Bankruptcy Probability: A Theoretical and Empirical Examination

A thesis presented

by

Maurice Peat

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Certificate

I certify that this thesis has not already been submitted for any degree and is not being submitted as part of candidature for any other degree.

I also certify that the thesis has been written by me and that any help that I have received in preparing this thesis, and all sources used, have been acknowledged in this thesis.

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Abstract

Early Bankruptcy classification models were developed to demonstrate the usefulness of information contained in financial statements. The majority of classification models developed have used a pool of financial ratios combined with statistical variable selection techniques to maximise the accuracy of the classifier being employed. Rather than follow an "ad hoc" variable selection process, this thesis seeks to provide an economic basis for the selection of variables for inclusion in bankruptcy models, which are based on accounting information. An implicit assumption underlying this work is that the probability of default is endogenous. That is, the decisions of a firm's management have a direct impact on the probability of bankruptcy. These decisions and their resultant effects can be identified through analysis of financial statements.

A model of a firm facing an uncertain environment with the possibility of bankruptcy is developed and analysed. In the model, a firm is created with given initial equity. These funds can be invested in productive resources or held as cash balances. The productive resources are used to earn random earnings in any period. If earnings are positive, they can be used to pay dividends to shareholders, invest in new productive resources, repay outstanding debt or increase the firm's cash balance. The firm is able to borrow and repay funds up to a credit limit. When the cash position of the firm falls to zero the firm is bankrupt. The firm attempts to maximise the stream of dividends paid to shareholders during its life. The solutions of the model and the associated bankruptcy probability expressions are derived by application of the dynamic programming algorithm.

The variables which differentiate the possible model solutions and those identified in the derived bankruptcy probability expressions, are 'proxied' by variables constructed from financial statement data. This data is derived from Annual Reports filed with the Australian Stock Exchange between 1966 and 1994. These proxy variables are used in the empirical validation of bankruptcy probability expressions derived from the model.

The random nature of the time horizon in the model for a single firm provides the rationale for the use of duration or hazard-based statistical methods in the validation of the derived bankruptcy probability expressions. The Cox (1972) proportional hazards model is used to estimate the coefficients and standard errors that are required for the validation of the derived bankruptcy probability expressions.

Results of the validation exercise confirm that the variables included in the empirical hazard formulation behave in a way that is consistent with the solutions of the model of the firm. Thus, the bankruptcy probability expressions derived from the model of the firm developed in this thesis provide a guide for the conduct of empirical investigations of the probability of corporate failure.

Chapter 1
Introduction

Continued interest in corporate bankruptcy is related to the frequency of its occurrence and the broad range of parities who are affected. Investors incur losses to their invested capital. Lenders lose the outstanding balances of loans that are defaulted. Auditors who have failed to issue a qualified report can be legally liable for losses. Labour and unions can lose accumulated entitlements and their jobs.

A continuing focus of academic interest has been bankruptcy classification models, which are capable of differentiating between a company which will become bankrupt and one which continues to operate[1]. A problem which is mentioned in a number of studies is the lack of a generally accepted theory which explains bankruptcy. In the absence of such a theory, researchers have used statistical variable selection methods on large sets of variables, derived from financial statements, to maximise the number of firms correctly classified. On the issue of the impact of economic theory on empirical bankruptcy studies Charitou and Trigeorgis (2000) observe,

> Despite the significant costs of business failure and the empirical research efforts of many academics, there has been little attempt to develop or apply theoretical models to identify the financial variables that might explain business failure more rigorously. The lack of a theoretical framework concerning the primary variables that are relevant in distinguishing between failing and non-failing firms has been a serious impediment to the development of a truly scientific approach to bankruptcy prediction. (pg. 1)

Foster (1986) notes,

> Economic theory has played a small role in the development of univariate or multivariate distress prediction models. The few attempts at having theoretical analysis guide empirical model development have drawn on statistics or mathematics

[1] Jones (1987), in a review article, explores classification models by considering, sample selection and experimental design, variable selection, statistical method employed and the evaluation of results.

literature. (pg. 559)

Jones (1987) in a survey of bankruptcy classification studies also observes,

> Overall, most bankruptcy researchers have not applied theoretical models to empirical research. As mentioned earlier, the more sophisticated models have been based on statistical or mathematical literature and have not provided economic guidelines to aide in variable selection. (pg. 135)

The aim of the thesis is to provide a sound economic basis for the selection of variables, derived from financial statements, for inclusion in empirical bankruptcy prediction models. An implicit assumption underlying the work is that the probability of default is endogenous. That is, the decisions of a firm's management have a direct impact on the probability of bankruptcy. These decisions and their resultant effects can be identified through analysis of financial statements.

. To achieve this aim a model of the firm facing the possibility of bankruptcy is analysed. Once the model has been developed and solved, its results will be used to define a set of variables which influence bankruptcy probabilities. These variables will be constructed from financial statement data and used in a model validation exercise on a sample of firms that contains both bankrupt and continuing firms.

1.1 Research Method

There are a number of events which can lead to a firm becoming bankrupt. They are, failure to make an interest payment on some part of the firm's debt, failure to make a repayment of principal on part of the firm's debt, applying to the courts for bankruptcy or liquidation of the firm.

The first two causes are indicative of a cash flow problem and they often lead to liquidation or bankruptcy. Cash flow problems arise when the firm does not have sufficient cash and liquid assets to meet its obligations to debt holders as they fall due. This inability to meet the conditions of loan contracts leads to some action by debt holders to recover their capital. Therefore, the cash position of the firm is an important determinant of distress. The model to be developed will focus on the cash position of the firm.

Firms exist in an uncertain environment. The uncertainties that firms face are related to economic conditions, economic policy settings and their impact on the operations of the firm. For example, the cost of debt to the firm will be affected by monetary policy, directly through interest payable and indirectly through flow-on cost factors. Demand for a firm's product will be dependent on the position of the economy in the business cycle and the actions of competitors. These and other uncertainties need to be summarised by a stochastic element in the model. This stochastic element should be defined in a way that allows the economic factors mentioned to influence the parameters of its associated probability distribution.

The process of bankruptcy takes time. Firms typically do not instantaneously become bankrupt. Studies that have concentrated on the empirical prediction of bankruptcy usually look at the financial status of a firm 1 year before failure, 3 years before failure and 5 years before failure. Studies, such as Beaver (1967) and Zavgren (1985), have found that bankruptcy can be predicted with decreasing accuracy out to 5 years before failure. These results demonstrate that there is a time path to corporate distress.

The management of the firm, acting as a single decision maker, make decisions which have a direct impact on the firm's survival. To make choices from the set of available alternatives some form of evaluation criterion will be required. In a certain environment, the principles of financial management suggest that the management of a firm will work to enhance shareholder wealth by investing in positive net present value projects. When the project cash flows have been properly defined to include the value of management flexibility[2], such investments will build the shareholders' equity in the firm. Microeconomics suggests that firms will seek to maximise profit. The survey article by Lesourne and Leban (1982) describes the criterion that have been in used in dynamic models of the firm in a certain environment. The presence of uncertainty complicates the specification of a criterion for decision making. Alchian (1950) argues that in a uncertain world, with the economic system acting as a mechanism for ensuring the "survival of the fittest" profit maximisation is an inappropriate decision criterion. It is argued that the precondition for firm survival is the generation of positive profit. This suggests that an appropriate criterion for management decision making is the maximisation of payments to shareholders, as measured by the dividend stream. This criterion is consistent with the argument of Alchian, as payment of dividends can not occur if the firm does not generate a positive profit.

Accordingly we see that a bankruptcy model should have the following properties. It should focus on the cash position, it should have a stochastic element, it should be dynamic and it should use a decision criteria implied by the generation of positive profit.

In this research, a model of the firm which has these properties is developed. In the model, a firm is created with a given initial equity and no new equity is issued within the life of the firm. These funds can be invested in productive resources or held as cash balances. The productive resources are used to generate random earnings in any period. If the profit, derived from these earnings, is positive it can be used to pay dividends to the shareholders, invest in new productive resources, repay outstanding debt or increase

[2] The incorporation of flexibility ito investment decisionmaking is known as the Real Options approach to investments. See Trigeorgis (1996) for a complete exposition of this approach.

the firm's cash balance. The firm is able to borrow and repay funds up to a credit limit. When the cash position of the firm falls to zero the firm is bankrupt. The firm attempts to maximise the expected value of the stream of dividends paid to shareholders during its life.

The solutions of the model and the associated bankruptcy probability conditions are derived by the application of the dynamic programming algorithm. Variables which differentiate the possible model solutions are identified, as are those which appear in the derived bankruptcy probability conditions. These variables will be 'proxied' by expressions constructed from the financial statement data of listed Australian companies. These proxy variables are then used in the empirical validation of bankruptcy probability expressions derived from the model. The random nature of the time horizon in the model for a single firm provides the rationale for the use of duration or hazard-based statistical methods in the validation of bankruptcy probability expressions.

To validate the model we need to derive testable hypotheses. The hypotheses formed from the analysis of the model relate to the significance of the variables which influence the probability of bankruptcy, and the direction of their effect. Tests of the significance of a variable's effect are based on the statistical properties of the probability estimator. The direction of a variable's effect on the probability of bankruptcy is captured in the sign of its estimated coefficient. To test the hypotheses, a statistical technique that can estimate failure probabilities, allow for explanatory variables and deal with the time series nature of financial statement data is required. Specifically, the Cox (1972) proportional hazards method will be used to estimate the coefficients and standard errors that are required to test the formulated hypotheses.

1.2 Organisation of thesis

The following chapter reviews the theoretical and empirical research into corporate bankruptcy. Prior explanatory models of bankruptcy behaviour are presented. Studies that have investigated the properties of financial statements, through the analysis of financial ratios, are reviewed next. Studies that classify firms as bankrupt, or a going concern, are then reviewed. The chapter ends with a summary of the studies reviewed.

Chapter Three presents the development and solution of a model of the firm facing the possibility of bankruptcy. The chapter opens with an simplified example which demonstrates the possibility of bankruptcy for a firm which is operating to maximize its value. The dynamic programing technique, which will be used to solve the model developed, is summarised in the next section. The structure of the model is then explained. Optimality conditions for the model are presented, then analysed to identify the possible solutions of the model. These solutions are then used to derive a number of explicit bankruptcy probability expressions.

Chapter Four is concerned with establishing the hypotheses and a test statistic to be used in the validation of bankruptcy probability expressions which were derived in Chapter Three. The chapter opens with an analysis of the expected direction of the effects of the variables defined in Chapter Three on the probability of bankruptcy. The definition of proxy variables, constructed from financial statement variables follows. A test statistic that is based on the random nature of the firm's lifetime is presented. It takes the form of a probabilistic regression, which provides estimates of the coefficients and the standard errors of the estimated coefficient for each variable included in the analysis.

The fifth chapter presents the results of the statistical testing of the hypotheses developed in Chapter Four. The first section of the chapter is a description of the sources and properties of the data set used. The second contains the results of the analysis of differences in univariate means, while the final part is a report of the findings from the Cox regression analysis of the hypothesised effects of the model variables on bankruptcy probability.

The thesis concludes with a summary of the results of the modeling and empirical validation exercises. Extensions to the modeling and empirical work are then suggested.

Chapter 2
Review of Bankruptcy Probability Literature

The objective of this study is the development of expressions for the probability of bankruptcy and their empirical verification. There are a number of existing explanatory models which allow bankruptcy as a possible outcome or allow the derivation of an expression for the probability of bankruptcy. These models are the focus of this review. The models are presented in some detail to allow comparison with the model developed in the next chapter. The bankruptcy conditions and the associated probability of bankruptcy will be derived where possible. In those cases where there has been some attempt to empirically validate the model, the method used will be analysed.

Development of a bankruptcy classification model is not the focus of this thesis. Therefore, the extensive classification literature[3] is not the primary focus of this review. However, we present a brief summary of this literature, along with its companion critical literature. This body of work is driven by the search for a method and a set of input variables which can best solve the dichotomous classification problem. The approach of these studies is to search the universe of possible statistical models and input variables to find a combination which will maximise the percentage of correct classifications made. A small number of studies use an empirical estimate of the probability of bankruptcy coupled with a decision rule to ultimately produce classification results. The probability estimation sections of these studies will be analysed.

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3 The early literature is surveyed in papers by Jones (1987) and Zavgren (1983). Recent empirical studies of alternative statistical approaches to bankruptcy classification, Lennox (1999) and Mossman, Bell, Swartz and Turtle (1998) provide a guide to recent work in this area.

The first section of this review covers descriptive models which include the possibility of bankruptcy. The second section outlines the characteristics of financial statement data that is widely used in empirical modeling of bankruptcy. It also presents the results of studies which have used the statistical technique of factor analysis to summarise the information contained in financial statements. These exercises derive a small number of factors, seven or eight, from the many financial ratios used in bankruptcy classification studies. Empirical studies which have classified firms as bankrupt or going concerns are then reviewed. These studies provide a framework for the empirical verification of the explanatory model to be developed in the next section. Finally, the common elements of the explanatory models and empirical studies will be highlighted.

2.1 Explanatory Models of Financial Distress

The connection between financial statement information and financial distress was explored by Beaver (1967). The aim of this work was the verification of the usefulness of financial statement information. The ability of ratios, constructed from financial statement data, to predict failure was the application chosen. Along with the statistical exploration of the ability of single ratios to classify firms as bankrupt or solvent, a descriptive model of the firm was outlined, Beaver (1967):

> The firm is viewed as a reservoir of liquid assets, which is supplied by inflows and drained by outflows. The reservoir serves as a cushion or buffer against variations in the inflows. The solvency of the firm can be defined in terms of the probability that the reservoir will be exhausted, at which point the firm will be unable to pay its obligations as they mature (i.e. failure). (p.80)

This description can be translated into a simple cash flow equation:

$$Cash_t = Cash_{t-1} + Inflow_t - Outflow_t \quad (2.1)$$

Where: $Cash_t$ is the reserve of cash at time t, $Inflow_t$ is the revenue of the firm at time t and $Outflow_t$ is the sum of the expenditures of the firm from operational and financing activities at time t. Inflow and Outflow are random variables[4].

When $Cash_t$ is less than or equal to zero then the "reservoir is exhausted" and the firm is insolvent. The interaction between the random Inflow and Outflow variables along with the initial cash position determine the viability of the firm. Making a simple algebraic transformation, the system results in (2.1) becoming a first order difference equation of the firm's cash position.

$$Cash_t - Cash_{t-1} = Inflow_t - Outflow_t \tag{2.2}$$

In this model, the firm is bankrupt when $Cash_t$ is less than or equal to zero. This occurs when

$$Cash_{t-1} + Inflow_t \leq Outflow_t \tag{2.3}$$

As the distributions of the $Inflow_t$ and $Outflow_t$ random variables are not specified, a functional form for the probability of bankruptcy cannot be formed. This descriptive approach captures the important elements to be modelled, that is the central position of the cash reserve and random flows of cash. This framework was used by Laitinen and Laitinen (2000) as the basis for variable selection in a logit-based modeling exercise.

2.1.1 Gambler's Ruin Approach

A simple probability model was proposed by Wilcox (1971, 1976) in order to make the interaction of the random inflows and outflows described by Beaver more explicit. In this model there are a finite number of states, S_j, $j = 0, 1, 2, \cdots, N$ that a firm

[4] A random variable has a value that is drawn from a probability distribution. The value of the variable is not know in advance. It is determined at the time that the drawing from the probability distribution occurs.

can attain at time t, where N is the largest value that the wealth of the firm can reach. The states are the wealth of the firm, which is the size of the reservoir in the Beaver formulation. The probability of being in state S_j is only dependent on the state of the firm at the previous time $t-1$. This is a Markov process. The transition probability $\Pr(S_j|S_i)$ defines the likelihood of moving from state S_i at $t-1$ to state S_j at time t. This process is generalised to a one-dimensional random walk with the constraint that once the firm enters the bankrupt state, S_0, it stays in that state. That is:

$$\begin{aligned}
\Pr(S_0|S_0) &= 1\text{, once the firm is bankrupt it stays bankrupt}\\
\Pr(S_j|S_i) &= p\text{, for } i = j-1, i \neq 0\\
\Pr(S_j|S_i) &= 1-p,\ \text{ for } i = j+1\\
\Pr(S_j|S_i) &= 0\text{, all other } i.
\end{aligned}$$

This is the definition of the gambler's ruin model, which is a one dimensional random walk with an absorbing barrier at one end and no barrier at the other. With $q = 1 - p$, the probability of the firm starting in state S_z and visiting state S_0, the bankruptcy state, is:

$$\Pr(\text{bankruptcy}) = \begin{cases} 1, & \text{if } p \leq q \\ \left(\frac{q}{p}\right)^z, & \text{otherwise.} \end{cases}$$

In order to generate a simple model of the probability of bankruptcy for a firm, let σ be a constant profit the firm can realise with probability, p. The firm can lose a fixed amount $-\sigma$ with probability $q = 1 - p$. The firm starts its life with wealth C. Suppose $p > q$, then the firm is more likely to make a profit than a loss. In this case

$$\Pr(\text{bankruptcy}) = \left(\frac{q}{p}\right)^{\frac{C}{\sigma}},$$

where $z = \frac{C}{\sigma}$ is the number of consecutive losses the firm can sustain before going bankrupt. To use this model to calculate a firm's probability of bankruptcy, the ratio of probability of profit to probability of loss needs to be defined. This ratio is related to the "drift" of the random walk which describes the firm's wealth. The drift is assumed to be a function of the return on capital invested. The drift in the random walk model is $(p-q)\sigma$. A measure of drift for a specific firm would be $A\theta\delta\gamma$, where A is the total assets employed, θ is the average return on total assets, $(1-\delta)$ is the dividend payout ratio and $(1-\gamma)$ is the fraction of net cash flow after dividends reinvested in productive assets.

Equating these drift expressions we get

$$(p-q)\sigma = A\theta\delta\gamma \tag{2.4}$$

Using $q = 1 - p$ and solving (2.4) we get,

$$\frac{q}{p} = \frac{1-(\frac{A\theta\delta\gamma}{\sigma})}{1+(\frac{A\theta\delta\gamma}{\sigma})}.$$

The probability of bankruptcy is then given by

$$\Pr(\text{Bankruptcy}) = \left(\frac{1-(\frac{A\theta\delta\gamma}{\sigma})}{1+(\frac{A\theta\delta\gamma}{\sigma})}\right)^{\frac{C}{\sigma}}.$$

To allow the calculation of failure probabilities, Wilcox proposed the following definitions for the variables in the model. C equals Realisable value of Assets minus Total Liabilities, σ equals the standard deviation of net cash flow less capital spending and dividends and $(\frac{A\theta\delta\gamma}{\sigma}) = \frac{\text{adjusted cash flow}}{\text{the standard deviation of (net cashflow less capital spending and dividends)}}$[5].

This formulation suggests that the variables, net income, the dividend payout ratio, the proportion of cash flow invested in illiquid assets, the standard deviation of net cashflow less capital spending and dividends and Realisable value of Assets mi-

[5] Where adjusted cash flow = (net income)(1 - dividend payout ratio)(1 - proportion of cash flow invested in illiquid assets)

nus Total Liabilities (Owners Equity), are important in the calculation of bankruptcy probabilities.

This modeling effort was important as it formalised the framework proposed by Beaver. A simple random walk model was used to describe the movements in the "reservoir of liquid assets". Three simplifying assumptions are required to get a random walk with a barrier, namely

1. the wealth of the firm is fixed (no borrowing or equity raising),
2. the level of profit to loss is fixed in advance, and
3. the probability of a profit to loss is also fixed.

Using these assumptions the model gives rise to a single formulation of the probability of bankruptcy. The management of the firm does not exercise any influence on the state of the firm's finances in this framework. There is no notion of optimising the value of the firm, its profit level or any other criterion. Wilcox (1976) tried to calculate the probability of bankruptcy for a sample of firms using the gambler's ruin formula directly. However the firm specific data violated the assumptions of the model for many firms. In his 1976 study, he did use the variables suggested by the model to construct a Multiple Discriminant Analysis (MDA)[6] classification model. It correctly classified 94% of firms in the estimation sample. The idea of using the component variables in the bankruptcy probability expression as explanatory variables in a statistical model as a method for verifying the proposed model has merit. If the variables are statistically significant and have the signs that the model suggests, then the model can be seen as

[6] The technique of MDA fits a linear discriminate function $Z = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_n X_n$, which transforms the values of a firms ratios into a single Z-score.

verified (in the context of the statistical method chosen). Use of the variables from the gambler's ruin framework, which is couched in terms of the probability of bankruptcy, in a probabilistic regression rather than MDA would have provided a preferable framework for the verification of the gambler's ruin model.

2.1.2 Firms with access to External Capital

What happens to the probability of bankruptcy when the firm has access to a capital market, but investment is irreversible? Scott (1981) addressed this problem for a firm with perfect, then imperfect access to external capital.

In the perfect access case[7], a firm can potentially continue indefinitely by meeting losses that occur through the sale of debt or equity. Purchases of real assets are irreversible as the market for used assets is assumed to be imperfect. The firm becomes bankrupt when stockholder wealth (measured by market value) becomes zero[8]. In the case of a firm incurring a loss, determination of bankruptcy follows by firstly, ignoring the loss and determining the financial plan to optimise the value of the firm, then optimising the value of the firm. After observing the value of equity, while still ignoring the loss, if the loss is greater than the value of equity then the firm fails.

Let X be next period's profit or loss, and S be the expected value of equity next period (ignoring any loss). The firm fails if $S + X < 0$. Let μ_x be average profit and σ_x the standard deviation of profit. The firm fails if

$$\frac{X - \mu_x}{\sigma_x} < \frac{-(\mu_x + S)}{\sigma_x}.$$

7 The following economic assumptions are required for the derived conditions to hold. Security markets are competitive with no transaction, flotation or information costs. There is no personal taxation. Both debt and equity are limited liability instruments. Investors are risk neutral. Expectations are homogenous and firms do not grow.

8 This is equivalent to the firms wealth going to zero in the gamblers ruin model.

If firms have a common two parameter profit distribution, $F[\bullet]$, then

$$\Pr(\text{Bankruptcy}) = F\left[\frac{-(\mu_x + S)}{\sigma_x}\right].$$

The sum of the expected value of equity, S, and the average profit, μ_x, are the main determinants of the probability of bankruptcy. The larger this sum the smaller the probability of bankruptcy. To calculate a bankruptcy probability the distribution of profit, $F[\bullet]$, must be specified and its parameters estimated.

By changing some assumptions, Scott (1981) derived a model with imperfect access to capital. He assumes that the firm has no access to debt, only equity issues. The issue of equity capital attracts a flotation cost. In this model the market for used real assets is perfect and investments are perfectly reversible. The model has three periods; namely, $0, 1$ and 2. The firm can survive until period 2, when it will be liquidated. The model contains an expression for the probability of bankruptcy in period 1.

If the variables in this model are defined as in Table 2.1, then the firm goes bankrupt if stockholder wealth goes to zero in period one.

Variable	Definition
K_i	stockholders equity at period i
$X_i[K_{i-1}]$	firm income when $i = 0$ or 1. The liquidation value occurs when $i = 2$. X_i is a random variable with values in the range $(-\infty, \infty)$. Each draw of the random variable is an increasing, concave function of K_{i-1}.
I_i	net investment at period i, $K_i = K_{i-1} + I_i$
$S_i[I_i]$	market value of firm's equity at period i
c	cost of issuing new equity

Table 2.1. **Scott Imperfect Access Model - Variables**

The value of the firm in period one, $S_1[I_1]$, is dependent on the amount invested in period 1, I_1. The level of income, $X_1[K_0]$, does not influence firm value. Scott

(1981) works out the value of I_1, using a back recursion from period two[9]. In period two, the value of the firm is the greater of firm income, $X_2[K_1]$, and the value of the firm if it failed in period 1 (zero). Stockholder wealth at the end of period two is the $\max[X_2[K_1], 0]$. Investors are risk neutral by assumption and they face a common interest rate, r. It follows that the value of the firm's equity in period one will be the discounted value of expected future income,

$$S_1[I_1] = \frac{E[X_2[K_0 + I_1]]}{1 + r}$$

where $E[\bullet]$, is the expectation operator. In the absence of an optimising criterion the level of investment, I_1, is not determined by the model. However, if the firm does invest in period one with I_1 greater than zero, the investment will require financing. The firm can fund the investment costlessly from retained earnings, or with a flotation fee in the equity market. Three cases are possible:

1. If the income level $X_2[K_0]$ is high, the firm finances the investment from earnings and has funds to pay a dividend.

2. The firm can set the investment to its current income, $I_1 = X_1[K_0]$, thus avoiding flotation costs.

3. If the investment is greater than current income, that is $I_1 > X_1[K_0]$, then the firm incurs the flotation cost, c, and uses external funds for the investment.

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9 The method of backwards recursion from the final state is the most common approach to the numerical solution of Dynamic Programming problems. Solving this model requires the back recursion approach even though there is no explicit optimising criterion in this model.

Let SW be stockholder wealth at period 1, then

$$SW = \begin{cases} S_1[I_1] + X_1[K_0] & \text{(case 1)} \\ S_1[X_1[K_0]] & \text{(case 2)} \\ S_1[I_1] + (1-c)(X_1[K_0] - I_1) & \text{(case 3)} \end{cases}$$

The firm is bankrupt in period one when stockholder wealth is zero. In case 3 this occurs when

$$S_1[I_1] + (1-c)(X_1[K_0] - I_1) \leq 0. \tag{2.5}$$

For (2.5) to hold the firm must make a loss in period one, that is $X_1[K_0] < 0$. As in the perfect access case, let μ_x be average profit and σ_x the standard deviation of profit. Note that, $I_1 = K_0 - K_1$, and by rearranging the bankruptcy condition (2.5) we get

$$\frac{X_1[K_0] - \mu_x}{\sigma_x} \leq \frac{-\mu_x - (K_0 - K_1) - \frac{S_1[I_1]}{(1+c)}}{\sigma_x}.$$

If firms have a common two parameter profit distribution, $F[\bullet]$, then

$$\Pr(\text{Bankruptcy}) = F\left[\frac{-\left(\mu_x + (K_0 - K_1) + \frac{S_1[I_1]}{(1+c)}\right)}{\sigma_x}\right].$$

This results indicates that the probability of bankruptcy is determined by the interaction of the average profit level, μ_x, the change in stockholders equity, $(K_0 - K_1)$, the market value of equity, $S_1[I_1]$, and the level of flotation costs, c. As the bankruptcy probability is measuring the area in the left tail of the profit distribution, low values of the expression, $\mu_x + (K_0 - K_1) + \frac{S_1[I_1]}{(1+c)}$, will be associated with high probabilities of failure.

In the absence of an optimising framework, the Scott (1981) models rely on the "conceptual experiment" described in the perfect access case, along with back recursion arguments to derive bankruptcy probability expressions. The dynamic structure of the imperfect access model allows three solutions, one of which has bankruptcy as a possible outcome. Common elements of the gambler's ruin and Scott approaches are the use of the value of the firm as the bankruptcy indicator and use of a profit distribution. The properties of the profit distribution are captured in the fixed profit or loss amount and the profit to loss probabilities in the gambler's ruin case, and the unspecified two parameter profit distribution, $F[\bullet]$, in the Scott (1981) models.

Scott (1981) made no attempt to statistically verify the functional forms or variables suggested by his models. The variables in the bankruptcy condition of the perfect access case were all divided by Total Assets. The three variables in this form were related to three of the seven variables used in the Altman, Haldeman and Narayannan, (1977), Zeta model. Scott (1981) then argued,

> Although the overlap between the empirical and theoretical models is imperfect, it provides empirical support for existing theory as well as theoretical justification for bankruptcy prediction models. (pg. 341)

Using a classification approach can demonstrate the usefulness of variables derived from the probability expressions of the Scott (1981) model, and from these results the validity of the explanatory model can be inferred. However, as noted in the case of the gambler's ruin model, the theory generates an expression for the probability of failure. To verify this expression directly, a probabilistic regression framework would be preferred.

2.1.3 Stability Analysis

Another approach to modeling failure is through the application of bifurcation analyses. The stability of equilibrium solutions in deterministic dynamic system are analysed. One approach used is Catastrophe Theory (CT)[10]. This method is concerned with the

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[10] A concise summary of the development of Catastrophe Theory is presented in Gregory-Allen and Henderson (1991).

stability of a dynamic system which has been perturbed[11] from a state of equilibrium. A dynamic system will generally be represented by a differential equation,

$$\frac{dx}{dt} = f(x, y, z).$$

Where x is the variable of interest, y, z are parameters of the system and t is time.

CT is concerned with explaining how sudden discontinuous changes (jumps) in the value of the system output, x, can arise as the result of smooth changes in the parameters of the system. To conduct this analysis, a functional form for the system equation is needed. Thom (1969) identified seven polynomial forms which are capable of generating jumps in their output. In finance applications, the form most commonly used is the "cusp catastrophe" described by the equation

$$f(x, a, b) = x^4 + 2ax^2 + 4bx$$

It has one state variable, x, and two control variables, a and b.

The set of parameter values associated with jumps in the system output is known as the bifurcation set. It represents the points of instability of the system. Figure 2.1 shows a "cusp catastrophe" in the upper section and its bifurcation set projected below. Note the "fold" in the surface. Follow trajectory T2 from point B on the upper section of the fold. When the system arrives at the edge of the fold a small smooth movement in the system causes the output to jump to the point A. Projecting the area of the surface where the system has multiple values (the folded section) onto the a,b plane defines the systems bifurcation set. At all points on the boundary of this set, jump behaviour in the system output is possible.

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[11] A perturbation is a small disturbance that is applied to a system that is in an equilibrium state.

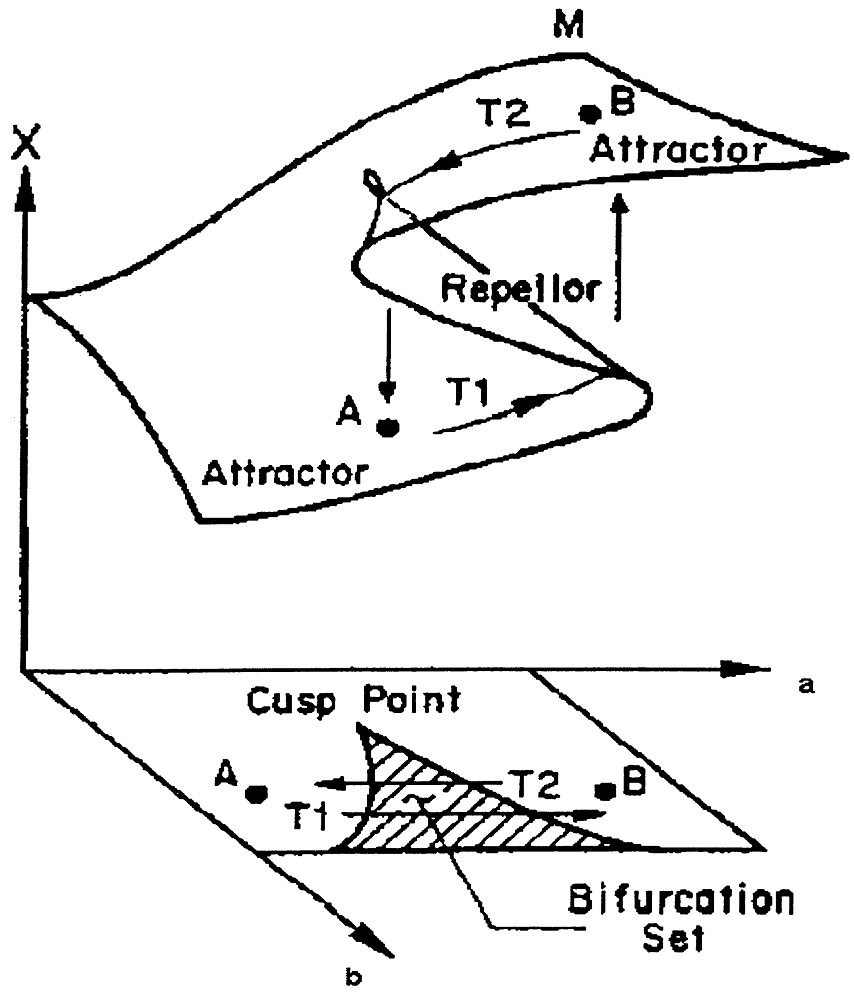

Fig. 2.1. **Bifurcation set of a Cusp Catastrophe**

To model bankruptcy using a cusp catastrophe all the relevant factors have to be represented by one dependent and two independent variables. In Scapens, Ryan and Flecther (1981), CT is used to connect a firm's financial position, as reflected by financial ratios, to the behaviour of creditors. Scapens, Ryan and Flecther (1981) base the selection of the dependent variable on the following argument.

It is assumed that creditors will want to extend credit to successful companies and withdraw credit from potential failures. (pg. 7)

Using the behaviour of creditors as the dependent variable, x, does not provide an expression for the probability of failure. However, a sudden withdrawal of credit or the calling in of a loan by a creditor can quickly lead to a liquidity problem that ultimately ends in bankruptcy. The independent variables used are a, which measures return, and b, that captures the operating risk of the firm. Return is defined in a general way. It could be measured by profit, cash flow or net income divided by total assets or debt.[12] Companies with a low return for their industry type may be seen as having a greater "failure potential" by lenders. This will lead to lenders having less confidence in the firm's ability to generate funds required to meet interest and principal repayments in future periods. The second independent variable is the company's operating risk, measured by the variability of the company's return. The return and operating risk are used by creditors to assign companies to "risk classes". Companies in a risk class will be expected to deliver returns within a predefined range. The width of this range will be determined by the company's operating risk. A return within the range will not cause the lender to revise their credit rating, while returns on the boundary of, or outside, the band will cause a revision (jump) in the credit rating. This work indicates that company returns and operating risk (variability of company returns) would be expected to have an impact on a firm's probability of bankruptcy, as they have been shown to influence the behaviour of lenders. In the absence of an explicit expression for bankruptcy probability, empirical verification would be difficult. Scapens, Ryan and Flecther did not attempt such a validation.

[12] That is, one of Profit / TA, Profit / Debt, Net Income / TA, Net Income / Debt, Cash Flow / TA or Cash Flow / Debt.

Francis, Fastings and Fabozzi (1983) also used CT to explain bankruptcy. In their formulation the variables are defined as the state variable, x, which measures the bankruptcy potential of the firm, the first control variable, a, representing the firm's earning power and the second control variable, b, the firm's liquidity.

The firm is defined to be bankrupt when x is less than zero. With this bankruptcy condition, the system equation becomes a nonlinear discriminant function. The variables are scaled so that the cutoff point for this discriminant function is zero.

The liquidity control variable is defined so that positive values indicate that the firm is solvent. This is achieved by defining b in terms of the current ratio[13],

$$b = \ln\left(\frac{\text{current assets}}{\text{current liabilities}}\right).$$

When the current ratio is one, current assets equal current liabilities, and the firm is just solvent. The value of b in this case is zero. The earnings control variable is defined to be a increasing function $q(\bullet)$ of the owner's rate of return, that is

$$a = q\left(\frac{\text{economic income}}{\text{equity}}\right), q(0) = 0, q' > 0.$$

When the firm earns no income, the value of a is zero. The use of the owner's equity to measure earnings is argued on the grounds that the equity holders bear the losses in the case of a bankruptcy, where they have the last claim on the assets of the firm. This formulation suggests that a liquidity measure and a return measure will be significant variables when used in a nonlinear discriminant function approach to bankruptcy classification. The authors did not validate the model.

The CT approach allows for an instantaneous jump from being a going concern to bankruptcy in a smooth, continuous system. This is an appealing feature. However, the

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13 The function ln() refers to the natural logarithm of the argument. q' denotes a first order ordinary derivative.

use of an arbitrary functional form to allow for instability in the system is a drawback. In the case of the cusp catastrophe only two independent variables are allowed, there is no guidance from CT on how to select these variables. As the system equations which give rise to CT models are deterministic, to go from these models to a probability of bankruptcy, a function that maps the state variable to a probability value is required. If such a function was available the jump behaviour displayed by the state variable would be evident in the probability of bankruptcy.

Schipper (1977) constructed an optimising, deterministic optimal control model of the financial behaviour of a private college. The thrust of Schipper's argument being that a college that fails is not on the optimal path, rather it is behaving in a sub-optimal manner. By determining the conditions for unstable regions in the solution space of the control model specified, this sub-optimal behaviour can be characterised. The stability conditions are found by developing the first order conditions for the Hamiltonian equation of the control problem to attain a maximum. The matrix of second order partial derivatives of the Hamiltonian equation is then analysed. If the matrix is positive definite then the system is stable. By analysing the principal minors and conditions required to make them positive, the differences between stable (going concern) and unstable (distressed) entities can be inferred. Using this approach, a set of testable hypotheses based on the outcome of the stability analysis were derived and then tested using the MDA approach.

2.1.4 Option theory approach to bankruptcy

This approach derives from studies that have applied option methods to the problem of the valuation of risky debt. Merton (1974) proposed that the equity of a firm with debt in its financial structure be viewed as a European call option on the value of the firms assets. The debt held is a single discount bond. The strike price of the option is the book value of the firms debt, the option expires when the debt falls due. The following assumptions are necessary if the Black and Scholes (1973) option pricing approach is to be used in pricing the call option.

1. Perfect capital markets. The markets are liquid, have continuous trading, there are no transaction costs or taxes and assets are perfectly divisible,
2. The value of the firm follows a geometric Brownian motion,
3. The risk free rate is constant over the life of the option,
4. Short selling is allowed,
5. There are no bankruptcy costs.

Using this approach the equity holder decides whether to exercise the call option. If the value of the assets are below the book value of the liabilities (the strike price of the option) on the expiration date the firm defaults the loan (the option is not exercised). When the firm defaults the loan it files for bankruptcy. Ownership of the firm is costliness transferred to the debt holders. The terminal payoff for the equity holders is the greater of the difference between the value of the firms assets and the face value of the debt and zero.

Solving the call option problem outlined gives the following expression for the value of equity if the firm is solvent:[14]

$$E = VN(d_1) - Be^{-rt}N(d_2) \tag{2.6}$$

In this equation E is the value of equity, V is the value of assets, B is the value of the debt maturing in t periods, r is the risk free rate and $N(d_1)$ and $N(d_2)$ are the values from the univariate cumulative normal distribution function from $-\infty$ to d_i. The values of the d_i are given by

$$d_1 = \frac{\ln\left[\frac{V}{B}\right] + \left(r + \frac{\sigma_V^2}{2}\right)t}{\sigma_V\sqrt{t}}$$

and

$$\begin{aligned} d_2 &= d_1 - \sigma_V\sqrt{t} \\ &= \frac{\ln\left[\frac{V}{B}\right] + \left(r - \frac{\sigma_V^2}{2}\right)t}{\sigma_V\sqrt{t}} \end{aligned}$$

σ_V is the standard deviation of asset value.

The first term in equation 2.6 is the expected value of the firm if it remains solvent. $N(d_2)$ in the second term is the risk-neutral probability that the firm will be solvent at maturity, the probability that the option expires "in the money". The risk-neutral probability that the firm will default on the debt is therefore given by

$$\begin{aligned} \Pr(\text{Bankruptcy}) &= \Pr(V < B) \\ &= 1 - N(d_2) \\ &= N(-d_2) \end{aligned}$$

14 The value of equity is derived from the solution of the PDE, $-rV + rVE_V + E_t + \frac{1}{2\sigma^2}V^2E_{VV} = 0$ subject to the boundary condition $E_T = \min(V - B, 0)$. Derivation of this PDE is discussed in Neftci (2000) pp 276-9. The solution of the PDE for the case of geometric Brownian motion, the Black and Scholes case, is presented in Neftci pp 296-7.

The risk-nuetral probability of bankruptcy is a function of the difference between the current asset value and the face value of the firms debt, relative to the standard deviation of the asset value. The number of standard deviations the firms asset value would have to fall to hit the default point D is known as the distance to default.

The probability of bankruptcy at the debt's maturity is determined by the variables that appear in the expression for d_2. The probability of bankruptcy will be higher when the current value of the firm is low, the value of debt outstanding is high, the firms leverage, $\frac{V}{B}$, is high, or the volatility of the firms asset value is high.

Derivation of an expression for the probability of bankruptcy has not been the primary focus of the contingent claims approach. The model was developed and has been extended in an attempt to provide an explanation for the term structure of risky corporate debt. Some examples of the extensions to the model are; Leland and Toft (1966) who allow the firm to select the default time and Longstaff and Schwartz (1995) who introduce a stochastic risk free rate. Bohn (2000) surveys the extensions to the basic model presented here.

Charitou and Trigeorgis (2000) extend the basic model in two directions. Firstly, they consider the case of a dividend paying firm. This introduces the dividend payment, D, into the expression for d_2.

$$d_2' = \frac{\ln\left[\frac{V}{B}\right] + \left(r - D - \frac{\sigma_V^2}{2}\right)t}{\sigma_V\sqrt{t}}$$

The dividend payment reduces the value of assets, which causes the probability of bankruptcy to increase by reducing the distance to default. Hillegeist et. al. (2002) move from a risk neutral approach in the case of a dividend paying firm by replacing the risk free rate, r, in d_2' with an expression for the expected market return on the firms assets, μ, to derive an actual probability of bankruptcy.

The second extension introduced by Charitou and Trigeorgis (2000) allows for periodic interest payments, I, with the possibility of default at each interest payment point, τ. This formulation is demonstrated to be equivalent to the valuation of a compound call option[15]. The default can be triggered by equity holders if the value of the option to continue is less than the interest payment, a voluntary liquidation. The debt-holders can wind up the company, involuntary liquidation, if the firm does not have sufficient cash or liquid assets to make the next interest payment. If the firm is assumed to hold a constant proportion, c, of its value in cash then

$$\begin{aligned} \Pr(\text{Bankruptcy due to insufficient cash}) &= \Pr(cV_\tau < I) \\ &= N(d_2'') \end{aligned}$$

where

$$d_2'' = \frac{\ln\left[\frac{cV}{I}\right] + \left(r - D - \frac{\sigma_V^2}{2}\right)t}{\sigma_V\sqrt{t}}.$$

The expression, d_2'', shows that the cash holdings of the firm and the level interest payments due will affect the probability of bankruptcy in the case of periodic interest payments.

Charitou and Trigeorgis (2000) used a logistic regression framework to test the explanatory power of the variables included in the option based models they analysed. They found that the variables from the expressions d_2' and d_2'', other than $(r - D)$, had explanatory power. Hillegeist et. al. (2002) tested a modified d_2' that included the expected market return on the firms assets rather than the risk free rate. The definition

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15 Geske (1979) provides a solution method for this type of option.

of the expected market return on the firms assets used in the Hillegeist et. al. (2002) study (pg. 7) is constructed from financial statement data. They use a discrete hazard function approach proposed by Shumway (2001), and find that all the variables have explanatory power.

One of the aims of tests of the option-based approach is to demonstrate that market-based variables are superior to financial statement information in bankruptcy prediction models. To this end Hillegeist et. al. (2002) add a Z-sore value as an explanatory variable to their model. This variable is found to be significant, as is the expected return on assets which is also based on financial statement information. These findings support the conclusion that appropriate financial statement information should be included in empirical bankruptcy models.

2.1.5 Optimising firms in a uncertain environment

The final model discussed is that of Bensoussan and Lesourne (1980). The extended version of this model (see Bensoussan and Lesourne ,1981), is the starting point for the derivation of bankruptcy conditions which follow. The model is of a firm controlled by its shareholders. The firm is attempting to maximise shareholder wealth. The model presented is of a "self financing" firm. All investments are financed out of retained earnings, with no borrowing permitted in the formulation. Investments made by the firm are completely irreversible.

The objective of the model is to maximise the expected discounted stream of dividends over the life of the firm, namely,

$$\max_{w(t),v(t)} E\left[\int_0^{\tau} w(t)e^{-it}dt\right], \tag{2.7}$$

where $w(t)$ is the total dividend payment by the firm in period t (a decision variable), $v(t)$ is the net investment by the firm in period t (a decision variable), and $m(t)$ is the firm's cash balance in period t. τ is length of the firm's life, that is, the first date t when the firm's cash position is negative. $m(t)$ is less than zero.

Optimisation of (2.6) is subject to,

$$\dot{m}(t) = [\lambda + \epsilon\gamma(t)]f(c(t)) - v(t) - w(t)$$

where $\dot{m}(t)$ is the change in the cash position of the firm at time t. It represents the firm's cash flow. The variable, $c(t)$, is the net investment of the firm or the amount of productive capital available to the firm at t. While $\lambda f(c(t))$ is the average profit. λ is a positive constant and $f(\bullet)$ a continuous concave function. Finally, $\epsilon\gamma(t)$ is a random variable where ϵ is a constant and $\gamma(t)$ is a standard normal variate. Profit in any time period is a random variable. The risk associated with each unit of profit is assumed to be proportional to the average profit in that period. This assumption is implemented through the use of a multiplicative random component in the profit function.

The firm can distribute profit to shareholders as a dividend , invest in productive equipment or increase its cash position,

$$\dot{c}(t) = v(t),$$

where $\dot{c}(t)$ is the increase in productive equipment by the firm at time t.

The behaviour of the decision maker is constrained by ensuring that the dividend at any time is greater than or equal to zero. That is,

$$w(t) \geq 0,$$

The firm has a perfectly irreversible investment policy,

$$v(t) \geq 0,$$

with assets able to be purchased but not resold to raise finance. It follows that

$$w(t) + v(t) \leq \lambda f(c(t)).$$

The above condition does not allow the manager to distribute a dividend and make investments that sum to more than the average profit level in any period. The firm starts its life with equity funding, K_0, which is held as cash or invested in productive equipment. These initial conditions are given by

$$m(0) = m_0; c(0) = c_0$$

This model is solved using dynamic programing[16]. The Hamilton-Jacobi-Bellman equation for the system is derived and analysed. This equation is a partially degenerate, second order, partial differential equation. Analysing the optimal feedback of this equation leads to three possible solution regimes. They are,

1. The cash regime where the firm keeps its expected cash flow without investing or distributing a dividend.

2. The investment regime where the firm uses its expected cash flow to invest without distributing dividends.

3. The dividend regime in which the firm distributes all the expected cash flow in the form of dividends.

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[16] See Ross (1983) for an introduction to Dynamic Programming techniques.

Bensoussan and Lesourne (1980) conducted two numerical experiments within their model's framework. The first was to map the boundaries of the three solution regimes. The second dealt with survival probabilities of the firm. The authors noted that these probabilities depended not only on the profit function chosen and the multiplier of the random profit element, ϵ, but also on the initial level of equity funding, K_0, and the proportion kept as cash, α. A series of simulations were carried out where the firm is followed until it is bankrupt, at which time the firm lifetime, τ, was recorded. These lifetimes were then used to construct a probability distribution, $P(t)$, which gave the probability of the firm going bankrupt at some date, τ, less than t[17]. The simulated distributions display the following properties. Bankruptcy occurs most commonly in the first years of operation. Further, an increase in the proportion of equity held as cash reduces the peak of the distribution and spreads the risk over time. Finally, an increase in initial equity has the same effect, although to a lesser extent.

2.2 Empirical Regularities

A set of financial statements conveys information about many aspects of a firm's financial position. All publicly traded firms are required to publish a Profit and Loss (P&L) statement and a Balance Sheet (BS) each year. The P&L records the firm's income and the expenses incurred in generating the profit over an accounting period. A Balance Sheet records the company's assets and liabilities at the end of an accounting period.

A simple P&L would follow the relationships as detailed in Table 2.2 below. All of the accounts that contribute to the P&L are accrued over the accounting period.

[17] See Bensoussan and Lesourne (1980, page 262-3) for graphics of the simulation results.

Revenue – Cost of generating revenue = EBDIT
EBDIT – Depreciation = EBIT
EBIT – Interest = EBT
EBT – Taxes = Net Income (Profit/ Loss)
Net Income – Dividend = Change in Owner's Equity

Table 2.2. **Structure of a Profit and Loss Statement**

The Balance Sheet is a snapshot of the firm's assets and the claims against those assets at the end of an accounting period. By convention the assets and liabilities are grouped by maturity. Current assets and liabilities can be realised, or mature, within the accounting period. Long term assets and liabilities have lives longer than a single accounting period. A simple Balance Sheet structure is presented in Table 2.3.

Assets	Liabilities
Cash + Marketable Securities + Accounts Receivable + Inventories = Current Assets Plant, Land & Buildings - Accumulated Depreciation + Intangibles = Non Current Assets	Short Term Debt + Trade Creditors = Current Liabilities Borrowings + Provisions = Non current Liabilities Owner's Equity
Total Assets = Current + Non Current	Total Liabilities = Current + Non Current + Owners Equity

Table 2.3. **A simple Balance Sheet Structure**

The assets and liabilities are brought into balance through the "residual claim"[18] of the shareholders on the assets of the firm.

To compare the financial positions of firms, or a single firm through time, financial statement data needs to be standardised. The most common form of standardisation is by the use of financial ratios. A ratio can be formed from any two items in the

[18] The residual claim is the difference between Total Assets and the sum of Current and Non Current Liabilities. This is the maximum amount that the shareholders could expect to recover from the firm in the event of bankruptcy. In most bankruptcy situations these book values are not realised.

financial statements[19], so there is a potentially large set of ratios available to work with. Chen and Shimerda (1981) list sixty-five different ratios that have been used in failure classification studies. The thrust of studies looking for empirical regularities in financial statement information is to group the ratios in some way, then select one ratio from each group as being representative of the factor which the group represents. The grouping method can be ad-hock or statistical. In Beaver's (1967) study, he collected thirty ratios by "frequency of appearance in nineteen financial-statement analysis texts". He then grouped them into the six "common element(s)", as listed in Table 2.4, in an ad-hoc manner.

Group Name
Cash Flow ratios
Net income ratios
Debt to total asset ratios
Liquid asset to total asset ratios
Liquid asset to current debt ratios
Turnover ratios

Table 2.4. **Ratio Groupings - Beaver (1966)**

Later studies, such as Pinches, Mingo & Caruthers (1973), Stevens (1973) and Libby (1975), applied statistical dimension reduction methods to sets of variables derived from failure classification studies. The aim of these studies was to extract a set of factors which can account for the information in the financial ratios. Pinches, Mingo & Caruthers (1973) applied Factor Analysis[20] to a set of forty-eight financial ratios[21] constructed for two hundred and twenty-one industrial firms. They found seven underlying

19 Noting that many of the combinations would be difficult to interpret in a financially meaningful way.

20 Factor analysis is a statistical technique to explain the correlations between observable variables in terms of underlying factors, which are themselves not directly observable.

21 Definitions of the components of the financial ratios mentioned in this Chapter can be found in the Chapter Appendix.

factors captured the information in these ratios. Their interpretation of the factors in financial terms is shown in Table 2.5. The stability of the factors was investigated by repeating the analysis at six yearly intervals from 1951 to 1969. The same seven factors were found each time, but the factor weights of the ratios changed over time. Ratios with factor weights of 0.7 or more[22] in all four periods are listed as possible representative ratios in Table 2.5. The changes in factor weights were explained by financial

Group Name	Representative Ratios
Return on Investment	Cash Flow / TA, Cash Flow / Net Worth, Total Income / TA, Net Income / TA, Net Income / Net Worth, EBIT/ TA, Cash Flow / Total Capital, Total Income / Total Capital
Capital Intensiveness	Net Worth / Sales, Sales / Total Capital
Inventory Intensiveness	Sales / Working Capital
Financial Leverage	Debt / Total Capital, Total Liabilities / Net Worth, Debt / TA, Total Liabilities / TA
Receivables Intensiveness	Receivables / Inventory, Receivables / Sales
Short Term Liquidity	Current Assets / Current Liabilities, Current Assets / TA, Quick Assets / Current Liabilities
Cash Position	Cash / TA, Cash / Current Liabilities, Cash / Fund Expenditures

Table 2.5. **Ratio Groupings -Pinches, Mingo and Caruthers (1973)**

trends over the period. The trends were increasing use of debt, increasing expenditure on capital assets and changes in receivables management practices. The stability of the factors over time establishes their importance as empirical regularities. Chen and Shimedra (1981) reconciled the results of factor studies by Stevens (1973) and Libby (1975) with the seven factors found by Pinches, Mingo & Caruthers (1973). There were ten ratios identified as being significant in failure studies which were not included in the Pinches, Mingo & Caruthers study. After applying a principal components analysis[23],

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22 Factor loadings enumerate the extent to which a variable is related to a factor. They are usually interpreted as the correlation between the variable and factor. So a 0.7 weight would be interpreted as a 70% correlation between the variable and the factor.

23 For investigations involving a large number of observed variables, it is often useful to simplify the analysis by considering a smaller number of linear combinations of the original variables. A linear combination of variables with weights summing to one is called a standardised linear combination. Principal

each of the ratios could be assigned to one of the seven factors. This result confirms the idea that the information in a set of financial statements can be reduced to seven factors. Finally, Chen and Shimedra (1981) observe:

> It is important then that a minimum number of ratios, one ratio in most cases, be selected to represent each factor for further statistical analysis. Still, the question of which ratio should represent a factor has yet to be resolved. (pg. 59)

Gombola and Ketz (1983) noted the definition of cash flow used in previous studies, as being Net Income plus Non Recurring Income on Expenses plus Depreciation. This definition resulted in an approximation for cash flow as cash flow from operations. A factor analysis was conducted using forty ratios, four of these having operating cash flow as their numerator, for each year from 1962 to1980. Eight distinct factors were found. They were the seven factors found by Pinches, Mingo & Caruthers, and a cash flow from operations factor. The ratios Cash flow from Operations to Total Assets (TA) and Cash flow from Operations to Equity were found to be consistently related to the cash flow factor. The existence of the seven factors found by Pinches, Mingo & Caruthers was again confirmed. The finding of the Cash flow form Operations factor is important from a corporate finance perspective, as analysis of cash flow rather than accounting profit is one of the basic premises of corporate finance.

Non-ratio variables such as, earnings and debt variability, or changes in financial statement items, have not been included in dimension reduction studies. Little is known about how this type of measure relates to the factors which have been found. Non-ratio variables have been included in failure classification studies. Altman et.al. (1977) included the errors about a ten year trend in the EBIT to TA ratio as a measure of the

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components analysis finds a set of standardised linear combinations, called the principal components, which are orthogonal and taken together explain all the variance of the original data.

volatility of earnings, Ohlson (1980) included the ratio, $\frac{(NI_t - NI_{t-1})}{(|NI_t| - |NI_{t-1}|)}$, to measure the change in Net Income. These variables attempt to measure some dynamic aspect of financial behaviour. Ratio analysis, being based on a single set of financial statements, is a form of static analysis. The results of the Altman Zeta MDA model and Ohlson's probit model, suggest that a combination of static and dynamic variables can be useful in failure classification.

Laitinen (1991) applied factor analysis to the ratios of a group of failed Finnish companies. The failed firms in his sample could be allocated into three groups on the basis of their factor loadings. Each of the groups was associated with a failure type. The first type was a "chronic failure" where the company's ratios were poor four years before ultimate failure. The second an "acute failure" where the ratios were good until the final year before failure, after which they deteriorated quickly. The final type was a "revenue failure". These companies had poor profitability and low revenue growth leading up to ultimate failure. These results suggest that explanatory models of failure should be capable of generating at least these three different paths to failure.

Eight factors have been derived from the variable sets used in classification studies. A small number of dynamic variables calculated from a sequence of financial statements have also been used. Three distinct failure processes have been identified using factor analysis on the ratios of failed companies. Explanatory models of failure behaviours need to yield results that can be related to these results. The development of explanatory models can also help to resolve the question of which ratio should be selected to represent a factor, and the nature of dynamic elements to be incorporated in empirical studies.

2.3 Classification Studies

Much empirical work has been concerned with the classification of a firm as a going concern or failure. The work in this area has largely been driven by the classification tool used. Inputs to the classification tool have usually been ad hoc, based on what has worked in previous studies. These studies have been criticised on the basis of their experimental design and inappropriate use of the various classification tools. However, they have provided the set of variables that have been analysed by factor analysis to derive patterns in financial statement data, as discussed in the previous section.

The first distress classification study was conducted by Beaver (1967). The aim of this work was the verification of the usefulness of financial statement information. The ability of ratios, constructed from financial statement data, to predict failure was the application chosen. A matched pairs experimental design was used, the cases were matched on industry and asset size. Using one ratio from each of the six groups identified in Table 2.4, a sequence of tests were carried out. A profile approach was adopted, which plots the means of the failed or non failed groups over a five year period, and conducts a classification test for each ratio. A cutoff point is selected (subjectively) to maximise the number of correct classifications on one half of the sample. This cutoff point is then used to classify the other half of the sample, and likelihood ratio analysis of distributions (histograms) of the ratios for the failed and non failed firms. The methods show that ratios (financial statement information) can discriminate between failed or non failed firms. Using the results from the classification tests (percentage misclassification one year prior to failure) the ratio groups can be ranked. The classification test results are presented in Table 2.7.

Group Name	Misclassification Rate
Cash Flow ratios	13%
Net income ratios	13%
Debt to total asset ratios	19%
Liquid asset to total asset ratios	24%
Liquid asset to current debt ratios	20%
Turnover ratios	23%

Table 2.6. **Classification Results - Beaver (1966)**

The logical extension of the univariate approach taken by Beaver was the inclusion of a number of ratios in a multivariate analysis. Altman (1968) used Multivariate Discriminant Analysis (MDA) to include the information from a number of ratios in a classification model. The technique of MDA fits a linear discriminant function, $Z = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_n X_n$, which transforms the values of a firm's ratios into a single Z-score. This score is then used to classify the firm. The β's are estimated by maximising the distance between the average failed and non failed firms, while minimising the distance of the observations from their group average value. The later Zeta model of Altman, Haldeman and Narayannan, (1977), will be analysed as a representative MDA model. This model was selected as it has been widely used in commercial situations. The experimental design was a matched pairs approach, industry and year were used as the matching variables. Seven variables were used in this model:

1. Earnings Before Interest and Taxes divided by Total Assets (EBIT/TA) representing **return on assets.**

2. A normalised standard error from a ten year trend in EBIT / TA measures **volatility of earnings**. This variable is not a standard ratio. It is a measure of earnings variability derived from a series of financial statements.

3. The inverse debt cover ratio (total interest payments divided by EBIT) measures the firm's ability to **service debt.**

4. Retained Earnings divided by Total Assets measures the firm's **cumulative profitability**, this variable will be influenced by the age of the firm, its dividend policy and the ability to generate profit over time.

5. The current ratio, Current Assets divided by Current Liabilities, measures the short term **liquidity** of the firm.

6. Five year average values of the market value of common equity over total capital is used to measure the firm's **capitalisation**

7. The log of total tangible assets is used to measure **size** of the firm.

The resulting MDA had a misclassification rate of 7.2% with data one year prior to failure. This was an improvement over the best misclassification rate from the univariate study of 13%. This increase in classification accuracy suggests that a number of different factors and their interaction are the cause of corporate failure. The results also show that a variable based on the dynamics of financial statement data, not in ratio form, can have significant discriminating power. Studies by Blum (1974), Libby (1975) and Schipper (1977) among others, also applied MDA[24].

The MDA approach, and Altman's model in particular, has continued to be a focus of investigation. For example, Grice and Ingram (2001) test the generalisability of the Altman model with respect to time frame, industry classification and other distress types. They find that the model does not generalise over time frame and industry clas-

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24 See Altman (1993, Chapter 12, pp.279) for a bibliography of non US failure classification studies.

sification. The model has also been widely compared with competing models (usually Ohlson's logit approach), examples of comparison studies are Begley et. al. (1996), Mossman et. al. (1998) and Lenox (1999). The logit approach is found to consistently outperform the Altman model.

Problems with the MDA approach to classification have been highlighted by a number of authors, see for example Joy and Tollefson (1975) or more recently Grice and Dugan (2001):

1. The proportion of firms from each of the populations (failed and continuing) included in a classification study should reflect the prior probabilities of population membership. The use of matched pair experimental designs causes failed firms to be overrepresented in samples.

2. The statistical requirements of linear MDA, that variance-covariance matrices for each group should be the same and the normality assumptions are not verified[25].

3. The result of MDA models is a score (Z-score) which provides an ordinal ranking of firms processed by the model. To find the score that best discriminates between the failed and non failed group the probability of occurrence in the population and the costs of misclassification are required. In most studies the costs of misclassification have been assumed (often implicitly) to be equal. In reality this is unlikely to be the case.

4. To validate a classification model the following steps are required. First, fit the model to a set of data. Then classify a holdout sample from the same time period

[25] Indeed it can be argued that the existence of a variance-covariance matrix is not assured when the variables are ratios with a common denominator, as is the case when financial ratios are used as explanatory variables. This issue is discussed is Pearson (1897).

as the estimation sample. Success in this step implies that the variables included in the discriminant function are important. Finally, classify a sample of firms from a later time period. This is a prediction. Most classification studies have not gone beyond the first step, a few the second.

5. The results are sample specific, they do not generalise over different time periods and industry classifications.

The problems identified cover all stages of a classification study, the experimental design, the application of the MDA statistical method through to validation of and forecasting. Most classification studies have one or more of these problems. The most common approach for avoiding the statistical assumptions of MDA has been the use of some other classification tool.

Ohlson (1980) applied a logistic regression[26] analysis to a sample of one hundred and five failed firms and two thousand and fifty-eight nonfailed firms. The use of a non matched design provides a sample of firms which is closer to observed population proportions. It also allows the size variable that had been used to match failed and non failed firms to be tested as an explanatory variable. The results in Ohlson (1980) show that

> ... the four factors derived from financial statement which are statistically significant for the purposes of assessing the probability of bankruptcy are: *i)* size; *ii)* the financial structure as reflected by a measure of leverage; *iii)* some performance measure or combination of performance measures; *iv)* some measure(s) of

[26] A logistic regression estimates a probability value. Let $\underset{-}{X}_i$ be the vector of firm specific variables and $\underset{-}{\beta}$ the vector of prarameters. $P(\underset{-}{X}_i, \underset{-}{\beta})$ is the probability of bankruptcy given $\underset{-}{X}_i$ and $\underset{-}{\beta}$, where $0 \leq P(\bullet) \leq 1$. The logistic function is $P = (1 + \exp(\underset{-}{\beta}'\underset{-}{X}_i))^{-1}$. The parameter values are found by maximising the following log likelihood function:

$$l(\beta) = \sum_{i \in S_1} \log(P(\underset{-}{X}_i, \underset{-}{\beta})) + \sum_{i \in S_2} \log(1 - P(\underset{-}{X}_i, \underset{-}{\beta}))$$

where S_1 is the set of failed firms and S_2 is the set of nonfailed firms.

current liquidity. (pg. 123)

The logistic regression model produces an estimate of the probability of failure for a firm with attributes X_i. Classification of a firm as failing or non failing requires a rule to transform this probability. To generate classification rates for comparison with other studies, a simple procedure of classification error minimisation (reminiscent of Beaver's approach) was used. The logit method was also used by Zavgren (1985) in a matched pairs experimental design. The cutoff point for failed or non failed classification was derived from entropy theory. The important contribution of this work was the use of the seven factors found by Pinches, Mingo & Caruthers (1973) as explanatory variables. The analysis was repeated for each of five years before failure. A tick ($\surd$) in Table 2.6 represents the variable indicating the factor[27] was significant at greater than 95% confidence up to five years before failure. Of the seven factors only

Years before Failure	Inventory Turnover	Receivables Turnover	Cash Position	Short Term Liquidity	Return on Investment	Financial Leverage	Capital Turnover
1			√	√		√	
2				√		√	
3	√			√		√	
4	√	√				√	
5	√		√			√	√

Table 2.7. **Logit Results - Zavgren (1985)**

Financial Leverage was significant in all years. Some factors, like Short Term Liquidity are significant close to failure while others, like Inventory Turnover are significant

27 The ratios used in the study were:
Total Income / Total Capital ⟶ **Return on Investment**
Sales / Net Plant → **Capital Turnover**
Inventory / Sales → **Inventory Turnover**
Debt / Total Capital → **Financial Leverage**
Receivables / Inventory → **Receivables Turnover**
Quick Assets / Current Liabilities → **Short Term Liquidity**
Cash / Total Assets → **Cash Position**

further out from failure. The Olson (1980) and Zavgren (1985) studies took a "cross section" approach to their analysis, using only one observation from each firm included in each regression. This ignores the information contained in a sequence of financial statements. These results show that fewer than the seven factors found in financial statement information are required to explain the probability of financial distress. They also introduced modeling financial distress in terms of probability models, even though the authors felt compelled to generate classification results for comparative purposes. Laitinen and Laitinen (2000) use the variables suggested by Beaver in a Logistic regression framework. They expand the analysis by taking a second-order Taylor's expansion of the logistic function and include the second-order terms in the modeling. They find that the interaction (cross product) terms add to the classification accuracy of the model. Another probabilistic approach is based on the estimation of hazard functions. The approach is suggested by the "reduced form" approach to the pricing of risky debt suggested in Jarrow and Turnbull (1995), where the probability of bankruptcy is taken to be a random occurrence. Shumway (2001) builds a discrete hazards model and evaluates market-based and financial statement variables, finding both classes of variables to be significant. Hillegeist et. al. (2002) use the same modeling approach and draw similar conclusions.

Several pattern recognition algorithms have been applied to failure classification problems. The algorithms are not based on statistical assumptions, thus overcoming the need to validate statistical assumptions. The method of Recursive Partitioning was applied to failure classification by Frydman, Altman and Kao (1985). Recursive partitioning is an iterative nonparametric method that makes no distributional assumptions. The method works by:

1. Selecting the single explanatory variable that classifies cases with minimal cost.

2. Establishing a cutoff point using this variable.

3. If there is substantial mixing of case types in either group, then select a variable for further partitioning.

4. Establish a cutoff point for this variable in the mixed group.

5. Proceed until all final nodes in the binary tree formed form the current variables are predominantly from a single group.

This method is a forward selection method which does not review previous variable selections at each splitting. Variables may enter the process more than once, making interpretation of the results difficult. The method does not provide an estimate of the probability of classification. A recent application of recursive partitioning is McKee and Greenstein (2000). A related approach known as Rough Set theory was applied to the problem of bankruptcy by Mckee (2000). Yang, Platt and Platt (1999) compared two Neural Network approaches to MDA. They found that the MDA was best at classifying bankrupt firms. Anandarajan et. al. (2001) survey recent studies using neural networks and confirm the finding that these models outperform MDA in classification exercises. There is a tendency for pattern recognition methods to over fit within sample, especially when a variable is used more than once. Recursive Partitioning could well fall victim to this problem without careful monitoring of the termination condition of the algorithm.

2.4 Summary

A number of authors have attempted to formulate a model which describes bankruptcy behaviour. Of the studies reviewed, the works of Wilcox (1976), Scapens, Ryan and Flecther (1981) and Francis, Fastings and Fabozzi (1983) are based on constructing a formulation of the bankruptcy problem that is consistent with the mathematical or statistical technique used. For example, the use of Catastrophe Theory by Scapens, Ryan and Flecther (1981) necessitates the use of a specific functional form to generate a cusp catastrophe. There is no attempt in either of the Catastrophe Theory studies to relate this functional form to the process of bankruptcy. The models of Scott (1980), Schipper (1977), the contingent claim approach and Bensoussan and Lesourne (1980) are based on development of an economic argument. As a result all of the models are dynamic formulations. Schipper (1977) and Bensoussan and Lesourne (1980) provide explicit optimising criteria and are solved by optimisation techniques. The Scott (1980), the contingent claims approach and Bensoussan and Lesourne (1980) models all incorporate uncertainty through a random element. Only the Bensoussan and Lesourne (1980) model achieves the aim of building a model that focuses on cash flows, attempts to optimise some economic criteria, has a stochastic element and is dynamic.

The many financial ratios that have been used as inputs to bankruptcy classification models were analysed by Pinches, Mingo & Caruthers (1973). They found that the forty eight ratios they analysed could be characterised by the seven factors described in Table 2.5. These factors were used as the explanatory variables in a bankruptcy classification study by Zavgren (1985). The results of this study show that only three of the seven factors, financial leverage, short term liquidity and the cash position, are significant in a probabilistic regression framework one year prior to failure. Altman, Haldeman and Narayannan, (1977) and Ohlson (1980) both find a dynamic variable significant in their models. The dynamic element in the Altman, Haldeman and Narayannan (1977) study is earnings volatility, a measure of the variability of the profit function. Ohlson (1980) included the ratio, $\frac{(NI_t - NI_{t-1})}{(|NI_t| - |NI_{t-1}|)}$, which is also an approximation for the variability of profit. These empirical results indicate that a dynamic structure, which involves the variability of profit, and some combination of the firm's debt, cash position and short term liquidity should appear in the development of models which seek to explain bankruptcy.

2.A Appendix: Financial Ratio Components

Cash Flow = Net Income + Non Recurring Income/Expenses + Depreciation;

Total Income = Net Income + Non Recurring Income/Expenses;

Total Liabilities = Current Liabilities + Long Term Debt;

Net Worth = Total Assets - Total Liabilities;

Total Capital = Total Invested Capital - Long Term Debt;

Working Capital = Current Assets - Current Liabilities;

Quick assets = Cash + Receivables;

Fund Expenditures for Operations = Operating Expenses - Depreciation;

Operating Expenses = Net Sales - (Cost of Goods Sold + Operating Income);

EBIT = Net Income + Fixed Charges + Income Tax.

Chapter 3
Theoretical Framework

In this section a dynamic, stochastic, constrained dividend maximising model of a firm facing the possibility of bankruptcy is developed. An optimality criterion for the model is presented and analysed. The model, and its solution concept, is based on the work of Bensoussan and Lesourne (1981) who were concerned with the policies which would lead to optimal growth of a firm facing an uncertain economic environment. An analysis of the optimising conditions identifies a number of policy regions, or actions that the firm may take. These policy regions are identified and the probability of the firm becoming bankrupt under each policy is derived.

The modelling framework in this section, a stochastic dynamic optimising approach, differs from the stability analysis approach of Schipper (1977) and the catastrophe theory approach of Scapens, Ryan and Fletcher (1981). These models rely on the analysis of the stability of solutions of a dynamic system. The stochastic models of Wilcox (1976), a gambler's ruin approach, along with imperfect access to capital model of Scott (1981) and the option approach concentrate on the liquidation value of the firm. This value is measured by the asset base in the gambler's ruin approach, the market value of equity in the Scott model or the value of the firm in the options approach. The bankruptcy conditions of these models are not based on the cash position of the firm, which is fundamental to the model presented here.

In the model, a firm is created with a given initial equity with no new equity issues made during the life of the firm. These funds can be invested in productive resources or held as cash balances. The productive resources are used to earn random earnings in any period. If the profit derived from these earnings is positive, it can be used to pay dividends to the shareholders, invest in new productive resources, repay outstanding debt or increase the firm's cash balance. The firm is able to borrow and repay funds up to a credit limit. When the cash position of the firm falls to zero the firm is bankrupt[28]. The firm attempts to maximise the stream of dividends paid to shareholders during its life. The solutions of the model and associated bankruptcy probability conditions are derived from a dynamic programming solution to the model.

The variables which differentiate between the possible model solutions identified by the analysis are proxied by variables constructed from financial statement data. These proxy variables will be used in the empirical validation of bankruptcy probability expressions derived from the model. The random nature of the time horizon in the model for a single firm provides the rationale for the use of duration or hazard-based statistical methods in the validation of bankruptcy probability expressions.

To get a feeling for how bankruptcy can occur in a firm facing uncertain earnings, a two period example is presented. The dynamic nature of the example shows the need for a dynamic solution approach. The firm is created at time zero with equity, K_0, with no new equity issued. This initial equity is split between cash, m_0, and productive equipment, c_0, both of which are held constant in this example. The cash balance of the firm defines its current state. The firm is bankrupt if it runs out of cash. Productive equipment is used to generate the firm's earnings. The production process is described by a production function, $\theta(c_t)^a$ where $0 \leq a < 1$.[29] These earnings are uncertain.

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[28] This is an inability to meet current commitments. Such events cause the firm to default on a loan agreement, triggering the bankruptcy process. See Giroux and Wiggins (1984) for a description of this sequence of events.

[29] This is a production function of the Cobb-Douglas type with a single factor of production. The function is continuous, twice differentiable and displays decreasing returns to scale.

The uncertainty is described by a simple binomial mechanism. The firm can have a positive earnings "surprise" with probability p, or a negative earnings "surprise" with probability $(1-p)$. The size of the "surprise", S, is constant in each period. This is similar to the approach used in the gambler's ruin model. The earnings of the firm in any period are described by the equation

$$\pi_t = \theta(c_t)^a + \begin{cases} +S \text{ with probability } p, \\ -S \text{ with probability } (1-p) \end{cases} .$$

The firm has a fixed borrowing limit, X. For this example the firm has borrowed up to this limit and is holding the funds aside pending some investment decision. The firm is required to pay interest at the rate of r per period on outstanding debt. The total interest payable in any period is rX. The decision available to management in any period is payment of a dividend, D_t, or increasing the cash balance of the firm, m_t. The cash position of the firm at the end of each period is given by the cash flow equation

$$m_t = m_{t-1} + \pi_t - rX - D_t.$$

Using this equation, the maximum dividend that can be paid in any period is given by,

$$D_t \leq m_{t-1} + \pi_t - rX$$

The objective is to maximise the discounted expected dividend payout to equity holders,

$$E\left[D_1 + \frac{1}{(1+i)}D_2\right], \tag{3.8}$$

where i is the shareholders' discount rate, D_1 is the dividend paid in period 1 and D_2 is the dividend paid in period 2. Using the definition of uncertain earnings and the cash flow equation, a bankruptcy condition and probability can be derived. Only a negative earnings surprise needs to be considered, as this is the only case where earnings can be negative. If the surprise is greater than the cash balance plus the certain part of earnings minus interest payable in any period, then

$$S \geq m_t + \theta(c_0)^a - rX,$$

and the firm is bankrupt. The probability of bankruptcy in any period is $(1-p)$, which is the probability of the firm having a negative earnings surprise.

Is it possible for management to make an optimal choice of the dividend to be paid at time zero for the two remaining periods that the example runs? One approach to making this decision is to work out the expected value of earnings in each period,

$$E(\pi_t) = \theta(c_o)^a + S(2p-1),$$[30]

and then use this value to construct a deterministic optimisation problem. Using this approach has the major drawback of ignoring the effect of the firm's decisions in one period on the possible actions in subsequent periods. For example, at period zero the split of initial equity between cash and productive equipment influences the earnings, cash flow and maximum dividend payable in the first period. If the firm is solvent after earnings are determined, management then has to choose the dividend payout. This decision determines the cash balance that is carried forward to the second period. The interaction of this cash balance and earnings determines the solvency of the firm in this second period. If the firm is solvent, management will set the dividend payment to the cash balance as the model terminates. In general terms, each decision management takes has two impacts on the objective function. A direct one which can be seen in the objective and an indirect one, through the influence of the dividend paid on the cash balance in the next period. Bellman's (1957) dynamic programming algorithm was developed to handle problems which have this type of dynamic interdependence.

30 The expected value of the earnings surprise is, $Sp - S(1-p)$, which simplifies to $S(2p-1)$.

Using a dynamic programming approach, it is possible for management to determine the optimal dividend policy for each period.

In the remainder of this section the dynamic programming approach is presented, followed by the structure of the model and the determination of the optimality condition. Finally, the possible policies in the model, along with the conditions which would cause a bankruptcy under each of the identified policies are derived.

3.1 Dynamic Programming

The firm's status is described by a state variable, x. While this is taken to be a one dimensional scalar, the method can be extended to a vector of states. At any period t, the value of this variable, x_t, is known and the future values, $x_{t+1}, x_{t+2}, \cdots$ are random variables. Suppose that the process driving x_t is Markov, that is, all the information required to determine x_{t+1} is summarised in the current state x_t.

At any period, t, choices are available to the firm. They are represented by the control variables, u. In the example above the only choice was the dividend payout in each period. The value, u_t, of the control at time t must be chosen using the information that is available at this time, x_t.

The state and the control at time t affect the immediate dividend, $V(x_t, u_t)$. Here the control variable is the dividend payout. The x_t and u_t of period t also affect the probability distribution of future states. Let $\Phi_t(x_{t+1}|x_t, u_t)$ be the cumulative probability distribution function of the state variable in the next period, conditional on the current information.

The aim is to choose a sequence of the control over time, $\{u_t\}$, so that the expected present value of the payoff is maximised. The discount rate used in the present value calculation is ρ. Sometimes the decision process ends after T periods, with a terminal payoff dependent on the final state reached. Let $\Omega_T(x_T)$ be this terminal value function.

The basic dynamic programming technique can be applied to this problem. The idea is to split the decision sequence into two parts, the immediate period and the continuation beyond this period. Suppose the current period is t and the state is x_t. Then $F_t(x_t)$ is the outcome, the expected present value of the firm's dividend policy when the firm makes all decisions optimally from this point onwards, that is $E_t\left[F_t\left(x_t\right)\right] = F_t\left(x_t\right)$.

When the control variable, u_t, is chosen there is an immediate return, $V(x_t, u_t)$. At the next period, $(t+1)$, the state will be x_{t+1}. Optimal decisions from this time on will generate $F_{t+1}(x_{t+1})$. This value is random at period t, so its expected value,[31] $E_t\left[F_{t+1}(x_{t+1})\right]$, can be calculated. This is the continuation value. Discounting back to period t, the sum of the immediate payoff and continuation value is

$$V(x_t, u_t) + \frac{1}{(1+\rho)} E_t\left[F_{t+1}(x_{t+1})\right].$$

The firm will choose u_t to maximise this, and the result is the continuation value from period t, $F_t(x_t)$. That is,

$$F_t(x_t) = \max_{u_t}\left\{V(x_t, u_t) + \frac{1}{(1+\rho)} E_t\left[F_{t+1}(x_{t+1})\right]\right\}. \tag{3.9}$$

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31 If X is a random variable defined on the probability space (Ω, F, P), the expectation of X is defined by

$$E(X) = \int_\Omega X dP.$$

The idea behind the decomposition of the decision sequence is derived from Bellman's Principal of Optimality. An optimal policy has the property that, whatever the initial conditions are, the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision.

The result of the decomposition, equation 3.9, is called the Bellman equation. In this equation the first term on the right hand side is the immediate return, the second term is the continuation value. The optimal action in the current period is the one which maximises the sum of these two values[32].

In a problem with a finite time horizon, T, an optimal value for u_t can be found by working back from the final period. In period T, the firm gets the terminal value, $\Omega_T(x_T)$. Then in period, $T-1$,

$$F_{T-1}(x_{T-1}) = \max_{u_{T-1}} \left\{ V(x_{T-1}, u_{T-1}) + \frac{1}{(1+\rho)} E_{T-1}\left[\Omega_T(x_T)\right] \right\}.$$

The value function $V(x_{T-1}, u_{T-1})$, is then known. This allows the solution of the maximisation problem to find u_{T-2}, and the continuation value function, $F_{T-2}(x_{T-2})$. This procedure is repeated until the optimal sequence of controls, $\{u_t\}$, is found.

If the problem has an infinite horizon there is no terminal value to work backwards from. Instead, the problem has a recursive structure that allows for analytical analysis and numerical simulation. Provided that the value function, $V(\bullet)$, the transition probability function, $\Phi_t(\bullet)$, and the discount rate, ρ, are independent of time, the calendar date, t, has no effect on the solution. In this form, the problem one period in the future looks the same as the problem now, except for the starting date. While the value function is common to all periods, it will be evaluated at different states x_t. The

32 The objective function of the example outlined above, equation 3.8, is a simple Bellman equation.

function, $F(x_t)$, is written without time subscripts. The Bellman equation for any t is,

$$F(x_t) = \max_{u_t} \left\{ V(x_t, u_t) + \frac{1}{(1+\rho)} E_t \left[F(x_{t+1}) \right] \right\}.$$

As x_t and x_{t+1} could be any of the possible states, they can be written in general form as x and x'. Then for all x,

$$F(x) = \max_{u} \left\{ V(x, u) + \frac{1}{(1+\rho)} E \left[F(x'|x, u) \right] \right\}, \tag{3.10}$$

and the expectation is now conditioned on knowledge of the current period's x and u. This is the Bellman equation for a recursive dynamic programming problem. To find the optimal sequence of controls, $\{u_t\}$, the continuation value function $F(\bullet)$ must be specified. As there is no terminal date to work back from, a new method of finding the function $F(\bullet)$ is needed.

The recursive Bellman equation, equation 3.10, can be transformed into a list of equations, one for each possible x,and with a list of unknowns, the $F(x)$. Equation 3.10 can be thought of as a functional equation, with the function $F(x)$ as its unknown. This equation is not linear. The optimal choice of u depends on all the values $F(x')$, weighted by their probabilities, in the expectation on the right hand side of the equation. When the optimal control is back-substituted the result is often non linear in the $F(x')$ values. There is, however, an iterative solution procedure that can be used to solve the functional equation. Start with any guess for the continuation value function, say $F^{(1)}(x)$. Use it on the right hand side of of equation 3.10 to find the corresponding optimal policy $u^{(1)}$, which can be expressed as a function of x alone. Substitute this policy back, then the right hand side becomes a new function of x, say $F^{(2)}(x)$. Now, use this function as the next guess at the value function, and repeat the procedure. The successive guesses at the value function, $F^{(3)}(x), F^{(4)}(x), \cdots$ will converge to the true function. The procedure is known to converge, no matter how bad the initial guess (see Quadrant, 1980).

In order to deal with continuous time in a dynamic programming framework, we let each period be of length Δt, and then take the limit where Δt goes to zero. Further, we write $V(x, u, t)$ for the rate of flow of the dividend, so that the total dividend over a period of length Δt is $V(x, u, t)\Delta t$[33]. If ρ is the discount rate per unit time, the total effect of discounting over the period Δt is $\frac{1}{(1+\rho\Delta t)}$. The Bellman equation (equation 3.10) becomes

$$F(x,t) = \max_u \left\{ V(x,u,t)\Delta t + \frac{1}{(1+\rho\Delta t)} E\left[F(x', t+\Delta t)|x, u\right] \right\}.$$

After multiplying by $(1 + \rho\Delta t)$ and rearranging we have

$$\begin{aligned} \rho\Delta t F(x,t) &= \max_u \left\{V(x,u,t)\Delta t(1+\rho\Delta t) + E\left[F(x', t+\Delta t) - F(x,t)\right]\right\} \\ &= \max_u \left\{V(x,u,t)\Delta t(1+\rho\Delta t) + E\left[\Delta F\right]\right\}. \end{aligned}$$

Dividing by Δt and taking the limit of Δt as it approaches zero results in

$$\rho F(x,t) = \max_u \left\{ V(x,u,t) + \frac{1}{dt} E\left[dF\right] \right\}, \tag{3.11}$$

where $\frac{1}{dt}E\left[dF\right]$ is the limit of $\frac{E[\Delta F]}{\Delta t}$. The expectation is still conditioned on the current x and u, and the influence of changes in both x and u need to be included when changes in $F(x, t)$ are calculated over the interval dt. The left hand side of the Bellman equation,

33 The dividend stream over a small interval can also be described as follows. Let $D(\bullet)$ be the dividend payment up to time t. Take a second order Taylor series expansion of the discount rate

$$e^{-\rho\Delta t} = 1 - \rho\Delta t + \frac{1}{2}(\rho\Delta t)^2$$

then the dividend stream is given by

$$\int_t^{t+\Delta t} e^{-\rho(t+\Delta t-s)} dD(s)$$

which is equivalent to the expression, $V(x, u, t)\Delta t$, above as $\Delta t \longrightarrow 0$.

equation 3.11, in this form can be seen as the value per unit of time that an owner, using ρ as their discount rate, would derive from the firm. On the right hand side of equation 3.11, the first term is the immediate dividend, the second term is the expected rate of capital gain.

The stochastic process that generates the random variables $x_{t+1}, x_{t+2}, \cdots$ needs to be specified, and if possible, used to construct a Bellman equation. The Ito process is a continuous time stochastic process which has Markov properties. It is characterised by

$$dx = a(x,u,t) + b(x,u,t)dz,$$

where the first term, $a(x,u,t)$, is a drift parameter, $b(x,u,t)$ is a diffusion parameter and dz is the increment of a standard Wiener process[34]. This class of stochastic processes can be thought of as continuous time versions of a random walk. As in the continuous time case, let $V(x,u,t)$ be the rate of flow of the dividend and $F(x,t)$, the value of the firm. Let x_t be the known starting state at time t, and x_{t+dt} the random state at the end of a small period, Δt. Applying Itô's lemma to the value function F, we have

$$\begin{aligned} &E\left[F(x+\Delta x, t+\Delta t)|x,u\right] \\ &\quad = F(x+t) + \left[F_t(x,t) + a(x,u,t)F_x(x,t) + \frac{1}{2}b(x,u,t)F_{xx}(x,t)\right]\Delta t + o(\Delta t), \end{aligned}$$

where $F_t(\bullet)$ is the first order partial derivative of the function $F(\bullet)$ with respect to t, $F_x(\bullet)$ is the first order partial derivative of the function $F(\bullet)$ with respect to x, $F_{xx}(\bullet)$ is the second order partial derivative of the function $F(\bullet)$ with respect to x and $o(\Delta t)$ represents all the higher order terms which go to zero faster than Δt. The Bellman

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34 If $z(t)$ is a Wiener process, then any change in z, Δz, corresponding to time interval Δt, satisfies:
1. The relationship between Δz and Δt is given by $\Delta z = \gamma_t\sqrt{\Delta t}$, with γ_t a standard normal variable.
2. The random variable γ_t is serially uncorrelated, $E[\gamma_t\gamma_s] = 0$ for $t \neq s$. The values of Δz for any two time intervals are independent. That is, $z(t)$ follows a Markov process with independent increments.

equation becomes

$$\rho F(x,t) = \max_u \left\{ V(x,u,t) + F_t(x,t) + a(x,u,t)F_x(x,t) + \frac{1}{2}b(x,u,t)F_{xx}(x,t) \right\}. \tag{3.12}$$

This expression is commonly referred to as the Hamilton-Jacobi-Bellman (H-J-B) equation. The optimal u can be expressed as a function of $F_t(x,t)$, $F_x(x,t)$, $F_{xx}(x,t)$ as well as x, t and the parameters that come from the functional form of $V(x,u,t)$. Substituting this expression for the optimal u back into the right hand side of equation 3.12, gives rise to a second order partial deferential equation with F as the dependent variable and x and t as independent variables. In general, this equation is complicated but it can be solved by analytical or numerical methods. Detailed derivations of the Hamilton-Jacobi-Bellman equation for Ito processes can be found in Mallarias and Brock (1982, Chapter 3) and Fleming and Rishel (1975, Chapter 5).

The Bellman equation of dynamic programming (3.12) will be used to analyse the model of the firm developed in the next section. Optimality conditions are found and bankruptcy probabilities are derived from these conditions.

3.2 Model of a Firm Facing a Risk of Bankruptcy

The firm is created at time $t = 0$, where t is a continuous variable on the positive part of a real number line, R^+. The model is developed in continuous time. When the firm is created it has a fixed amount of equity capital, K_0. No new equity raising is permitted. Initial equity capital is split between the purchase of productive resources, c_0, and holding a cash balance, m_0, such that $K_0 = c_0 + m_0$.

3.2.1 Choice of Objective

The objective of the firm is the maximisation of the expected value of discounted dividend distribution over the life of the firm, namely

$$\phi = \max E\left[\int_0^{\tau} w_t e^{-it} dt\right], \tag{3.13}$$

where w_t is the total dividend paid by the firm at time t, i is the required rate of return of the shareholders, and τ is the life time of the firm.

How is the lifetime of the firm determined? A firm can be considered to be bankrupt when it does not have cash available to meet obligations, such as debt repayments, as they fall due. Let m_t be the amount of cash (or very liquid assets) available to the firm at time t. A firm goes bankrupt at time τ, $0 \leq \tau \leq \infty$, which is the first date at which the firm's cash balance is strictly negative, $m_t < 0$, an absorbing barrier. The value of τ is not known in advance. It is a random variable conditioned on the decisions made by the management of the firm and the realised value of uncertain earnings.

This objective is consistent with the firm's management acting to maximise shareholder wealth.

3.2.2 Earnings uncertainty in the model

The earnings of the firm[35] are uncertain, they evolve according the following process[36]

$$d\pi_t = (\theta f[c_t])\, dt + \epsilon dz_t.$$

[35] These earnings are measured by EBIDT (Earnings Before Interest Depreciation and Taxes) in the reported financial statements. See section 2.2.

[36] Goldstein, Ju and Leland (2001) take a similar approach. They model the contingent claims against EBIT, where EBIT is modeled as a geometric Brownian motion, in a study of the dynamics of the firm's capital structure.

The term $\theta f[c_t]$ is the drift of the process. This term determines how, on average, the process will behave in the next small increment of time. In this case it represents the earnings that the firm will generate using the current stock of productive equipment, c_t. The function $f(\bullet)$ is concave[37] with $f(0) = 0$. The function is scaled by a constant parameter θ. The value $\theta f[c_t]$ is equivalent to the expected component of the earnings of the firm.

The constant, ϵ, is the diffusion coefficient of the process. It defines the extent of earnings uncertainty for the firm. This constant capturers both firm specific and economy-wide influences on the earnings of the firm. A firm with a high level of earnings uncertainty will be characterised by high values of ϵ. That is, a large ϵ is consistent with a large amount of variability in the observed earnings of the firm. dz_t[38] is the Wiener process which drives the earnings. The product ϵdz_t is the extent of the earnings surprise observed at time t. At time t, the value of both $\theta f[c_t]$ and ϵ will be observed, and adapted to the information set available, I_t.

By using a Wiener process to characterise the uncertainty in earnings we rule out the possibility of discontinuous jumps in the earnings process[39]. In small intervals of length h, unexpected earnings which are driven by a Wiener process, have a variance of $\epsilon^2 h$. So the extent of earnings surprise is governed by the diffusion coefficient, ϵ.

3.2.3 Definition of variables and constraints

The financial decisions that the firm can make describe the control variables in the model. In any period t, the firm can invest in new productive resources, v_t, change the

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37 f is a continuous and twice differentiable function of $c(t)$ with $f'_c \geq 0$ and $f''_c \leq 0$.

38 Where $z(t)$ is a Weiner process as defined in footnote 34.

39 Neftci (2000) Chapter 8 discusses this issue in detail.

level of debt by u_t, or pay a dividend, w_t, to shareholders. These three variables are the control variables for the model.

The current financial position of the firm is recorded in the "balance sheet"[40]. This position defines the current state of the firm. The balance sheet relationship in any period is defined as

$$c_t + m_t = x_t + OE_t \tag{3.14}$$

By definition the firm's assets, productive resources, c_t, and cash, m_t, are equal to the firm's liabilities, the amount of outstanding debt, x_t[41], and shareholders equity, OE_t. The shareholder's equity is the balancing item in this simple balance sheet[42].

i) State Transition Equations

The values of the variables c_t, m_t and x_t define the current state of the firm. For each of these variables there is a mechanism which describes how the interaction of the current state and decision variables generate a new value of the state variable. This mechanism is known as a state transition equation.

The first state transition equation describes changes in the cash position of the firm

$$dm_t = [\theta f(c_t) - rx_t - v_t - w_t + u_t]dt + \epsilon dz_t, \tag{3.15}$$

where r is the rate of interest payable on outstanding loan balances, and rx_t is total interest payable.

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40 The balance sheet is one of the firm's financial statements presented in section 2.3.

41 In the description of the dynamic programming approach in the previous section x_t was defined to be the state of the system at time t. In this section the level of debt, which is one of three state variables, is denoted by x_t.

42 As is common in financial statement analysis the Owner's Equity, OE_t, is the residual value in the balance sheet. It is $OE_t = c_t + m_t$ - x_t.

Equation 3.15 is a cash flow definition. The cash flow of a firm is derived from the operating activities of the firm. These activities are the cash transactions arising from the day-to-day running of the business. They are typically recorded in a firm's profit and loss statement. The difference between the uncertain profit and interest payable, $[\theta f(c_t) + \epsilon dz_t] - rx_t$, describes operating cash flow[43]. Financing activities, which cover changes to the level of debt and the payment of dividends to equity holders, $u_t - w_t$ are the second component of cash flow. Investing activities, including the purchase and sale of assets by the firm, v_t, are the final cash flow component. Changes to the firm's cash position can be positive or negative.

The transition equation for debt is

$$\frac{dx}{dt} = u_t. \tag{3.16}$$

The firm can choose to change its debt level by u_t in any period. When the firm borrows more funds, u_t is positive. If the firm repays debt then u_t is negative.

The final state transition equation is for productive equipment it is given by

$$\frac{dc}{dt} = v_t, \tag{3.17}$$

the change in the stock of capital equipment is the level if investment at time t. This assumes that there is no physical deterioration of productive equipment and all additions increase earnings generating productive equipment. That is, the firm's investment decisions, v_t, add to the amount of productive resources.

The initial values of the state variables define the starting point for the development of the firm. These initial conditions are required so that a particular solution of the model can be found. Let c_0, a constant, be the initial value for productive equip-

43 In the absence of corporate taxation and depreciation charges this is the Net Profit or Loss as defined in AASB 1018, Statement of Financial Performance.

ment, m_0, the initial value for the cash balance and x_0 be the initial amount borrowed. These initial values have to be feasible. That is, they have to obey the constraints of the model.

ii) Constraints

The decisions that the management of the firm may make in any period are constrained in a number of ways.

From the definition of bankruptcy, cash or very liquid assets held at any time t must be non negative. That is,

$$m_t \geq 0. \tag{3.18}$$

It is assumed that investments are irreversible. There is no secondary market for productive resources. Therefore, the firm cannot sell assets and investment is always positive. That is,

$$v_t \geq 0. \tag{3.19}$$

Dividends are, by definition, a payment to shareholders and as such cannot be non positive. As the equity capital of the firm is assumed to be constant the firm cannot raise new equity by asking for a shareholder payment, which would appear as a negative dividend. Both of these considerations imply that dividend payments are bounded from below such that

$$w_t \geq 0. \tag{3.20}$$

There is a limit on the amount of debt that the firm can carry. This limit is set by constraining the debt to assets ratio of the firm, such that

$$0 \leq \frac{x_t}{(c_t + m_t)} \leq h \text{ where } 0 \leq h \leq 1. \tag{3.21}$$

This constraint, in conjunction with the interest rate on debt, r, provides a mechanism that financial institutions can use to manage their exposure to the risky earnings of the firm. For firms with highly variable earnings (that is, ϵ large), financial institutions will set a small value for h. This minimises their exposure to default. They will also set a high interest rate to achieve a return that compensates for risk. This type of constraint is consistent with financial institutions assigning firms to a 'risk class'. For the firm to borrow more than the value of $h\,[c_t + m_t]$, the interest rate, r, would have to be renegotiated. These changes would move the firm into a new 'risk class'. This constraint is a simplification of the operational practices of financial institutions, as historic and projected cash flows will also be part of their decision making process.

This constraint also provides a bound on the amount that can be borrowed or repaid in any period. If the firm has not borrowed ($x_t = 0$), then the maximum it can borrow is $h[c_t + m_t]$. If the firm has borrowed up to its limit, the maximum repayment that the firm will make will be $h[c_t + m_t]$. So the change in the firm's level of debt is bounded by

$$|u_t| \leq h[c_t + m_t], \tag{3.22}$$

which is derived from the non negativity constraint on debt and the upper bound on borrowing[44].

The firm is not allowed to use borrowed funds to finance a dividend distribution. Dividends can only be paid from expected earnings after interest expenses have been

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[44] For convenience in the analysis of possible solutions, the constraint

$$|u_t| \leq a_t \text{ where } a_t = h[c_t + m_t] \text{ and } 0 \leq a \leq \infty$$

will be used.

paid[45], which results in the following inequality,

$$w_t \leq \theta[f(c_t)] - rx_t. \tag{3.23}$$

This constraint has two effects. First, it stops the firm from borrowing up to its limit in one period in order to distribute the funds in a subsequent period as a dividend. The second effect is to allow the firm to use its cash reserve to smooth the dividend stream. The random component of earnings, ϵdz_t, adds to the cash reserve when it is positive or reduces the cash reserve when negative. This process of adding to and withdrawing from the cash reserve serves to smooth the dividend paid. Dividend smoothing[46] is common amongst firms, as many firms announce dividend policy in advance and then meet the announced policy in order to avoid adverse market reactions associated with dividend "surprise".

Investments in productive resources can be financed through debt, from the cash reserve, or by some combination of these. A constraint that takes these financing options into account is

$$v_t \leq m_t + \theta[f(c_t)] + u_t - rx_t - w_t. \tag{3.24}$$

The amount invested in productive resources is bounded by the free cash position of the firm. That is, the sum of the cash reserve, expected earnings and increases in debt, minus interest payable on debt and dividends distributed. The constraint in this form includes the cash reserve as a source of internal financing of new investments.

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45 Expected Profit / Loss is equivalent to earnings after interest as there are no corporate taxes in the model as formulated.

46 See, Chen, Chung and Chunchi Wu (1999) for an analysis of the interaction of earnings dynamics and dividend smoothing.

3.2.4 The full model

The model is defined by the combination of the objective function, equation 3.13, the state transition equations, equations 3.15 to 3.17 and the constraints given by equations 3.18 to 3.24. That is,

$$\phi = \max E\left[\int_0^T w_t e^{-it} dt\right]$$

Subject to

$$dm_t = [\theta f(c_t) - rx_t - v_t - w_t + u_t]dt + \epsilon dz_t$$

$$\frac{dx}{dt} = u_t$$

$$\frac{dc}{dt} = v_t$$

$$0 \leq \frac{x_t}{(c_t + m_t)} \leq h$$

$$|u_t| \leq a$$

$$w_t \leq \theta[f(c_t)] - rx_t$$

$$v_t \leq m_t + \theta[f(c_t)] + u_t - rx_t - w_t$$

$$m_t \geq 0$$

$$v_t \geq 0$$

$$w_t \geq 0$$

and initial values

$$c_0, m_0, x_0$$

The state variables can be observed at any time t. All the control variables are bounded by deterministic constants in all time periods. Accordingly, the objective is strictly bounded. Under these conditions Bertsekas (1976 Chapter 6.1) demonstrates that the objective function exists and that the Dynamic Programming algorithm will find a unique solution.

3.3 Analysis of the Firm's Policy Regimes

The model outlined in the previous section is a continuous time, stochastic dynamic model. The uncertain element of the model is the earnings of the firm. The earnings process is Markov by definition. Optimal policies for models of this class are found by applying the dynamic programming technique as outlined in Section 1 of this chapter. The Hamilton-Jacobi-Bellman equation is formulated. Analysis of the H-J-B equation for the model presented leads to the identification of a number of policy regimes which describe the optimal settings of the control variables. For each of the policy regimes identified, the bankruptcy condition and a probability of bankruptcy are derived.

3.3.1 Optimality conditions

The objective of the model,

$$\phi(m, c, x) = \max E\left[\int_0^T w_t e^{-it} dt\right]$$

represents the expected maximum value of future dividend payments over the life of the firm. At time, t, the current state of the firm is described by the cash position, m_t, the stock of productive resources, c_t, and the level of debt, x_t[47].

The H-J-B equation which corresponds to the model defined in the previous section, when the state variable, x, is at an interior point, $0 \leq x \leq h(c+m)$, is

$$i\phi = \max_{\substack{\text{subject to} \\ v,w\geq 0 \\ w\leq \theta f(c)-rx \\ v\leq \theta f(c)+m+u-w-rx \\ |u|\leq a}} \left[\begin{array}{c} w + \frac{\partial\phi}{\partial x}u + \frac{\partial\phi}{\partial c}v + \\ + \frac{\partial\phi}{\partial m}\left(\theta f(c) - v - w - rx + u\right) \end{array}\right] + \frac{\epsilon^2}{2}\frac{\partial^2\phi}{\partial m^2} \quad (3.25)$$

[47] For convenience the time arguments will be dropped in the following analysis.

The partial derivatives in the model, which correspond to the co-state variables in a optimal control problem, have the following economic interpretation:

$$\begin{aligned}
1 &= \text{the marginal contribution of each dollar increase in dividend payments,}\\
\frac{\partial\phi}{\partial x} &= \text{the marginal contribution of each additional dollar borrowed,}\\
\frac{\partial\phi}{\partial m} &= \text{the marginal contribution of an additional dollar kept as cash, and}\\
\frac{\partial\phi}{\partial c} &= \text{the marginal contribution of an additional dollar spent on productive resources.}
\end{aligned}$$

Rearranging terms in equation (3.25) we derive

$$i\phi = \max_{\substack{\text{subject to}\\ v,w\geq 0\\ w\leq\theta f(c)-rx\\ v\leq\theta f(c)+m+u-w-rx\\ |u|\leq a}} \left[\begin{array}{c} w\left(1-\frac{\partial\phi}{\partial m}\right)+u\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial m}\right)+v\left(\frac{\partial\phi}{\partial c}-\frac{\partial\phi}{\partial m}\right)\\ +\left(\theta f(c)-rx\right)\frac{\partial\phi}{\partial m}\end{array}\right]+\frac{\epsilon^2}{2}\frac{\partial^2\phi}{\partial m^2} \tag{3.26}$$

After appending the boundary condition for bankruptcy,

$$\phi(0,c,x)=0, \tag{3.27}$$

we complete the system.

In this form, the interpretation of the expression is intuitive. For every dollar of dividends distributed, the objective is increased by the direct contribution of the dividend payment minus the contribution of that dollar as if it had been held as cash, namely $\left(1-\frac{\partial\phi}{\partial m}\right)$. Each dollar borrowed increases the objective by the sum of the contributions of debt and cash to the objective, $\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial m}\right)$. A dollar invested in productive resources contributes to the sum of the positive effect of a dollar increase in productive resources, and the negative effect of not allocating a dollar to an increase in the cash balance, $\left(\frac{\partial\phi}{\partial c}-\frac{\partial\phi}{\partial m}\right)$. Finally, earnings after interest expenses contribute to the cash position of the firm. The objective is increased by $\frac{\partial\phi}{\partial m}$ for each dollar of earnings after interest expenses.

Analysis of the values of the partial derivatives, $\left(1, \frac{\partial\phi}{\partial m}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial c}\right)$, and the sign of $\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial m}\right)$, gives seven possible policy regimes. Each of these regimes will be explained and the condition for a bankruptcy to occur under each regime explored.

3.3.2 Policy regimes

The Bellman equation, 3.26, gives rise to seven possible policy regimes. They are:

Regime 1. The firm borrows, improves the cash position and distributes dividends

Within this regime, holding debt makes a positive contribution to the objective, $\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial m}\right) \geq 0$. The contribution to the objective of dividend payments is greater than the benefit to be gained from holding cash or investing, $1 \geq \frac{\partial\phi}{\partial m} \geq \frac{\partial\phi}{\partial c}$. So the firm pays the largest possible dividend, borrows up to its credit limit, then uses what remains after paying the dividend to increase the cash position.

That is, the firm pays the maximum allowable dividend,

$$w = \theta f(c) - rx$$

If the firm borrows up to its debt limit, then

$$u = a$$

Further the firm does not invest, so

$$v = 0$$

and, consequently, the stock of productive resources remains constant,

$$\frac{dc}{dt} = 0.$$

The cash position of the firm changes by the random component of earnings plus the amount borrowed, resulting in

$$dm = adt + \epsilon dz.$$

Combining these results a probability of bankruptcy for this regime can be found.

Proposition 1 *In regime 1, Pr(bankruptcy)* $= \Pr\left(\Delta z < -\left[\frac{M_t + a}{\epsilon}\right]\right)$.

Proof. The condition for bankruptcy to occur at any time t is $m_t < 0$. Applying the Euler-Mauryama approximation method[48] to the equation for the cash position, we derive the stochastic difference equation in the variable M, which is the discrete equivalent of the continuous variable m_t.

$$M_{t+1} = M_t + a\Delta + \epsilon\Delta z,$$

where Δ is the length of the discretisation interval, Δz is the increment of the Wiener process and the initial condition $M_0 = m_0$ holds. By definition the increments of a Wiener process are independent normal variates with zero mean and variance Δ.

That is, the cash position at the end of the interval is made up of the cash position at the beginning of the interval plus the cash generated over the interval.

When $M_{t+1} < 0$ the firm is bankrupt. As a first step look at the case when $M_{t+1} = 0$. This implies that

$$M_t + a\Delta + \epsilon\Delta z = 0$$

[48] This method of making a discrete time approximation of a continuous stochastic process is described in Chapter 9 of Kloden and Platen (1992).

Let $\Delta = 1$. That is, the discretisation interval is one time period. Then

$$M_t + a + \epsilon\Delta z = 0$$

Rearranging the terms to determine the extent of the negative earnings surprise that would be required to drive cash to zero results in

$$\epsilon\Delta z = -\left[M_t + a\right]$$

Therefore, for bankruptcy to occur at the end of the interval

$$\epsilon\Delta z < -\left[M_t + a\right]$$

As Δz is a normally distributed random variable, the probability of bankruptcy, $\Pr(M_{t+1} < 0)$, will be the left tail probability

$$\Pr\left(M_{t+1} < 0\right) = \Pr\left(\Delta z < \frac{-\left[M_t + a\right]}{\epsilon}\right).$$

■

This condition is most likely to occur when the firm has low cash reserves and a low borrowing limit. The constraint on borrowing is described in equation 3.22 of the model. A low borrowing limit can come from a low h value imposed by lenders, or a small asset base, $(c + m)$. A large negative random component of earnings could lead to bankruptcy under these conditions.

Regime 2. The firm repays debt and distributes dividends

In this regime, holding debt makes a negative contribution to the objective, $\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial m}\right) \leq 0$. The contribution to the objective of dividend payments is greater than the benefit to be gained from holding cash, $\frac{\partial\phi}{\partial m} \leq 1$, and the contribution to the objective of dividend payments is greater than the benefit to be gained from investing, $\frac{\partial\phi}{\partial c} \leq 1$. Thus, the firm pays the maximum allowable dividend, pays back as much debt as possible and makes no investment in productive equipment,

The firm pays the maximum allowable dividend,

$$w = \theta f(c) - rx.$$

Additionally, the firm repays as much debt as possible, namely

$$u = -a.$$

The firm does not invest so,

$$v = 0,$$

and, consequently, the stock of productive resources remains constant,

$$\frac{dc}{dt} = 0.$$

The cash position of the firm changes by the random component of earnings minus the amount of debt repaid

$$dm = -adt + \epsilon dz.$$

To find the probability of bankruptcy for this regime, we combine these results with the specification of earnings uncertainty in the model.

Proposition 2 *In regime 2, Pr(bankruptcy)* $= \Pr\left(\Delta z < -\left[\frac{M_t - a}{\epsilon}\right]\right)$.

Proof. The condition for bankruptcy to occur at any time t is $m_t < 0$. Applying the Euler-Mauryama approximation method to the equation for the cash position, we derive the stochastic difference equation,

$$M_{t+1} = M_t - a\Delta + \epsilon\Delta z,$$

where Δ is the length of the discretisation interval, Δz is the increment of the Wiener process and the initial condition $M_0 = m_0$ holds.

When $M_{t+1} < 0$ the firm is bankrupt. Look at the case when $M_{t+1} = 0$. This implies

$$M_t - a\Delta + \epsilon \Delta z = 0$$

Let $\Delta = 1$. Then

$$M_t - a + \epsilon \Delta z = 0$$

Rearranging terms to determine the extent of the negative earnings surprise that would be required to drive cash to zero results in

$$\epsilon \Delta z = -\left[M_t - a\right]$$

Therefore, for bankruptcy to occur at the end of the interval,

$$\epsilon \Delta z < -\left[M_t - a\right]$$

As Δz is a normally distributed random variable by definition, the probability of bankruptcy will be the left tail probability

$$\Pr\left(M_{t+1} < 0\right) = \Pr\left(\Delta z < \frac{-\left[M_t - a\right]}{\epsilon}\right).$$

■

This condition is most likely to occur when the firm has low cash reserves and a large debt level. In this case the debt repayment depletes the cash reserve, a large negative random earnings element would then lead to bankruptcy.

Regime 3. The firm borrows and builds up the cash reserve

In this regime, holding debt makes a positive contribution to the objective, $\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial m}\right) \geq 0$. The contribution to the objective of holding cash is greater than the benefit to be gained from dividend payments, $\frac{\partial \phi}{\partial m} \geq 1$, and the contribution to the objective of holding cash is greater than the benefit to be gained from investing, $\frac{\partial \phi}{\partial m} \geq \frac{\partial \phi}{\partial c}$. Here the firm's primary aim is to improve the cash position. It borrows up to its limit and stops investment and dividend distributions to achieve this aim.

That is, the firm pays no dividend, so

$$w = 0.$$

The firm borrows as much debt as possible,

$$u = a.$$

The firm does not invest,

$$v = 0,$$

and, consequently, the stock of productive resources remains constant,

$$\frac{dc}{dt} = 0.$$

The cash position of the firm is increased by the level of earnings plus new borrowings less interest payable. That is,

$$dm = [\theta f(c) + a - rx]\, dt + \epsilon dz.$$

To find the probability of bankruptcy for this regime, we combine these results with the specification of earnings uncertainty in the model.

Proposition 3 *In regime 3, Pr(bankruptcy)* $= \Pr\left(\gamma < -\left[\Delta z < \frac{-[M_t + (\theta f(c) + a - rx)]}{\epsilon}\right]\right)$.

Proof. The condition for bankruptcy to occur at any time t is $m_t < 0$. Applying the Euler-Mauryama approximation method to the equation for the cash position, we derive the stochastic difference equation,

$$M_{t+1} = M_t + [\theta f(c) + a - rx] \Delta + \epsilon \Delta z,$$

where Δ is the length of the discretisation interval, Δz is the increment of the Wiener process and the initial condition $M_0 = m_0$ holds.

When $M_{t+1} < 0$ the firm is bankrupt. Look at the case when $M_{t+1} = 0$. This implies

$$M_t + [\theta f(c) + a - rx] \Delta + \epsilon \Delta z = 0$$

Let $\Delta = 1$. Then

$$M_t + [\theta f(c) + a - rx] + \epsilon \Delta z = 0$$

Rearranging terms to determine the extent of the negative earnings surprise that would be required to drive cash to zero results in

$$\epsilon \Delta z = - [M_t + (\theta f(c) + a - rx)]$$

Therefore, for bankruptcy to occur at the end of the interval,

$$\epsilon \Delta z < - [M_t + (\theta f(c) + a - rx)]$$

As Δz is a normally distributed random variable by definition, the probability of bankruptcy will be the left tail probability

$$\Pr (M_{t+1} < 0) = \Pr \left(\Delta z < \frac{- [M_t + (\theta f(c) + a - rx)]}{\epsilon} \right).$$

■

This condition is most likely to occur when the firm has low cash reserves and a low level of productive resources. In this case the ability of the firm to borrow is limited by the small asset base, $(c + m)$. Earnings, $\theta f(c)$, are also limited by the level of productive resources. The interest payment will also deplete the cash reserve.

Regime 4. The firm repays debt and builds up the cash reserve

In regime 4, holding debt makes a negative contribution to the objective, $\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial m}\right) \leq 0$. The contribution to the objective of holding cash is greater than the benefit to be gained from dividend payments, $\frac{\partial \phi}{\partial m} \geq 1$. The contribution to the objective of holding cash is greater than the benefit to be gained from investing, $\frac{\partial \phi}{\partial m} \geq \frac{\partial \phi}{\partial c}$. In this regime, the firm pays back as much debt as it can. To free up the funds to retire debt, the firm stops investment and the distribution of dividends.

In terms of the model, the firm pays no dividend, so

$$w = 0\,.$$

The firm repays at the maximum allowable rate so

$$u = -a.$$

The firm does not invest with

$$v = 0\,,$$

and, consequently, the stock of productive resources remains constant,

$$\frac{dc}{dt} = 0.$$

The cash position of the firm is increased by the level of earnings less interest payable and the amount of debt repaid. This is given by,

$$dm = [\theta f(c) - a - rx]\, dt + \epsilon dz\ .$$

Combining these results with the specification of earnings uncertainty in the model, a probability of bankruptcy for this regime can be found.

Proposition 4 *In regime 4, Pr(bankruptcy)* = $\Pr\left(\Delta z < \frac{-[M_t + (\theta f(c) - a - rx)]}{\epsilon}\right)$.

Proof. The condition for bankruptcy to occur at any time t is $m_t < 0$. Applying the Euler-Mauryama approximation method to the equation for the cash position, we derive the stochastic difference equation,

$$M_{t+1} = M_t + [\theta f(c) - a - rx]\, \Delta + \epsilon \Delta z,$$

where Δ is the length of the discretisation interval, Δz is the increment of the Wiener process and the initial condition $M_0 = m_0$ holds.

When $M_{t+1} < 0$ the firm is bankrupt. Look at the case when $M_{t+1} = 0$. This implies

$$M_t + [\theta f(c) - a - rx]\, \Delta + \epsilon \Delta z = 0$$

Let $\Delta = 1$. Then

$$M_t + [\theta f(c) - a - rx] + \epsilon \Delta z = 0$$

Rearranging terms to determine the extent of the negative earnings surprise that would be required to drive cash to zero results in

$$\epsilon \Delta z = -\left[M_t + (\theta f(c) - a - rx)\right]$$

Therefore, for bankruptcy to occur at the end of the interval,

$$\epsilon \Delta z < -\left[M_t + (\theta f(c) - a - rx)\right]$$

As Δz is a normally distributed random variable by definition, the probability of bankruptcy will be the left tail probability

$$\Pr\left(M_{t+1} < 0\right) = \Pr\left(\Delta z < \frac{-\left[M_t + (\theta f(c) - a - rx)\right]}{\epsilon}\right).$$

■

This condition is most likely to occur when the debt repayment and interest payment are large relative to the cash reserve and level of expected earnings. A large negative random component of earnings can then wipe out the remaining cash and cause bankruptcy.

Regime 5. The firm borrows and invests

In this regime, holding debt makes a positive contribution to the objective, $\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial m}\right) \geq 0$. The contribution to the objective of investing is greater than the benefit to be gained from dividend payments, $\frac{\partial \phi}{\partial c} \geq 1$, and the contribution to the objective of investing is greater than the benefit to be gained from holding cash, $\frac{\partial \phi}{\partial c} \geq \frac{\partial \phi}{\partial m}$. So the firm borrows up its credit limit, does not distribute a dividend and invests at the maximum possible rate.

That is, the firm pays no dividend so

$$w = 0\,.$$

The firm borrows at the maximum allowable rate with

$$u = a\,.$$

The firm invests at the maximum allowable rate, given by

$$v = m + \theta f(c) + a - rx,$$

and, consequently, the stock of productive resources changes by

$$\frac{dc}{dt} = v\,.$$

The cash position of the firm changes by the random component of earnings minus the portion of the cash reserve that has been allocated to investment in productive equipment,

$$dm = -mdt + \epsilon dz.$$

To find the probability of bankruptcy for this regime, we combine these results with the specification of earnings uncertainty in the model.

Proposition 5 *In regime 5, Pr(bankruptcy)* $= \Pr(\Delta z < 0)$.

Proof. The condition for bankruptcy to occur at any time t is $m_t < 0$. Applying the Euler-Mauryama approximation method to the equation for the cash position, we derive the stochastic difference equation,

$$M_{t+1} = M_t - M_t\Delta + \epsilon\Delta z,$$

where Δ is the length of the discretisation interval, Δz is the increment of the Wiener process and the initial condition $M_0 = m_0$ holds.

When $M_{t+1} < 0$ the firm is bankrupt. Look at the case when $M_{t+1} = 0$. This implies

$$M_t - M_t\Delta + \epsilon\Delta z = 0$$

Let $\Delta = 1$. Then

$$M_t - M_t + \epsilon\Delta z = 0$$

Rearranging terms to determine the extent of the negative earnings surprise that would be required to drive cash to zero results in

$$\epsilon\Delta z = 0$$

Therefore, for bankruptcy to occur at the end of the interval,

$$\epsilon\Delta z < 0$$

As Δz is a normally distributed random variable by definition, the probability of bankruptcy will be the left tail probability

$$\Pr\left(M_{t+1} < 0\right) = \Pr\left(\Delta z < 0\right).$$

■

As the firm is devoting all possible resources to increasing its stock of productive resources, any unexpected negative earnings result will cause bankruptcy. This type of bankruptcy can be thought of as a working capital shortage in an environment of rapid growth.

Regime 6. The firm repays debt and invests

In regime 6, holding debt makes a negative contribution to the objective, $\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial m}\right) \leq 0$. The contribution to the objective of investing is greater than the benefit to be gained from dividend payments, $\frac{\partial\phi}{\partial c} \geq 1$, and the contribution to the objective of investing is greater than the benefit to be gained from holding cash, $\frac{\partial\phi}{\partial c} \geq \frac{\partial\phi}{\partial m}$. Here, the firm redeems the maximum amount of debt. It does not distribute a dividend and invests whatever funds are available after the repayment of debt.

That is, the firm pays no dividend, so

$$w = 0 \, .$$

The firm repays as much debt as possible with

$$u = -a \, .$$

The firm invests at the maximum allowable rate,

$$v = m + \theta f(c) - a - rx \, .$$

Consequently, the stock of productive resources changes by

$$\frac{dc}{dt} = v \, .$$

The cash position of the firm changes by the random component of earnings minus the portion of the cash reserve that has been allocated to investment in productive equipment, which is given by

$$dm = -mdt + \epsilon dz \, .$$

Combining these results with the specification of earnings uncertainty in the model, a probability of bankruptcy for this regime can be found.

Proposition 6 *In regime 6, Pr(bankruptcy)* $= \Pr(\Delta z < 0)$.

Proof. The condition for bankruptcy to occur at any time t is $m_t < 0$. Applying the Euler-Mauryama approximation method to the equation for the cash position, we derive the stochastic difference equation,

$$M_{t+1} = M_t - M_t \Delta + \epsilon \Delta z,$$

where Δ is the length of the discretisation interval, Δz is the increment of the Wiener process and the initial condition $M_0 = m_0$ holds.

When $M_{t+1} < 0$ the firm is bankrupt. Look at the case when $M_{t+1} = 0$. This implies

$$M_t - M_t\Delta + \epsilon\Delta z = 0$$

Let $\Delta = 1$. Then

$$M_t - M_t + \epsilon\Delta z = 0$$

Rearranging terms to determine the extent of the negative earnings surprise that would be required to drive cash to zero results in

$$\epsilon\Delta z = 0$$

Therefore, for bankruptcy to occur at the end of the interval,

$$\epsilon\Delta z < 0$$

As Δz is a normally distributed random variable by definition, the probability of bankruptcy will be the left tail probability

$$\Pr\left(M_{t+1} < 0\right) = \Pr\left(\Delta z < 0\right).$$

■

As the firm is devoting all possible resources to increasing its stock of productive resources and repaying debt, any unexpected negative earnings result will cause a bankruptcy. This was also the case in regime 5, where the firm was attempting to grow at the maximum rate. This type of bankruptcy can be thought of as a working capital shortage in the case of rapid growth.

Regime 7. The firm borrows, invests and pays dividends

In this regime, holding debt makes a positive contribution to the objective, $(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial m}) \geq 0$. The contribution to the objective of dividend payments and investing are greater than the benefit to be gained from holding cash, $1 \geq \frac{\partial\phi}{\partial c} \geq \frac{\partial\phi}{\partial m}$. Thus, the firm distributes the maximum allowable dividend. It also borrows up to its credit limit and invests as much as possible.

In terms of the model, the firm pays the maximum allowable dividend,

$$w = \theta f(c) - rx \,.$$

The firm borrows at the maximum allowable rate, which is

$$u = a \,.$$

The firm invests as much as possible

$$v = m + a \,,$$

and, consequently, the stock of productive resources changes by

$$\frac{dc}{dt} = v \,.$$

The cash position of the firm changes by the random component of earnings minus the portion of the cash reserve that has been allocated to investment in productive equipment,

$$dm = -mdt + \epsilon dz \,.$$

To find the probability of bankruptcy for this regime, we combine these results with the specification of earnings uncertainty in the model.

Proposition 7 *In regime 7, Pr(bankruptcy)* $= \Pr(\Delta z < 0)$.

Proof. The condition for bankruptcy to occur at any time t is $m_t < 0$. Applying the Euler-Mauryama approximation method to the equation for the cash position, we derive the stochastic difference equation,

$$M_{t+1} = M_t - M_t\Delta + \epsilon\Delta z,$$

where Δ is the length of the discretisation interval, Δz is the increment of the Wiener process and the initial condition $M_0 = m_0$ holds.

When $M_{t+1} < 0$ the firm is bankrupt. Look at the case when $M_{t+1} = 0$. This implies

$$M_t - M_t\Delta + \epsilon\Delta z = 0$$

Let $\Delta = 1$. Then

$$M_t - M_t + \epsilon\Delta z = 0$$

Rearranging terms to determine the extent of the negative earnings surprise that would be required to drive cash to zero results in

$$\epsilon\Delta z = 0$$

Therefore, for bankruptcy to occur at the end of the interval,

$$\epsilon\Delta z < 0$$

As Δz is a normally distributed random variable by definition, the probability of bankruptcy will be the left tail probability

$$\Pr\left(M_{t+1} < 0\right) = \Pr\left(\Delta z < 0\right).$$

■

This condition is most likely to occur because the cash reserve has been allocated to investing for future growth. Under these circumstances a negative random component of earnings will then cause bankruptcy. This regime describes firms that have embarked on an aggressive growth program. This type of failure is related to working capital management problems.

3.4 Summary

How the management of the firm would use the seven regimes to determine the probability of bankruptcy can be summed up as follows. When the firm has a strong cash position and a large stock of productive resources it will follow the policy described in regime one. That is, it will not invest, distribute dividends and borrow funds. If it is very close to its credit limit, then it will switch to the policy of regime four, repaying debt and accumulating cash. When the cash position is strong the firm follows the policy of regime seven, investing and distributing a dividend while borrowing. In the case where the cash position is strong and the firm is close to its credit limit, it will follow the policy of regime six, by repaying debt and investing in new productive equipment. When the firm has a good cash position and a low level of productive equipment it follows the policy of regime five, borrowing and investing. If the firm is close to its credit limit while cashed up and having a low level of equipment, it will have to follow the policy of regime six, paying off debt while investing. When the firm has a low cash position and a reasonable level of equipment it attempts to improve its cash position. Under these conditions it follows the policy of regime three, borrowing and increasing the cash position. If the firm is close to its borrowing limit it will follow the policy of regime four, redeeming debt and increasing the cash position if possible. When the firm has a low level of cash and a low level of productive equipment, the probability of bankruptcy becomes large. In this case management will follow the policy of regime two, saving the maximum to offset negative earnings while distributing a dividend. If the firm is also close to its credit limit it will attempt to build up its cash buffer by following the policy of regime four, by redeeming debt and building up its cash position.

The five unique bankruptcy probability expressions that can be derived from the model are collected in Table 3.8. Each of these expressions is conditional on the firm being a going concern at the beginning of the period of interest.

Failure Probability Expressions
$\Pr\left(\Delta z < -\left[\frac{M_t+a}{\epsilon}\right]\right)$
$\Pr\left(\Delta z < -\left[\frac{M_t-a}{\epsilon}\right]\right)$
$\Pr\left(\Delta z < \frac{-[M_t+(\theta f(c)+a-rx)]}{\epsilon}\right)$
$\Pr\left(\Delta z < \frac{-[M_t+(\theta f(c)-a-rx)]}{\epsilon}\right)$
$\Pr\left(\Delta z < 0\right)$

Table 3.8. **Five unique Failure Probability Expressions**

The variables that make up these expressions are summarised in Table 3.9.

Variable	**Description**
a	maximum change in debt at t
M_t	cash balance at t
rx	interest paid in period t
$\theta f(c)$	expected earnings in period t
ϵ	diffusion coefficient of the earnings process.

Table 3.9. **Variables in Failure Probability Formula**

A comparison of the results derived from the model developed in this chapter to those derived in previous studies follows. The comparison will highlight the common elements in the approaches, as well as pointing out the assumptions which lead to differences. The results of the gambler's ruin and Scott's imperfect and perfect access to capital models are summarised in Table 3.10.

Model	**Bankruptcy Expression**
Gambler's Ruin	$\Pr(\text{Bankruptcy}) = \left(\frac{1-(\frac{A\theta\delta\gamma}{\sigma})}{1+(\frac{A\theta\delta\gamma}{\sigma})}\right)^{\frac{C}{\sigma}}$
Scott (Imperfect Access)	$\Pr(\text{Bankruptcy}) = F\left[\frac{-(\mu_x+S)}{\sigma_x}\right]$
Scott (Perfect Access)	$\Pr(\text{Bankruptcy}) = F\left[\frac{-\left(\mu_x+(K_0-K_1)+\frac{S_1[I_1]}{(1+c)}\right)}{\sigma_x}\right]$
Option Pricing	$\Pr(\text{Bankruptcy}) = \Pr(cV_\tau < I)$ $= N\left(\frac{\ln[\frac{cV}{Int}]+\left(r-D-\frac{\sigma_V^2}{2}\right)t}{\sigma_V\sqrt{t}}\right)$

Table 3.10. **Summary Results from Prior Studies**

The variables which make up these expressions are defined in Table 3.11.

The common elements of the probability expressions relate to the interaction of the parameters of the random earnings stream distribution with the other elements in the models. All of the models incorporate a measure of expected profitability and its dispersion.

The gambler's ruin framework shares a number of common features with the model developed in this chapter. Both feature a cash flow notion in their bankruptcy probability expression. The bankruptcy conditions in the model developed in this chapter are based on the cash position of the firm, so only cash flow elements appear in the bankruptcy probability expressions developed. The variable $A\theta\delta\gamma$ in the gambler's ruin model is also a cash flow variable. The initial equity of the firm is a primary determinant of bankruptcy in the gambler's ruin model, with its effect on the probability of bankruptcy captured by the influence of the variable C on the probability of bankruptcy. As noted in Section 2 of this chapter, the initial equity of the firm is split between productive equipment and cash holdings. Although the size of the initial equity does not

Variable	Description
C	Realisable value of Assets - Total Liabilities
$A\theta\delta\gamma$	(total assets employed)*(average return on total assets)* (1 - dividend payout ratio)*(1 - proportion of cash flow invested in illiquid assets)
σ	the standard deviation of net cashflow less capital spending and dividends
X	next period's results
S	the expected value of equity next period ignoring losses
μ_x	average profit
σ_x	standard deviation of profit
K_i	stockholders' equity at period i
$X_i[K_{i-1}]$	firm income in period i, a random, increasing, concave function of K_{i-1}
I_i	net investment at period i
$S_i[I_i]$	market value of firm's equity at period i
c	the cost of issuing new equity
cV -	the firms cash position as fixed proportion of value
Int	interest payment due at t
r	the risk free rate
D	the dividend payment
σ_V	standard deviation of changes in the value of the firm

Table 3.11. **Variables in Gambler's Ruin and Scott Probability Formula**

appear in the expressions in Table 3.8, the nature of this split affects the probability of bankruptcy in the early life of the firm. If inadequate cash reserves are held, the probability of bankruptcy will be high.

The notion of bankruptcy in the Scott formulations is based on the value of the firm. The market value of equity variable, $S_i[I_i]$, maps changes in the asset base into a value of stockholder equity. The average profit, μ_x, is also included in the probability expressions from the Scott models. The probability expressions from both Scott models seek an approximate value of the firm, if the random loss in any period is greater than this value, the firm is bankrupt. The perfect access model incorporates reversible investments and allows the issue of new equity. There is no debt financing in the model. The model developed in this chapter assumes irreversible investment, no equity issues and allows debt financing. Due to these differences in assumptions, the bankruptcy probability expressions of the model developed in this chapter and presented in Table 3.8, along with those for the Scott models as presented in Table 3.10, have only variables which describe the properties of the profit distribution in common.

The option pricing approach provides the closest match to the results developed. The probability of bankruptcy in the case of a firm making periodic interest payments is based on the value of the firm being a geometric Brownian motion, this is a risk neutral probability. In the model developed, the earnings of the firm are the stochastic element. This difference is fundamental in empirical work as earnings are easily observed where value has to be inferred. The variability of the uncertain element is fundamental in both approaches. Both approaches include the interest payable and the firms cash position. The option approach includes the dividend as a constant payment. The decisions of management play no part in the determination of the value of the firm, bankruptcy occurs when random events drive the value of the firm down to a level where the interest payments cannot be made. In the model developed, the investment and financing decisions of firm management are prime determinants of the probability of bankruptcy.

Laitinen (1991) found three failure types using factor analysis in a sample of failed companies. They are "chronic failure", where the companies ratios were poor, "acute failure", where the ratios were good until the final year before failure, and "revenue failure", where the company had poor profitability and low revenue growth over its life. The analysis of the solution of the model supports this finding. A company failing in regime two or four can be thought of as having suffered a "chronic failure". Firms that fail while engaged in growth activities, that is, regimes five, six and seven, will have suffered an "acute failure". Failure in regimes one and three is consistent with Laitinen's notion of a "revenue failure".

In the next section these bankruptcy probability conditions and their component variables will guide the definition of proxy variables for empirical testing. The proxy variables will be calculated from financial statement data. The decision context of the model, specifically the conditional decision sequence attributable to the random time horizon, will be used to suggest an appropriate statistical approach to be combined with the proxy variables. The statistical method will be used to verify the usefulness of the variables suggested in the bankruptcy probability expressions for determining the probability of bankruptcy. This empirical work will be found in Chapter 5.

Chapter 4
Empirical Validation of Model Results

The analysis of the model developed in Chapter 3 resulted in the five unique expressions for the probability of bankruptcy, presented in Table 3.8. The objective of this chapter is to empirically validate these expressions and, by induction, the model developed in Chapter 3. To validate the model we need to derive some testable hypotheses. Once the hypotheses are formulated, a statistical structure which allows for their testing needs to be established. If the hypotheses are supported, we will conclude that the probability expressions presented in Table 3.8 provide a useful framework for the conduct of empirical studies of bankruptcy phenomena. The hypotheses formed from the analysis of the model presented in Chapter 3, relate to the significance of variables which influence the probability of bankruptcy, and the direction of their effect. Tests of the significance of a variable's effect are based on the statistical properties of the probability estimator. The direction of a variable's effect on the probability of bankruptcy is captured in the sign of its estimated coefficient. Then, the primary aim of this chapter is to formulate a set of hypotheses relating to the significance and signs of variables which make up the expressions for the probability of bankruptcy developed in Chapter 3.

To test hypotheses, availability of data and a statistical method which generates coefficient estimates and their standard errors is required. Regression style methods typically provide the statistics required to generate tests for the signs and significance of an explanatory variable. A regression method that specifically measures the effect of explanatory variables on the probability of failure will be the vehicle for the validation of the model.

This chapter has the following structure. A set of testable hypotheses are developed from the failure probability analysis of Chapter 3. Then, the variables in the probability expressions are mapped into expressions formed from variables contained in financial statements described in Section 2 of Chapter 2. The chapter concludes with the development of a test statistic that can be used to test the hypotheses developed.

4.1 Derivation of Testable Hypotheses

Analysis of the optimality conditions of the model, presented in Section 2 of chapter 3, suggests two groups of variables that need to be tested as part of model verification. The first group are the control variables of the model that indicate which of the seven regimes the firm is operating in. The second group are those variables which appear in the expressions for the probability of bankruptcy.

4.1.1 Control Variables

Values of control variables, the first group of variables to be considered in the validation of the model, determine which policy regime the firm is operating in. This information is important as the probability of bankruptcy expressions are regime specific. Combinations of the control variables; change in the level of debt, u, investment in productive equipment, v, and the dividend paid, w, determine the current regime. The values taken by the control variables in each regime is presented in Table 4.12.

There are three regimes where a dividend is paid. The constraint on dividend payments, Equation 3.23, sets an upper bound on the dividend payment allowed. In all three regimes the dividend paid is the maximum allowed under this constraint. A dividend will only paid if expected earnings are larger than interest payable. Thus, firms

	Regime	Control Variable		
		u	v	w
Value	R1	a	0	$\theta f(c) - rx$
	R2	$-a$	0	$\theta f(c) - rx$
	R3	a	0	0
	R4	$-a$	0	0
	R5	a	$m + \theta f(c) + a - rx$	0
	R6	$-a$	$m + \theta f(c) - a - rx$	0
	R7	a	$m + a$	$\theta f(c) - rx$

Table 4.12. **Regime Specific Values of Control Variables**

paying dividends are generating earnings that are larger than the expenses incurred. Management of these firms has judged the cash balance of the firm to be large enough to offset negative earnings suprises, otherwise the funds available would have been allocated to increasing the cash balance. That is, profitable firms will have adequate cash reserves and be less likely to fail. The ability to pay a dividend will reduce the probability of a firm going bankrupt.

The effect of a change in the level of debt on the probability of failure will depend on whether the change is an increase or decrease in debt. A firm increasing its debt level within its debt constraint, a positive change, will be judged by the lending institution as being able to repay these funds. This positive assessment by lending institutions can be taken as a sign of financial health. The implication is that an increase in the level of debt should reduce the probability of bankruptcy in the short term. Over the longer term the increased interest commitments may increase the probability of bankruptcy. When a firm pays back debt, it does so out of its free cash. When free cash is used to repay debt it is not available to invest in new productive capacity. Forgoing these investments reduces the pool of productive equipment available in later periods. This equipment generates the certain part of the random earnings stream. If the certain part of the random earnings is falling, the chance of a large negative earnings shock sending the firm bankrupt increases. So a decrease in debt should increase the probability of bankruptcy. Combining these two effects we can conclude that the sign of the coefficient of the change in debt will be negative. That is, an increase in debt multiplied by the negative coefficient indicates that the probability of failure is reduced. Negative changes in debt, resulting from repayments, are multiplied by the negative coefficient to produce a positive effect increasing the probability of failure.

Sound financial management practices dictate that firms should not undertake investments that have nonpositive value, when the project cash flows have been properly defined to include the value of management flexibility. That is, the discounted stream of revenue from the project must be greater than the cost of the investment. Investments in productive equipment increase the certain part of the random earnings stream, providing a larger buffer against negative earnings shocks. This effect helps to reduce the probability of failure. Thus, the coefficient of the investment variable should be negative.

From this analysis of the effects of the control variables on the probability of bankruptcy, we can hypothesise:

Hypothesis 1

1.1 Dividend payments reduce the probability of failure (significant negative coefficient)

1.2 Changes in debt are negatively related to the probability of failure (significant negative coefficient)

1.3 Investment activity reduces the probability of failure (significant negative coefficient)[49]

4.1.2 Probability Expressions

Once the policy regime that the firm is operating in is revealed, the regime specific probability of bankruptcy is given by the expressions in Table 4.13.

Regime	Failure Probability
1	$\Pr\left(\Delta z < -\left[\frac{M_t+a}{\epsilon}\right]\right)$
2	$\Pr\left(\Delta z < -\left[\frac{M_t-a}{\epsilon}\right]\right)$
3	$\Pr\left(\Delta z < \frac{-[M_t+(\theta f(c)+a-rx)]}{\epsilon}\right)$
4	$\Pr\left(\Delta z < \frac{-[M_t+(\theta f(c)-a-rx)]}{\epsilon}\right)$
5	$\Pr(\Delta z < 0)$
6	$\Pr(\Delta z < 0)$
7	$\Pr(\Delta z < 0)$

Table 4.13. **Failure Probabilities by Regime**

The second group of variables to be included as explanatory variables are those appearing in these probability expressions, as listed in Table 4.14.

In the model developed in Chapter 3 the value of the upper limit on borrowing, a, is constrained by the limit on debt:

$$|a_t| \leq h\left[c_t + m_t\right] .$$

When testing the model, the actual change in debt are observed and will be used as the value of a_t. Thus, the actual value of the change in debt control variable, u, will be used in place of the variable a in the expressions listed in Table 4.12.

A list of the variables suggested by analysing the failure probability expressions in Table 4.13 can be found in Table 4.14.

49 Having the funds to engage in investment activity will reduce the probability of failure. Once the funds are committed the cash buffer is reduced which increases the probability of bankruptcy.

Variable	Description
M_t	cash balance at t
rx	total interest paid on current debt, x, in period t
u	change in debt at t
$\theta f(c)$	expected earnings in period t
ϵ	diffusion coefficient of the earnings process.

Table 4.14. **Component Variables from Failure Probabilities**

The effect of these variables on the probability of bankruptcy can be determined through an analysis of the impact on the value of the bankruptcy probability expression for a given regime when the variable is changed, holding the value of all other variables constant. All of these expressions describe the area in the left tail of a normal probability distribution, the shaded area in Figure 4.2. The larger the value of the expression the smaller the left tail area, which is the probability of failure.

The cash position of the firm, M_t, is negatively related to the probability of failure. Cash holdings provides a buffer against negative values of the random component of earnings. The larger the cash holding, the greater the buffer.

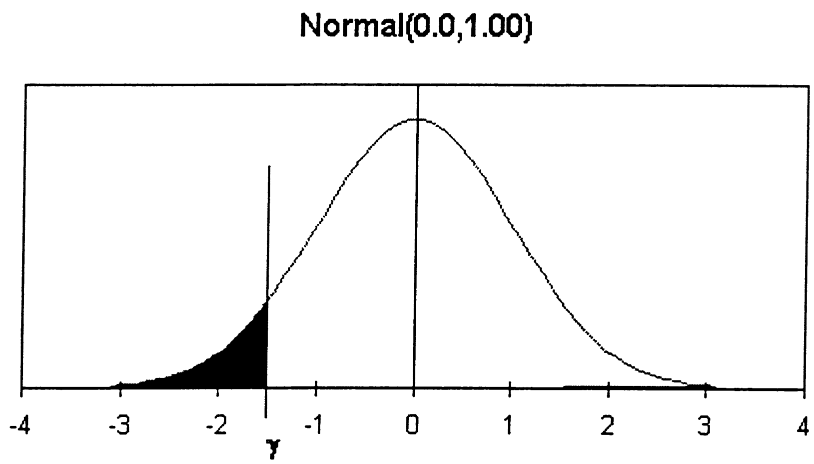

Fig. 4.2. **Bankruptcy Probability - Left Tail**

This buffer effect reduces the probability of failure. This effect can be further explored by looking at the failure probability for regime one. The larger the value of $\frac{[M_t+a]}{\epsilon}$, the smaller the probability of failure. Holding ϵ and a constant, an increase in the cash position of the firm makes the expression $\frac{[M_t+a]}{\epsilon}$ larger, decreasing the area in the left tail of the distribution. The cash position of the firm has a negative relationship with the probability of bankruptcy.

The nature of the random component of earnings is described in Section 2.2 of Chapter 3. A Wiener process is multiplied by a constant, ϵ, the result is the unexpected component of earnings. If the constant is large the possibility of a negative earnings surprise increases. A large enough negative earnings result will wipe out the cash holdings of the firm, leading to bankruptcy. The probability of bankruptcy increases with increases in this constant. Now, if holding the cash position, M_t, and change in debt, a, constant in the bankruptcy probability expression from regime one, $\frac{[M_t+a]}{\epsilon}$, then increasing ϵ causes the expression to become smaller, increasing the left tail area of the probability distribution. The multiplier variable, ϵ, is positively related to the probability of bankruptcy.

Interest payable on outstanding debt has to be paid from the expected part of earnings, before the firm can distribute a dividend or invest in productive equipment. Interest payable is the product of the debt level and the lending interest rate. When the interest payable is larger than the expected part of earnings, it has to be funded from the cash reserve. The greater the value of interest payable, the larger the potential drain on the cash reserve. Reductions in the cash reserve decrease its ability to act as a buffer against negative values of the random component of earnings, leading to an increase in the probability of bankruptcy. This effect can be confirmed using the bankruptcy probability expression from regime three. By keeping cash, M_t, the expected part of earnings, $\theta f(c)$, and the change in debt, a, all constant in the expression, $\frac{[M_t+(\theta f(c)+a-rx)]}{\epsilon}$, then the effect of an increase in interest payable, rx, can be determined. An increase in interest payable, all else constant, makes the value of the expression smaller. This increases the area in the left tail of the distribution, increasing the probability of bankruptcy. Interest payable is positively related to the probability of bankruptcy.

The expected part of earnings after interest payments, $\theta f(c) - rx$, provides funds that the firm can use to distribute as a dividend, invest in productive equipment, increase the cash holding or repay debt. The larger the expected part of earnings, $\theta f(c)$, relative to interest payable, rx, the greater the funds available. The availability of funds to offset negative values of the random component of earnings will decrease the probability of bankruptcy. In the expression, $\frac{[M_t+(\theta f(c)+a-rx)]}{\epsilon}$, from regime three, let the expected part of earnings, $\theta f(c)$, increase while holding all other variables in the expression constant. Increases in $\theta f(c)$ will cause the value of the expression to increase, decreasing the area in the left tail of the distribution. The expected part of earnings, $\theta f(c)$, is negatively related to the probability of bankruptcy.

The effect of changes in the level of debt, u, on the probability of bankruptcy are described above in section 4.1.1, where we discuss the effect that the control variables in the model have on the probability of bankruptcy.

The analysis of the bankruptcy probability expressions can be summarised into the following hypothesis.

Hypothesis 2

2.1 The cash position is negatively related to the probability of bankruptcy (significant negative coefficient)

2.2 Interest payable is positively related to the probability of bankruptcy (significant positive coefficient)

2.3 The expected part of earnings is negatively related to the probability of bankruptcy (significant negative coefficient)

2.4 The constant, ϵ, is positively related to the probability of bankruptcy (significant positive coefficient)

4.1.3 Variables for Statistical Testing

To test Hypothesis 1 and 2, the variables defined in the model have to be approximated using the financial statements of actual companies. The accuracy of the approximations will determine the extent to which the model can be validated. A high correlation between the approximated variables and their counterparts in the model will facilitate a strong validation of model results, while low correlation will only allow for a weak validation. A factor which may affect the quality of the approximations is the split of earnings into expected and random components. Earnings are not broken up in this way in standard financial statements. As well, the model has no corporate tax, unlike the actual corporate environment.

The proxy variables for the control variables flow directly from the definitions in Section 2 of Chapter 3. The dividend paid, w, is measured by the actual dividend paid by the firm in the period of the financial statements. Investments in productive equipment, v, are made to increase the earnings of the firm. Investing in the stock of productive equipment of the company is required to allow for the generation of larger expected earnings, $\theta f(c)$. The productive equipment of firms is recorded in the Plant, Land and Equipment account of the balance sheet. Changes in this account from one year to the next provide a measure of the amount that a company is investing. Debt is measured by the total debt of the firm, which is the sum of short term and long term borrowings. The change in the debt control variable, u, is therefore measured by the change in total debt from one balance sheet to the next.

In section 2.1 of Chapter 3 a definition of bankruptcy was outlined. A firm is bankrupt when its cash position, m, falls below zero. By this definition, bankruptcy is caused by the firm not having enough liquid assets to meet current commitments. The definition of liquidity is limited in the model by the assumption of irreversible investment (see equation 3.19), and the prohibition on equity raising (section 2 of Chapter 3). These limitations on the notion of the liquid assets of the firm suggest the use of a liquidity measure based on short term balance sheet quantities. The liquidity measure, working capital, calculated by Current Assets minus Current Liabilities, will be used to represent the cash, M_t, in testing Hypothesis 2 (section 1.2 of this chapter). Negative values of working capital indicate the firm is facing a short term liquidity problem. The inability to meet debt servicing requirements that arise from a liquidity problem is often an important factor in the instigation of bankruptcy proceedings.

The interest paid variable, rx, has a direct counterpart in a firm's financial statements. The amount of interest that a firm has paid is recorded in the Profit and Loss statement. This value will be used in the hypothesis testing exercises.

The earnings of the firm in any period are described in Section 2.2 of chapter 3. This earnings equation has two components, a certain element based on a production function which maps productive resources to dollars and a random component which may increase or decrease realised earnings. The first line of Table 2.2 describes the calculation of Earnings Before Interest Depreciation and Taxes (EBDIT). As this is the item which records the results of use of productive resources in monetary terms, it will be used to represent the certain portion of earnings, $\theta f(c)$, which is the drift coefficient of the earnings stochastic differential equation.

The multiplier of the random component of earnings, ϵ, will have to be approximated from some combination of financial statement elements. In Section 2.2 of Chapter 3, a Wiener process is multiplied by ϵ. Therefore, ϵ influences the variability of earnings. A measure of the variability of earnings will be used to represent ϵ, the diffusion coefficent of the earnings stochastic differential equation. A simple dispersion measure, the deviation from a two period average $\left(EBDIT_t - \frac{(EBDIT_t + EBDIT_{t-1})}{2}\right)$, will be used to measure the variability of profit. This measure is a simplification derived from the Mean Absolute Deviation of earnings, $MAD = \frac{1}{n}\sum_{i=1}^{n}\left|EBDIT_i - \overline{EBDIT}\right|$. This measure is used for two reasons. Firstly, the calculation of the measure only requires two sets of financial statements, so it is economical on data and, secondly, the measure behaves in a logical manner. Defining the proxy for the variable, ϵ, in this way will have an impact on the sign of the coefficient of this variable in a probabilistic regression. Firms with EBDIT that is above the two period moving average of EBDIT, have experienced a positive earnings surprise. Such values will decrease the probability of failure. Where the observed EBDIT is below the two period moving average, the firm has experienced a negative earnings surprise and is more likely to fail. These two effects suggest that the appropriate sign for the coefficient of this proxy variable is negative. Hypothesis 2.4, which relates the the sign of ϵ, is modified to reflect this measurement effect.

Table 4.15 summarises how these variables are operationalised using the standard financial statements described in Table 2.2 and Table 2.3 of Section 2 of Chapter 2.

Variable	Proxy Variable Constructed from Financial Statements
w	The dividend paid in period t.
$\bar{v}$	The difference in Plant, Land & Buildings(PLB) over the last 2 Balance Sheets, (PLB_t - PLB_{t-1}).
u	The difference in total debt (Short Term Debt + Borrowings) over the last 2 Balance Sheets, (Total $Debt_t$ − Total $Debt_{t-1}$).
M_t	Working Capital (Current Assets - Current Liabilities) from the Balance Sheet.
rx	Interest paid from the P&L.
$\theta f(c)$	Earnings Before Depreciation Interest & Taxes (EBDIT) from the P&L.
ϵ	A simple earnings dispersion measure, the deviation from a 2 period average, is used. $\left(EBDIT_t - \frac{(EBDIT_t+EBDIT_{t-1})}{2}\right)$.

Table 4.15. **Proxy Variables from Financial Statements**

All these variables are divided by Total Assets, as this is the simplest consistent method available to standardise financial statement information. Using Total Assets, that is a measure of the size of a company, as a standardising variable rules out its inclusion as an explanatory variable. As Total Assets do not appear in the expressions for bankruptcy probability in Table 4.13, this would not appear to present a problem in the analysis.

In a probabilistic regression framework, the coefficients represent the effect of the explanatory variable on the probability of failure. A negative coefficient indicates that the variable reduces the probability of failure, a positive coefficient indicates that the variable increases the probability of failure.

The two hypothesis are summarised in Table 4.16.

Variable	Name of Standardised Proxy Variable	Hypothesised sign of Proxy Variable
w	DIV / TA	-
v	INVEST / TA	-
u	DDEBT / TA	-
M_t	WC / TA	-
rx	INT / TA	+
$\theta f(c)$	EBDIT / TA	-
ϵ	DISP / TA	-

Table 4.16. **Summary of Hypotheses to be Tested**

4.2 Development of a Test Statistic

To verify the results, a statistical technique is required that can estimate failure probabilities, allow for explanatory variables and deal with interval-censored data. The Cox (1972) proportional hazard method, as extended by Fleming and Harrington (1991), is a probabilistic regression model that allows for censored data. This approach to the modeling and estimation of failure probabilities will be used in this study. The hazards approach is not attempting to estimate the parameters of the earnings process, with all the associated biases discussed in Lo (1998). In this study the drift, $\theta f(c)$, and diffusion, ϵ, coefficients of the earnings process will not be estimated, rather variables that approximate them will be input variables in the hazard model.

A hazard rate is the probability that a subject (firm) will fail in the next increment of time. It is derived from the distribution of firm durations (lifetimes). The hazard is defined as a function of explanatory variables and estimated parameters.

The model specification problem that presents itself here has two distinct parts. The probability of bankruptcy is defined as

$$\Pr(Bankruptcy = true) = f(\underset{\sim}{\beta} \underset{\sim}{X}), \tag{4.28}$$

where $f(\bullet)$ is a function that returns a value between 0 and 1, with $\underset{\sim}{\beta}$ a vector of parameters to be estimated and $\underset{\sim}{X}$ a vector of firm-specific variables. Then the first part of the model specification process is to identify the functional form of $f(\bullet)$. Once the function $f(\bullet)$ has been defined, the second task is to find a statistical technique to estimate the unknown parameters, $\underset{\sim}{\beta}$.

4.2.1 Hazard functions

As a starting point, we define the probability of bankruptcy in more detail.

At any time, t, the firm can be in one of two states: a going concern (the firm has a positive cash balance, $m_t \geq 0$[50]) or the firm is bankrupt (the firm has a negative cash balance, $m_t < 0$). The quantity of interest is the probability that the firm will switch from being in the going concern state, to one of bankruptcy. This probability is conditional, the state transition can only occur if the firm is a going concern at the beginning of the period of interest. The definition of bankruptcy becomes,

$$\begin{aligned} \Pr(Bankruptcy) &= \Pr(\text{Bankrupt in period } t \mid \text{going concern at beginning of period } t). \\ &= \lambda(t)\Delta \end{aligned}$$

The function, $\lambda(t)$, is known as a hazard function for the random variable firm lifetime, τ[51], and Δ is the length of the time period. Roughly, $\lambda(t)$ is the rate at which firms will fail at time t, given that they last until t. This function, like the probability of bankruptcy, is conditional, as a firm can only be at risk of failing at time t if it was a going concern at the end of period $t-1$.

Further development of a statistical argument requires the availability of a sample containing both going concerns and bankrupt firms. In such a sample of firms, there

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50 Using the variables defined in the explanatory model developed in Chapter 3, m_t is the firm's cash balance at time t.

51 The variable τ denotes the first time that the cash balance falls below zero, $m_t < 0$. That is, the length or duration of the firm's lifetime.

is a distribution of firm lifetimes. The lifetime of any firm is a realisation of the random variable, τ. The probability distribution of these lifetimes can be specified by the distribution function.

$$F(t) = \Pr(\tau < t). \tag{4.29}$$

It specifies that the probability of the random lifetime, τ, is less than some given value, t. This is the probability that the firm goes bankrupt before time t. The corresponding probability density function for this distribution is

$$f(t) = \frac{dF(t)}{dt}.$$

The probability of surviving longer than t periods is given by the survival function

$$\begin{aligned} S(t) &= 1 - F(t) \qquad (4.30) \\ &= \Pr(\tau \geq t), \end{aligned}$$

which defines the upper tail area of the distribution of firm lifetimes.

The definition of the probability of bankruptcy can now be made more precise in terms of the probability distributions[52] that have been introduced

$$\lambda(t) = \lim_{\delta \to 0} \frac{P(t \leq \tau \leq t + \delta \mid \tau \leq t)}{\delta}. \tag{4.31}$$

This is the probability that the firm will become bankrupt in the next instant, given that it was a going concern at time t. The statistical analysis[53] of this hazard function will form the basis for the verification of the explanatory model developed in Chapter 3.

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52 The hazard function can also be written as $\lambda(t) = \frac{f(t)}{S(t)}$.

53 The statistical theory underlying the analysis of lifetime data can be found in Kalbflesch and Prentice (1980).

4.2.2 Explanatory variables and the form of the hazard function

The Proportional Hazard specification of Cox (1972) will be used to estimate bankruptcy probabilities. This model includes explanatory variables and partitions the hazard function in a way that lends itself to the verification of the variables derived from the explanatory model.

In the Cox (1972) proportional hazards model, the hazard function depends on a vector of explanatory variables, $\underset{\sim}{X}$, with unknown coefficients $\underset{\sim}{\beta}$, and λ_0. It is defined as

$$\lambda(t, X) = \phi(\underset{\sim}{X}, \underset{\sim}{\beta})\lambda_0(t), \tag{4.32}$$

where $\lambda_0(t)$ is a "baseline", or underlying, hazard faced by all firms at time t. It is conventional when working with this model, to scale the $\underset{\sim}{X}'s$ so that $\phi(\bullet)$ equals one at their mean values[54]. Under these conditions, $\lambda_0(t)$ can be interpreted as the hazard function for the average firm. The baseline hazard is an unknown which has to be estimated before the model can be used to generate bankruptcy probabilities. In this form, the effect of the firm specific variables is to multiply the baseline hazard, $\lambda_0(t)$, by a factor $\phi(\bullet)$ which does not depend on time. The form of $\phi(\bullet)$ which was specified by Cox (1972) is

$$\phi(X, \beta) = \exp(\underset{\sim}{\beta}'\underset{\sim}{X}). \tag{4.33}$$

This allows $\phi(\bullet)$ to be non negative without requiring restrictions on the $\underset{\sim}{\beta}'s$ in the estimation process. To interpret the parameters, $\underset{\sim}{\beta}$, take the derivative of the log of the

54 The transformation used to scale the explanatory varialbes is $x = X - \overline{X}$

hazard function with respect to $\underset{\sim}{X}$. Then,

$$\begin{aligned}\frac{\partial \ln \lambda(t)}{\partial \underset{\sim}{X}} &= \frac{\partial}{\partial \underset{\sim}{X}} \ln \left(\exp(\underset{\sim}{\beta}' \underset{\sim}{X}) \right) + \frac{\partial}{\partial \underset{\sim}{X}} \lambda_0(t) \\ &= \underset{\sim}{\beta}.\end{aligned} \quad (4.34)$$

The underlying hazard drops out as it does not depend on $\underset{\sim}{X}$. The coefficients, $\underset{\sim}{\beta}$, describe the constant proportional effect of the explanatory variables on the conditional probability of bankruptcy.

4.2.3 Estimation

Coefficients are estimated using a partial likelihood approach suggested by Cox (1972)[55]. The estimation method is developed for a sample of firms in which there are D failed firms. These failed firms are ordered, based on the length of their lifetimes, $\tau_1 < \tau_2 \cdots < \tau_D$, where τ_i is the completed lifetime of the i^{th} failed firm. Parameter estimation is based on the concept of risk sets. These sets are all the firms that are at risk of failure at any time. The risk set for time, t_i, is denoted by $R(t_i)$. When a firm fails, the probability of that firm failing at that time is computed. The probability that firm, i, with attributes, $\underset{\sim}{X}_i$, fails at time t_i given that one firm in $R(t_i)$ fails at this time, is given

[55] The method presented assumes that there are no ties in the failure times. Where there are tied failure times an approximation proposed by Breslow (1974) is used in estimating $\underset{\sim}{\beta}$. This approximation works well when the number of ties in the risk set at each failure time is not large.

by:

$$\begin{aligned}
& \Pr(\text{ firm } i \text{ fails at } t_i | \text{ one failure at } t_i) \quad (4.35) \\
= & \frac{\Pr(\text{firm } i \text{ fails at } t_i | \text{ survival to } t_i)}{\Pr(\text{one firm fails at } t_i | \text{ survival to } t_i)} \\
= & \frac{\lambda(t_i, \underset{-}{X}_i)}{\sum_{j \in R(t_i)} \lambda(t_i, \underset{-}{X}_j)} \\
= & \frac{\lambda_0(t_i). \exp(\underset{-}{\beta}' \underset{-}{X}_i)}{\sum_{j \in R(t_i)} \lambda_0(t_i). \exp(\underset{-}{\beta}' \underset{-}{X}_j)} \\
= & \frac{\exp(\underset{-}{\beta}' \underset{-}{X}_i)}{\sum_{j \in R(t_i)} \exp(\underset{-}{\beta}' \underset{-}{X}_j)}
\end{aligned}$$

The product of these probabilities over all D failure times is the partial likelihood function,

$$L(\beta) = \prod_{i=1}^{D} \frac{\exp(\underset{-}{\beta}' \underset{-}{X}_i)}{\sum_{j \in R(t_i)} \exp(\underset{-}{\beta}' \underset{-}{X}_j)}. \quad (4.36)$$

The values of the parameters that maximise this function are then found. This process can be thought of as finding the parameters which maximise the distance (in probability terms) between the failed and surviving companies[56].

The log likelihood function is needed in order to generate the standard errors for the parameters estimated in the proportional hazards framework. Let $LL(\beta) = \ln[L(\beta)]$, then when there are p coefficients to be estimated

$$LL(\beta) = \sum_{i=1}^{D} \sum_{k=1}^{p} \beta_k X_{i,k} - \sum_{i=1}^{D} \ln \left[\sum_{j \in R(t_i)} \exp \left(\sum_{k=1}^{p} \beta_k X_{j,k} \right) \right]. \quad (4.37)$$

The information matrix for this likelihood function is given by the negative of the matrix of second derivatives of $LL(\beta)$. A typical element of this information matrix, $I_{g,h}(\beta)$,

[56] The approach is analogous to Multiple Descriminate Analysis (MDA) in that MDA attempts to find parameters which maximise the distance between the means of two groups.

is given by,

$$I_{g,h}(\beta) = \sum_{i=1}^{D} \frac{\sum_{j\in R(t_i)} X_{j,g} X_{j,h} \exp\left(\sum_{k=1}^{p} \beta_k X_{j,k}\right)}{\sum_{j\in R(t_i)} \exp\left(\sum_{k=1}^{p} \beta_k X_{j,k}\right)}$$

$$-\sum_{i=1}^{D} \left[\frac{\sum_{j\in R(t_i)} X_{j,g} \exp\left(\sum_{k=1}^{p} \beta_k X_{j,k}\right)}{\sum_{j\in R(t_i)} \exp\left(\sum_{k=1}^{p} \beta_k X_{j,k}\right)}\right]\left[\frac{\sum_{j\in R(t_i)} X_{j,h} \exp\left(\sum_{k=1}^{p} \beta_k X_{j,k}\right)}{\sum_{j\in R(t_i)} \exp\left(\sum_{k=1}^{p} \beta_k X_{j,k}\right)}\right]$$

The variance-covariance matrix of the estimated coefficients is estimated by taking the inverse of the estimated information matrix, $\underset{\sim}{I}^{-1}(\beta)$. The standard errors of the estimated parameters are the appropriate diagonal elements extracted from this matrix.

Hypothesis tests of the estimated parameters are based on the asymptotic normality of the partial likelihood estimator. The standard errors and the estimated parameters are combined to form a z-test,

$$z_i = \frac{\hat{\beta}_i}{se(\hat{\beta}_i)}. \tag{4.38}$$

This test statistic is used to test the hypothesis that the estimated parameter, $\hat{\beta}_i$, is equal to zero. This test is equivalent to a Wald chi-squared test applied to a single parameter in the context of the Cox (1972) model. The critical values of the statistic are compared to a standard normal distribution.

Estimates of the coefficients, $\underset{\sim}{\hat{\beta}}$, along with the estimated z-scores for each coefficient are the tests statistics that will be used in the next chapter to test Hypothesis 1 and 2 as summarised in Table 4.16.

Chapter 5
Results

The purpose of this chapter is to report the results of the empirical analysis of the two hypotheses described in Chapter 4.1. These hypotheses are presented in Table 5.17.

Number	Hypotheses
1.1	Dividend payments reduce the probability of failure
1.2	Changes in debt are negatively related to the probability of failure
1.3	Investment activity reduces the probability of failure
2.1	The cash position is negatively related to the probability of bankruptcy
2.2 -	Interest payable is positively related to the probability of bankruptcy
2.3	The expected part of earnings is negatively related to the probability of bankruptcy
2.4	The constant ϵ is positively related to the probability of bankruptcy

Table 5.17. **Hypotheses to be Validated**

To test Hypothesis 1 and 2, the variables mentioned in the hypotheses are approximated using the financial statement data of companies. The definitions of the approximations used, the proxy variables, can be found in Table 4.15. When the definition of the proxy variables is taken into account, the hypothesised sign for the multiplier of the random part of the earnings variable, ϵ, changes, see Section 1.3 of Chapter 4. Table 5.18 summarises the hypothesised signs that will be tested in this chapter.

The results in this section are presented in three parts. The first part is a description of the sources and properties of the data set used. The second contains results from the analysis of differences in univariate means for each of the variables which proxy for the variables mentioned in Hypotheses 1 and 2, which are summarised in Table 5.18. The final part is a report of the findings from the Cox (1972) regression analysis of the hypothesised effects of the model variables on bankruptcy probability.

Hypothesis	Variable	Standardised Proxy Variable	Hypothesised sign
1.1	Dividend	DIV.TA	-
1.2	Change in Debt	DDEBT.TA	-
1.3	Investment	INVEST.TA	-
2.1	Cash Position	WC.TA	-
2.2	Interest Payable	INT.TA	+
2.3	Expected Part of Earnings	EBDIT.TA	-
2.4	Multiplier of Unexpected Earnings	DISP.TA	-

Table 5.18. **Summary of Hypotheses for Proxy Variables**

5.1 Data and Sample Properties

5.1.1 Data

The published financial statements of listed firms has been the source of data for the classification and data reduction studies reported in the literature. Financial data for Australian firms derived from Annual Reports filed with the Australian Stock Exchange(ASX) between 1966 and 1994 are used in this study. Firms are able to enter the study from the time they are listed. Each firm has one set of financial statements for each year, a P&L statement and a Balance Sheet[57]. When the first set of financial statements becomes available, the firm enters the study.

The end of a firm's lifetime can be caused by a number of different events. These include delisting caused by financial distress or the appointment of a liquidator, voluntary liquidation, takeover, or a voluntary delisting. Only firms that are surviving at the end of 1994, or have been delisted due to financial distress or appointment of a liquidator, are used in this analysis[58]. Companies which have been delisted are flagged

57 See section 2.2 for a description of these statments.

58 The Cox (1972) proportional hazards model, the model used in this study, can be used when there is more than one event that can remove a subject from the study. Such models are said to have competing risks. A hazard function is estimated for each event type, then the interrelation between the failure types is found. Chapter Seven of Kalbfleisch and Prentice (1980) presents the statistics of competing risk models.

through the inclusion of a delisting date in their ASX company header information. Those companies with a delisting date were then cross checked against the Nothman (1999) Record of Delistings. Only those firms identified as being distressed or having a liquidator appointed are included as bankrupt. This definition of bankruptcy is a legal definition, based on the listing rules of the Australian Stock Exchange. The bankruptcy definition in the model developed in Section 2.1 of Chapter 3 is that the firm is bankrupt when its cash holding is negative. This is an economic definition based on liquidity. Severe liquidity problems in a firm are often the trigger to have liquidators appointed. This is a first step in the formal delisting process.

The analysis is run as time-on-study, not in calendar time. The time-on-study is the difference between the current balance sheet date and the firm's listing date. This means that firms of equal age are compared no matter what date their statements were released[59]. There are 146 delisted companies and 1505 companies that are going concerns in the sample.

A number of variables in the model are expressed in first-difference terms. To calculate these variables, two consecutive financial statements are required. Firms not having two consecutive statements to work with have been excluded from the study. This requirement may introduce a bias. Companies which fail in the first two years of their lifetimes, that is companies who only report once then fail, will not appear in the study.

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59 For example, in 1995 a company which was listed in 1990 will have a lifetime of 5 years, a company listed in 1985 will have a lifetime of 10 years. The 1995 statements of these companies will not be analysed together, but with other 5 year old statements and 10 year old statements, respectively.

5.1.2 Censoring and Time Variation

In the sample of financial statements for the companies included in the study there are 146 records with known failure times. Each of the companies that has failed contributes one of these records. For the companies that are still going concerns at the study termination date, December 31, 1993, we do not know whether they fail. The information on these companies is said to be right-censored. The listing date, and the date of their first financial statement included in the sample, is known for all companies. That is, there are no companies which are left-censored in the sample.

Each set of statements contain the public information about a company, until the issue of the next statement or exit of the firm from the study due to failure. The information in these statements is unknown before the statement date and not applicable following the release of the next financial statements, or delisting of the company. Observations which apply to a specific interval of time are known as interval-censored data. Intervals commence on the release of a set of financial statements. They end when a new set of financial statements is released or the company is delisted from the exchange. All the observations used in model validation are interval-censored. This interval-censoring allows for the introduction of time varying effects. The financial position of companies in the sample is repeatedly measured over the period they remain on study. Each of these measures captures the firm in a different stage of evolution. The use of all the observed intervals allows the full history of the financial status of the firm to be incorporated into the statistical analysis of failure probabilities[60]. There are 7987 intervals not ending in delisting, and 146 intervals ending in delisting available for

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60 The use of repeated measures in the Proportional Hazards model is analogous to the analysis of Panel Data by regression analysis. Chapter 12 of Johnston and DiNardo (1997) describes regression methods for panel data.

use in the model validation exercise. The formation of the risk sets used in the estimation of the Cox (1972) proportional hazards model, described in Chapter 4.2, allows for the inclusion of interval-censored observations. At each failure time the risk set constructed will contain all the records in the sample whose time on study matches the age of the company that has failed. Construction of the risk sets in this way allows for time variation in the estimation of the Cox (1972) model.

5.1.1 Descriptive Statistics

Descriptive statistics of seven variables identified in the probability expressions derived from the model developed in Section 2 of Chapter 3 are described in Figure 5.3. The statistics reported for each variable[61] are the minimum value, the quartile values, the mean, median, the maximum value and its standard deviation.

The statistics presented in the top panel of the display, labeled Continuing, relate to those intervals which do not end in a delisting. The results in the bottom panel of the display, labeled delisted, present the statistics of the variables for firms which were delisted at the end of an interval. In the next section each of the variables will be tested for significant differences in the mean values for the two groups of firms.

61 INT.TA represents interest paid, DDEBT.TA represents the change in debt, INVEST.TA represents the firms investment in productive equipment, DIV.TA represents the dividend paid, EBIDT.TA represents the certain part of the random earnings, WC.TA represents the firm's cash holding and DISP.TA represents the multiplier of the random component of earnings. All these variables have been standardised.

Continuing

	INT.TA	DDEBT.TA	INVEST.TA	DIV.TA	EBDIT.TA	WC.TA	DISP.TA
Min:	-0.203	-5332.105	-5007.316	0	-31.919	-0.46296	-1393.87
1st Qu.:	0.003	-0.013	-0.011	0	0.038	0.157858	-0.012
Mean:	0.026	-7.347	-16.734	0.024	0.059	0.358682	1.809
Median:	0.018	0	0.005	0.017	0.106	0.343121	0.007
3rd Qu.:	0.036	0.026	0.052	0.033	0.158	0.541968	0.023
Max:	4.013	115.686	0.94	3.889	394.197	1	5170.094
Std Dev.:	0.063	102.525	104.982	0.065	4.149	0.235372	96.321

Delisted

	INT.TA	DDEBT.TA	INVEST.TA	DIV.TA	EBDIT.TA	WC.TA	DISP.TA
Min:	-1.136	-853.983	-1781.777	0	-32.032	0	-92.8078
1st Qu.:	0.011	-0.058	-0.052	0	-0.126	0.084051	-0.0741
Mean:	0.083	-18.277	-31.182	0.013	-0.458	0.31603	-1.75456
Median:	0.037	0	0	0	0.042	0.241061	-0.00196
3rd Qu.:	0.073	0.081	0.051	0.017	0.107	0.498638	0.03551
Max:	2.76	0.48	0.815	0.149	3.905	0.998729	92.05049
Std Dev.:	0.28	93.945	166.278	0.026	3.334	0.267208	16.33181

Fig. 5.3. **Sample properties of Continuing and Delisted Firms**

5.2 Tests for Differences in Univariate Means

Analysis of the results of univariate tests for a significant difference in the group means are presented in Figure 5.4. The test used is a standard t-test for a significant difference in the mean of two groups with unequal variance[62]. If the p-value reported in Figure 5.4 is less than 0.05, the alternate hypothesis that the difference in the group means is not

[62] Let $\bar{y}_1$ be the estimated mean of the first group and $\bar{y}_2$ be the estimated mean for the second group. n_1 is the size of the first group, n_2 is the size of the second group. The variance of group one is S_1^2, for group two it is S_1^2.
The hypothesis tested is $\mathrm{H}_0 : (\mu_1 - \mu_2) = 0$ vs's $\mathrm{H}_1 : (\mu_1 - \mu_2) \neq 0$.
The test statistic for this hypotheses test is

$$t = \frac{(\bar{y}_1 - \bar{y}_2)}{\sqrt{S_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where the pooled variance, S_p^2, is defined to be,

$$s_p^2 = \frac{(n_1 - 1)\, S_1^2 + (n_2 - 1)\, S_2^2}{n_1 + n_2 - 2}.$$

equal to zero is supported at the 95% level of confidence. The closer the p-value is to zero the greater the difference between the means of the two groups.

Difference in Group Means

	INT.TA	DDEBT.TA	INVEST.TA	DIV.TA	EBDIT.TA	WC.TA	DISP.TA
t-score	-2.5399	1.3909	1.0461	4.8008	1.9274	1.9998	2.0615
p-value	0.0121	0.1663	0.2973	0	0.0556	0.0472	0.0399

Fig. 5.4. **Results of t-tests for Difference in Group Means.**

Where the test statistic reported is negative, the mean value of the variable for continuing firms is smaller that the mean value for the delisted firms. Only the interest paid variable has a negative t-score. That is, on average firms that are delisted pay significantly more interest than do continuing firms. The remaining six variables have positive test statistics. Hypothesis 1 and 2, summarised in Table 5.18, assert that the sign of the regression coefficient for interest paid will have the opposite sign to the coefficients of the remaining variables. This pattern of signs is born out in the t-statistics listed in Figure 5.4.

Five of the seven variables have significant differences in their group means at the 95% level of confidence. The variables which represent two of the control variables, namely change in debt, DDEBT.TA, and investment in productive equipment, INVEST.TA, do not demonstrate a significant difference between the mean value of the continuing and delisted groups. It is reasonable to expect the five variables which demonstrate significant differences in their means will have significant coefficients in the probabilistic regression framework. The variables which do not show significant differences in their group means are unlikely to have significant coefficients in a probabilistic regression. The results of the probabilistic regression analysis are presented in the following section.

5.3 Cox Regression Results

The results of the Cox regressions necessary to validate Hypotheses 1 and 2 are presented in this section. All results reported in this section were generated using the S+ software package and the data set described in Section 1.1 of this Chapter.

Figure 5.5 presents the results from the estimation of the $\underset{\sim}{\beta}$ coefficients of the Cox (1972) proportional hazards model, $\lambda(t, X) = \exp(\underset{\sim}{\beta}'\underset{\sim}{X})\lambda_0(t)$. The results reported do not arise from a stepwise variable selection approach to the Cox model. All variables that are included are specified in advance. The statistics reported are, the estimated coefficient, β_i, for each variable included, an estimated relative risk for a one unit change in the variable, $\exp(\beta_i)$, the standard error of the estimated coefficient, a z-test which tests the null hypothesis, $H_0 : \beta_i = 0, i = 1, 2, \cdots, 7$, and its p-value. The likelihood ratio test presented is analogous to the F-test for significant coefficients in the regression framework. It tests the null hypothesis $H_0 : \beta_1 = \beta_2 = \cdots = \beta_7 = 0$, using the test statistic, $\chi^2_{LR} = 2[\ln(L(\underset{\sim}{\beta})) - \ln(L(0))]$[63], which is distributed chi-squared with m degrees of freedom, where m is the number of estimated parameters. The p-value for this test is also presented.

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63 The function $L(\bullet)$ used in the calculation of this test statistic is the partial likelihood function of the Cox proportional hazards model presented in equation 4.36.

	coef	exp(coef)	se(coef)	z	p
DIV.TA	-15.46997	0.0000	4.7235	-3.275	0.0011
DDEBT.TA	-0.000855	0.9990	0.0006	-1.373	0.1700
INVEST.TA	-0.000314	1.0000	0.0004	-0.745	0.4600
WC.TA	-0.483116	0.6170	0.3639	-1.327	0.1800
INT.TA	0.834069	2.3000	0.2345	3.557	0.0004
EBDIT.TA	-0.051663	0.9500	0.0264	-1.954	0.0510
DISP.TA	-0.002412	0.9980	0.0013	-1.872	0.0610

Likelihood ratio test= 38 on 7 df, p=2.99e-006

Fig. 5.5. **Results of Cox Regression - All Variables**

The likelihood ratio test for this regression has a p-value of 2.99e-6, which is less than the critical value of 0.05 that arises from a 95% confidence level. This value allows the rejection of the null hypothesis, that all the estimated coefficients are equal to zero. The signs of all the variables in this regression are in agreement with the signs hypothesised in Table 5.19. Hypotheses 1 and 2, summarised in Table 5.18, are also concerned with the significance of the coefficients. The p-values derived from the z-tests for the hypothesis, $H_0 : \beta_i = 0, i = 1, 2, \cdots, 7$, provide a means to evaluate the significance of the variables included.

The variable with the largest p-value, 0.46, making it the least significant, is the investment variable, INVEST.TA. As this variable did not exhibit a significant difference in group means in univariate testing, reported in Section 2 of this Chapter, this is not surprising. The other variable which exhibited no difference in group means in univariate testing was the change in debt, DDEBT.TA. The p-value associated with the z-test of the coefficient of this variable is 0.17. This value indicates that the coefficient is not significantly different from zero. Both of these variables exhibit low values for their estimated coefficients and standard errors. This suggests that there may be a degree of correlation between these variables which will require further investigation. The z-test for the coefficient of the firm's cash position, WC.TA, has a p-value of 0.18. In a two tailed test we would conclude that the probability of the coefficient being zero is 64%. Although the signs of the coefficients of these variables agree with Hypotheses 1 and 2, they are not statistically significant at the 95% or 90% levels.

The remaining coefficients are all significant. The coefficient of DISP.TA, which represents the multiplier of the random component of earnings, ϵ, is significant at the 94% level. The coefficient of the certain part of earnings, $\theta f(c)$, represented by EBDIT.TA, is significant at the 95% level. The coefficients of both the dividend, DIV.TA, and interest paid, INT.TA, variables are significant at the 99% level.

Table 5.19 summarises these results. The results of this Cox regression support Hypotheses 1.1, 2.2, 2.3 and 2.4.

Hypothesis	Variable Name	Expected sign	Sign Correct	Variable Significant
1.1	DIV.TA	-	yes	99%
1.2	INVEST.TA	-	yes	no
1.3	DDEBT.TA	-	yes	no
2.1	WC.TA	-	yes	no
2.1	INT.TA	+	yes	99%
2.3	EBDIT.TA	-	yes	95%
2.4	DISP.TA	-	yes	94%

Table 5.19. **Summary of Results Hypotheses Tests - Full Variable Set**

The behaviour of the coefficients and standard errors of the change in debt and investment variables indicate that some further analysis is warranted. The correlation structure of the variables included in the initial regression model is presented in Figure 5.6.

	DDEBT.TA	INVEST.TA	DIV.TA	EBDIT.TA	WC.TA	INT.TA	DISP.TA
DDEBT.TA	1.0000	0.3439	0.0154	0.0023	0.0266	-0.08306	-0.27886
INVEST.TA	0.3439	1.0000	0.0046	0.0029	0.0161	-0.02725	-0.08102
DIV.TA	0.0154	0.0046	1.0000	0.0161	0.0129	-0.04325	-0.02621
EBDIT.TA	0.0023	0.0029	0.0161	1.0000	0.0401	0.264113	-0.00288
WC.TA	0.0266	0.0161	0.0129	0.0401	1.0000	-0.0112	-0.04459
INT.TA	-0.0831	-0.0273	-0.0432	0.2641	-0.0112	1.0000	0.02469
DISP.TA	-0.2789	-0.0810	-0.0262	-0.0029	-0.0446	0.02469	1.0000

Fig. 5.6. **Correlation Matrix - Proxy Variables used in Cox Regression**

This correlation matrix points to a potential multicollinearity problem in the Cox regression. The variables DDEBT.TA and INVEST.TA are correlated, with $\rho = 0.34$. To correct for this correlation the variable INVEST.TA, which was the least significant in the previous Cox regression, was excluded and the regression rerun. The results of this regression are reported in Figure 5.7.

	coef	exp(coef)	se(coef)	z	p
DDEBT.TA	-0.001	0.99900	0.000503	-1.99	0.0470
DIV.TA	-15.52096	0.00000	4.726968	-3.28	0.0010
WC.TA	-0.4872	0.61400	0.363804	-1.34	0.1800
INT.TA	0.83405	2.30000	0.234408	3.56	0.0004
EBDIT.TA	-0.05158	0.95000	0.026444	-1.95	0.0510
DISP.TA	-0.00254	0.99700	0.001238	-2.05	0.0400

Likelihood ratio test= 37.6 on 6 df, p=1.35e-006

Fig. 5.7. **Cox Regression Results - Reduced Variable Set.**

The results of the likelihood ratio test, with a p-value of 1.35e-6, indicate that at least one of the coefficients is significantly different to zero. The signs of the coefficients for all variables included are in agreement with the hypothesised direction of the effect of the variables. In this regression the variable representing the cash position of the firm, WC.TA, is not significant. The result for this variable remaining constant in both regressions. The coefficients of all other variables included are significant. The coefficient of DISP.TA, which represents the multiplier of the random section of earnings, ϵ, is significant at the 95% level. The coefficient of the certain part of earnings, $\lambda f(c)$, represented by EBDIT.TA, is significant at the 95% level. The coefficient of the change in debt, DDEBT.TA and the coefficients of both the dividend and interest paid variables are also significant.

Table 5.20 summarises these results. The results of this Cox regression support Hypotheses 1.1, 1.3, 2.2, 2.3 and 2.4.

Hypothesis	Variable Name	Expected sign	Sign Correct	Variable Significant
1.1	DIV.TA	-	yes	99%
1.2	INVEST.TA	-		
1.3	DDEBT.TA	-	yes	95%
2.1	WC.TA	-	yes	no
2.1	INT.TA	+	yes	99%
2.3	EBDIT.TA	-	yes	95%
2.4	DISP.TA	-	yes	94%

Table 5.20. **Summary of Results Hypotheses Tests - Reduced Variable Set**

5.4 Summary

The Cox regression results are summarised in Table 5.21. Hypothesis 1 tests for the direction and significance of the relationship between the control variables of the explanatory model and the probability of bankruptcy. The effect on the probability of bankruptcy of the dividend paid, w, is tested in Hypothesis 1.1. Hypothesis 1.2 tests for the effect of the change in debt, u, and Hypothesis 1.3 for the effect of investment, v. Two of these hypotheses can be fully supported. They are Hypothesis 1.1, relating to the dividend paid and Hypothesis 1.2, relating to the change in debt.

Hypothesis 2 tests for the direction and significance of the relationship between variables included in the bankruptcy probability expressions derived in Section 3.2 of

Hypothesis	Variable	Proxy Variable	Hypothesised Sign	Estimated Sign	Significance Level
1.1	Dividend	DIV.TA	-	-	99%
1.2	Change in Debt	DDEBT.TA	-	-	95%.
1.3	Investment	INVEST.TA	-	-	n.a.
2.1	Cash Position	WC.TA	-	-	82%
2.2	Interest Payable	INT.TA	+	+	99%
2.3	Expected Part of Earnings	EBDIT.TA	-	-	95%
2.4	Multiplier of Unexpected Earnings	DISP.TA	-	-	94%

Table 5.21. **Results of Hypotheses Testing**

Chapter 3. The effect on the probability of bankruptcy of the firm's cash position, m, is tested in Hypothesis 2.1. Hypothesis 2.2 tests for the effect of interest payable, rx, Hypothesis 2.3 tests for the effect of the expected part of earnings, $\theta f(c)$, and Hypothesis 2.4 for the multiplier of the random component of earnings, ϵ. Three of these hypotheses are fully supported. They are hypothesis 2.2 relating to interest payable, hypothesis 2.3 relating to the expected part of earnings and hypothesis 2.4 regarding the multiplier of the random component of earnings.

The results of two hypotheses that were not supported, Hypothesis 1.3 relating to investment and Hypothesis 2.1 relating to the firm's cash position, warrant some explantation. The definition of investment in productive equipment, INVEST.TA, in terms of the Plant, Land and Buildings balance sheet item includes investment in real property as part of the definition of productive equipment. If investments in real property are large relative to the plant component, this variable may not be an effective measure of investment in productive equipment. The correlation between INVEST.TA and the change in debt variable, DDEBT.TA, also makes it difficult to draw a conclusion for Hypothesis 1.2. The result for the cash position of the firm, WC.TA, which is a working capital measure, needs to be considered in conjunction with the result for the interest paid variable INT.TA. Firstly, the definition of bankruptcy in the model as developed in

Section 2.1 of Chapter 3, has the firm failing when its cash holdings become negative. In terms of balance sheet items, the equivalent condition occurs when the working capital of the firm is insufficient to meet debt servicing requirements. So the effect of the cash position may be "swamped" by the interest payable factor. Secondly, provisions of the corporations law which prohibit trading while insolvent work to limit the spread of this variable between the two groups. This will reduce the explanatory power of the variable in a hazards regression. The results of the Cox regressions in this chapter show that in the sample of firms used, the level of interest payable is more important than the cash holding in the determination of bankruptcy probabilities.

Five of the seven hypotheses developed from the analysis of the model developed in Chapter 3 are fully supported by the data. The results on the signs of the estimated coefficients of all variables are in agreement with the effects predicted by the analysis of the descriptive model of Chapter 3. These results are conditional, as they depend on the adequacy of the proxy variables to represent the variables defined in Chapter 3.

Chapter 6
Conclusion

The objective of this research is the construction and validation of a model of the firm, based on economics, that is capable of characterising the probability of bankruptcy.

The model developed in Section 2 of Chapter 3 draws on financial economics for its objective and structure. The objective of the model is to maximise the value of the firm over its lifetime. The value of the firm is derived from the expected value of the discounted stream of dividend payments that the owners of the firm receive. Shareholder wealth maximisation is the objective of management, and the notion that the value of a financial asset is the discounted value of the cash flow that the asset generates is a basic concept in financial economics. The dividend that is paid by the firm is part of the financing decision of the firm. The other part of the financing decision the firm can make concerns changes to the level of debt it carries. Firms also make investment decisions. In the model developed, the firm decides on the amount of productive equipment to be acquired. The model is based on financial economics, the objective is the maximisation of shareholder wealth with the investment and financing decisions of the firm included as decision variables.

The actions available to the firm can be described by a set of constraints on the operation of the firm. The first set of constraints govern the movement of the variables which are used to characterise the current state of the firm. They are the state transition equations. The variables that are used to characterise the state of the firm are its cash position, the stock of productive capital and the level of debt. The state transition equation for the cash position of the firm is a cash flow equation. The cash flow of the firm is stochastic, as the cash flow from operations, specifically the earnings component, is assumed to be random. Changes in the stock of productive equipment are the result of new investment. Debt changes by the amount the firm chooses to borrow or repay. The remaining constraints restrict the range of investment and financing decisions available to the firm. They limit the payment of dividends, the amount of investment the firm can undertake and the amount the firm can borrow or repay. These constraints are based on the types of constraints that firms would normally face. Solutions of the model were found through the application of dynamic programming in Section 3 of Chapter 3. Analysis of the optimality condition of the model gives rise to seven possible policy regimes. An expression for the probability of bankruptcy was derived for each of these regimes. This analysis results in the five unique bankruptcy probability expressions listed below.

Failure Probability Expressions
$\Pr\left(\Delta z < -\left[\frac{M_t+a}{\epsilon}\right]\right)$
$\Pr\left(\Delta z < -\left[\frac{M_t-a}{\epsilon}\right]\right)$
$\Pr\left(\Delta z < \frac{-[M_t+(\theta f(c)+a-rx)]}{\epsilon}\right)$
$\Pr\left(\Delta z < \frac{-[M_t+(\theta f(c)-a-rx)]}{\epsilon}\right)$
$\Pr\left(\Delta z < 0\right)$

Table 6.22. **Five Unique Failure Probability Expressions**

The variables that make up these expressions are listed in Table 6.23.

Variable	Description
a	maximum change in debt at t
M_t	cash balance at t
rx	interest paid in period t
$\theta f(c)$	expected profit in period t
ϵ	diffusion coefficient of the earnings process.

Table 6.23. **Variables in Failure Probability Formula**

The remainder of the thesis was concerned with validating these results in an empirical setting. The first step in the validation exercise was the derivation of testable hypotheses from the model solutions. Two sets of hypotheses were proposed, the first set relate to the firm's decision variables. The second set relate to the effect that each of the variables in the bankruptcy probability expressions will have on the probability of bankruptcy. The hypotheses propose the direction and significance of the effect of each of the variables on the probability of bankruptcy. To test these hypotheses, data from the financial statements of a sample of listed Australian companies was used. The next step in the validation exercise was to use the financial statement data to construct a set of proxy variables. These proxy variables are a representation of the variables specified in the model of Chapter 3. These proxy variables are

Variable	Proxy Variable Constructed from Financial Statements
w	The dividend paid in period t.
v	The difference in Plant, Land & Buildings(PLB) over the last 2 Balance Sheets, (PLB_t - PLB_{t-1}).
u	The difference in total debt (Short Term Debt + Borrowings) over the last 2 Balance Sheets, (Total $Debt_t$ − Total $Debt_{t-1}$).
M_t	Working Capital (Current Assets - Current Liabilities) from the Balance Sheet.
rx	Interest paid from the P&L.
$\theta f(c)$	Earnings Before Depreciation Interest & Taxes (EBDIT) from the P&L.
ϵ	A simple earnings dispersion measure, the deviation from a 2 period average, is used. $\left(EBDIT_t - \frac{(EBDIT_t+EBDIT_{t-1})}{2}\right)$.

Table 6.24. **Proxy Variables from Financial Statements**

The hypotheses to be tested using these proxy variables are summarised in Table 4.16. To carry out a hypotheses test, a test statistic is required. The final part of Chapter 4 develops a probabilistic regression framework, the Cox (1972) proportional hazards model, which generates estimates of the coefficients and standard errors that are needed to test the hypotheses proposed.

The results of the hypotheses tests are summarised below.

Hypothesis	Variable	Proxy Variable	Hypothesised Sign	Estimated Sign	Significance Level
1.1	Dividend	DIV.TA	-	-	99%
1.2	Change in Debt	DDEBT.TA	-	-	95%.
1.3	Investment	INVEST.TA	-	-	n.a.
2.1	Cash Position	WC.TA	-	-	82%
2.2	Interest Payable	INT.TA	+	+	99%
2.3	Expected Part of Earnings	EBDIT.TA	-	-	95%
2.4	Multiplier of Unexpected Earnings	DISP.TA	-	-	94%

Table 6.25. **Results of Hypotheses Testing**

The direction of the effect of the variables on the probability is in agreement with the hypothesised result for all of the variables. Five of the seven variables are significant at the 95% level of confidence in the reduced variable set case. The results for the remaining variables are explained in terms of the model structure and the adequacy of the proxy variables developed. Thus, we can conclude that the model developed in Chapter 3 has been validated.

6.1 Extensions

The aim of this research was to develop a model of bankruptcy behaviour, then to derive hypotheses and a test statistic which allowed the model to be evaluated. The following suggestions for further work cover the validation of the model results, validation of the assumptions made in constructing the model and extensions using the proportional hazards modeling framework.

The validation of the model is dependent on the data set used. It would be instructive to repeat the validation exercise with data from other markets. This exercise would provide a view of the general applicability of the model results.

The description of the uncertainty of earnings in Equation 3.11 needs to be benchmarked against earnings data from a sample of firms to establish the adequacy of the additive noise assumption. The relationship of the multiplier of the random component, ϵ, to economic indicators, such as the interest rate and variables which are coincident indicators of the business cycle, needs to be investigated to establish the claim that the random component of earnings is related to variability in the economic environment.

An investigation of the competing risks to bankruptcy such as takeover and name changes, using the variables suggested by the probability expressions derived from the model, would be useful in understanding the relationship between the various events that can lead to the end of a company's life.

Firms from all sectors have been included in the estimation of the Cox regression model. There is reason to suspect that firms in different sectors will have different financial structures. The use of sectors to stratify the estimation and analysis of the various Cox models that result may be important in empirical applications.

To go from testing hypotheses to the generation of bankruptcy probabilities the underlying hazard parameter, $\lambda_0(t)$, in the Cox regression model, $\lambda(t, X) = \phi(\underset{\sim}{X}, \underset{\sim}{\beta})\lambda_0(t)$, needs to be estimated. Altman (1993) presents a brief analysis of the age of failing businesses by way of a cumulative histogram. This suggests using the distribution of the ages of firms at delisting as the underlying hazard. The product of the results from the Cox regressions in Chapter Five and the estimated distribution would result in an estimate of the probability of bankruptcy. The results of the probability estimator could then be used in the construction of bankruptcy probability time series for various companies. These "profiles" of bankruptcy probabilities would have many potential uses in risk management applications.

In conclusion, this thesis has derived a set of expressions for the probability of bankruptcy from an economic model of a firm. Hypotheses about the signs and significance of the variables that appear in the bankruptcy probability expressions were formulated. A probabilistic regression framework, suggested by the random horizon of the explanatory model, was used to generate the coefficient estimates and associated standard errors that were needed for hypotheses testing. The results of the hypotheses tests on a sample of listed Australian firms demonstrate that the variables suggested by the model developed are useful in the empirical study of bankruptcy behaviour.

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