundary-value problem. The application of these results to tomographic reconstruction problems is discussed.

Key words: boundary value problems, propagators (analytical), Green's function, ray perturbation theory, tomographic reconstructions.

1 INTRODUCTION
In recent years, a number of papers have presented results in ray-perturbation theory using both Hamiltonian (Chapman 1985; Farra & Madariaga 1987; Farra, Virieux & Madariaga 1989; Virieux 1991) and Lagrangian formulations [Moore (1991) and Snieder & Sambridge (1992) which assign different meanings to the arclength parameter in the problem]. Moore (1991) used a two-step approach, initially using distance along the perturbed ray as independent parameter and then (in Section 4) applying the necessary corrections to account for the fact that it is a different ray which passes through a particular receiver in the presence of perturbations and that the arclength from source to receiver along this reference ray will differ from the arclength between source and receiver in the absence of perturbations. On the other hand, Snieder & Sambridge (1992, to be referred to hereafter as S&S) apply such a correction at every instant along the ray path and take account of the differences in arclength between source and receiver in perturbed and unperturbed media when setting up the basic ray-tracing equations for perturbations to the ray trajectory. As they note, such corrections are incorporated into the Hamiltonian formalism of Farra & Madariaga (1987) through a non-trivial differentiation of the geometric term $h_o$ in their Hamiltonian although few choose to use such a parametrization of the ray.

Most previous papers have focused on the numerical determination of the propagators for the problem rather than analytical approaches. In this paper, we present a simple closed-form solution to the problem of ray propagation through media which have weak lateral inhomogeneities superimposed on an unperturbed velocity field which varies with depth only. We adopt the approach of S&S in as much as we incorporate the corrections for differences in arclength in perturbed and unperturbed media at every instant along the ray path. Although the basic ray-tracing equations for perturbations to the ray trajectory appear much more complicated than in Moore (1991), we show that the solutions to the initial value problem are, in fact, greatly simplified in this formulation. Furthermore, we show how it is possible to reduce the first-order solution for the two-point boundary-value problem to the solution of a single initial-value problem. In this way, the total Green's function $G$ of S&S for two-point boundary-value problems can be related to the propagator $\Pi$ for initial-value problems. Thus the analytical expressions to be derived for the propagators in this paper may also be used to determine analytical expressions for the Green's function of the corresponding two-point boundary-value problem.

2 PERTURBATIONS TO RAY GEOMETRY
We consider the propagation of seismic rays through an isotropic elastic medium whose velocity field deviates only slightly from the depth-varying field $a_{0i}(x_3) = a_{i0}^{-1}(x_3)$ where
\(x_2\) denotes either the vertical coordinate of position in plane geometry or \(|\mathbf{x}|\) in spherical geometry. In this case, we represent the slowness field by
\[
\mathbf{u}(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}_0)[1 + \varepsilon \mathbf{f}(\mathbf{x})], \quad |\varepsilon| < 1
\] (1)
where \(\varepsilon\) is a small parameter measuring the deviation of the medium from its unperturbed state.

Like S&S, we shall use \(s\) to denote arclength on the perturbed ray and \(s_0\) to denote arclength on the unperturbed ray. These quantities are related to first order by the equation
\[
\frac{\partial s_0}{\partial s} = 1 - \varepsilon \frac{d\mathbf{x}^0}{ds_0} \cdot \frac{d\mathbf{x}'}{ds} + O(\varepsilon^2)
\] (2)
when we represent \(\mathbf{x}\) by means of its Taylor series in \(\varepsilon\):
\[
\mathbf{x}(\mathbf{s}_0;\varepsilon) = \mathbf{x}^0(\mathbf{s}_0) + \varepsilon \mathbf{x}'(\mathbf{s}_0) + O(\varepsilon^2).
\] (3)
It should be noted that whilst \(s_0\) has unit scale length on the unperturbed ray, it has a different scale length on the perturbed ray because it no longer measures arclength along that ray.

Using \(s_0\) rather than \(s\) as the parameter along the ray introduces several extra terms into eq. (18) of Moore (1991). The standard ray equation becomes (to first order in small quantities)
\[
\frac{d\mathbf{x}}{ds} \cdot \left(\frac{d\mathbf{x}}{ds}\right) = \nabla \mathbf{u}
\] (4)
becomes (to first order in small quantities)
\[
(1 - \varepsilon \mathbf{x}' \cdot \mathbf{x}') \frac{d}{ds_0} \left[ \mathbf{u}(1 - \varepsilon \mathbf{x}' \cdot \mathbf{x}') \frac{d\mathbf{x}'}{ds_0} \right] = \nabla \mathbf{u}
\] (5)
where \(\cdot\) denotes differentiation with respect to \(s_0\).

Substituting eqs (1) and (3) into eq. (5) gives a zeroth-order solution \(\mathbf{x}^0(\mathbf{s}_0)\) identical to the ray geometry in the unperturbed depth-varying medium.

The first-order perturbation to the solution satisfies the equation
\[
\frac{d}{ds_0} \left( \frac{d\mathbf{x}^0}{ds_0} \right) + \frac{d}{ds_0} \left[ \mathbf{x}^0 \cdot \nabla \mathbf{u}_0 \right] \frac{d\mathbf{x}'}{ds_0} - \mathbf{x}' \cdot \nabla (\nabla \mathbf{u}_0) = - \left[ \frac{\varepsilon \mathbf{x}' \cdot \mathbf{x}^0}{ds_0} \right] \frac{d}{ds_0} \left[ \mathbf{u}_0 \frac{d\mathbf{x}'}{ds_0} + \frac{d}{ds_0} \left( \mathbf{x}' \cdot \mathbf{x}^0 \right) \frac{d\mathbf{x}'}{ds_0} \right]
\] (6)
where
\[
\nabla^T = \nabla - \frac{d\mathbf{x}^0}{ds_0} \left( \frac{d\mathbf{x}^0}{ds_0} \right) \cdot \nabla.
\]
This is the same as eq. (24) in S&S. It is not difficult to show that this equation has no component directed along the ray as the scalar product of each side of the equation with \(\mathbf{x}'\) is identically zero. As noted in S&S, the use of \(s_0\) as a parameter on the ray makes perturbations in ray position along the reference ray irrelevant. Strictly speaking, it is not the choice of the independent parameter \(s_0\) that defines the ray perturbation of S&S but the prescription that the ray perturbation has no component along the reference ray.

Thus the solution of eq. (6) is subject to the constraint
\[
\mathbf{x}' \cdot \frac{d\mathbf{x}^0}{ds_0} = 0.
\] (7)

The terms within the brackets in eq. (6) are those which are additional to eq. (18) of Moore (1991) and result from the change in the parameter along the ray from \(s\) to \(s_0\). Despite the apparent complexity of eq. (6), the solution of the initial value problem turns out to be quite simple.

Since we wish to study the geometry of the ray which passes through a particular reference point \(\mathbf{x}'(0)\) in a particular direction \(d\mathbf{x}'/ds_0(0)\), the appropriate initial conditions to use with eq. (6) are:
\[
\mathbf{x}'(0) = \mathbf{0} \quad \text{and} \quad \frac{d\mathbf{x}'}{ds_0}(0) = \mathbf{0}.
\] (8)

However, eq. (6) with \(f_j = 0\) effectively becomes the equation for paraxial rays in the unperturbed medium. It describes how small variations in the initial position and direction of the ray are propagated along the ray. (In this context, \(\varepsilon\) should be interpreted as a small parameter measuring the variation in the initial conditions rather than its usual meaning of measuring the deviation of the medium from its unperturbed state.) Therefore, eq. (6) is solved in Appendices A and B subject to non-zero initial conditions:
\[
\mathbf{x}'(0) = \mathbf{q} \quad \text{with} \quad \mathbf{q} \cdot \frac{d\mathbf{x}^0}{ds_0} = 0
\] (9)
and
\[
\frac{d\mathbf{x}^0}{ds_0}(0) = \delta.
\]

In fact, only two components of \(\delta\) are arbitrary as it follows from eq. (7) that the tangential component must satisfy
\[
\mathbf{x}' \cdot \delta = - \frac{1}{u_0}(\mathbf{x}' \cdot \nabla \mathbf{u}_0).
\] (10)

Without loss of generality, we choose a particular orientation of the coordinate axes so that the zeroth-order solution is confined to the plane \(x_2 = 0\).

Thus, we may write
\[
\mathbf{u}_0 \frac{d\mathbf{x}^0}{ds_0} = \begin{cases} \hat{\mathbf{p}} + p_0(\mathbf{s}_0) \hat{\mathbf{k}}, & \text{in plane geometry} \\ \frac{1}{r_0} [p\hat{\mathbf{r}}(\mathbf{s}_0) + p_0(\mathbf{s}_0) \hat{\mathbf{r}}(\mathbf{s}_0)], & \text{in spherical geometry} \end{cases}
\] (11)
where
\[
p_0^2 = \frac{\sqrt{u_0^2(\mathbf{x}_0^2) - p_s^2}}{r_0 \sin^2(r_0)} = \frac{p_s^2}{r_0^2}, \quad \text{in spherical geometry}
\] (12)
and \(p\) is the ray parameter for the reference ray.

It is also convenient to introduce a coordinate \(n\) measuring distance perpendicular to the unperturbed ray in the plane \(x_2 = 0\). The unit vector \(\hat{\mathbf{n}}\) in this direction is given by
\[
u_0 \hat{\mathbf{n}} = \begin{cases} \mathbf{p} - p_0(\mathbf{s}_0) \hat{\mathbf{k}}, & \text{for plane geometry} \\ \left[ p\hat{\mathbf{r}}(\mathbf{s}_0) - p_0(\mathbf{s}_0) \hat{\mathbf{r}}(\mathbf{s}_0) \right], & \text{for spherical geometry} \end{cases}
\] (13)
The solution to eq. (6) is written most succinctly with the aid
of propagator notation (cf. Farra & Madariaga 1987):

\[
\begin{bmatrix}
\mathbf{x}'(s_0) \\
\frac{d\mathbf{x}'}{ds_0}
\end{bmatrix} = \int_0^{s_0} \Pi(s_0, \sigma) \left\{ \mathbf{v}^T f_1(x^0(\sigma)) \right\} d\sigma + \Pi(s_0, 0) \left( \frac{\mathbf{q}}{\delta r} \right)
\]

(14)

where the propagator \( \Pi \) contains four 2-D submatrices \( P_k \) which are diagonal

\[ \Pi(s, \sigma) = \begin{pmatrix} P_1(s, \sigma) & P_2(s, \sigma) \\ P_3(s, \sigma) & P_4(s, \sigma) \end{pmatrix} \]

(15)

These submatrices are determined in the appendices.

In fact, the propagator \( P_2 \) which describes how deflections are propagated along the ray is unchanged from the results of Moore (1991). Once the geometry of the unperturbed ray is known, the components of \( \Pi \) are completely specified by calculating just one of the integrals given in Moore (1991):

\[ S_1(s_0, 0) = \int_0^{s_0} \frac{p\delta(0)u_0^2(s_0)}{u_0(s_0)[p_0^2(s_0)]^{1/2}} ds^0 \]

(16)

\( \Pi \) also depends on the curvature of the unperturbed ray at \( x^0(s_0) \) which is given by

\[ K(s_0) = \left| \frac{d^2 \mathbf{x}^0}{ds_0^2} \right| = \left| \frac{d^2 \mathbf{x}^0}{ds_0} \right| \]

\[ \begin{cases} pu_0^2(s_0) & \text{for plane geometry} \\
\frac{p}{u_0^2(s_0)} & \text{for spherical geometry} \end{cases} \]

(17)

Writing \( \mathbf{e} \) for the initial direction of the ray and listing the \( x \) component before the two component, the submatrices in eq. (15) are the diagonal matrices given by:

\[ P_1(s_0, 0) = \text{diag} \left( R_1(s_0, 0), R_2(s_0, 0) \right) \]

(18)

where

\[ R_1(s_0, 0) = \frac{u_0(0)p_0^2(s_0)}{u_0(s_0)p_0^2(0)} - F(0)S_1(s_0, 0) \]

with

\[ F(0) = \begin{cases} pK(0) & \text{for plane geometry} \\
\frac{p_0K(0)}{p_0^2(0)} & \text{for spherical geometry} \end{cases} \]

and

\[ R_2(s_0, 0) = \begin{cases} 1 & \text{for plane geometry} \\
\frac{u_0(0)r_0(s_0)}{p} \hat{\theta}(s_0) \cdot \hat{\mathbf{e}} & \text{for spherical geometry} \end{cases} \]

(19)

\[ P_2(s_0, 0) = \text{diag} \left( S_1(s_0, 0), S_2(s_0, 0) \right) \]

where

\[ S_2(s_0, 0) = \begin{cases} \int_0^{s_0} \frac{u_0(0)}{u_0(s^0)} ds^0 = \frac{u_0(0)}{p} \left[ x^0_1(s_0) - x^0_1(0) \right] & \text{for plane geometry} \\
- \frac{u_0(0)r_0(s_0)}{p} \hat{\theta}(s_0) \cdot \hat{\mathbf{e}} & \text{for spherical geometry} \end{cases} \]

and

\[ P_2(s_0, 0) = \text{diag} \left[ S_1(s_0, 0), S_2(s_0, 0) \right] \]

(20)

where

\[ R_3(s_0, 0) = F(s_0) \frac{u_0(0)p_0^2(s_0)}{u_0(s_0)p_0^2(0)} - F(0)S_1(s_0, 0) \]

\[ R_4(s_0, 0) = \begin{cases} 0 & \text{for plane geometry} \\
- \frac{u_0(0)}{p} \hat{\mathbf{n}}(s_0) \cdot \mathbf{e} & \text{for spherical geometry} \end{cases} \]

(21)

The elements of the propagator \( \Pi \) are particularly simple in the case of a depth-varying model which has constant gradient in the quadratic slowness:

\[ [u_0(x_3)]^2 = B + Cx_3. \]

(22)

Then we have an exact polynomial solution for the unperturbed-ray geometry,

\[ x^0(w) = x^0(w_0) + p^0(w_0)(w - w_0) + C \frac{1}{2} (w - w_0)^2 \mathbf{k} \]

(23)

where

\[ w - w_0 = \int_0^{s_0} \frac{ds^0}{u_0(s^0)} = \frac{x^0_1(s_0) - x^0_1(0)}{p} = \frac{2}{C} \left[ p_0^2(s_0) - p_0^2(0) \right] \]

(24)

In this case

\[ S_1(s_0, 0) = \frac{p_0^2(0)p_0^2(s_0)}{u_0(s_0)} (w - w_0) \]

and

\[ K(s_0) = - \frac{p}{2} \frac{u_0(s_0)}{u_0(0)} \]

(25)

When \( C \neq 0 \), the \( n \) components of these results differ from the 3-D propagators for the problem quoted at eq. (76) of Farra et al. (1989), eq. (13) of Farra (1992) and at eq. (41) of Virieux (1991). The main difference arises from the choice of coordinates used in this paper. Basically, the result
derived at eq. (14) of this paper gives the ray perturbation in the surface \( s_0 = \text{constant} \) [defined by the constraint at eq. (7)] whereas Farra et al. (1989) give the ray perturbation in the surface \( w = \text{constant} \), which may be quite different when \( C \neq 0 \). In particular, it should be noted that the 3-D propagators quoted in Farra et al. (1989) do not permit any variation in scale length between the parameters \( w \) on perturbed and unperturbed rays; that is, they require corresponding points on perturbed and unperturbed rays to be located at the same \( w \) length from the source. This means that the ray equation implicitly has been solved subject to the constraint \( \mathbf{x} \cdot \mathbf{x} = 1/\|\mathbf{x} \cdot \mathbf{V}_{\beta 0}\| \) which restricts the allowed values of \( q \) and \( \delta \) to be used in paraxial ray-tracing applications of that propagator. With such a choice of parameter, special care also is required in determining the \( w \) value at the endpoint of the ray in the two-point boundary-value problem (it is generally different from the \( w \) value of the endpoint on the unperturbed ray). However, in ray shooting problems where the endpoint location is somewhat different, e.g. continue until the ray crosses a particular surface and the \( w \) value at the endpoint is determined numerically, then one choice of parameter should work as well as the other. Indeed, Farra (1992) gives expressions for adjusting the parameter value in such situations. Nevertheless, as we shall see below, there are times when it is advantageous to have a parameter which corresponds to the simpler geometric constraint presented in this paper.

3 APPLICATION TO TWO-POINT BOUNDARY-VALUE PROBLEMS

The closed-form solution to the initial-value problem given at eq. (14) is easily adapted to the two-point boundary-value problem. We simply use \( \mathbf{x}(\Sigma_0) = 0 \) with \( q = 0 \) to determine \( \delta \), the correction required to the initial direction of the ray to ensure that the perturbed ray passes through the required endpoint:

\[
\delta = -\mathbf{P}^{-1}_2(\Sigma_0, 0) \int_{\Sigma_0}^{\Sigma_0} \mathbf{P}_2(\Sigma_0, \sigma) \mathbf{V}_f \left[ k^0(\sigma) \right] d\sigma
\]

(26)

where \( \Sigma_0 \) denotes the total arclength of the unperturbed ray and the path of integration is the ray which would join source and receiver in the absence of any perturbations.

When the assumptions of first-order perturbation theory are valid for the entire length of the ray joining source and receiver, there is no need to use an iterative procedure to continually update corrections to the initial direction of the ray. Instead, the components of \( \Pi \) should be calculated by integration along the ray which would join source and receiver in the absence of any perturbations; then paraxial ray-tracing terms in eq. (14) are used to propagate the correction \( \delta \) all the way along the ray. Provided the assumptions of first-order perturbation theory are not violated, the method of calculating \( \delta \) in eq. (26) guarantees that the perturbed ray will pass through the desired endpoint. Thus, it is only necessary to calculate the components of \( \Pi \) once for each source-receiver pair. In this way, we obtain a solution of the two-point boundary-value problem studied by S&S. It follows that their total Green's function \( G \) for the two-point boundary-value problem can be related to the propagator \( \Pi \) for initial value problems in this case.

3.1 Comparison with solution of S&S

The coordinates introduced above correspond to setting \( \Omega = 0 \) in eq. (50) of S&S. It is not difficult to show that the \( n \) component and the two component of eq. (6) above are in agreement with this equation:

\[
\frac{d}{ds} (u_0 \mathbf{v}_n^0) + u_0 \mathbf{v}_n^0 \cdot \mathbf{V}_n \frac{1}{u_0} = u_0 \mathbf{V}_n f_1
\]

(27)

\[
\frac{d}{ds} (u_0 \mathbf{v}_2^0) + u_0 \mathbf{v}_2^0 \cdot \mathbf{V}_2 \frac{1}{u_0} = u_0 \mathbf{V}_2 f_1
\]

The Green's function solution to the two-point boundary-value problem given by S&S may be written in the notation of this paper as:

\[
s^*(s_0) = \int_{s_0}^{\Sigma_0} G(s_0, \sigma) [u_0(\sigma) \mathbf{V}_f \left[ k^0(\sigma) \right]] d\sigma,
\]

(28)

where \( j = 1, 2 \) and \( i = 1, 2 \)

\[
H(t) = \begin{cases} 
0, & t < 0 \\
1, & t \geq 0
\end{cases}
\]

Of course, analytical expressions for the Green's function can give considerable savings compared with numerical determinations of such functions.

4 APPLICATION TO TOMOGRAPHIC RECONSTRUCTION PROBLEMS

A major problem encountered in trying to form tomographic reconstructions of heterogeneous media is that the ray paths required in the integrations depend on the heterogeneities themselves and thus are unknowns of the problem. It follows from Section 3 of this paper that, in problems where one needs to determine small perturbations to be superimposed on a known 'average' depth-varying velocity model in order to better model the heterogeneous medium and reproduce the data, one can use the ray paths joining source and receiver in the unperturbed medium as the path of integration in the reconstruction problem. It is well known that the 'first-order' travelltime perturbations...
depend on the integral of $u_0 f_i$ along the unperturbed ray path. This is an iteration of Fermat's theorem and the first-order ray deflections are not needed for this result. One can obtain more accurate expressions for the perturbation to the traveltime either by integrating numerically along the perturbed ray or by using the second-order traveltime perturbation. Having found analytical expressions for S&S's Green's function for this particular problem at eq. (29) above, it is possible to use expression (62) of S&S to determine the second-order traveltime perturbation in terms of the analytical propagator $P$. More recently, Hu & Menke (1992) have presented a method of using $P$-wave polarization data to invert for laterally heterogeneous velocity structure. They note that because the integral describing the direction of propagation (polarization) of $P$ waves is not stationary about the ray like traveltime, the polarization data are more sensitive to the exact position of the velocity heterogeneities than the traveltime data. Thus, polarization data can probe the transverse-velocity gradient (the velocity-gradient perpendicular to the profile cross-section) as well as the vertical and radial gradients.

This property of polarization data is apparent from the solution given at eq. (14) of this paper. We have shown that

\[
\frac{d\mathbf{x}}{ds} = x_0^a \mathbf{\hat{a}} + x_0^2 \mathbf{k} - x_0^a K(s_0) \mathbf{\hat{i}}.
\]

(30)

where

\[
\mathbf{\hat{i}} = \frac{d\mathbf{x}^0}{ds_0}.
\]

We know that the direction with which the perturbed ray approaches the surface is given by

\[
\frac{d\mathbf{x}}{ds} = (1 - \mathbf{e} \cdot \mathbf{k})(\frac{d\mathbf{x}^0}{ds_0} + \varepsilon \frac{d\mathbf{x}^0}{ds_0}) + o(\varepsilon^2).
\]

(31)

Thus, the first-order correction to the direction of propagation is simply

\[
\frac{d\mathbf{x}^0}{ds} = x_0^a \mathbf{\hat{a}} + x_0^2 \mathbf{k} = \left(\frac{d\mathbf{x}^0}{ds_0}\right)^T
\]

\[
= \int_0^\infty \mathbf{P}_d(s_0, \sigma) \mathbf{\nabla}_{\mathbf{f}^i}[u^0(\sigma)] d\sigma + \mathbf{P}_d(s_0, 0)\mathbf{q} + \mathbf{P}_d(s, 0)\delta.
\]

(32)

It follows that fluctuations in the polarization data for $P$ waves depend on the transverse gradients of the velocity perturbation. More specifically, they depend on the integral of $\mathbf{P}_d \nabla f_i$ along the unperturbed ray path.

The results of this paper are easily adapted to the calculation of the partial derivative matrix for polarization inversion problems considered in Hu & Menke (1992). In this application, $f_i u_0$ should represent the fluctuation in the velocity model due to their perturbation to the model $\delta m$; then eq. (32) gives the corresponding deviation in the direction of propagation of the ray (to first order). Once again, we require $\mathbf{q} = 0$ and $\delta$ to be given by eq. (26) to ensure that the ray satisfies the two-point boundary-value problem under consideration.

The results of this paper also may be adapted to tomographic reconstruction problems which require the relocation of one endpoint (say an earthquake) provided that this change is sufficiently small. The results of eq. (14) are directly applicable when the relocation lies in the plane perpendicular to the reference ray at the source. When the relocation contains a component in the direction of the ray at the source, it is necessary to change the lower limit of integration from 0 to $s_q = q \cdot \mathbf{e}$ to account for the difference in path length from the new source.

The presence of interfaces in the more complex velocity distributions usually associated with tomographic reconstruction problems requires careful attention to ensure that appropriate boundary conditions for ray tracing are satisfied. The principles outlined in Farra et al. (1989) and Farra (1992) may be applied in this case.

5 CONCLUSIONS

A simple closed-form solution has been obtained for the problem of ray propagation through media which have weak lateral inhomogeneities superimposed on an unperturbed velocity field which varies with depth only. Analytical expressions have been found for the total Green's function for two-point boundary-value problems in such media. Furthermore, we have indicated how these results may be used with perturbations in observed traveltimes and polarization of $P$ waves to obtain tomographic reconstructions of $f_i$ and its transverse derivatives when an 'average' depth-varying model of the medium is known.

ACKNOWLEDGMENTS

The author would like to thank two anonymous reviewers for useful and constructive comments. This work has been facilitated by the University of Technology, Sydney in appointing the author to a three-year term as University Reader.

REFERENCES


APPENDIX A: SOLUTION TO EQUATION (6) IN PLANE GEOMETRY

Under different conditions, eq. (6) describes both the first-order corrections to ray geometry resulting from small
fluctuations in the velocity field about a vertically varying model and the corrections to ray geometry resulting from small changes to the initial conditions when there are no fluctuations in the velocity field. Therefore \( z \) is used rather than \( x' \) as the variable describing the correction to position on the ray and the equation is solved subject to non-zero initial conditions thus allowing the solution to both problems to be determined simultaneously. Specifically, our initial conditions will be

\[
z(0) = q \quad \text{with} \quad q \cdot \frac{dx^0}{ds_0} = 0
\]

and

\[
\frac{dz}{ds_0}(0) = \delta.
\]  

(A1)

From eq. (6), it is seen that the two component of \( z \) satisfies:

\[
\frac{d}{ds_0} \left( u_0 \frac{dz}{ds_0} \right) = u_0 \nabla f_1(x_0) + u_0 \nabla f_1(x_0)
\]

and thus

\[
z_2(s_0) = \int_0^{s_0} S_2(s_0, \sigma) \nabla f_1(x_0') d\sigma + S_2(s_0, 0) \delta_2 + q_2  
\]  

(A3)

where

\[
S_2(s_0, \sigma) = \int_0^{s_0} u_0(\sigma) \frac{ds}{d\sigma} = \frac{u_0(\sigma)}{p} [x_1'(\sigma) - x_1'(\sigma)].
\]

That is, \( S_2(s_0, \sigma) \) is simply proportional to the horizontal displacement of the reference ray as the arclength parameter varies from \( \sigma \) to \( s_0 \).

The equation for the three-component of \( z \) is simplified by noting that as \( u_0 \) varies only with \( x_2^0 \),

\[
\frac{d}{ds_0} u_0 = \frac{u_0}{p} \frac{dx_3}{ds_0}
\]

and so the basic equation for the three-component of \( z \) is:

\[
\frac{d^2}{ds_0^2} (u_0 z_3) - z_3 u_0 = u_0 \nabla f_1(x_0') + 2A u_5 + p_3^0 \frac{dA}{ds_0}
\]  

(A4)

where

\[
A = \mathbf{z} \cdot \mathbf{x}^0 = \frac{1}{u_0} (p_4^0 + p_4^0 z_3).
\]

Now

\[
\frac{d^2 p_3^0}{ds_0^2} = \frac{d}{ds_0} \left( u_0 \frac{dx_3}{ds_0} \right) = u_0 \frac{dx_3}{ds_0}.
\]

(A5)

So provided \( dx_3^0/ds_0 \neq 0 \), eq. (A4) may be multiplied throughout by \( p_3^0 \) to obtain:

\[
\frac{d}{ds_0} \left( p_3^0 \frac{dz}{ds_0} \right) = \frac{d}{ds_0} \left( u_0 \frac{dx_3}{ds_0} \right) - u_0 \frac{dz_3}{ds_0} \frac{dp_3^0}{ds_0} 
\]

\[
= p_3^0 u_0 \nabla f_1(x_0') + \frac{d}{ds_0} \left( [p_3^0(s_0)]^2 A \right).
\]

(A6)

Thus

\[
\frac{d}{ds_0} \left( p_3^0 \frac{dz}{ds_0} \right) = \frac{\phi_3(s_0)}{[p_3^0(s_0)]^2} + A, \quad c_3 \quad \text{constant}
\]

(A7)

where

\[
\phi_3(s_0) = \int_0^{s_0} p_3^0(\sigma) u_0(\sigma) \nabla f_1(x_0') d\sigma.
\]

Now

\[
\frac{d}{ds_0} \left( \frac{dz_3}{p_3^0} \right) - A = p \frac{d}{ds_0} \left( \frac{z_3}{p_3^0} \right)
\]

(A8)

where

\[
z_3 = z \cdot \mathbf{a}.
\]

Thus

\[
\frac{d}{ds_0} \left( \frac{z_3}{p_3^0} \right) = \frac{u_0(s_0) \phi_3(s_0) + c_3}{p \left[ p_3^0(s_0) \right]^2}
\]

(A7')

where

\[
c_3 = p[\phi_3(0) \delta_3 - \kappa(0) q_3]
\]

with

\[
\kappa(s_0) = \frac{d^2 x^0}{ds_0^2} = \frac{pu_0(s_0)}{u_0(s_0)},
\]

the curvature of the ray at \( x^0(s_0) \).

Further integration gives

\[
z_3(s_0) = \int_0^{s_0} S_3(s_0, \sigma) \nabla f_1(x_0') d\sigma + S_3(s_0, 0) \delta_3 + R_3(s_0, 0) q_3
\]

(A9)

where

\[
S_3(s_0, \sigma) = \int_0^{s_0} p_3^0(\sigma) u_0(\sigma) \left[ p_3^0(s_0) \right]^2 d\sigma
\]

\[
R_3(s_0, 0) = \frac{u_0(0) p_3^0(s_0)}{u_0(s_0) p_3^0(0)} - \kappa(0) S_3(s_0, 0).
\]

Appendix B: Solution of Equation (6) in Spherical Geometry

As in Appendix A, \( z \) is used rather than \( x' \) as the variable describing the correction to position on the ray and our initial conditions are:

\[
z(0) = q \quad \text{with} \quad q \cdot \frac{dx^0}{ds_0} = 0
\]

\[
\phi_3(s_0) = \int_0^{s_0} p_3^0(\sigma) u_0(\sigma) \nabla f_1(x_0') d\sigma.
\]
Analytical ray perturbations

287

\[ \frac{dz}{ds_0} (0) = \delta. \]  

(B1)

We introduce spherical polar coordinates \( r, \theta \) and \( \phi \) such that in the plane of the unperturbed ray, \( \phi \) remains constant and only \( r \) and \( \theta \) vary with position on the ray. It may be shown that

\[ \frac{d\hat{r}}{ds_0} = \frac{\partial \phi}{\partial z} \quad \frac{d\hat{z}}{ds_0} = -\frac{\partial \phi}{\partial r} \quad \frac{d\hat{\rho}}{ds_0} = 0 \]  

(B2)

where \( \cdot \) denotes differentiation with respect to arclength along the unperturbed ray.

First, we take the scalar product of eq. (6) with \( x^0 = \hat{r} \) to obtain:

\[ \frac{d}{ds_0} \left( u_0 \frac{dz}{ds_0} - u_0 z \right) = \frac{d}{ds_0} \left( u_0 \frac{dx^0}{ds_0} - u_0 z \right) = u_0 \frac{dz}{ds_0} \cdot \nabla_T f_1 (x^0) + 2 A \frac{dp_3^0}{ds_0} + \frac{dA}{ds_0} p_3^0 \]  

(B3)

where \( A = \hat{z} \cdot \hat{r} \) as before.

Now, from eq. (12), we have

\[ \frac{dx^0}{ds_0} = -u_0 \hat{z}, \]  

(B4)

and from eq. (7) we find that

\[ \frac{dx^0}{ds_0} = \frac{d}{ds_0} \left( z \cdot x^0 \right). \]  

(B5)

Thus, eq. (B3) may be written

\[ \frac{d^2}{ds_0^2} \left( u_0 \frac{dz}{ds_0} - u_0 z \right) = u_0 \frac{dx^0}{ds_0} \cdot \nabla_T f_1 (x^0) + 2 A \frac{dp_3^0}{ds_0} + \frac{dA}{ds_0} p_3^0. \]  

(B6)

Noting that

\[ \frac{2u_0}{r_0} + \frac{u_0^2}{r_0 r_0} = \frac{(r_0 u_0)^2}{r_0 u_0} \]  

it is seen that eq. (B6) does, in fact, have essentially the same form as eq. (A6) and the solution proceeds in a similar way. Provided that \( r_0 \neq 0 \), it is possible to multiply eq. (B6) throughout by \( p_3^0 = u_0 \frac{dz}{ds_0} - u_0 z \) which gives:

\[ \frac{d}{ds_0} \left( u_0 \frac{dz}{ds_0} - u_0 z \right) = \Phi_3 (s_0) + C, \quad C: \text{constant} \]  

(B7)

where

\[ \Phi_3 (s_0) = \int_0^{s_0} p_3^3 (\sigma) u_0 (\sigma) r_0 (\sigma) \nabla_T f_1 [x^0 (\sigma)] \ d\sigma. \]

In this case, it may be shown that

\[ \frac{d}{ds_0} \left( u_0 \frac{dz}{ds_0} \right) = \frac{A p}{u_0 ds_0} \left( \frac{u_0 \frac{dz}{ds_0}}{p_3^0} \right) \]  

(B8)

Thus

\[ \frac{d}{ds_0} \left( u_0 \frac{dz}{ds_0} \right) = \frac{u_0 (s_0) \Phi_3 (s_0) + C}{p} \left( \frac{p_3 (s_0)}{p_3^0} \right)^2 \]  

(B7')

where

\[ C_r = \frac{p p_3^0 (0) \delta_n - [p u_0 (0) + p^2 \kappa (0)] q_n}{ds_0} \]

with

\[ \kappa (s_0) = \frac{\frac{d^2}{ds_0^2} x^0}{\frac{ds_0^2}{ds_0^2}} = \frac{p u_0 (s)}{r_0 (s) u_0 (s_0)} \quad \text{for spherical geometry.} \]

Further integration gives

\[ z_n (s_0) = \int_0^{s_0} S_1 (s_0, \sigma) \nabla_T f_1 [x^0 (\sigma)] \ d\sigma + S_3 (s_0, 0) \delta_n + R_1 (s_0, 0) q_n \]  

(B9)

where

\[ S_1 (s_0, \sigma) = \int_0^{s_0} p_3^3 (\sigma) p_3^0 (s_0) u_0 (s) \]  

and

\[ R_1 (s_0, 0) = \frac{u_0 (0) p_3^3 (s_0)}{u_0 (s_0) p_3^0 (0)}, \quad p_3^0 (0). \]

In spherical polar coordinates where \( \hat{r} = \sin \theta \cos \phi \hat{\phi} + \sin \theta \sin \phi \hat{\theta} + \cos \theta \hat{\rho} \) and the scale factors are \( h_r = 1, h_\theta = r \) and \( h_\rho = r \sin \theta \), it may be shown that

\[ \frac{\partial}{\partial \phi} = r \frac{\partial}{\partial \phi}. \]

It follows from eqs (6) and (B2) that the \( \phi \) component of \( x \) (denoted by \( z_2 \)) satisfies the equation:

\[ r_0 \frac{d}{ds_0} \left( u_0 \frac{dz_2}{ds_0} - z_2 \right) = r_0 u_0 \nabla_T f_1 (x^0). \]  

(B10)

Multiplying throughout by the unit vector \( \hat{r} \) gives:

\[ \frac{d}{ds_0} \left( x^0 u_0 \frac{dz_2}{ds_0} - z_2 u_0 \frac{dx^0}{ds_0} = \hat{x}^0 u_0 \nabla_T f_1 (x^0). \]  

(B11)

Integrating with respect to \( s_0 \) and taking the \( \hat{\rho} (s_0) \) component of eq. (B11), using the orthogonality of the unit vectors \( \hat{r} (s_0) \) and \( \hat{\theta} (s_0) \) gives:

\[ z_\rho (s_0) = \int_0^{s_0} S_2 (s_0, \sigma) \nabla_T f_1 [x^0 (\sigma)] \ d\sigma + S_3 (s_0, 0) \delta_3 + R_2 (s_0, 0) q_2. \]  

(B12)

where

\[ S_2 (s_0, \sigma) = -\rho_0 (s_0) r_0 (s_0) \frac{\delta (s_0) \cdot \hat{r} (s_0)}{p} \]  

and

\[ R_2 (s_0, 0) = \frac{u_0 (s_0) r_0 (s_0)}{p} \]  

Integrating with respect to \( s_0 \) and taking the \( \hat{\rho} (s_0) \) component of eq. (B11), using the orthogonality of the unit vectors \( \hat{r} (s_0) \) and \( \hat{\theta} (s_0) \) gives:

\[ z_\rho (s_0) = \int_0^{s_0} S_3 (s_0, \sigma) \nabla_T f_1 [x^0 (\sigma)] \ d\sigma + S_3 (s_0, 0) \delta_3 + R_3 (s_0, 0) q_3. \]  

(B13)

Expressions for the transverse components of \( dz/ds_0 \) may be obtained from (B9) and (B12). They are
where

\[ S_4(s_0, \sigma) = \frac{p_3^0(\sigma)}{p_3(s_0)} + F(s_0)S_4(s_0, \sigma) \]

with

\[ F(s_0) = \frac{\rho_3(s_0) + u_0(s_0)}{p_3^0(s_0)} \]

\[ S_4(s_0, \sigma) = \frac{u_0(\sigma)\eta_0(\sigma)}{p} \tilde{n}(s_0) \cdot \tilde{t}(\sigma) \]

\[ R_4(s_0, 0) = F(s_0) \left( \frac{u_0(0)p_3^0(s_0)}{u_0(s_0)p_3^0(0)} - F(0)S_4(s_0, 0) \right) \]

and

\[ R_4(s_0, 0) = -\frac{u(0)}{p} \tilde{n}(s_0) \cdot e. \]