

AN INTEGRAL EQUATION APPROACH FOR ANALYSIS OF CONTROL CHARTS

YUPAPORN AREEPONG

A thesis submitted for the degree of

Doctor of Philosophy

in Mathematical Sciences

Department of Mathematical Sciences, Faculty of Science

University of Technology, Sydney, Australia

April 2009

Principal Supervisor: Professor Alexander Novikov



Copyright[®] by Yupaporn Areepong, 2009

UNIVERSITY OF TECHNOLOGY, SYDNEY
CERTIFICATE OF AUTHORSHIP/ORIGINALITY

Date: April 2009

Author: Yupaporn Areepong

Title: An Integral Equation Approach for Analysis of Control Charts

Department: Mathematical Sciences

Degree: Ph.D.

Convocation: Mar

Year: 2006

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Production Note:

Signature removed prior to publication.

Signature of Candidate



Dedication

To Dad & Mom

Acknowledgments

I would like to express my deep and sincere gratitude to my supervisor, Professor Alexander Novikov, for his enthusiasm and inspiration, and for his great efforts to explain things to me clearly and simply. He has helped to make mathematics and statistics fun for me. His wide knowledge and logical way of thinking have been of great value to me and it has been a great privilege to work with him. I warmly thank Dr. Nino Kordazakhia from Macquarie University for all her support and guidance. I would also like to thank my examiners Professor Vladimir Mazalov, Dr. Boris Choy and Associate Professor Sorachai Bhisalbutra who make many suggestions to improve the thesis and have been particularly inspiring to the future directions.

I am deeply grateful to Professor Dr. Tony Moon for his support, for his help during the difficult times and for warmly looking after me during my stay in Australia.

During this work I have collaborated with many colleagues, specially, Eka Shinjikashvili and Gabriel Mititelu for whom I have great regards. I wish to extend my warmest thanks to all of those who have helped me with my work in the Department of Mathematical Sciences, University of Technology, Sydney (UTS).

I am indebted to Dr. Elvin James Moore, Lecturer at Department of Mathematics, King Mongkut's University of Technology, North Bangkok (KMUTNB), Thailand for his careful proof-reading and helpful suggestions for my thesis.

I would like to express my deepest gratitude to the Government of Thailand for offering me a scholarship during my Ph.D. study. This thesis is also sponsored by Faculty of Science, University of Technology, Sydney through a small grant. I am also grateful to the Department of Applied Statistics, King Mongkut's University of Technology North Bangkok, Thailand, where I will work after finishing my Ph.D., for giving me this opportunity to study for a Ph.D. in Australia.

Lastly, and most importantly, I wish to thank my parents, Narong and Nipa Areepong, for all of their support for me, all that they have taught me and their unconditional love

for me. Without their encouragement and understanding it would have been impossible for me to finish this work. My special gratitude is also due to my sisters for their loving support. Finally, to my special friend, Kik, thank you all for being there.

Table of Contents

Table of Contents	vi
List of Figures	viii
List of Tables	ix
Abstract	x
1 Introduction	1
1.1 Background to Problem	2
1.2 Research Objectives and Contributions	3
1.3 Scope of Study	4
1.4 Outline of Thesis	4
1.5 List of Publications and Presentations	5
2 Statistical Process Control Charts	6
2.1 Chart Characteristics	6
2.2 Shewhart Procedure	7
2.3 EWMA Procedure	9
2.4 CUSUM Procedure	13
2.5 Shiriyayev-Roberts Procedure	15
2.6 Discussion	15
3 Methods for Evaluating the Performance of EWMA	17
3.1 Markov Chain Approach	17
3.2 Integral Equations	23
3.3 Monte Carlo Simulations	27
3.4 Discussion	29
4 EWMA Control Charts for Change in Exponential Distribution	30
4.1 The Exponential Family of Distributions	30
4.2 Explicit Formulas for One-sided Exponential EWMA	32
4.3 Comparison of Numerical Results for $E_x(\theta^{Tb})$	40
4.4 Comparison of Numerical Results for ARL and AD with Other Methods . .	43

4.5	Comparison of EWMA, CUSUM and Shirayayev-Roberts Charts: Exponential case	44
4.6	Choices of Optimal Parameters for Exponential EWMA Charts	46
4.7	Explicit Formulas for One-sided Pareto EWMA	49
5	Numerical Solutions for Integral Equations	54
5.1	The Nyström Method	55
5.2	Integral Equations for EWMA Chart	68
5.3	Integral Equations for CUSUM Chart	77
6	Conclusion and Recommendations for Further Research	81
6.1	Overall Conclusion	81
6.2	Recommendations for Further Research	82
A	Codes for Simulation and Calculation by Mathematica[®]	83
A.1	Exponential case	83
A.2	Pareto case	85
A.3	Gamma case	88
A.4	Weibull case	90
B	Mathematica[®] Codes for Algorithms of Obtaining Optimal Designs of EWMA	94
B.1	Procedure for obtaining optimal parameter values	94
B.2	Calculation codes for obtaining numerical method for Integral Equations . .	96
B.3	Calculation codes for obtaining numerical results of MCA	100
	Bibliography	102

List of Figures

2.1	Shewhart chart	8
2.2	EWMA charts	12
3.1	In-control region divided into N subintervals of the Markov Chain	21
4.1	Curves of AD for EWMA, CUSUM and Shirayev-Roberts charts: exponential case	46
4.2	Curves of AD for optimal exponential EWMA designs	47
4.3	Curves of AD for EWMA, CUSUM and Shirayev-Roberts charts: Pareto case	52
4.4	Curves of AD for optimal Pareto EWMA designs	52
5.1	Curves of AD for EWMA, CUSUM and Shirayev-Roberts charts: gamma case	70
5.2	Curves of AD for optimal gamma EWMA designs	72
5.3	Curves of AD for EWMA, CUSUM and Shirayev-Roberts charts: Weibull case	74
5.4	Curves of AD for optimal Weibull EWMA designs	76

List of Tables

4.1	Comparison of the numerical results for $E_x(\theta^{Tb})$	43
4.2	Comparison of ARL and AD values with other methods	44
4.3	The numerical results for ARL and AD obtained from formula (4.33) and (4.39) and simulation results of EWMA, CUSUM and Shiryayev-Roberts charts: exponential case	45
4.4	Optimal design parameters and AD for one-sided exponential EWMA	48
4.5	The numerical results for ARL and AD obtained from formulas (4.59) and (4.60) and simulation results of EWMA, CUSUM and Shiryayev-Roberts charts: Pareto case	51
4.6	Optimal design parameters and AD for one-sided Pareto EWMA	53
5.1	Comparison of approximations for ARL and AD from Integral Equations and Monte Carlo Simulation	67
5.2	Comparison of approximations ARL and AD obtained from Integral Equations with MCA and Monte Carlo Simulation: gamma case	69
5.3	The numerical results for ARL and AD obtained from Integral Equation and simulation results of EWMA, CUSUM and Shiryayev-Roberts charts: gamma case	70
5.4	Optimal design parameters and AD for one-sided gamma EWMA	71
5.5	Comparison of approximate ARL and AD values obtained from Integral Equations with MCA and Monte Carlo Simulation: Weibull case	73
5.6	The numerical results for ARL and AD obtained from Integral Equation and simulation results of EWMA, CUSUM and Shiryayev-Roberts charts: Weibull case	74
5.7	Optimal design parameters and AD for one-sided Weibull EWMA	75
5.8	Comparison of approximation ARL and AD values obtained from Integral Equation for two-sided EWMA with values obtained from Monte Carlo Simulation: exponential case	77
5.9	Comparison of approximation ARL and AD values obtained from Integral Equation with values from Monte Carlo Simulation: exponential CUSUM case	79
5.10	Comparison of approximation ARL and AD values obtained from Integral Equation with values from Monte Carlo Simulation: gamma CUSUM case	80

Abstract

This thesis is concerned with the use of Statistical Process Control (SPC) charts for detection of change-point in distributions in quality control and surveillance problems. We derive explicit analytical formulas and develop numerical algorithms for evaluating important characteristics of “Exponentially Weighted Moving Average” (EWMA) control charts for a range of distributions.

The most popular characteristics of a control chart are Average Run Length (ARL) - the mean of observations/times that are taken before a system is signalled to be out-of-control when it is actually still in-control, and Average Delay (AD) time - the mean of delay of true alarm times before a system that is actually out-of-control is signalled to be out-of-control. An important property required of ARL is that it should be sufficiently large when the process is in-control to reduce a number of false alarms. On the other side, if the process is actually out-of-control then its AD should be as small as possible. Traditional methods that are used for evaluating chart characteristics include Markov Chain Approach (MCA), Integral Equation (IE) and Monte Carlo simulation (MC) methods. Some crucial features of the methods are as follows: the MCA requires many matrix inversions and there is no theoretical proof of convergence of the method; the IE is most advanced method and it was used before only for Gaussian distribution; the MC is very time consuming.

In this thesis, we find explicit formulas for ARL and AD of EWMA in the case when observations are exponentially distributed. These explicit formulas can be applied to some other distributions, e.g. the Pareto distribution. The numerical results obtained from our explicit formulas are compared with results obtained from the Monte Carlo simulation (MC) and Markov Chain Approach (MCA). We also compare the performance of the EWMA procedure with charts obtained with the CUSUM and Shiriyayev-Roberts procedures. The technique that we use to derive the formulas for an exponential distribution cannot be used to derive formulas for gamma and Weibull distributions. However, we have developed a different method for evaluating the ARL and AD for the case of gamma and

Weibull distributions. This method is based on a numerical solution of Integral Equations based on Gauss-Legendre integration rules to approximate the integrals. Numerical results for these distributions are compared with results from other approaches.

Chapter 1

Introduction

Statistical Process Control (SPC) charts play a vital role in monitoring, measuring, controlling and improving quality in areas such as industrial statistics and manufacturing (Mason and Antony, 2000). In addition, SPC charts have been used in finance and economics (Ergashev, 2003; Golosnoy and Schmid, 2006), computer sciences and telecommunications (Mazalov and Zhuravlev, 2002; Ye et al., 2002, 2003), epidemiology (Sitter et al., 1990; Frisén, 1992), environmental statistics (Basseville and Nikiforov, 1993), and in other areas of applications.

Most of the popular charts such as *Shewhart*, *Exponential Weighted Moving Average* (EWMA), *Cumulative Sum* (CUSUM) and *Shiryayev-Roberts* (SR) charts have been developed for detecting changes in process means. The traditional Shewhart chart, which was first introduced by Shewhart (1931), is still widely used in many applications as it is useful for detecting large changes in process means. However, the Shewhart chart has been found to be inadequate for detecting small shifts in parameters.

In the past few decades, CUSUM, Shiryayev-Roberts and EWMA charts have been proposed as good alternatives to the Shewhart chart for detecting small shifts. The CUSUM chart was initially presented by Page (1954). It has been shown that the CUSUM charts are asymptotically optimal under minimax type criteria (see details Lorden, 1971; Moustakides, 1986; Shiryayev, 1996). The SR chart was first formulated for a continuous-time model by Shiryayev (1963) and Roberts (1966) considered a discrete-time model analogous of the former. Recently, this chart has attracted attention of many authors (see on Pollak and Siegmund, 1985, 1991; Mevorach and Pollak, 1991; Srivastava and Wu, 1993). The EWMA chart was initially introduced by Roberts (1959). It is a very flexible and effective chart for detecting small changes and has the advantage of showing robustness to non-

normality (Borror et al., 1999; Stoumbos and Reynolds, 2000; Montgomery, 2005). Borror et al. (1999) have compared the Average Run Length (ARL) performance between Shewhart and EWMA charts for the case of non-normal distributions using Markov Chain Approach and shown that EWMA chart is more robust to the assumption of normality.

In real applications, there are many situations in which the process data come from non-normal distributions such as exponential, Student-t or gamma (see for detail Borror et al., 1998, 1999; Stoumbos and Reynolds, 2000; Somerville et al., 2002). Processes with data from these non-normal distributions need to be monitored by appropriate control charts.

1.1 Background to Problem

A control chart is a sequential procedure for detecting a change-point in sequentially observed data. In general, these data present observations of some industrial or manufacturing process which will be regarded as being in an in-control state if the mean of the data lies within target values. Then, if the mean falls above or below this target values the process will be regarded as being in an out-of-control state. An important property of a control chart is that it should detect a change to an out-of-control state as soon as possible, but at the same time it should not give false alarms when the process is still in-control. A common characteristic used for comparing the performance of control charts is *Average Run Length* (ARL) - the expected number of observations taken from an in-control process until the control chart falsely signals out-of-control. An ARL will be regarded as acceptable if it is large enough to keep the level of false alarms at an acceptable level. A second common characteristic used for comparing performance is traditionally called *Average Delay* (AD) time - the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control. Ideally, the AD time should be small as possible.

Many methods for evaluating ARL and AD of EWMA charts have been discussed in the literature, e.g. *Markov Chain Approach* (MCA), *Integral Equation approach* (IE) and *Monte Carlo simulation* (MC). Roberts (1959), who was the first researcher to introduce the ARL for EWMA charts, used simulation to estimate the ARL. Later, Robinson and Ho (1978) numerically evaluated the ARL of EWMA chart using Edgeworth expansion for the probability density function (pdf) and cumulative distribution function (cdf) of the process. Crowder (1987) used Integral Equations to find both ARL and AD for Gaussian

case. Lucas and Saccucci (1990) have evaluated the ARL by using a finite-state Markov Chain approximation. They provided tables of optimal parameters for EWMA designs and claimed that EWMA charts perform better than CUSUM charts for small and moderate shifts of Gaussian distribution.

As discussed earlier, MCA, IE and MC are the most popular methods for evaluating the characteristics of EWMA. However, these methods have the following features: the IE method is a most advanced method but it requires a lot of programming and computation. MCA requires a discretisation of the continuity of the process into many steps and a large number of calculations of matrix inverse. MC is a simple to program and good for checking accuracy but it requires a large number of sample trajectories. Therefore this method is usually very time consuming; in addition, it is also difficult and laborious to find the optimal designs.

The limitations of the MCA, IE and MC methods provide the motivation for finding explicit analytical formulas for evaluating ARL and AD. Since ARL of the EWMA chart depends on a control limit H and a smoothing parameter λ there is the problem of finding a set of optimal parameters (λ^*, H^*) for EWMA designs which will give a minimum of AD.

1.2 Research Objectives and Contributions

The objective of this thesis is concerned with the use of SPC charts for detecting change-points in distributions. In particular, we focus on the performance of control charts for detecting change-point in distributions. We study analytical and numerical methods for the derivation of formulas for ARL and AD of EWMA charts.

One of the main aims of this thesis is to evaluate the characteristics of EWMA charts for detecting change-point in sequential data from exponential distribution. We derive analytical formulas for ARL and AD of EWMA charts by Integral Equation technique. Some of these results have already been published in Areepong and Novikov (2008).

The suggested explicit formulas can be extended to some other distributions, e.g. the Pareto distribution, by a simple transformation. Explicit formulas are obtained in the form of infinite series. To use it we need to truncate the series. To check the accuracy of truncated series the results obtained from them are compared with the results obtained from MC and from some other methods.

Unfortunately, the technique that we use to derive analytical expressions for the expo-

ponential distribution cannot be used when the observations come from gamma or Weibull distributions. Consequently, we use numerical methods to solve the Integral Equations for ARL and AD for the case of gamma and Weibull distributions and two-sided exponential EWMA. To find a numerical solution of the Integral Equations, we use Gauss-Legendre integration rules to approximate the integrals (see details in Chapter 5 for details). The numerical results obtained from the Integral Equations for ARL and AD of EWMA and CUSUM charts are compared with results obtained from the MC and MCA methods. The results obtained for the EWMA chart are also compared with results for the CUSUM and Shiriyayev-Roberts charts. The explicit formulas for ARL and AD that we have derived have been found to be accurate, fast and easy to calculate in comparison with MCA, IE and MC. Tables of a set of optimal parameters for the design of EWMA charts are provided.

1.3 Scope of Study

The scope of study is as follows:

1. We derived analytical explicit formulas for the case of exponential distribution.
2. We used numerical methods to find solution of the Integral Equations for ARL and AD for the case of gamma and Weibull distributions.
3. Comparison of results

We compared the accuracy of the suggested analytical expressions for finding ARL and AD for EWMA charts with other standard methods - IE, MCA and MC. The performance of EWMA, CUSUM and Shiriyayev-Roberts charts are compared using simulation studies.

4. The Computer programmes

We used the Mathematica[®] package for numerical calculations with the suggested explicit formulas. To find the numerical solution of Integral Equation we used Visual C.net package. We used the package R to simulate sample trajectories.

1.4 Outline of Thesis

The thesis is divided into six chapters:

- Chapter I is the introductory chapter of the thesis. It discusses the objectives and contributions of the thesis and gives this outline.
- Chapter II introduces the statistical process control charts - Shewhart, EWMA, CUSUM and Shiriyayev-Roberts charts for change-point detection and discusses properties and constraints for ARL and AD.
- Chapter III presents definitions and basic results for standard methods for finding ARL and AD, such as Integral Equations, Markov Chain Approach and Monte Carlo Simulation.
- Chapter IV discusses the derivations and proofs of the explicit formulas for ARL and AD for an EWMA chart when observations are from the exponential distribution. This chapter also contains a discussion of the explicit formulas for Pareto distribution. The numerical results are compared with MCA, IE and MC and optimal designs of EWMA are illustrated.
- Chapter V discusses numerical methods for solving Integral Equations to obtain approximations for ARL and AD when observations are from gamma or Weibull distributions. To solve the equations, we use Gauss-Legendre Quadrature rules to approximate integrals. The comparisons of the chart performance are studied between EWMA, CUSUM and Shiriyayev-Roberts procedures using simulation.
- Chapter VI summarizes the conclusions of the thesis and gives suggestions for further research.

1.5 List of Publications and Presentations

1. Areepong, Y. & Novikov, A. 2007. EWMA Control Charts for Change in exponential distribution. In *Proceedings of International Conference of Mathematical Sciences, 28-29 November*, Bangi-Putrajaya, Malaysia (ICMS 2007).
2. Areepong, Y. & Novikov, A. 2008. Martingale Approach to EWMA Control Chart for Changes in exponential distribution. *Journal of Quality Measurement and Analysis*, 4(1): 197-203.

Chapter 2

Statistical Process Control Charts

Statistical process control (SPC) charts are widely used in many areas of applications. In this chapter, we present some popular control charts and their basic characteristics under the conditions used in change-point detection.

In Section 2.1, we discuss the most popular chart characteristics such as the ARL and AD. The charts under consideration are designed for independent random sequential observations. The Shewhart chart is described in Section 2.2, the Exponentially Weighted Moving Average (EWMA) chart is presented in Section 2.3, the Cumulative Sum (CUSUM) chart is discussed in Section 2.4 and the Shiriyayev-Roberts (SR) is presented in Section 2.5.

2.1 Chart Characteristics

Let $\xi_1, \xi_2, \dots, \xi_t$, $t = 1, 2, \dots$ be sequentially observed independent random variables with a distribution $F(x, \alpha)$, where α is a parameter. It is usually assumed that there is an “in-control” state with parameter value $\alpha = \alpha_0$, and an “out-of-control” state with parameter value $\alpha \neq \alpha_0$. It is assumed that the change from the in-control state to the out-of-control state occurs at some unknown time ν which is called the *change-point* time ($\nu \leq \infty$). We assume that the parameters for the in-control and out-of-control states are known.

An important measure of the performance of chart characteristics is the speed of detection of changes in a parameter. A good control chart should not give a “false alarm” signal that a process is out-of-control when it is still in-control, but it should quickly give a “true alarm” signal when a process goes out-of-control.

Two measures that are commonly used to analyse the chart characteristics are the Average Run Length (ARL) and the Average Delay (AD) time. The ARL is the average number of observations that will occur before an in-control process falsely gives an out-

of-control signal. To reduce the number of false out-of-control signals a sufficiently large ARL is required. The AD is a measure of the average number of observations that will occur before an out-of-control process correctly gives an out-of-control signal. To reduce the time that the process is out-of-control, a small AD is required. Therefore the ARL and AD are two conflicting criteria that must be balanced to give an optimal control chart.

All popular charts such as the Shewhart, CUSUM, Shiriyayev-Roberts and EWMA charts (see e.g. Page, 1954; Woodall and Adams, 1993; Hawkins and Olwell, 1998) use a first passage time (τ) over a boundary as the alarm signal.

For the case of a process that is always in the in-control state (i.e. there is no change in parameter value), we set $\nu = \infty$. Then we define:

$$ARL = \hat{E}_{\infty}(\tau) = T, \quad (2.1)$$

where T is a given number (usually large) and $\hat{E}_{\infty}(\cdot)$ is the expectation under the assumption that observations ξ_t have the distribution $F(x, \alpha_0)$ (no change-points occurs). In the literature on quality control the quantity $\hat{E}_{\infty}(\tau)$ is called *Average Run Length* (ARL) of the chart.

Another typical characteristic of a chart is obtained by minimizing the quantity

$$\hat{E}_{\nu}(\tau - \nu + 1 | \tau \geq \nu), \quad (2.2)$$

where $\hat{E}_{\nu}(\cdot)$ is the expectation under the assumption that a change-point occurs at time ν . In practice, the condition (2.2) is usually calculated when $\nu = 1$. The expectation in (2.2) is usually called *Average Delay* (AD) time. A control chart is regarded as having a near optimal performance if its AD is close to a minimum value.

2.2 Shewhart Procedure

The most commonly used control procedure is the *Shewhart* chart. This chart was first introduced by Shewhart (1931). In this chart, only the last observation is used to determine the alarm signal and all earlier observations are ignored. In general, this procedure is good for detecting a large change but is poor for detecting small changes in parameter values. Note that a large change observation will have very low probability of occurring if the process is in the in-control state. Therefore, one “out-of-control” observation should be sufficient to detect that the process is out-of-control. However, for small changes in

parameter, there will be a comparatively high probability that a single, small “out-of-control” observation occurs when the process is still in an in-control state. Therefore, more than one out-of-control observation will be required to confirm that the process is actually out-of-control.

A typical control chart is shown in Figure 2.1. The three horizontal lines are called the *lower control limit* (LCL), the *central line* (CL) and the *upper control limit* (UCL). Typically, the LCL and UCL are defined as:

$$LCL = \alpha_0 - K\sigma \quad \text{and} \quad UCL = \alpha_0 + K\sigma,$$

where σ is the standard deviation of the distribution and K is a constant. K is usually chosen as 3 for the case of standard Gaussian distribution.

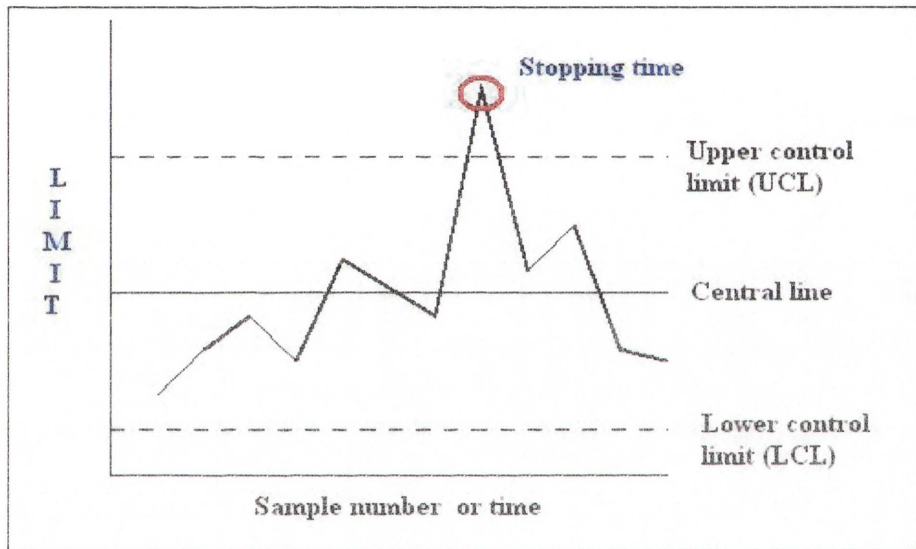


Figure 2.1: Shewhart chart

Let $\xi_1, \xi_2, \dots, \xi_t$ be independent and identically distributed random variables/observations with mean α_0 and variance σ^2 .

The stopping time of the Shewhart procedure is:

$$\tau = \inf\{t > 0 : \xi_t < LCL \quad \text{or} \quad \xi_t > UCL\}.$$

The ARL and AD characteristics of the Shewhart chart can be easily calculated and the error probabilities are as follows:

- The probability of *Type I* error (P_I); it is a probability to produce an alarm signal

when there is no real change (false alarm),

$$ARL = \hat{E}_\infty(\tau) = \frac{1}{P_I}.$$

- The probability of *Type II* error (P_{II}); it is a probability to have no signal when there is a real change (false negative),

$$AD = \hat{E}_1(\tau) = \frac{1}{(1 - P_{II})}.$$

If the ξ_t are normally distributed with mean μ and standard deviation σ and if 3σ limits are used, then $P_I = P(|X - \mu| > 3\sigma) = 0.0027$ is the probability that any observation exceeds the control limits of an in-control process and $ARL = (1/0.0027) = 370$.

2.3 EWMA Procedure

The Exponentially Weighted Moving Average (EWMA) procedure was initially proposed by Roberts (1959). It is usually used to monitor and detect a small change in a process mean. More recent references on evaluation of characteristics of EWMA charts are Crowder (1987); Gan (1990b); Lucas and Saccucci (1990); Knoth (2007).

The EWMA control chart is based on a weighted average Z_t of current and previous data which is defined through the recurrence relation:

$$Z_t = (1 - \lambda)Z_{t-1} + \lambda\xi_t, \quad t = 1, 2, \dots, \quad (2.3)$$

where ξ_t is a sequence of independent identically distributed random variables and λ is a weighting constant with $0 < \lambda < 1$. The mean of the data distribution is $\hat{E}_\infty(\xi_t)$. Typically, the target mean is defined to be $\hat{E}_\infty(\xi_t) = \alpha_0$ and the initial value Z_0 is usually chosen to be this target mean, i.e. $Z_0 = \alpha_0$.

By recursion of Equation (2.3) we see that the weights for each observation decrease geometrically by a factor $(1 - \lambda)$ for each time step:

$$\begin{aligned} Z_t &= \lambda\xi_t + (1 - \lambda)[\lambda\xi_{t-1} + (1 - \lambda)Z_{t-2}] \\ &= \lambda\xi_t + (1 - \lambda)\lambda\xi_{t-1} + (1 - \lambda)^2 Z_{t-2} \\ &= \lambda \{ \xi_t + (1 - \lambda)\xi_{t-1} + (1 - \lambda)^2 \xi_{t-2} \} + (1 - \lambda)^3 Z_{t-3}. \end{aligned}$$

Continuing the recursion for t steps, we obtain

$$Z_t = \lambda \sum_{j=0}^{t-1} (1-\lambda)^j \xi_{t-j} + (1-\lambda)^t Z_0. \quad (2.4)$$

Using Equation (2.4) we can show that $E(Z_t) = \alpha_0$ as follows.

Using the formula for the sum of a geometric progression, we obtain

$$\lambda \sum_{j=0}^{t-1} (1-\lambda)^j = \lambda \left[\frac{1 - (1-\lambda)^t}{1 - (1-\lambda)} \right] = 1 - (1-\lambda)^t.$$

After substituting this formula into Equation (2.4) and using the fact that $E(\xi_{t-j}) = \alpha_0$ for $j = 0, 1, 2, \dots, t-1$, we obtain

$$\begin{aligned} E(Z_t) &= \lambda \sum_{j=0}^{t-1} (1-\lambda)^j E(\xi_{t-j}) + (1-\lambda)^t \alpha_0 \\ &= (1 - (1-\lambda)^t) \alpha_0 + (1-\lambda)^t \alpha_0 = \alpha_0. \end{aligned}$$

If the observations ξ_t are independent random variables with variance σ^2 , then the variance of EWMA statistics Z_t is

$$\sigma_{Z_t}^2 = \sigma^2 \left(\frac{\lambda}{2-\lambda} \right) [1 - (1-\lambda)^{2t}], \quad t = 1, 2, \dots \quad (2.5)$$

To derive this equation, we note that

$$\begin{aligned} \sigma_{Z_t}^2 &= \text{Var} \left[\lambda \sum_{j=0}^{t-1} (1-\lambda)^j \xi_{t-j} + (1-\lambda)^t \alpha_0 \right] \\ &= \text{Var} \left[\lambda \sum_{j=0}^{t-1} (1-\lambda)^j \xi_{t-j} \right] \\ &= \lambda^2 \left[\sum_{j=0}^{t-1} \text{Var}((1-\lambda)^j \xi_{t-j}) + 2 \sum_{j=0}^{t-2} \sum_{k=j+1}^{t-1} \text{Cov}((1-\lambda)^j \xi_{t-j}, (1-\lambda)^k \xi_{t-k}) \right] \\ &= \sigma^2 \lambda^2 \sum_{j=0}^{t-1} (1-\lambda)^{2j}. \end{aligned} \quad (2.6)$$

The expression $\sum_{j=0}^{t-1} (1-\lambda)^{2j}$ is the sum of t terms of a geometric progression with ratio

$(1 - \lambda)^2$. Therefore the sum is:

$$\begin{aligned} \sum_{j=0}^{t-1} (1 - \lambda)^{2j} &= \sum_{j=0}^{t-1} [(1 - \lambda)^2]^j = \frac{1 - ((1 - \lambda)^2)^t}{1 - (1 - \lambda)^2} \\ &= \frac{1 - (1 - \lambda)^{2t}}{\lambda(2 - \lambda)}, \quad \lambda(2 - \lambda) \neq 0, \quad 0 < \lambda < 1. \end{aligned} \quad (2.7)$$

Then, substitution of Equation (2.7) into Equation (2.6) leads to Equation (2.5).

Since $0 < 1 - \lambda < 1$, we have that $(1 - \lambda)^{2t} \rightarrow 0$ as $t \rightarrow \infty$, and therefore from Equation (2.5) the asymptotic value of the variance is

$$\sigma_{Z_t}^2 = \sigma^2 \frac{\lambda}{2 - \lambda}. \quad (2.8)$$

When constant upper and lower limits are preferred for detecting change-points, the standard deviation used in the limits is usually the asymptotic value. Using the expression in Equation (2.5), the upper control limit of the EWMA chart is the following:

$$UCL = H_U = \alpha_0 + L\sigma \sqrt{\frac{\lambda}{2 - \lambda}},$$

and the lower control limit is:

$$LCL = H_L = \alpha_0 - L\sigma \sqrt{\frac{\lambda}{2 - \lambda}},$$

where L is a constant to be chosen (see Figure 2.2(b)). The process will be declared to be in an out-of-control state when $Z_t > H_U$ or $Z_t < H_L$.

The first passage time of an EWMA chart is:

$$\tau_{L,U} = \inf\{t > 0 : Z_t < H_L \text{ or } Z_t > H_U\}. \quad (2.9)$$

In this thesis, we mainly discuss the case of a positive change in distribution which crossing the upper limit raises alarm. This is called an ‘‘upper-sided EWMA’’ procedure. We then use the notation $H_U = H$.

The first passage time of an upper-sided EWMA chart is:

$$\tau_H = \inf\{t > 0 : Z_t > H\}, \quad (2.10)$$

where H is the control limit.

The ARL of the EWMA chart depends on the control limit H and the smoothing parameter λ used in defining Z_t . An EWMA chart will be regarded as having an optimal design if the values of the parameters (λ, H) give a minimum AD after change in a process mean.

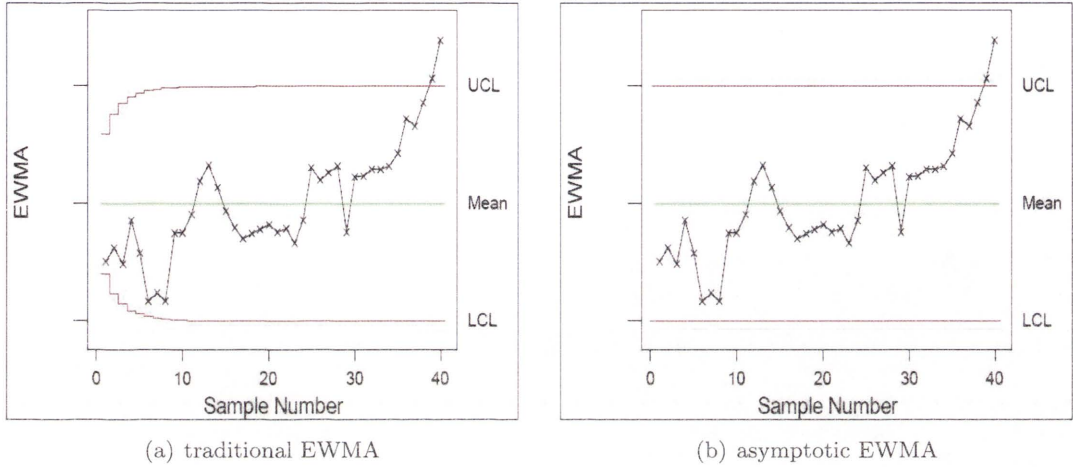


Figure 2.2: EWMA charts

The equations of the EWMA procedure can also be written in other forms. For example, Moustakides et al. (2008) used the following equations:

$$M_t^r = (M_{t-1}^r)^\kappa \eta_t, \quad M_0^r = r,$$

where M_t^r are the EWMA statistics, typically $\eta_t = q(\xi_t)$ where $q(\xi_t)$ is likelihood ratio statistics and $0 < \kappa < 1$ is a weighted smoothing parameter.

The first passage time of an EWMA chart is:

$$\tau_{H_L, H_U}^r = \inf\{t > 0 : M_t^r \notin (H_L, H_U)\},$$

where τ_{H_L, H_U}^r is the corresponding two-sided passage time and $0 < H_L < 1 < H_U$ are the lower and upper boundaries. The case $H_L = 0$ corresponds to the one-sided EWMA procedure.

2.4 CUSUM Procedure

A CUSUM chart was proposed by Page (1954). This procedure is defined by the statistics Y_t which satisfies the following equation:

$$Y_t = \max(Y_{t-1} + q(\xi_t), 0), \quad t = 1, 2, \dots, \quad Y_0 = 0, \quad (2.11)$$

where

$$q(x) = \log \frac{dF(x, \alpha)}{dF(x, \alpha_0)}. \quad (2.12)$$

The first passage time of a CUSUM chart is given by:

$$\tau_A = \inf\{t > 0 : Y_t > A\}, \quad (2.13)$$

where A is the control limit.

Many modifications of CUSUM algorithms have been given in the literature (see e.g. Hawkins and Olwell, 1998; Montgomery, 2005). In this thesis, we consider CUSUM charts when observations are from exponential, Pareto, Weibull and gamma distributions. The CUSUM formulas for these four distributions are as follows:

Exponential distribution. The density function of the exponential distribution is:

$$f(x) = \begin{cases} \frac{1}{\alpha} e^{-\frac{x}{\alpha}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Then, from Equation (2.12), we can find the function $q(x)$ to be as follows:

$$\begin{aligned} q(x) &= \log \frac{dF(x, \alpha)}{dF(x, \alpha_0)} \\ &= \log \left[\frac{\alpha_0}{\alpha} e^{x(\frac{1}{\alpha_0} - \frac{1}{\alpha})} \right] \\ &= \log \frac{\alpha_0}{\alpha} + x \left(\frac{1}{\alpha_0} - \frac{1}{\alpha} \right). \end{aligned}$$

Pareto distribution. The density function of the Pareto distribution is:

$$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}}, \quad x \geq k, \quad \alpha > 0 \text{ and } k > 0.$$

We can find the function $q(x)$ to be:

$$\begin{aligned} q(x) &= \log \frac{dF(x, \alpha)}{dF(x, \alpha_0)} \\ &= \log \left[\frac{\alpha k^\alpha}{x^{(\alpha+1)}} \frac{x^{(\alpha_0+1)}}{\alpha_0 k^{\alpha_0}} \right] \\ &= \log \left[\frac{\alpha}{\alpha_0} k^{(\alpha-\alpha_0)} x^{(\alpha_0-\alpha)} \right] \\ &= \log \left[\frac{\alpha}{\alpha_0} \right] + (\alpha - \alpha_0) \log k + (\alpha_0 - \alpha) \log x. \end{aligned}$$

Weibull distribution. The density function of the Weibull distribution is:

$$f(x) = \begin{cases} \frac{k}{\alpha} \left(\frac{x}{\alpha}\right)^{k-1} e^{-\left(\frac{x}{\alpha}\right)^k}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

We can find the function $q(x)$ to be:

$$\begin{aligned} q(x) &= \log \frac{dF(x, \alpha)}{dF(x, \alpha_0)} \\ &= \log \left[\left(\frac{k}{\alpha}\right) \left(\frac{\alpha_0}{k}\right) \left(\frac{\alpha_0}{\alpha}\right)^{k-1} e^{-\left(\frac{x}{\alpha}\right)^k + \left(\frac{x}{\alpha_0}\right)^k} \right] \\ &= k \log \left(\frac{\alpha_0}{\alpha}\right) + x^k \left(\frac{1}{\alpha_0^k} - \frac{1}{\alpha^k}\right). \end{aligned}$$

Gamma distribution. the density function of a gamma distribution is:

$$f(x) = \begin{cases} \frac{x^{k-1} e^{-\frac{x}{\alpha}}}{\Gamma(k) \alpha^k}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

We can find the function $q(x)$ to be:

$$\begin{aligned} q(x) &= \log \left[\left(\frac{\alpha_0}{\alpha}\right)^k e^{x \left(\frac{1}{\alpha} - \frac{1}{\alpha_0}\right)} \right] \\ &= k \log \left(\frac{\alpha_0}{\alpha}\right) + x \left(\frac{1}{\alpha_0} - \frac{1}{\alpha}\right). \end{aligned}$$

CUSUM is usually considered as a candidate for an optimal chart under minimax criterion (Lorden, 1971; Shiriyayev, 1996). In reality its performance (for moderate values of ARL and small changes in parameters) can be inferior to the performance of EWMA as already noted by Lucas and Saccucci (1990) for the case of a Gaussian distribution.

2.5 Shiriyayev-Roberts Procedure

The discrete time Shiriyayev-Roberts procedure was first introduced by Roberts (1966). Pollak and Siegmund (1985) compared SR and CUSUM procedures for detecting a change in the drift of a Brownian motion process based on the conditional average delay times. In addition, Srivastava and Wu (1993) compared the EWMA, CUSUM and SR procedures based on the steady-state distribution as advocated by Shiriyayev (1963). In a more recent paper, Mahmoud et al. (2008) evaluated and compared the performance of CUSUM and Shiriyayev-Roberts procedures using the Markov chain Approach.

The Shiriyayev-Roberts procedure is defined by the statistics R_t which satisfies the following equation

$$R_t = (1 + R_{t-1})q(\xi_t), \quad t = 1, 2, \dots, \quad R_0 = 0,$$

where

$$q(x) = \frac{dF(x, \alpha)}{dF(x, \alpha_0)}.$$

The first passage time of a Shiriyayev-Roberts procedure is given by:

$$\tau_B = \inf\{t > 0 : R_t > B\}, \quad (2.14)$$

where B is the control limit.

2.6 Discussion

Common characteristics of control charts which have been used to evaluate and compare the performance of different control charts are average run length (ARL) and average delay time (AD). Examples of the interpretation of ARL and AD are as follows. An ARL=1000 means that a false out-of-control signal will be given on average every 1000 observations when the process is actually in-control. An AD=20 means that on average 20 observations will be required of a process that is actually out-of-control before a true out-of-control signal is given. Therefore, in a good control chart, the ARL should be sufficiently long that false out-of-control signals are kept at an acceptable level, whereas the AD should be sufficiently short.

The Shewhart procedure is well-known to be useful when large changes in mean between the in-control and out-of-control states are considered. The CUSUM, SR and

EWMA procedures perform better for the case of small changes in parameters. However, the analysis of CUSUM, SR and EWMA charts is much more complicated than the analysis of the traditional Shewhart procedure (Lucas and Saccucci, 1990; Srivastava and Wu, 1993). CUSUM is known to be an efficient tool under some conditions (Lorden, 1971; Shirayayev, 1996). The difference in performance of SR and CUSUM charts is significant only for detection of small changes in a mean of Gaussian distribution, for large changes they have the same performance (Moustakides et al., 2009). EWMA procedures are inherently simpler than CUSUM and SR. Moreover, it is also known to be more robust with respect to assumptions about process distributions than CUSUM (Lucas and Saccucci, 1990; Ergashev, 2003).

Chapter 3

Methods for Evaluating the Performance of EWMA

Various methods for evaluating the performance of the EWMA procedure have been studied in the literature (see Yashchin, 1987; Gan, 1990a, 1991; Srivastava and Wu, 1993; Borror et al., 1998; Ye et al., 2002). Three standard methods that are often used to evaluate Average Run Length (ARL) and Average Delay (AD) time are the Markov Chain Approach (MCA), the Integral Equation (IE) and the Monte Carlo simulation (MC) methods. This chapter gives a survey of these popular methods and gives a derivation of formulas for the ARL for the three methods. Section 3.1 contains a discussion of MCA, Section 3.2 gives a discussion of IE and Section 3.3 gives a discussion of MC.

In this chapter, we present the basic theory, derivations and discussions based on the following references: Lucas and Saccucci (1990); Champ and Rigdon (1991); Ross (1996); Borror et al. (1998); Fu et al. (2002); Borovkov (2003); Montgomery (2005).

3.1 Markov Chain Approach

The Markov Chain Approach is one of the most effective methods for studying the characteristics of CUSUM and EWMA procedures. This approach has been discussed by many authors (see for detail Brook and Evans, 1972; Lucas and Saccucci, 1990; Vardeman and Ray, 1985; Champ and Rigdon, 1991).

Definition 1. *A Markov Chain is a sequence of random variables X_1, X_2, X_3, \dots with the*

Markov property

$$P(X_{n+1} = x | X_n = x_n, \dots, X_1 = x_1) = P(X_{n+1} = x | X_n = x_n).$$

3.1.1 Properties of Markov Chains

We define the probability of going from state x_i to state x_j in N steps as

$$P_{ij}^{(N)} = P(X_N = x_j | X_0 = x_i)$$

and the single-step probability as

$$P_{ij} = P(X_1 = x_j | X_0 = x_i).$$

We call each move a step. If the chain is in state x_i , then the probability of moving to state x_j at the next step is denoted by P_{ij} . These probabilities are assumed to be independent of time and to be independent of any past states that the chain may have passed through to reach state x_i . These single-step probabilities are called *transition probabilities* and P_{ij} is called the transition probability from state x_i to state x_j .

A state x_i of a Markov Chain is called “absorbing” if it is impossible to leave this state. Therefore, the state x_i is absorbing if and only if

$$P_{ii} = 1 \quad \text{and} \quad P_{ij} = 0, \quad \text{for } i \neq j.$$

3.1.2 Derivation of the ARL for Markov Chain

Let X_t be a homogeneous Markov Chain with states x_j , $j = 1, 2, \dots, N, N + 1$. We assume that states x_j , $j = 1, 2, \dots, N$ are in-control states and state x_{N+1} is an out-of-control state. The absorbing time τ_i is the time such that, starting at an in-control state x_i , the chain reaches state x_{N+1} for the first time, that is:

$$\tau_i = \inf\{t > 0 : X_t = x_{N+1}\}, \quad X_0 = x_i, \quad i \leq N.$$

We first consider the case when the state x_{N+1} is absorbing and the states $x_i, i \leq N$ are all transient states, then the matrix of transition probabilities has the form

$$\mathbf{P} = \begin{bmatrix} \mathbf{R} & (\mathbf{I}_N - \mathbf{R})\mathbf{1}_N \\ \mathbf{0}_N^T & \mathbf{I}_N \end{bmatrix}$$

where

\mathbf{R} is the submatrix of transition probabilities (P_{ij}) for states $1, \dots, N$,

\mathbf{I}_N is the unit $N \times N$ matrix,

$\mathbf{1}_N$ is the unit $N \times 1$ vector (all elements are 1),

$\mathbf{0}_N$ is zero $N \times 1$ vector.

The k stage transition probability matrix \mathbf{P}^k is useful for evaluating ARL because it contains the probability that the chain goes from one state to another state in k steps.

This matrix is

$$\mathbf{P}^k = \begin{bmatrix} \mathbf{R}^k & (\mathbf{I}_N - \mathbf{R}^k)\mathbf{1}_N \\ \mathbf{0}_N^T & \mathbf{I}_N \end{bmatrix}$$

The vector $(\mathbf{I}_N - \mathbf{R}^k)\mathbf{1}_N$ is the vector of transition probabilities from states $i < N + 1$ to the state $N + 1$ in k steps. Hence,

$$\begin{aligned} P(\tau_i \leq k | X_0 = x_i) &= \text{element}[(\mathbf{I}_N - \mathbf{R}^k)\mathbf{1}_N](i) \\ &= \text{sum of row } (i)(\mathbf{I}_N - \mathbf{R}^k) \\ &= \mathbf{p}_N^{(i)T} (\mathbf{I}_N - \mathbf{R}^k)\mathbf{1}_N, \end{aligned}$$

where $\mathbf{p}_N^{(i)T}$ is the initial probability vector with 1 at i^{th} position and 0 otherwise. Then

$$\begin{aligned} P(\tau_i = k | X_0 = x_i) &= P(\tau_i \leq k | X_0 = x_i) - P(\tau_i \leq k - 1 | X_0 = x_i) \\ &= \mathbf{p}_N^{(i)T} (\mathbf{I}_N - \mathbf{R}^k - (\mathbf{I}_N - \mathbf{R}^{k-1}))\mathbf{1}_N \\ &= \mathbf{p}_N^{(i)T} (\mathbf{R}^{k-1} - \mathbf{R}^k)\mathbf{1}_N. \end{aligned} \tag{3.1}$$

Denote ARL of N approximation to $ARL(N)$ based on Markov Chain $N + 1$ states it

describes above. Then using Equation (3.1), we obtain

$$\begin{aligned}
 ARL(N) &= \sum_{k=0}^{\infty} kP(\tau_i = k | X_0 = x_i) \\
 &= \sum_{k=0}^{\infty} k \mathbf{p}_N^{(i)T} (\mathbf{R}^{k-1} - \mathbf{R}^k) \mathbf{1}_N \\
 &= \sum_{k=0}^{\infty} \mathbf{p}_N^{(i)T} ((k+1)\mathbf{R}^k - k\mathbf{R}^k) \mathbf{1}_N \\
 &= \sum_{k=0}^{\infty} \mathbf{p}_N^{(i)T} \mathbf{R}^k \mathbf{1}_N \\
 &= \mathbf{p}_N^{(i)T} (\mathbf{I}_N - \mathbf{R})^{-1} \mathbf{1}_N,
 \end{aligned} \tag{3.2}$$

One may expect that when $N \rightarrow \infty$, $ARL(N)$ and $AD(N)$ converge to true values ARL and AD. However, we have failed to find any theoretical results of this type in the literature.

In deriving Equation (3.2), we have used the following theorem:

Theorem 3.1.1. *If $\sum_{k=0}^{\infty} \mathbf{R}^k$ converges absolutely, then $\sum_{k=0}^{\infty} \mathbf{R}^k = (\mathbf{I}_N - \mathbf{R})^{-1}$.*

Proof. If $\sum_{k=0}^{\infty} \mathbf{R}^k$ converges absolutely then we can multiply each term of the series by another matrix and, in particular, we have the following relation

$$\begin{aligned}
 (\mathbf{I}_N - \mathbf{R}) \sum_{k=0}^{\infty} \mathbf{R}^k &= \sum_{k=0}^{\infty} (\mathbf{R}^k - \mathbf{R}^{k+1}) \\
 &= \sum_{k=0}^{\infty} \mathbf{R}^k - \sum_{k=1}^{\infty} \mathbf{R}^k \\
 &= \mathbf{R}^0 = \mathbf{I}.
 \end{aligned}$$

Therefore, $\sum_{k=0}^{\infty} \mathbf{R}^k = (\mathbf{I}_N - \mathbf{R})^{-1}$.

□

3.1.3 The Procedure to Obtain ARL Using Markov Chain Approach

For the Markov Chain Approach, the in-control states are defined as a set of discrete states in the region between a lower limit H_L and an upper limit H_U . The region below H_L and above H_U is regarded as the absorbing state. The procedure to obtain the ARL using the MCA is then as follows:

- Divide the interval between lower and upper bounds for in-control parameter values

(H_L, H_U) into N subintervals. These subintervals are used as the N in-control states of a Markov Chain.

- Denote the midpoints of the N subintervals as m_j , $j = 1, 2, \dots, N$. Calculate a lower L_j and upper U_j bound of the j^{th} subinterval as follows:

$$L_j = H_L + \frac{(j-1)(H_U - H_L)}{N},$$

$$U_j = H_L + \frac{j(H_U - H_L)}{N},$$

The midpoint of the j^{th} subinterval is then

$$m_j = \frac{1}{2}(L_j + U_j) = H_L + \frac{(2j-1)(H_U - H_L)}{2N}.$$

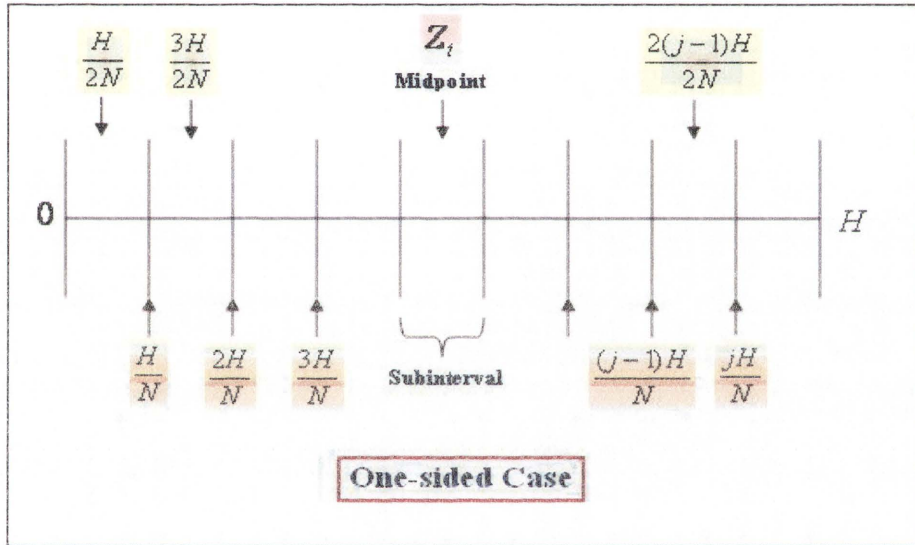


Figure 3.1: In-control region divided into N subintervals of the Markov Chain

In this thesis, we are interested only in the one-sided case, i.e. the in-control region is between 0 and $H_U = H$. Then the in-control states of the Markov Chain are defined as follows (see Figure 3.1):

$$L_j = \frac{(j-1)H}{N}, \quad (3.3)$$

$$U_j = \frac{jH}{N} \quad (3.4)$$

and the midpoint is defined as

$$m_i = \frac{(2i-1)H}{2N}. \quad (3.5)$$

- The next step is to construct the matrix \mathbf{R} and the transition probabilities P_{ij} . For EWMA statistics, the transition probability P_{ij} is the probability that Z_t will move from interval i to interval j . For the MCA, Z_{t-1} is assumed to be the midpoint, m_i of the i^{th} interval. Therefore, P_{ij} is the probability of a transition from the midpoint of the i^{th} interval to anywhere in the j^{th} interval. This can be defined as

$$P_{ij} = P(L_j < Z_t < U_j | Z_{t-1} = m_i).$$

Then, by using the recurrence relation for the EWMA procedure, the probability can be expressed as

$$P_{ij} = P(L_j < (1-\lambda)Z_{t-1} + \lambda\xi_t < U_j | Z_{t-1} = m_i).$$

The probability can be reduced to

$$\begin{aligned} P_{ij} &= P(L_j < (1-\lambda)m_i + \lambda\xi_t < U_j) \\ &= P\left(\frac{L_j - (1-\lambda)m_i}{\lambda} < \xi_t < \frac{U_j - (1-\lambda)m_i}{\lambda}\right). \end{aligned}$$

Then, substituting for L_j, U_j using Equation (3.3) and (3.4), we obtain

$$\begin{aligned} P_{ij} &= P\left[\frac{\frac{(j-1)H}{N} - (1-\lambda)m_i}{\lambda} < \xi_t < \frac{\frac{jH}{N} - (1-\lambda)m_i}{\lambda}\right] \\ &= P\left[\frac{(j-1)H}{N\lambda} - \frac{(1-\lambda)m_i}{\lambda} < \xi_t < \frac{jH}{N\lambda} - \frac{(1-\lambda)m_i}{\lambda}\right]. \end{aligned}$$

Substituting m_i from Equation (3.5), we obtain

$$\begin{aligned} P_{ij} &= P\left[\frac{(j-1)H}{N\lambda} - \frac{(1-\lambda)}{\lambda} \left(\frac{(2i-1)H}{2N}\right) < \xi_t < \frac{jH}{N\lambda} - \frac{(1-\lambda)}{\lambda} \left(\frac{(2i-1)H}{2N}\right)\right], \\ &= P\left[\frac{H(2(j-1) - (1-\lambda)(2i-1))}{2N\lambda} < \xi_t < \frac{H(2j - (1-\lambda)(2i-1))}{2N\lambda}\right] \end{aligned}$$

then

$$P_{ij} = F\left(\frac{H(2j - (1 - \lambda)(2i - 1))}{2N\lambda}\right) - F\left(\frac{H(2(j - 1) - (1 - \lambda)(2i - 1))}{2N\lambda}\right),$$

where $F(\cdot)$ is the cumulative distribution function.

- Construct the vector $\mathbf{p}_i(N) = [0, \dots, 0, 1, 0, \dots, 0]$, $i = 1, 2, \dots, N$ by a $1 \times N$ matrix with 1 at i^{th} coordinate and zeros elsewhere.
- Calculate the ARL by using Equation (3.2).

3.2 Integral Equations

Crowder (1987) used the Integral Equation method to develop an approximation for the ARL of a Gaussian EWMA chart and obtained a Fredholm Integral Equation of the second kind. Champ and Rigdon (1991) used this method to evaluate the ARL for both the CUSUM and EWMA charts and compared the results with results from MCA. Intensive studies of the Integral Equation have been carried out for the case of continuous-time processes by Srivastava and Wu (1993) and for the case of discrete processes by Srivastava and Wu (1997). In this section we will give a summary of the main properties of the Integral Equation method that we will use in this thesis.

3.2.1 The Integral Equation Method for Calculating ARL

We assume that the EWMA statistic Z_t is defined as in Equation (2.3). We first consider the EWMA procedure when ξ_1, ξ_2, \dots are i.i.d. with a probability density function $f(y)$. We assume that the system is in-control at time t if the EWMA statistic Z_t is in the range $H_L \leq Z_t \leq H_U$ and out-of-control if $Z_t > H_U$ or $Z_t < H_L$, where H_L is a constant lower bound and H_U is a constant upper bound. We also assume that the system is initially in an in-control state u , i.e. $Z_0 = u$ and $H_L \leq u \leq H_U$.

We now define a function $L(u)$ as follows:

$$L(u) = E_\infty(\tau_{L,U}), \quad Z_0 = u, \quad (3.6)$$

where $\tau_{L,U}$ is the stopping time defined in Equation (2.9). Then $L(u)$ is the ARL for initial value u . There are two possibilities for Z_1 after the first observation, ξ_1 , is made.

If ξ_1 gives an out-of-control value for Z_1 , then

$$Z_1 = (1 - \lambda)Z_0 + \lambda\xi_1 > H_U \quad \text{or} \quad Z_1 = (1 - \lambda)Z_0 + \lambda\xi_1 < H_L.$$

In this case the run length will be 1 because there will be an immediate out-of-control signal. If ξ_1 gives Z_1 in an in-control state, then

$$H_L < (1 - \lambda)Z_0 + \lambda\xi_1 < H_U. \quad (3.7)$$

Then one observation will have been made and on average $L(Z_1) = L((1 - \lambda)Z_0 + \lambda\xi_1)$ more observations will be made before an out-of-control signal occurs. The inequality (3.7) can be rewritten in the form

$$\frac{H_L - (1 - \lambda)Z_0}{\lambda} < \xi_1 < \frac{H_U - (1 - \lambda)Z_0}{\lambda}. \quad (3.8)$$

The probability that ξ_1 satisfies the bounds in Equation (3.8) for a probability distribution function $f(\xi_1)$ is given by:

$$P\left(\frac{H_L - (1 - \lambda)Z_0}{\lambda} < \xi_1 < \frac{H_U - (1 - \lambda)Z_0}{\lambda}\right) = \int_{\frac{H_L - (1 - \lambda)Z_0}{\lambda}}^{\frac{H_U - (1 - \lambda)Z_0}{\lambda}} f(y) dy, \quad (3.9)$$

where $f(y)$ is the probability density function.

Then, following the method given in Champ and Rigdon (1991), if we let initial $Z_0 = u$ and make the substitution $y = \xi_t$, we can rewrite the formula for the $L(u)$ defined in Equation (3.6) as follows:

$$\begin{aligned} L(u) &= (1 - P\left[\frac{H_L - (1 - \lambda)u}{\lambda} < \xi_1 < \frac{H_U - (1 - \lambda)u}{\lambda}\right]) \\ &\quad + \int_{\frac{H_L - (1 - \lambda)u}{\lambda}}^{\frac{H_U - (1 - \lambda)u}{\lambda}} (1 + L((1 - \lambda)u + \lambda y)) f(y) dy \\ &= 1 + \int_{\frac{H_L - (1 - \lambda)u}{\lambda}}^{\frac{H_U - (1 - \lambda)u}{\lambda}} L((1 - \lambda)u + \lambda y) f(y) dy, \end{aligned}$$

or finally, on changing the integration variable, we obtain

$$L(u) = 1 + \frac{1}{\lambda} \int_{H_L}^{H_U} L(y) f\left(\frac{y - (1 - \lambda)u}{\lambda}\right) dy. \quad (3.10)$$

In this thesis we consider mainly the upper-sided case for nonnegative random variable ξ_t . Since $Z_t \geq 0$ for the case of nonnegative ξ_t we can assume $H_L = 0$ and $H_U = H$. Then we get

$$L(u) = 1 + \frac{1}{\lambda} \int_0^H L(y) f\left(\frac{y - (1 - \lambda)u}{\lambda}\right) dy. \quad (3.11)$$

Note that the AD also can be found by Equation (3.11) where $f(y)$ is the probability density function with an out-of-control parameter α .

3.2.2 Numerical Methods for Solving the Integral Equation

In general, the Integral Equations (3.10) and (3.11) cannot be solved analytically for $L(u)$ and it is necessary to use numerical methods to solve them. We shall use a quadrature rule to approximate the integral by a finite sum. Any quadrature rule is defined by a set of points $\{a_j, j = 1, 2, \dots, n\}$ on the interval $[0, H]$ and a set of constant weights $\{w_j, j = 1, 2, \dots, n\}$. The approximation for an integral is of the form:

$$\int_0^H W(y) F(y) dy \approx \sum_{j=1}^n w_j F(a_j), \quad (3.12)$$

where $W(y)$ and $F(y)$ are given functions. Different choices for the function $W(y)$ and the sets of points and weights define different quadrature rules. The quadrature rules that we use in this thesis will be discussed in Chapter 5.

Using the quadrature rule (3.12), a numerical approximation $\tilde{L}(u)$ for Integral Equation (3.11) can be found as a solution of system of algebraic linear equations:

$$\tilde{L}(a_i) = 1 + \frac{1}{\lambda} \sum_{j=1}^n w_j \tilde{L}(a_j) f\left(\frac{a_j - (1 - \lambda)a_i}{\lambda}\right), \quad i = 1, 2, \dots, n. \quad (3.13)$$

That is

$$\begin{aligned}\tilde{L}(a_1) &= 1 + \frac{1}{\lambda} \sum_{j=1}^n w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)a_1}{\lambda}\right) \\ \tilde{L}(a_2) &= 1 + \frac{1}{\lambda} \sum_{j=1}^n w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)a_2}{\lambda}\right) \\ &\vdots \\ \tilde{L}(a_n) &= 1 + \frac{1}{\lambda} \sum_{j=1}^n w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)a_n}{\lambda}\right).\end{aligned}$$

The above set of n equations in n unknowns can be rewritten in a matrix form as follows.

We define a column vector L with components:

$$L_j = \tilde{L}(a_j), \quad j = 1, 2, \dots, n$$

and an $n \times n$ matrix R with matrix entries given by:

$$[\mathbf{R}]_{ij} = \frac{1}{\lambda} w_j f\left(\frac{a_j - (1-\lambda)a_i}{\lambda}\right).$$

The system of algebraic linear equations can then be written as the matrix equation:

$$\mathbf{L} = \mathbf{1} + \mathbf{R}\mathbf{L},$$

where $\mathbf{1}$ is a column vector of n 1's. Therefore

$$\mathbf{L} - \mathbf{R}\mathbf{L} = \mathbf{1},$$

or, equivalently,

$$(\mathbf{I} - \mathbf{R})\mathbf{L} = \mathbf{1}. \quad (3.14)$$

This equation can be easily solved numerically to obtain the vector \mathbf{L} and therefore the components $\tilde{L}(a_j) = L_j$. After substituting the values for $\tilde{L}(a_j)$ into Equation (3.13) and replacing the a_i by u , we obtain an approximation for the function $L(u)$ as

$$\tilde{L}(u) = 1 + \frac{1}{\lambda} \sum_{j=1}^n w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)u}{\lambda}\right). \quad (3.15)$$

One of the simplest quadrature rules is the composite trapezoidal rule in which the function

$W(y) = 1$ and the points a_j are n equally spaced points on the interval $[0, H]$ given by $a_1 = 0$, $a_n = H$, $a_j = (j-1)h$, $j = 2, 3, \dots, n-1$, where $h = \frac{H}{n-1}$. The weights w_j are $w_1 = w_n = \frac{1}{2}h$ and $w_j = h$, $2 \leq j \leq n-1$. On substituting these values into Equation (3.15), we obtain an approximate formula for the function $L(u)$ in Equation (3.11):

$$\begin{aligned} \tilde{L}(u) = & 1 + \frac{h}{2\lambda} \left[\tilde{L}(0)f\left(\frac{-(1-\lambda)u}{\lambda}\right) + \tilde{L}(H)f\left(\frac{H-(1-\lambda)u}{\lambda}\right) \right] \\ & + \frac{h}{\lambda} \sum_{j=2}^{n-1} \tilde{L}(a_j)f\left(\frac{a_j-(1-\lambda)u}{\lambda}\right), \end{aligned} \quad (3.16)$$

where the values of $\tilde{L}(a_j)$ are the components of the solution vector \mathbf{L} of Equation (3.14).

3.3 Monte Carlo Simulations

Simulation studies can be used when exact analytical formulas are not available. MC simulation studies are also very useful for comparing the performance of different charts, such as Shewhart, EWMA, CUSUM and SR, and for checking the accuracy of approximate formulas, e.g. those obtained for EWMA charts by the Markov Chain or Integral Equation methods described in sections 3.1 and 3.2. However, the simulation process is well-known to be very time consuming. In this thesis, we use MC simulation in order to compare the performance of EWMA, CUSUM and SR charts and to test the accuracy of the formulas and numerical results that we obtain for EWMA charts.

The MC method is based on the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT) (Ross, 1997; Borovkov, 2003).

Theorem 3.3.1. (*Strong Law of Large Numbers (SLLN)*). Let Y_1, Y_2, \dots, Y_n be a sequence of independent identically distributed (i.i.d.) random variables with finite expected value $E(Y_i) = m$ and finite variance $\text{Var}(Y_i) = \sigma^2$. If $\bar{Y}_n = \frac{1}{n} \sum_{i=1}^n Y_i$, then

$$P\left\{\lim_{n \rightarrow \infty} \bar{Y}_n = m\right\} = 1.$$

Let

$$\xi_n = \frac{\sum_{i=1}^n (Y_i - m)}{\sqrt{n}\sigma}, \quad \Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y \exp\left\{-\frac{y^2}{2}\right\} dy.$$

Theorem 3.3.2. (*Central Limit Theorem (CLT)*). Let Y_1, Y_2, \dots be a sequence of i.i.d.

random variables such that $E(Y_i) = m$, $EY_i^2 < \infty$ and $\text{Var}(Y_i) = \sigma^2 > 0$. Then

$$\xi_n \xrightarrow{d} \xi \sim N(0, 1)$$

or, equivalently, for all $y \in (-\infty, \infty)$

$$P(\xi_n \leq y) \rightarrow \Phi(y).$$

Suppose that we want to compute the number

$$H = E[g(Y)],$$

where $g(Y)$ is a given function and Y is a random variable. If we can generate a sequence of independent random variables Y_1, Y_2, \dots, Y_n , such that all Y_i have the same distribution as Y , then according to SLLN we have

$$H_n := \frac{1}{n} \sum_{i=1}^n g(Y_i) \xrightarrow{a.s.} E[g(Y)].$$

To estimate an accuracy of the approximation, we assume

$$\text{Var}(g(Y)) := \sigma^2(g) < \infty$$

and note

$$\text{Var}(H_n) = \frac{\text{Var}(g(Y))}{n} = \frac{\sigma^2(g)}{n}.$$

Then, applying CLT we have a bound:

$$|H_n - H| \leq 3 \frac{\sigma(g)}{\sqrt{n}}$$

with probability approximately equal to 0.997... (for large n). The constant $\sigma^2(g)$ is usually unknown but it can be estimated by:

$$\hat{\sigma}_n^2(g) := \frac{1}{n} \sum_{i=1}^n g^2(Y_i) - (H_n)^2 \xrightarrow{P} E(g^2(Y)) - H^2 = \sigma^2(g).$$

3.4 Discussion

A variety of methods for obtaining numerical approximations for ARL and AD for EWMA charts have been given in the literature. In this thesis, we shall obtain explicit formulas for the ARL and AD of EWMA charts for observations from an exponential distribution. Explicit formulas are unknown when observations are from gamma and Weibull distributions. In Chapter 5 we develop numerical approximations for these distributions by use of the Integral Equation method discussed in Section 3.2.

Chapter 4

EWMA Control Charts for Change in Exponential Distribution

In this chapter, we study EWMA control charts designed to test changes in the Exponential distribution. The structure of this chapter is as follows. In Section 4.1, we define the Exponential family of distributions. In Section 4.2, we use the method of Integral Equations to derive explicit formulas for ARL and AD of EWMA charts for observations from an exponential distribution. In Section 4.3, we compare explicit formulas for $E_x(\theta^{r_0})$ for EWMA charts with the results of Jacobsen (2007). In Section 4.4, we compare results obtained from the suggested formulas for ARL and AD for EWMA charts with results obtained from Monte Carlo simulations and other methods such as Markov Chain and Integral Equation methods. In Section 4.5, we compare the results obtained for the EWMA charts with results obtained for CUSUM and Shirayev-Roberts charts. In Section 4.6, we find optimal designs for EWMA charts for fixed values of ARL and parameter change (α). In Section 4.7, we present explicit formulas for the EWMA characteristics for observations from Pareto distribution.

4.1 The Exponential Family of Distributions

Definition 2. A density function is a member of the exponential family if it can be written in the form:

$$f(x; \gamma) = h(x)c(\gamma)e^{(g(\gamma)t(x))}, \quad -\infty < x < \infty, \quad (4.1)$$

where h, c, g, t are known functions and γ is a constant parameter. Further, if $t(x) = x$, then the distribution is said to be in canonical form. The function $g(\gamma)$ is called a natural parameter when the distribution is in canonical form.

Many of the well-known distributions can be put into this form. One example is shown in Example 1

Example 1. *Exponential distribution:*

$$f(x) = \begin{cases} \frac{1}{\alpha} e^{-\frac{x}{\alpha}}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad (4.2)$$

This can be written in the form 4.1 with $h(x) = 1\{x > 0\}$, $c(\alpha) = \frac{1}{\alpha}$, $g(\alpha) = -\frac{1}{\alpha}$, $t(x) = x$.

Note that $1\{\cdot\}$ is the indicator function, equal to one if the condition is true and 0 otherwise. Furthermore, the natural parameter is $g(\alpha) = -\frac{1}{\alpha}$.

1. Similarly, Bernoulli, Poisson, Normal, gamma, and inverse Gaussian distributions all belong to the exponential family and they can all be written in the canonical form.
2. Lognormal and Weibull distributions also belong to the exponential family, but they can not be written in the canonical form.

Let ξ_t , $t = 1, 2, \dots$ be sequentially observed independent random variables with a distribution $F(x, \alpha)$, where α is a parameter. We study the change-point detection problem, i.e. of detecting if and when the value of the parameter α changes. Recently, Sukparungsee and Novikov (2006) have used the martingale approach to derive approximate analytical formulas for ARL and AD in the case of a Gaussian distribution and some non-Gaussian distributions (see details Sukparungsee and Novikov, 2007). In this chapter, we are interested in obtaining explicit formulas for the case when the observations are from an exponential distribution.

The change-point model for the exponential distribution given in Equation (4.2) may be stated as follows. We assume that:

$$\xi_t \sim \begin{cases} \text{Exp}(\alpha_0), & t = 1, 2, \dots, \nu - 1 \\ \text{Exp}(\alpha), & t = \nu, \nu + 1, \dots, \quad \alpha > \alpha_0, \end{cases} \quad (4.3)$$

where α_0 and α are known parameters. Usually, the parameter value α_0 is assumed to define the in-control state and the parameter value α to define an out-of-control state. We assume that the value α_0 is maintained up to some unknown time $\nu - 1$ and that at time ν the parameter value changes to the new value $\alpha > \alpha_0$. The time ν is called the “change-point time”.

4.2 Explicit Formulas for One-sided Exponential EWMA

In this section we will first discuss the explicit analytical formulas obtained by Novikov (2006) for expectation of stopping times of a first-order autoregressive ($AR(1)$) process generated by observations from an exponential distribution. In his derivation of these formulas, Novikov used the theory of martingales. We will show how Novikov’s explicit formulas for $AR(1)$ processes can be derived by the method of Integral Equations. We will then derive explicit formulas for the expectation of stopping times for a one-sided exponential EWMA process by adapting the $AR(1)$ formulas.

4.2.1 Expectation of stopping times for a first-order autoregressive process

Novikov (2006) studied a first-order autoregressive process $AR(1)$ defined by:

$$X_t = \beta X_{t-1} + \eta_t, \quad t = 1, 2, \dots, \quad X_0 = x, \quad (4.4)$$

where η_t is an independent identically distributed (i.i.d.) exponential variables with parameter 1 ($\eta_t \sim Exp(1)$), where x and $\beta \in (0, 1)$ are given constants. The stopping time is given by

$$\tau_b = \inf\{t > 0 : X_t > b\}, \quad X_0 = x, \quad b > x. \quad (4.5)$$

In this chapter notations $E_x(\cdot)$ denote the expectation given the initial condition $X_0 = x$. Novikov (2006) derived explicit formulas for $E_x(\theta^{\tau_b})$ and $E_x(\tau_b)$, $\theta \in (0, 1)$, using the theory of martingales and obtained the following results for the case $\eta_t \sim Exp(1)$.

Theorem 4.2.1. *Let $0 < \beta < 1, 0 < x < b$. Then for any $\theta \in (0, 1)$*

$$E_x(\theta^{\tau_b}) = \frac{\theta T(x, \theta)}{T(b/\beta, \theta)}, \quad (4.6)$$

where

$$T(x, \theta) = \sum_{k=0}^{\infty} \frac{(\beta x)^k (\theta, \beta)_k}{k!}, \quad 0 < \theta < 1. \quad (4.7)$$

Corollary 4.2.2.

$$E_x(\tau_b) = G\left(\frac{b}{\beta}\right) + 1 - G(x). \quad (4.8)$$

where

$$G(x) = \sum_{k=1}^{\infty} \frac{(\beta x)^k (\beta, \beta)_{k-1}}{k!}. \quad (4.9)$$

The functions $(\theta, \beta)_k$ and $(\beta, \beta)_{k-1}$ in Equations (4.7) and (4.9) are q-Pochhammer symbols from the theory of q-series (see e.g. Andrews et al., 1999):

$$\prod_{j=1}^m (1 - \theta \beta^{j-1}) := (\theta, \beta)_m, \quad (\theta, \beta)_0 = 1. \quad (4.10)$$

Using this notation, we have

$$(\beta, \beta)_m = \prod_{j=1}^m (1 - \beta^j).$$

We will now give another proof of Corollary 4.2.2.

Proof. We define a function $Q(x)$ as follows:

$$Q(x) := E_x(\tau_b) = E_x(I\{X_1 > b\}) + E_x(I\{X_1 \leq b\}\tau_b), \quad (4.11)$$

where $I\{X_1 > b\}$ and $I\{X_1 \leq b\}$ are indicator functions. Note that on the set $\{X_1 \leq b\}$, $E(\tau_b|X_1) = 1 + Q(X_1)$. Then, we have

$$E_x(I\{X_1 \leq b\}\tau_b) = E_x[I\{X_1 \leq b\}E(\tau_b|X_1)] = E_x[I\{X_1 \leq b\}] + E_x[I\{X_1 \leq b\}Q(X_1)].$$

Hence, for $x \leq b$, the function $Q(x)$ is a solution of the following Integral Equation:

$$Q(x) = 1 + E_x(I\{X_1 \leq b\}Q(X_1)).$$

Since $P\{X_1 > y\} = P\{\xi_1 > y - \beta x\} = e^{\beta x - y}$, $X_1 = \beta x + \xi_1 > \beta x$ this equation can be written in the following form

$$Q(x) = 1 + \int_{\beta x}^b Q(y) e^{\beta x - y} dy \quad (4.12)$$

or, equivalently,

$$e^{-\beta x} Q(x) = e^{-\beta x} + \int_{\beta x}^b Q(y) e^{-y} dy. \quad (4.13)$$

Obviously, $Q(x)$ is a continuous function for $x < b$. The last expression implies that $Q(y)$ is also a continuously differentiable function for $x < b$.

Differentiating the left hand side of (4.13), we obtain

$$\frac{d}{dx} e^{-\beta x} Q(x) = -\beta e^{-\beta x} Q(x) + e^{-\beta x} Q'(x).$$

Differentiating the right hand side of (4.13), we have

$$\frac{d}{dx} [e^{-\beta x} + \int_{\beta x}^b Q(y) e^{-y} dy] = -\beta e^{-\beta x} - \beta Q(\beta x) e^{-\beta x}$$

and hence we obtain

$$-\beta e^{-\beta x} Q(x) + e^{-\beta x} Q'(x) = -\beta e^{-\beta x} - \beta Q(\beta x) e^{-\beta x}$$

or, equivalently,

$$Q'(x) = \beta Q(x) - \beta Q(\beta x) - \beta. \quad (4.14)$$

To solve this equation we try a power series solution of the form

$$Q(x) = \sum_{k=0}^{\infty} \frac{c_k x^k}{k!} = C + \sum_{k=1}^{\infty} \frac{c_k x^k}{k!}, \quad (4.15)$$

where the coefficients C and c_k , $k = 1, 2, \dots$ are to be chosen so that $Q(x)$ is a solution of Equation (4.14).

Differentiating Equation (4.15) we obtain

$$Q'(x) = \sum_{k=1}^{\infty} \frac{c_k x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{c_{k+1} x^k}{k!} = c_1 + \sum_{k=1}^{\infty} \frac{c_{k+1} x^k}{k!}. \quad (4.16)$$

Next, we substitute Equations (4.16) and (4.15) into Equation (4.14) and obtain

$$\begin{aligned} c_1 + \sum_{k=1}^{\infty} \frac{c_{k+1} x^k}{k!} &= \beta C + \beta \sum_{k=1}^{\infty} \frac{c_k x^k}{k!} - \beta C - \beta \sum_{k=1}^{\infty} \frac{c_k (\beta x)^k}{k!} - \beta \\ &= \sum_{k=1}^{\infty} \frac{c_k \beta (1 - \beta^k) x^k}{k!} - \beta \end{aligned}$$

and therefore we must have $c_1 = -\beta$ and

$$\sum_{k=1}^{\infty} \frac{c_{k+1}x^k}{k!} = \sum_{k=1}^{\infty} \frac{c_k\beta(1-\beta^k)x^k}{k!}. \quad (4.17)$$

Since the coefficients of x^k on the left and right hand sides of Equation (4.17) must be equal, we obtain the following recurrence relation for the coefficients c_k , $k = 1, 2, \dots$,

$$c_{k+1} = c_k\beta(1-\beta^k). \quad (4.18)$$

The solution of Equation (4.18) can be obtained by induction for $k \geq 1$ given $c_1 = -\beta$.

We obtain

$$\begin{aligned} c_2 &= c_1\beta(1-\beta) = -\beta^2(1-\beta) \\ c_3 &= c_2\beta(1-\beta) = -\beta^3(1-\beta)(1-\beta^2) \\ c_4 &= c_3\beta(1-\beta) = -\beta^4(1-\beta)(1-\beta^2)(1-\beta^3) \\ &\vdots \\ c_k &= -\beta^k(1-\beta)(1-\beta^2)\dots(1-\beta^{k-1}) \end{aligned}$$

and therefore

$$c_k = -\beta^k \prod_{j=1}^{k-1} (1-\beta^j) = -\beta^k (\beta, \beta)_{k-1}, \quad k = 1, 2, \dots, \quad (4.19)$$

where $(\beta, \beta)_{k-1}$ is the q-Pochhammer symbol defined in Equation (4.10).

Then, on substituting Equation (4.19) into Equation (4.15), we obtain the result

$$E_x(\tau_b) = Q(x) = C - \sum_{k=1}^{\infty} \frac{(\beta x)^k (\beta, \beta)_{k-1}}{k!} = C - G(x), \quad (4.20)$$

where

$$G(x) = \sum_{k=1}^{\infty} \frac{(\beta x)^k (\beta, \beta)_{k-1}}{k!} \quad (4.21)$$

and $G(0) = 0$.

Now we show that

$$C = G(b/\beta) + 1. \quad (4.22)$$

To see this, we note that substitution of $x = 0$ into Equation (4.12) gives

$$\begin{aligned} E_0(\tau_b) &= C = 1 + \int_0^b Q(y)e^{-y} dy, \\ C &= 1 + \int_0^b [C - G(y)]e^{-y} dy, \\ C &= 1 + C(1 - e^{-b}) - \int_0^b e^{-y}G(y)dy, \\ Ce^{-b} &= 1 - \int_0^b e^{-y}G(y)dy \end{aligned}$$

and therefore

$$E_0(\tau_b) = C = e^b - e^b \int_0^b e^{-y}G(y)dy. \quad (4.23)$$

Next, we need to find $\int_0^b e^{-y}G(y)dy$. Now, from Equation (4.20), we have $G(x) = C - Q(x)$ and therefore, from Equation (4.14)

$$\frac{d}{dx}G(x) = \beta G(x) - \beta G(\beta x) + \beta$$

which imply that

$$\begin{aligned} \frac{d}{dx}(e^{-\beta x}G(x)) &= -\beta e^{-\beta x}G(x) + e^{-\beta x}G'(x) \\ &= -\beta e^{-\beta x}G(x) + e^{-\beta x}[\beta G(x) - \beta G(\beta x) + \beta] \\ &= -\beta e^{-\beta x}G(\beta x) + \beta e^{-\beta x}. \end{aligned}$$

Integrating with respect to x on the interval $[0, b/\beta]$ both parts of the last equation we obtain

$$\int_0^{b/\beta} \frac{d}{dx}(e^{-\beta x}G(x))dx = -\beta \int_0^{b/\beta} e^{-\beta x}G(\beta x)dx + \beta \int_0^{b/\beta} e^{-\beta x} dx.$$

After changing the integration variable to $y = \beta x$, we have

$$e^{-b}G(b/\beta) = - \int_0^b e^{-y}G(y)dy + 1 - e^{-b}$$

or, equivalently,

$$\int_0^b e^{-y}G(y)dy = 1 - e^{-b}G(b/\beta) - e^{-b}. \quad (4.24)$$

Hence, substituting Equation (4.24) into Equation (4.23), we find

$$\begin{aligned} E_0(\tau_b) = C &= e^b - e^b \int_0^b e^{-y} G(y) dy \\ &= e^b - (e^b - G(b/\beta) - 1) \\ &= G(b/\beta) + 1. \end{aligned} \tag{4.25}$$

Finally, Equations (4.25) and (4.21) together with (4.20) imply

$$E_x(\tau_b) = Q(x) = C - G(x) = G(b/\beta) + 1 - G(x),$$

where

$$G(x) = \sum_{k=1}^{\infty} \frac{(\beta x)^k (\beta, \beta)_{k-1}}{k!}.$$

□

The proof of convergence of the series

We have that $0 < \beta < 1$. Since

$$(\beta, \beta)_{k-1} = \prod_{j=1}^{k-1} (1 - \beta^j) < 1$$

then

$$|G(x)| \leq \sum_{k=1}^{\infty} \frac{(\beta|x|)^k}{k!} (\beta, \beta)_{k-1} \leq \sum_{k=1}^{\infty} \frac{(\beta|x|)^k}{k!} = e^{\beta|x|}$$

and the series $\sum_{k=1}^{\infty} \frac{(\beta|x|)^k}{k!} = e^{\beta|x|}$ is convergent for $\forall x \in \mathbb{R}$. Then based on the comparison test we can claim that the series $G(x)$ is absolute convergent for $\forall x \in \mathbb{R}$ and $0 < \beta < 1$.

The proof of Corollary 4.2.2 is complete.

4.2.2 Expectation of Stopping Times for a One-sided EWMA Procedure

In this subsection, we adapt the explicit formulas of Corollary 4.2.2 to obtain formulas of ARL and AD for an EWMA procedure for the change in the exponential distribution. For convenience, we will let the in-control parameter $\alpha_0 = 1$ and assume that the out-of-control parameter $\alpha > 1$, i.e. the in-control parameter has the exponential distribution e^{-t} and the out-of-control parameter has the distribution $\frac{1}{\alpha} e^{-t/\alpha}$. Note that, for the $AR(1)$ process considered in Subsection 4.2.1, the random variable η_t is assumed to have the exponential distribution $Exp(1)$.

The definition of an EWMA process Z_t is as follows (see Equation (2.3))

$$Z_t = (1 - \lambda)Z_{t-1} + \lambda\xi_t, \quad t = 1, 2, \dots, \quad Z_0 = \alpha_0 = 1 \quad (4.26)$$

and the stopping time for a one-sided case is considered as the following form

$$\tau_H = \inf\{t > 0 : Z_t > H\}. \quad (4.27)$$

It is convenient to define a new variable $\tilde{Z}_t = \frac{Z_t}{\lambda}$. Then (4.26) can be written as follows:

$$\tilde{Z}_t = (1 - \lambda)\tilde{Z}_{t-1} + \xi_t, \quad \tilde{Z}_0 = \frac{1}{\lambda}. \quad (4.28)$$

Equation (4.28) can be expressed in the form of the $AR(1)$ formula in Corollary 4.2.2 by setting $(1 - \lambda) = \beta$, $\xi_t = \eta_t \sim \text{Exp}(1)$, $x = \frac{1}{\lambda}$ and $\frac{H}{\lambda} = b$. The stopping time for the \tilde{Z}_t process is then:

$$\tau_H = \inf\{t > 0 : \tilde{Z}_t > \frac{H}{\lambda}\}, \quad (4.29)$$

where by Corollary 4.2.2,

$$E_{\frac{1}{\lambda}}(\tau_H) = G\left(\frac{H}{\lambda(1 - \lambda)}\right) + 1 - G\left(\frac{1}{\lambda}\right) \quad (4.30)$$

where

$$G(x) = \sum_{k=1}^{\infty} \frac{((1 - \lambda)x)^k (1 - \lambda, 1 - \lambda)_{k-1}}{k!}, \quad (4.31)$$

then we obtain

$$E_{\frac{1}{\lambda}}(\tau_H) = 1 + \sum_{k=1}^{\infty} \frac{(1 - \lambda)^k \left(\frac{1}{\lambda}\right)^k (1 - \lambda, 1 - \lambda)_{k-1}}{k!} \left(\left(\frac{H}{1 - \lambda}\right)^k - 1\right). \quad (4.32)$$

The expectation of stopping time in Equation (4.30) corresponds to the ARL defined for an EWMA process in Equation (2.1). In terms of the EWMA statistics, Corollary 4.2.2 can be restated as follows (note that the notation E_x for the $AR(1)$ process has been changed to the notation \hat{E}_{∞} for the EWMA process):

Proposition 4.2.3. *The ARL of one-sided exponential EWMA is:*

$$ARL = \hat{E}_{\infty}(\tau_H) = G\left(\frac{H}{\lambda(1 - \lambda)}\right) + 1 - G\left(\frac{1}{\lambda}\right), \quad (4.33)$$

where $G(x)$ is defined in Equation (4.31).

We now show that, by a suitable modification of the $AR(1)$ formulas, we can obtain an EWMA formula for AD in the case of observations from the exponential distribution. We assume that the process is initially in the in-control state $\alpha_0 = 1$, i.e. the out-of-control EWMA process has $Z_0 = 1$. We also assume that the observations from the out-of-control distribution are given by $\xi_t \sim Exp(\alpha)$. The $AR(1)$ formulas assume that the distribution is $Exp(1)$. To obtain the formulas for an $Exp(\alpha)$ distribution from the formulas for an $Exp(1)$, we note that if $\xi_t \sim Exp(\alpha)$, then $\eta_t = \frac{\xi_t}{\alpha} \sim Exp(1)$. Therefore, we replace ξ_t in Equation (4.26) by $\alpha\eta_t$ and obtain:

$$Z_t = (1 - \lambda)Z_{t-1} + \lambda\alpha\eta_t, \quad t = 1, 2, \dots \quad Z_0 = \alpha_0 = 1 \quad (4.34)$$

and the stopping time for a one-sided case is typically

$$\tau_H = \inf\{t > 0 : Z_t > H\}. \quad (4.35)$$

Equation (4.34) can be transformed into the same form as the $AR(1)$ relation in Equation (4.4) by making the substitutions $\tilde{Z}_t = \frac{Z_t}{\lambda\alpha}$ and $\beta = 1 - \lambda$. The EWMA equation is then:

$$\tilde{Z}_t = (1 - \lambda)\tilde{Z}_{t-1} + \eta_t, \quad \tilde{Z}_0 = \frac{1}{\alpha\lambda}, \quad \eta_t \sim Exp(1), \quad (4.36)$$

with the stopping time

$$\tau_H = \inf\{t > 0 : \tilde{Z}_t > \frac{H}{\alpha\lambda}\}, \quad (4.37)$$

where from Corollary 4.2.2,

$$E_{\frac{1}{\lambda}}(\tau_{\frac{H}{\alpha\lambda}}) = G\left(\frac{H}{\alpha\lambda(1-\lambda)}\right) + 1 - G\left(\frac{1}{\alpha\lambda}\right) \quad (4.38)$$

and $G(x)$ is defined in Equation (4.31).

The expectation of stopping time in Equation (4.38) corresponds to the AD defined for the EWMA process in Equation (2.2) with $\nu = 1$. In terms of the EWMA statistics, Corollary 4.2.2 can be restated as follows (note that the notation E_x for the $AR(1)$ process has been changed to the notation \hat{E}_1 for the EWMA process):

Proposition 4.2.4. *The AD for one-sided exponential EWMA is:*

$$AD = \hat{E}_1(\tau_H) = G\left(\frac{H}{\alpha\lambda(1-\lambda)}\right) + 1 - G\left(\frac{1}{\alpha\lambda}\right), \quad (4.39)$$

where $G(x)$ is defined in Equation (4.31).

4.3 Comparison of Numerical Results for $E_x(\theta^{\tau_b})$

In this part, we shall discuss the case when η_1 in Equation (4.4) is exponentially distributed and we shall compare the explicit formula in Theorem 4.2.1 for $E_x(\theta^{\tau_b})$ with results obtained by Jacobsen (2007).

Jacobsen (2007) considered an autoregressive process for the case where random variables η_t have a distribution function of the form:

$$F(dt) = pF^+(dt) + qF^-(dt), \quad (4.40)$$

where $0 < p \leq 1$ and $q = 1 - p$. He then assumed that the functions F^+ and F^- are linear combinations of exponentials, where F^+ is a distribution function concentrated on $[0, \infty)$ given by:

$$F^+(dt) = \sum_{k=1}^r \alpha_k \mu_k e^{-\mu_k t} dt \quad (t > 0), \quad \text{with} \quad \sum_{k=1}^r \alpha_k = 1 \quad (4.41)$$

and F^- is a distribution function concentrated on $(-\infty, 0]$ given by

$$\int_{[-\infty, 0]} |t| F^-(dt) < \infty. \quad (4.42)$$

Jacobsen (2007) used the Laplace transform method to derive explicit formulas for $E_x(\theta^{\tau_b})$. The Laplace transform L^+ of F^+ is given by

$$L^+(\gamma) = \int_{\mathbb{R}^+} e^{-\gamma t} F^+(dt) = \sum_{k=1}^r \alpha_k \frac{\mu_k}{\mu_k + \gamma} \quad (\gamma \geq 0). \quad (4.43)$$

The Laplace transform L^- of F^- is given by

$$L^-(\gamma) = \int_{[-\infty, 0]} e^{\gamma t} F^-(dt) \quad (\gamma \geq 0). \quad (4.44)$$

Jacobsen (2007) stated and proved Theorem 4.3.1 (formulated below) for an autoregressive process

$$X_t = \beta X_{t-1} + \eta_t, \quad X_0 = x, \quad \eta_t \sim F(dt), \quad (4.45)$$

where $0 < \beta < 1$, F is defined in Equations (4.40), (4.41) and (4.42), the stopping time is

$$\tau_b = \inf\{t > 0 : X_t > b\}$$

and the overshoot $\chi_b := X_{\tau_b} - b$.

Theorem 4.3.1. *Assume that $m \leq r$, $0 < \theta < 1$ and $0 < \beta < 1$. Then for $\zeta \geq 0$ and $x \leq b$*

$$E_x(\theta^{\tau_b} e^{-\zeta \chi_b}) = \sum_{m=1}^r \sum_{j=1}^{\infty} c_{mj}(\theta) e^{\beta^j \mu_m x}, \quad (4.46)$$

where the coefficients $c_{mj}(\theta)$ are defined as follows:

For $j \geq 2$ and $c_{m0}^* = 1$,

$$c_{mj}(\theta) = c_{m1}(\theta) \theta^{j-1} c_{m,j-1}^*,$$

where

$$c_{m,j-1}^* = \prod_{l=1}^{j-1} (pL^+(-\beta^l \mu_m) + qL^-(\beta^l \mu_m)) \quad \text{with } p + q = 1 \quad (4.47)$$

and the column vector $C_1(\theta) := (c_m(\theta))_{1 \leq m \leq r}$ is given by

$$C_1(\theta) = M^{-1}(\theta) (p\theta \frac{\alpha_k \mu_k}{\mu_k + \zeta} e^{-\mu_k b})_{1 \leq k \leq r} \quad (4.48)$$

with $M(\theta)$ the $r \times r$ -matrix with entries

$$M_{km}(\theta) = \delta_{km} + p\alpha_k \theta \sum_{j=0}^{\infty} \theta^j c_{mj}^* \frac{\mu_k}{\mu_k - \beta^{j+1} \mu_m} e^{(\beta^{j+1} \mu_m - \mu_k) b}. \quad (4.49)$$

In particular, the formula in Theorem 4.2.1 for $E_x(\theta^{\tau_b})$ is applicable for the case when random variables in (4.45) are exponentially distributed. To compare the results we substitute the values $\alpha_k = \mu_k = p = r = 1$ and $\zeta = 0$ into the formulas in Theorem 4.3.1.

If we substitute these values into Equation (4.46), we obtain

$$E_x(\theta^{\tau_b}) = \sum_{j=1}^{\infty} c_{1j}(\theta) e^{\beta^j x}.$$

Then substitution of Equation (4.47) into Equation (4.46) gives the result:

$$\begin{aligned} E_x(\theta^{\tau_b}) &= c_{11}(\theta)e^{\beta x} + \sum_{j=2}^{\infty} c_{1j}(\theta)e^{\beta^j x} \\ &= c_{11}(\theta)e^{\beta x} + \sum_{j=2}^{\infty} c_{11}(\theta)\theta^{j-1} \prod_{l=1}^{j-1} \frac{1}{(1-\beta^l)} e^{\beta^j x}. \end{aligned}$$

Then from Equation (4.48), we obtain

$$E_x(\theta^{\tau_b}) = \theta e^{-b} M^{-1}(\theta) e^{\beta x} + \theta e^{-b} \sum_{j=2}^{\infty} M^{-1}(\theta) \theta^{j-1} \prod_{l=1}^{j-1} \frac{1}{(1-\beta^l)} e^{\beta^j x}. \quad (4.50)$$

Then for the value of $M^{-1}(\theta) = M_{11}(\theta)$ in Equation (4.49), we obtain

$$\begin{aligned} M_{11}(\theta) &= 1 + \theta \sum_{j=0}^{\infty} \theta^j c_{1,j}^* \frac{1}{(1-\beta^{j+1})} e^{(\beta^{j+1}-1)b} \\ &= 1 + \theta c_{1,0}^* \frac{1}{(1-\beta)} e^{(\beta-1)b} + \theta \sum_{j=1}^{\infty} \theta^j c_{1,j}^* \frac{e^{(\beta^{j+1}-1)b}}{(1-\beta^{j+1})}. \end{aligned}$$

Note that $c_{1,0}^* = 1$, then

$$M_{11}(\theta) = 1 + \theta \frac{e^{(\beta-1)b}}{(1-\beta)} + \theta \sum_{j=1}^{\infty} \theta^j \prod_{l=1}^j \frac{1}{(1-\beta^l)} \frac{e^{(\beta^{j+1}-1)b}}{(1-\beta^{j+1})}. \quad (4.51)$$

Then, on substituting Equation (4.51) into Equation (4.50), we find

$$\begin{aligned} E_x(\theta^{\tau_b}) &= \frac{\theta e^{-b} M^{-1}(\theta) e^{\beta x} + \theta e^{-b} \sum_{j=1}^{\infty} \theta^j \frac{e^{\beta^{j+1} x}}{\prod_{l=1}^j (1-\beta^l)}}{1 + \theta \frac{e^{(\beta-1)b}}{(1-\beta)} + \theta \sum_{j=1}^{\infty} \theta^j \prod_{l=1}^j \frac{1}{(1-\beta^l)} \frac{e^{(\beta^{j+1}-1)b}}{(1-\beta^{j+1})}} \\ &= \frac{\theta(e^{\beta x} + \sum_{j=1}^{\infty} \theta^j \frac{e^{\beta^{j+1} x}}{\prod_{l=1}^j (1-\beta^l)})}{e^b + \theta \frac{e^{\beta b}}{(1-\beta)} + \sum_{j=1}^{\infty} \theta^{j+1} \frac{e^{\beta^{j+1} b}}{\prod_{l=1}^j (1-\beta^l)}} \\ &= \frac{\theta(e^{\beta x} + \sum_{j=1}^{\infty} \theta^j \frac{e^{\beta^{j+1} x}}{\prod_{l=1}^j (1-\beta^l)})}{e^b + \sum_{j=1}^{\infty} \theta^j \frac{e^{\beta^j b}}{\prod_{l=1}^j (1-\beta^l)}}. \quad (4.52) \end{aligned}$$

We can now compare the results from the AR(1) formula for $E_x(\theta^{\tau_b})$ given in Equa-

4.4. Comparison of Numerical Results for ARL and AD with Other Methods

tion (4.6) with the results from the adapted Jacobsen formula given in Equation (4.52). For the comparison we used numerical methods to compute 200 terms of the sums in the two formulas. The results are shown in Table 4.1. It can be seen that the results from the two formulas are the same even though the two formulas have very different forms.

Table 4.1: Comparison of the numerical results for $E_x(\theta^{\tau_b})$

θ	x	b	β	$E_x(\theta^{\tau_b})$	
				Formula (4.6)	Formula (4.52)
0.25	1	2	0.01	0.043552	0.043552
	2	3.5	0.2	0.014324	0.014324
0.5	1	2	0.01	0.120118	0.120118
	2	3.5	0.2	0.040315	0.040315
0.75	1	2	0.01	0.290416	0.290416
	2	3.5	0.2	0.107763	0.107763

The Jacobsen formula given in this subsection is valid for a one-sided $AR(1)$ process in the case of a distribution consisting of a linear combination of exponential random variables. Jacobsen (2007) gives a generalization of the one-sided results and obtains formula for a two-sided $AR(1)$ process for linear combinations of exponential random variables.

4.4 Comparison of Numerical Results for ARL and AD with Other Methods

In this section, we compute the numerical values for ARL and AD from Equation (4.33) and Equation (4.39) with parameters ($\lambda = 0.01, H = 1.1071$) and compare them with values obtained from the Integral Equation (IE), Markov Chain (MCA) and Monte Carlo (MC) methods. We also compare the CPU times required to compute the numerical values for ARL and AD. The results of the comparison are shown in Table 4.2 for a range of values of the parameter value α . Note that $\alpha = 1$ is the value assumed for the in-control parameter and therefore this row gives the values of the ARL. The rows for $\alpha > 1$ correspond to values of the out-of-control parameter and therefore these rows give values for AD. The numbers in each cell in the table represent the value of ARL or AD and in parentheses () the CPU time for the calculation. The table shows that the results from the suggested formulas and the IE and MCA methods are close to the MC simulation results. However, the table also shows that the computational time for evaluating the suggested formula is much less than the CPU times required for the other methods. For example,

when $\alpha = 1$, the CPU time required to evaluate the suggested expression is 0.13 seconds compared with CPU times required for the IE, MCA and MC runs of 130, 158 and 5074 seconds, respectively.

Table 4.2: Comparison of ARL and AD values with other methods
Row 1 gives ARL values and the other rows give AD values, $\lambda=0.01$ and $H=1.1071$.
The numbers in parentheses () are CPU times in seconds.

α	Formulas (4.33) and (4.39)	Integral Equation	MCA	Monte Carlo (10^6)
1.0	500.03 (0.13)	500.397 (130)	500.276 (158)	499.87±0.508 (5074)
1.1	135.029 (0.14)	137.475 (131)	135.211 (158)	134.85±0.12 (1403)
1.2	68.670 (0.14)	69.275 (131)	68.788 (158)	68.663±0.052 (711)
1.3	44.956 (0.14)	44.009 (135)	45.042 (154)	44.944±0.031 (475)
1.4	33.281 (0.13)	32.717 (129)	33.347 (159)	33.282±0.021 (352)
1.5	26.429 (0.14)	26.045 (129)	26.483 (158)	26.434±0.016 (282)
1.6	21.949 (0.14)	21.667 (130)	21.995 (159)	21.933±0.013 (234)
1.7	18.799 (0.13)	18.581 (130)	18.839 (158)	18.804±0.011 (205)
1.8	16.468 (0.14)	16.293 (130)	16.502 (159)	16.473±0.009 (179)
1.9	14.673 (0.14)	14.529 (130)	14.704 (158)	14.666±0.008 (163)
2.0	13.250 (0.14)	13.129 (130)	13.278 (159)	13.251±0.007 (147)

4.5 Comparison of EWMA, CUSUM and Shiriyayev-Roberts Charts: Exponential case

In Table 4.3 and Figure 4.1 we present results for ARL and AD calculated for EWMA from our Equations (4.33) and (4.39) and compare them with results of Monte Carlo simulations for EWMA, CUSUM and Shiriyayev-Roberts charts. The optimal parameter values used for the EWMA were $H = 1.33379$ and $\lambda = 0.03024$, the boundary value used for CUSUM was $A = 3.84$ and the boundary value used for Shiriyayev-Roberts was $B = 700$. The Monte Carlo simulation results were obtained from a sequence of 10^6 runs.

Obviously, the numerical results from the suggested formulas are very close to Monte

Carlo simulation results for all values of α . A comparison of the AD values for EWMA, CUSUM and Shiriyayev-Roberts procedures shows that the EWMA values are less than the CUSUM and Shiriyayev-Roberts values for $\alpha < 1.3$, slightly less than CUSUM values for $1.4 \leq \alpha < 1.7$ and slightly greater than CUSUM values for $\alpha \geq 1.7$. That is, the EWMA chart performs better than CUSUM for small values of α . The performance of EWMA is superior to Shiriyayev-Roberts for all of the range of parameter changes α .

Table 4.3: The numerical results for ARL and AD obtained from formula (4.33) and (4.39) and simulation results of EWMA, CUSUM and Shiriyayev-Roberts charts: exponential case
Row 1 contains ARL values and the other rows contain AD values.

α	ARL and AD			
	Formulas (4.33) and (4.39)	Monte Carlo simulation		
		EWMA	CUSUM	SR
boundary	1.33379	1.33379	3.84	700.00
1.0	999.877	999.372±.942	998.165±.991	999.658±.305
1.1	251.711	251.098±.234	300.358±.286	285.440±.081
1.2	109.684	109.655±.093	129.174±.116	129.203±.032
1.3	64.205	64.188±.049	72.276±.059	77.286±.017
1.4	44.153	44.156±.031	47.700±.036	54.396±.010
1.5	33.363	33.384±.022	36.860±.025	42.078±.007
1.6	26.755	26.732±.017	27.225±.019	34.572±.005
1.7	22.335	22.308±.014	22.187±.015	29.264±.004
1.8	19.186	19.185±.011	18.633±.012	25.533±.003
1.9	16.835	16.837±.010	16.036±.010	22.748±.003
2.0	15.017	15.020±.009	14.832±.009	20.451±.002
2.5	9.884	9.888±.005	8.479±.006	13.951±.001
3.0	7.496	7.510±.004	7.113±.004	10.735±.001
5.0	4.126	4.125±.002	3.416±.002	5.969±.001

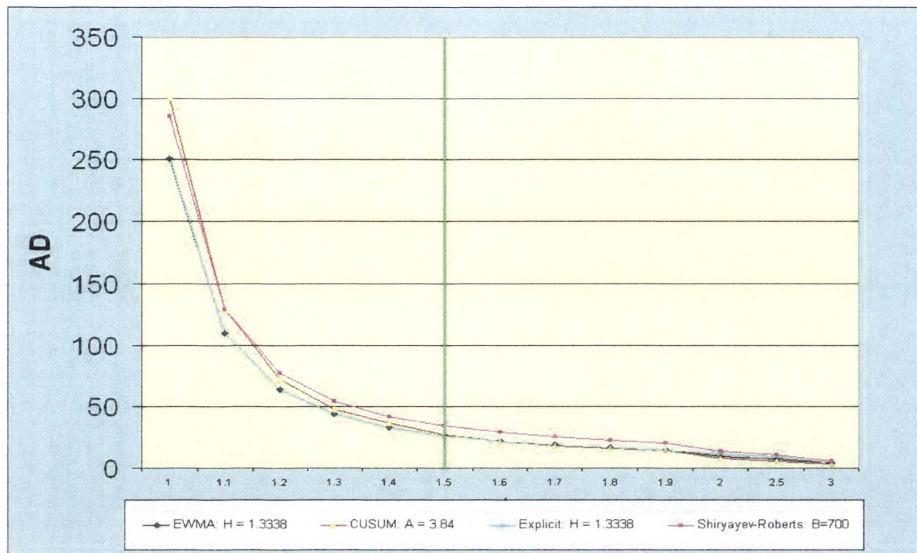


Figure 4.1: Curves of AD for EWMA, CUSUM and Shiryayev-Roberts charts: exponential case

4.6 Choices of Optimal Parameters for Exponential EWMA Charts

In this section, we first describe a procedure for obtaining optimal designs for exponential EWMA charts. The criterion used for choosing optimal values for the EWMA weight parameter (λ) and alarm boundary parameter (H) is minimisation of AD for a given in-control parameter value $\alpha_0 = 1$, $ARL = T$ and a given out-of-control parameter value (α). We compute optimal (λ^* , H^*) values for $T = 500, 1000, 3000, 5000$ and magnitudes of change. Plots of the optimal parameter values are shown in Figure 4.2 and a table of the values is given in Table 4.4.

The numerical procedure for obtaining optimal parameters for EWMA designs

1. Select an acceptable in-control value of ARL and decide on the change parameter value (α) for an out-of-control state.
2. For given α and T , find optimal values of λ^* and H^* to minimise the AD values given by Equation (4.39) subject to the constraint that $ARL=T$ in Equation (4.33), i.e. λ^* and H^* are solutions of the optimality problem:

$$\min_{\lambda, H} AD = G\left(\frac{H}{\alpha\lambda(1-\lambda)}\right) + 1 - G\left(\frac{1}{\alpha\lambda}\right), \quad (4.53)$$

subject to:

$$ARL = T = G\left(\frac{H}{\lambda(1-\lambda)}\right) + 1 - G\left(\frac{1}{\lambda}\right), \quad (4.54)$$

where $G(x)$ is given by Equation 4.31.

The numerical results in terms of optimal EWMA weight (λ^*), optimal alarm boundary (H^*) and minimal AD (AD^*) are shown in Table 4.4 and Figure 4.2. For example, if we want to detect a parameter change from $\alpha_0 = 1$ to $\alpha = 1.5$ and the ARL value is $T = 500$, then the optimality procedure given above will give optimal parameter values $\lambda^* = 0.02648$ and $H^* = 1.25116$. On substituting the values for α , λ^* and H^* into Equation (4.39) we obtain an optimal AD value of $AD^* = 26.569$. This optimal AD value agrees with the value of $AD^* = 26.57$ that we computed from MC for the same parameter values. Obviously, the suggested explicit formulas for evaluating ARL and AD are very effective alternatives to MC and other approaches. As shown in Table 4.2, the use of the suggested explicit formulas for ARL and AD can greatly reduce the computation times as compared with IE, MCA and MC approaches.

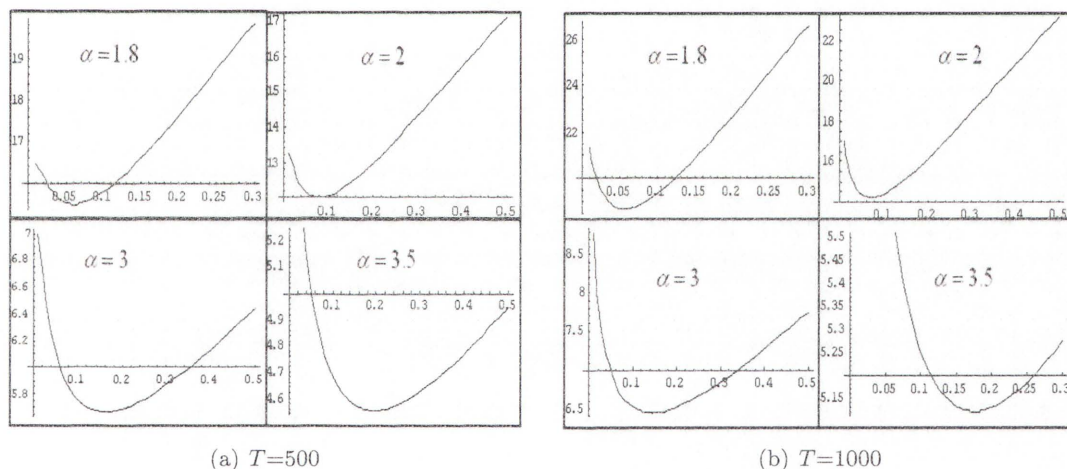


Figure 4.2: Curves of AD for optimal exponential EWMA designs

Table 4.4: Optimal design parameters and AD for one-sided exponential EWMA

α	$T=500$			
	λ^*	H^*	AD^*	AD (Simulation)
1.5	0.02648	1.25116	26.569	26.570±0.056
1.6	0.03902	1.34459	20.176	21.761±0.047
1.7	0.05044	1.41958	18.083	18.089±0.039
1.8	0.06063	1.48419	15.499	15.511±0.034
1.9	0.07042	1.54385	13.534	13.568±0.030
2.0	0.07991	1.59985	11.998	12.006±0.027
2.5	0.12346	1.84285	7.645	7.627±0.167
3.0	0.16213	2.04427	5.662	5.686±0.013
3.5	0.19625	2.21739	4.552	4.557±0.010
4.0	0.22673	2.36935	3.853	3.852±0.009
5.0	0.27912	2.62673	3.028	3.027±0.007
α	$T=1000$			
1.5	0.03024	1.33379	33.364	33.205±0.070
1.6	0.03891	1.40302	26.636	26.661±0.018
1.7	0.04745	1.46715	22.012	21.992±0.046
1.8	0.05588	1.52751	18.674	18.684±0.040
1.9	0.06417	1.58483	16.169	16.494±0.011
2.0	0.07233	1.63953	14.231	14.211±0.030
2.5	0.11079	1.88362	8.845	8.834±0.006
3.0	0.14533	2.08819	6.446	6.449±0.014
3.5	0.17630	2.26679	5.122	5.124±0.004
4.0	0.20423	2.42482	4.294	4.292±0.003
5.0	0.25268	2.69488	3.327	3.329±0.002
α	$T=3000$			
1.5	0.02552	1.36332	44.856	44.814±0.089
1.6	0.03223	1.42811	35.122	35.154±0.070
1.7	0.03906	1.48974	28.553	28.600±0.057
1.8	0.04578	1.54863	23.970	23.970±0.048
1.9	0.05253	1.60510	20.548	20.550±0.041
2.0	0.05922	1.65938	17.931	17.914±0.036
2.5	0.09218	1.91015	10.810	10.812±0.007
3.0	0.12178	2.12150	7.724	7.726±0.005
3.5	0.14718	2.29718	5.664	6.045±0.013
4.0	0.17311	2.47226	5.010	5.012±0.003
5.0	0.21604	2.75752	3.810	3.811±0.003
α	$T=5000$			
1.5	0.02348	1.37026	50.378	50.394±0.097
1.6	0.02967	1.43500	39.179	39.163±0.024
1.7	0.03594	1.49682	31.732	31.722±0.019
1.8	0.04191	1.55319	26.485	26.475±0.016
1.9	0.04853	1.61299	22.624	22.619±0.014
2.0	0.05477	1.66781	19.684	19.732±0.039
2.5	0.08487	1.91556	11.742	11.741±0.007
3.0	0.11256	2.12897	8.329	8.324±0.005
3.5	0.13781	2.31680	6.485	6.491±0.004
4.0	0.16087	2.48471	5.347	5.346±0.004
5.0	0.20152	2.77539	4.037	4.034±0.003

4.7 Explicit Formulas for One-sided Pareto EWMA

4.7.1 EWMA for Pareto Distribution

Heavy-tailed distributions appear in many real world applications, e.g. in finance and economics (Mandelbrot, 1963), hydrology (Anderson and Meerschaert, 1998) and teletraffic (Resnick and Starica, 1995).

Definition 3. *The distribution of a random variable X_1, X_2, \dots with distribution function F is said to have a heavy tail if*

$$\lim_{x \rightarrow \infty} e^{\lambda x} P(X > x) = \infty \quad \text{for all } \lambda > 0. \quad (4.55)$$

A typical example of a heavy-tailed distribution is a power-tailed distribution, i.e. a distribution for which $P(X > x) = ax^{-\alpha}$ for constants $a > 0$ and parameter $\alpha > 0$. For this power-tailed distribution $\lim_{x \rightarrow \infty} e^{\lambda x} ax^{-\alpha} = \infty$ for all $\lambda > 0$.

The simplest case of a power-tailed distribution is the Pareto distribution which is defined by the following function

$$P(\xi_t > x) = \left(\frac{k}{x}\right)^\alpha, \quad x \geq k, \quad \alpha \quad \text{and} \quad k > 0.$$

The change-point problem for a Pareto distribution is as follows. Let $\xi_t, t = 1, 2, \dots$ be sequentially observed independent random variables with Pareto distribution given by:

$$\xi_t \sim \begin{cases} \text{Pareto}(k, \alpha_0), & t = 1, 2, \dots, \nu - 1 \\ \text{Pareto}(k, \alpha), & t = \nu, \nu + 1, \dots, \quad \alpha < \alpha_0. \end{cases}$$

Mazalov and Zhuravlev (2002) have derived analytical expressions for the basic characteristics of the CUSUM chart for the Pareto case. However, no approximate formulas for the basic characteristics of the EWMA chart have so far been published.

In this section we derive the basic characteristics for a Pareto distribution. A common method used to transform a Pareto distribution to an exponential distribution is to take logarithms of the Pareto observations. For example, if the Pareto distribution has the form:

$$P(\xi_t > x) = \left(\frac{k}{x}\right)^\alpha,$$

then taking the logarithm we get:

$$\begin{aligned} P(\log(\frac{\xi_t}{k}) > x) &= P(\frac{\xi_t}{k} > e^x) = P(\xi_t > ke^x) \\ &= (\frac{k}{ke^x})^\alpha = e^{-\alpha x} \end{aligned} \quad (4.56)$$

and therefore $\log(\frac{\xi_t}{k})$ has the exponential distribution with $E(\log(\frac{\xi_t}{k})) = \frac{1}{\alpha}$. Therefore, if $\xi_t \sim \text{Pareto}(k, \alpha)$ then $\log(\frac{\xi_t}{k}) \sim \text{Exp}(1/\alpha)$ and $\eta_t = \alpha \log(\frac{\xi_t}{k}) \sim \text{Exp}(1)$.

For the Pareto distribution, we define the EWMA recurrence relation by:

$$Z_t = (1 - \lambda)Z_{t-1} + \lambda \log(\frac{\xi_t}{k}), \quad Z_0 = 1 \quad (4.57)$$

and the stopping time by

$$\tau_H = \inf(t > 0 : Z_t > H). \quad (4.58)$$

Equation (4.57) can be transformed to the standard $AR(1)$ form of Equation (4.4) with $\eta_t \sim \text{Exp}(1)$ by the substitutions

$$\tilde{Z}_t = \frac{Z_t}{(1/\alpha)\lambda}, \quad \tilde{Z}_0 = \frac{1}{(1/\alpha)\lambda}, \quad \beta = (1 - \lambda), \quad \eta_t = \alpha \log(\frac{\xi_t}{k}),$$

where for the in-control Pareto distribution $\alpha = \alpha_0 = 1$.

Therefore, it can be seen that the ARL and AD for the case of a Pareto distribution can also be calculated by using the closed-form formulas for the exponential distribution. The explicit formulas are as follows:

Proposition 4.7.1. *Explicit formulas for ARL and AD of one-sided EWMA for Pareto distribution:*

$$ARL = \hat{E}_\infty(\tau_H) = G(\frac{H}{\lambda(1 - \lambda)}) + 1 - G(\frac{1}{\lambda}), \quad (4.59)$$

$$AD = \hat{E}_1(\tau_H) = G(\frac{H}{(1/\alpha)\lambda(1 - \lambda)}) + 1 - G(\frac{1}{(1/\alpha)\lambda}), \quad (4.60)$$

where

$$G(x) = \sum_{k=1}^{\infty} \frac{((1 - \lambda)x)^k (1 - \lambda, 1 - \lambda)_{k-1}}{k!}.$$

4.7.2 Comparison for EWMA, CUSUM and Shiriyayev-Roberts Charts: Pareto case

In this subsection, we compare the numerical results for ARL and AD calculated from Equation (4.59) and Equation (4.60) for EWMA with the results of Monte Carlo simulations. In the Monte Carlo simulations we use EWMA, CUSUM and Shiriyayev-Roberts charts to obtain values for ARL and AD. For the EWMA chart we used optimal parameter values $\lambda = 0.03024$ and $H = 1.33379$, for CUSUM the boundary value $A = 3.84$ and for Shiriyayev-Roberts the boundary value $B = 700$. As for the exponential case in Section 4.5 the parameter values for EWMA, CUSUM and Shiriyayev-Roberts charts were chosen so that the ARL has the given value $T = 1000$ for an in-control parameter value $\alpha_0 = 1$.

The results of the comparison are shown in Table 4.5 and Figure 4.3. The results show that the performances of the EWMA, CUSUM and Shiriyayev-Roberts charts with the Pareto observation are similar to their performances for exponential observations. For example, for $T = 1000$ and $\alpha = 1/2.0$, CUSUM gives $AD = 14.832$, EWMA gives $AD = 15.02$ and Shiriyayev-Roberts gives $AD = 20.451$ and therefore CUSUM is better for this change. However, EWMA shows a better performance than CUSUM and Shiriyayev-Roberts charts for small values of change.

Table 4.5: The numerical results for ARL and AD obtained from formulas (4.59) and (4.60) and simulation results of EWMA, CUSUM and Shiriyayev-Roberts charts: Pareto case

Row 1 contains ARL values and the other rows contain AD values.

(k, α)	ARL and AD			
	Formulas (4.59) and (4.60)	Monte Carlo simulation		
		EWMA	CUSUM	SR
boundary	1.33379	1.33379	3.84	700.00
(1,1.0)	999.877	999.372 \pm .942	998.165 \pm .991	999.658 \pm .305
(1,1/1.1)	251.711	251.098 \pm .234	300.358 \pm .286	285.440 \pm .081
(1,1/1.2)	109.684	109.655 \pm .093	129.174 \pm .116	129.203 \pm .032
(1,1/1.3)	64.205	64.188 \pm .049	72.276 \pm .059	77.286 \pm .017
(1,1/1.4)	44.153	44.156 \pm .031	47.700 \pm .036	54.396 \pm .010
(1,1/1.5)	33.363	33.384\pm.022	36.860\pm.025	42.078\pm.007
(1,1/1.7)	22.335	22.308 \pm .014	22.187 \pm .015	29.264 \pm .004
(1,1/2.0)	15.017	15.020 \pm .009	14.832 \pm .009	20.451 \pm .002
(1,1/2.3)	11.412	11.407 \pm .006	10.468 \pm .002	15.933 \pm .002
(1,1/2.5)	9.884	9.888 \pm .005	8.479 \pm .006	13.951 \pm .001
(1,1/2.7)	8.746	8.748 \pm .005	8.055 \pm .002	12.405 \pm .002
(1,1/3.0)	7.496	7.510 \pm .004	7.113 \pm .004	10.735 \pm .001
(1,1/3.5)	6.118	6.121 \pm .004	6.014 \pm .001	8.839 \pm .001
(1,1/4.0)	5.222	5.226 \pm .003	5.010 \pm .001	7.569 \pm .001
(1,1/5.0)	4.126	4.125 \pm .002	3.416 \pm .002	5.969 \pm .001

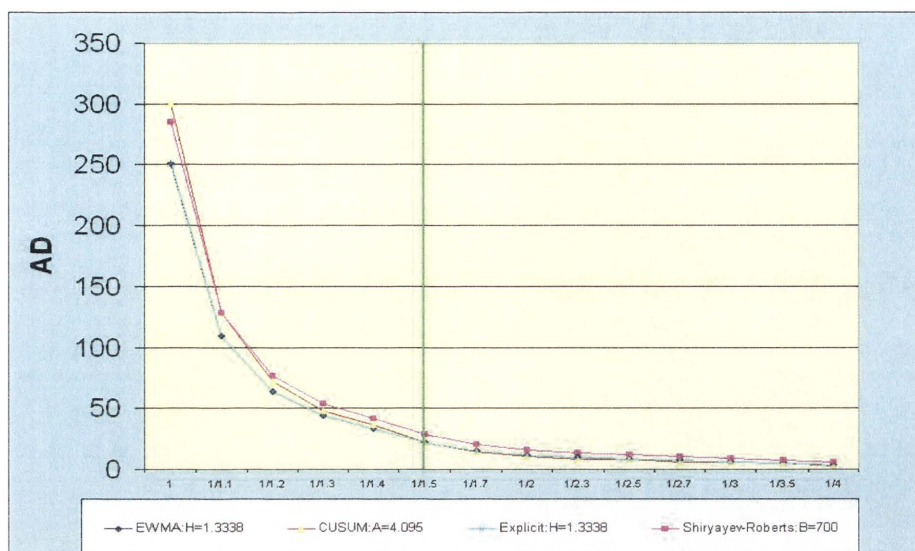


Figure 4.3: Curves of AD for EWMA, CUSUM and Shiriyayev-Roberts charts: Pareto case

4.7.3 Choices of Optimal Parameters for Pareto EWMA Procedure

The optimal designs for Pareto EWMA procedures can be obtained by the same optimization procedure that was used in Section 4.6. The results are presented in Table 4.6 and Figure 4.4 for ARL values of $T = 500, 1000, 3000, 5000$, in-control parameter $\alpha = 1$ and a range of out-of-control (α) values. For example, given $T = 1000$ and $\alpha = 1/2.5$, the optimal design for those parameters is for optimal smoothing parameter $\lambda^* = 0.11079$ and alarm boundary $H^* = 1.88262$. The minimal value of the AD is $AD^* = 8.845$. For the same parameter values the AD value from MC simulation is $AD = 8.834$.

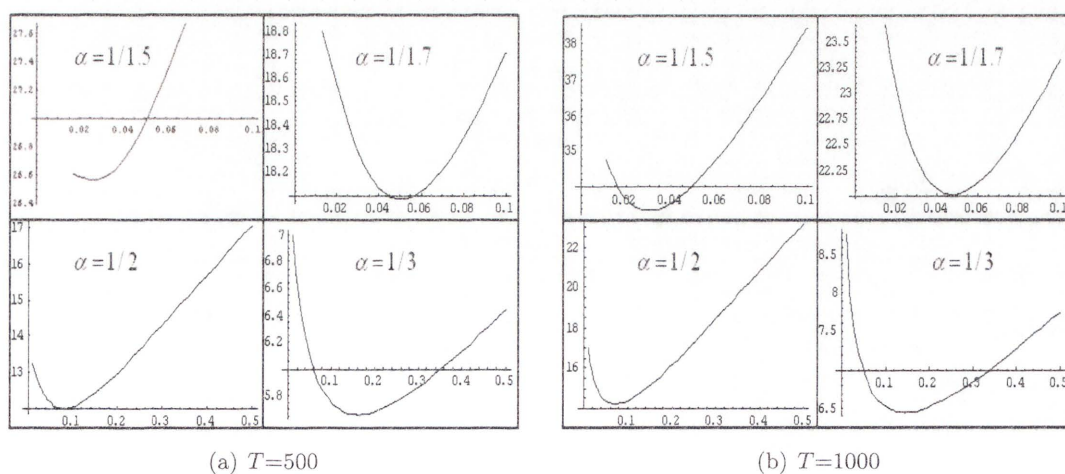


Figure 4.4: Curves of AD for optimal Pareto EWMA designs

Table 4.6: Optimal design parameters and AD for one-sided Pareto EWMA

(k, α)	$T=500$			
	λ^*	H*	AD*	AD (Simulation)
(1,1/1.5)	0.02648	1.25116	26.569	26.570±0.056
(1,1/1.7)	0.05044	1.41958	18.083	18.089±0.039
(1,1/2.0)	0.07991	1.59985	11.998	12.006±0.027
(1,1/2.3)	0.10684	1.75180	8.936	8.937±0.019
(1,1/2.5)	0.12364	1.84285	7.645	7.627±0.167
(1,1/2.7)	0.13961	1.92745	6.693	6.704±0.015
(1,1/3.0)	0.16213	2.04427	5.662	5.686±0.013
(1,1/3.5)	0.19625	2.21739	4.552	4.557±0.010
(1,1/4.0)	0.22673	2.36935	3.853	3.852±0.009
(1,1/4.5)	0.25420	2.50473	3.374	3.379±0.007
(1,1/5.0)	0.27912	2.62673	3.028	3.027±0.007
(k, α)	$T=1000$			
(1,1/1.5)	0.03024	1.33379	33.364	33.205±0.070
(1,1/1.7)	0.04745	1.46715	22.012	21.992±0.046
(1,1/2.0)	0.07233	1.63953	14.231	14.211±0.030
(1,1/2.3)	0.09588	1.79072	10.427	10.432±0.007
(1,1/2.5)	0.11079	1.88262	8.845	8.834±0.006
(1,1/2.7)	0.12506	1.96863	7.689	7.692±0.005
(1,1/3.0)	0.14533	2.08819	6.446	6.449±0.014
(1,1/3.5)	0.17630	2.26679	5.122	5.124±0.004
(1,1/4.0)	0.20423	2.42482	4.294	4.292±0.003
(1,1/4.5)	0.22956	2.56650	3.732	3.731±0.003
(1,1/5.0)	0.25268	2.69488	3.327	3.329±0.002
(k, α)	$T=3000$			
(1,1/1.5)	0.02552	1.36332	44.856	44.814±0.089
(1,1/1.7)	0.03906	1.48974	28.553	28.600±0.057
(1,1/2.0)	0.05922	1.65938	17.931	17.914±0.036
(1,1/2.3)	0.07955	1.81649	12.873	12.872±0.008
(1,1/2.5)	0.09218	1.91015	10.810	10.812±0.007
(1,1/2.7)	0.10436	1.99828	9.315	9.318±0.006
(1,1/3.0)	0.12178	2.12150	7.724	7.726±0.005
(1,1/3.5)	0.14718	2.29718	5.664	6.045±0.013
(1,1/4.0)	0.17311	2.47226	5.010	5.012±0.003
(1,1/4.5)	0.19356	2.60906	4.034	4.308±0.009
(1,1/5.0)	0.21604	2.75752	3.810	3.811±0.003
(k, α)	$T=5000$			
(1,1/1.5)	0.02348	1.37026	50.378	50.394±0.097
(1,1/1.7)	0.03594	1.49682	31.732	31.722±0.019
(1,1/2.0)	0.05477	1.66781	19.684	19.732±0.039
(1,1/2.3)	0.07310	1.82123	14.036	14.032±0.009
(1,1/2.5)	0.08487	1.91556	11.742	11.741±0.007
(1,1/2.7)	0.09625	2.00446	10.086	10.084±0.006
(1,1/3.0)	0.11256	2.12897	8.329	8.324±0.005
(1,1/3.5)	0.13781	2.31680	6.485	6.491±0.004
(1,1/4.0)	0.16087	2.48471	5.347	5.346±0.004
(1,1/4.5)	0.18052	2.62604	4.583	4.585±0.003
(1,1/5.0)	0.20152	2.77539	4.037	4.034±0.003

Chapter 5

Numerical Solutions for Integral Equations

As mentioned in Chapters 3 and 4, there are many methods that have been used for evaluating ARL and AD for EWMA charts. In Chapter 4 we have obtained explicit formulas for ARL and AD for EWMA charts for observations from the exponential and Pareto distributions. However, in other cases explicit formulas for ARL and AD are unknown. In these cases, values for ARL and AD can be obtained by numerical methods, such as the Integral Equation or Markov Chain method, or by Monte Carlo simulation.

In this chapter, we use the Integral Equations method described in Section 3.2 to find numerical values for ARL and AD for EWMA charts for observations from gamma and Weibull distributions and for CUSUM charts for observations from exponential and gamma distributions. In real applications, particularly in industry and manufacturing, the observations have often gamma and Weibull distributions, e.g. time between failure in reliability trials and the cumulative number of items inspected (Xie et al., 2002; Borrer et al., 2003; Liu et al., 2007; Zhang et al., 2007).

Integral Equations were first used by Crowder (1987) for approximating ARL and AD for Gaussian distribution. Later, Champ and Rigdon (1991) intensively studied them by comparing the run length distributions obtained from the Integral Equation technique and MCA for the case of Gaussian distribution (see detail Gan, 1990a, 1992; Srivastava and Wu, 1993, 1997; Rao et al., 2001).

The outline of this chapter is as follows:

- In Section 5.1 we present the Nyström method for obtaining numerical solutions to Integral Equations. The method is based on numerical quadrature rules to evaluate

integrals. We discuss some standard quadrature rules for approximating integrals such as the composite Midpoint rule, composite Trapezoidal rule, composite Simpson's rule and the Gaussian rules.

- In Subsection 5.2.1 we use the Nyström method to calculate the ARL for the EWMA chart for the gamma distribution. We then compare the values for ARL with the values obtained by other methods such as MCA and MC.
- In Subsection 5.2.3 we use the Nyström method to calculate the ARL for the EWMA chart for the Weibull distribution. We compare the values for ARL with the values obtained from other methods such as MCA and MC. In Section 5.3 we study the Integral Equation for CUSUM chart and study some examples when observations are from exponential and gamma distributions.

5.1 The Nyström Method

The Nyström method was developed for obtaining numerical solutions of Integral Equations in Press et al. (1992); Atkinson (1997); Atkinson and Han (2005). The Integral Equations method for calculating the ARL are of the type called Fredholm Integral Equations of the second kind (see for example, Wieringa, 1999).

Definition 4. *A Fredholm Integral Equation of the second kind is of the form:*

$$\lambda x(t) - \int_D K(t,s)x(s)ds = y(t), \quad t \in D, \quad (5.1)$$

where λ is a scalar, $K(s,t)$ is a given function called the kernel of the equation and $y(t)$ is a given function. D is a closed and bounded set in \mathbb{R}^m , $m \geq 1$. The problem is to find values of the scalar λ and the function $x(t)$ such that the Fredholm Integral Equation (5.1) holds.

Several authors have investigated numerical methods for finding solution of Equation (5.1) with use of numerical quadrature rules to approximate the integral (see for detail Atkinson, 1997; Baker, 1977; Kress, 1999).

The method is as follows. Let $K(t,s)$ be continuous for all $t,s \in \mathcal{D}$. A numerical quadrature rule is then used to approximate the integral in Equation (5.1). As explained in Subsection 3.2.2, a quadrature rule for an interval D is defined by the choice of a function $W(s)$, a set of points $\{s_k, k = 1, 2, \dots, n\}$ and a set of weights $\{w_k, k = 1, 2, \dots, n\}$ such

that:

$$\int_D W(s)g(s)ds \approx \sum_{k=1}^n w_k g(s_k), \quad (5.2)$$

where $g(s)$ is a function to be integrated. The points $\{s_k\}$ are often called the nodes of the rule and the function $W(s)$ is often called a weight function.

Using the quadrature scheme of Equation (5.2) to approximate the integral in Equation (5.1), we obtain

$$\lambda x_n(t) - \sum_{k=1}^n w_k K(t, s_k) x_n(s_k) = y(t), \quad t \in \mathcal{D}, \quad (5.3)$$

where $x_n(s)$ is an approximation for $x(s_k)$

To find the solution at the quadrature node points $\{s_i, i = 1, 2, \dots, n\}$, we need to solve the following system of linear algebraic equations:

$$\lambda x_n(s_i) - \sum_{k=1}^n w_k K(s_i, s_k) x_n(s_k) = y(s_i), \quad i = 1, \dots, n. \quad (5.4)$$

This system of equations is called the *Nyström interpolation formula* (Atkinson, 1997). In the next subsection, we will describe some of the standard quadrature rules that we will use to obtain numerical solutions of the Integral Equations for the ARL of EWMA and CUSUM charts.

5.1.1 Numerical Quadrature

In this thesis, we will use the composite midpoint rule, the composite trapezoidal rule, the composite Simpson's rule and the Gauss-Legendre rule to obtain numerical solutions of the Nyström interpolation formula. We will now give a summary of each of these quadrature rules.

The main criteria used in selecting the function $W(s)$, the set of points $\{s_k, k = 1, 2, \dots, n\}$ and the weights $\{w_k, k = 1, 2, \dots, n\}$ to integrate $\int_a^b W(s)g(s)ds$ are as follows. First, the function $W(s)$ is chosen so that a set of polynomials will give a good approximation to the function $g(s)$ to be integrated. The sets of points and weights are then chosen so that the quadrature rule is exact if $g(s)$ is replaced by the highest possible degree polynomial for the given choice of points.

5.1.2 Midpoint Rule

Consider the problem of integrating a function $g(s)$ over $[a, b]$, where $[a, b]$ is a finite interval. The value $W(s) = 1$ is chosen and a set of equally spaced points is used. The interval $[a, b]$ is subdivided into n subintervals $\{[s_{k-1}, s_k], k = 1, 2, \dots, n\}$ of equal width $h = \frac{b-a}{n}$ by using equally spaced points $s_k = s_0 + kh$ for $k = 0, 1, 2, \dots, n$, where $s_0 = a$ and $s_n = b$. For the midpoint rule the set of nodes is chosen as the midpoints m_k of the subintervals. The midpoints are given by $m_k = \frac{1}{2}(s_{k-1} + s_k) = a + (k - \frac{1}{2})h$. The weights w_k for each midpoint are chosen to be 1. On each subinterval, the midpoint rule will integrate a constant function exactly. For integrating a general function $g(s)$ over an interval $[a, b]$, the composite midpoint rule for n subintervals is obtained by combining the rules for the subintervals and the result is:

$$M(g, h) = h \sum_{k=1}^n g(a + (k - \frac{1}{2})h).$$

The algorithm for numerical approximation to the integral $g(s)$ over $[a, b]$ is given by:

Composite Midpoint Rule Algorithm. If a function $g(s)$ is sampled at the n midpoints of the subintervals defined by the $n + 1$ equally spaced points $s_k = a + kh$ for $k = 0, 1, \dots, n$, where $h = \frac{b-a}{n}$ and $s_0 = a$ and $s_n = b$, then the composite midpoint rule approximation for the integral is given by:

$$\int_a^b g(s) ds \approx h \sum_{k=1}^n g(a + (k - \frac{1}{2})h).$$

Remark. It has been shown by Champ and Rigdon (1991) that the midpoint rule and MCA produced the closed approximations for ARL.

5.1.3 Trapezoidal Rule

For trapezoidal rule, the function $W(s) = 1$, the interval of integration $[a, b]$ is finite and the set of points is equally spaced. The interval $[a, b]$ is subdivided into n subintervals $\{[s_{k-1}, s_k], k = 1, 2, \dots, n\}$ of equal width $h = \frac{b-a}{n}$ by using equally spaced points $s_k = s_0 + kh$ for $k = 0, 1, 2, \dots, n$, where $s_0 = a$ and $s_n = b$. The weights are chosen so that on each subinterval of width h , a polynomial of degree 1 (i.e. the monomials 1 and s) will be integrated exactly by the rule. This choice gives equal weights $\frac{1}{2}h$ at the end points of a subinterval. The composite trapezoidal rule for n subintervals is obtained by combining

the rules for the subintervals. The composite rule is:

$$T(g, h) = \frac{h}{2}(g(a) + g(b)) + h \sum_{k=1}^n g(s_k) \quad (5.5)$$

and the algorithm for numerical approximation to the integral $g(s)$ over $[a, b]$ is given by:

Composite Trapezoidal Rule Algorithm. If a function $g(s)$ is sampled at the $n + 1$ equally spaced points $s_k = a + kh$ for $k = 0, 1, \dots, n$, where $h = \frac{b-a}{n}$ and $s_0 = a$ and $s_n = b$, then the composite trapezoidal rule approximation for the integral is given by:

$$\int_a^b g(s) ds \approx \frac{h}{2}(g(a) + g(b)) + h \sum_{k=1}^{n-1} g(s_k). \quad (5.6)$$

5.1.4 Simpson's Rule

For Simpson's rule, the function $W(s) = 1$, the interval of integration $[a, b]$ is finite and the set of points is equally spaced. The interval $[a, b]$ is subdivided into $2n$ subintervals $\{[s_{k-1}, s_k], k = 1, 2, \dots, 2n\}$ of equal width $h = \frac{b-a}{2n}$ by using equally spaced points $\{s_k = s_0 + kh, k = 1, 2, \dots, 2n\}$, where $s_0 = a$ and $s_{2n} = b$. The basic Simpson's rule applies to 2 subintervals and is obtained by choosing the weights at the 3 points $\{s_{2k}, s_{2k+1} = s_{2k} + h, s_{2k+2} = s_{2k} + 2h\}$ so that the monomials $\{1, s, s^2\}$ are integrated exactly by the rule. The weights at the 3 points are $\{\frac{1}{3}h, \frac{4}{3}h, \frac{1}{3}h\}$. It can be shown that the rule then also integrates a monomial s^3 exactly. The composite Simpson's rule is obtained by combining the rules for the subintervals of width $2h$. The composite rule is

$$S(g, h) = \frac{h}{3}(g(a) + g(b)) + \frac{4h}{3} \sum_{k=1}^n g(s_{2k-1}) + \frac{2h}{3} \sum_{k=1}^{n-1} g(s_{2k}) \quad (5.7)$$

and the algorithm for numerical approximation to the integral $g(s)$ over $[a, b]$ is given by:

Composite Simpson's Rule Algorithm. If a function $g(s)$ is sampled at the $2n + 1$ equally spaced points $s_k = a + kh$ for $k = 0, 1, \dots, 2n$, where $h = \frac{b-a}{2n}$ and $s_0 = a$ and $s_{2n} = b$, then the composite Simpson's rule approximation for the integral is given by:

$$\int_a^b g(s) ds \approx \frac{h}{3}(g(a) + g(b)) + \frac{4h}{3} \sum_{k=1}^n g(s_{2k-1}) + \frac{2h}{3} \sum_{k=1}^{n-1} g(s_{2k}). \quad (5.8)$$

5.1.5 Gaussian Rules

In the midpoint, trapezoidal, and Simpson's rules, the integration interval $[a, b]$ is finite, the function $W(s) = 1$ and the set of points $\{s_k, k = 1, 2, \dots, n\}$ is equally spaced. In Gaussian rules, the integration interval can be infinite, the weight function $W(s)$ might not equal 1, and the set of points $\{s_k, k = 1, 2, \dots, n\}$ is not equally spaced. In a Gaussian rule of the form:

$$\int_a^b W(s)g(s)ds \approx \sum_{k=1}^n w_k g(s_k) \quad (5.9)$$

the positions of the n points and the n weights $\{w_k, k = 1, 2, \dots, n\}$ are chosen so that the rule will integrate exactly the $2n$ monomials $\{1, s, s^2, \dots, s^{2n-1}\}$, i.e. every polynomial in s of degree less than or equal to $2n - 1$ will be integrated exactly.

A basic result used in the theory of Gaussian integration (David and Rabinowitz, 1967; Press et al., 1992) is that for each interval $[a, b]$ and function $W(s)$, there exists a set of orthogonal polynomials $\{p_k(s), k = 0, 1, 2, \dots\}$ such that $p_k(s)$ is a polynomial in s of degree k that has exactly k distinct real zeros on the interval $[a, b]$ and such that

$$\int_a^b W(s) p_k(s) p_l(s) ds \begin{cases} = 0 & \text{if } k \neq l \\ \neq 0 & \text{if } k = l \end{cases}$$

The Gaussian rule for a given interval $[a, b]$ and weight function $W(s)$ is usually named after the set of orthogonal polynomials for the interval and function. The most commonly used Gaussian rules are as follows:

Gauss-Legendre:

$$W(s) = 1, \quad -1 \leq s \leq 1.$$

The orthogonal polynomials are the Legendre polynomials $\{P_k(s), k = 0, 1, 2, \dots\}$.

Gauss-Chebyshev:

$$W(s) = (1 - s^2)^{-1/2}, \quad -1 \leq s \leq 1.$$

The orthogonal polynomials are the Chebyshev polynomials $\{T_k(s), k = 0, 1, 2, \dots\}$.

Gauss-Hermite:

$$W(s) = e^{-s^2}, \quad -\infty < s < \infty.$$

The orthogonal polynomials are the Hermite polynomials $\{H_k(s), k = 0, 1, 2, \dots\}$.

Gauss-Laguerre:

$$W(s) = s^\alpha e^{-s}, \quad 0 \leq s < \infty.$$

The orthogonal polynomials are the Laguerre polynomials $\{L_k^\alpha(s), k = 0, 1, 2, \dots\}$.

5.1.6 Theory of Gaussian Integration

We will now give a summary of the theory of Gaussian integration rules. This summary is based on Stroud and Secrest (1966); Isaacson and Keller (1966); Atkinson (1989, 1997); Press et al. (1992).

Using the weight function $W(s)$, we define an inner product

$$\langle f, g \rangle = \int_a^b W(s) f(s) g(s) ds.$$

A set of orthogonal polynomials $p_k(s)$ satisfies

$$\langle p_k, p_l \rangle \begin{cases} = 0, & k \neq l \\ \neq 0, & k = l \end{cases}$$

A set of orthogonal polynomials can be constructed for any given interval $[a, b]$ and weight function $W(s)$ by applying the Gram-Schmidt orthonalization process to the monomials $1, s, s^2, \dots$. However, the Legendre, Chebyshev, Hermite and Laguerre polynomials have been extensively studied because of their importance in many fields and there are now many well-known methods available for constructing these polynomials that are much more efficient than the Gram-Schmidt process. These methods include the three-term recurrence relations, the Rodrigues' formulas or the generating functions (see detail, Abramowitz and Stegun, 1964).

As stated above, the goal in constructing a Gaussian rule is to find a set of n nodes and n weights such that

$$\int_a^b W(s) g(s) ds = \sum_{k=1}^n w_k g(s_k) + \epsilon \quad (5.10)$$

is exact ($\epsilon = 0$) if $g(s)$ is a polynomial of degree less than or equal to $2n - 1$. Note that a polynomial of degree $2n - 1$ is a linear combination of $2n$ monomials $\{1, s, s^2, \dots, s^{2n-1}\}$ and therefore the exactness condition consists of $2n$ conditions that can be solved to obtain values for the n nodes and n weights.

The first step in constructing a Gaussian rule is the selection of the n nodes $\{s_k, k = 1, 2, \dots, n\}$. The method is as follows. Let $g(s)$ be a polynomial of degree $2n - 1$. Dividing

by the orthogonal polynomial $p_n(s)$ gives

$$g(s) = p_n(s)Q_{n-1}(s) + R(s), \quad (5.11)$$

where both $Q_{n-1}(s)$ and $R(s)$ are polynomials of degree less than or equal to $n - 1$. If we multiply by the weight function and integrate both sides, we get

$$\int_a^b W(s)g(s)ds = \int_a^b W(s)p_n(s)Q_{n-1}(s)ds + \int_a^b W(s)R(s)ds. \quad (5.12)$$

The first integral on the right of Equation (5.12) is zero because any polynomial of degree less than or equal to $n - 1$ can be expressed as a linear combination of the orthogonal polynomials $\{p_0, p_1, \dots, p_{n-1}\}$, and so must be orthogonal to p_n . Now, we look at the Gauss rule for this integral. We find that:

$$0 = \int_a^b W(s)p_n(s)Q_{n-1}(s)ds = \sum_{k=1}^n w_k p_n(s_k)Q_{n-1}(s_k).$$

The rule will be exact for this integral if the n nodes are chosen so that $p_n(s_k) = 0$, $k = 1, 2, \dots, n$, i.e. if the n nodes can be chosen as zeros of the orthogonal polynomial $p_n(s)$. As noted above, it is known that an orthogonal polynomial $p_n(s)$ always has exactly n distinct real zeros on the interval $[a, b]$ and therefore it is always possible to choose n distinct real nodes for a Gaussian rule as the zeros of the orthogonal polynomial $p_n(s)$. Note also that the n nodes $\{s_k, k = 1, 2, \dots, n\}$ are the zeros of $p_n(s)$ and therefore the polynomial $p_n(s)$ can be factorized in the form $p_n(s) = A_n \prod_{k=1}^n (s - s_k)$, where A_n is the coefficient of s^n in $p_n(s)$.

The next step is to choose the n weights. From Equation (5.12), the rule will be exact for all polynomials of degree less than or equal to $2n - 1$ if:

$$\int_a^b W(s)g(s)ds = \int_a^b W(s)R(s)ds = \sum_{k=1}^n w_k R(s_k), \quad (5.13)$$

where $g(s_k) = p_n(s_k)Q_{n-1}(s_k) + R(s_k) = R(s_k)$ for all polynomials $R(s)$ of degree less than or equal to $n - 1$.

We now introduce another special set of polynomials, the Lagrange interpolating poly-

nomials, defined by:

$$L_k(s) = \prod_{l=1, l \neq k}^n \frac{s - s_l}{s_k - s_l}, \quad k = 1, 2, \dots, n.$$

The Lagrange polynomials are of degree $n - 1$ and $L_k(s)$ is equal to one at $s = s_k$, and zero at all the other points s_l for $l \neq k$. Because of these properties, they are called interpolating polynomials of degree $n - 1$.

Now, since $R(s)$ is of degree less than or equal to $n - 1$, we can write it as a sum of Lagrange polynomials, so that

$$R(s) = \sum_{k=1}^n R(s_k)L_k(s) = \sum_{k=1}^n g(s_k)L_k(s).$$

Then, combining this expression with Equation (5.10) and (5.13), we obtain

$$\int_a^b W(s)g(s)ds = \sum_{k=1}^n g(s_k) \left[\int_a^b W(s)L_k(s)ds \right],$$

where we can identify the term in square brackets as the weight w_k . To simplify the computation of the weights, we can express the Lagrange polynomials in terms of the orthogonal polynomial $p_n(s)$ and its derivative $p'_n(s)$ as follows. As noted above, we have:

$$p_n(s) = A_n \prod_{l=1}^n (s - s_l) \quad \text{and therefore} \quad p'_n(s_k) = A_n \prod_{l=1, l \neq k}^n (s_k - s_l).$$

Then, we have for $k = 1, 2, \dots, n$ that

$$L_k(s) = \prod_{l=1, l \neq k}^n \frac{s - s_l}{s_k - s_l} = \frac{p_n(s)}{A_n(s - s_k) \prod_{l=1, l \neq k}^n (s_k - s_l)} = \frac{p_n(s)}{(s - s_k)p'_n(s_k)}.$$

An algorithm for finding the weights and nodes of a Gaussian quadrature rule is as follows:

1. Find the set of orthogonal polynomials $\{p_k(s), k = 0, 1, 2, \dots\}$ with respect to the weight function $W(s)$ on the interval $[a, b]$.
2. Nodes: For a rule with n nodes, choose the n nodes as the n zeros $\{s_k, k = 0, 1, 2, \dots, n - 1\}$ of the orthogonal polynomial $p_n(s)$ of degree n .

3. Weights: Calculate the n weights from the formula:

$$w_k = \int_a^b W(s) \frac{p_n(s)}{(s - s_k)p'_n(s_k)} ds.$$

It has been shown by Isaacson and Keller (1966, pp.333-334) and Atkinson (1989, p.275) that an equivalent formula for the weights is:

$$w_k = -\frac{A_n \langle p_n, p_n \rangle}{p_{n+1}(s_k) p'_n(s_k)}, \quad (5.14)$$

where A_n is the coefficient of s^n in $p_n(s)$.

Gauss-Legendre Quadrature

If $W(s) = 1$ and $[a, b] = [-1, 1]$, the orthogonal polynomials are the Legendre polynomials:

$$\begin{aligned} P_0(s) &= 1 \\ P_1(s) &= s \\ P_2(s) &= \frac{1}{2}(3s^2 - 1) \\ &\vdots \\ P_n(s) &= \frac{1}{2^n n!} \frac{d^n}{ds^n} (s^2 - 1)^n. \end{aligned}$$

The nodes for the rule can be obtained using a zero finding algorithm such as Newton's method. A useful formula for the weights can be obtained by substituting the identity

$$\langle P_n, P_n \rangle = \int_{-1}^1 |P_n(s)|^2 ds = \frac{2}{2n + 1}$$

into Equation (5.14) to obtain:

$$w_k = -\frac{2}{(n + 1)P'_n(s_k)P_n(s_k)}, \quad k = 1, 2, \dots, n. \quad (5.15)$$

Extensive tables of values for the nodes and weights of Gauss-Legendre rules are available in the literature. For example, Abramowitz and Stegun (1964) give tables of values of nodes and weights for the standard Gaussian quadrature rules.

In this thesis we are using the Mathematica[®] package for computations. The "Gaussian-Quadrature" package in Mathematica[®] gives values for both the nodes and weights for n point quadrature rules for a given interval $[a, b]$.

5.1.7 Properties and Error Analysis of the Nyström Method

In this subsection we discuss some basic properties of the Nyström method, including sufficient conditions for convergence, rate of convergence, and errors for the quadrature rules described in Subsection 5.1.1.

The Nyström method is implemented with the finite linear system in Equation (5.4), but the error analysis must be carried out using the system of Integral Equations in Equation (5.3). Following (Atkinson, 1997), we write the original Fredholm Integral Equation (5.1) in an operator form as $(\lambda - K)x = y$ and the Integral Equation (5.3) as $(\lambda - K_n)x_n = y$. Atkinson (1997) then proves the following theorem which gives sufficient conditions for the convergence of the Nyström method.

Theorem 5.1.1. (Atkinson, 1997, p.107) *Let D be a closed and bounded set in \mathbb{R}^m , some $m \geq 1$, and let $K(t, s)$ be continuous for $t, s \in D$. Assume the quadrature scheme of Equation (5.9) is convergent for all continuous functions on D . Further, assume that the Integral Equation (5.1) is uniquely solvable for given $y \in C(D)$, with $\lambda \neq 0$. Then for all sufficiently large n , say $n \geq N$, the approximate inverses $(\lambda - K_n)^{-1}$ exist and are uniformly bounded,*

$$\|(\lambda - K_n)^{-1}\| \leq \frac{1 + \|(\lambda - K)^{-1}\| \|K_n\|}{|\lambda| - \|(\lambda - K)^{-1}\| \|K - K_n\|} \leq c_s, \quad n \geq N$$

with a suitable constant $c_s < \infty$. For the equation $(\lambda - K)x = y$ and $(\lambda - K_n)x_n = y$,

$$\begin{aligned} \|x - x_n\|_\infty &\leq \|(\lambda - K_n)^{-1}\| \|K - K_n\| \|x\|_\infty \\ &\leq c_s \|K - K_n\| \|x\|_\infty, \quad n \geq N. \end{aligned} \tag{5.16}$$

From the error bound in Equation (5.16), the speed with which $\|x - x_n\|_\infty$ converges to zero is bounded by the numerical integration error given by:

$$\|(K - K_n)x\|_\infty = \max_{t \in D} \left| \int_D K(t, s)x(s)ds - \sum_{j=1}^n w_j K(t, t_j)x(t_j) \right|.$$

Now, we have $(\lambda - K)x = y$ and $(\lambda - K_n)x_n = y$ and therefore $(\lambda - K)x = (\lambda - K_n)x_n$.

Therefore, we have

$$\begin{aligned}(\lambda - K_n)(x - x_n) &= \lambda x - K_n x - (\lambda - K_n)x_n \\ &= \lambda x - K_n x - (\lambda - K)x \\ &= (K - K_n)x\end{aligned}$$

and then,

$$\|(K - K_n)x\|_\infty = \|(\lambda - K_n)(x - x_n)\|_\infty \leq \|\lambda - K_n\|_\infty \|x - x_n\|_\infty. \quad (5.17)$$

Then combining Equation (5.17) with the error bound in Equation (5.16), we obtain

$$\frac{1}{\|\lambda - K_n\|_\infty} \|(K - K_n)x\|_\infty \leq \|x - x_n\|_\infty \leq c_s \|(K - K_n)x\|_\infty, \quad \text{for all } n > N,$$

which shows that $\|x - x_n\|_\infty$ and $\|(K - K_n)x\|_\infty$ have the same rate of convergence to zero. Thus the rate of convergence of $\|x - x_n\|_\infty$ is the same as the rate of convergence of a quadrature rule of the form in Equation (5.9) to the exact value of the integral:

$$\int_D K(t, s)x(s)ds.$$

Error bounds for quadrature rules

The error bounds for the quadrature rules given in this subsection were obtained from Atkinson (1989, 1997). The error bounds are obtained by finding the lowest degree monomial that is not integrated exactly by the rule. For the trapezoidal rule, the lowest degree not integrated exactly is s^2 , for Simpson's rule the lowest degree is s^4 and for a Gaussian rule with n nodes the lowest degree is s^{2n} .

Composite Trapezoidal Rule For a function $g \in C^2([a, b])$ the error in the trapezoidal rule approximation to $\int_a^b g(s)ds$ for a uniform grid of $(n + 1)$ points with $h = \frac{b-a}{n}$ is given by:

$$\int_a^b g(s)ds - T(g, h) = -\frac{b-a}{12} h^2 f''(\xi_n) \quad \text{for some } \xi_n \in (a, b),$$

where $T(g, h)$ is the composite trapezoidal rule formula given in Equation (5.5).

Composite Simpson's Rule For a function $g \in C^4([a, b])$ the error in the Simpson's rule approximation to $\int_a^b g(s)ds$ for a uniform grid of $(2n + 1)$ points with $h = \frac{b-a}{2n}$ is given

by:

$$\int_a^b g(s)ds - S(g, h) = -\frac{b-a}{180}h^4 f^4(\xi_n) \quad \text{for some } \xi_n \in (a, b),$$

where $S(g, h)$ is the composite Simpson's rule formula given in Equation (5.7).

Gaussian Quadrature Rules For a Gaussian rule with n nodes w_k and weights s_k for a weight function $W(s)$ and integration interval $[a, b]$, a Gaussian rule is of the form:

$$\int_a^b W(s)g(s)ds = \sum_{k=1}^n w_k g(s_k) + E_n,$$

where E_n is the error term. It can be shown (see e.g. Atkinson, 1997) that an error bound in an asymptotic sense is:

$$|E_n| \leq e_n M_{2n}$$

with

$$M_l = \frac{1}{l!} \left(\frac{b-a}{2}\right)^{l+1} \|g^{(l)}\|_\infty$$

and

$$e_n \approx \frac{\pi}{4^n} \quad \text{as } n \rightarrow \infty.$$

For infinitely differentiable integrands, M_l is bounded or tends to zero as $l \rightarrow \infty$. As pointed out by Atkinson (1997), this gives some intuition for why Gaussian rules can be very accurate and lead to a very rapidly convergent method for solving Fredholm Integral Equations.

A comparison of the error terms for the trapezoidal, Simpson's and Gaussian rules suggests that, for the same number of nodes, Gaussian rules will give higher accuracy than Simpson's rule which in turn will give higher accuracy than the trapezoidal rule. We have not given the error term for the midpoint rule. However, it has similar accuracy to the trapezoidal rule.

In the next subsection, we give numerical comparisons of the different rules. For the Gaussian rule, we will only consider the Gauss-Legendre rule as this is the appropriate Gaussian rule for approximation ARL.

5.1.8 Numerical Comparisons

In this part, we compare the numerical results obtained by Integral Equations with results by Monte-Carlo simulations. We solve the integration equations using the four quadrature rules, composite midpoint rule, composite trapezoidal rule, composite Simpson's rule and

Gauss-Legendre rule, discussed above. In the numerical tests we assume that observations are from gamma distribution with parameter $k = 2$ and that a typical set of EWMA chart parameters is $\lambda = 0.05$ and $H = 2.45$. Numerical values for ARL are computed for an in-control parameter value $\alpha_0 = 1$ and numerical values for AD are computed for a range of out-of-control parameter values, $\alpha = 1.1, 1.3, 1.5, 1.7, 1.9, 2.0$. The CPU times have been obtained for each computation. The results of the tests are shown in Table 5.1.

It can be seen from Table 5.1 that all of the Integral Equation methods give results of similar accuracy for ARL and AD for the gamma distribution and that the ARL and AD values are in good agreement with values from the MC simulations. The CPU times for the different quadrature rules are similar with the CPU times in increasing order being Gauss-Legendre, Simpson's, Midpoint, and Trapezoidal rule. This ordering of CPU times agrees with the error estimates for the different quadrature rules given in subsection 5.1.7. Because the Gauss-Legendre rule is expected to give higher accuracy for a given number of nodes than the other rules, we will use it in all further computations.

Table 5.1: Comparison of approximations for ARL and AD from Integral Equations and Monte Carlo Simulation

The $\alpha = 1$ row gives ARL, the other rows give AD values.

The numbers in parentheses () are CPU times in seconds.

(k, α)	Integral Equations				Monte Carlo Simulations
	Midpoint	Trapezoidal	Simpson's rule	Gauss-Legendre	
(2,1.0)	218.789 (31.141)	218.793 (32.110)	218.834 (30.719)	218.854 (27.018)	218.735±0.217
(2,1.05)	114.005 (31.875)	114.017 (33.906)	114.03 (30.075)	114.054 (27.688)	114.021±0.086
(2,1.1)	69.652 (31.735)	69.662 (32.407)	69.667 (30.516)	69.666 (27.006)	69.969±0.064
(2,1.3)	22.541 (31.844)	22.545 (34.969)	22.545 (30.515)	22.544 (26.826)	22.562±0.017
(2,1.5)	12.836 (33.313)	12.837 (33.078)	12.838 (31.703)	12.837 (26.356)	12.833±0.009
(2,1.7)	9.001 (32.291)	9.002 (33.140)	9.002 (31.531)	9.002 (27.145)	9.002±0.006
(2,1.9)	6.988 (32.275)	6.989 (32.890)	6.989 (31.625)	6.989 (27.366)	6.985±0.004
(2,2.0)	6.306 (32.204)	6.307 (33.453)	6.307 (30.672)	6.306 (27.015)	6.309±0.004

5.2 Integral Equations for EWMA Chart

We have not been able to apply the same methods to the gamma and Weibull distributions that we used in Chapter 4 to derive explicit formulas for ARL and AD for the exponential distribution. In this chapter we have developed alternative techniques to compute ARL and AD for these distributions based on numerical solution of Integral Equations. In this section we evaluate the ARL and AD for EWMA charts for the case of gamma and Weibull distributions using the algorithm discussed in Section 3.2. For the integration nodes and weights in Equation (3.13), we use Gauss-Legendre quadrature rule values.

5.2.1 EWMA for Gamma Distribution

In this part, we shall discuss the case when ξ_t have a gamma distribution. The probability density function is then defined by the following function:

$$f(x) = \frac{x^{k-1} e^{-\frac{x}{\alpha}}}{\Gamma(k)\alpha^k}, \quad x \geq 0 \quad \text{and} \quad k, \alpha > 0.$$

The change-point model is the following:

$$\xi_t \sim \begin{cases} \text{Gamma}(k, \alpha_0), & t = 1, 2, \dots, \nu - 1 \\ \text{Gamma}(k, \alpha), & t = \nu, \nu + 1, \dots, \quad \alpha > \alpha_0. \end{cases}$$

1. Comparison of numerical results for ARL and AD for gamma EWMA chart by the Integral Equation method with other methods

In Table 5.2, we compare the numerical results for ARL and AD for a $\text{Gamma}(2, 1)$ obtained from the Integral Equation method with results obtained from MCA and Monte Carlo simulation for parameter values $\lambda=0.01$ and $H=2.15$. The table shows that the outputs obtained by IE and MCA methods are very similar and close to MC results. The choice of method for calculating ARL values should therefore be made based on other factors (e.g. CPU times, available software or ease of programming).

Table 5.2: Comparison of approximations ARL and AD obtained from Integral Equations with MCA and Monte Carlo Simulation: gamma case

(k, α)	Integral Equations	MCA	Monte Carlo Simulations
(2,1.0)	495.915	496.365	495.662±0.509
(2,1.05)	181.825	181.528	181.924±0.215
(2,1.1)	96.237	96.418	96.364±0.078
(2,1.2)	46.821	46.932	46.836±0.031
(2,1.3)	30.486	30.562	30.277±0.177
(2,1.4)	22.591	22.648	22.597±0.012
(2,1.5)	17.978	18.025	18.001±0.092
(2,1.6)	14.962	14.999	14.968±0.007
(2,1.7)	12.840	12.871	12.828±0.062
(2,1.8)	11.267	11.299	11.266±0.005
(2,1.9)	10.054	10.084	10.055±0.005
(2,2.0)	9.091	9.118	9.096±0.041

2. Comparison of EWMA, CUSUM and Shiriyayev-Roberts Charts: Gamma case

For gamma distribution, we have compared results obtained for ARL and AD from the Integral Equation method with the results of Monte Carlo simulations for EWMA, CUSUM and Shiriyayev-Roberts charts. The results are shown in Table 5.3 and Figure 5.1 for a value of $T=1000$, for an in-control parameter value $\alpha = 1.0$ and for out-of-control parameter values α from 1.05 to 2.0. The ARL and AD values for the Integral Equation method were calculated from Equation (3.15). For the gamma EWMA we obtain a pair of optimal parameter values of $\lambda=0.05$, $H=2.6588$, for the CUSUM procedure we used a boundary value $A=4.3105$ and for the Shiriyayev-Roberts procedure we used a boundary value $B=615.5$. The results in Table 5.3 show that for small changes ($\alpha < 1.6$), the AD values obtained from the Integral Equation method and the MC of EWMA charts are less than the values obtained from either CUSUM or Shiriyayev-Roberts. For larger changes, $\alpha > 1.6$, the CUSUM values for AD are less than the other procedures. For the smaller values of α , the EWMA chart gives a smaller value of AD than the Shiriyayev-Roberts chart. For the larger values of α , the values of AD from the EWMA are slightly less than Shiriyayev-Roberts. For given $T = 1000$, the performance of the EWMA chart is superior to CUSUM and Shiriyayev-Roberts charts for small shifts, but that CUSUM performs better than EWMA and Shiriyayev-Roberts for moderate to large shifts. Further, EWMA performs appreciably better than Shiriyayev-Roberts for small shifts and has comparable performance for large shifts.

Table 5.3: The numerical results for ARL and AD obtained from Integral Equation and simulation results of EWMA, CUSUM and Shiriyayev-Roberts charts: gamma case

(k, α)	ARL and AD			
	Integral Equations	Monte Carlo simulations		
		EWMA	CUSUM	SR
boundary	2.6588	2.6588	4.3105	615.5
(2,1.0)	1000.237	999.215±.306	1000.560±.320	1001.480±.308
(2,1.05)	381.025	381.717±.275	456.420±.306	445.750±.295
(2,1.1)	191.260	191.277±.193	373.906±.228	216.365±.062
(2,1.2)	73.578	74.037±.051	103.848±.043	84.017±.021
(2,1.3)	40.824	41.061±.037	51.023±.031	47.099±.010
(2,1.4)	27.404	27.362±.015	31.464±.022	31.989±.006
(2,1.5)	20.463	20.557±.015	21.995±.014	24.373±.004
(2,1.6)	16.311	16.314±.010	16.574±.011	19.739±.003
(2,1.7)	13.573	13.557±.007	13.191±.008	16.634±.003
(2,1.8)	11.640	11.653±.006	10.896±.007	14.458±.002
(2,1.9)	10.207	10.200±.005	9.253±.006	12.752±.002
(2,2.0)	9.104	9.132±.005	8.045±.005	11.482±.002
(2,3.0)	4.588	4.602±.003	3.581±.002	5.997±.001
(2,4.0)	3.237	3.247±.002	2.469±.002	4.219±.001

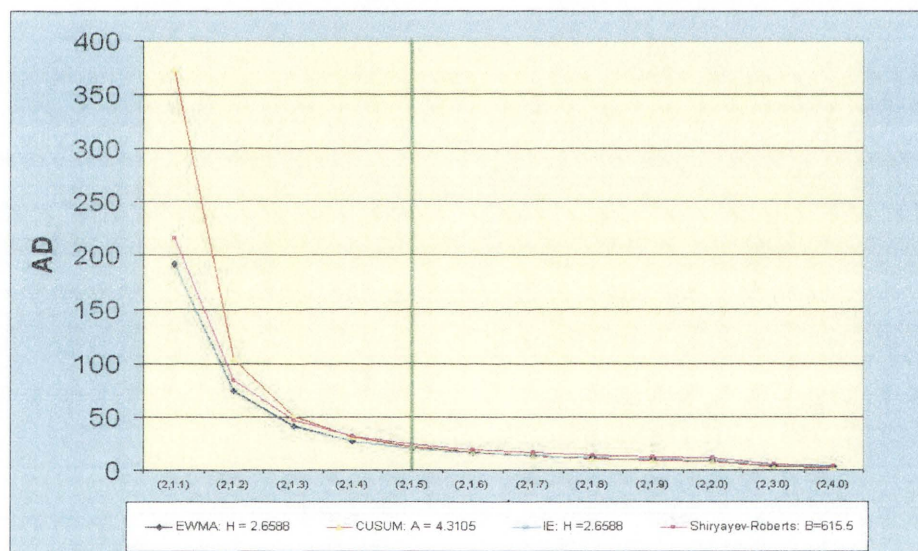


Figure 5.1: Curves of AD for EWMA, CUSUM and Shiriyayev-Roberts charts: gamma case

5.2.2 Choices of Optimal Parameters of Gamma EWMA Procedure

Using the Integral Equation technique, we can easily find the approximation for optimal parameters (λ^*, H^*) of EWMA designs. The parameter values for approximated optimal EWMA designs for values of $T=500, 750$ and 1000 , magnitudes of change $\alpha=1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2$ and in-control parameter $\alpha = 1$ are shown in Table 5.4. This

table gives values for the optimal pairs of parameters (λ^*, H^*) , and the minimal AD^* for each size of changes. Figure 5.2 shows the curves of optimal AD when fixed in-control $T=500$ and 1000 and magnitudes of change $\alpha=1.3, 1.5$ and 1.8.

Table 5.4: Optimal design parameters and AD for one-sided gamma EWMA

(k, α)	$T=500$			
	λ^*	H^*	AD^*	MC
(2,1.1)	0.01	2.1343	84.239	84.297±0.220
(2,1.2)	0.02	2.2770	48.949	48.078±0.111
(2,1.3)	0.03	2.3844	31.053	31.105±0.066
(2,1.4)	0.04	2.4808	22.121	22.120±0.045
(2,1.5)	0.06	2.6535	16.888	16.872±0.035
(2,1.6)	0.08	2.810	13.523	13.529±0.028
(2,1.7)	0.09	2.8841	11.207	11.251±0.023
(2,1.8)	0.11	3.0267	9.539	9.578±0.020
(2,1.9)	0.13	3.1632	8.288	8.282±0.017
(2,2.0)	0.14	3.230	7.320	7.307±0.015
(k, α)	$T=750$			
(2,1.1)	0.01	2.1570	103.675	103.239±0.236
(2,1.2)	0.02	2.3133	58.328	58.582±0.141
(2,1.3)	0.03	2.4272	35.890	36.073±0.075
(2,1.4)	0.04	2.529	25.114	25.176±0.051
(2,1.5)	0.06	2.7115	18.957	18.983±0.039
(2,1.6)	0.07	2.7960	15.055	15.084±0.031
(2,1.7)	0.09	2.9555	12.401	12.406±0.025
(2,1.8)	0.10	3.0320	10.502	10.514±0.021
(2,1.9)	0.12	3.1795	9.084	9.084±0.018
(2,2.0)	0.13	3.2512	7.992	8.003±0.016
(k, α)	$T=1000$			
(2,1.1)	0.01	2.1713	117.315	117.361±0.289
(2,1.2)	0.02	2.3375	65.385	65.351±0.144
(2,1.3)	0.03	2.4558	39.463	39.593±0.081
(2,1.4)	0.04	2.5614	27.290	27.293±0.055
(2,1.5)	0.05	2.6588	20.463	20.454±0.040
(2,1.6)	0.07	2.8383	16.159	16.197±0.033
(2,1.7)	0.08	2.9226	13.261	13.237±0.026
(2,1.8)	0.10	3.0835	11.194	11.182±0.023
(2,1.9)	0.11	3.1610	9.656	9.699±0.019
(2,2.0)	0.12	3.2368	8.477	8.493±0.017

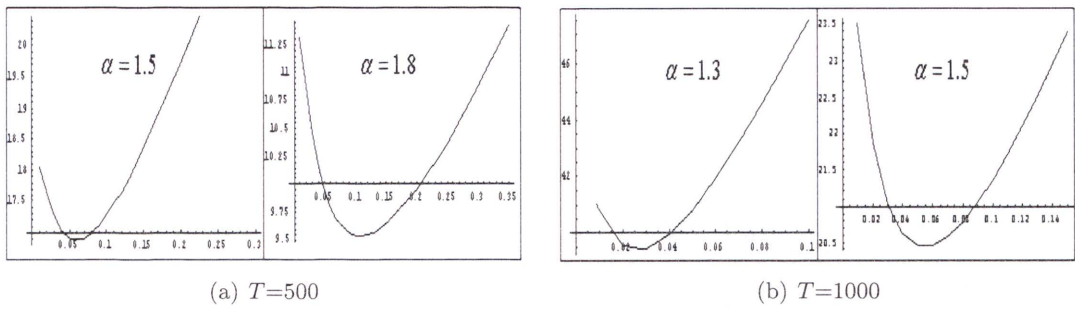


Figure 5.2: Curves of AD for optimal gamma EWMA designs

5.2.3 EWMA for Weibull Distribution

The Weibull distribution has been used to model the lifetimes of objects, including physical system components (Beldica et al., 2002) and also to model distributions of available computer resources (Xu et al., 1999).

Let $\xi_t, t = 1, 2, \dots$ be sequentially observed independent random variables. The change-point model is the following:

$$\xi_t \sim \begin{cases} Weibull(k, \alpha_0), & t = 1, 2, \dots, \nu - 1 \\ Weibull(k, \alpha), & t = \nu, \nu + 1, \dots, \quad \alpha > \alpha_0. \end{cases}$$

1. Comparison of numerical results for ARL and AD for Weibull EWMA chart by the Integral Equation with other methods

The numerical results shown in Table 5.5 for ARL and AD were obtained for a $Weibull(2, 1)$. The parameter values used were $\lambda=0.01, H=0.9351$ and in-control parameter $\alpha=1$. The results obtained by IE method were computed from Equation (3.15). As for the gamma case results shown in Table 5.2, the numerical results for the different methods are all approximately the same.

Table 5.5: Comparison of approximate ARL and AD values obtained from Integral Equations with MCA and Monte Carlo Simulation: Weibull case

(k, α)	Integral Equations	MCA	Monte Carlo Simulations
(2,1.0)	499.577	499.346	492.264±0.504
(2,1.05)	136.815	137.086	136.752±0.178
(2,1.1)	69.386	69.581	69.433±0.051
(2,1.2)	33.198	33.331	33.174±0.019
(2,1.3)	21.686	21.778	21.671±0.011
(2,1.4)	16.142	16.210	16.143±0.008
(2,1.5)	12.896	12.951	12.892±0.005
(2,1.6)	10.769	10.814	10.772±0.004
(2,1.7)	9.268	9.307	9.268±0.004
(2,1.8)	8.153	8.187	8.151±0.003
(2,1.9)	7.293	7.322	7.298±0.003
(2,2.0)	6.609	6.635	6.607±0.003

2. Comparison for EWMA, CUSUM and Shiriyayev-Roberts Charts: Weibull case

We have compared results obtained for ARL and AD from the Integral Equation method with the results of Monte Carlo simulations for EWMA, CUSUM and Shiriyayev-Roberts charts. The results are shown in Table 5.6 and Figure 5.3 for a value of $T=1000$, for an in-control parameter value $\alpha = 1.0$ and for out-of-control parameter values α from 1.05 to 4. The ARL and AD values for the Integral Equation method were calculated from Equation (3.15). For the Weibull EWMA we obtain a pair of optimal parameter values of $\lambda=0.15$, $H=1.3061$, for the CUSUM procedure we used a boundary value $A=4.4931$ and for the Shiriyayev-Roberts procedure we used a boundary value $B=450.20$. The numerical results show that EWMA performs better than Shiriyayev-Roberts and CUSUM for detection of very small shifts, but when $\alpha > 1.3$ CUSUM charts have the best performance with Shiriyayev-Roberts having the next best and EWMA having the worst performance.

Table 5.6: The numerical results for ARL and AD obtained from Integral Equation and simulation results of EWMA, CUSUM and Shiriyayev-Roberts charts: Weibull case

(k, α)	ARL and AD			
	Integral Equations	Monte Carlo simulations		
		EWMA	CUSUM	SR
boundary	1.3061	1.3061	4.4931	450.2
(2,1.0)	999.623	998.842±.313	1000.310±.319	1001.00±.316
(2,1.05)	354.691	353.897±.202	374.604±.219	343.181±.248
(2,1.1)	160.904	157.753±.049	162.191±.050	149.909±.045
(2,1.2)	52.805	51.801±.015	51.842±.015	49.740±.013
(2,1.3)	26.181	25.867±.007	25.489±.006	25.859±.058
(2,1.4)	16.356	16.262±.003	15.926±.004	16.967±.035
(2,1.5)	11.670	11.667±.003	11.370±.002	12.503±.024
(2,1.6)	9.042	9.063±.002	8.802±.002	9.879±.018
(2,1.7)	7.385	7.432±.002	7.157±.001	8.127±.015
(2,1.8)	6.258	6.317±.001	6.033±.001	6.926±.012
(2,1.9)	5.446	5.501±.001	5.195±.001	6.015±.011
(2,2.0)	4.834	4.905±.001	4.585±.001	5.327±.001
(2,3.0)	2.464	2.499±.001	2.247±.001	2.574±.001
(2,4.0)	1.810	1.827±.001	1.650±.001	1.832±.001

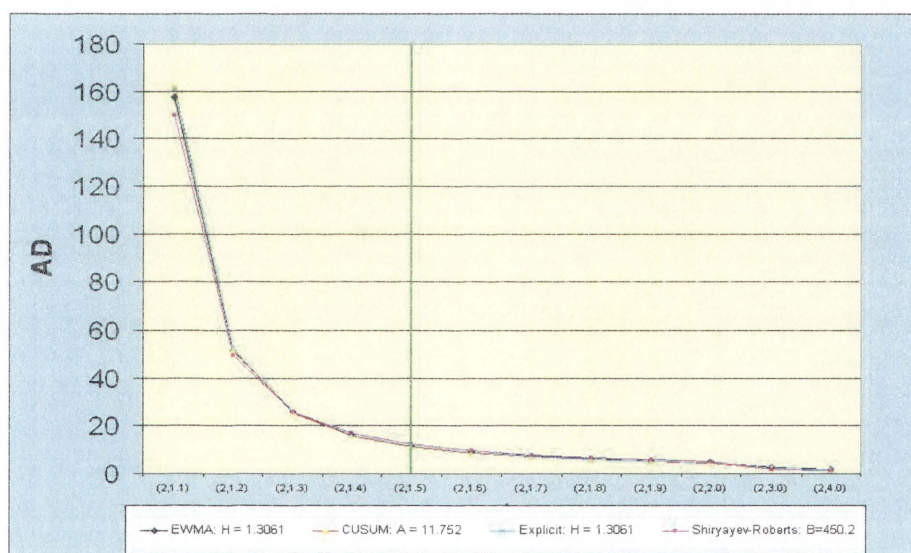


Figure 5.3: Curves of AD for EWMA, CUSUM and Shiriyayev-Roberts charts: Weibull case

5.2.4 Choices of Optimal Parameters of Weibull EWMA Procedure

Table 5.7 contains approximations for optimal values of parameters (λ^*, H^*) when observations are from a Weibull distribution. The values were calculated numerically for the one-sided EWMA case from Equation (3.15). These optimal values were obtained by minimising AD values when fixed ARL values of 500, 750 and 1000, in-control parameter

$\alpha_0=1$ and the sizes of parameter change, $\alpha= 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2$. The numerical results from the Integral Equation approximations closed to the results from the Monte Carlo simulations. A curves of AD for a range of magnitudes of change are shown in Figure 5.4.

Table 5.7: Optimal design parameters and AD for one-sided Weibull EWMA

(k, α)	$T=500$			
	λ^*	H^*	AD*	MC
(2,1.1)	0.01	0.9351	69.386	69.315±0.161
(2,1.2)	0.03	1.0081	33.120	33.084±0.070
(2,1.3)	0.08	1.1331	19.143	19.762±0.044
(2,1.4)	0.12	1.2124	13.443	13.422±0.029
(2,1.5)	0.18	1.3160	9.926	9.942±0.023
(2,1.6)	0.23	1.3947	7.756	7.747±0.018
(2,1.7)	0.28	1.4689	6.308	6.280±0.014
(2,1.8)	0.33	1.5401	5.289	5.285±0.012
(2,1.9)	0.39	1.6232	4.545	4.562±0.01
(2,2.0)	0.45	1.6913	3.985	3.968±0.009
(k, α)	$T=750$			
(2,1.1)	0.01	0.9438	85.986	86.252±0.192
(2,1.2)	0.03	1.0208	38.234	38.259±0.079
(2,1.3)	0.07	1.1286	22.251	22.254±0.048
(2,1.4)	0.11	1.2144	14.940	15.028±0.033
(2,1.5)	0.15	1.2898	10.939	10.952±0.023
(2,1.6)	0.19	1.3459	8.134	8.120±0.018
(2,1.7)	0.26	1.4707	6.858	6.852±0.015
(2,1.8)	0.31	1.5463	5.721	5.722±0.013
(2,1.9)	0.34	1.6049	4.893	4.901±0.011
(2,2.0)	0.40	1.6768	4.271	4.271±0.009
(k, α)	$T=1000$			
(2,1.1)	0.01	0.9501	99.577	99.223±0.268
(2,1.2)	0.03	1.0293	42.001	42.007±0.087
(2,1.3)	0.07	1.1402	24.087	24.123±0.052
(2,1.4)	0.10	1.2077	16.038	16.046±0.034
(2,1.5)	0.15	1.3061	11.670	11.682±0.025
(2,1.6)	0.19	1.3772	9.008	9.015±0.019
(2,1.7)	0.25	1.4764	7.257	7.250±0.016
(2,1.8)	0.29	1.5391	6.032	6.056±0.013
(2,1.9)	0.35	1.6301	5.145	5.145±0.011
(2,2.0)	0.39	1.6893	4.476	4.479±0.010

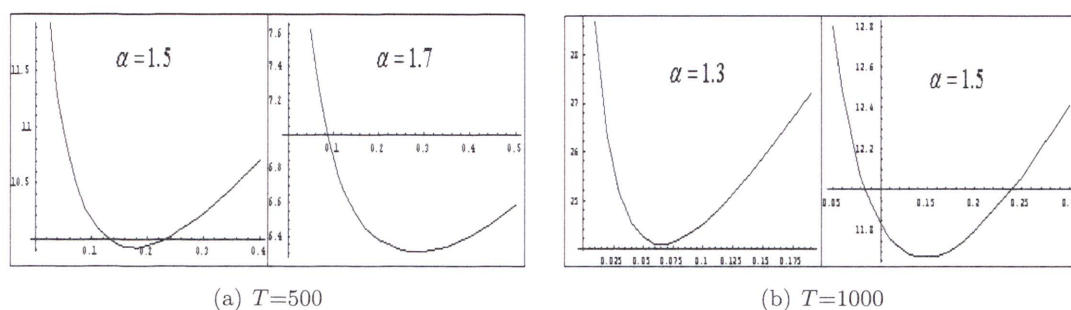


Figure 5.4: Curves of AD for optimal Weibull EWMA designs

5.2.5 EWMA for Two-sided Exponential Distribution

The EWMA control chart for the exponential Distribution was introduced by Gan (1998) who calculated the ARL by using a differential equation. Calzada and Scariano (2003) studied the Integral Equation and Markov Chain Approach for computing the ARL for a two-sided EWMA.

In this section, we develop numerical algorithms for evaluating ARL and AD of the two-sided EWMA for the exponential case.

Recall Equation (3.11)

$$L(u) = 1 + \frac{1}{\lambda} \int_{H_L}^{H_U} L(y) f\left(\frac{y - (1 - \lambda)u}{\lambda}\right) dy.$$

The function $L(u)$ can be approximated by using a quadrature rule as in Equation (3.15).

As for the gamma and Weibull distributions, we have solved the approximate Integral Equations using the Gauss-Legendre rule. In Table 5.8 we compare the results for ARL and AD from the Integral Equation method with the results of Monte Carlo simulation for an example of a two-sided EWMA for an exponential distribution. The results show that the Integral Equation approach can be an effective alternative to MC.

Table 5.8: Comparison of approximation ARL and AD values obtained from Integral Equation for two-sided EWMA with values obtained from Monte Carlo Simulation: exponential case

α	ARL and AD	
	Integral Equations	Monte Carlo Simulation
boundary	$H_L = 0.5, H_U = 1.142$	
1.0	1000.4330	997.3360±0.061
1.1	212.1590	212.5500±0.057
1.2	98.3429	99.2717±0.023
1.3	62.1264	62.1938±0.012
1.4	45.0981	45.0486±0.008
1.5	35.3701	35.3208±0.006
1.6	29.1173	29.1217±0.005
1.7	24.7726	24.7870±0.004
1.8	21.5829	21.5834±0.003
1.9	19.1439	19.1297±0.003
2.0	17.2193	17.2474±0.003

5.3 Integral Equations for CUSUM Chart

We assume that the CUSUM statistic Y_t is defined as in Equation (2.11) where ξ_1, ξ_2, \dots are i.i.d. with a continuous distribution given by a probability density $f(x)$. For example, in the case of exponential distribution it can be shown that

$$Y_t = \max(Y_{t-1} + \log\left(\frac{\alpha_0}{\alpha}\right) + x\left(\frac{1}{\alpha_0} - \frac{1}{\alpha}\right), 0), \quad t = 1, 2, \dots, \quad Y_0 = 0,$$

where α_0 and α are in-control and changed parameters and the stopping time is defined in Equation (2.13). In general, the CUSUM chart can be rewritten as a sequence with

$$U_t = \max(U_{t-1} + \xi_t - k, 0), \quad t = 1, 2, \dots, \quad U_0 = 0.$$

The first passage time of a CUSUM chart is given by:

$$\tau_h = \inf\{t > 0 : Y_t > h\},$$

where h is the control limit, then $k = -\frac{\log(\frac{\alpha_0}{\alpha})}{(\frac{1}{\alpha_0} - \frac{1}{\alpha})}$ and $h = \frac{A}{(\frac{1}{\alpha_0} - \frac{1}{\alpha})}$ when the observations are exponential distributed.

Let $L(u) = \hat{E}_\infty(\tau_h)$ be the ARL of the CUSUM procedure after it is reset at $u \in [0, h]$.

It can be shown (Page, 1954; Rao et al., 2001) that the ARL satisfies the Integral Equation:

$$L(u) = 1 + L(0)F(k - u) + \int_0^h L(x)f(x + k - u)dx, \quad (5.18)$$

where $F(\cdot)$ is the distribution function of the sample statistic.

As shown in Section 3.2 a quadrature rule can be used to approximate the integral equation (5.18) by a solution of system of algebraic linear equations of the form:

$$\tilde{L}(a_i) = 1 + \tilde{L}(a_1)F(k - a_i) + \sum_{j=1}^n w_j \tilde{L}(a_j)f(a_j + k - a_i), \quad i = 1, 2, \dots, n. \quad (5.19)$$

That is

$$\begin{aligned} \tilde{L}(a_1) &= 1 + \tilde{L}(a_1)[F(k - a_1) + w_1 f(k)] + \sum_{j=2}^n \tilde{L}(a_j)w_j f(a_j + k - a_1) \\ \tilde{L}(a_2) &= 1 + \tilde{L}(a_1)[F(k - a_2) + w_1 f(a_1 + k - a_2)] + \sum_{j=2}^n \tilde{L}(a_j)w_j f(a_j + k - a_2) \\ &\vdots \\ \tilde{L}(a_n) &= 1 + \tilde{L}(a_1)[F(k - a_n) + w_1 f(a_1 + k - a_n)] + \sum_{j=2}^n \tilde{L}(a_j)w_j f(a_j + k - a_n). \end{aligned}$$

The above set of n equations in n unknowns can be rewritten in a matrix form as follows.

We define a column vector L with components:

$$L_j = \tilde{L}(a_j), \quad j = 1, 2, \dots, n$$

and an $n \times n$ matrix R with matrix entries given by:

$$\mathbf{R} = \begin{pmatrix} F(k - a_1) + w_1 f(k) & w_2 f(a_2 + k - a_1) & \cdots & w_n f(a_n + k - a_1) \\ F(k - a_2) + w_1 f(a_1 + k - a_2) & w_2 f(a_2 + k - a_2) & \cdots & w_n f(a_n + k - a_2) \\ \vdots & \vdots & \ddots & \vdots \\ F(k - a_n) + w_1 f(a_1 + k - a_n) & w_2 f(a_2 + k - a_n) & \cdots & w_n f(a_n + k - a_n) \end{pmatrix}$$

Equation (5.19) can then be written in the matrix form:

$$(\mathbf{I} - \mathbf{R})\mathbf{L} = \mathbf{1}, \quad (5.20)$$

where \mathbf{I} is an $n \times n$ identity matrix and $\mathbf{1}$ is a column vector of n 1's. On substituting

the values for $\tilde{L}(a_j)$ obtained from solving Equation (5.20) into the right-hand side of Equation (5.19) and replacing the a_i by u in the left-hand side, we obtain an approximation for the function $L(u)$ as

$$\tilde{L}(u) = 1 + \tilde{L}(a_1)F(k - u) + \sum_{j=1}^n w_j \tilde{L}(a_j) f(a_j + k - u). \quad (5.21)$$

Table 5.9 shows a comparison of the numerical results for ARL and AD for exponential CUSUM obtained from Equation (5.21) with the results of a Monte Carlo simulation. Table 5.10 shows a similar comparison for gamma CUSUM. The approximate values for ARL and AD obtained from the Integral Equation method are reasonable when compared with MC values. The numerical Integral Equation method is a useful approach when explicit expressions are not obtainable for ARL and AD.

Table 5.9: Comparison of approximation ARL and AD values obtained from Integral Equation with values from Monte Carlo Simulation: exponential CUSUM case

α	ARL and AD	
	Integral Equations	Monte Carlo Simulation
boundary	$H = 2.0$	
1.0	125.5580	127.4410±0.391
1.1	68.6524	68.5035±0.202
1.2	43.2107	42.8563±0.121
1.3	30.2241	29.9420±0.080
1.4	22.8092	22.6519±0.058
1.5	18.1793	18.1058±0.045
1.6	14.9756	14.9756±0.035
1.7	12.8263	12.8263±0.029
1.8	11.2188	11.2188±0.026
1.9	9.9715	9.9715±0.022
2.0	9.0037	9.0037±0.019

Table 5.10: Comparison of approximation ARL and AD values obtained from Integral Equation with values from Monte Carlo Simulation: gamma CUSUM case

(k, α)	ARL and AD	
	Integral Equations	Monte Carlo Simulation
boundary	$H = 2.5$	
(2,1.0)	145.7420	145.2710±0.447
(2,1.1)	62.2807	63.6040±0.189
(2,1.2)	34.7026	34.8817±0.097
(2,1.3)	22.5154	22.4857±0.058
(2,1.4)	16.2347	16.2674±0.034
(2,1.5)	12.5820	12.5659±0.029
(2,1.6)	10.2526	10.2325±0.023
(2,1.7)	8.6592	8.6400±0.018
(2,1.8)	7.5094	7.5112±0.016
(2,1.9)	6.6442	6.6137±0.013
(2,2.0)	5.9713	5.9647±0.012

Chapter 6

Conclusion and Recommendations for Further Research

6.1 Overall Conclusion

In this thesis, we have presented explicit formulas for ARL and AD of one-sided EWMA charts for the case of an exponential distribution. We have shown that the suggested formulas are easy to calculate and program. The explicit formulas obviously take the computational time much less than other methods such as Markov Chain Approach (MCA), Integral Equation method (IE) and Monte Carlo (MC) simulation methods. The explicit formulas for the exponential distribution can be applied to some other distributions, e.g. the Pareto distribution. Using the explicit formulas, we have been able to provide tables for the optimal weights, boundaries and approximations for ARL and AD for one-sided EWMA charts for the exponential and Pareto distributions.

For gamma and Weibull distributions it is not possible to obtain explicit formulas for ARL and AD by the methods we used for the exponential and Pareto distributions. For the gamma and Weibull distributions we obtained values for ARL and AD by numerical solution of the Integral Equations. We have examined the efficiency of different quadrature rules that can be used for numerical integration. These rules are the midpoint rule, the trapezoidal rule, Simpson's rule and the Gauss-Legendre rule. For most of our calculations we have used the Gauss-Legendre rule because this rule has better error bounds than the midpoint, trapezoidal and Simpson's rules. Using the midpoint rule to solve the equations of the Integral Equation method is equivalent to the Markov Chain Approach (MCA). We have also applied the Integral Equation method to evaluate ARL and AD for two-

sided EWMA charts for the exponential distribution. These numerical results have been compared with MC and found to be very close to MC.

We have compared the effectiveness of the EWMA, CUSUM and Shiriyayev-Roberts procedures for detecting small, moderate and large changes in sequential observations for the exponential, Pareto, gamma and Weibull distributions. The comparison of the control charts has been based on Average Run Length (ARL) and Average Delay (AD) criteria. For all distributions considered in this thesis, we have shown that the performance of EWMA charts is superior to CUSUM and Shiriyayev-Roberts charts for small changes and that the performance of CUSUM charts is superior to EWMA and Shiriyayev-Roberts charts for moderate to large changes.

6.2 Recommendations for Further Research

In this thesis, we have presented explicit analytical formulas and developed numerical algorithms for evaluating important characteristics of EWMA control charts for some distributions. However, there are further studies that can be made in areas of sequential change-point detection. Some possible areas of future study are as follows:

- Explicit formulas could be developed for two-sided EWMA charts when observations are from the exponential distribution. In addition, explicit formulas for finding ARL and AD for the multivariate control chart should be considered.
- In this thesis we have assumed that a change in the process occurs at the very beginning, the so-called zero-state. However, other changes should be considered including the so-called steady-state.
- In this thesis, we have assumed that observations are i.i.d. random variables. However, in real applications they could be serially-correlated observations such as in $AR(1)$, $MA(1)$ etc. Serially-correlated data often appear in finance and insurance. The Integral Equations could be used to analyse this serially-correlated data.

Appendix A

Codes for Simulation and Calculation by Mathematica[®]

The appendix provides all the Mathematica[®] codes used in simulations for the expectation of the first alarm time of EWMA chart. For Monte Carlo simulations the expectation can be approximated by the average of first alarm times observed in trials.

A.1 Exponential case

A.1.1 Simulation ARL for one-sided exponential EWMA chart

```
Clear[ee, X, Y, j, H]; H = 1.1405; n = 10^6; lamp = 0.01;
ee := -Log[N[Random[], 20]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For[i = 1; Y = 1, Y < H, i++, Y = Y(1 - lamp) + lamp1ee], j = j + i - 1,
X = X + (i - 1)^2}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0
```

A.1.2 Simulation AD for one-sided exponential EWMA chart

```
Clear[ee, X, Y, j, H]; H = 1.1405; n = 10^6; lamp = 0.01;
ee := -Log[N[Random[], 20]]; SeedRandom[]; t0 = TimeUsed[];
```

```

X = 0; j = 0;
Do[
{For[i = 1; Y = 1, Y < H , i++, Y = Y(1 - lamp) + lamp1.5ee], j = j + i - 1,
X = X + (i - 1)^2}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.1.3 Simulation ARL for one-sided exponential CUSUM chart

```

Clear[ee, X, Y, j]; H = 3.84; n = 10^6; cc = -Log[1.5]; aa = 1 - 1/1.5;
ee := - Log[Random[]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For[i = 1; Y = 0, Y < H , i++, Y = Max[0, Y + eeaa + cc]], X = X + (i - 1)^2,
j = j + i - 1}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.1.4 Simulation AD for one-sided exponential CUSUM chart

```

Clear[ee, X, Y, j]; H = 3.84; n = 10^6; cc = -Log[1.5]; aa = 1 - 1/1.5;
ee := - Log[Random[]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For[i = 1; Y = 0, Y < H , i++, Y = Max[0, Y + 1.1eeaa + cc]], X = X + (i - 1)^2,
j = j + i - 1}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.1.5 Simulation ARL for one-sided exponential Shiriyayev-Roberts chart

```

Clear[ee, X, Y, j,  $\alpha_0$ ,  $\alpha$ , b];  $\alpha_0 = 1$ ;  $\alpha = 1.5$ ; b = 700.001; n = 10^6;
ee := -Log[Random[]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For [i = 1; Y = 0, Y < b, i++, Y = (1 + Y) ( $\frac{\alpha_0}{\alpha}$ ) Exp [1.0ee ( $\frac{1}{\alpha_0} - \frac{1}{\alpha}$ )]] ,
X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.1.6 Simulation AD for one-sided exponential Shiriyayev-Roberts chart

```

Clear[ee, X, Y, j,  $\alpha_0$ ,  $\alpha$ , b];  $\alpha_0 = 1$ ;  $\alpha = 1.5$ ; b = 700.001; n = 10^6;
ee := -Log[Random[]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For [i = 1; Y = 0, Y < b, i++, Y = (1 + Y) ( $\frac{\alpha_0}{\alpha}$ ) Exp [1.2ee ( $\frac{1}{\alpha_0} - \frac{1}{\alpha}$ )]] ,
X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.2 Pareto case

A.2.1 Simulation ARL for one-sided Pareto EWMA chart

```

<< Statistics`ContinuousDistributions`
Clear[ee, X, Y, j, b]; H = 1.142; n = 10^6; lamp = 0.01; alf = 1; t0 = TimeUsed[];
ee := Random[ParetoDistribution[1, 1/alf]];
SeedRandom[];
X = 0; Y = 1; j = 0;
Do[

```



```

{For[i = 1; Y = 1, Y < H, i++, Y = (1 - lamp)Y + lampLog[ee], X = X + (i - 1)^2,
j = j + i - 1}, {k, 1, n}]
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.2.2 Simulation AD for one-sided Pareto EWMA chart

```

<< Statistics`ContinuousDistributions`
Clear[ee, X, Y, j, H]; b = 1.142; n = 10^6; lamp = 0.01; alf = 1.1; t0 = TimeUsed[];
ee:=Random[ParetoDistribution[1, 1/alf]];
SeedRandom[];
X = 0; j = 0;
Do[
{For[i = 1; Y = 1, Y < b, i++, Y = (1 - lamp)Y + lampLog[ee], X = X + (i - 1)^2,
j = j + i - 1}, {k, 1, n}]
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.2.3 Simulation ARL for one-sided Pareto CUSUM chart

```

<< Statistics`ContinuousDistributions`
Clear[ee, X, Y, j]; H = 4.096; n = 10^6; α0 = 1; α1 = 1/1.5; r = 1; cc = Log[α1];
ee:=Random[ParetoDistribution[r, α0]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For[i = 1; Y = 0, Y < H, i++, Y = Max[0, Y + cc + (α1 - α0)r
+(α0 - α1)Log[ee]], X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}]; av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.2.4 Simulation AD for one-sided Pareto CUSUM chart

```
<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j]; H = 4.096; n = 10^6; α0 = 1; α1 = 1/1.5; r = 1; cc = Log[α1];
ee:=Random[ParetoDistribution[r, α1]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For[i = 1; Y = 0, Y < H, i++, Y = Max[0, Y + cc + (α1 - α0)r
+(α0 - α1)Log[ee]], X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}]; av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0
```

A.2.5 Simulation ARL for one-sided Pareto Shiriyayev-Roberts chart

```
<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j]; b = 4.095; n = 10^6; alf0 = 1; alf1 = 1/1.5; cc = Log [alf1/alf0] ;
ee:=Random[ParetoDistribution[1, alf0]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For [i = 1; Y = 0, Y < b, i++, Y = (1 + Y) (alf1/alf0) ee^(alf0-alf1)] ,
X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
AbsoluteTime[] - a;
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0
```

A.2.6 Simulation AD for one-sided Pareto Shiriyayev-Roberts chart

```
<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j]; b = 4.095; n = 10^6; alf0 = 1; alf1 = 1/1.5; cc = Log [alf1/alf0] ;
ee:=Random[ParetoDistribution[1, alf1]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For [i = 1; Y = 0, Y < b, i++, Y = (1 + Y) (alf1/alf0) ee^(alf0-alf1)] ,
```

```

X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
AbsoluteTime[] - a;
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.3 Gamma case

A.3.1 Simulation ARL for one-sided Gamma EWMA chart

```

<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j, H]; H = 2.1964; n = 10^6; α0 = 1; lamp = 0.01;
ee:=Random[GammaDistribution[2, α0]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[{For[i = 1; Y = 2, Y < H, i++, Y = Y(1 - lamp) + lampee], X = X + (i - 1)^2,
j = j + i - 1}, {k, 1, n}]; AbsoluteTime[] - a;
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.3.2 Simulation AD for one-sided Gamma EWMA chart

```

<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j, H]; H = 2.1964; n = 10^6; α0 = 1.5; lamp = 0.01;
ee:=Random[GammaDistribution[2, α0]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For[i = 1; Y = 2, Y < H, i++, Y = Y(1 - lamp) + lampee], X = X + (i - 1)^2,
j = j + i - 1}, {k, 1, n}]; av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.3.3 Simulation ARL for one-sided Gamma CUSUM chart

```

<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j]; H = 4.3105; n = 10^6; r = 2; α0 = 1; α1 = 1.5;
ee:=Random[GammaDistribution[2, α0]];
SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For[i = 1; Y = 0, Y < H, i++, Y = Max[0, Y + rLog[α0/α1] - ee(1/α1 - 1/α0)]},
X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.3.4 Simulation AD for one-sided Gamma CUSUM chart

```

<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j]; H = 4.3105; n = 10^6; r = 2; α0 = 1; α1 = 1.5;
ee:=Random[GammaDistribution[2, α1]];
SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For[i = 1; Y = 0, Y < H, i++, Y = Max[0, Y + rLog[α0/α1] - ee(1/α1 - 1/α0)]},
X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.3.5 Simulation ARL for one-sided Gamma Shiriyayev-Roberts chart

```

<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j, av]; b = 615; n = 10^6; r = 2; α0 = 1; α1 = 1.5;
ee:=Random[GammaDistribution[2, α0]];
SeedRandom[]; t0 = TimeUsed[];

```

```

X = 0; j = 0;
Do[
{For [i = 1; Y = 0, Y < b, i++, Y = (1 + Y) ( $\frac{\alpha_0}{\alpha_1}$ )r Exp [ee ( $\frac{1}{\alpha_0} - \frac{1}{\alpha_1}$ )]] ,
X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.3.6 Simulation AD for one-sided Gamma Shiriyayev-Roberts chart

```

<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j, av]; b = 615; n = 10^6; r = 2;  $\alpha_0 = 1$ ;  $\alpha_1 = 1.5$ ;
ee:=Random[GammaDistribution[2,  $\alpha_1$ ]];
SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For [i = 1; Y = 0, Y < b, i++, Y = (1 + Y) ( $\frac{\alpha_0}{\alpha_1}$ )r Exp [ee ( $\frac{1}{\alpha_0} - \frac{1}{\alpha_1}$ )]] ,
X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.4 Weibull case

A.4.1 Simulation ARL for one-sided Weibull EWMA chart

```

<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j, H]; H = 0.9501; n = 10^6; lamp = 0.01;  $\alpha = 1$ ; r = 2;
ee:=Random[WeibullDistribution[r,  $\alpha$ ]]; SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For[i = 1; Y = 0.886227, Y < H, i++, Y = Y(1 - lamp) + lampee], j = j + i - 1,
X = X + (i - 1)^2}, {k, 1, n}];
av = N[j/n]

```

```
Sqrt[(X/n - av^2)/n]
```

```
"TimeUsed"
```

```
TimeUsed[] - t0
```

A.4.2 Simulation AD for one-sided Weibull EWMA chart

```
<< Statistics'ContinuousDistributions'
```

```
Clear[ee, X, Y, j, H]; H = 0.9501; n = 10^6; lamp = 0.01;  $\alpha$  = 1.5; r = 2;
```

```
ee:=Random[WeibullDistribution[r,  $\alpha$ ]]; SeedRandom[]; t0 = TimeUsed[];
```

```
X = 0; j = 0;
```

```
Do[
```

```
{For[i = 1; Y = 0.886227, Y < H, i++, Y = Y(1 - lamp) + lampee], j = j + i - 1,
```

```
X = X + (i - 1)^2}, {k, 1, n}];
```

```
av = N[j/n]
```

```
Sqrt[(X/n - av^2)/n]
```

```
"TimeUsed"
```

```
TimeUsed[] - t0
```

A.4.3 Simulation ARL for one-sided Weibull CUSUM chart

```
<< Statistics'ContinuousDistributions'
```

```
Clear[ee, X, Y, j, av]; b = 4.4931; n = 10^6; r = 2;  $\alpha_0$  = 1;  $\alpha_1$  = 1.5;
```

```
ee:=Random[WeibullDistribution[r, 1]];
```

```
SeedRandom[]; t0 = TimeUsed[];
```

```
X = 0; j = 0;
```

```
Do[
```

```
{For [i = 1; Y = 0, Y < b, i++, Y = Max [0, Y + rLog [ $\frac{\alpha_0}{\alpha_1}$ ] + ee^r ( $\frac{1}{\alpha_0^r}$  -  $\frac{1}{\alpha_1^r}$ )]] ,
```

```
X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
```

```
av = N[j/n]
```

```
Sqrt[(X/n - av^2)/n]
```

```
"TimeUsed"
```

```
TimeUsed[] - t0
```

A.4.4 Simulation AD for one-sided Weibull CUSUM chart

```

<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j, av]; b = 4.4931; n = 10^6; r = 2; α0 = 1; α1 = 1.5;
ee:=Random[WeibullDistribution[r, 1.1]];
SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For [i = 1; Y = 0, Y < b, i++, Y = Max [0, Y + r Log [(α0/α1)] + ee^r (1/α0^r - 1/α1^r)]],
X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.4.5 Simulation ARL for one-sided Weibull Shiriyayev-Roberts chart

```

<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j, av]; b = 460; n = 10^6; r = 2; α0 = 1; α1 = 1.5;
ee:=Random[WeibullDistribution[r, α0]];
SeedRandom[]; t0 = TimeUsed[];
X = 0; j = 0;
Do[
{For [i = 1; Y = 0, Y < b, i++, Y = (1 + Y) (α0/α1)^r Exp [(ee)^r (1/α0^r - 1/α1^r)]],
X = X + (i - 1)^2, j = j + i - 1}, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av^2)/n]
"TimeUsed"
TimeUsed[] - t0

```

A.4.6 Simulation AD for one-sided Weibull Shiriyayev-Roberts chart

```

<< Statistics'ContinuousDistributions'
Clear[ee, X, Y, j, av]; b = 460; n = 10^6; r = 2; α0 = 1; α1 = 1.5;
ee:=Random[WeibullDistribution[r, α1]];
SeedRandom[]; t0 = TimeUsed[];

```

```
X = 0; j = 0;
Do[
  {For [i = 1; Y = 0, Y < b, i++, Y = (1 + Y) ( $\frac{\alpha_0}{\alpha_1}$ )r Exp [(ee)r ( $\frac{1}{(\alpha_0)^r} - \frac{1}{(\alpha_1)^r}$ )]]},
  X = X + (i - 1)2, j = j + i - 1, {k, 1, n}];
av = N[j/n]
Sqrt[(X/n - av2)/n]
"TimeUsed"
TimeUsed[] - t0
```


Appendix B

Mathematica[®] Codes for Algorithms of Obtaining Optimal Designs of EWMA

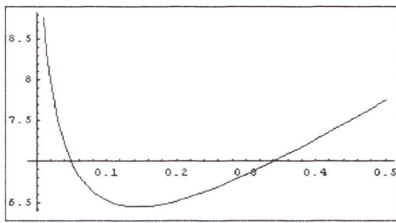
B.1 Procedure for obtaining optimal parameter values

In Appendix B, we show the codes for finding the optimal parameters of EWMA designs (λ^*, H^*) from the explicit expression for the exponential distribution.

One-sided exponential EWMA chart

Step 1: Needs["Statistics"]

```
Clear[lamp, H, alpha, T, QQ, FF]; T = 1000; alpha = 3;
QQ[H_, lamp_, alpha_] := N[Sum[(H/(alpha lamp))^k/(k!) Product[j=1, k-1] (1 - (1 - lamp)^j), {k, 1, 100}]]
+ 1 - N[Sum[((1 - lamp)/(alpha lamp))^k/(k!) Product[j=1, k-1] (1 - (1 - lamp)^j), {k, 1, 100}]];
myinv[lamp_] := H/.FindRoot[N[QQ[H, lamp, 1], 20] == T, {H, 1, 3}];
FF[lamp_] := N[Sum[(myinv[lamp]/(alpha lamp))^k/(k!) Product[j=1, k-1] (1 - (1 - lamp)^j), {k, 1, 100}]]
+ 1 - N[Sum[((1 - lamp)/(alpha lamp))^k/(k!) Product[j=1, k-1] (1 - (1 - lamp)^j), {k, 1, 100}]];
Plot[FF[lamp], {lamp, .01, .5}]
FM = FindMinimum[FF[lamp], {lamp, .01, .5}]
AD = First[FM]
ww = lamp/.Last[FM]
hh = myinv[ww]
```



```
{6.44636, {λ → 0.145325}}
```

```
6.44636
```

```
0.145325
```

```
2.08819
```

Step 2: << **Statistics**'ContinuousDistributions'

```
Clear[ee, Y, j, alf, av, n]; n = 10^6; a = ww; m = α;
```

```
ee := - Log[Random[]]; SeedRandom[]; t0 = TimeUsed[];
```

```
X = 0; j = 0;
```

```
Do[{For[i = 1; Y = 1, Y < hh, i++, Y = Y(1 - a) + amee], X = X + (i - 1)^2,
```

```
j = j + i - 1}, {k, 1, n}]; av = N[j/n]
```

```
Sqrt[(X/n - av^2)/n]
```

```
"TimeUsed"
```

```
TimeUsed[] - t0
```

```
6.44996
```

```
0.0142089
```

```
TimeUsed=6.438
```

Step 3: << **Statistics**'ContinuousDistributions'

```
Clear[ee, Y, j, alf, av, n]; n = 10^6; a = ww; m = 1;
```

```
ee := - Log[Random[]]; SeedRandom[]; t0 = TimeUsed[];
```

```
X = 0; j = 0;
```

```
Do[{For[i = 1; Y = 1, Y < hh, i++, Y = Y(1 - a) + aleee], X = X + (i - 1)^2,
```

```
j = j + i - 1}, {k, 1, n}]; av = N[j/n]
```

```
Sqrt[(X/n - av^2)/n]
```

```
"TimeUsed"
```

```
TimeUsed[] - t0
```

```
994.244
```

3.15602

TimeUsed=815.391

B.2 Calculation codes for obtaining numerical method for Integral Equations

B.2.1 Integral Equation by Midpoint rule for Gamma EWMA

```
<< Statistics'ContinuousDistributions'
Clear[fdist, L, Lu, La, r, s, m, H, a, w, λ, i, j, R]; H = 2.45;
m = 1000; k = 2; α = 2.0; λ = 0.05; h = H/m; t0 = TimeUsed[];
a = Table[(j - 1/2)h, {j, 1, m}];
weights = Table[H/m, {j, 1, m}];
f[x_] := If[x > 0, (x^(k - 1) Exp[-x/α]) / (Gamma[k] α^k), 0];
r[i_, j_] :=  $\frac{\text{weights}[[j]] f[(a[[j]] - (1 - \lambda)a[[i]])/\lambda]}{\lambda}$ ;
R = Table[r[i, j], {i, 1, m}, {j, 1, m}];
newR = IdentityMatrix[m] - R;
R2 = Inverse[newR];
vector1 = Table[1, {i, 1, m}];
La = R2.vector1;
L[u_] = 1 + (1/λ)  $\sum_{j=1}^m \text{weights}[[j]] \text{La}[[j]] f[(a[[j]] - (1 - \lambda)u)/\lambda]$ ;
L[2]
"TimeUsed"
TimeUsed[] - t0
Midpoint
```

6.30613

TimeUsed

32.204

B.2.2 Integral Equation by Gauss-Legendre for Gamma EWMA

```
<< Statistics'ContinuousDistributions'
<< NumericalMath'GaussianQuadrature'
```

```

Clear[L, Lu, La, r, m, H, a, w, λ, i, j, R, k, α]; H = 2.15; m = 500; λ = 0.1; k = 2; α = 1;
t0 = TimeUsed[]; GaussianQuadratureWeights[m, 0, H];
a = GaussianQuadratureWeights[m, 0, H][[All, 1]];
weights = GaussianQuadratureWeights[m, 0, H][[All, 2]];
f[x_] := If[x > 0, (x^(k - 1) Exp[-x/α]) / (Gamma[k] α^k), 0];
r[i_, j_] :=  $\frac{\text{weights}[[j]] f[(a[[j]] - (1 - \lambda) a[[i]]) / \lambda]}{\lambda}$ ;
R = Table[r[i, j], {i, 1, m}, {j, 1, m}];
newR = IdentityMatrix[m] - R;
R2 = Inverse[newR];
vector1 = Table[1, {i, 1, m}];
La = R2.vector1;
L[u_] = 1 + (1/λ)  $\sum_{j=1}^m$  weights[[j]] La[[j]] f[(a[[j]] - (1 - λ) u) / λ];
L[2]
"TimeUsed"
TimeUsed[] - t0
14.3582
TimeUsed=55.484
    
```

B.2.3 Integral Equation by Trapezoidal for Gamma EWMA

```

<< Statistics'ContinuousDistributions'
Clear[Lu, La, r, m, H, a, w, λ, i, j, R]; H = 2.15; m = 500; k = 2; α = 1; λ = 0.1;
h = H/m; t0 = TimeUsed[]; a = Table[jh, {j, 0, m}];
weights = Table[If[1 ≤ j ≤ m - 1, h, h/2], {j, 0, m}];
f[x_] := If[x > 0, (x^(k - 1) Exp[-x/α]) / (Gamma[k] α^k), 0];
r[i_, j_] :=  $\frac{\text{weights}[[j]] f[(a[[j]] - (1 - \lambda) a[[i]]) / \lambda]}{\lambda}$ ;
R = Table[r[i, j], {i, 1, m + 1}, {j, 1, m + 1}];
newR = IdentityMatrix[m + 1] - R;
R2 = Inverse[newR];
vector1 = Table[1, {i, 1, m + 1}];
La = R2.vector1;
L[u_] = 1 + (1/λ)  $\sum_{j=1}^m$  weights[[j]] La[[j]] f[(a[[j]] - (1 - λ) u) / λ];
L[2]
"TimeUsed"
    
```

TimeUsed[] - t0

14.3494

TimeUsed=46.797

B.2.4 Integral Equation by Simpson's rule for Gamma EWMA

```
<< Statistics\ContinuousDistributions
```

```
Clear[fdist, L, Lu, La, r, s, m, H, a, w, λ, i, j, R]; H = 2.45; m = 500; k = 2; α = 2.0;
```

```
λ = 0.05; h = H/(2m); t0 = TimeUsed[];
```

```
a = Table[jh, {j, 0, 2m}];
```

```
weights = Table[If[j==0, h/3, If[j == 2m, h/3,
```

```
If[FractionalPart[j/2] == 0, (2h)/3, (4h)/3]], {j, 0, 2m}];
```

```
f[x_] := If[x > 0, (x^(k - 1) Exp[-x/α]) / (Gamma[k] α^k), 0];
```

```
r[i_, j_] :=  $\frac{\text{weights}[[j]] f[(a[[j]] - (1 - \lambda) a[[i]]) / \lambda]}{\lambda}$ ;
```

```
R = Table[r[i, j], {i, 1, (2m) + 1}, {j, 1, (2m) + 1}];
```

```
newR = IdentityMatrix[(2m) + 1] - R;
```

```
R2 = Inverse[newR];
```

```
vector1 = Table[1, {i, 1, (2m) + 1}];
```

```
La = R2.vector1;
```

```
L[u_] = 1 + (1/λ)  $\sum_{j=1}^{2m+1}$  weights[[j]] La[[j]] f[(a[[j]] - (1 - λ)u)/λ];
```

```
L[2]
```

```
"TimeUsed"
```

TimeUsed[] - t0

Simpson

6.30662

TimeUsed=30.672

B.2.5 Integral Equation by Gauss-Legendre for exponential CUSUM

```
<< Statistics\ContinuousDistributions
```

```
<< NumericalMath\GaussianQuadrature
```

```
Clear[edist, L, Lu, La, r, rr, k, s, m, H, a, w, i, j, R]; H = 3.5/c; m = 400; α0 = 1; α = 1.3;
```

```
k = -Log[α0/α] / ( $\frac{1}{\alpha_0} - \frac{1}{\alpha}$ ); c = ( $\frac{1}{\alpha_0} - \frac{1}{\alpha}$ );
```

```
edist = ExponentialDistribution[1/1.9];
```

```

GaussianQuadratureWeights[m, 0, H];
a = GaussianQuadratureWeights[m, 0, H][[All, 1]];
weights = GaussianQuadratureWeights[m, 0, H][[All, 2]];
f[x_]:=If[x > 0, (1/1.9)Exp[-x/1.9], 0];
rr[i_,j_]:=CDF[edist, (k - a[[i]])] + weights[[j]](f[a[[j]] + k - a[[i]]);
r[i_,j_]:=weights[[j]]f[a[[j]] + k - (a[[i]]/1)];
R = Table[If[j<=1, rr[i, j], r[i, j]], {i, 1, m}, {j, 1, m}];
newR = IdentityMatrix[m] - R;
R2 = Inverse[newR];
vector1 = Table[1, {i, 1, m}];
La = R2.vector1;
L[u_] = 1 + La[[1]]CDF[edist, (k - u)] + Sum_{j=1}^m weights[[j]]La[[j]]f[a[[j]] + k - u];
L[0]
8.49154

```

B.2.6 Integral Equation by Gauss-Legendre for Gamma CUSUM

```

<<Statistics'ContinuousDistributions'
<<NumericalMath'GaussianQuadrature'
Clear[edist, L, Lu, La, r, rr, k, s, m, H, a, w, i, j, R]; p = 2; H = 2.5/c; m = 400; α0 = 1;
α1 = 1.5; k = -(pLog[α0/α1]) / (1/α0 - 1/α1); c = (1/α0 - 1/α1);
edist = GammaDistribution[p, α0];
GaussianQuadratureWeights[m, 0, H];
a = GaussianQuadratureWeights[m, 0, H][[All, 1]];
weights = GaussianQuadratureWeights[m, 0, H][[All, 2]];
f[x_]:=If[x > 0, (x^(p - 1)Exp[-x/α0])/(Gamma[p]α0^p), 0];
rr[i_,j_]:=CDF[edist, (k - a[[i]])] + weights[[j]](f[a[[j]] + k - a[[i]]);
r[i_,j_]:=weights[[j]]f[a[[j]] + k - (a[[i]]/1)];
R = Table[If[j<=1, rr[i, j], r[i, j]], {i, 1, m}, {j, 1, m}];
newR = IdentityMatrix[m] - R;
R2 = Inverse[newR];
vector1 = Table[1, {i, 1, m}];
La = R2.vector1;
L[u_] = 1 + La[[1]]CDF[edist, (k - u)] + Sum_{j=1}^m weights[[j]]La[[j]]f[a[[j]] + k - u];

```

$L[0]$

144.432

B.3 Calculation codes for obtaining numerical results of MCA

B.3.1 Exponential case

```
<< Statistics'ContinuousDistributions'
Clear[n, i, j, x, ARL];
 $\alpha = 1; n = 500; \lambda = 0.1; H = 1.1; t0 = \text{TimeUsed}[];$ 
fdist = ExponentialDistribution[ $\alpha$ ];
transmat1 = Table[CDF[fdist,  $H(2j - (1 - \lambda)(2i - 1)) / (2 * n * \lambda)$ ]
-CDF[fdist,  $H(2(j - 1) - (1 - \lambda)(2i - 1)) / (2 * n * \lambda)$ ], {i, 1, n}, {j, 1, n}];
transmat1//TableForm;
newR1 = IdentityMatrix[n] - transmat1;
R1 = Inverse[newR1];
vectorP = Table[If[j == Floor[n/H] + 1, 1, 0], {j, 1, n}];
vector1 = Table[1, {j, 1, n}];
ARL = vectorP.R1.vector1
"TimeUsed"
TimeUsed[] - t0
14.7308
TimeUsed=24.86
```

B.3.2 Weibull case

```
<< Statistics'ContinuousDistributions'
Clear[n, i, j, x, ARL];
 $\alpha = 1; n = 900; \lambda = 0.01; H = 0.9351; t0 = \text{TimeUsed}[];$ 
fdist = WeibullDistribution[2,  $\alpha$ ];
transmat1 = Table[CDF[fdist,  $H(2j - (1 - \lambda)(2i - 1)) / (2 * n * \lambda)$ ]
-CDF[fdist,  $H(2(j - 1) - (1 - \lambda)(2i - 1)) / (2 * n * \lambda)$ ], {i, 1, n}, {j, 1, n}];
transmat1//TableForm;
```

```
newR1 = IdentityMatrix[n] - transmat1;
R1 = Inverse[newR1];
vectorP = Table[If[j == Floor[0.886227n/H] + 1, 1, 0], {j, 1, n}];
vector1 = Table[1, {j, 1, n}];
ARL = vectorP.R1.vector1
"TimeUsed"
TimeUsed[] - t0
499.346
TimeUsed=995.422
```


Bibliography

- Abramowitz, M. and Stegun, I. A. (1964). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, ninth dover printing, tenth gpo printing edition.
- Anderson, P. and Meerschaert, M. (1998). Modeling river flows with heavy tails. *Water Resources Research*, 34:2271–2280.
- Andrews, G. E., Askey, R., and Roy, R. (1999). *Special Functions*. Cambridge University Press.
- Areepong, Y. and Novikov, A. A. (2008). Martingale approach to EWMA control chart for changes in Exponential distribution. *Journal of Quality Measurement and Analysis*, 4:197–203.
- Atkinson, K. E. (1989). *An Introduction to Numerical Analysis*. Wiley, New York.
- Atkinson, K. E. (1997). *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press.
- Atkinson, K. E. and Han, W. (2005). *Theoretical Numerical Analysis: A Functional Analysis Framework*. Springer Science+Business Media, LLC.
- Baker, C. T. H. (1977). *The Numerical Treatment of Integral Equations*. Oxford, UK:, Clarendon Press.
- Basseville, M. and Nikiforov, I. (1993). *Detection of Abrupt Changes: Theory and Applications*. Prentice-Hall, Englewood Cliffs.
- Beldica, C. E., Hilton, H. H., and Hinrichsen, R. L. (2002). Viscoelastic beam damping and piezoelectric control of deformations, probabilistic failures and survival

- times analytical and massively parallel computational simulations. In *Working Paper*, Bologna. Seventh International Conference on High Performance Computing in Engineering (HPC 2002).
- Borovkov, K. (2003). *Elements of Stochastic Modelling*. World Scientific, New Jersey-Singapore.
- Borrer, C. M., Champ, C. W., and Rigdon, S. E. (1998). Poisson EWMA control charts. *Journal of Quality Technology*, 30:352–361.
- Borrer, C. M., Keats, J. B., and Montgomery, D. C. (2003). Robustness of the time between events CUSUM. *International Journal of Production Research*, 41:3435–3444.
- Borrer, C. M., Montgomery, D. C., and Runger, G. C. (1999). Robustness of the EWMA control chart to Non-normality. *Journal of Quality Technology*, 31:309–316.
- Brook, D. and Evans, D. A. (1972). An approach to the probability distribution of Cusum run length. *Biometrika*, 59:539–548.
- Calzada, M. E. and Scariano, S. M. (2003). Reconciling the Integral Equation and Markov Chain Approach for computing EWMA average run lengths. *Communications in Statistics: Simulation and Computation*, 32:591–604.
- Champ, C. W. and Rigdon, S. E. (1991). A comparison of the Markov chain and the integral equation approaches for evaluating the run length distribution of quality control charts. *Communications in statistics. Simulation and computation*, 20:191–204.
- Crowder, S. V. (1987). A simple method for studying run length distributions of exponentially weighted moving average charts. *Technometrics*, 29:401–407.
- David, P. J. and Rabinowitz, P. (1967). *Numerical Integration*. Blaisdell Publishing Company.
- Ergashev, B. A. (2003). On a CAPM monitoring based on the EWMA procedure. In *Working Paper*. Presented at 9-th International Conference of the Society for Computational Economics and Finance.

- Frisén, M. (1992). Evaluations of methods for statistical surveillance. *Statistics in Medicine*, 11:1489–1502.
- Fu, J. C., Spiring, F. A., and Xie, H. (2002). On the average run lengths of quality control schemes using a Markov chain approach. *Statistics & Probability Letters*, 56:369–380.
- Gan, F. F. (1990a). Monitoring observations generated from a binomial distribution using modified exponentially weighted moving average control chart. *Journal of Statistical Computation and Simulation*, 37:45–60.
- Gan, F. F. (1990b). Monitoring poisson observations using modified exponentially weighted moving average control charts. *Communications in Statistics-Simulation and Computational*, 19:103–124.
- Gan, F. F. (1991). An optimal design of CUSUM quality control charts. *Journal of Quality Technology*, 23:279–286.
- Gan, F. F. (1992). Exact run length distributions for one-sided exponential CUSUM schemes. *Statistica Sinica*, 2:297–312.
- Gan, F. F. (1998). Designs of one-and two-sided exponential EWMA chart. *Journal of Quality Technology*, 30:55–69.
- Golosnoy, V. and Schmid, W. (2006). EWMA control charts for monitoring optimal portfolio weights. *Sequential Analysis*, 26:195–224.
- Hawkins, D. M. and Olwell, D. H. (1998). *Cumulative sum charts and charting for quality improvement*. Springer Verlag, New York, NY.
- Isaacson, E. and Keller, H. B. (1966). *Analysis of Numerical Methods*. Wiley, New York.
- Jacobsen, M. (2007). Exit times for a class of autoregressive sequences and random walks. *Dept. of Applied mathematics and Statistics, University of Copenhagen. Working paper*.
- Knoth, S. (2007). Accurate ARL calculation for EWMA control charts monitoring normal mean and variance simultaneously. *Sequential Analysis*, 26:251–263.
- Kress, R. (1999). *Linear Integral Equations*. Springer-Verlag, New York, 2nd edition.

- Liu, J. Y., Xie, M., Goh, T. N., and Chan, L. Y. (2007). A study of EWMA chart with transformed exponential data. *International Journal of Production Research*, 45:743–763.
- Lorden, G. (1971). Procedures for reacting to a change in distribution. *Annals of Mathematical Statistics*, 42:1897–1908.
- Lucas, J. M. and Saccucci, M. S. (1990). Exponentially weighted moving average control schemes: properties and enhancements. *Technometrics*, 32:1–29.
- Mahmoud, M. A., Woodall, W. H., and Davis, R. E. (2008). Performance comparison of some likelihood ratio-based statistical surveillance methods. *Journal of Applied Statistics*, 35:783–798.
- Mandelbrot, B. (1963). The variation of certain speculative price. *Journal of Business*, 36:394–419.
- Mason, B. and Antony, J. (2000). Statistical process control: an essential ingredient for improving service and manufacturing quality. *Managing Service Quality*, 10:233–238.
- Mazalov, V. V. and Zhuravlev, D. N. (2002). A method of Cumulative sums in the problem of detection of traffic changes in computer networks. *Programming and Computer Software*, 28:342–348.
- Mevorach, Y. and Pollak, M. (1991). A small sample size comparison of the CUSUM and Shirayev-Roberts approaches to change point detection. *American Journal of Mathematical and Management Sciences*, 11:277–298.
- Montgomery, D. C. (2005). *Introduction to Statistical Quality Control*. Chichester: Wiley, New York, 5 th edition.
- Moustakides, G. V. (1986). Optimal stopping times for detecting changes in distributions. *Annals of Statistics*, 14:1379–1387.
- Moustakides, G. V., Polunchenko, A. S., and Tartakovsky, A. G. (2008). A numerical approach to comparative efficiency analysis of quickest change-point detection procedures. *Research paper (see <http://www-scf.usc.edu/~polunche/pdf/Moustakides&Polunchenko&Tartakovsky-SS08.pdf>)*.

- Moustakides, G. V., Polunchenko, A. S., and Tartakovsky, A. G. (2009). Numerical comparison of CUSUM and Shiriyayev-Roberts procedures for detecting changes in distributions. *Communications in Statistics, Theory and Methods*, submitted.
- Novikov, A. A. (2006). Levy-driven Ornstein-Uhlenbeck processes: survey of results on first passage times. Lecture notes presented at the conference “Stochastic Calculus with Jumps”. University of Angers, May 3-9, 2006.
- Page, E. S. (1954). Continuous inspection schemes. *Biometrika*, 41:100–114.
- Pollak, M. and Siegmund, D. (1985). A diffusion process and its applications to detecting a change in the drift of Brownian motion. *Biometrika*, 72:267–280.
- Pollak, M. and Siegmund, D. (1991). Sequential detection of a change in a normal mean when initial value is unknown. *Annals of Statistics*, 19:394–416.
- Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P. (1992). *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press.
- Rao, B. V., Disney, R. L., and Pignatiello, J. J. (2001). Uniqueness and convergence of solutions to average run length integral equations for cumulative sum and other control charts. *IIE Transactions*, 33:463–469.
- Resnick, S. and Starica, C. (1995). Consistency of Hill’s estimator for dependent data. *Journal of Applied probability*, 32:139–167.
- Roberts, S. W. (1959). Control chart tests based on geometric moving average. *Technometrics*, 1:239–250.
- Roberts, S. W. (1966). A comparison of some control chart procedures. *Technometrics*, 8:411–430.
- Robinson, P. B. and Ho, T. Y. (1978). Average run lengths of Geometric moving average charts by numerical methods. *Technometrics*, 20:85–93.
- Ross, S. M. (1996). *Stochastic Processes*. Wiley, New York, 2nd edition.
- Ross, S. M. (1997). *Introduction to Probability Models*. Academic Press.
- Shewhart, W. A. (1931). *Economic Control of Quality of Manufactured Product*. Van Nostrand, New York.

- Shiryayev, A. N. (1963). On optimum methods in quickest detection problems. *Theory Probability and Its Applications*, 8:22–46.
- Shiryayev, A. N. (1996). Minimax optimality of the method of cumulative sums (CUSUM) in the case of continuous time. *Russian Mathematical Surveys*, 51:750–751.
- Sitter, R. R., Hanrahan, L., DeMets, D., and Anderson, H. (1990). A monitoring system to detect increased rates of cancer incidence. *American Journal of Epidemiology*, 132:123–130.
- Somerville, S. E., Montgomery, D. C., and Runger, G. C. (2002). Filtering and smoothing methods for mixed particle count distributions. *International Journal of Production Research*, 40:2991–3013.
- Srivastava, M. S. and Wu, Y. (1993). Comparison of EWMA, CUSUM and Shiryayev-Roberts procedure for detecting a shift in the mean. *Annals of Statistics*, 21:645–670.
- Srivastava, M. S. and Wu, Y. (1997). Evaluation of optimum weights and average run lengths in EWMA control schemes. *Communications in Statistics: Theory and Methods*, 26:1253–1267.
- Stoumbos, Z. G. and Reynolds, M. R. (2000). Robustness to non-normality and autocorrelation of individuals control charts for monitoring the process mean and variance. *Journal of Statistical Computation and Simulation*, 66:145–187.
- Stroud, A. H. and Secrest, D. (1966). *Gaussian Quadrature Formulas*. Prentice-Hall, Englewood Cliffs, N.J.
- Sukparungsee, S. and Novikov, A. A. (2006). On EWMA procedure for detection of a change in observations via martingale approach. *KMITL Science Journal; An International Journal of Science and Applied Science*, 6:373–380.
- Sukparungsee, S. and Novikov, A. A. (2007). Analytical approximations for average run lengths in EWMA charts in case of light-tailed distributions. In *International Conference of Mathematical Sciences*, Bangi-Putrajaya, Malaysia (ICMS 2007).
- Vardeman, S. and Ray, D. (1985). Average run lengths for CUSUM when observations are exponentially distributed. *Technometrics*, 27:145–150.

- Wieringa, J. E. (1999). *Statistical process control for serially correlated data*. Phd thesis, University of Groningen, Groningen, the Netherlands.
- Woodall, W. H. and Adams, B. M. (1993). The statistical design of CUSUM charts. *Quality Engineering*, 5:559–570.
- Xie, M., Goh, T. N., and Ranjan, P. (2002). Some effective control chart procedures for reliability monitoring. *Reliability Engineering and System Safety*, 77:143–150.
- Xu, J., Kalbarczyk, Z., and Iyer, R. K. (1999). Networked windows NT system field failure data analysis. In *PROC '99: Proceedings of the 1999 Pacific Rim International Symposium on Dependable Computing*, page 178, Washington, DC, USA. IEEE Computer Society.
- Yashchin, E. (1987). Some aspects of the theory of statistical control schemes. *IBM Journal of Research and Development*, 31:199–205.
- Ye, N., Borrer, C., and Zhang, Y. (2002). EWMA techniques for computer intrusion detection through anomalous changes in event intensity. *Quality and Reliability Engineering International*, 18:443–451.
- Ye, N., Vilbert, S., and Chen, Q. (2003). Computer intrusion detection through EWMA for autocorrelated and uncorrelated data. *IEEE Transactions on Reliability*, 52:75–82.
- Zhang, C. W., Xie, M., Liu, J. Y., and Goh, T. N. (2007). A control chart for the Gamma distribution as model of time between events. *International Journal of Production Research*, 45:5649–5666.