# Lie Symmetry Methods for Multidimensional Linear Parabolic PDEs and Diffusions 

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## Certificate of Authorship/ Originality

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text. I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

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## Introduction

Sophus Lie first developed and applied the theory of symmetry groups to differential equations in the 1880s, see for example [57]. A symmetry group of a system of differential equations is a group of transformations that maps solutions of the system to other solutions, allowing complex solutions to be obtained from trivial solutions. ${ }^{1}$ The books by Olver [62], Hydon[40] and Bluman and Kumei [12], provide a rigorous account of Lie group methods for differential equations. These methods provide a systematic, mechanical computational algorithm which allows us to determine explicitly the symmetry group of any system of differential equations. In fact, software packages exist which partially automate such computations. See the book [15] by Cantwell for an example as well as [7] and [61] for a discussion of the computation of symmetries.

The applications of Lie symmetry groups have been extensive. G. Birkhoff did famous work on hydrodynamics using techniques which are now considered a part of Lie symmetry analysis in [6]; there have also been applications in general relativity (e.g. [34] and[41]); quantum mechanics [51] and many other areas. See the references in [62] and the extensive list of applications in the CRC Handbooks [43], [44], [45] for some idea of the scope of the subject. For applications to ordinary differential equations see Ibragimov's book $[\mathbf{4 6}]$ as well as [12] and [62].

The major aim of this thesis is to use symmetry group methods to construct fundamental solutions for classes of parabolic equations in higher dimensions.

[^0]Symmetries have been successfully applied to the problem of finding fundamental solutions for parabolic equations on the real line. For onedimensional problems, a range of methods are available, each of which has its own strengths and limitations. Bluman used group-invariant solutions to obtain fundamental solutions, see the book [11] for the details. Ibragimov and Gazizov found the fundamental solution of the Black-Scholes equation using group invariant solutions in [31]. See also Ibragimov's paper [42] and the book by Hill [37].

Lie proved that any parabolic PDEs on the line with a six-dimensional Lie symmetry algebra can be reduced to the heat equation and Bluman used symmetry analysis to explicitly compute the change of variables, [10]. See alsó [7] for more recent developments in this area. Goard has given an extensive treatment of the calculation of fundamental solutions using reduction to canonical form in [33]. For parabolic PDEs in one space dimension, Craddock [26], Craddock and Platen [23], and Craddock and Lennox [22], developed techniques that allow for the construction of integral transforms of fundamental solutions by symmetry analysis.

Finding closed-form expressions for fundamental solutions for higherdimensional PDEs is a significantly harder problem. The standard techniques that are applied in the one-dimensional case typically do not work in higher dimensions for the classes of equations we consider in this thesis.

The method of group-invariant solutions relies on reducing the number of variables in the equation. Typically we can reduce the number of variables by one for each one-parameter group of symmetries. For a
two-variable PDE it is easy to reduce the equation to an ODE. However, if we have a four-variable PDE and only a two-parameter group action, we can only reduce the problem to solving a PDE with two variables by classical group invariance techniques.

In this thesis we are interested in PDEs of the form

$$
\begin{equation*}
u_{t}=\Delta u+A(x) u, x \in \Omega \subseteq \mathbb{R}^{n} \tag{0.0.1}
\end{equation*}
$$

Here $\Delta$ is the $\mathbb{R}^{n}$ Laplacian and

$$
\begin{equation*}
A(x)=\frac{1}{x_{1}^{2}} K\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right)+\sum_{i=1}^{n}\left(c_{i} x_{i}^{2}+a_{i} x_{i}\right)+E, a_{i}, c_{i}, E \in \mathbb{R} \tag{0.0.2}
\end{equation*}
$$

where $K$ is an arbitrary continuous function and $c_{1}, \ldots, c_{n},, a_{1}, \ldots, a_{n}$ and $E$ are arbitrary constants. Note that we can take $E=0$ without loss of generality, by letting $v=e^{E t} u$ in the equation. If $A$ is of the form (0.0.2) then (0.0.1) has nontrivial symmetries. The Lie symmetries for (0.0.1) for $n=2$ were calculated by Finkel, [30].

The change of variables $u=e^{\phi} v$ converts (0.0.1) to

$$
\begin{equation*}
v_{t}=\Delta v+2 \nabla \phi \cdot \nabla v+B(x) v \tag{0.0.3}
\end{equation*}
$$

where

$$
\begin{equation*}
B(x)=\Delta \phi+|\nabla \phi|^{2}+A(x) \tag{0.0.4}
\end{equation*}
$$

Equations of this form are of importance in the theory of stochastic processes, since the transition probability density for the processes governed by the stochastic differential equations

$$
\begin{equation*}
d X_{t}^{i}=2 \phi_{x_{i}}\left(X_{t}^{1}, \ldots, X_{t}^{n}\right) d t+\sqrt{2} d W_{t}^{i}, i=1, \ldots, n \tag{0.0.5}
\end{equation*}
$$

is a fundamental solution of the Kolmogorov backward equation (0.0.3) for $B=0$. We provide some applications in stochastic calculus later in the thesis.

Other equations can be studied by different changes of variables. We also note that the methods introduced in this thesis can also be applied to other types of equations, but we will restrict our analysis to the study of (0.0.1).

Finding fundamental solutions has proved problematic for equations of the form (0.0.1) for $n \geq 2$, since typically we have only $S L(2, \mathbb{R}) \times \mathbb{R}$ as the symmetry group. For the case when $a_{i}=c_{i}=0, i=1, \ldots, n$ the symmetry group of the $\operatorname{PDE}(0.0 .1)$ is $S L(2, \mathbb{R}) \times \mathbb{R}$ for all $n$. For such equations, we do not have enough one-parameter subgroups to obtain fundamental solutions using group invariance techniques, or construct integral transforms of fundamental solutions.

The major contribution of this thesis is to present a method for constructing explicit fundamental solutions for higher-dimensional PDEs for the cases where there are only $S L(2, \mathbb{R})$ symmetries. In two space dimensions the problem can be regarded as completely solved and we can obtain useful expressions in $n$ dimensions.

In Chapter 1 we provide the background material used to obtain the results in the thesis. The material covered includes Lie groups and symmetry groups, PDEs, Fourier transforms, fundamental solutions, Sturm-Liouville theory, stochastic calculus and representation theory. For the reader unfamiliar with Lie symmetry analysis, we provide a detailed derivation of the symmetries in three dimensions for the type of equations we are considering. There is also a discussion of work
done by previous authors on obtaining fundamental solutions by Lie symmetry analysis.

Chapter 2 covers some of the results established for the one- dimensional problem. This chapter contains results already published jointly with Craddock and provides essential background for the extension of the integral transform method which is discussed in Chapter 3. We show how to obtain closed-form expressions for fundamental solutions for parabolic PDEs on the line using the integral transform method. We also obtain some expressions for fundamental solutions using group invariance methods. The functionals computed at the end of the chapter appear however to be new.

In Chapter 3 the first substantive new results appear. We present some special cases where we can obtain Fourier transforms of fundamental solutions for higher-dimensional problems. We also introduce a new $n$-dimensional process which generalises Bessel processes to higher dimensions.

Chapter 4 presents the major results of the thesis. In this chapter we develop new methods for obtaining fundamental solutions for a rich class of multidimensional equations in the case when we only have $S L(2, \mathbb{R})$ symmetries. For the class of PDEs under study we obtain a series expansion of the desired solution. The most complete results are obtained for the $n=2$ case. We also consider higher dimensional examples in detail.

Chapter 5 contains some new applications. We extend results of Craddock and Craddock and Dooley connecting Lie symmetries with classical representation theory, to higher-dimensional problems. We also present some results on the pricing of derivatives. We conclude
with some applications to the calculation of functionals for Itô diffusions. Examples are provided throughout to illustrate the results. Unless otherwise specified, the results presented in the body of this thesis are due to the author.

## Mathematical Preliminaries

### 1.1. Lie Symmetry Groups

In this section we introduce the basic ideas and methods of Lie symmetry analysis. The presentation is not intended as a detailed discussion of either Lie groups or Lie symmetries. Rather we present only the material that is needed for analysis in the thesis. For a more thorough discussion of these topics, we refer the reader to texts such as Olver [62], Hydon [40] or Bluman and Kumei [12]. A Lie group is a group which possesses the structure of a smooth manifold. Lie groups often arise as groups of transformations. Elementary examples of the types of transformations which are associated with Lie groups are scalings, translations and rotations. In this context we are interested in local groups of transformations acting on objects, such as a system of differential equations, or symmetry groups, which are defined on open subsets of $\mathbb{R}^{m}$. We now provide the definition of a Lie group taken from [62].

Definition 1.1.1. An $r$-parameter Lie group is a group which is also an $r$-dimensional smooth manifold, such that the group operation

$$
m: G \times G \rightarrow G, \quad m(g, h)=g \cdot h, \quad g, h \in G
$$

and the inversion

$$
i: G \rightarrow G, \quad i(g)=g^{-1}, \quad g \in G
$$

are smooth maps between manifolds.

We now give some examples of Lie groups.

Example 1.1.1. The group $G=\mathbb{R}^{n}$, where the group multiplication is given by addition is a Lie group. The inversion map is $x \rightarrow-x$ and the identity element is $x=0$.

Example 1.1.2. Let $G=G L(n, \mathbb{R})$, the set of all invertible $n \times n$ matrices with real entries. The group operation is given by matrix multiplication, the identity element is the identity matrix $\mathbb{I}$, and the inverse of a matrix $A$ is the ordinary matrix inverse, which again has real entries.

Example 1.1.3. The $2 n+1$-dimensional Heisenberg group is defined as

$$
H_{2 n+1}=\mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R}
$$

If

$$
(\mathbf{a}, \mathbf{b}, c) \in H, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^{n}, \quad c \in \mathbb{R}
$$

and

$$
\left(\mathbf{a}^{\prime}, \mathbf{b}^{\prime}, c^{\prime}\right) \in H
$$

then multiplication in this group is given by

$$
(\mathbf{a}, \mathbf{b}, c) \cdot\left(\mathbf{a}^{\prime}, \mathbf{b}^{\prime}, c^{\prime}\right)=\left(\mathbf{a}+\mathbf{a}^{\prime}, \mathbf{b}+\mathbf{b}^{\prime}, c+c^{\prime}+\frac{1}{2}\left(\mathbf{a} \cdot \mathbf{b}^{\prime}-\mathbf{a}^{\prime} \cdot \mathbf{b}\right)\right)
$$

Here, $\mathbf{a} \cdot \mathbf{b}$ is the usual dot product. This group is important in Quantum mechanics and many other areas. See [39].

Example 1.1.4. The group $S L(2, \mathbb{R})$ is the group of $2 \times 2$ matrices of determinant 1. This is also a Lie group. The $S L(2, \mathbb{R})$ group plays an important role in this thesis.
1.1.0.1. Calculating Symmetry Groups of PDEs. A symmetry group of a system of differential equations is a group of transformations $G$ acting on the independent and dependent variables of the system such that it maps solutions of the equation to other solutions.

Definition 1.1.2 (Symmetry of a Differential Equation). If $\mathcal{H}_{P}$ denotes the space of all solutions of the PDE

$$
P\left(x, D^{\alpha} u\right)=0
$$

then a symmetry $\mathcal{S}$ is an epimorphism of $\mathcal{H}_{P}$, i.e $\mathcal{S}: \mathcal{H}_{P} \rightarrow \mathcal{H}_{P}$. Thus if $u \in \mathcal{H}_{P}$, then we must have $\mathcal{S} u \in \mathcal{H}_{P}$.

An elementary example of a symmetry of the heat equation is given below.

Example 1.1.5. If $u(x, t)$ is a solution of the heat equation

$$
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}
$$

then $u(x+\epsilon, t)$ is also a solution. We will see that PDEs can have many more complex symmetries.

We will consider a PDE of order $n$ in $m$ variables, defined on a simply connected subset $\Omega \subset \mathbb{R}^{m}$. The PDE takes the form

$$
P\left(x, D^{\alpha} u\right)=0
$$

where $P$ is a differential operator on $\Omega \times \mathbb{R}$,

$$
D^{\alpha} u=\frac{\partial^{|\alpha|} u}{\partial x_{1}^{\alpha_{1}} \ldots \partial x_{n}^{\alpha_{m}}}
$$

and $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right)$ is a multi-index.

We would like to determine a symmetry group of this PDE. To this end we introduce the notion of a vector field. For our purposes, a vector field is a first order differential operator, which can be written as

$$
\begin{equation*}
\mathbf{v}=\sum_{k=1}^{m} \xi_{k}(x, u) \frac{\partial}{\partial x_{k}}+\phi(x, u) \frac{\partial}{\partial u} \tag{1.1.1}
\end{equation*}
$$

A more general form of a vector field is used in Olver's Theorem below, but it is only vector fields of the form (1.1.1) which we will need. Symmetries of the type generated by vector fields of this form are called point symmetries. There exist other types of symmetries, such as generalised symmetries and nonlocal symmetries, however we do not consider them in this thesis. See [62] and [7] for more on this topic.

Associated with every Lie group is a Lie algebra.

Definition 1.1.3. A Lie algebra $\mathbf{g}$ is a vector space which is closed under the Lie bracket. That is, if $\mathbf{g}$ is a Lie algebra, and $\mathbf{v}, \mathbf{w} \in \mathbf{g}$, then $[\mathbf{v}, \mathbf{w}] \in \mathbf{g}$, where $[\mathbf{v}, \mathbf{w}]=\mathbf{v w}-\mathbf{w} \mathbf{v}$. Thus for vectors of the form (1.1.1),

$$
[\mathbf{v}, \mathbf{w}] f=\mathbf{v}(\mathbf{w}(f))-\mathbf{w}(\mathbf{v}(f)) .
$$

The product $[\mathbf{v}, \mathbf{w}]$ is called the Lie bracket of $\mathbf{v}$ and $\mathbf{w}$. Here $f$ is a smooth function.

See [49] for more on Lie algebras. Elements of the Lie algebra generate elements of the group by the process of exponentiation, which we will introduce below.

EXAMPLE 1.1.6. Let $M_{2}$ be the space of $2 \times 2$ matrices with real entries. Then the space

$$
\mathfrak{s l}_{2}=\left\{A \in M_{2}: \operatorname{tr}(A)=0\right\}
$$

is a Lie algebra. It can be shown that if $k \in \mathfrak{s l}_{2}$ then $g=\sum_{n=0}^{\infty} \frac{k^{n}}{n!} \in$ $S L(2, \mathbb{R})$. A basis for the Lie algebra is

$$
X_{1}=\left(\begin{array}{cc}
0 & 0 \\
-1 & 0
\end{array}\right), X_{2}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right), X_{3}=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) .
$$

We are interested in Lie algebras of vector fields of the form (1.1.1). The reason for this will be made clear shortly. Every vector field $\mathbf{v}$ generates a one-parameter local Lie group, which is called the flow of $\mathbf{v}$. The flow of $\mathbf{v}$ is often written as $\exp (\epsilon \mathbf{v})$. Consider a function $u$ which depends upon $x$. If we have a vector field of the form (1.1.1), then the group it generates can be thought of as acting on the graph of $u$, which we write $(x, u)$. That is, the flow acts on the graph $(x, u)$, transforming it in some manner. It is not hard to determine exactly how $x$ and $u$ are transformed. If we denote the transformed variables by $(\tilde{x}, \tilde{u})$, then we have

$$
\begin{aligned}
\frac{d \tilde{x}_{k}}{d \epsilon} & =\xi_{k}(\tilde{x}, \tilde{u}), \quad k=1, \ldots, m \\
\frac{d \tilde{u}}{d \epsilon} & =\phi(\tilde{x}, \tilde{u})
\end{aligned}
$$

and $\tilde{x}_{k}(0)=x_{k}, k=1, \ldots, m$ and $\tilde{u}(0)=u$. It is standard practice to write $\exp (\epsilon \mathbf{v})(x, u)=(\tilde{x}, \tilde{u})$. It is this sense in which a vector field generates a one-parameter group action. In this case, the parameter is $\epsilon$. The process of calculating the group action generated by an infinitesimal symmetry is known as exponentiating the symmetry.

Let $\mathcal{G}$ denote the group generated by $\mathbf{v}$. We introduce the $n$th prolongation of $\mathcal{G}$, denoted $\operatorname{pr}^{n} \mathcal{G}$. It is the natural extension of the action of $\mathcal{G}$, to $(x, u)$, and all the derivatives of $u$, up to order $n$. In other words the $n$th prolongation acts on the collection $\left(x, u, u_{x_{1}}, \ldots, u_{x_{m}, \ldots, x_{m}}\right)$, where
the order of the highest derivative is $n$. More formally, we define the $n$th prolongation as follows.

Definition 1.1.4. To determine $\operatorname{pr}^{n} \mathcal{G}$, let $\mathcal{D}^{n}$ be the mapping

$$
\mathcal{D}^{n}:(x, u) \longmapsto\left(x, u, u_{x_{1}}, \ldots, u_{x_{m}, \ldots, x_{m}}\right)
$$

Then the $n$th prolongation must satisfy

$$
\mathcal{D}^{n} \circ \mathcal{G}=\operatorname{pr}^{n} \mathcal{G} \circ \mathcal{D}^{n}
$$

The technical definition of the $n$th prolongation is essentially a statement that the action of the group and differentiation commute with one another in a certain sense. That is, if we act with $\mathcal{G}$ on $x$ and $u(x)$, then write down all the derivatives of the new function $\tilde{u}(\tilde{x})$ up to order $n$, the result should be the same as writing down the derivatives of $u(x)$ up to order $n$, then acting on this set with the $n$th prolongation of $\mathcal{G}$. This condition requires that the chain rule of multivariable calculus holds. The $n$th prolongation of $\mathcal{G}$ also has an infinitesimal generator, which we denote $\operatorname{pr}^{n} \mathbf{v}$. The following definition is taken from [62].

Definition 1.1.5. Let $\mathbf{v}$ be a vector field with corresponding local one-parameter group $\exp (\epsilon \mathbf{v})$. The $n$th prolongation of $\mathbf{v}$, denoted $\mathrm{pr}^{(n)} \mathbf{v}$ is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $\operatorname{pr}^{(n)}[\exp (\epsilon \mathbf{v})]$. That is,

$$
\left.\operatorname{pr}^{(n)} \mathbf{v}\right|_{\left(x, u^{(n)}\right)}=\left.\frac{d}{d \epsilon}\right|_{\epsilon=0} \operatorname{pr}^{(n)}[\exp (\epsilon \mathbf{v})]\left(x, u^{(n)}\right)
$$

The reason for introducing prolongations is that the concept is fundamental to the calculation of symmetry groups, as we shall now see.

The central result of the theory of symmetry groups is Lie's theorem. It provides the necessary and sufficient conditions for a Lie group $\mathcal{G}$, with infinitesimal generator (1.1.1), to be a symmetry group.

Theorem 1.1.6 (Lie). Let

$$
\begin{equation*}
P_{q}\left(x, D^{\alpha} u\right)=0, \quad q=1, \ldots, d \tag{1.1.2}
\end{equation*}
$$

be a system of $d$, nth-order partial differential equations. Let $\mathbf{v}$ be a vector field of the form

$$
\mathbf{v}=\sum_{i=1}^{p} \xi^{i}(x, u) \frac{\partial}{\partial x^{i}}+\sum_{\alpha=1}^{q} \phi_{\alpha}(x, u) \frac{\partial}{\partial u^{\alpha}}
$$

Then $\mathbf{v}$ generates a one-parameter group of symmetries of (1.1.2) if and only if

$$
\begin{equation*}
\operatorname{pr}^{n} \mathbf{v}\left[P\left(x, D^{\alpha} u\right)\right]=0 \tag{1.1.3}
\end{equation*}
$$

whenever $P\left(x, D^{\alpha} u\right)=0$.

See [62] for the proof. The symmetries of this type are known as point symmetries. Applying Lie's Theorem to a system of PDEs yields a system of determining equations for the functions $\xi$ and $\phi$. These determining equations can sometimes be solved by inspection, however in most cases a solution of the system can require considerable analysis. Although this theorem is given for systems of differential equations, we are only concerned with a single differential equation. That is, the case $d=1$. The vector fields satisfying (1.1.3) are referred to as infinitesimal symmetries.
1.1.0.2. The Prolongation Formula. The determination of the symmetry groups for a system of differential equations relies upon the prolongation of a group action to the dependent and independent variables
of the system as well as the derivatives of the system. Lie derived an algorithm for computing the prolongation of a vector field. An explicit formula for $\mathrm{pr}^{n} \mathbf{v}$, due to Olver [62], allows for the systematic calculation of symmetry groups of a system of differential equations.

Theorem 1.1.7 (Olver: The General Prolongation Formula). Let

$$
\begin{equation*}
\mathbf{v}=\sum_{i=1}^{p} \xi^{i}(x, u) \frac{\partial}{\partial x^{i}}+\sum_{\alpha=1}^{q} \phi_{\alpha}(x, u) \frac{\partial}{\partial u^{\alpha}} \tag{1.1.4}
\end{equation*}
$$

be a vector field defined on an open subset $M \subset X \times U$. The nth prolongation of $\mathbf{v}$ is the vector field

$$
\begin{equation*}
\operatorname{pr}^{(n)} \mathbf{v}=\mathbf{v}+\sum_{\alpha=1}^{q} \sum_{\boldsymbol{J}} \phi_{\alpha}^{\boldsymbol{J}}\left(x, u^{(n)}\right) \frac{\partial}{\partial u_{\boldsymbol{J}}^{\alpha}} \tag{1.1.5}
\end{equation*}
$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, the second summation being over all (unordered) multi-indices $J=\left(j_{1}, \ldots, j_{k}\right)$, with $1 \leq j_{k} \leq p, 1 \leq k \leq n$. The coefficient functions $\phi_{\alpha}^{\boldsymbol{J}}$ of $\mathrm{pr}^{(n)} \boldsymbol{v}$ are given by the following formula:

$$
\begin{equation*}
\phi_{\alpha}^{\boldsymbol{J}}\left(x, u^{(n)}\right)=\boldsymbol{D}_{\boldsymbol{J}}\left(\phi_{\alpha} \sum_{i=1}^{p} \xi^{i} u_{i}^{\alpha}\right)+\sum_{i=1}^{p} \xi^{i} u_{J, i}^{\alpha}, \tag{1.1.6}
\end{equation*}
$$

where $u_{i}^{\alpha}=\frac{\partial u^{\alpha}}{\partial x^{i}}$, and $u_{J, i}^{\alpha}=\frac{\partial u_{J}^{\alpha}}{\partial x^{i}}$, and $\boldsymbol{D}_{J}$ is the total differentiation operator.

Although the technical details involved in the construction of Lie's theory of symmetry groups are quite sophisticated, application of the main theorems is actually straightforward. Let us illustrate the prolongation formula by an example. Let

$$
\mathbf{v}=\xi(x, t, u) \frac{\partial}{\partial x}+\tau(x, t, u) \frac{\partial}{\partial t}+\phi(x, t, u) \frac{\partial}{\partial u}
$$

be a general vector field on $X \times U \simeq \mathbb{R}^{2} \times \mathbb{R}$. We take $p=2$ and $q=1$ in the prolongation formula (1.1.5), giving

$$
\operatorname{pr}^{1} \mathbf{v}=\mathbf{v}+\phi^{x} \frac{\partial}{\partial u_{x}}+\phi^{t} \frac{\partial}{\partial u_{t}}
$$

This is the first prolongation of $\mathbf{v}$.
The determining functions $\phi^{x}$ and $\phi^{t}$ are found using (1.1.6). This is simply an exercise in differentiation. We have

$$
\begin{aligned}
\phi^{x} & =D_{x}\left(\phi-\xi u_{x}-\tau u_{t}\right)+\xi u_{x x}+\tau u_{x t} \\
& =\left(\phi_{x}+\phi_{u} u_{x}-\xi_{x} u_{x}-\xi_{u} u_{x}^{2}-\xi u_{x x}\right. \\
& \left.-\tau_{x} u_{t}-\tau_{u} u_{x} u_{t}-\tau u_{x t}\right)+\xi u_{x x}+\tau u_{x t} \\
& =\phi_{x}+\left(\phi_{u}-\xi_{x}\right) u_{x}-\xi_{u} u_{x}^{2}-\tau_{x} u_{t}-\tau_{u} u_{x} u_{t}
\end{aligned}
$$

and

$$
\phi^{t}=\phi_{t}-\xi_{t} u_{x}+\left(\phi_{u}-\tau_{t}\right) u_{t}-\xi_{u} u_{x} u_{t}-\tau_{u} u_{t}^{2}
$$

Further, the second prolongation of $\mathbf{v}$ is the vector field

$$
\operatorname{pr}^{2} \mathbf{v}=\mathbf{v}+\phi^{x} \frac{\partial}{\partial u_{x}}+\phi^{t} \frac{\partial}{\partial u_{t}}+\phi^{x x} \frac{\partial}{\partial u_{x x}}+\phi^{x t} \frac{\partial}{\partial u_{x t}}+\phi^{t t} \frac{\partial}{\partial u_{t t}}
$$

where for example

$$
\begin{aligned}
\phi^{x x} & =D_{x x}\left(\phi-\xi u_{x}-\tau u_{t}\right)+\xi u_{x x x}+\tau u_{x x t} \\
& =\phi_{x x}+\left(2 \phi_{x u}-\xi_{x x}\right) u_{x}-\tau_{x x} u_{t} \\
& +\left(\phi_{u u}-2 \xi_{x u}\right) u_{x}^{2}-2 \tau_{x u} u_{x} u_{t}-\xi_{u u} u_{x}^{3} \\
& -\tau_{u u} u_{x}^{2} u_{t}+\left(\phi_{u}-2 \xi_{x}\right) u_{x x} \\
& -2 \tau_{x} u_{x t}-3 \xi_{u} u_{x} u_{x x}-\tau_{u} u_{t} u_{x x}-2 \tau_{u} u_{x} u_{x t} .
\end{aligned}
$$

The coefficients $\phi^{t t}$ and $\phi^{x t}$ can be computed by the same method. Before we present an example of a symmetry calculation, we point out that the set of all infinitesimal generators of symmetries for a PDE forms a Lie algebra. This result was proved by Lie.

Theorem 1.1.8 (Lie). Let

$$
\begin{equation*}
P_{q}\left(x, D^{\alpha} u\right)=0 \quad q=1, \ldots d \tag{1.1.7}
\end{equation*}
$$

be a system of d, nth-order partial differential equations. Let the set of all infinitesimal generators of symmetries of (1.1.7) be $\boldsymbol{g}$. Then $\boldsymbol{g}$ is a Lie algebra. Note that $\boldsymbol{g}$ may be infinite-dimensional.
1.1.1. Symmetries of the $\operatorname{PDE} u_{t}=\Delta u+A(x) u$. In this thesis we will be concerned with equations of the form

$$
\begin{equation*}
u_{t}=\Delta u+A(x) u \tag{1.1.8}
\end{equation*}
$$

on $\mathbb{R}^{n}$. For a linear parabolic PDE of the form (1.1.8) on $\mathbb{R}$, there are only a few possible linearly independent forms that $A$ can take if the PDE is to have nontrivial Lie symmetries. When we move to higher dimensions, this is no longer true. There are now infinitely many possible linearly independent choices of $A$ that will lead to a nontrivial symmetry group. The $n=2$ case was studied by Finkel, see [30]. For completeness, we present here the calculations for the $n=3$ case.
1.1.2. The Lie Symmetries of $u_{t}=\Delta u+f(x, y, z) u$. In accordance with Lie's method, we let

$$
\mathbf{v}=\xi \partial_{x}+\eta \partial_{y}+\zeta \partial_{z}+\tau \partial_{t}+\phi \partial_{u}
$$

then compute the second prolongation and apply it to the PDE to obtain

$$
\begin{equation*}
\phi^{t}=\phi^{x x}+\phi^{y y}+\phi^{z z}+f \phi+\left(f_{x} \xi+f_{y} \eta+f_{z} \zeta\right) u . \tag{1.1.9}
\end{equation*}
$$

This holds whenever $u_{t}=\Delta u+f(x, y, z) u$. Here

$$
\begin{gathered}
\phi^{t}=\phi_{t}+\left(\phi_{u}-\tau-t\right) u_{t}-\xi_{t} u_{x}-\eta_{t} u_{y}-\zeta_{t} u_{z} \\
\phi^{x x}=\phi_{x x}+\left(2 \phi_{x u}-\xi_{x x}\right) u_{x}+\left(\phi_{u}-2 \xi_{x}\right) u_{x x}-\eta_{x x} u_{y}-2 \eta_{x} u_{x y}-\zeta_{x x} u_{z} \\
-2 \zeta_{x} u_{x z}-\tau_{x x} u_{t}-2 \tau_{x} u_{x t}+\phi_{u u} u_{x}^{2}
\end{gathered}
$$

etc. From (1.1.9) we can read off the defining equations. We immediately find as expected that $\tau$ is independent of $x, y, z$ and $u, \xi, \eta, \zeta$ do not depend on $u$, and $\phi$ is linear in $u$. We then have

$$
\begin{aligned}
\phi_{t}+f\left(\phi_{u}-\tau_{t}\right) & =\phi_{x x}+\phi_{y y}+\phi_{z z}+f \phi+\left(f_{x} \xi+f_{y} \eta+f_{z} \zeta\right) u \\
\phi_{u}-\tau_{t} & =\phi_{u}-2 \xi_{x}, \phi_{u}-\tau_{t}=\phi_{u}-2 \eta_{y}, \phi_{u}-\tau_{t}=\phi_{u}-2 \zeta_{z} \\
-\xi_{t} & =2 \phi_{u x}-\xi_{x x}-\xi_{y y}-\xi_{z z} \\
-\eta_{t} & =2 \phi_{u y}-\eta_{x x}-\eta_{y y}-\eta_{z z} \\
-\zeta_{t} & =2 \phi_{u z}-\zeta_{x x}-\zeta_{y y}-\zeta_{z z}
\end{aligned}
$$

Finally $-2 \eta_{x}-2 \xi_{y}=0,-2 \zeta_{y}-2 \eta_{z}=0,-2 \zeta_{x}-2 \xi_{z}=0$. These last equations imply that $\xi_{x x}=\xi_{y y}=\xi_{z z}=0$ and the same holds for $\eta$ and $\zeta$. From $\xi_{x}=\frac{1}{2} \tau_{t}$ we get

$$
\begin{equation*}
\xi=\frac{1}{2} x \tau_{t}+\rho^{1}(y, z, t) \tag{1.1.10}
\end{equation*}
$$

where $\rho^{1}$ is an as yet arbitrary function. And we also find

$$
\begin{equation*}
\eta=\frac{1}{2} y \tau_{t}+\rho^{2}(x, z, t), \zeta=\frac{1}{2} z \tau_{t}+\rho^{3}(x, y, t) \tag{1.1.11}
\end{equation*}
$$

Since $\xi_{x x}=\xi_{y y}=\xi_{z z}=0$, it follows that $\xi$ is linear in $x, y, z$. Similarly for $\eta$ and $\zeta$, so that

$$
\begin{align*}
\xi & =\frac{1}{2} x \tau_{t}+y A^{1}(t)+z A^{2}(t)+A^{3}(t)  \tag{1.1.12}\\
\eta & =\frac{1}{2} y \tau_{t}+x B^{1}(t)+z B^{2}(t)+B^{3}(t)  \tag{1.1.13}\\
\zeta & =\frac{1}{2} z \tau_{t}+x C^{1}(t)+y C^{2}(t)+C^{3}(t) \tag{1.1.14}
\end{align*}
$$

where $A^{1}(t), \ldots, C^{3}(t)$ are functions of $t$ to be determined. We also know that $\xi_{y}=-\eta_{x}, \eta_{y}=-\eta_{z}$ and $\xi_{z}=-\zeta_{x}$ which implies that

$$
\xi_{y}=A^{1}(t)=-\eta_{x}=-B^{1}(t)
$$

and so on, so that

$$
\begin{align*}
\xi & =\frac{1}{2} x \tau_{t}+y A^{1}(t)+z A^{2}(t)+A^{3}(t)  \tag{1.1.15}\\
\eta & =\frac{1}{2} y \tau_{t}-x A^{1}(t)+z B^{2}(t)+B^{3}(t)  \tag{1.1.16}\\
\zeta & =\frac{1}{2} z \tau_{t}-x A^{2}(t)+y B^{2}(t)+C^{3}(t) \tag{1.1.17}
\end{align*}
$$

Next we have $-\xi_{t}=2 \phi_{u x}$, since $\Delta \xi=0$. Since $\phi=\alpha(x, y, z, t) u+$ $\beta(x, y, z, t)$ we immediately get

$$
\begin{equation*}
2 \alpha_{x}=-\frac{1}{2} x \tau_{t t}-y A_{t}^{1}-z A_{t}^{2}-A_{t}^{3} \tag{1.1.18}
\end{equation*}
$$

and from this we obtain

$$
\begin{equation*}
\alpha=-\frac{1}{8} x^{2} \tau_{t t}-\frac{1}{2} x y A_{t}^{1}-\frac{1}{2} z x A_{t}^{2}-\frac{1}{2} x A_{t}^{3}+D^{1}(y, z, t) \tag{1.1.19}
\end{equation*}
$$

From $-\eta_{t}=2 \phi_{y u}$ and $-\zeta_{t}=2 \phi_{z u}$ we can compute alternative expressions for $\alpha$ :

$$
\begin{align*}
& \alpha=-\frac{1}{8} y^{2} \tau_{t t}+\frac{1}{2} x y A_{t}^{1}-\frac{1}{2} y z B_{t}^{2}-\frac{1}{2} y B_{t}^{3}+D^{2}(x, z, t)  \tag{1.1.20}\\
& \alpha=-\frac{1}{8} z^{2} \tau_{t t}+\frac{1}{2} x z A_{t}^{2}+\frac{1}{2} z y B_{t}^{2}-\frac{1}{2} z C_{t}^{3}+D^{3}(x, y, t) \tag{1.1.21}
\end{align*}
$$

Now we calculate $\alpha_{x y}$ using the first and second of these expressions and conclude that $A_{t}^{1}=-A_{t}^{1}$ which implies that $A^{1}$ is a constant. Similarly, $A^{2}$ and $B^{2}$ must also be constants. This yields

$$
\begin{align*}
\xi & =\frac{1}{2} x \tau_{t}+y A_{1}+z A_{2}+A^{3}(t)  \tag{1.1.22}\\
\eta & =\frac{1}{2} y \tau_{t}-x A_{1}+z B_{2}+B^{3}(t)  \tag{1.1.23}\\
\zeta & =\frac{1}{2} z \tau_{t}-x A_{2}-y B_{2}+C^{3}(t) \tag{1.1.24}
\end{align*}
$$

Comparison of the forms of $\alpha$ now shows that

$$
\begin{equation*}
\alpha=-\frac{1}{8}\left(x^{2}+y^{2}+z^{2}\right) \tau_{t t}-\frac{1}{2} x A_{t}^{3}-\frac{1}{2} y B_{t}^{3}-\frac{1}{2} z C_{t}^{3}+\sigma(t), \tag{1.1.25}
\end{equation*}
$$

where $\sigma$ is an arbitrary function of $t$. Now using

$$
\begin{equation*}
\phi_{t}+f\left(\phi_{u}-\tau_{t}\right)=\Delta \phi+\left(f_{x} \xi+f_{y} \eta+f_{z} \zeta\right) u+f u \tag{1.1.26}
\end{equation*}
$$

and so we are led to the final set of equations

$$
\begin{align*}
- & \frac{1}{8}\left(x^{2}+y^{2}+z^{2}\right) \tau_{t t t}-\frac{1}{2} x A_{t t}^{3}-\frac{1}{2} y B_{t t}^{3}-\frac{1}{2} z C_{t t}^{3}+\sigma_{t} \\
= & -\frac{3}{4} \tau_{t t}+\frac{1}{2}\left(x f_{x}+y f_{y}+z f_{z}+2 f\right) \tau_{t}+A_{1}\left(y f_{x}-x f_{y}\right) \\
& +A_{2}\left(z f_{x}-x f_{z}\right)+B_{2}\left(z f_{y}-y f_{z}\right)+f_{x} A^{3}(t)+f_{y} B^{3}(t)+f_{z} C^{3}(t) \tag{1.1.27}
\end{align*}
$$

This final equation allows us to determine which forms of $f$ lead to equations with nontrivial symmetries and they allow us to determine the Lie algebra. For example, it is immediate that if $f$ is invariant under rotations in $\mathbb{R}^{3}$, then we have

$$
\begin{equation*}
A_{1}\left(y f_{x}-x f_{y}\right)+A_{2}\left(z f_{x}-x f_{z}\right)+B_{2}\left(z f_{y}-y f_{z}\right)=0 \tag{1.1.28}
\end{equation*}
$$

so the symmetry group of the PDE $u_{t}=\Delta u+f(x, y, z) u$ contains at least $S O(3)$, a fact which does not require Lie's method to discover. The $n$-dimensional case can be easily deduced from the structure of the three-dimensional case. We now list the symmetries which are of relevance in the thesis.
1.1.2.1. The equation $u_{t}=\Delta u+\frac{1}{x^{2}} k\left(\frac{y}{x}, \frac{z}{x}\right) u$. For this PDE, the Lie algebra of symmetries has basis

$$
\begin{aligned}
& \mathbf{v}_{1}=\partial_{t}, \mathbf{v}_{2}=x \partial_{x}+y \partial_{y}+z \partial_{z}+2 t \partial_{t} \\
& \mathbf{v}_{3}=4 x t \partial_{x}+4 y t \partial_{y}+4 z t \partial_{z}+4 t^{2} \partial_{t}-\left(x^{2}+y^{2}+z^{2}+6 t\right) u \partial_{u} \\
& \mathbf{v}_{4}=u \partial_{u}, \mathbf{v}_{\beta}=\beta(x, y, z, t) \partial_{u}
\end{aligned}
$$

in which $\beta$ is an arbitrary solution of the PDE.
1.1.2.2. The equation $u_{t}=\Delta u+\left(\frac{1}{x^{2}} k\left(\frac{y}{x}, \frac{z}{x}\right)-\frac{1}{4} c\|x\| u+^{2}+\frac{E}{2}\right) u$. Here $\|x\|^{2}=x^{2}+y^{2}+z^{2}$. The Lie algebra of symmetries has basis

$$
\begin{aligned}
\mathbf{v}_{1} & =\partial_{t}, \mathbf{v}_{2}=\frac{1}{2} x e^{2 \sqrt{c} t} \partial_{x}+\frac{1}{2} y e^{2 \sqrt{c} t} \partial_{y}+\frac{1}{2} z e^{2 \sqrt{c} t} \partial_{z}+\frac{1}{2 \sqrt{c}} e^{2 \sqrt{c t}} \partial_{t} \\
& -\frac{1}{4 \sqrt{c}} e^{2 \sqrt{c} t}\left(c\left(x^{2}+y^{2}+z^{2}\right)+3 \sqrt{c}-E\right) u \partial_{u} \\
\mathbf{v}_{3} & =\frac{1}{2} x e^{-2 \sqrt{c} t} \partial_{x}+\frac{1}{2} y e^{-2 \sqrt{c} t} \partial_{y}+\frac{1}{2} z e^{-2 \sqrt{c} t} \partial_{z}+\frac{1}{2 \sqrt{c}} e^{-2 \sqrt{c t}} \partial_{t} \\
& +\frac{1}{4 \sqrt{c}} e^{-2 \sqrt{c} t}\left(c\left(x^{2}+y^{2}+z^{2}\right)-3 \sqrt{c}-E\right) u \partial_{u}, \mathbf{v}_{\beta}=\beta(x, y, z, t) \partial_{u}
\end{aligned}
$$

Here $\|x\|^{2}=x^{2}+y^{2}+z^{2}$. We can easily show that in 1.1.2.1 and 1.1.2.2 $\left\{\mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}\right\}$ generate the Lie algebra $\mathfrak{s l}_{2}$.
1.1.2.3. The equation $u_{t}=\Delta u+(a x+b y+c z+d) u$. A basis for the Lie algebra is

$$
\begin{aligned}
& \mathbf{v}_{1}= \frac{1}{2}\left(x t-a t^{3}\right) \partial_{x}+\frac{1}{2}\left(y t-b t^{3}\right) \partial_{y}+\frac{1}{2}\left(z t-c t^{3}\right) \partial_{z}+\frac{1}{2} t^{2} \partial_{t}- \\
& \frac{1}{8}\left(x^{2}+y^{2}+z^{2}+\left(a^{2}+b^{2}+c^{2}\right) t^{4}+\frac{3}{4} t+\frac{3}{4}(a x+b y+c z) t^{2}+d t^{2}\right) u \partial_{u} \\
& \mathbf{v}_{2}= \frac{1}{2}\left(x-3 a t^{2}\right) \partial_{x}+\frac{1}{2}\left(y-3 b t^{2}\right) \partial_{y}+\frac{1}{2}\left(z-c t^{2}\right) \partial_{z}+t \partial_{t}+ \\
&+\left(d t-\frac{1}{2}\left(a^{2}+b^{2}+c^{2}\right) t^{3}+\frac{3}{4} t(a x+b y+c z)\right) u \partial_{u}, \mathbf{v}_{3}=\partial_{t} \\
& \\
& \mathbf{v}_{4}= t \partial_{x}+\frac{1}{2}\left(a t^{2}-x\right) u \partial_{u}, \mathbf{v}_{5}=\partial_{x}+a t u \partial_{u}, \mathbf{v}_{6}=t \partial_{y}+\frac{1}{2}\left(b t^{2}-y\right) u \partial_{u} \\
& \mathbf{v}_{7}= \partial_{y}+b t u \partial_{u}, \mathbf{v}_{8}=t \partial_{z}+\frac{1}{2}\left(c t^{2}-z\right) u \partial_{u}, \mathbf{v}_{9}=\partial_{z}+c t u \partial_{u}, \mathbf{v}_{10}=u \partial_{u} \\
& \mathbf{v}_{\beta}= \beta(x, y, z, t) \partial_{u} .
\end{aligned}
$$

The vector fields $\left\{\mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}\right\}$ span the Heisenberg lie algebra, and the remaining vector fields span $\mathfrak{s l}_{2}$. If $a=b=c=d=0$, this gives the symmetries of the heat equation. Similarly we can show that the PDE $u_{t}=\Delta u-\left(a x^{2}+b y^{2}+c z^{2}\right) u, a \neq b \neq c$, has only Heisenberg group symmetries of the form

$$
\begin{aligned}
& \mathbf{v}_{1}=e^{\sqrt{a} t} \partial_{x}-\frac{1}{2} x \sqrt{a} e^{\sqrt{a} t} u \partial_{u}, \mathbf{v}_{2}=e^{-\sqrt{a} t} \partial_{x}+\frac{1}{2} x \sqrt{a} e^{-\sqrt{a} t} u \partial_{u} \\
& \mathbf{v}_{3}=e^{\sqrt{b} t} \partial_{y}-\frac{1}{2} y \sqrt{b} e^{\sqrt{b} t} u \partial_{u}, \mathbf{v}_{4}=e^{-\sqrt{b} t} \partial_{y}-\frac{1}{2} y \sqrt{b} e^{-\sqrt{b} t} u \partial_{u} \\
& \mathbf{v}_{5}=e^{\sqrt{c} t} \partial_{z}-\frac{1}{2} z \sqrt{c} e^{\sqrt{c} t} u \partial_{u}, \mathbf{v}_{6}=e^{-\sqrt{c t}} \partial_{z}-\frac{1}{2} z \sqrt{c} e^{-\sqrt{c} t} u \partial_{u}, \mathbf{v}_{7}=u \partial_{u} .
\end{aligned}
$$

If $a=b=c$, then there is also an $S L(2, \mathbb{R})$ symmetry group. See [18]. We will see in Chapter 3 that Heisenberg group symmetries lead to Fourier transforms of fundamental solutions.

The final class of equation with non-trivial symmetries is as follows. If

$$
u_{t}=\Delta u+k(a x+b y+c z) u
$$

where $k \neq 0$ is a function of a single variable, then a basis for the Lie algebra of symmetries is

$$
\begin{aligned}
& \mathbf{v}_{1}=c t \partial_{x}-a t \partial_{z}-\frac{1}{2}(c x-a z) u \partial_{u}, \mathbf{v}_{2}=c t \partial_{y}-b t \partial_{z}-\frac{1}{2}(c y-b z) u \partial_{u} \\
& \mathbf{v}_{3}=\partial_{t}, \mathbf{v}_{4}=u \partial_{u}
\end{aligned}
$$

We have not been able to obtain fundamental solutions for this final case. There is a special case $u_{t}=\Delta u+\left(\frac{d}{2}+\frac{A}{(a x+b y+c z)^{2}}\right) u, A \neq 0, d \in \mathbb{R}$ with the additional vector fields

$$
\begin{aligned}
& \mathbf{v}_{5}=x \partial_{x}+y \partial_{y}+z \partial_{z}+2 t \partial_{t} \\
& \mathbf{v}_{6}=4 x t \partial_{x}+4 y t \partial_{y}+4 z t \partial_{z}+4 t^{2} \partial_{t}-\left(\|x\|^{2}+6 t-2 d t^{2}\right) u \partial_{u}
\end{aligned}
$$

Our methods do work for this final equation.

### 1.2. Other Mathematical Preliminaries

1.2.1. Fundamental Solutions. In this thesis we are interested in obtaining fundamental solutions by Lie symmetry methods. Fundamental solutions play a major role in the study of PDEs and stochastic analysis. We provide a definition of a fundamental solution below.

Definition 1.2.1. Set $|\alpha|=\alpha_{1}+\cdots+\alpha_{n}$, where $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ is a multi-index and $\alpha_{i} \in \mathbb{N}$ for each $i=1, \ldots n$. Then denote

$$
D^{\alpha} u=\frac{\partial^{|\alpha|} u}{\partial x_{1}^{\alpha_{1}} \ldots x_{n}^{\alpha_{n}}}
$$

Let $P\left(x, D^{\alpha}\right) u$ be a linear differential operator on $\Omega \subseteq \mathbb{R}^{n}$. Then $p(t, x, y)$ is a fundamental solution of

$$
\begin{equation*}
u_{t}=P\left(x, D^{\alpha}\right) u \tag{1.2.1}
\end{equation*}
$$

if for every $y \in \Omega, p(t, x, y)$ is a solution of (1.2.1) and for all $f \in L^{1}(\Omega)$

$$
\begin{equation*}
u(x, t)=\int_{\Omega} f(y) p(t, x, y) d y \tag{1.2.2}
\end{equation*}
$$

is a solution of (1.2.1) such that $\lim _{t \rightarrow 0} u(x, t)=f(x)$. Integration against the kernel $p(t, x, y)$ defines an operator from $L^{1}(\Omega)$ into the space of solutions of (1.2.1).

The condition that $f \in L^{1}(\Omega)$ may be varied in certain situations. For example, we may require that the integral (1.2.2) converges for all $f \in \mathcal{D}(\Omega)$, the space of compactly supported smooth functions. Fundamental solutions are closely related to Green's functions, but Green's functions usually incorporate boundary conditions. See [69] for a history and survey of fundamental solutions.

Example 1.2.1. The fundamental solution of the heat equation $u_{t}=\Delta u$ on $\mathbb{R}^{n}$ is also called the heat kernel. It is

$$
\begin{equation*}
K(t, x, y)=\frac{1}{(4 \pi t)^{n / 2}} e^{-\frac{\|x-y\|^{2}}{4 t}}, \tag{1.2.3}
\end{equation*}
$$

where $\|x\|=\sum_{i=1}^{n} x_{i}^{2}$. Then the solution of the problem $u_{t}=\Delta u$ with $u(x, 0)=f(x)$ is given by $u(x, t)=\int_{\mathbb{R}^{n}} f(y) \frac{1}{(4 \pi t)^{n / 2}} e^{-\frac{\|x-y\|^{2}}{4 t}} d y$.

Example 1.2.2. In Chapter 3 we show that the $n$-dimensional PDE

$$
\begin{equation*}
u_{t}=\Delta u+\left(\sum_{i=1}^{n} a_{i} x_{i}+c\right) u \tag{1.2.4}
\end{equation*}
$$

has a fundamental solution

$$
p(t, x, y)=\frac{e^{c t}}{(4 \pi t)^{\frac{n}{2}}} \exp \left(\frac{1}{12} \sum_{i=1}^{n} a_{i}^{2} t^{3}-\frac{\|x-y\|^{2}}{4 t}+\frac{t}{2} \sum_{i=1}^{n} a_{i}\left(x_{i}+y_{i}\right)\right) .
$$

Note that fundamental solutions are not unique. It is possible for a PDE to have more than one fundamental solution and our methods will lead to ways of obtaining multiple fundamental solutions for a given PDE.

### 1.2.2. Group-Invariant Solutions and Fundamental Solu-

 tions. Group-invariant solutions can be used in the construction of fundamental solutions. Lie established the theory of group-invariant solutions and the theory has been developed by Olver and others, [62]. A group-invariant solution is a solution which is invariant under the action of a group transformation. We provide a key result below, which combines two theorems in Bluman and Kumei's text [12].Theorem 1.2.2. Consider the $n$ th-order boundary value problem (BPV)
$P\left(x, D^{\alpha} u\right)=0$ subject to the conditions $B_{j}\left(x, u, u^{(n-1)}\right)=0$ on the surface $\omega_{j}(x)=0$. A vector field $\mathbf{v}$ is admitted by the BVP if
(1) $\operatorname{pr}^{n} \mathbf{v}\left[P\left(x, D^{\alpha} u\right)\right]=0$ when $P\left(x, D^{\alpha} u\right)=0$.
(2) $\mathbf{v}\left(\omega_{j}(x)\right)=0$ when $\omega_{j}(x)=0$.
(3) $\operatorname{pr}^{n-1} \mathbf{v}\left[B_{j}\left(x, u, u^{(n-1)}\right)\right]=0$ when $B_{j}\left(x, u, u^{(n-1)}\right)=0$ on the surface $\omega_{j}(x)=0$.

Suppose that a BVP admits a vector field $\mathbf{v}$. Then the solution of the $B V P$ is a group-invariant solution with respect to the symmetries generated by $\mathbf{v}$.

Most studies of fundamental solutions using Lie groups published up to now have relied on group invariance methods. Bluman in $[8]$ and [9], studied the Fokker-Planck equation $u_{t}=u_{x x}+(f(x) u)_{x}$ subject to different boundary conditions. See also [11] and [37]. He obtained explicit fundamental solutions in the case when $f$ satisfies a Riccati equation of the form $2 f^{\prime}-f^{2}+\beta^{2} x^{2}-\gamma+\frac{16 \nu^{2}-1}{x^{2}}=0$, in which $\beta, \nu$ and $\gamma$ are arbitrary constants.

Bluman has also considered some $n$-dimensional examples. Using group-invariant solutions, in fundamental solutions for the $n$-dimensional wave equation and the Poisson kernel for the Laplace equation are obtained in [12]. In addition, the equation

$$
u_{t}=d \Delta u+u_{y}-y u_{x}, \quad(x, y) \in \mathbb{R}^{2}, d>0
$$

is studied.
Ibragimov has made extensive use of group-invariant solution methods to obtain fundamental solutions. See the papers [42] and [4] as well as the CRC handbooks $[\mathbf{4 3}],[44]$ and $[45]$. In [42], Lie group actions are extended to spaces of distributions and new derivations of the fundamental solutions of the heat, wave and Laplace equations of mathematical physics are given. With Gazizov, Ibragimov also obtained fundamental solutions of some important PDEs arising in financial mathematics [31]. Specifically the fundamental solution of the Black-Scholes equation of option pricing is obtained, as well as groupinvariant solutions of the Jacob-Jones equation.

Laurence and Wang [55] obtained fundamental solutions for PDEs of the form

$$
\begin{equation*}
u_{t}=\Delta u+A(x) u, x \in \mathbb{R}^{2}, \tag{1.2.5}
\end{equation*}
$$

by using group-invariant solution methods developed by Ibragimov. They are able to obtain fundamental solutions only in some special cases. In most of these cases the resulting fundamental solutions could actually be obtained as products of the corresponding one-dimensional fundamental solution. To illustrate, Laurence and Wang give the example of the PDE

$$
u_{t}=u_{x x}+u_{y y}+\left(\frac{a}{x^{2}}+b y\right) u
$$

Using group-invariant solution methods, they obtain a fundamental solution, however this fundamental solution is simply the product of fundamental solutions of the one dimensional equations $u_{t}=u_{x x}+\frac{a}{x^{2}} u$ and $u_{t}=u_{y y}+b y u$. For equations of the form

$$
\begin{equation*}
u_{t}=\Delta u+\frac{1}{x^{2}} k\left(\frac{y}{x}\right) u \tag{1.2.6}
\end{equation*}
$$

they are only able to obtain the value of the fundamental solution $p(t, x, y, \xi, \eta)$ at $(\xi, \eta)=(0,0)$.

In Chapter 4 we obtain fundamental solutions in the case when the symmetry group is only $S L(2, \mathbb{R}) \times \mathbb{R}$ and the fundamental solutions are not products of one-dimensional fundamental solutions. In particular, we obtain the full fundamental solution for (1.2.6).

Methods of obtaining fundamental solutions by reducing an equation on the line to a so-called canonical form have also been used extensively. This idea goes back to Lie. Bluman found the explicit transformation mapping a PDE with a six-dimensional symmetry group to the heat equation in [10]. Goard obtained large classes of fundamental solutions by this approach in [33].

Although group-invariant solutions are a powerful tool, there are certain limitations. An example due to Craddock [26] illustrates. Consider the SDE

$$
d X_{t}=\frac{2 a X_{t}}{2+a X_{t}} d t+\sqrt{2} d W_{t}
$$

Suppose we wish to obtain a transition density for this process. We must solve

$$
\begin{equation*}
u_{t}=u_{x x}+\frac{2 a x}{2+a x} u_{x} \tag{1.2.7}
\end{equation*}
$$

If we use group-invariant solution methods, we obtain the fundamental solution of (1.2.7) given by

$$
\begin{equation*}
q(x, y, t)=\frac{1}{t} \frac{2+a y}{(2+a x)} \sqrt{\frac{x}{y}} e^{-\frac{(x+y)}{t}} I_{1}\left(\frac{2 \sqrt{x y}}{t}\right) \tag{1.2.8}
\end{equation*}
$$

But this is not a transition density, because it does not have total integral equal to one. The transition density is actually

$$
\begin{equation*}
p(t, x, y)=\frac{e^{-\frac{(x+y)}{t}}}{\left(1+\frac{1}{2} a x\right) t}\left[\left(\sqrt{\frac{x}{y}}+\frac{1}{2} a \sqrt{x y}\right) I_{1}\left(\frac{2 \sqrt{x y}}{t}\right)+\delta(y)\right] \tag{1.2.9}
\end{equation*}
$$

The delta function term cannot be obtained from a group-invariant solution, because the Dirac delta is not the solution of any secondorder ODE. The same problem exists for reduction to canonical form, [26]. Fundamental solutions containing delta function terms are common and if we use group-invariant solution methods, they have to be
added in after the fact. The methods of Chapter 2 avoid this problem altogether, because they automatically produce fundamental solutions with these delta function terms. See also Example 2.0.5.
1.2.3. Fourier Transforms. Fourier transforms provide one of the most important tools of analysis. In this thesis Fourier transforms of fundamental solutions arise in the case when the PDE has Heisenberg group symmetries. We collect here the relevant material.

Definition 1.2.3 (Fourier Transform). Let $f \in L^{1}\left(\mathbb{R}^{n}\right)$. Then the Fourier transform of $f$ is defined by

$$
\begin{equation*}
\widehat{f}(y)=\int_{\mathbb{R}^{n}} f(x) e^{-i y \cdot x} d x \tag{1.2.10}
\end{equation*}
$$

We can recover the original function by using the Fourier Inversion Theorem.

Theorem 1.2.4 (Fourier Inversion). Let $f \in L^{1}\left(\mathbb{R}^{n}\right)$ and further suppose that $\widehat{f} \in L^{1}\left(\mathbb{R}^{n}\right)$. Then we may recover $f$ from $\widehat{f}$ by

$$
\begin{equation*}
f(x)=\frac{1}{(2 \pi)^{n}} \int_{\mathbb{R}^{n}} f(x) e^{i y \cdot x} d y \tag{1.2.11}
\end{equation*}
$$

It is important to know when a function can be represented as a Fourier integral. Under what conditions can we say that a given function is the the Fourier transform of another function? This problem was solved by Bochner.

Theorem 1.2.5 (Bochner). A necessary and sufficient condition for a continuous function $\phi$ to be represented as a Fourier transform of a finite Borel measure is that it be positive definite. That is for all
complex numbers $\rho_{1}, \ldots, \rho_{N}$ and $\alpha^{1}, \ldots, \alpha^{N} \in \mathbb{R}^{n}$

$$
\begin{equation*}
\sum_{p, q=1}^{N} \phi\left(\alpha^{p}-\alpha^{q}\right) \rho_{p} \overline{\rho_{q}} \geq 0 \tag{1.2.12}
\end{equation*}
$$

Proof. See Bochner [13].

If we impose stronger conditions on $f$ then we can guarantee that the Fourier transform is an $L^{1}$ function.

Theorem 1.2.6. Suppose that $f \in L^{1}\left(\mathbb{R}^{n}\right) \cap C^{2}\left(\mathbb{R}^{n}\right)$. Then $\widehat{f} \in$ $L^{1}\left(\mathbb{R}^{n}\right)$.

Proof. See [70].
In the study of Fourier analysis there are natural spaces which arise. We define below the spaces relevant to the results in this thesis.

Definition 1.2.7. Let $\Omega \subseteq \mathbb{R}^{n}$ be open.
(1) The space of infinitely differentiable, compactly supported functions on $\Omega$ is denoted $\mathcal{D}(\Omega)$.
(2) The Schwartz space of rapidly decreasing smooth functions is denoted by $\mathcal{S}(\Omega)$.
$\mathcal{S}(\Omega)=\left\{f \in C^{\infty}(\Omega), \sup _{|\alpha| \leq N} \sup x \in \Omega\left|\left(1+|x|^{2}\right)^{N}\left(D^{\alpha} f\right)(x)\right|<\infty, \forall N \in \mathbb{N}\right\}$.
That is $\left|P(x)\left(D^{\alpha} f\right)(x)\right|$ is bounded for every polynomial $P$.

Definition 1.2.8. The space of distributions $\mathcal{D}^{\prime}(\Omega)$, the dual of $\mathcal{D}(\Omega)$ is defined by

$$
\begin{equation*}
\mathcal{D}^{\prime}(\Omega)=\{T: \mathcal{D}(\Omega) \rightarrow \mathbb{R}, T \text { linear }\} \tag{1.2.13}
\end{equation*}
$$

That is, $\mathcal{D}^{\prime}(\Omega)$ is the set of linear functionals on $\mathcal{D}(\Omega)$. Elements of $\mathcal{D}^{\prime}(\Omega)$ are continuous in the topology of $\mathcal{D}(\Omega)$. For a description of the topology of $\mathcal{D}(\Omega)$, see page 138 of [67].

For more details on distributions see Chapter 6 of Rudin's book [67].
1.2.4. Stochastic Calculus. We provide in this section material from the theory of stochastic calculus to be used in subsequent chapters. A general reference for stochastic calculus is [60]. See also [53]. We begin with the definition of Brownian motion.

Definition 1.2.9. A one-dimensional Brownian motion is a stochastic process $B=\{B(t) ; t \geq 0\}$ with the following properties.
(a) Independent increments: $B(t)-B(s)$ for $t>s$, is independent of the past, that is, of $B(u), 0 \leq u \leq s$, or of $\mathcal{F}_{s}$, the $\sigma$-field generated by $B(u), u \leq s$.
(b) Normal increments: $B(t)-B(s)$ has a normal distribution with mean 0 and variance $t-s$. So $B(t)-B(0)$ has the distribution $N(0, t)$. In addition this implies that $B(t)-B(s)$ has the same distribution as $B(t-s)$.
(c) Continuity of Paths: $B(t), t \geq 0$ are continuous functions of $t$ with probability 1 . That is, almost surely.

An $n$-dimensional Brownian motion is a vector $B(t)=\left(B_{1}(t), \ldots, B_{n}(t)\right)$ where each $B_{i}$ is a one-dimensional Brownian motion.

A common notation which we use throughout the thesis is $B_{t}=$ $B(t)$. Larger classes of stochastic processes can be constructed from Brownian motion. This construction is due to Itô.

More general definitions of Itô diffusions are possible, but for our purposes, the following is sufficient.

Definition 1.2.10. A time-homogenous Itô diffusion is a stochastic process $X=\left\{X_{t}: t \geq 0\right\}$ satisfying a stochastic differential equation of the form

$$
\begin{equation*}
d X_{t}=b\left(X_{t}\right) d t+\sigma\left(X_{t}\right) d B_{t}, t \geq s, X_{s}=x \tag{1.2.14}
\end{equation*}
$$

where $B_{t}$ is an $m$-dimensional Brownian motion and $b: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}, \sigma$ : $\mathbb{R}^{n} \rightarrow \mathbb{R}^{n \times m}$ are Lipschitz-continuous.

In this thesis, all Itô diffusions we consider are homogeneous. The most important result is the Itô formula. Although we do not use it in this thesis, we include it because of its central role in the theory of stochastic processes.

Theorem 1.2.11. The Itô formula for the $n$-dimensional diffusion $\mathbf{X}(t)=\left(X_{1}(t), \ldots, X_{n}(t)\right)$ satisfying the $S D E$

$$
\begin{equation*}
d X_{i}(t)=a_{i}(t, X(t)) d t+\sum_{j=1}^{n} b_{i j}\left(t,(\mathbf{X}(t)) d B_{j}(t), i=1, \ldots, n\right. \tag{1.2.15}
\end{equation*}
$$

is given by

$$
\begin{aligned}
& d f(t, \mathbf{X}(t))=\sum_{i=1}^{n} \frac{\partial}{\partial x_{i}} f(t, \mathbf{X}(t)) d X_{i}(t) \\
& +\frac{\partial f}{\partial t}(t, \mathbf{X}(t)) d t+\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^{2}}{\partial x_{i} \partial x_{j}} f(t, \mathbf{X}(t)) d\left[X_{i}, X_{j}\right](t)
\end{aligned}
$$

Remark 1.2.12. Here $d\left[X_{i}, X_{j}\right](t)=\sigma_{i j} d t$ with $\sigma_{i j}$ the $i j$ th-component of the matrix $\sigma=\mathbf{b} \mathbf{b}^{T}$, and the $i j$ th-component of $\mathbf{b}$ is $b_{i j}$.

Definition 1.2.13. Let $X=\left\{X_{t}: t \geq 0, X_{0}=x\right\}$ be an Itô diffusion. The generator of $X$ is the operator defined by

$$
\begin{align*}
A f(x) & =\lim _{t \rightarrow 0} \frac{E\left[f\left(X_{t}\right)\right]-A f(x)}{t}  \tag{1.2.16}\\
f & \in C^{2}\left(\mathbb{R}^{n}\right) \tag{1.2.17}
\end{align*}
$$

The generator can be written as a partial differential operator. This provides the link between the theory of diffusions and the theory of PDEs.

Definition 1.2.14. Let $X=\left\{X_{t}: t \geq 0\right\}$ be the Itô diffusion

$$
d X_{t}=b\left(X_{t}\right) d t+\sigma\left(X_{t}\right) d B_{t}
$$

Then the generator of $X$ is

$$
\begin{equation*}
A f=\frac{1}{2} \sum_{i, j}\left(\sigma \sigma^{T}\right)_{i j} \frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}+\sum_{i=1}^{n} b_{i}(x) \frac{\partial f}{\partial x_{i}}, f \in C^{2}\left(\mathbb{R}^{n}\right) \tag{1.2.18}
\end{equation*}
$$

Theorem 1.2.15. Let $X=\left\{X_{t}: t \geq 0, X_{0}=x\right\}$ be an Itô diffusion with generator $A$. Then the functional $u(x, t)=\mathbb{E}\left[f\left(X_{t}\right)\right]$ is a solution of the PDE

$$
\begin{equation*}
u_{t}(x, t)=A u(x, t), u(x, 0)=f(x) . \tag{1.2.19}
\end{equation*}
$$

As a consequence of this result, to find the transition density for $X=\left\{X_{t}: t \geq 0, X_{0}=x\right\}$ we need to find a fundamental solution of Kolmogorov's equation (1.2.19) which is positive and has total integral one. In later chapters we calculate functionals of Itô diffusions. The key tool is the Feynman-Kac formula.

Theorem 1.2.16 (The Feynman-Kac formula). Let $f \in C^{2}\left(\mathbb{R}^{n}\right)$ and $q \in C\left(\mathbb{R}^{n}\right)$. Assume that $q$ is lower-bounded.
(i) Put

$$
v(t, x)=E_{x}\left[e^{-\int_{0}^{t} q\left(X_{s}\right) d s} f\left(X_{s}\right)\right]
$$

Then

$$
\begin{align*}
& v_{t}=A v-q v, t>0, x \in \mathbb{R}^{n},  \tag{1.2.20}\\
& v(0, x)=f(x)
\end{align*}
$$

(ii) Moreover, if $w(t, x) \in C^{1,2}\left(\mathbb{R} \times \mathbb{R}^{n}\right)$ is bounded on $K \times \mathbb{R}^{n}$ for each compact $K \subset \mathbb{R}$ and $w$ solves (1.2.20) with $w(0, x)=$ $f(x)$, then $w(t, x)=v(t, x)$.

Proof. See Oksendal [60], p143.
1.2.5. Sturm-Liouville Theory. The expansion results for fundamental solutions require some basic results from Sturm-Liouville theory. We present the relevant material below.

Definition 1.2.17. A Sturm-Liouville problem can be written as

$$
\begin{equation*}
L(y)=-r \lambda y \tag{1.2.21}
\end{equation*}
$$

in which the operator $L$ is

$$
\begin{equation*}
L y=\left(p y^{\prime}\right)^{\prime}+q y \tag{1.2.22}
\end{equation*}
$$

and the boundary conditions are

$$
\begin{align*}
& U_{1}(y)=\alpha_{11} y(a)+\alpha_{12} y^{\prime}(a)=0  \tag{1.2.23}\\
& U_{2}(y)=\alpha_{21} y(a)+\alpha_{22} y^{\prime}(a)=0 \tag{1.2.24}
\end{align*}
$$

where the constants $\alpha_{i j} \in \mathbb{R}$. We assume that the boundary conditions described by $U_{1}$ and $U_{2}$ are linearly independent. That is, $U_{1}$ is not a
constant multiple of $U_{2}$. We will assume that $p, q$ and $r$ are continuous on $[a, b]$ and that $p>0, r(x)>0$ on $[a, b]$.

The following results about Sturm-Liouville theory are all classical and can be found in [2] for example. The most pertinent facts about Sturm-Liouville problems and their eigenvalues and eigenfunctions are as follows.

Proposition 1.2.18. The eigenvalues of the Sturm-Liouville problem defined above are real.

Theorem 1.2.19. The eigenfunctions for the Sturm-Liouville problem defined in this section form an orthogonal basis for $L^{2}([a, b], r(x))$.

The eigenvalues of the Sturm-Liouville problem 1.2.17 can be shown under suitable assumptions to form an increasing sequence $\lambda_{0}<\lambda_{1}<$ $\lambda_{2}<\lambda_{3}<\cdots$ and $\lim _{n \rightarrow \infty} \lambda_{n}=\infty$. See [2]. Estimates for the asymptotic behaviour of the eigenvalues have been obtained by a number of authors.

Proposition 1.2.20. For large $n$ the $n$ th-positive eigenvalue of 1.2.17 is $\lambda_{n}=\bar{b}+\bar{a} n^{2}+o(1)$, where $\bar{a}$ and $\bar{b}$ are constants depending on $p, q, r$.

Note the term $o(1)$ can be expanded in powers of $1 / n$ under further assumptions on the smoothness of $p, q, r$. See [5].

Under further assumptions we can show that the eigenvalues are always positive. There exist many different conditions implying that the Sturm-Liouville problem has positive eigenvalues. A simple and well known result follows.

Theorem 1.2.21. The eigenvalues of the Sturm-Liouville problem of definition 1.2.17 will be positive if $q<0$ on $[a, b]$ and the first eigenfunction $L_{0}$ satisfies $-\left.p(x) L_{0}^{\prime}(x)\right|_{a} ^{b} \geq 0$.

Proof. Multiply the Sturm-Liouville equation by $L_{0}$ and integrate by parts to obtain the relation

$$
\begin{equation*}
\lambda_{0}=\frac{-\left.p(x) \frac{d L_{0}}{d x}\right|_{a} ^{b}+\int_{a}^{b}\left[p(x)\left(L_{1}^{\prime}(x)\right)^{2}-q(x) L_{0}^{2}(x)\right] d x}{\int_{a}^{b} L_{0}^{2}(x) r(x) d x} \tag{1.2.25}
\end{equation*}
$$

The given conditions clearly guarantee that $\lambda_{0}$ is positive.
A more useful result can be easily established.

Theorem 1.2.22. Consider the Sturm-Liouville problem

$$
\begin{aligned}
\left(p u^{\prime}\right)^{\prime}+(q+\lambda r) u & =0, a \leq x \leq b \\
\alpha u(a)-\beta u^{\prime}(b) & =0, \gamma u(b)+\delta u^{\prime}(b)=0 .
\end{aligned}
$$

If $q(x) \leq 0$ on $[a, b] \alpha, \beta \geq 0, \gamma, \delta \geq 0$ then all eigenvalues of the given Sturm-Liouville problem are positive.

A proof is in [54], p208. Convergence of the eigenfunction expansion $\sum_{n=0}^{\infty}\left(f, L_{n}\right) L_{n}$ to $f$ can be made uniform under suitable assumptions.

Theorem 1.2.23 (Mishoe-Ford). Suppose that $f^{\prime}$ exists and is of bounded variation and suppose that $f(a)=f(b)=0$. Then the eigenfunction expansion of $f$ converges uniformly to $f$ on $(a, b)$.

See [59] for a proof. More generally,

Theorem 1.2.24. If $f$ is a smooth function on $(a, b)$, satisfying the boundary conditions of 1.2 .17 then the eigenfunction expansion of $f$ converges uniformly to $f$ on $(a, b)$.

Suppose that $G(x, y)$ is the Green's function for the problem $\left(p y^{\prime}\right)^{\prime}+$ $q y=f$, subject to the boundary conditions of 1.2.17.

Theorem 1.2.25. Let $f$ be continuous on $[a, b]$ and suppose that the eigenfunctions of 1.2 .17 are $L_{n}$ and $\lambda_{n}$ are the eigenvalues. Then the series $\sum_{n=1}^{\infty} \frac{\left(f, L_{n}\right) L_{n}}{\lambda_{n}}$ converges uniformly to $G f$, where $G$ is the Green's function for the equation $\left(p y^{\prime}\right)^{\prime}+q y=f$ subject to the given Sturm-Liouville boundary conditions.

See [38].
We will need a generalisation of Sturm-Liouville theory to eigenfunctions of the Laplacian on the $n$-dimensional sphere. This will be used in the $n$-dimensional case in Chapter 4 . We use the following result.

Theorem 1.2.26. Let $\Delta_{S}$ denote the Laplace-Beltrami operator on the sphere $S^{n-1}$. That is, $\Delta_{S}$ is the angular part of the Laplacian on $\mathbb{R}^{n}$. Suppose that $G$ is a continuous function. Let $\Psi_{\lambda_{n}}$ be the eigenfunctions of the problem $\Delta_{S}+G=0$, subject to the boundary conditions $\alpha(\boldsymbol{\Theta}) \Psi(\boldsymbol{\Theta})+(1-\alpha(\boldsymbol{\Theta})) \frac{\partial \Psi}{\partial n}=0$, with $\alpha$ a continuous function and $\frac{\partial \Psi}{\partial n}$ the normal derivative on the surface of the unit sphere $S^{n-1}$, and $\lambda_{n}$ the eigenvalues. Then $\left\{\Psi_{\lambda_{n}}\right\}_{n=1}^{\infty}$ forms an orthogonal basis for $L^{2}\left(S^{n-1}\right)$.

Proof. See [48].
1.2.6. Representation Theory. One of the applications of the thesis will be to problems in representation theory. In the final chapter we will show that for certain important classes of multidimensional parabolic equations, the Lie symmetries are actually equivalent to representations of the underlying group. Therefore we need to present
some elementary background details on representation theory. Some general references for representation theory are [64], [58] and [36].

Definition 1.2.27. Let $V$ be a locally convex topological vector space (LCTVS). Let $\pi$ be a homomorphism from a group $G$ to the space of linear operators on $V$, denoted $L(V)$, with topology $\tau$. If the mapping $G \times V \rightarrow V$ given by $(g, v) \rightarrow \pi(g) v$ is continuous, then we say that $\pi$ is a continuous representation of $G$. The representation is denoted $(\pi, V)$.

We are really interested in a variant of this idea. The symmetries are projective representations.

Definition 1.2.28. Let $V$ be a LCTVS. Let $\pi$ be a mapping from a group $G$ to the space of linear operators on $V$, denoted $L(V)$, with the property that for all $v \in V$ and for all $g, h \in G$

$$
\pi(g h) v=c(g, h) \pi(g) \pi(h) v
$$

where the cocycle $c$ satisfies $|c(g, h)|=1$ and the cocycle equation $c(h, k) c(g, h k)=c(g, h) c(g h, k)$. Then $(\pi, V)$ is called a projective representation of $G$.

In Chapter 5 we will be interested in irreducible representations.

Definition 1.2.29. Let $(\pi, V)$ be a representation of $G$. We say that the representation is irreducible if there are no closed, invariant subspaces of $V$ under $\pi(G)$ other than $\{0\}$ and $V$ itself.

For the first case we consider, the representations are unitary.

Definition 1.2.30. Suppose that $V$ is a Hilbert space. We say that the representation is unitary if $(\pi(g) v, \pi(g) v)=(v, v)$ for all $v \in$ $V, g \in G$. Here $(a, b)$ is the inner product of $a, b \in V$.

Definition 1.2.31. Two representations $(\pi, V)$ and $(\rho, W)$ are said to be equivalent if there exists an operator $A: V \rightarrow W$ such that for all $g \in G, v \in V, A \pi(g) v=\rho(g) A v$. The operator $A$ is known as an intertwining operator.

In the final chapter we will apply our results on fundamental solutions of multidimensional PDEs to the problem of constructing intertwining operators between lie symmetries and group representations. We present some examples of irreducible representations below.

Example 1.2.3. Consider $G=\mathbb{R}^{n}$. Let $V=\mathbb{C}$ and for each $\lambda \in$ $\mathbb{R}^{*}=\mathbb{R}-\{0\}$, where $\pi_{\lambda}(x)=e^{i \lambda x}$. Then for each $\lambda, \pi_{\lambda}$ is an irreducible representation of $\mathbb{R}$. These one-dimensional representations are known as characters.

Example 1.2.4. Anther example is the Heisenberg group, described in Example 1.1.3. Let $G=H_{2 n+1}$. If $f \in L^{2}\left(\mathbb{R}^{2 n}\right)$, then for all $\lambda \in \mathbb{R}^{*}$,

$$
\left(\pi_{\lambda}(a, b, c) f\right)(\xi)=e^{i \lambda(c+a \cdot(\xi-b / 2))} f(\xi-b)
$$

is an irreducible representation of $H_{3}$. All other irreducible representations act trivially on the centre of $H_{2 n+1}$. These representations are onedimensional (i.e. characters) and are of the form $\pi_{\mu, \nu}(a, b, c)=e^{i(\mu \cdot a+\nu \cdot b)}$ and act on $\mathbb{C}$.

In the final chapter we present another important irreducible, projective representation of $S L(2, \mathbb{R})$, the modified Segal-Shale-Weil representation. This is an important representation for Lie symmetry analysis.
1.2.7. Applications in Stochastic Calculus. We focus on the PDE

$$
\begin{equation*}
u_{t}=\Delta u+A(x) u, x \in \Omega \subseteq \mathbb{R}^{n} \tag{1.2.26}
\end{equation*}
$$

In many applications the potential $A<0, \forall x \in \Omega$. This is because a negative potential physically corresponds to the killing rate of a process. There are two immediate ways we can apply our results to stochastic calculus. By the Feynman-Kac formula, we can calculate the functional

$$
\begin{equation*}
u(x, t)=\mathbb{E}_{x}\left[f\left(B_{t}\right) e^{\int_{0}^{t} A\left(B_{s}\right) d s}\right] \tag{1.2.27}
\end{equation*}
$$

where $B_{t}(\cdot) \in \Omega$ is an $n$-dimensional Brownian motion, by solving (1.2.26) subject to $u(x, 0)=f(x)$. We may also convert (1.2.26) to

$$
\begin{equation*}
v_{t}=\Delta v+2 \nabla \phi \cdot \nabla v+\left(\Delta \phi+|\nabla \phi|^{2}+A(x)\right) v \tag{1.2.28}
\end{equation*}
$$

by the substitution $u=e^{\phi} v$. If we suppose that

$$
\begin{equation*}
\Delta \phi+|\nabla \phi|^{2}+A(x)=K(x) \tag{1.2.29}
\end{equation*}
$$

then we are led to the PDE

$$
\begin{equation*}
v_{t}=\Delta v+2 \nabla \phi \cdot \nabla v+K(x) v \tag{1.2.30}
\end{equation*}
$$

The quasilinear equation (1.2.29) becomes linear if we set $w=e^{\phi} v$. The result is $\Delta w+A(x) w=0$. The functions $2 \phi_{x_{1}}, 2 \phi_{x_{2}}$ etcetera are called
the drift functions. Obviously finding drifts is equivalent to finding stationary solutions of (1.2.26). If

$$
\begin{equation*}
d X_{t}^{i}=2 \nabla \phi\left(X_{t}^{1}, \ldots, X_{t}^{n}\right) d t+\sqrt{2} d W_{t}^{i}, i=1, \ldots, n \tag{1.2.31}
\end{equation*}
$$

then we may compute

$$
\begin{equation*}
u(x, t)=\mathbb{E}_{x}\left[f\left(X_{t}\right) e^{\int_{0}^{t} K\left(X_{s}\right) d s}\right], \tag{1.2.32}
\end{equation*}
$$

by solving (1.2.30) subject to $u(x, 0)=f(x)$.
The potentials in (1.2.26) that we are concerned with have the structure

$$
\begin{equation*}
A(x)=\frac{1}{x_{1}^{2}} k\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right)+C\|x\|^{2}+\sum_{i=1}^{n} a_{i} x_{i}+E, \tag{1.2.33}
\end{equation*}
$$

where $k$ is an arbitrary continuous function and $C, a_{1}, \ldots, a_{n}$ and $E$ are arbitrary constants. If $A$ is of this form then (1.2.26) has nontrivial symmetries. In fact we have the following simple result.

Proposition 1.2.32. Let $u$ be a solution of (1.2.26) and let

$$
A(x)=\frac{1}{x_{1}^{2}} k\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right),
$$

for some function $k$. Then for $\epsilon>-1 / 4 t$, so is

$$
\tilde{u}_{\epsilon}\left(x_{1}, \ldots, x_{n}, t\right)=\frac{\exp \left(-\frac{\epsilon\|x\|^{2}}{1+4 \epsilon t}\right)}{(1+4 \epsilon t)^{\frac{n}{2}}} u\left(\frac{x_{1}}{1+4 \epsilon t}, \ldots, \frac{x_{n}}{1+4 \epsilon t}, \frac{t}{1+4 \epsilon t}\right) .
$$

Proof. Earlier we saw the calculation for $n=3$. One may easily show for general $n$ that the PDE has an infinitesimal symmetry $\mathbf{v}=$ $\sum_{i=1}^{n} 4 x_{i} t \partial_{x_{i}}+4 t^{2} \partial_{t}-\left(\|x\|^{2}+2 n t\right) u \partial_{u}$. Exponentiating this symmetry completes the proof.

To obtain a symmetry of (1.2.30), let $u=e^{\phi} v$, then apply the previous symmetry to $u$ and then multiply by $e^{-\phi}$ to obtain a symmetry of (1.2.30).

COROLLARY 1.2.33. Suppose that $\Delta \phi+|\nabla \phi|^{2}+\frac{1}{x_{1}^{2}} k\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right)=0$ and $u$ satisfies

$$
\begin{equation*}
u_{t}=\Delta u+2 \nabla \phi \cdot \nabla u \tag{1.2.34}
\end{equation*}
$$

Then for $\epsilon>-1 / 4 t$,

$$
\begin{aligned}
\tilde{u}_{\epsilon}\left(x_{1}, \ldots, x_{n}, t\right)= & \frac{1}{(1+4 \epsilon t)^{\frac{n}{2}}} e^{-\frac{\epsilon\|x\|^{2}}{1+4 \epsilon t}+\phi\left(\frac{x_{1}}{1+4 \epsilon t}, \ldots, \frac{x_{n}}{1+4 \epsilon t}\right)-\phi\left(x_{1}, \ldots, x_{n}\right)} \\
& \times u\left(\frac{x_{1}}{1+4 \epsilon t}, \ldots, \frac{x_{n}}{1+4 \epsilon t}, \frac{t}{1+4 \epsilon t}\right),
\end{aligned}
$$

is also a solution.
1.2.7.1. Finding Drifts. For the applications considered later in the thesis, we need to be able to compute drift functions $\phi$. For simplicity we focus on the $n=2$ case. We need to find solutions of

$$
\begin{equation*}
\phi_{x x}+\phi_{y y}+\phi_{x}^{2}+\phi_{y}^{2}+\frac{1}{x^{2}} k\left(\frac{y}{x}\right)=0 . \tag{1.2.35}
\end{equation*}
$$

The change of variables $\phi=\ln w$, leads to a second order linear PDE. An easier way is to use group-invariant solutions. Setting $r=y / x$ gives $\phi_{x}=-\frac{y}{x^{2}} \phi_{r}, \phi_{y}=\frac{1}{x} \phi_{r}, \phi_{x x}=\frac{y^{2}}{x^{4}} \phi_{r r}+2 \frac{y}{x^{3}} \phi_{r}$ and $\phi_{y y}=\frac{1}{x^{2}} \phi_{r r}$. So that

$$
\begin{equation*}
\frac{1}{x^{2}}\left(\left(1+\frac{y^{2}}{x^{2}}\right) \phi_{r r}+2 \frac{y}{x} \phi_{r}+\left(1+\frac{y^{2}}{x^{2}}\right) \phi_{r}^{2}+k\left(\frac{y}{x}\right)\right)=0 \tag{1.2.36}
\end{equation*}
$$

which is the nonlinear ODE

$$
\begin{equation*}
\left(1+r^{2}\right) \phi_{r r}+2 r \phi_{r}+\left(1+r^{2}\right) \phi_{r}^{2}+k(r)=0 \tag{1.2.37}
\end{equation*}
$$

The substitution $F(r)=\phi_{r}$ gives the Riccati equation

$$
\begin{equation*}
\left(1+r^{2}\right) F^{\prime}+2 r F+\left(1+r^{2}\right) F^{2}+k(r)=0 \tag{1.2.38}
\end{equation*}
$$

Every Riccati equation can be linearised, and taking $F=G^{\prime} / G$ will do this here. Then

$$
\begin{equation*}
\left(1+r^{2}\right)\left(\frac{G^{\prime \prime}}{G}-\left(\frac{G^{\prime}}{G}\right)^{2}\right)+2 r \frac{G^{\prime}}{G}+\left(1+r^{2}\right)\left(\frac{G^{\prime}}{G}\right)^{2}+k(r)=0 \tag{1.2.39}
\end{equation*}
$$

or

$$
\begin{equation*}
\left(1+r^{2}\right) G^{\prime \prime}+2 r G^{\prime}+k(r) G=0 \tag{1.2.40}
\end{equation*}
$$

We then have the drifts given by

$$
\begin{equation*}
\phi_{x}=-\frac{y}{x^{3}} \frac{G^{\prime}(y / x)}{G(y / x)}, \phi_{y}=\frac{1}{x} \frac{G^{\prime}(y / x)}{G(y / x)} . \tag{1.2.41}
\end{equation*}
$$

For example, the choice $k(r)=-\frac{2}{r^{2}}$ leads to

$$
\begin{equation*}
G(r)=\frac{c_{1}}{r}+c_{2} \frac{\left(r-\tan ^{-1}(r)\right)}{r} \tag{1.2.42}
\end{equation*}
$$

and so

$$
\begin{gather*}
\phi_{x}=\frac{\left(c_{1} x^{2}+c_{2} y x+c_{1} y^{2}-c_{2}\left(x^{2}+y^{2}\right) \tan ^{-1}\left(\frac{y}{x}\right)\right)}{x\left(x^{2}+y^{2}\right)\left(c_{1} x+c_{2} y-c_{2} x \tan ^{-1}\left(\frac{y}{x}\right)\right)}  \tag{1.2.43}\\
\phi_{y}=-\frac{x\left(c_{1} x^{2}+c_{2} y x+c_{1} y^{2}-c_{2}\left(x^{2}+y^{2}\right) \tan ^{-1}\left(\frac{y}{x}\right)\right)}{y\left(x^{2}+y^{2}\right)\left(c_{1} x+c_{2} y-c_{2} x \tan ^{-1}\left(\frac{y}{x}\right)\right)} . \tag{1.2.44}
\end{gather*}
$$

For drift equations of the form $\phi_{x x}+\phi_{y y}+\phi_{x}^{2}+\phi_{y}^{2}=C\left(x^{2}+y^{2}\right)$ the change of variables $r=x^{2}+y^{2}$ will also lead to a Riccati equation.

Suppose that we want

$$
\phi_{x x}+\phi_{y y}+\phi_{x}^{2}+\phi_{y}^{2}=\frac{A}{x^{2}+y^{2}}
$$

With $r=x^{2}+y^{2}$ take $\phi(x, y)=\Phi\left(x^{2}+y^{2}\right)$ then

$$
4 r \Phi^{\prime \prime}+4 \Phi^{\prime}+4 r\left(\Phi^{\prime}\right)^{2}=\frac{A}{r}
$$

So if we put $\Phi^{\prime}=\frac{h^{\prime}}{h}$, we have

$$
\begin{equation*}
4 r^{2} h^{\prime \prime}+4 r h^{\prime}-A h=0 \tag{1.2.45}
\end{equation*}
$$

which has solutions $h(r)=c_{1} r^{\frac{1}{2} \sqrt{A}}+c_{2} r^{-\frac{1}{2} \sqrt{A}}$ from which drifts may be obtained. Simply by including a constant term, that is working with the PDE

$$
u_{t}=\Delta u-\left(\frac{A}{x^{2}+y^{2}}+E\right) u
$$

we discover more drifts. We solve

$$
4 r \Phi^{\prime \prime}+4 \Phi^{\prime}+4 r\left(\Phi^{\prime}\right)^{2}=\frac{A}{r}+E
$$

This leads to the Bessel equation

$$
\begin{equation*}
4 r^{2} h^{\prime \prime}+4 r h^{\prime}-(A+E r) h=0 \tag{1.2.46}
\end{equation*}
$$

with solutions $h(r)=c_{1} I_{\sqrt{A}}(\sqrt{E r})+c_{2} I_{-\sqrt{A}}(\sqrt{E r})$. These types of drifts will be explored in the body of the thesis.

## Integral Transform Methods in One Dimension

Symmetry analysis has proved a useful tool for the problem of finding fundamental solutions for parabolic equations on the line. Bluman and Cole used group-invariant solutions to obtain fundamental solutions, see for example [12]. Methods involving reduction to canonical form have also been employed, see [33]. See also Rosinger and Walus [65] on group invariant generalized solutions in the context of nonlinear PDEs. Here we present results using Lie symmetry methods to obtain integral transforms of fundamental solutions in one space dimension. In this chapter we discuss some methods for computing fundamental solutions for PDEs of the form

$$
\begin{equation*}
u_{t}=\sigma x^{\gamma} u_{x x}+f(x) u_{x}-g(x) u, \sigma>0, \gamma \in \mathbb{R} \tag{2.0.47}
\end{equation*}
$$

which possesses a sufficiently large symmetry group. Although there are many ways of obtaining fundamental solutions, the integral transform method has a number of advantages over alternative methods. We can obtain closed-form expressions for the fundamental solutions that do not require any changes of variables or measure. As we will see in the following chapter, another important advantage is that the method can be extended in some cases to higher dimensions. Craddock and Platen [23] developed the technique for the $\gamma=1, g(x)=0$ case, and these results were extended in [22] and [26]. Their method reduces the problem to the evaluation of a single inverse Laplace transform, which
is given as an explicit function of the drift $f$. Craddock and Platen showed that for a large class of PDEs, one of the multipliers in the Lie point symmetry group is the Laplace transform of the fundamental solution. It was then shown in [22] that for at least one of the vector fields in the $\mathfrak{s l}_{2}$ part of the Lie symmetry algebra, the multiplier of the symmetry, or a slight modification of the multiplier, is always a classical integral transform of the fundamental solution of the PDE. Then in $[\mathbf{1 8}]$ it was proved that if the PDE

$$
u_{t}=A(x, t) u_{x x}+B(x, t) u_{x}+C(x, t) u
$$

has at least a four-dimensional symmetry group, we can always find a point symmetry which maps a nonzero solution to a Fourier or Laplace transform of a fundamental solution.
2.0.7.2. Finding Integral Transforms. We provide an overview of the transform method. For simplicity we consider the single linear equation

$$
\begin{equation*}
u_{t}=P\left(x, u^{(n)}\right), \quad x \in \Omega \subseteq \mathbb{R} \tag{2.0.48}
\end{equation*}
$$

Using Lie's method we find vector fields of the form

$$
\begin{equation*}
\mathbf{v}=\xi(x, t, u) \partial_{x}+\tau(x, t, u) \partial_{t}+\phi(x, t, u) \partial_{u} \tag{2.0.49}
\end{equation*}
$$

which generate one-parameter groups preserving solutions of (2.0.48). We denote by $\tilde{u}_{\epsilon}=\rho(\exp \epsilon \mathbf{v}) u(x, t)$ the action on solutions generated by $\mathbf{v}$. Typically we have

$$
\begin{equation*}
\rho(\exp \epsilon \mathbf{v}) u(x, t)=\sigma(x, t ; \epsilon) u\left(a_{1}(x, t ; \epsilon), a_{2}(x, t ; \epsilon)\right) \tag{2.0.50}
\end{equation*}
$$

for some functions $\sigma, a_{1}$ and $a_{2}$. We call $\sigma$ the multiplier and $a_{1}$ and $a_{2}$ the change of variables of the symmetry. Now suppose that (2.0.48)
has a fundamental solution $p(t, x, y)$. Then the function

$$
\begin{equation*}
u(x, t)=\int_{\Omega} f(y) p(t, x, y) d y \tag{2.0.51}
\end{equation*}
$$

solves the initial value problem for (2.0.48) with appropriate initial data $u(x, 0)=f(x)$. The idea behind the transform method is to connect the solutions (2.0.50) and (2.0.51). To do this, let us consider a stationary solution $u=u_{0}(x)$. Applying the symmetry gives

$$
\begin{equation*}
\rho(\exp \epsilon \mathbf{v}) u_{0}(x)=\sigma(x, t ; \epsilon) u_{0}\left(a_{1}(x, t ; \epsilon)\right) \tag{2.0.52}
\end{equation*}
$$

Then taking $t=0$ and using (2.0.51), leads to

$$
\begin{equation*}
\int_{\Omega} \sigma(y, 0 ; \epsilon) u_{0}\left(a_{1}(y, 0 ; \epsilon) p(t, x, y) d y=\sigma(x, t ; \epsilon) u_{0}\left(a_{1}(x, t ; \epsilon)\right)\right. \tag{2.0.53}
\end{equation*}
$$

Since $\sigma$ and $a_{1}$ are known, we have an equation for $p(t, x, y)$. As an example we consider the one-dimensional heat equation $u_{t}=u_{x x}$. If $u(x, t)$ solves the heat equation, then for $\epsilon$ small enough, so does

$$
\begin{equation*}
\tilde{u}_{\epsilon}=e^{-\epsilon x+\epsilon^{2} t} u(x-2 \epsilon t, t) . \tag{2.0.54}
\end{equation*}
$$

Taking $u_{0}=1$, equation (2.0.52) gives

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{-\epsilon y} p(t, x-y) d y=e^{-\epsilon x+\epsilon^{2} t} \tag{2.0.55}
\end{equation*}
$$

where $p(t, x)$ is the one-dimensional heat kernel. Then the multiplier in the symmetry (2.0.54) is the two-sided Laplace transform of $p(t, x-y)$. We can recover $p(t, x-y)$ by inverting (2.0.55). It turns out this result can be generalised to more complex problems. We present a result telling us when the relevant equations have nontrivial symmetries. A
proof of the following proposition is in $[\mathbf{2 0}]$. The $\gamma=1$ case was proved by Lennox [56].

Proposition 2.0.34 (Craddock-Lennox). If $\gamma \neq 2$, the $P D E$

$$
\begin{equation*}
u_{t}=\sigma x^{\gamma} u_{x x}+f(x) u_{x}-g(x) u, \quad x \geq 0 \tag{2.0.56}
\end{equation*}
$$

has a nontrivial Lie symmetry group if and only if $f$ is a solution of one of the following families of drift equations.

$$
\begin{align*}
& L f=A x^{2-\gamma}+B  \tag{2.0.57}\\
& L f=A x^{2-\gamma}+B x^{1-\gamma / 2}-\frac{3}{8(2-\gamma)} \sigma^{2} \tag{2.0.58}
\end{align*}
$$

where

$$
\begin{equation*}
L f=\sigma x^{\gamma}\left(\frac{x^{1-\gamma} f(x)}{2 \sigma(2-\gamma)}\right)^{\prime \prime}+f(x)\left(\frac{x^{1-\gamma} f(x)}{2 \sigma(2-\gamma)}\right)^{\prime}+g(x)+\frac{x g^{\prime}(x)}{2-\gamma} \tag{2.0.59}
\end{equation*}
$$

If $\gamma=2$ then the PDE has a nontrivial Lie group of symmetries if and only if

$$
\begin{align*}
& U f=A  \tag{2.0.60}\\
& U f=A \ln x+B \tag{2.0.61}
\end{align*}
$$

With $v(x)=\frac{f(x) \ln x}{x}$ we have

$$
\begin{equation*}
U f=\frac{x^{2}}{4} v^{\prime \prime}(x)+\frac{f(x)}{4 \sigma} v^{\prime}(x)-\frac{f(x)}{4 x}+\frac{x g^{\prime}(x) \ln x}{2}+g(x) \tag{2.0.62}
\end{equation*}
$$

For equations with nontrivial symmetries we can find integral transforms of fundamental solutions. The first result was proved in [56].

Theorem 2.0.35 (Lennox). Let $f$ be a solution of the Riccati equation

$$
\begin{equation*}
\sigma x f^{\prime}-\sigma f+\frac{1}{2} f^{2}+2 \mu \sigma x^{2}=A x+B \tag{2.0.63}
\end{equation*}
$$

Let $U_{0}(x)$ be a stationary solution of

$$
\begin{equation*}
u_{t}=\sigma x u_{x x}+f(x) u_{x}-\mu x u . \tag{2.0.64}
\end{equation*}
$$

Then there is a fundamental solution $p(t, x, y)$ of (2.0.64) such that

$$
U_{\lambda}(x, t)=\int_{0}^{\infty} U_{0}(y) p(t, x, y) e^{-\lambda y} d y,
$$

where

$$
\begin{align*}
U_{\lambda}(x, t)= & \exp \left\{\frac{1}{2 \sigma}\left(F\left(\frac{x}{(1+\sigma \lambda t)^{2}}\right)-F(x)\right)-\frac{\lambda\left(2 x+A t^{2}\right)}{2(1+\sigma \lambda t)}\right\} \times \\
& U_{0}\left(\frac{x}{(1+\sigma \lambda t)^{2}}\right) \tag{2.0.65}
\end{align*}
$$

Here $F^{\prime}(x)=f(x) / x$.

Note that if $\mu=1$ in (2.0.64) then the solution satisfying $u(x, T)=$ 1 gives the price of a zero-coupon bond in the case where the instantaneous short rate $X_{t}$ satisfies the SDE

$$
d X_{t}=f\left(X_{t}\right) d t+\sqrt{2 \sigma X_{t}^{\gamma}} d W_{t}
$$

See the book $[\mathbf{2 7}]$ for more details.
An extension of this theorem was proved by Craddock and Lennox in [22].

Theorem 2.0.36 (Craddock-Lennox). Suppose that $h(x)=x^{1-\gamma} f(x)$ is a solution of the Riccati equation

$$
\begin{equation*}
\sigma x h^{\prime}-\sigma h+\frac{1}{2} h^{2}+2 \sigma x^{2-\gamma} g(x)=2 \sigma A x^{2-\gamma}+B . \tag{2.0.66}
\end{equation*}
$$

Then the PDE

$$
\begin{equation*}
u_{t}=\sigma x^{\gamma} u_{x x}+f(x) u_{x}-g(x) u, x \geq 0 \tag{2.0.67}
\end{equation*}
$$

has a symmetry of the form

$$
\begin{aligned}
& \bar{U}_{\epsilon}(x, t)=\frac{1}{(1+4 \epsilon t)^{\frac{1-\gamma}{2-\gamma}}} \exp \left\{\frac{-4 \epsilon\left(x^{2-\gamma}+A \sigma(2-\gamma)^{2} t^{2}\right)}{\sigma(2-\gamma)^{2}(1+4 \epsilon t)}\right\} \times \\
& \exp \left\{\frac{1}{2 \sigma}\left(F\left(\frac{x}{(1+4 \epsilon t)^{\frac{2}{2-\gamma}}}\right)-F(x)\right)\right\} u\left(\frac{x}{(1+4 \epsilon t)^{\frac{2}{2-\gamma}}}, \frac{t}{1+4 \epsilon t}\right),
\end{aligned}
$$

where $F^{\prime}(x)=f(x) / x^{\gamma}$ and $u$ is a solution of the PDE. That is, $\bar{U}_{\epsilon}$ is a solution of (2.0.67) whenever $u$ is a solution. If $u(x, t)=u_{0}(x)$ with $u_{0}$ an analytic, stationary solution then there is a fundamental solution $p(t, x, y)$ of (2.0.67) such that

$$
\begin{equation*}
\int_{0}^{\infty} e^{-\lambda y^{2-\gamma}} u_{0}(y) p(t, x, y) d y=U_{\lambda}(x, t) \tag{2.0.68}
\end{equation*}
$$

Here $U_{\lambda}(x, t)=\bar{U}_{\frac{1}{4} \sigma(2-\gamma)^{2} \lambda}$.
The proof we present below is taken from [22].

Proposition 2.0.37. The solution $U_{\lambda}(x, t)$ in Theorem 2.0.36 is the Laplace transform of a distribution.

Proof. This follows from the observation that $U_{\lambda}(x, t)$ can be written as a product of $\lambda^{\nu}$ for some value $\nu$ and an analytic function $G(1 / \lambda)$. Any function which is analytic in $1 / \lambda$ is a Laplace transform. Further, $\lambda^{\nu}$ is the Laplace transform of a distribution. The product of two

Laplace transforms is a Laplace transform. Hence $U_{\lambda}(x, t)$ is a Laplace transform. See the table in [71], p348 for the inverse Laplace transform of $\lambda^{\nu}$ for different values of $\nu$ and Chapter 10 for general results on when a distribution can be represented as an inverse Laplace transform.

Now we proceed to the proof of the main result.

Proof. Lie's method shows that (2.0.67) has an infinitesimal symmetry of the form

$$
\begin{equation*}
\mathbf{v}=\frac{8 x t}{2-\gamma} \partial_{x}+4 t^{2} \partial_{t}-\left(\frac{4 x^{2-\gamma}}{\sigma(2-\gamma)^{2}}+\frac{4 x^{1-\gamma} t f(x)}{\sigma(2-\gamma)}+\beta t+4 A t^{2}\right) u \partial_{u} \tag{2.0.69}
\end{equation*}
$$

where $\beta=\frac{4(1-\gamma)}{2-\gamma}$. Exponentiating this symmetry and applying it to a solution $u(x, t)$ yields $\bar{U}_{\epsilon}$. By the remarks at the beginning of the chapter we write

$$
U_{\lambda}(x, t)=\int_{0}^{\infty} U_{\lambda}(y, 0) p(t, x, y) d y
$$

Now $U_{\lambda}(x, 0)=e^{-\lambda x^{2-\gamma}} u_{0}(x)$. This implies that

$$
\int_{0}^{\infty} e^{-\lambda y^{2-\gamma}} u_{0}(y) p(t, x, y) d y=U_{\lambda}(x, t)
$$

But this is the generalised Laplace transform of $u_{0} p$. We then need to show that $U_{\lambda}$ is a generalised Laplace transform of some distribution $u_{0} p$. Since

$$
\int_{0}^{\infty} e^{-\lambda y^{2-\gamma}} u_{0}(y) p(t, x, y) d y=\int_{0}^{\infty} e^{-\lambda z} u_{0}\left(z^{\frac{1}{2-\gamma}}\right) p\left(t, x, z^{\frac{1}{2-\gamma}}\right) \frac{z^{\frac{1}{2-\gamma}-1} d z}{2-\gamma}
$$

we must show that $U_{\lambda}$ is the Laplace transform of some distribution $u_{0} p$. This follows from Proposition 2.0.37. Now to show that $p$ is a fundamental solution of the PDE, we integrate a test function $\varphi(\lambda)$
with sufficiently rapid decay against $U_{\lambda}$, then the function $u(x, t)=$ $\int_{0}^{\infty} U_{\lambda}(x, t) \varphi(\lambda) d \lambda$ is a solution of (2.0.67). Note that we have
$u(x, 0)=\int_{0}^{\infty} U_{\lambda}(x, 0) \varphi(\lambda) d \lambda=\int_{0}^{\infty} u_{0}(x) e^{-\lambda x^{2-\gamma}} \varphi(\lambda) d \lambda=u_{0}(x) \Phi(x)$,
where $\Phi$ is the generalised Laplace transform of $\varphi$. Next observe that by Fubini's Theorem

$$
\begin{aligned}
\int_{0}^{\infty} u_{0}(y) \Phi(y) p(t, x, y) d y & =\int_{0}^{\infty} \int_{0}^{\infty} u_{0}(y) \varphi(\lambda) p(t, x, y) e^{-\lambda y^{2-\gamma}} d \lambda d y \\
& =\int_{0}^{\infty} \int_{0}^{\infty} u_{0}(y) \varphi(\lambda) p(t, x, y) e^{-\lambda y^{2-\gamma}} d y d \lambda \\
& =\int_{0}^{\infty} \varphi(\lambda) U_{\lambda}(x, t) d x=u(x, t)
\end{aligned}
$$

We know that $u(x, 0)=u_{0}(x) \Phi(x)$. Thus integrating the initial data $u_{0} \Phi$ against $p$ solves the Cauchy problem for (2.0.67), with this initial data. Hence $p$ is a fundamental solution.

An important fact is that this theorem can always be used to produce a fundamental solution which is a probability density. This is in contrast to other methods, which do not always achieve this; see the remarks in Chapter 1, subsection 1.2.2. However we have the following corollary.

Corollary 2.0.38. If $g=0$ in (2.0.67), then there is a fundamental solution $p(t, x, y)$ with the property

$$
\begin{aligned}
\int_{0}^{\infty} e^{-\lambda y^{2-\gamma}} p(t, x, y) d y= & \frac{1}{(1+4 \epsilon t)^{\frac{1-\gamma}{2-\gamma}}} \exp \left\{\frac{-4 \epsilon\left(x^{2-\gamma}+A \sigma(2-\gamma)^{2} t^{2}\right)}{\sigma(2-\gamma)^{2}(1+4 \epsilon t)}\right\} \\
& \times \exp \left\{\frac{1}{2 \sigma}\left(F\left(\frac{x}{(1+4 \epsilon t)^{\frac{2}{2-\gamma}}}\right)-F(x)\right)\right\}
\end{aligned}
$$

and $\int_{0}^{\infty} p(t, x, y) d y=1$. Here $\epsilon=\frac{1}{4} \sigma(2-\gamma)^{2} \lambda$.

Proof. Since $g(x)=0$ we may take $u_{0}(x)=1$ in 2.0.36. Observe that $U_{0}(x, t)=1$. Thus $\int_{0}^{\infty} p(t, x, y) d y=1$.

For the $\gamma=2$ case we have a similar result.

Theorem 2.0.39 (Craddock-Lennox). Suppose that

$$
\begin{equation*}
\frac{x^{2}}{4}\left(\frac{f(x) \ln x}{x}\right)^{\prime \prime}+\frac{f(x)}{4 \sigma}\left(\left(\frac{f(x) \ln x}{x}\right)^{\prime}-\frac{\sigma}{x}\right)+\frac{x \ln x g^{\prime}(x)}{2}+g(x)=A \tag{2.0.70}
\end{equation*}
$$

Let $u_{0}(x)$ be a stationary solution of

$$
\begin{equation*}
u_{t}=\sigma x^{2} u_{x x}+f(x) u_{x}-g(x) u \tag{2.0.71}
\end{equation*}
$$

which is analytic near zero. Then there is a fundamental solution of (2.0.71) such that

$$
\begin{equation*}
\int_{0}^{\infty} e^{-\frac{\epsilon}{\sigma}(\ln y)^{2}} p(t, x, y) u_{0}(y) d y=U_{\epsilon}(x, t) \tag{2.0.72}
\end{equation*}
$$

where

$$
\begin{aligned}
U_{\epsilon}(x, t)= & \frac{1}{\sqrt{1+4 \epsilon t}} \exp \left\{-\frac{\left.\epsilon\left((\ln x)^{2}+2 \sigma t \ln x+(4 A+\sigma) \sigma\right) t^{2}\right)}{\sigma(1+4 \epsilon t)}\right\} \\
& \times \exp \left\{\frac{1}{2 \sigma}\left(F\left(\frac{\ln x}{1+4 \epsilon t}\right)-F(x)\right)\right\} u_{0}\left(x^{\frac{1}{1+4 \epsilon t}}\right)
\end{aligned}
$$

and $F^{\prime}(x)=e^{-x} f(x)$.

Proof. The proof is similar to the previous result. If the drift $f$ satisfies (2.0.70), then the PDE has an infinitesimal symmetry of the form

$$
\mathbf{v}=4 x t \ln x \partial_{x}+4 t^{2} \partial_{t}-\left(\frac{(\ln x)^{2}}{\sigma}+\ln x^{2} g(x) t+2 t+(4 A+\sigma) t^{2}\right) u \partial_{u}
$$

with $g(x)=\left(\frac{f(x)}{\sigma x}-1\right)$. Exponentiating $\mathbf{v}$ and applying it to $u_{0}$ produces $U_{\epsilon}(x, t)$. This has the initial value $U_{\epsilon}(x, 0)=u_{0}(x) e^{-\frac{\epsilon}{\sigma}(\ln x)^{2}}$. The proof follows the same lines as previously.

Results for obtaining fundamental solutions for the other Riccati equations were established in Craddock [26] and Craddock and Lennox's papers [22] and [20]. In particular, Craddock constructed Laplace transforms of fundamental solutions for the other Riccati equations which give nontrivial symmetries. These results can be used to obtain transition densities and functionals for many one-dimensional processes. We present some examples below.
2.0.7.3. Finding Drifts. To obtain drift functions, we set

$$
f(x)=2 \sigma x y^{\prime}(x) / y(x)
$$

This transforms equation (2.0.63) to the linear ODE

$$
\begin{equation*}
2 \sigma^{2} x^{2} y^{\prime \prime}(x)+\left(2 \mu \sigma x^{2}-A x-B\right) y(x)=0 \tag{2.0.73}
\end{equation*}
$$

A substitution of the form $y=x^{m} e^{n x} u$ reduces this to the confluent hypergeometric equation, so (2.0.73) has the general solution

$$
\begin{equation*}
y(x)=e^{\frac{-i x \sqrt{\mu}}{\sqrt{\sigma}}} x^{\beta / 2}\left(a_{1} F_{1}\left(\alpha, \beta, \frac{2 i x \sqrt{\mu}}{\sqrt{\sigma}}\right)+b \Psi\left(\alpha, \beta, \frac{2 i x \sqrt{\mu}}{\sqrt{\sigma}}\right)\right) \tag{2.0.74}
\end{equation*}
$$

where $a$ and $b$ are constants, $\alpha=\frac{-\left(i A-2 \sqrt{\mu} \sigma^{\frac{3}{2}}-2 \sqrt{\mu} \sqrt{1+\frac{2 B}{\sigma^{2}}} \sigma^{\frac{3}{2}}\right)}{4 \sqrt{\mu} \sigma^{\frac{3}{2}}}, \beta=1+$ $\sqrt{1+\frac{2 B}{\sigma^{2}}},{ }_{1} F_{1}$ is Kummer's confluent hypergeometric function and $\Psi$ is Tricomi's confluent hypergeometric function, given by formula 13.1.6 in [1]. These are solutions of the confluent hypergeometric equation. This implies that $f$ is analytic. To obtain a stationary solution $u_{0}$ we
solve

$$
\begin{equation*}
\sigma x u_{x x}+f(x) u_{x}-\mu x u=0 \tag{2.0.75}
\end{equation*}
$$

We set $u=\tilde{u}(x) e^{\int \varphi(x) d x}$ with $\varphi(x)=-\frac{1}{2 \sigma x} f(x)$. Making the substitution gives

$$
\begin{equation*}
2 \sigma^{2} x^{2} \tilde{u}_{x x}-(A x+B) \tilde{u}=0 \tag{2.0.76}
\end{equation*}
$$

Equation (2.0.76) is solved in terms of Bessel functions. Hence $u_{0}$ is also analytic.

Example 2.0.5. If $A=0$, then

$$
\begin{equation*}
y(x)=a \sqrt{x} J_{\eta}(x \sqrt{\mu / \sigma})+b \sqrt{x} Y_{\eta}(x \sqrt{\mu / \sigma}) . \tag{2.0.77}
\end{equation*}
$$

Here $\eta=\frac{1}{2} \sqrt{1+2 B / \sigma^{2}}, J_{\eta}, Y_{\eta}$ are Bessel functions of the first and second kinds and $a$ and $b$ are arbitrary constants. If $B=a=0$ and $b=$ 1, we obtain the drift function $f(x)=2 x \sqrt{\mu \sigma} \cot \left(x \sqrt{\frac{\mu}{\sigma}}\right)$. A stationary solution of (2.0.75) is $u_{0}(x)=\csc \left(x \sqrt{\frac{\mu}{\sigma}}\right)$. Applying Theorem (2.0.35) we obtain

$$
U_{\lambda}(x, t)=\exp \left\{\frac{-\lambda x}{1+t \lambda \sigma}\right\} \csc \left(x \sqrt{\frac{\mu}{\sigma}}\right)
$$

Computing the inverse Laplace transform gives the fundamental solution,

$$
p_{\mu}(t, x, y)=\exp \left\{\frac{-(x+y)}{\sigma t}\right\}\left(\frac{\sqrt{x}}{\sqrt{y} \sigma t} I_{1}\left(\frac{2 \sqrt{x y}}{\sigma t}\right)+\delta(y)\right) \frac{\sin \left(y \sqrt{\frac{\mu}{\sigma}}\right)}{\sin \left(x \sqrt{\frac{\mu}{\sigma}}\right)} .
$$

Example 2.0.6. Setting $A=0, B=4 \sigma^{2}, a=1$ and $b=1$ in the general solution (2.0.74) we obtain the drift function

$$
f(x)=\frac{-2 \sigma}{g(x)}\left(g(x)+\frac{\mu x^{2}}{\sqrt{\sigma}}\left(\sin \left(x \sqrt{\frac{\mu}{\sigma}}\right)-\cos \left(x \sqrt{\frac{\mu}{\sigma}}\right)\right)\right)
$$

where $g(x)=(x \sqrt{\mu}+\sqrt{\sigma}) \cos \left(x \sqrt{\frac{\mu}{\sigma}}\right)+(x \sqrt{\mu}-\sqrt{\sigma}) \sin \left(x \sqrt{\frac{\mu}{\sigma}}\right)$. Omitting the details, we obtain via Theorem 2.0.35 a fundamental solution
of $u_{t}=\sigma x u_{x x}+f(x) u_{x}-\mu x u$. It is

$$
p_{\mu}(t, x, y)=\frac{1}{\sigma t} \exp \left\{-\frac{x+y}{\sigma t}\right\}\left(\frac{x}{y}\right)^{\frac{3}{2}} I_{3}\left(\frac{2 \sqrt{x y}}{\sigma t}\right) \frac{g(y)}{g(x)}
$$

These classes of equations arise in bond pricing.
2.0.8. Applications to Stochastic Calculus. We present some applications to stochastic calculus. The results below show how to calculate expectations of the form $E_{x}\left(e^{-\lambda X_{t}-\mu \int_{0}^{t} \frac{d s}{X_{s}}}\right)$. We wish to compute these expectations when $f$ satisfies the Riccati equation

$$
\sigma x f^{\prime}-\sigma f+\frac{1}{2} f^{2}=\frac{1}{2} A x^{2}+B x+C
$$

The approach taken in this section is to use group-invariant solution techniques. Recall from Chapter 1, that Bluman has used this method to find fundamental solutions of Fokker-Planck equations of the form $u_{t}=u_{x x}+(f(x) u)_{x},([\mathbf{1 1}])$. Some of the fundamental theorems obtained could in principle be deduced from Bluman's results by deriving a suitable point transformation. The analysis here is presented with an aim to calculate functionals of certain square root processes directly, without requiring any change of variables. Applications and examples will be presented below.

Theorem 2.0.40 (Lennox). Suppose that

$$
\sigma x f^{\prime}-\sigma f+\frac{1}{2} f^{2}=\frac{1}{2} A x^{2}+B x+C, A>0
$$

Then the PDE

$$
u_{t}=\sigma x u_{x x}+f(x) u_{x}-\frac{\mu}{x} u, \quad \mu \geq 0
$$

has a fundamental solution of the form

$$
\begin{aligned}
p(t, x, y)= & \frac{\sqrt{A} e^{\frac{1}{2 \sigma}(F(y)-F(x))}}{2 \sigma \sinh \left(\frac{\sqrt{A} t}{2}\right)} \sqrt{\frac{x}{y}} \exp \left\{-\frac{B t}{2 \sigma}-\frac{\sqrt{A}(x+y)}{2 \sigma \tanh \left(\frac{\sqrt{A} t}{2}\right)}\right\} \\
& \times\left(C_{1}(y) I_{\nu}\left(\frac{\sqrt{A x y}}{\sigma \sinh \left(\frac{\sqrt{A} t}{2}\right)}\right)+C_{2}(y) I_{-\nu}\left(\frac{\sqrt{A x y}}{\sigma \sinh \left(\frac{\sqrt{A} t}{2}\right)}\right)\right)
\end{aligned}
$$

in which $F^{\prime}(x)=f(x) / x$ and $\nu=\frac{\sqrt{2 C+4 \mu \sigma+\sigma^{2}}}{\sigma}$ and we interpret $I_{-\nu}(z)$ to be $K_{\nu}(z)$ if $\nu$ is an integer.

Proof. Lennox proved in [56] that the PDE has a Lie algebra of symmetries spanned by

$$
\begin{aligned}
& \mathbf{v}_{1}=\partial_{t}, \mathbf{v}_{2}=x e^{\sqrt{A} t} \partial_{x}+\frac{e^{\sqrt{A} t}}{\sqrt{A}} \partial_{t}-\frac{1}{2 \sigma}\left(\sqrt{A} x+f(x)+\frac{B}{\sqrt{A}}\right) e^{\sqrt{A} t} u \partial_{u} \\
& \mathbf{v}_{3}=-x e^{-\sqrt{A} t} \partial_{x}+\frac{e^{-\sqrt{A} t}}{\sqrt{A}} \partial_{t}-\frac{1}{2 \sigma}\left(\sqrt{A} x-f(x)+\frac{B}{\sqrt{A}}\right) e^{-\sqrt{A} t} u \partial_{u} \\
& \mathbf{v}_{4}=u \partial_{u}, \mathbf{v}=\beta(x, t) \partial_{u} .
\end{aligned}
$$

Using Theorem 1.2.24 from Chapter 1, we find invariants of $\mathbf{v}=$ $\sum_{k=1}^{4} c_{k} \mathbf{v}_{k}$ which preserve the boundary conditions. The invariants are

$$
\begin{aligned}
& \eta=\frac{x}{4 \sinh ^{2}\left(\frac{\sqrt{A} t}{2}\right)}, \\
& u=\exp \left(-\frac{(B t+F(x)-F(y))}{2 \sigma}-\frac{\sqrt{A}(x+y)}{2 \sigma \tanh \left(\frac{\sqrt{A} t}{2}\right)}\right) v\left(\frac{x}{4 \sinh ^{2}\left(\frac{\sqrt{A} t}{2}\right)}\right) .
\end{aligned}
$$

Using these invariants the PDE becomes

$$
4 \sigma^{2} \eta^{2} v^{\prime \prime}(\eta)-(2 C+4 A y \eta+4 \mu \sigma) v(\eta)=0
$$

Solving this ODE proves the result.
The functions $C_{1}(y)$ and $C_{2}(y)$ depend on the boundary conditions. The dependence of $C_{1}, C_{2}$ is often overlooked in the literature. Usually
it is sufficient to take $C_{1}(y)=1, C_{2}(y)=0$. The proofs for the following corollaries can be found in [20]. As an application we calculate the Laplace transform of the joint density of $\left(X_{t}, \int_{o}^{t} \frac{d s}{X_{s}}\right)$ for an important process.

Corollary 2.0.41. The PDE

$$
\begin{equation*}
u_{t}=\sigma x u_{x x}+(a-b x) u_{x}-\frac{\mu}{x} u, \quad \mu \geq 0, a \geq 0 \tag{2.0.78}
\end{equation*}
$$

has a fundamental solution

$$
\begin{align*}
p(t, x, y)= & \frac{b}{2 \sigma \sinh \left(\frac{b t}{2}\right)}\left(\frac{y}{x}\right)^{\frac{a}{2 \sigma}-\frac{1}{2}} \exp \left(\frac{b}{2 \sigma}\left(a t+(x-y)-\frac{x+y}{\tanh \left(\frac{b t}{2}\right)}\right)\right) \\
& \times I_{\nu}\left(\frac{b \sqrt{x y}}{\sigma \sinh \left(\frac{b t}{2}\right)}\right) \tag{2.0.79}
\end{align*}
$$

Here $\nu=\frac{1}{\sigma} \sqrt{(a-\sigma)^{2}+4 \mu \sigma}$.

This next result gives the two-dimensional Laplace transform of the joint density of the two-dimensional process $\left(X_{t}, \int_{0}^{t} \frac{d s}{X_{s}}\right)$.

Corollary 2.0.42. Let

$$
k=\frac{a}{2 \sigma}, \quad \alpha=\frac{b}{2 \sigma}\left(1+\operatorname{coth}\left(\frac{b t}{2}\right)\right)+\lambda, \quad \beta=\frac{b \sqrt{x}}{2 \sigma \sinh \left(\frac{b t}{2}\right)} .
$$

and $M_{s, r}(z)$ be the Whittaker functions of the first kind. For the CIR process $d X_{t}=\left(a-b X_{t}\right) d t+\sqrt{2 \sigma X_{t}} d W_{t}$ we have

$$
\begin{align*}
E_{x}\left(e^{-\lambda X_{t}-\mu \int_{0}^{t} \frac{d s}{X_{s}}}\right)= & \frac{\Gamma\left(k+\frac{\nu}{2}+\frac{1}{2}\right)}{\Gamma(\nu+1)} \beta x^{-k} \exp \left(\frac{b}{2 \sigma}\left(a t+x-\frac{x}{\tanh \left(\frac{b t}{2}\right)}\right)\right) \\
& \times \frac{1}{\beta \alpha^{k}} e^{\frac{\beta^{2}}{2 \alpha}} M_{-k, \frac{\nu}{2}}\left(\frac{\beta^{2}}{\alpha}\right) . \tag{2.0.80}
\end{align*}
$$

Proof. By the Feynman-Kac formula, Theorem 1.2.16 of Chapter 1, the expectation $u(x, t)=E_{x}\left(e^{-\lambda X_{t}-\mu \int_{0}^{t} \frac{d s}{X_{s}}}\right)$ is a solution of (2.0.78),
with $u(x, 0)=e^{-\lambda x}$. This solution is obtained by integration against the fundamental solution and so is given by

$$
\begin{equation*}
E_{x}\left(e^{-\lambda X_{t}-\mu \int_{0}^{t} \frac{d s}{X_{s}}}\right)=\int_{0}^{\infty} e^{-\lambda y} p(t, x, y) d y \tag{2.0.81}
\end{equation*}
$$

The result follows from the fact that

$$
\int_{0}^{\infty} y^{k-\frac{1}{2}} e^{-\alpha y} I_{2 \gamma}(2 \beta \sqrt{y}) d y=\frac{\gamma\left(k+\gamma+\frac{1}{2}\right)}{\Gamma(2 \gamma+1)} \frac{1}{\beta \alpha^{k}} e^{\frac{\beta^{2}}{2 \alpha}} M_{-k, \gamma}\left(\frac{\beta^{2}}{\alpha}\right)
$$

which is formula 6.643 .2 of [35].
Craddock and Lennox proved further results along these lines in [20]. We refer the reader to this paper for further details.
2.0.9. Square root process functional. In Chapter 5 , we present some applications in finance. We give two examples of the calculation of functionals. The first result will be used to price a volatility swap.

Proposition 2.0.43. Let $X=\left\{X_{t}: t \geq 0\right\}$ satisfy the $S D E$

$$
d X_{t}=\left(a-b X_{t}\right) d t+\sqrt{2 \sigma X_{t}}
$$

Let $\beta=1+m-\alpha+\frac{\nu}{2}, m=\frac{1}{2}\left(\frac{a}{\sigma}-1\right)$ and $\nu=\frac{1}{\sigma} \sqrt{(a-\sigma)^{2}+4 \mu \sigma}$. Then if $m>\alpha-1$,
$\mathbb{E}_{x}\left[\frac{\int_{0}^{t} \frac{d s}{X_{s}}}{X_{t}^{\alpha}}\right]=-x^{-m} e^{-\frac{b x}{\sigma\left(e^{b t}-1\right)}+b m t} \frac{d}{d \mu}\left(\left(\frac{b e^{b t}}{\left(e^{b t}-1\right) \sigma}\right)^{-m+\alpha-\frac{\nu}{2}} \times\right.$

$$
\left.\left(\frac{b^{2} x}{4 \sigma^{2} \sinh ^{2}\left(\frac{b t}{2}\right)}\right)^{\nu / 2} \frac{\Gamma\left(1+m-\alpha+\frac{\nu}{2}\right)}{\Gamma(1+\nu)}{ }_{1} F_{1}\left(\beta, 1+\nu, \frac{b x}{\sigma\left(e^{b t}-1\right)}\right)\right)\left.\right|_{\mu=0}
$$

Proof. The PDE $u_{t}=\sigma x u_{x x}+(a-b x) u_{x}-\frac{\mu}{x}$ has fundamental solution

$$
p(t, x, y)=\frac{1}{2 \sigma \sinh \left(\frac{b t}{2}\right)} b e^{\frac{b\left(a t+x-y-(x+y) \operatorname{coth}\left(\frac{b t}{2}\right)\right)}{2 \sigma}}\left(\frac{y}{x}\right)^{m} I_{\nu}\left(\frac{b \sqrt{x y}}{\sigma \sinh \left(\frac{b t}{2}\right)}\right) .
$$

Then $\mathbb{E}_{x}\left[\frac{\int_{0}^{t} \frac{d s}{X_{s}}}{X_{t}^{\alpha}}\right]=-\left.\frac{d}{d \mu} \mathbb{E}\left[X_{t}^{-\alpha} e^{-\mu \int_{0}^{t} \frac{d s}{X_{s}}}\right]\right|_{\mu=0}$. Now by the Feynman-
Kac formula

$$
\begin{equation*}
\mathbb{E}\left[X_{t}^{-\alpha} e^{-\mu \int_{0}^{t} \frac{d s}{X_{s}}}\right]=\int_{0}^{\infty} y^{-\alpha} p(t, x, y) d y \tag{2.0.82}
\end{equation*}
$$

The integral can be done in Mathematica or use 6.631 in [35]. The result follows.

Another example of this type follows.

Proposition 2.0.44. Let $X=\left\{X_{t}: t \geq 0\right\}$ satisfy the $S D E$

$$
d X_{t}=\left(a-b X_{t}\right) d t+\sqrt{2 \sigma X_{t}} d W_{t}
$$

Let $A=b^{2}+4 \mu \sigma, m=\frac{1}{\sigma} \sqrt{(a-\sigma)^{2}+4 \sigma \nu}, \beta=\frac{\sqrt{A x}}{\sigma \sinh \left(\frac{\sqrt{A} t}{2}\right.}$ and $k=$ $\frac{A+b \tanh \left(\frac{\sqrt{A} t}{2}\right)}{2 \sigma \tanh \left(\frac{\sqrt{A} t}{2}\right)}$. Then if $a>(2 \alpha-3) \sigma$,
$\mathbb{E}_{x}\left[X_{t}^{-\alpha} e^{-\mu \int_{0}^{t} \frac{d s}{X_{s}}-\nu \int_{0}^{t} X_{s} d s}\right]=\frac{\sqrt{A} \beta^{m}}{2 \sigma x^{\frac{a}{2^{m+2} \sigma}} \sinh \left(\frac{\sqrt{A} t}{2}\right)} e^{a b t-\frac{\sqrt{A} x}{2 \sigma \tanh \left(\frac{\sqrt{A} t}{2}\right)}+\frac{1}{2 \sigma} b x}$
$\times \frac{\Gamma\left(\frac{a+(3+m-2 \alpha) \sigma}{2 \sigma}\right)}{\Gamma(1+m)} k^{-\frac{a+(3+m-2 \alpha) \sigma}{2 \sigma}}{ }_{1} F_{1}\left(\frac{a+(3+m-2 \alpha) \sigma}{2 \sigma}, 1+m, \frac{\beta^{2}}{4 k}\right)$,
and

$$
\begin{equation*}
\mathbb{E}_{x}\left[\frac{\int_{0}^{t} X_{s} d s}{X_{t}^{\alpha}}\right]=-\left.\frac{d}{d \nu}\left(\mathbb{E}_{x}\left[X_{t}^{-\alpha} e^{-\mu \int_{0}^{t} \frac{d s}{X_{s}}-\nu \int_{0}^{t} X_{s} d s}\right]\right)\right|_{\mu=\nu=0} \tag{2.0.83}
\end{equation*}
$$

Proof. We proceed as for the previous proof. The necessary fundamental solution can be found using a similar approach to Theorem 2.0 .40 , see $[\mathbf{2 0}]$. We find the fundamental solution

$$
p(t, x, y)=\frac{\sqrt{A x y}}{2 \sigma \sinh \left(\frac{\sqrt{A} t}{2}\right)} e^{\frac{b(x-y+2 a t \sigma)-\sqrt{A}(x+y) \operatorname{coth}\left(\frac{\sqrt{A} t}{2}\right)}{2 \sigma}}\left(\frac{y}{x}\right)^{\frac{a}{2 \sigma}} I_{m}(\beta \sqrt{y})
$$

and the expectation is

$$
\mathbb{E}_{x}\left[X_{t}^{-\alpha} e^{-\mu \int_{0}^{t} \frac{d s}{X_{s}}-\nu \int_{0}^{t} X_{s} d s}\right]=\int_{0}^{\infty} y^{-\alpha} p(t, x, y) d y
$$

## Integral Transform Methods in Higher Dimensions

For one-dimension problems we can obtain one-dimensional Laplace and other classical transforms of fundamental solutions. We would like to extend this technique to higher-dimensional problems, however in higher dimensions we usually do not have enough one-parameter subgroups to construct these integral transforms. ${ }^{1}$ Even so, we show in this chapter that for certain subclasses of equations, specifically when there are Heisenberg group symmetries, we can in fact find such transforms. We also show how to compute transition densities for these classes of two-dimensional processes using Lie symmetry methods. We illustrate with the two-dimensional heat equation.

### 3.0.10. Multidimensional Fourier transforms. The two-

 dimensional heat equation has infinitesimal symmetries of the form$$
\begin{equation*}
\mathbf{v}_{1}=2 t \partial_{x}-x u \partial_{u}, \quad \mathbf{v}_{2}=2 t \partial_{y}-y u \partial_{u} . \tag{3.0.84}
\end{equation*}
$$

These symmetries, together with the vector fields $\mathbf{v}_{3}=\partial_{x}, \mathbf{v}_{4}=\partial_{y}$ and $\mathbf{v}_{5}=u \partial_{u}$ span a copy of the five-dimensional Heisenberg Lie algebra. Exponentiating, we see that if $u(x, y, t)$ is a solution of

$$
\begin{equation*}
u_{t}=u_{x x}+u_{y y}, \tag{3.0.85}
\end{equation*}
$$

[^1]then so are
\[

$$
\begin{equation*}
\rho\left(\exp \epsilon \mathbf{V}_{1}\right) u(x, y, t)=e^{-\epsilon x+\epsilon^{2} t} u(x-2 \epsilon t, y, t) \tag{3.0.86}
\end{equation*}
$$

\]

and

$$
\begin{equation*}
\rho\left(\exp \delta \mathbf{v}_{2}\right) u(x, y, t)=e^{-\delta y+\delta^{2} t} u(x, y-2 \delta t, t) \tag{3.0.87}
\end{equation*}
$$

If we apply these symmetries one after the other we obtain the solution

$$
\begin{equation*}
U_{\epsilon, \delta}(x, y, t)=e^{-\epsilon x-\delta y+\left(\delta^{2}+\epsilon^{2}\right) t} u(x-2 \epsilon t, y-2 \delta t, t) \tag{3.0.88}
\end{equation*}
$$

Now we use the right hand side of (3.0.88) and let $u=u_{0}$ be a stationary solution. Let $p(t, x, y, \xi, \eta)$ be a fundamental solution of the two-dimensional heat equation. As in the one-dimensional case we argue that

$$
\begin{equation*}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_{\epsilon, \delta}(\xi, \eta, 0) p(t, x, y, \xi, \eta) d \xi d \eta=U_{\epsilon, \delta}(x, y, t) \tag{3.0.89}
\end{equation*}
$$

which is the same as

$$
\begin{align*}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\epsilon \xi-\delta \eta} u_{0}(\xi, \eta) p & (t, x, y, \xi, \eta) d \xi d \eta \\
& =e^{-\epsilon x-\delta y+\left(\delta^{2}+\epsilon^{2}\right) t} u_{0}(x-2 \epsilon t, y-2 \delta t) \tag{3.0.90}
\end{align*}
$$

Now (3.0.90) is the two-dimensional, two-sided Laplace transform of $p$. Suppose we take $u_{0}=1$ and replace $\epsilon$ with $i \epsilon$ and $\delta$ with $i \delta$. Then we have

$$
\begin{equation*}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i \epsilon \xi-i \delta \eta} p(t, x, y, \xi, \eta) d \xi d \eta=e^{-i \epsilon x-i \delta y-\left(\delta^{2}+\epsilon^{2}\right) t} \tag{3.0.91}
\end{equation*}
$$

We have thus obtained the two-dimensional Fourier transform of the fundamental solution. Inverting gives

$$
p(t, x, y, \xi, \eta)=\frac{1}{4 \pi t} e^{\frac{-\left((x-\xi)^{2}+(y-\eta)^{2}\right)}{4 t}}
$$

It is possible to compute these types of Fourier transforms whenever there is a Heisenberg group of symmetries. There are two cases where this occurs. The first result is the following.

Theorem 3.0.45. We consider the PDE

$$
\begin{equation*}
u_{t}=\Delta u+2 \nabla \phi \cdot \nabla u+B(x) u, \quad x \in \mathbb{R}^{n} \tag{3.0.92}
\end{equation*}
$$

where $\phi$ is a solution of the quasi-linear PDE

$$
\Delta \phi+|\nabla \phi|^{2}+A(x)=B(x)
$$

and $A(x)=\sum_{i=1}^{n} a_{i} x_{i}+a_{n+1}$. Suppose also that $u_{0}$ is a nonzero solution such that as a function of $\epsilon$,

$$
K(t, x, \epsilon)=e^{-i \sum_{k=1}^{n} \epsilon_{k}\left(x_{k}-a_{k} t^{2}\right)-\sum_{k=1}^{n} \epsilon_{k}^{2} t+z(x, \epsilon)} \tilde{u_{0}}(x, \epsilon, t)
$$

is in $L^{1}\left(\mathbb{R}^{n}\right) \cap C^{2}\left(\mathbb{R}^{n}\right)$. Here $\tilde{u_{0}}(x, \epsilon, t)=u_{0}\left(x_{1}-2 i \epsilon_{1} t, \ldots, x_{n}\right.$ $\left.-2 i \epsilon_{n} t, t\right)$ and $z(x, \epsilon)=\phi\left(x_{1}-2 i \epsilon_{1} t, \ldots, x_{n}-2 i \epsilon_{n} t\right)-\phi\left(x_{1}, \ldots, x_{n}\right)$. Then there is a fundamental solution $p(t, x, y)$ of (3.0.92) such that

$$
\begin{equation*}
\int_{\mathbb{R}^{n}} e^{-i \sum_{k=1}^{n} \epsilon_{k} y_{k}} p(t, x, y) u_{0}(y, 0) d y=K(t, x, \epsilon) \tag{3.0.93}
\end{equation*}
$$

Proof. The PDE (3.0.92) has Lie symmetries coming from an action of the Heisenberg group given by

$$
\begin{aligned}
\rho\left(\exp \left(i \epsilon_{1} \mathbf{v}_{1}\right)\right) \cdots \rho\left(\exp \left(i \epsilon_{n} \mathbf{v}_{n}\right)\right) u(x, t) & =e^{-i \sum_{k=1}^{n} \epsilon_{k}\left(x_{k}-a_{k} t^{2}\right)-\sum_{k=1}^{n} \epsilon_{k}^{2} t+z(x, \epsilon)} \\
& \times u\left(x_{1}-2 i \epsilon_{1} t, \ldots, x_{n}-2 i \epsilon_{n} t, t\right),
\end{aligned}
$$

and so $K(t, x, \epsilon)$ is a solution of the PDE. Observe that

$$
K(0, x, \epsilon)=e^{-i \sum_{k=1}^{n} \epsilon_{k} x_{k}} u_{0}(x, 0)
$$

Since $K(t, x, \epsilon)$ is integrable define

$$
\begin{equation*}
P(t, x, y)=\frac{1}{(2 \pi)^{n}} \int_{\mathbb{R}^{n}} e^{i \sum_{k=1}^{n} \epsilon_{k} y_{k}} K(t, x, \epsilon) d \epsilon \tag{3.0.94}
\end{equation*}
$$

Since $K \in L^{1}\left(\mathbb{R}^{n}\right) \cap C^{2}\left(\mathbb{R}^{n}\right)$, the Fourier transform of $P$ exists, (see Theorem 1.2.6, Chapter 1). Now we let $u(x, t)=\int_{\mathbb{R}^{n}} \varphi(\epsilon) K(t, x, \epsilon) d \epsilon$, where $\varphi$ is a test function of suitably rapid decay. Notice that

$$
\begin{aligned}
u(x, 0) & =\int_{\mathbb{R}^{n}} \varphi(\epsilon) e^{-i \sum_{k=1}^{n} \epsilon_{k} x_{k}} u_{0}(x, 0) d \epsilon \\
& =u_{0}(x, 0) \Phi(x)
\end{aligned}
$$

where $\Phi$ is the Fourier transform of $\varphi$.
An application of Fubini's Theorem then shows that

$$
\begin{aligned}
\int_{\mathbb{R}^{n}} \Phi(y) P(t, x, y) d y & =\int_{\mathbb{R}^{2 n}} \varphi(\epsilon) P(t, x, y) e^{-i \sum_{k=1}^{n} \epsilon_{k} y_{k}} d \epsilon d y \\
& =\int_{\mathbb{R}^{2 n}} \varphi(\epsilon) P(t, x, y) e^{-i \sum_{k=1}^{n} \epsilon_{k} y_{k}} d y d \epsilon \\
& =\int_{\mathbb{R}^{n}} \varphi(\epsilon) K(t, x, \epsilon) d \epsilon=u(x, t)
\end{aligned}
$$

Thus integrating $\Phi$ against $P$ produces a solution with $u(x, 0)=u_{0} \Phi$. Hence $P(t, x, y)=p(t, x, y) u_{0}(y, 0)$, where $p$ is a fundamental solution.

Example 3.0.7. We will compute a fundamental solution of the PDE

$$
u_{t}=u_{x x}+u_{y y}+(a x+b y+c) u,(x, y) \in \mathbb{R}^{2}
$$

We use the exponential solution $u(x, y, t)=e^{\frac{1}{3} t\left(\left(a^{2}+b^{2}\right) t^{2}+3 c+3(a x+b y)\right)}$. The PDE has a symmetry

$$
\begin{equation*}
\tilde{u}_{\epsilon, \delta}(x, y, t)=e^{\frac{\left(a^{2}+b^{2}\right) t^{3}}{3}+c t+b(y-i t \delta) t+a(x-i t \epsilon) t-\left(\delta^{2}+\epsilon^{2}\right) t-i y \delta-i x \epsilon .} \tag{3.0.95}
\end{equation*}
$$

So by Theorem 3.0.45, we have

$$
\begin{aligned}
p(t, x, y, \xi, \eta) & =\frac{e^{\frac{\left(a^{2}+b^{2}\right) t^{3}}{3}+c t}}{(2 \pi)^{2}} \int_{\mathbb{R}^{2}} e^{i \xi \epsilon+i \eta \delta} e^{b(y-i t \delta) t+a(x-i t \epsilon) t-\left(\delta^{2}+\epsilon^{2}\right) t-i y \delta-i x \epsilon} d \epsilon d \delta \\
& =\frac{e^{c t}}{4 \pi t} e^{-\frac{(x-\xi)^{2}+(y-\eta)^{2}}{4 t}+\frac{1}{2} t(a(x+\xi)+b(y+\eta))+\frac{1}{12}\left(a^{2}+b^{2}\right) t^{3}}
\end{aligned}
$$

Similarly, one can show that the $n$-dimensional PDE

$$
\begin{equation*}
u_{t}=\Delta u+\left(\sum_{i=1}^{n} a_{i} x_{i}+c\right) u \tag{3.0.96}
\end{equation*}
$$

has a fundamental solution

$$
p(t, x, y)=\frac{e^{c t}}{(4 \pi t)^{\frac{n}{2}}} \exp \left(\frac{1}{12} \sum_{i=1}^{n} a_{i}^{2} t^{3}-\frac{\|x-y\|^{2}}{4 t}+\frac{t}{2} \sum_{i=1}^{n} a_{i}\left(x_{i}+y_{i}\right)\right) .
$$

There is a second case where we can extract fundamental solutions by Fourier inversion. These also arise from the Heisenberg group symmetries.

Theorem 3.0.46. We consider the PDE

$$
\begin{equation*}
u_{t}=\Delta u+2 \nabla \phi \cdot \nabla u+B(x) u, \quad x \in \mathbb{R}^{n} \tag{3.0.97}
\end{equation*}
$$

where $\phi$ is a solution of the quasi-linear PDE

$$
\Delta \phi+|\nabla \phi|^{2}+A(x)=B(x)
$$

and $A(x)=-\frac{1}{4} \sum_{k=1}^{n} c_{k} x_{k}^{2}, c_{k}>0$. Let $z(x, \epsilon)=\phi\left(x_{1}-\frac{2 i \epsilon_{1} \sinh \left(\sqrt{c_{1}} t\right)}{\sqrt{c_{2}}}, \ldots\right.$, $\left.x_{n}-\frac{2 i \epsilon_{n} \sinh \left(\sqrt{c_{n}} t\right)}{\sqrt{c_{n}}}\right)-\phi\left(x_{1}, \ldots, x_{n}\right)$. Suppose also that $u_{0}$ is a nonzero solution such that as a function of $\epsilon$

$$
\begin{aligned}
K(t, x, \epsilon) & =e^{-i \sum_{k=1}^{n} \epsilon_{k} x_{k} \cosh \left(\sqrt{c_{k}} t\right)-\sum_{k=1}^{n} \epsilon_{k} \frac{\sinh \left(2 \sqrt{c_{k}} t\right)}{2 \sqrt{c_{k}}}+z(x, \epsilon)} \\
& \times u_{0}\left(x_{1}-\frac{2 i \epsilon_{1} \sinh \left(\sqrt{c_{1}} t\right)}{\sqrt{c_{1}}}, \ldots, x_{n}-\frac{2 i \epsilon_{n} \sinh \left(\sqrt{c_{n}} t\right)}{\sqrt{c_{n}}}, t\right),
\end{aligned}
$$

is in $L^{1}\left(\mathbb{R}^{n}\right) \cap C^{2}\left(\mathbb{R}^{n}\right)$. Then there is a fundamental solution $p(t, x, y)$ of (3.0.97) such that

$$
\begin{equation*}
\int_{\mathbb{R}^{n}} e^{-i \sum_{k=1}^{n} \epsilon_{k} y_{k}} p(t, x, y) u_{0}(y, 0) d y=K(t, x, \epsilon) \tag{3.0.98}
\end{equation*}
$$

Proof. Using Lie's algorithm, we can show that there is a symmetry of the PDE of the form

$$
\begin{aligned}
& \Pi_{k=1}^{n} \rho\left(\exp \left(i \epsilon_{k} \mathbf{v}_{k}\right) u(x, t)=\right. e^{-i \sum_{k=1}^{n} \epsilon_{k} x_{k} \cosh \left(\sqrt{c_{1}} t\right)-\sum_{k=1}^{n} \frac{\sinh \left(2 \sqrt{c_{k}} t\right)}{2 \sqrt{c_{k}}} \epsilon_{k}^{2}+z(x, \epsilon)} \\
& \times u\left(x_{1}-\frac{2 i \epsilon_{1} \sinh \left(\sqrt{c_{1}} t\right)}{\sqrt{c_{1}}}, \ldots, x_{n}-\frac{2 i \epsilon_{n} \sinh \left(\sqrt{c_{n}} t\right)}{\sqrt{c_{n}}}, t\right) .
\end{aligned}
$$

The remainder of the proof proceeds along the same lines as the proof of Theorem 3.0.45.

We can handle the case $A(x)=-\sum_{k=1}^{n}\left(c_{k} x_{k}^{2}+a_{k} x_{k}\right)+b$ by making a change of variables in theorem 3.0.46. As in the one-dimensional case, we can always obtain a probability density.

Corollary 3.0.47. Let $p(t, x, y)$ be a fundamental solution obtained from Theorem 3.0.45 or Theorem 3.0.46. If $B=0$ and $u_{0}=1$, then

$$
\int_{\mathbb{R}^{n}} p(t, x, y) d y=1
$$

Proof. From Theorem 3.0.45, $\int_{\mathbb{R}^{n}} e^{-i \sum_{k=1}^{n} \epsilon_{k} y_{k}} p(t, x, y) d y=K(t, x, \epsilon)$, so that $\int_{\mathbb{R}^{n}} p(t, x, y) d y=K(t, x, 0)=1$ provided $u_{0}=1$. Similarly for Theorem 3.0.46.

Example 3.0.8. We obtain a fundamental solution for

$$
u_{t}=\Delta u-\frac{1}{4} c\left(x^{2}+y^{2}\right) u, c>0
$$

for $x, y \in \mathbb{R}^{2}$. We use the solution

$$
u_{1}(x, t)=e^{-\frac{1}{4} \sqrt{c}\left(x^{2}+y^{2}\right)-\frac{1}{2} \sqrt{c} t} .
$$

Applying Theorem 3.0.46 we obtain the following Fourier transform

$$
\begin{equation*}
\widehat{p}(t, x, y, \epsilon, \delta)=e^{\frac{-c\left(x^{2}+y^{2}+4 t\right)-4 i \sqrt{c} e^{-\sqrt{c t}}(y \delta+x \epsilon)-2\left(1-e^{-2 \sqrt{c t}}\right)\left(\delta^{2}+\epsilon^{2}\right)}{4 \sqrt{c}}} \tag{3.0.99}
\end{equation*}
$$

Then there is a fundamental solution such that

$$
\begin{aligned}
& u_{1}(x, y, 0) p(t, x, y, \xi, \eta)= \\
& \frac{1}{(2 \pi)^{2}} \int_{\mathbb{R}^{2}} e^{\frac{-c\left(x^{2}+y^{2}+4 t\right)-4 i \sqrt{c} e^{-\sqrt{c} t}(y \delta+x \epsilon)-2\left(1-e^{-2 \sqrt{c} t}\right)\left(\delta^{2}+\epsilon^{2}\right)}{4 \sqrt{c}}} d \epsilon d \delta .
\end{aligned}
$$

The integrals are standard Gaussians and the inversion may be carried out easily. For a discussion of Fourier inversion see [68]. We arrive at the fundamental solution

$$
\begin{equation*}
p(t, x, y, \xi, \eta)=\frac{\sqrt{c} e^{t}}{\pi\left(e^{2 \sqrt{c} t}-1\right)} e^{\left.\left.\frac{-\sqrt{c}\left(\left(y-e^{\sqrt{c}} t\right.\right.}{}\right)^{2}+\left(x-e^{\sqrt{c} t} \xi\right)^{2}\right]} 2\left(e^{2 \sqrt{c} t}-1\right), \tag{3.0.100}
\end{equation*}
$$

Next we compute a transition density for a two-dimensional Itô process.

Example 3.0.9. Suppose we are interested in the process

$$
\begin{align*}
d X_{t} & =\frac{X_{t}\left(6-X_{t}^{2}-Y_{t}^{2}\right)}{2\left(X_{t}^{2}+Y_{t}^{2}\right)-4} d t+\sqrt{2} d W_{t}^{1}  \tag{3.0.101}\\
d Y_{t} & =\frac{Y_{t}\left(6-X_{t}^{2}-Y_{t}^{2}\right)}{2\left(X_{t}^{2}+Y_{t}^{2}\right)-4} d t+\sqrt{2} d W_{t}^{2} \tag{3.0.102}
\end{align*}
$$

To obtain the transition density we must solve

$$
\begin{equation*}
u_{t}=\Delta u+\frac{x\left(6-x^{2}-y^{2}\right)}{2\left(x^{2}+y^{2}\right)-4} u_{x}+\frac{y\left(6-x^{2}-y^{2}\right)}{2\left(x^{2}+y^{2}\right)-4} u_{y}, x^{2}+y^{2} \neq 2 \tag{3.0.103}
\end{equation*}
$$

Take the stationary solution $u_{0}(x, y)=1$. Let

$$
\begin{equation*}
K(t, x, y, \xi, \eta)=\frac{e^{2 t}}{2\left(e^{2 t}-1\right) \pi} e^{-\frac{x^{2}-2 e^{t} \xi x+y^{2}-2 e^{t} y \eta+e^{2 t}\left(\eta^{2}+\xi^{2}\right)}{2\left(e^{2 t}-1\right)}} \tag{3.0.104}
\end{equation*}
$$

Then applying Theorem 3.0.46 gives the fundamental solution

$$
\begin{equation*}
p(t, x, y, \xi, \eta)=L K(t, x, y, \xi, \eta) \tag{3.0.105}
\end{equation*}
$$

where

$$
L=1-\frac{4 \sinh t}{x^{2}+y^{2}-2}\left(x \frac{\partial}{\partial \xi}+y \frac{\partial}{\partial \eta}\right)+\frac{4 \sinh ^{2} t}{x^{2}+y^{2}-2}\left(\frac{\partial^{2}}{\partial \xi^{2}}+\frac{\partial^{2}}{\partial \eta^{2}}\right)
$$

One strength of this methodology is that it allows for the calculation of many different fundamental solutions. To illustrate take the stationary solution

$$
u_{0}(x, y)=\sqrt{e} \operatorname{Ei}\left(\frac{1}{4}\left(x^{2}+y^{2}-2\right)\right)-\operatorname{Ei}\left(\frac{1}{4}\left(x^{2}+y^{2}\right)\right)
$$

where Ei is the exponential integral, [1], p227. According to Theorem 3.0.46

$$
\begin{aligned}
& p_{1}(t, x, y, \xi, \eta)=\frac{1}{(2 \pi)^{2}} \int_{\mathbb{R}^{2}} \frac{e^{-e^{-t}\left(i(y \delta+x \epsilon)+\left(\delta^{2}+\epsilon^{2}\right) \sinh (t)\right)}}{u_{0}(x, y)\left(x^{2}+y^{2}-2\right)} \\
& \times\left(x^{2}+y^{2}-4 \sinh (t)\left(i(y \delta+x \epsilon)+\left(\delta^{2}+\epsilon^{2}\right) \sinh (t)\right)-2\right) \\
& \times \sqrt{e} \operatorname{Ei}\left(\frac{1}{4}\left((x-2 i \sinh (t))^{2}+(y-2 i \sinh (t))^{2}-2\right)\right) \\
& -\operatorname{Ei}\left(\frac{1}{4}\left((x-2 i \sinh t)^{2}+(y-2 i \sinh t)^{2}\right)\right) d \delta d \epsilon
\end{aligned}
$$

is also a fundamental solution, different from the one above. We have not attempted to invert the Fourier transform. Many other fundamental solutions can be found by our methods.
3.0.11. A Generalization of Bessel Processes. There are some cases where we do not have Heisenberg group symmetries, but we can still obtain a fundamental solution by inverting a transform. Let us briefly present such an example. We consider the process

$$
\begin{equation*}
d X_{t}^{i}=\frac{2 a_{i}}{a_{1} X_{t}^{1}+\cdots+a_{n} X_{t}^{n}} d t+\sqrt{2} d W_{t}^{i}, X_{0}^{i}=x^{i}, i=1, \ldots, n \tag{3.0.106}
\end{equation*}
$$

with $a_{i}>0, i=1,2, \ldots$, which may be regarded as a multivariable analogue of a Bessel process. There are other known generalisations of Bessel processes, such as Wishart processes, see [14].

Proposition 3.0.48 (Craddock-Lennox). The transition density for the $n$-dimensional process $X_{t}=\left(X_{t}^{1}, \ldots, X_{t}^{n}\right)$ satisfying (3.0.106),
may be found by inverting the $n$-dimensional generalised Laplace transform

$$
\begin{aligned}
U_{\epsilon_{1}, \ldots, \epsilon_{n}}\left(x_{1}, \ldots, x_{n}, t\right) & =\frac{1}{\sqrt{1+4 \epsilon_{1} t} \cdots \sqrt{1+4 \epsilon_{n} t}} \exp \left(-\sum_{i=1}^{n} \frac{\epsilon_{i} x_{i}^{2}}{1+4 \epsilon_{1} t}\right) \\
& \times \exp \left(\phi\left(\frac{x_{1}}{1+4 \epsilon_{1} t}, \ldots, \frac{x_{n}}{1+4 \epsilon_{n} t}\right)-\phi\left(x_{1}, \ldots, x_{n}\right)\right),
\end{aligned}
$$

where $\phi\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\log \left(a_{1} x_{1}+\cdots+a_{n} x_{n}\right)$.

Proof. It is easy to see that $U_{\epsilon, \ldots, \epsilon}\left(x_{1}, \ldots, x_{n}, t\right)$ is a solution of the Kolmogorov backwards equation

$$
\begin{equation*}
u_{t}=\Delta u+\sum_{i=1}^{n} \frac{2 a_{i}}{\sum_{j=1}^{n} a_{j} x_{j}} . \tag{3.0.107}
\end{equation*}
$$

It is also a generalised Laplace transform. To see this, the change of variables $x_{i}^{2} \rightarrow y_{i}$ reduces it to a Laplace transform, and the resulting expression is a product of one-dimensional Laplace transforms and hence a Laplace transform. (See [71] for conditions under which a function is a Laplace transform). Arguing as in the Fourier transform case, we suppose that

$$
\int_{\mathbb{R}_{+}^{n}} e^{-\sum_{i=1}^{n} \epsilon_{i} \xi_{i}^{2}} p\left(t, x_{1}, \ldots, x_{n}, \xi_{1}, \ldots, \xi_{n}\right) d \xi_{1} \cdots d \xi_{n}=U_{\epsilon_{1}, \ldots, \epsilon_{n}}\left(x_{1}, \ldots, x_{n}, t\right)
$$

for some $p$. Setting $\epsilon_{i}=0, i=1, \ldots, n$ gives $\int_{\mathbb{R}_{+}^{n}} p=1$. The proof that $p$ is a fundamental solution is now exactly as in the Fourier transform case. We integrate a test function of sufficiently rapid decay against $U_{\epsilon_{1}, \ldots, \epsilon_{n}}$ to obtain a solution

$$
\begin{equation*}
u\left(x_{1}, \ldots, x_{n}, t\right)=\int_{\mathbb{R}_{+}^{n}} \phi\left(\epsilon_{1}, \ldots, \epsilon_{n}\right) U_{\epsilon, \ldots, \epsilon}\left(x_{1}, \ldots, x_{n}, t\right) d \epsilon_{1} \cdots d \epsilon_{n} \tag{3.0.108}
\end{equation*}
$$

with $u\left(x_{1}, \ldots, \xi_{n}, 0\right)=\Phi\left(x_{1}^{2}, \ldots, x_{n}^{2}\right)$. Using the same Fubini's Theorem argument as in the Fourier transform case we find that

$$
u\left(x_{1}, \ldots, x_{n}, t\right)=\int_{\mathbb{R}_{+}^{n}} \Phi\left(\xi_{1}^{2}, \ldots, \xi_{n}^{2}\right) p\left(t, x_{1}, \ldots, x_{n}, \xi_{1}, \ldots, \xi_{n}\right) d \xi_{1} \cdots d \xi_{n}
$$

Thus integrating $\Phi\left(\xi_{1}^{2}, \ldots, \xi_{n}^{2}\right)$ against $p$ produces a solution of (3.0.107), with initial data $u\left(x_{1}, \ldots, \xi_{n}, 0\right)=\Phi\left(x_{1}^{2}, \ldots, x_{n}^{2}\right)$. This shows that $p$ is a fundamental solution of the Kolmogorov backwards equations and it has total integral one.

This generalised Laplace transform is actually constructed by taking the symmetry of Corollary 1.2.33, and writing it as a product of $n$ one- dimensional symmetries, each with a different group parameter. It seems that this approach can be extended to other equations but we do not consider it here. Inversion of the $n$-dimensional generalised Laplace transform is easily accomplished. For example, for $n=2$ we have

$$
\begin{aligned}
p(t, x, y, \xi, \eta) & =\frac{1}{\pi t\left(a_{1} x+a_{2} y\right)} e^{-\frac{x^{2}+y^{2}+\xi^{2}+\eta^{2}}{4 t}} \times \\
& {\left[a_{1} \xi \sinh \left(\frac{x \xi}{2 t}\right) \cosh \left(\frac{y \eta}{2 t}\right)+a_{2} \eta \cosh \left(\frac{x \xi}{2 t}\right) \sinh \left(\frac{y \eta}{2 t}\right)\right] . }
\end{aligned}
$$

## Expansions of Fundamental Solutions

Standard techniques used to obtain fundamental solutions in the one-dimensional case often cannot be extended to higher-dimensional problems of the type we are concerned with in this thesis because there is not enough symmetry. Equations of the form $u_{t}=\Delta u+A(x) u$ on $\mathbb{R}^{n}$ for $n \geq 2$ typically have only $S L(2, \mathbb{R}) \times \mathbb{R}$ as the symmetry group. As the dimension of the PDE grows, the dimension of the Lie point symmetry group remains unchanged. As a result we do not have enough one-parameter subgroups to construct integral transforms. In this chapter we present the major contribution of the thesis. For a rich class of $n$-dimensional equations, we show that we can obtain fundamental solutions by Lie symmetry analysis if the symmetry group contains $S L(2, \mathbb{R})$. For the class of PDEs under study we obtain a series expansion of the desired solution. The method exploits the fact that the PDEs of interest have infinitely many linearly independent stationary solutions. The expansions we obtain for fundamental solutions are technically not eigenfunction expansions but they use eigenfunction expansions in their derivation. The two-dimensional problem can be regarded as completely solved and we can obtain some useful expressions for the $n$-dimensional problem. There are many cases where the eigenvalue problem can be solved analytically, and we present some examples in this chapter. However in most cases we can only obtain a numerical solution of the eigenvalue problem. We illustrate the idea
for the case $n=2$ and consider higher dimensions later. In Chapter 1 we saw that every PDE of the form

$$
\begin{equation*}
u_{t}=u_{x x}+u_{y y}+\frac{1}{x^{2}} k\left(\frac{y}{x}\right) u \tag{4.0.109}
\end{equation*}
$$

has a Lie point symmetry
$\tilde{u}_{\epsilon}(x, y, t)=\frac{1}{(1+4 \epsilon t)} \exp \left(-\frac{\epsilon\left(x^{2}+y^{2}\right)}{1+4 \epsilon t}\right) u\left(\frac{x}{1+4 \epsilon t}, \frac{y}{1+4 \epsilon t}, \frac{t}{1+4 \epsilon t}\right)$.

Now we integrate a test function $\varphi$ against $u_{\epsilon}$. Then, as long as $\varphi$ has sufficient decay

$$
\begin{equation*}
U(x, y, t)=\int_{0}^{\infty} \varphi(\epsilon) \tilde{u}_{\epsilon}(x, y, t) d \epsilon \tag{4.0.111}
\end{equation*}
$$

is a solution of (4.0.109), which satisfies

$$
\begin{equation*}
U(x, y, 0)=u(x, y, 0) \Phi\left(x^{2}+y^{2}\right) \tag{4.0.112}
\end{equation*}
$$

with $\Phi$ the Laplace transform of $\varphi$. By linearity, if $\left\{u_{k}\right\}$ are stationary solutions and $\left\{\varphi_{k}\right\}$ are test functions, then

$$
\begin{equation*}
U(x, y, t)=\sum_{k=1}^{\infty} \int_{0}^{\infty} \varphi_{k}(\epsilon) u_{\epsilon}^{k}(x, y, t) d \epsilon \tag{4.0.113}
\end{equation*}
$$

is a solution satisfying

$$
\begin{equation*}
U(x, y, 0)=\sum_{k=1}^{\infty} u_{k}(x, y) \Phi_{k}\left(x^{2}+y^{2}\right) \tag{4.0.114}
\end{equation*}
$$

Here
$\tilde{u}_{\epsilon}^{k}(x, y, t)=\frac{1}{(1+4 \epsilon t)} \exp \left(-\frac{\epsilon\left(x^{2}+y^{2}\right)}{1+4 \epsilon t}\right) u_{k}\left(\frac{x}{1+4 \epsilon t}, \frac{y}{1+4 \epsilon t}, \frac{t}{1+4 \epsilon t}\right)$.

If the stationary solutions are sufficiently rich, we may recover essentially any reasonable initial condition. This is the basis for the next result. We treat the problem in polar coordinates.

Theorem 4.0.49. Suppose that $K$ is continuous and that the SturmLiouville problem

$$
\begin{align*}
L^{\prime \prime}(\theta)+(K(\theta)+\lambda) L(\theta) & =0  \tag{4.0.116}\\
\alpha_{1} L(a)+\alpha_{2} L^{\prime}(a) & =0  \tag{4.0.117}\\
\beta_{1} L(b)+\beta_{2} L^{\prime}(b) & =0 \tag{4.0.118}
\end{align*}
$$

has a complete set of eigenfunctions and eigenvalues, and that the eigenvalues are all positive. Consider the initial and boundary value problem

$$
\begin{gather*}
u_{t}=u_{r r}+\frac{1}{r} u_{r}+\frac{1}{r^{2}} u_{\theta \theta}+\frac{K(\theta)}{r^{2}} u  \tag{4.0.119}\\
r>0, a \leq \theta \leq b, a, b \in[0,2 \pi] \\
u(r, \theta, 0)=f(r, \theta), f \in \mathcal{D}(\Omega) \\
\alpha_{1} u(r, a, t)+\alpha_{2} u_{\theta}(r, a, t)=0 \\
\beta_{1} u(r, b, t)+\beta_{2} u_{\theta}(r, b, t)=0
\end{gather*}
$$

Here $\Omega=[0, \infty) \times[a, b]$ in polar coordinates. Then there is a solution of the form

$$
\begin{equation*}
u(r, \theta, t)=\int_{0}^{\infty} \int_{a}^{b} f(\rho, \phi) p(t, r, \theta, \rho, \phi) \rho d \phi d \rho \tag{4.0.120}
\end{equation*}
$$

where

$$
\begin{equation*}
p(t, r, \theta, \rho, \phi)=\frac{1}{2 t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{n} \overline{L_{n}(\phi)} L_{n}(\theta) I_{\sqrt{\lambda_{n}}}\left(\frac{r \rho}{2 t}\right) \tag{4.0.121}
\end{equation*}
$$

in which $L_{n}(\theta), \lambda_{n}, n=1,2,3 \ldots$ are the normalised eigenfunctions and corresponding eigenvalues for the given Sturm-Liouville problem.

Proof. The PDE has a Lie group symmetry, the action of which in polar coordinates is given by

$$
\begin{equation*}
\tilde{u}_{\epsilon}(r, \theta, t)=\frac{1}{1+4 \epsilon t} e^{-\frac{\epsilon r^{2}}{1+4 \epsilon t}} u\left(\frac{r}{1+4 \epsilon t}, \theta, \frac{t}{1+4 \epsilon t}\right), \tag{4.0.122}
\end{equation*}
$$

for $\epsilon>-\frac{1}{4 t}$. This is obtained by taking the symmetry of Corollary 1.2.33 and converting it to polar coordinates. We first argue formally. The idea is to use superposition and symmetry integration to produce a solution of the form

$$
\begin{equation*}
u(r, \theta, t)=\sum_{n} \int_{0}^{\infty} \psi_{n}(\epsilon) \frac{1}{1+4 \epsilon t} e^{-\frac{\epsilon r^{2}}{1+4 \epsilon t}} u_{n}\left(\frac{r}{1+4 \epsilon t}, \theta\right) d \epsilon \tag{4.0.123}
\end{equation*}
$$

in which each $u_{n}$ is a stationary solution of (4.0.119). The functions $\psi_{n}$ are chosen to guarantee that the integrals and sums are convergent. We select separable stationary solutions such that

$$
u_{n}(r, \theta)=R_{n}(r) \Theta_{n}(\theta)
$$

Substitution into (4.0.119) shows that we require

$$
\begin{align*}
r^{2} R_{n}^{\prime \prime}+r R_{n}^{\prime}-\lambda R_{n} & =0  \tag{4.0.124}\\
\Theta_{n}^{\prime \prime}(\theta)+(K(\theta)+\lambda) \Theta_{n}(\theta) & =0 \tag{4.0.125}
\end{align*}
$$

We choose $\lambda_{n}$ and $\Theta_{n}(\theta)=L_{n}(\theta)$ to be the eigenvalues and normalised eigenfunctions of the given Sturm-Liouville problem. We also choose $R_{n}(r)=r^{\sqrt{\lambda_{n}}}$. Then our stationary solution is

$$
u_{n}(r, \theta)=r^{\sqrt{\lambda_{n}}} L_{n}(\theta)
$$

Now from solution (4.0.123), we have with this choice of stationary solution

$$
\begin{equation*}
u(r, \theta, 0)=\sum_{n} \Psi_{n}\left(r^{2}\right) r^{\sqrt{\lambda_{n}}} L_{n}(\theta) \tag{4.0.126}
\end{equation*}
$$

We require $u(r, \theta, 0)=f(r, \theta)$. Then we must have

$$
\begin{equation*}
f(r, \theta)=\sum_{n} \Psi_{n}\left(r^{2}\right) r^{\sqrt{\lambda_{n}}} L_{n}(\theta) \tag{4.0.127}
\end{equation*}
$$

Now since the eigenfunctions $L_{n}$ are complete, we may write

$$
\begin{equation*}
f(r, \theta)=\sum_{n} c_{n}(r) L_{n}(\theta) \tag{4.0.128}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{n}(r)=\int_{a}^{b} f(r, \phi) \overline{L_{n}(\phi)} d \phi \tag{4.0.129}
\end{equation*}
$$

This implies that we must choose

$$
\begin{equation*}
\Psi_{n}\left(r^{2}\right)=\frac{1}{r^{\sqrt{\lambda_{n}}}} \int_{a}^{b} f(r, \phi) \overline{L_{n}(\phi)} d \phi \tag{4.0.130}
\end{equation*}
$$

The solution we are working with has the form

$$
\begin{equation*}
u(r, \theta, t)=\int_{0}^{\infty} \sum_{n} \psi_{n}(\epsilon) \frac{r^{\sqrt{\lambda_{n}}}}{(1+4 \epsilon t)^{1+\sqrt{\lambda_{n}}}} e^{-\frac{\epsilon r^{2}}{1+4 \epsilon t}} L_{n}(\theta) d \epsilon \tag{4.0.131}
\end{equation*}
$$

We rewrite (4.0.131) in terms of the Laplace transform. We observe that

$$
\begin{equation*}
\frac{r^{\sqrt{\lambda_{n}}}}{(1+4 \epsilon t)^{1+\sqrt{\lambda_{n}}}} e^{-\frac{\epsilon r^{2}}{1+4 \epsilon t}}=\int_{0}^{\infty} \frac{z^{\frac{1}{2} \sqrt{\lambda_{n}}}}{4 t} e^{-\frac{r^{2}+z}{4 t}} I_{\sqrt{\lambda_{n}}}\left(\frac{r \sqrt{z}}{2 t}\right) e^{-\epsilon z} d z \tag{4.0.132}
\end{equation*}
$$

Assume that we may reverse the order of summation and integration. Using this expression, we find that

$$
\begin{align*}
u(r, \theta, t) & =\int_{0}^{\infty} \int_{0}^{\infty} \sum_{n} \psi_{n}(\epsilon) \frac{z^{\frac{1}{2} \sqrt{\lambda_{n}}}}{4 t} e^{-\frac{r^{2}+z}{4 t}} I_{\sqrt{\lambda_{n}}}\left(\frac{r \sqrt{z}}{2 t}\right) e^{-\epsilon z} L_{n}(\theta) d z d \epsilon \\
& =\int_{0}^{\infty} \sum_{n} \Psi_{n}(z) \frac{z^{\frac{1}{2} \sqrt{\lambda_{n}}}}{4 t} e^{-\frac{r^{2}+z}{4 t}} I_{\sqrt{\lambda_{n}}}\left(\frac{r \sqrt{z}}{2 t}\right) L_{n}(\theta) d z \tag{4.0.133}
\end{align*}
$$

where we have reversed the order of integration and evaluated the $\epsilon$ integral. Then we set $z=\rho^{2}$ to obtain

$$
\begin{align*}
u(r, \theta, t) & =\int_{0}^{\infty} \frac{\rho}{2 t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{n} \Psi_{n}\left(\rho^{2}\right) \rho^{\sqrt{\lambda_{n}}} L_{n}(\theta) I_{\sqrt{\lambda_{n}}}\left(\frac{r \rho}{2 t}\right) d \rho \\
& =\int_{0}^{\infty} \int_{a}^{b} f(\rho, \phi) \frac{\rho}{2 t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{n} L_{n}(\theta) \overline{L_{n}(\phi)} I_{\sqrt{\lambda_{n}}}\left(\frac{r \rho}{2 t}\right) d \phi d \rho \tag{4.0.134}
\end{align*}
$$

Here we have replaced $\Psi_{n}\left(\rho^{2}\right)$ with the value given by (4.0.130). By construction this function satisfies both the initial and boundary conditions and by symmetry it is a solution of the PDE (4.0.119). To complete the proof we establish convergence of the series

$$
\begin{equation*}
p(t, r, \theta, \rho, \phi)=\frac{1}{2 t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{n} \overline{L_{n}(\phi)} L_{n}(\theta) I_{\sqrt{\lambda_{n}}}\left(\frac{r \rho}{2 t}\right) \tag{4.0.135}
\end{equation*}
$$

We have the representation $I_{\nu}(z)=\frac{z^{\nu}}{\sqrt{\pi} 2^{\nu} \Gamma\left(\nu+\frac{1}{2}\right)} \int_{0}^{\pi} \cosh (z \cos \alpha) \sin ^{2 \nu}(\alpha) d \alpha$ (see 8.431 of $[\mathbf{3 5 ]}]$ ). Since $\left|\sin ^{2 \nu}(\alpha)\right| \leq 1$ it is obvious that

$$
\begin{equation*}
I_{\nu}(z) \leq \frac{\sqrt{\pi} z^{\nu}}{2^{\nu} \Gamma\left(\nu+\frac{1}{2}\right)} I_{0}(z) \tag{4.0.136}
\end{equation*}
$$

Thus

$$
\begin{aligned}
\sum_{n} \overline{L_{n}(\phi)} L_{n}(\theta) I_{\sqrt{\lambda_{n}}}(z) & \leq \sum_{n} \overline{L_{n}(\phi)} L_{n}(\theta) \frac{\sqrt{\pi} z^{\sqrt{\lambda_{n}}}}{2^{\nu} \Gamma\left(\sqrt{\lambda_{n}}+\frac{1}{2}\right)} I_{0}(z) \\
& =I_{0}(z) \sum_{n} \frac{\overline{L_{n}(\phi)} L_{n}(\theta)}{\lambda_{n}} \frac{\lambda_{n} \sqrt{\pi} z^{\sqrt{\lambda_{n}}}}{2^{\nu} \Gamma\left(\sqrt{\lambda_{n}}+\frac{1}{2}\right)}
\end{aligned}
$$

where $z=\frac{r \rho}{2 t}$. Now for each fixed $z$, an application of the ratio test shows that the series $\sum_{n=1}^{\infty} \frac{\lambda_{n} \sqrt{\pi} z \sqrt{\lambda_{n}}}{2^{\nu} \Gamma\left(\sqrt{\lambda_{n}}+\frac{1}{2}\right)}$ is absolutely convergent. Since $\sum_{n} \frac{\overline{L_{n}(\phi)} L_{n}(\theta)}{\lambda_{n}}$ is convergent it follows that $\sum_{n} \frac{\overline{L_{n}(\phi)} L_{n}(\theta)}{\lambda_{n}} \frac{\lambda_{n} \sqrt{\pi} z \sqrt{\lambda_{n}}}{2^{\nu} \Gamma\left(\sqrt{\lambda_{n}}+\frac{1}{2}\right)}$ is convergent and hence the series (4.0.135) converges for each fixed $z$. Since $L^{\prime \prime}(\theta)=-(K(\theta)+\lambda) L(\theta)$ and

$$
I_{\nu}^{\prime \prime}(z)=\frac{1}{4}\left(I_{\nu-2}(z)+2 I_{\nu}(z)+I_{\nu+2}(z)\right),
$$

a similar argument shows that series of derivatives are also convergent. So we conclude that (4.0.135) converges and defines a solution of the PDE. We also have the representation due to Weber

$$
\begin{equation*}
\frac{1}{2 t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} I_{\sqrt{\lambda_{n}}}\left(\frac{r \rho}{2 t}\right)=\int_{0}^{\infty} \xi e^{-t \xi^{2}} J_{\sqrt{\lambda_{n}}}(r \xi) J_{\sqrt{\lambda_{n}}}(\rho \xi) d \xi \tag{4.0.137}
\end{equation*}
$$

(formula 6.633 .2 of [35]); and the orthogonality relation

$$
\begin{equation*}
\int_{0}^{\infty} \xi J_{\sqrt{\lambda_{n}}}(r \xi) J_{\sqrt{\lambda_{n}}}(\rho \xi) d \xi=\frac{\delta(\rho-r)}{\rho}, \tag{4.0.138}
\end{equation*}
$$

see ( $\mathrm{p} 144,[70]$ ). Now assuming that $f$ is such that the integrals converge, it follows that

$$
u_{N}(r, \theta, t)=\sum_{n=1}^{N} \int_{a}^{b} \int_{0}^{\infty} f(\rho, \phi) \overline{L_{n}(\phi)} L_{n}(\theta) \frac{\rho}{2 t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} I_{\sqrt{\lambda_{n}}}\left(\frac{r \rho}{2 t}\right) d \phi d \rho,
$$

defines a solution of the PDE (4.0.119)which satisfies the Sturm-Liouville boundary conditions of the theorem. By Weber's result

$$
u_{N}(r, \theta, t)=\sum_{n=1}^{N} \int_{\Omega} \int_{0}^{\infty} f(\rho, \phi) \overline{L_{n}(\phi)} L_{n}(\theta) \xi e^{-t \xi^{2}} j_{\sqrt{\lambda_{n}}}(r, \rho, \xi) d \xi \rho d \phi d \rho
$$

in which $j_{\sqrt{\lambda_{n}}}(r, \rho, \xi)=J_{\sqrt{\lambda_{n}}}(r \xi) J_{\sqrt{\lambda_{n}}}(\rho \xi)$. Taking $t=0$ gives

$$
\begin{align*}
u_{N}(r, \theta, 0) & =\sum_{n=1}^{N} \int_{a}^{b} \int_{0}^{\infty} f(\rho, \phi) \overline{L_{n}(\phi)} L_{n}(\theta) \frac{\delta(\rho-r)}{\rho} \rho d \phi d \rho \\
& =\sum_{n=1}^{N} \int_{a}^{b} f(r, \phi) \overline{L_{n}(\phi)} L_{n}(\theta) d \phi \tag{4.0.139}
\end{align*}
$$

So we have $u_{N}(r, \theta, 0)=\sum_{n=1}^{N} c_{n}(r) L_{n}(\theta) \rightarrow f(r, \theta)$ as $N \rightarrow \infty$. If $f$ has compact support then we may apply the dominated convergence theorem to interchange the order of integration and summation as $N \rightarrow$ $\infty$ and conclude that

$$
\begin{equation*}
u(r, \theta, t)=\int_{0}^{\infty} \int_{a}^{b} f(\rho, \phi) \frac{\rho}{2 t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{n} L_{n}(\theta) \overline{L_{n}(\phi)} I_{\sqrt{\lambda_{n}}}\left(\frac{r \rho}{2 t}\right) d \phi d \rho \tag{4.0.140}
\end{equation*}
$$

This completes the proof.

Remark 4.0.50. We have specified that $f$ in the theorem has compact support. However we may easily extend the result to larger classes of functions. For a given function, we simply integrate against the fundamental solution and verify that we do have a solution of the initial value problem.

Let us recover the two-dimensional heat kernel. We are lead to the Sturm-Liouville problem $L^{\prime \prime}(\theta)=\lambda L(\theta)$ with $L(0)=L(2 \pi)$. The eigenvalues are $\lambda=n^{2}, n=0, \pm 1, \pm 2, \ldots$ and the eigenfunctions are
$L_{n}(\theta)=\frac{1}{\sqrt{2 \pi}} e^{i n \theta}$. This gives us the following representation of the solution:

$$
\begin{equation*}
u(r, \theta, t)=\int_{0}^{\infty} \int_{0}^{2 \pi} f(\rho, \phi) \frac{\rho}{4 \pi t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{n \in \mathbb{Z}} e^{i n(\theta-\phi)} I_{|n|}\left(\frac{r \rho}{2 t}\right) d \phi d \rho \tag{4.0.141}
\end{equation*}
$$

The Neumann-type series

$$
\begin{equation*}
e^{a \cos y}=I_{0}(a)+2 \sum_{n=1}^{\infty} I_{n}(a) \cos (n y) \tag{4.0.142}
\end{equation*}
$$

can be found on page 376 of [1]. Since $e^{i n(\theta-\phi)}+e^{-i n(\theta-\phi)}=2 \cos (n(\theta-$ $\phi)$ ), we can write

$$
\begin{align*}
\sum_{n \in \mathbb{Z}} e^{i n(\theta-\phi)} I_{|n|}\left(\frac{r \rho}{2 t}\right) & =I_{0}\left(\frac{r \rho}{2 t}\right)+2 \sum_{n=1}^{\infty} \cos (n(\theta-\phi)) I_{n}\left(\frac{r \rho}{2 t}\right) \\
& =e^{\frac{r \rho}{2 t} \cos (\theta-\phi)} \tag{4.0.143}
\end{align*}
$$

We conclude that

$$
\begin{equation*}
u(r, \theta, t)=\int_{0}^{\infty} \int_{0}^{2 \pi} f(\rho, \phi) \frac{\rho}{4 \pi t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} e^{\frac{r \rho}{2 t} \cos (\theta-\phi)} d \phi d \rho \tag{4.0.144}
\end{equation*}
$$

To convert (4.0.144) to Cartesian coordinates, let $x=r \cos \theta, y=$ $r \sin \theta, \xi=\rho \cos \phi$ and $\eta=\rho \sin \phi$. Then we have

$$
\begin{equation*}
U(x, y, t)=\int_{\mathbb{R}^{2}} \tilde{f}(\xi, \eta) \frac{1}{4 \pi t} e^{-\frac{(x-\xi)^{2}-(y-\eta)^{2}}{4 t}} d \xi d \eta \tag{4.0.145}
\end{equation*}
$$

Here $U$ is the solution in Cartesian coordinates and the initial value of the solution becomes $\tilde{f}(\xi, \eta)=f\left(\sqrt{\xi^{2}+\eta^{2}}, \tan ^{-1} \frac{\xi}{\eta}\right)$.

We can also obtain fundamental solutions restricted to different domains, with different boundary conditions. We consider some examples below.

Example 4.0.10. We solve

$$
\begin{aligned}
& u_{t}=\Delta u \\
& u(r, \theta, 0)=f(r, \theta) \\
& u(r, 0, t)=u\left(r, \frac{\pi}{2}, t\right)=0
\end{aligned}
$$

We require $\theta \in\left[0, \frac{\pi}{2}\right]$. Then we choose stationary solutions of the heat equation of the form $u_{n}(r, \theta)=r^{2|n|} \sin (2 n \theta)$. This gives
$u(r, \theta, t)=\int_{0}^{\infty} \int_{0}^{\frac{\pi}{2}} f(\rho, \phi) \frac{2 \rho}{\pi t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{n=1}^{\infty} \sin (2 n \theta) \sin (2 n \phi) I_{2|n|}\left(\frac{r \rho}{2 t}\right) d \Omega$,
where $d \Omega=d \phi d \rho$, which is valid on $0 \leq r<\infty, 0 \leq \theta \leq \frac{\pi}{2}$. The solution satisfies $u(r, \theta, 0)=f(r, \theta)$ for all $\theta \in\left[0, \frac{\pi}{2}\right]$ and moreover $u(r, 0, t)=0$. Similarly the problem

$$
\begin{aligned}
& u_{t}=\Delta u \\
& u(r, \theta, 0)=f(r, \theta) \\
& u_{\theta}(r, 0, t)=u_{\theta}\left(r, \frac{\pi}{2}, t\right)=0
\end{aligned}
$$

has solution

$$
\begin{aligned}
u(r, \theta, t) & =\int_{0}^{\infty} \int_{0}^{\frac{\pi}{2}} f(\rho, \phi)\left[\frac{\rho}{\pi t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} I_{0}\left(\frac{r \rho}{2 t}\right)+\right. \\
& \left.\frac{2 \rho}{\pi t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{n=1}^{\infty} \cos (2 n \theta) \cos (2 n \phi) I_{2|n|}\left(\frac{r \rho}{2 t}\right)\right] d \phi d \rho,
\end{aligned}
$$

Example 4.0.11. We will solve the equation

$$
\begin{equation*}
u_{t}=u_{x x}+u_{y y}-\frac{A}{x^{2}+y^{2}} u,(x, y) \in \mathbb{R}^{2}, A>0 \tag{4.0.146}
\end{equation*}
$$

subject to the initial condition $u(x, y, 0)=f(x, y), u(r, 0, t)=u(r, 2 \pi, t)$. In polar coordinates the equation is

$$
\begin{equation*}
u_{t}=u_{r r}+\frac{1}{r} u_{r}+\frac{1}{r^{2}} u_{\theta \theta}-\frac{A}{r^{2}} u, A>0 \tag{4.0.147}
\end{equation*}
$$

We solve the eigenvalue problem

$$
L^{\prime \prime}-A L=-\lambda L
$$

with periodic boundary conditions $L(0)-L(2 \pi)=0$. The eigenvalues are $\lambda=n^{2}+A$ and once more $L_{n}(\theta)=e^{i n \theta}$. From this we find that (4.0.147) has a fundamental solution

$$
\begin{equation*}
p(r, \theta, t, \rho, \phi)=\frac{1}{4 \pi t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{n \in \mathbb{Z}} e^{i n(\theta-\phi)} I_{\sqrt{n^{2}+A}}\left(\frac{r \rho}{2 t}\right) \tag{4.0.148}
\end{equation*}
$$

It does not seem possible to obtain a closed-form expression for this series, however approximations can be found. Suppose that $n$ is large compared to $A$. Then $\sqrt{n^{2}+A} \approx n$. As $n$ increases, the approximation improves. Obviously

$$
\begin{aligned}
& \sum_{n=-\infty}^{\infty} e^{i n(\theta-\phi)} I_{\sqrt{n^{2}+A}}(z)=I_{\sqrt{A}}(z)+2 \sum_{n=1}^{N} \cos (n(\theta-\phi)) I_{\sqrt{n^{2}+A}}(z) \\
& \quad+2 \sum_{n=N+1}^{\infty} \cos (n(\theta-\phi)) I_{\sqrt{n^{2}+A}}(z)
\end{aligned}
$$

For $N$ sufficiently large

$$
\sum_{n=N+1}^{\infty} \cos (n(\theta-\phi)) I_{\sqrt{n^{2}+A}}(z) \approx \sum_{n=N+1}^{\infty} \cos (n(\theta-\phi)) I_{n}(z)
$$

Also we have $\sum_{n=N+1}^{\infty} a_{n}=\sum_{n=1}^{\infty} a_{n}-\sum_{n=1}^{N} a_{n}$. This gives

$$
\begin{align*}
& \sum_{n=-\infty}^{\infty} e^{i n(\theta-\phi)} I_{\sqrt{n^{2}+A}}\left(\frac{r \rho}{2 t}\right) \approx \exp \left(\frac{r \rho}{2 t} \cos (\theta-\phi)\right)+I_{\sqrt{A}}\left(\frac{r \rho}{2 t}\right) \\
& -I_{0}\left(\frac{r \rho}{2 t}\right)+2 \sum_{n=1}^{N} \cos (n(\theta-\phi))\left(I_{\sqrt{n^{2}+A}}\left(\frac{r \rho}{2 t}\right)-I_{n}\left(\frac{r \rho}{2 t}\right)\right) \tag{4.0.149}
\end{align*}
$$

Different fundamental solutions can be found using different boundary conditions. Let us solve

$$
\begin{aligned}
& u_{t}=u_{x x}+u_{y y}-\frac{A}{x^{2}+y^{2}} u,(x, y) \in \mathbb{R}^{2}, A>0 \\
& u(x, y, 0)=e^{-x^{2}-y^{2}}
\end{aligned}
$$

Since $\int_{0}^{2 \pi} e^{i n(\theta-\phi)} d \phi=0, n= \pm 1, \pm 2, \ldots$ there is only one term in the series. So we have

$$
\begin{aligned}
& u(x, y, t)=\int_{\mathbb{R}^{2}} e^{-\left(\xi^{2}+\eta^{2}\right)} p(t, x, y, \xi, \eta) d \xi d \eta=\frac{\sqrt{\pi\left(x^{2}+y^{2}\right)}}{4 \sqrt{t}\left(1+4 t^{2}\right)^{3 / 2}} e^{-\frac{\left(1+8 t^{2}\right)\left(x^{2}+y^{2}\right)}{8 t(1+4 t)}} \\
& \times\left(I_{\frac{\sqrt{A}-1}{2}}\left(\frac{x^{2}+y^{2}}{8 t+32 t^{2}}\right)+I_{\frac{\sqrt{A}+1}{2}}\left(\frac{x^{2}+y^{2}}{8 t+32 t^{2}}\right)\right) .
\end{aligned}
$$

Expanding the series for the fundamental solution gives

$$
\begin{aligned}
& p(t, x, y, \xi, \eta)=\frac{1}{4 \pi t} e^{-\frac{x^{2}+y^{2}+\xi^{2}+\eta^{2}}{4 t}}\left(I_{\sqrt{A}}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)+\right. \\
& \frac{2(x \xi+y \eta)}{\sqrt{\left(x^{2}+y^{2}\right)\left(\xi^{2}+\eta^{2}\right)}} I_{\sqrt{1+A}}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)+ \\
& \left.\frac{2\left(4 x y \xi \eta+y^{2}\left(\eta^{2}-\xi^{2}\right)+x^{2}\left(\xi^{2}-\eta^{2}\right)\right)}{\left(x^{2}+y^{2}\right)\left(\xi^{2}+\eta^{2}\right)} I_{\sqrt{4+A}}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)+\cdots\right) .
\end{aligned}
$$

We plot the fundamental solution for $A=1, t=0.25, \xi=\eta=1$.


Figure 1. Fundamental Solution at $t=0.25, \xi=\eta=1$.

Example 4.0.12. We now consider the PDE

$$
\begin{equation*}
u_{t}=u_{x x}+u_{y y}-\frac{\mu x^{2}}{\tan ^{-1}\left(\frac{y}{x}\right)^{2}\left(x^{2}+y^{2}\right)} u, \quad \mu>0 \tag{4.0.150}
\end{equation*}
$$

In polar coordinates this becomes

$$
\begin{equation*}
u_{t}=u_{r r}+\frac{1}{r} u_{r}+\frac{1}{r^{2}} u_{\theta \theta}-\frac{\mu}{r^{2} \theta^{2}} u \tag{4.0.151}
\end{equation*}
$$

and we suppose that $\theta \in(0,2 \pi]$ and impose the two boundary conditions that $u(r, 2 \pi, t)=0$ and $u\left(r, 0^{+}, t\right)$ is finite. The Sturm-Liouville problem in this case is

$$
\begin{equation*}
L^{\prime \prime}-\left(\frac{\mu}{\theta^{2}}-\lambda\right) L=0 \tag{4.0.152}
\end{equation*}
$$

where $L\left(0^{+}\right)$is finite and $L(2 \pi)=0$. The general solution of (4.0.152) is

$$
L(\theta)=c_{1} \sqrt{\theta} J_{\frac{1}{2} \sqrt{1+4 \mu}}(\sqrt{\lambda} \theta)+c_{2} \sqrt{\theta} Y_{\frac{1}{2} \sqrt{1+4 \mu}}(\sqrt{\lambda} \theta) .
$$

To satisfy the finiteness condition, we set $c_{2}=0$ and choose $\lambda_{n}$ so that $2 \pi \sqrt{\lambda_{n}}=\alpha_{n}$ is the $n$th positive zero of $J_{\frac{1}{2} \sqrt{1+4 \mu}}(\theta)$. So if $\alpha_{n}$ is the $n$th
zero of $J_{\frac{1}{2} \sqrt{1+4 \mu}}(\theta)$, then $\lambda_{n}=\alpha_{n}^{2} / 4 \pi^{2}$ and we take

$$
L_{n}(\theta)=\sqrt{\theta} J_{\frac{1}{2} \sqrt{1+4 \mu}}\left(\frac{\alpha_{n}}{2 \pi} \theta\right)
$$

This leads to the fundamental solution

$$
p(r, \theta, t, \rho, \phi)=\frac{\rho}{2 t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{n=1}^{\infty} \frac{\sqrt{\theta \phi}}{c_{n}} J_{k}\left(\frac{\alpha_{n}}{2 \pi} \theta\right) J_{k}\left(\frac{\alpha_{n}}{2 \pi} \phi\right) I_{\sqrt{\frac{\alpha_{n}}{2 \pi}}}\left(\frac{r \rho}{2 t}\right)
$$

with $k=\frac{1}{2} \sqrt{1+4 \mu}$ and $c_{n}=\int_{0}^{2 \pi} \theta J_{k}\left(\frac{\alpha_{n}}{2 \pi} \theta\right)^{2} d \theta$. Converting this back to Cartesian coordinates is straightforward.

From this example, we could calculate a new functional of planar Brownian motion $B(t)=\left(B_{1}(t), B_{2}(t)\right)$. Specifically we may determine the expectation $\mathbb{E}\left[e^{-\mu \int_{0}^{t} \frac{d s}{\left(\theta_{s} r_{s}\right)^{2}}}\right]$, where $r_{t}=\sqrt{\left(B_{1}(t)^{2}+\left(B_{2}(t)\right)^{2}\right.}$ and $\theta_{t}$ is the angular part of $B(t)$. See Chapter 5 for more on this.

The fundamental solutions we have derived can be easily converted back to Cartesian coordinates, and we can find fundamental solutions of other types of problems, such as those which give transition probability densities for two-dimensional stochastic processes. Here is an example.

Example 4.0.13. Here we look at the Itô process

$$
\begin{equation*}
d X_{t}=\frac{2 X_{t}}{X_{t}^{2}+Y_{t}^{2}} d t+\sqrt{2} d W_{t}^{1}, d Y_{t}=\frac{2 Y_{t}}{X_{t}^{2}+Y_{t}^{2}} d t+\sqrt{2} d W_{t}^{2} \tag{4.0.153}
\end{equation*}
$$

where $W_{t}^{1,2}$ are independent Wiener processes. In order to obtain the transition density for this process we must find a fundamental solution of the Kolmogorov equation

$$
\begin{equation*}
v_{t}=\Delta v+\frac{2 x}{x^{2}+y^{2}} v_{x}+\frac{2 y}{x^{2}+y^{2}} v_{y} . \tag{4.0.154}
\end{equation*}
$$

The change of variables $u=e^{\frac{1}{2} \ln \left(x^{2}+y^{2}\right)} v$ converts this to

$$
u_{t}=\Delta u-\frac{1}{x^{2}+y^{2}} u
$$

This is a PDE we solved in Example 4.0.11. A solution of equation (4.0.154), with initial condition $v(x, y, 0)=f(x, y)$ is then

$$
v(x, y, t)=\int_{\mathbb{R}^{2}} e^{\phi(\xi, \eta)-\phi(x, y)} f(\xi, \eta) p(t, x, y, \xi, \eta) d \xi d \eta
$$

Here $p$ is the fundamental solution found above. This leads to the fundamental solution of equation (4.0.154)

$$
\begin{align*}
p(t, x, y, \xi, \eta) & =\frac{1}{4 \pi t} \sqrt{\frac{\xi^{2}+\eta^{2}}{x^{2}+y^{2}}} e^{-\frac{x^{2}+y^{2}+\xi^{2}+\eta^{2}}{4 t}} \sum_{n=-\infty}^{\infty}\left(\frac{(x+i y)(\xi-i \eta)}{(x-i y)(\xi+i \eta)}\right)^{\frac{n}{2}} \\
& \times I_{\sqrt{n^{2}+1}}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right) \tag{4.0.155}
\end{align*}
$$

We must check that this is a probability density. This is easiest in polar coordinates. Because $\int_{0}^{2 \pi} e^{i n \theta} d \theta=0$ for $n \neq 0$, there is only one term in the series that contributes to the integral. From [35] we find

$$
\int_{0}^{\infty} \int_{0}^{2 \pi} \frac{\rho^{2}}{4 \pi t r} e^{-\frac{r^{2}+\rho^{2}}{4 t}} I_{1}\left(\frac{r \rho}{2 t}\right) d \rho d \theta=1
$$

which confirms that (4.0.155) is indeed a probability density.

Example 4.0.14. We consider the two-dimensional process

$$
\begin{align*}
d X_{t} & =\frac{2 X_{t}\left(c_{1}\left(X_{t}^{2}+Y_{t}^{2}\right)^{2}-c_{2}\right)}{\left(X_{t}^{2}+Y_{t}^{2}\right)\left(c_{1}\left(X_{t}^{2}+Y_{t}^{2}\right)+c_{2}\right)} d t+\sqrt{2} d W_{t}^{1}  \tag{4.0.156}\\
d Y_{t} & =\frac{2 Y_{t}\left(c_{1}\left(X_{t}^{2}+Y_{t}^{2}\right)^{2}-c_{2}\right)}{\left(X_{t}^{2}+Y_{t}^{2}\right)\left(c_{1}\left(X_{t}^{2}+Y_{t}^{2}\right)+c_{2}\right)} d t+\sqrt{2} d W_{t}^{2} \tag{4.0.157}
\end{align*}
$$

where $c_{1}, c_{2}$ are arbitrary constants. Now we solve
$v_{t}=\Delta v+\frac{2 x\left(c_{1}\left(x^{2}+y^{2}\right)^{2}-c_{2}\right)}{\left(x^{2}+y^{2}\right)\left(c_{1}\left(x^{2}+y^{2}\right)+c_{2}\right)} v_{x}+\frac{2 y\left(c_{1}\left(x^{2}+y^{2}\right)^{2}-c_{2}\right)}{\left(x^{2}+y^{2}\right)\left(c_{1}\left(x^{2}+y^{2}\right)+c_{2}\right)} v_{y}$,
where $c_{1}, c_{2}$ are constants. Similar to the previous example, we can reduce this to $u_{t}=\Delta u-\frac{4}{x^{2}+y^{2}} u$, which shows that (4.0.158) has a fundamental solution

$$
\begin{aligned}
K_{t}(x, y, \xi, \eta) & =\frac{1}{4 \pi t} \frac{\left(c_{1}\left(\xi^{2}+\eta^{2}\right)^{2}+c_{2}\right)\left(x^{2}+y^{2}\right)}{\left(c_{1}\left(x^{2}+y^{2}\right)^{2}+c_{2}\right)\left(\xi^{2}+\eta^{2}\right)} e^{-\frac{x^{2}+y^{2}+\xi^{2}+\eta^{2}}{4 t}} \\
& \times \sum_{n=-\infty}^{\infty}\left(\frac{(x+i y)(\xi-i \eta)}{(x-i y)(\xi+i \eta)}\right)^{\frac{n}{2}} I_{\sqrt{n^{2}+4}}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right) .
\end{aligned}
$$

However this fundamental solution is not a transition density, since it does not have total integral 1:

$$
\begin{equation*}
\int_{0}^{2 \pi} \int_{0}^{\infty} \frac{e^{\frac{-r^{2}-\rho^{2}}{4 t}} r^{2}\left(c_{1} \rho^{4}+c_{2}\right) I_{2}\left(\frac{r \rho}{2 t}\right)}{4 \pi\left(c_{1} r^{4}+c_{2}\right) t \rho} \rho d \rho d \theta=1-\frac{c_{2} e^{-\frac{r^{2}}{4 t}}\left(r^{2}+4 t\right)}{4\left(c_{1} r^{4}+c_{2}\right) t} \tag{4.0.159}
\end{equation*}
$$

We may find a fundamental solution which is a probability density as follows. Subtracting 1 from the right side of (4.0.159) we set

$$
p(t, x, y, \xi, \eta)=K_{t}(x, y, \xi, \eta)+\frac{c_{2} e^{-\frac{x^{2}+y^{2}}{4 t}}\left(x^{2}+y^{2}+4 t\right)}{4\left(c_{1}\left(x^{2}+y^{2}\right)^{2}+c_{2}\right) t} \delta(x, y)
$$

where $\delta(x, y)$ is the Dirac delta on $\mathbb{R}^{2}$. If $f \in L^{1}\left(\mathbb{R}^{2}\right)$ and is continuous, then

$$
\begin{aligned}
\int_{\mathbb{R}^{2}} f(\xi, \eta) p(t, x, y, \xi, \eta) d \xi d \eta & =\int_{\mathbb{R}^{2}} f(\xi, \eta) K_{t}(x, y, \xi, \eta) d \xi d \eta \\
& +f(0,0) \frac{c_{2} e^{-\frac{x^{2}+y^{2}}{4 t}}\left(x^{2}+y^{2}+4 t\right)}{4\left(c_{1}\left(x^{2}+y^{2}\right)^{2}+c_{2}\right) t}
\end{aligned}
$$

Since

$$
\lim _{t \rightarrow 0} f(0,0) \frac{c_{2} e^{-\frac{x^{2}+y^{2}}{4 t}}\left(x^{2}+y^{2}+4 t\right)}{4\left(c_{1}\left(x^{2}+y^{2}\right)^{2}+c_{2}\right) t}=0
$$

holds for $x, y \neq 0$ and

$$
\lim _{t \rightarrow 0} \int_{\mathbb{R}^{2}} f(\xi, \eta) K_{t}(x, y, \xi, \eta) d \xi d \eta=f(x, y)
$$

this shows that $K_{t}$ is a fundamental solution. Also the integral

$$
\int_{\mathbb{R}^{2}} p(t, x, y, \xi, \eta) d \xi d \eta=1
$$

is now obvious.

We present another result for the two-dimensional case.

Theorem 4.0.51. Suppose that the Sturm-Liouville problem

$$
\begin{align*}
L^{\prime \prime}(\theta)+(K(\theta)+\lambda) L(\theta) & =0  \tag{4.0.160}\\
\alpha_{1} L(a)+\alpha_{2} L^{\prime}(a) & =0  \tag{4.0.161}\\
\beta_{1} L(b)+\beta_{2} L^{\prime}(b) & =0 \tag{4.0.162}
\end{align*}
$$

has a complete set of eigenfunctions and eigenvalues, and that the eigenvalues are all positive. Consider the initial and boundary value problem

$$
\begin{aligned}
& u_{t}=u_{r r}+\frac{1}{r} u_{r}+\frac{1}{r^{2}} u_{\theta \theta}+\frac{K(\theta)}{r^{2}} u-\frac{1}{4} c r^{2} u \\
& r>0, a \leq \theta \leq b, a, b \in[0,2 \pi], c>0 \\
& u(r, \theta, 0)=f(r, \theta), f \in \mathcal{D}\left(\mathbb{R}^{2}\right) \\
& \alpha_{1} u(r, a, t)+\alpha_{2} u_{\theta}(r, a, t)=0 \\
& \beta_{1} u(r, b, t)+\beta_{2} u_{\theta}(r, b, t)=0
\end{aligned}
$$

Then there is a solution of the form

$$
\begin{equation*}
u(r, \theta, t)=\int_{0}^{\infty} \int_{a}^{b} f(\rho, \varphi) K_{t}(r, \theta, \rho, \varphi) d \phi d \rho \tag{4.0.163}
\end{equation*}
$$

Here

$$
\begin{align*}
K_{t}(r, \theta, \rho, \varphi)= & \frac{2 \sqrt{c} \rho}{\sqrt{8(\cosh (2 \sqrt{c} t)-1)}} e^{-\frac{\sqrt{c}\left(r^{2}+\rho^{2}\right) \sinh (2 \sqrt{c} t)}{4(\operatorname{coshh}(2 \sqrt{c t} t)-1)}} \times \\
& \sum_{n} \frac{1}{I_{\frac{\sqrt{\lambda_{n}}}{2}}\left(\frac{\sqrt{c} \rho^{2}}{4}\right)} \Gamma_{n}\left(r, \rho^{2}, t\right) \overline{L_{n}(\theta)} L_{n}(\varphi), \tag{4.0.164}
\end{align*}
$$

where

$$
\Gamma_{n}(r, \rho, t)=\mathcal{L}^{-1}\left(\frac{1}{\sqrt{\varepsilon^{2}-\frac{c}{16}}} e^{\left(\frac{a \varepsilon r^{2}}{\varepsilon^{2}-\frac{c}{16}}\right)} I_{\frac{\sqrt{n}}{2}}\left(\frac{b r^{2}}{\varepsilon^{2}-\frac{c}{16}}\right)\right)
$$

$a=\frac{c}{8(\cosh (2 \sqrt{c} t)-1)}, b=\frac{c^{3 / 2}}{32(\cosh (2 \sqrt{c} t)-1)}$ and $L_{n}(\theta)$ and $\lambda_{n}, n=1,2,3 \ldots$ are the normalised eigenfunctions and corresponding eigenvalues for the Sturm-Liouville problem

$$
\begin{aligned}
L^{\prime \prime}(\theta)+(K(\theta)+\lambda) L(\theta) & =0 \\
\alpha_{1} L(a)+\alpha_{2} L^{\prime}(a) & =0 \\
\beta_{1} L(b)+\beta_{2} L^{\prime}(b) & =0
\end{aligned}
$$

Proof. In polar coordinates the PDE

$$
\begin{equation*}
u_{t}=u_{x x}+u_{y y}+\left(\frac{1}{x^{2}} k\left(\frac{y}{x}\right)-\frac{1}{4} c\left(x^{2}+y^{2}\right)\right) u \tag{4.0.165}
\end{equation*}
$$

becomes

$$
\begin{equation*}
u_{t}=u_{r r}+\frac{1}{r} u_{r}+\frac{1}{r^{2}} u_{\theta \theta}+\left(\frac{1}{r^{2}} k(\tan \theta)-\frac{1}{4} c r^{2}\right) u . \tag{4.0.166}
\end{equation*}
$$

We use stationary solutions of the PDE, namely solutions of

$$
\begin{equation*}
u_{r r}+\frac{1}{r} u_{r}+\frac{1}{r^{2}} u_{\theta \theta}+\left(\frac{1}{r^{2}} k(\tan \theta)-\frac{1}{4} c r^{2}\right) u=0 \tag{4.0.167}
\end{equation*}
$$

In Chapter One symmetries for the three-dimensional case were calculated. From these we obtain symmetries for the case $n=2$. A basis for the Lie symmetry algebra is

$$
\begin{aligned}
\mathbf{v}_{1} & =\partial_{t}, \mathbf{v}_{2}=\frac{1}{2} x e^{2 \sqrt{c t}} \partial_{x}+\frac{1}{2} y e^{2 \sqrt{c} t} \partial_{y}++\frac{1}{2 \sqrt{c}} e^{2 \sqrt{c t}} \partial_{t} \\
& -\frac{1}{4 \sqrt{c}} e^{2 \sqrt{c t}}\left(c\left(x^{2}+y^{2}\right)+2 \sqrt{c}\right) u \partial_{u}, \\
\mathbf{v}_{3} & =\frac{1}{2} x e^{-2 \sqrt{c} t} \partial_{x}+\frac{1}{2} y e^{-2 \sqrt{c t}} \partial_{y}++\frac{1}{2 \sqrt{c}} e^{-2 \sqrt{c t}} \partial_{t} \\
& +\frac{1}{4 \sqrt{c}} e^{-2 \sqrt{c t}}\left(c\left(x^{2}+y^{2}\right)-2 \sqrt{c}\right) u \partial_{u},
\end{aligned}
$$

From the infinitesimal symmetry $\mathbf{v}_{2}-\mathbf{v}_{3}-\frac{1}{\sqrt{c}} \mathbf{v}_{1}$ we find the symmetry

$$
\begin{align*}
& U_{\epsilon}(r, \theta, t)=\frac{e^{\frac{-\epsilon c r^{2}(2 \sqrt{c} \sinh (2 \sqrt{c} t)+\cosh (2 \sqrt{c} t)}{1+8 c \epsilon^{2}(\cosh (2 \sqrt{c} t)-1)+4 \sqrt{c} \sinh (2 \sqrt{c} t)}}}{\sqrt{1+8 c \epsilon^{2}(\cosh (2 \sqrt{c} t)-1)+4 \sqrt{c} \sinh (2 \sqrt{c} t)}} \times \\
& u\left(\frac{r}{\sqrt{1+8 c \epsilon^{2}(\cosh (2 \sqrt{c} t)-1)+4 \sqrt{c} \sinh (2 \sqrt{c} t)}}, \theta, T(\epsilon, t)\right) \tag{4.0.168}
\end{align*}
$$

where $T(\epsilon, t)=\frac{\operatorname{coth}^{-1}(4 \sqrt{c} \epsilon+\operatorname{coth}(\sqrt{c} t))}{\sqrt{c}}$. We let $u(r, \theta)=R(r) \Theta(\theta)$ and $K(\theta)=\sec ^{2} \theta k(\tan \theta)$. We then find

$$
\begin{equation*}
\frac{r^{2}}{R}\left(R^{\prime \prime}+\frac{1}{r} R^{\prime}-\frac{1}{4} c r^{2}\right) R=-\frac{1}{\Theta}\left(\Theta^{\prime \prime}+K(\theta) \Theta\right)=\lambda \tag{4.0.169}
\end{equation*}
$$

with $\lambda$ a constant. Then

$$
\begin{equation*}
R(r)=c_{1} I_{\frac{-\sqrt{\lambda}}{2}}\left(\frac{\sqrt{c} r^{2}}{4}\right)+c_{2} I_{\frac{\sqrt{\lambda}}{2}}\left(\frac{\sqrt{c} r^{2}}{4}\right) \tag{4.0.170}
\end{equation*}
$$

and $\Theta$ is a solution of the equation $\Theta^{\prime \prime}(\theta)+(K(\theta)+\lambda) \Theta(\theta)=0$. We choose $\Theta$ so that it satisfies the Sturm-Liouville boundary conditions and let the eigenvalues be $\lambda_{n}$ and the corresponding eigenfunctions be $L_{n}(\theta)$. We then take the stationary solutions

$$
\begin{equation*}
u_{n}(r, \theta)=I_{\frac{\sqrt{\lambda}}{2}}\left(\frac{\sqrt{c} r^{2}}{4}\right) L_{n}(\theta) \tag{4.0.171}
\end{equation*}
$$

We apply the symmetry (4.0.168) to these stationary solutions to obtain new solutions $U_{\epsilon, n}$ and form the solution

$$
\begin{equation*}
u(r, \theta, t)=\int_{0}^{\infty} \sum_{n} \varphi_{n}(\epsilon) U_{\epsilon, n}(r, \theta, t) d \epsilon \tag{4.0.172}
\end{equation*}
$$

As in the previous expansion theorem, we have

$$
\begin{equation*}
u(r, \theta, 0)=\sum_{n} \int_{0}^{\infty} \varphi_{n}(\epsilon) e^{-c \epsilon r^{2}} I_{\frac{\lambda_{n}}{2}}\left(\frac{\sqrt{c} r^{2}}{4}\right) L_{n}(\theta) d \theta \tag{4.0.173}
\end{equation*}
$$

Now make the substitution $\epsilon \rightarrow \epsilon / c$. This becomes

$$
\begin{equation*}
u(r, \theta, 0)=\sum_{n} \Phi_{n}\left(r^{2}\right) I_{\frac{\lambda_{n}}{2}}\left(\frac{\sqrt{c} r^{2}}{4}\right) L_{n}(\theta) \tag{4.0.174}
\end{equation*}
$$

and we require $u(r, \theta, 0)=f(r, \theta)$. We write as before

$$
\begin{equation*}
f(r, \theta)=\sum_{n} \widehat{f}(r, n) L_{n}(\theta), \tag{4.0.175}
\end{equation*}
$$

where

$$
\begin{equation*}
\widehat{f}(r, n)=\int_{0}^{2 \pi} f(r, \phi) \overline{L_{n}(\phi)} d \phi \tag{4.0.176}
\end{equation*}
$$

If

$$
\begin{equation*}
\sum_{n} \Phi_{n}\left(r^{2}\right) I_{\frac{\sqrt{\lambda_{n}}}{2}}\left(\frac{\sqrt{c} r^{2}}{4}\right) L_{n}(\theta)=\sum_{n} \widehat{f}(r, n) L_{n}(\theta) \tag{4.0.177}
\end{equation*}
$$

then we must have

$$
\begin{equation*}
\Phi_{n}\left(r^{2}\right)=\frac{1}{I_{\frac{\sqrt{n}}{2}}}\left(\frac{\sqrt{c} r^{2}}{4}\right) \int_{0}^{2 \pi} f(r, \phi) \overline{L_{n}(\phi)} d \phi \tag{4.0.178}
\end{equation*}
$$

After some algebraic simplifications, we can as in the first theorem write the solution in terms of the Laplace transform,

$$
\begin{gather*}
\Gamma_{n}(r, \rho, t)=\mathcal{L}^{-1}\left(\frac{1}{\sqrt{\varepsilon^{2}-\frac{c}{16}}} e^{\left(\frac{a \varepsilon r^{2}}{\varepsilon^{2}-\frac{c}{16}}\right)} I_{\frac{\sqrt{n}}{2}}\left(\frac{b r^{2}}{\varepsilon^{2}-\frac{c}{16}}\right)\right) \\
u(r, \theta, t)=\int_{0}^{\infty} \int_{0}^{\infty} \varphi_{n}(\epsilon) \frac{\sqrt{c} L_{n}(\theta) \Gamma_{n}(r, z, t)}{\sqrt{8(\cosh (2 \sqrt{c} t)-1)}} e^{-\frac{\sqrt{c}\left(r^{2}+z^{2}\right) \sinh (2 \sqrt{c t})}{4 \cosh (2 \sqrt{c} t)-1}} e^{-\epsilon z} d z d \epsilon . \tag{4.0.179}
\end{gather*}
$$

The inverse Laplace transform is continuous and $f$ has compact support. So the integral converges and we may apply Fubini's Theorem and reverse the order of integration. Then letting $z \rightarrow \rho^{2}$, we have

$$
\begin{equation*}
u(r, \theta, t)=\int_{0}^{\infty} \int_{0}^{2 \pi} f(\rho, \phi) K_{t}(r, \theta, \rho, \phi) d \phi d \rho \tag{4.0.180}
\end{equation*}
$$

where

$$
\begin{equation*}
K_{t}(r, \theta, \rho, \phi)=\frac{2 \sqrt{c} \rho e^{-\frac{\sqrt{c}\left(r^{2}+z^{2}\right) \sinh (2 \sqrt{c} t)}{4 \cosh (2 \sqrt{c} t)-1}}}{\sqrt{8(\cosh (2 \sqrt{c} t)-1)}} \sum_{n} \frac{\Gamma_{n}\left(r, \rho^{2}, t\right)}{I_{\frac{\sqrt{\lambda}}{2}}\left(\frac{\sqrt{c} \rho^{2}}{4}\right)} L_{n}(\theta) \overline{L_{n}(\phi)} \tag{4.0.181}
\end{equation*}
$$

The remainder of the proof proceeds as for the previous result.

The inverse Laplace transform $\Gamma_{n}(r, \rho, t)$ used in the preceding theorem, does not seem to be known analytically, however it can be computed numerically. See [19] for the numerical inversion of Laplace transforms. Also, since the theorem uses the same Sturm-Liouville
problem as the first expansion theorem, given any example for the first expansion theorem, there is a corresponding example for the second.
4.0.12. Numerical Solutions of Eigenvalue Problems. Sophisticated methods for computing large numbers of eigenvalues for Sturm-Liouville problems have been developed, see $[\mathbf{4 7}]$ and the references therein, or the SLEIGN project and [3]. We present an elementary method. We focus only on the absorbing boundary conditions $L(a)=L(b)=0$.

We replace the equation $L^{\prime \prime}+(K+\lambda) L=0$ with the centraldifference approximation

$$
L^{\prime \prime}\left(\theta_{i}\right) \approx \frac{L_{i-1}-2 L_{i}+L_{i+1}}{h^{2}}
$$

where $L_{i}=L\left(\theta_{i}\right)$ and $\theta_{i}=a+h i, i=1, \ldots, n-1$, and $h=\frac{b-a}{n}$. The eigenvalues of the Sturm-Liouville problem are approximated by the eigenvalues of the $(n-1) \times(n-1)$ system

$$
\begin{equation*}
\frac{L_{i-1}-2 L_{i}+L_{i+1}}{h^{2}}+K\left(\theta_{i}\right) L_{i}+\lambda L_{i}=0 \tag{4.0.182}
\end{equation*}
$$

where $L_{0}=L_{n}=0, i=1, \ldots, n-1$. That is to say, we want the eigenvalues for the $(n-1) \times(n-1)$ matrix $M$, with entries $M_{i i}=K\left(\theta_{i}\right)-$ $\frac{2}{h^{2}}, M_{i i+1}=M_{i+1 i}=\frac{1}{h^{2}}$ and all other entries zero. The negative of the eigenvalues of the tridiagonal matrix $M$ will provide approximations to the eigenvalues of the Sturm-Liouville problem. The eigenfunctions may then be obtained from the eigenvectors by interpolating the vector by a polynomial, or some other method. We present two examples to illustrate the method. All calculations were performed in Mathematica.
4.0.12.1. The equation $u_{t}=\Delta u-\frac{\tan ^{-1}\left(\frac{y}{x}\right)}{\left(x^{2}+y^{2}\right)\left(1+\tan ^{-1}\left(\frac{y}{x}\right)\right)} u$. The eigenfunction problem is

$$
\begin{equation*}
L^{\prime \prime}+\left(-\frac{\theta}{1+\theta^{2}}+\lambda\right) L=0, L(0)=L(2 \pi) \tag{4.0.183}
\end{equation*}
$$

The first three eigenvalues $l_{1}=0.544, \lambda_{2}=1.291, \lambda_{3}=2.405$. The first three eigenfunctions

$$
L_{1}(\theta)=4.4317 \times 10^{-8} \theta^{10}-1.4486 \times 10^{-6} \theta^{9}+0.00002 \theta^{8}-0.0002 \theta^{7}
$$

$$
+0.0008 \theta^{6}-0.0016711 \theta^{5}-0.0023 \theta^{4}+0.0135 \theta^{3}-0.0208 \theta^{2}+0.1802 \theta
$$

$$
L_{2}(\theta)=-1.4717 \times 10^{-8} \theta^{10}-1.4039 \times 10^{-6} \theta^{9}+0.00005 \theta^{8}-0.0007 \theta^{7}
$$

$$
+0.0036 \theta^{6}-0.00598 \theta^{5}-0.00214 \theta^{4}-0.0194 \theta^{3}-0.0538 \theta^{2}+0.4398 \theta
$$

$$
L_{3}(\theta)=6.2529 \times 10^{-6} \theta^{10}-0.0002 \theta^{9}+0.0028 \theta^{8}-0.019632 \theta^{7}
$$

$$
+0.0727 \theta^{6}-0.1478 \theta^{5}+0.2258 \theta^{4}-0.3648 \theta^{3}+0.0117 \theta^{2}+0.6701 \theta
$$



Figure 2. Plot of first three eigenfunctions

We normalise the eigenfunctions by computing

$$
\begin{aligned}
\frac{1}{c_{1}^{2}} & =\int_{0}^{2 \pi} L_{1}(\theta)^{2} d \theta=0.628328, \frac{1}{c_{2}^{2}}=\int_{0}^{2 \pi} L_{2}(\theta)^{2} d \theta=0.62838 \\
\frac{1}{c_{3}^{2}} & =\int_{0}^{2 \pi} L_{3}(\theta)^{2} d \theta=0.628365
\end{aligned}
$$

We check the orthogonality of these eigenfunctions. For example $\int_{0}^{2 \pi} L_{1}(\theta) L_{3}(\theta) d \theta=0.00002$.

The fundamental solution is then

$$
\begin{aligned}
& p(t, x, y, \xi, \eta)=\frac{1}{2 t} e^{-\frac{x^{2}+y^{2}+\xi^{2}+\eta^{2}}{4 t}}\left(c_{1}^{2} L_{1}\left(\tan ^{-1}\left(\frac{y}{x}\right)\right) L_{1}\left(\tan ^{-1}\left(\frac{\eta}{\xi}\right)\right)\right. \\
& \times I_{0.738}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)+c_{2}^{2} L_{2}\left(\tan ^{-1}\left(\frac{y}{x}\right)\right) L_{2}\left(\tan ^{-1}\left(\frac{\eta}{\xi}\right)\right) \\
& \times I_{1.136}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)+c_{3}^{2} L_{3}\left(\tan ^{-1}\left(\frac{y}{x}\right)\right) L_{3}\left(\tan ^{-1}\left(\frac{\eta}{\xi}\right)\right) \\
& \left.\times I_{1.55}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)+\cdots\right)
\end{aligned}
$$

4.0.12.2. The equation $u_{t}=\Delta u-\frac{\tan ^{-1}\left(\frac{y}{x}\right)}{\left(x^{2}+y^{2}\right) \sqrt{1+\tan ^{-1}\left(\frac{y}{x}\right)^{2}}} u$. The eigenvalue problem is $L^{\prime \prime}+\left(-\frac{\theta}{\sqrt{1+\theta^{2}}}+\lambda\right) L=0, L(0)=L(2 \pi)=0$. The eigenvalues are $\lambda_{1}=1.169, \lambda_{2}=1.856, \lambda_{3}=2.962, \ldots$ and the first three eigenfunctions are

$$
\begin{aligned}
& L_{1}(\theta)=0.308 \theta-0.0224 \theta^{2}-0.0427 \theta^{3}+0.0275 \theta^{4}-0.0111 \theta^{5}+0.00297 \theta^{6} \\
& -0.0005 \theta^{7}+0.00006 \theta^{8}-3.77486 \times 10^{-6} \theta^{9}+1.0679 \times 10^{-7} \theta^{10} \\
& L_{2}(\theta)=-0.53613 \theta+0.0399 \theta^{2}+0.143694 \theta^{3}-0.06021 \theta^{4}+0.01918 \theta^{5} \\
& -0.0057 \theta^{6}+0.0011 \theta^{7}-0.00013 \theta^{8}+7.93082 \times 10^{-6} \theta^{9}-2.0883 \times 10^{-7} \theta^{10} \\
& L_{3}(\theta)=-0.7608 \theta+0.04496 \theta^{2}+0.4082 \theta^{3}-0.15997 \theta^{4}+0.0444339 \theta^{5} \\
& -0.02423 \theta^{6}+0.00848 \theta^{7}-0.00146 \theta^{8}+0.00012 \theta^{9}-3.94676 \times 10^{-6} \theta^{10}
\end{aligned}
$$

We have $\int_{0}^{2 \pi} L_{1}(\theta) L_{2}(\theta) d \theta=-0.00004$ etc, so these approximate eigenfunctions are approximately orthogonal.

The normalisation constants are

$$
\begin{aligned}
& \frac{1}{c_{1}^{2}}=\int_{0}^{2 \pi} L_{1}(\theta)^{2} d \theta=0.62834, \frac{1}{c_{2}^{2}}=\int_{0}^{2 \pi} L_{2}(\theta)^{2} d \theta=0.628396 \\
& \frac{1}{c_{3}^{2}}=\int_{0}^{2 \pi} L_{1}(\theta)^{2} d \theta=0.628427
\end{aligned}
$$

The fundamental solution is then

$$
\begin{aligned}
& p(t, x, y, \xi, \eta)=\frac{1}{2 t} e^{-\frac{x^{2}+y^{2}+\xi^{2}+\eta^{2}}{4 t}}\left(c_{1}^{2} L_{1}\left(\tan ^{-1}\left(\frac{y}{x}\right)\right) L_{1}\left(\tan ^{-1}\left(\frac{\eta}{\xi}\right)\right)\right. \\
& \times I_{1.0812}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)+c_{2}^{2} L_{2}\left(\tan ^{-1}\left(\frac{y}{x}\right)\right) L_{2}\left(\tan ^{-1}\left(\frac{\eta}{\xi}\right)\right) \\
& \times I_{1.362}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)+c_{3}^{2} L_{3}\left(\tan ^{-1}\left(\frac{y}{x}\right)\right) L_{3}\left(\tan ^{-1}\left(\frac{\eta}{\xi}\right)\right) \\
& \left.\times I_{1.72105}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)+\cdots\right)
\end{aligned}
$$



Figure 3. Plot of first three eigenfunctions
4.0.12.3. Some Explicitly Solvable Examples. We present below a list of some exactly solvable examples.
(1) (i) The equation: $u_{t}=\Delta u-\frac{1}{x^{2}} \frac{x^{2}+\mu y^{2}}{x^{2}+y^{2}} u, \mu>0, \mu \neq 1$.
(ii) The Sturm-Liouville Problem: $L^{\prime \prime}+\left(\lambda-1-\mu \tan ^{2} \theta\right) L=$ $0, L(0)=0, L\left(\frac{\pi}{2}\right)=0$.
(iii) The eigenvalues: $\lambda_{n}=1-\mu+\left(2 n+\frac{3}{2}+\sqrt{\mu+1 / 4}\right)^{2}, n=$ $0,1,2,3, \ldots$.
(iv) The eigenfunctions: $L_{n}(\theta)=\cos ^{\alpha} \theta_{2} F_{1}\left(-n-\frac{1}{2}, \beta ; \gamma ; \cos ^{2} \theta\right)$, where $\alpha=\sqrt{\mu+1 / 4}+\frac{1}{2}, \beta=n+1+\sqrt{\mu+1 / 4}, \gamma=$ $1+\sqrt{\mu+1 / 4}$.
(v) The fundamental solution:

$$
\begin{aligned}
& p(t, x, y, \xi, \eta)=\frac{1}{2 t} e^{-\frac{x^{2}+y^{2}+\xi^{2}+\eta^{2}}{4 t}} \frac{(x \xi)^{\alpha}}{\left(\left(x^{2}+y^{2}\right)\left(\xi^{2}+\eta^{2}\right)\right)^{\frac{\alpha}{2}}} \\
& \times \sum_{n=0}^{\infty} c_{n 2} F_{1}\left(-n-\frac{1}{2}, \beta, \gamma, \frac{x^{2}}{r^{2}}\right){ }_{2} F_{1}\left(-n-\frac{1}{2}, \beta, \gamma, \frac{\xi^{2}}{\rho^{2}}\right) I_{\sqrt{\lambda_{n}}}\left(\frac{r \rho}{2 t}\right), \\
& \quad \text { with } r=\sqrt{x^{2}+y^{2}}, \rho=\sqrt{\xi^{2}+\eta^{2}} \cdot \frac{1}{c_{n}^{2}}=\int_{0}^{\frac{\pi}{2}} L_{n}(\theta)^{2} d \theta
\end{aligned}
$$

(2) (i) The PDE : $u_{t}=\Delta u-\frac{A}{x^{2}+y^{2}} u-\frac{1}{4} c\left(x^{2}+y^{2}\right) u$.
(ii) The Sturm-Liouville problem: $L^{\prime \prime}+(\lambda-A) l=0, L(0)=$ $L(2 \pi)$.
(iii) The eigenvalues: $\lambda_{n}=n^{2}+A$.
(iv) The eigenfunctions: $L_{n}(\theta)=\frac{1}{\sqrt{2 \pi}} e^{i n \theta}$.
(v) The fundamental solution:

$$
p(t, r, \theta, \rho, \phi)=\frac{\sqrt{c} e^{-\frac{\sqrt{c}\left(r^{2}+\rho^{2}\right) \sinh (2 \sqrt{c} t)}{4(\cosh (2 \sqrt{c} t)-1)}}}{2 \pi \sqrt{2(\cosh (2 \sqrt{c} t)-1)}} \sum_{n \in \mathbb{Z}} \frac{\Gamma_{n}(r, \rho, t) e^{i n(\theta-\phi)}}{\frac{\sqrt{n^{2}+A}}{2}\left(\frac{\sqrt{c} \rho^{2}}{4}\right)},
$$

where $r=\sqrt{x^{2}+y^{2}}, \rho=\sqrt{\xi^{2}+\eta^{2}}$, etc.
(3) (i) The PDE: $u_{t}=\Delta u-\frac{x^{2}}{\left(x^{2}+y^{2}\right)^{2}} u,(x, y) \in \mathbb{R}_{+}^{2}$, $u(x, y, 0)=f(x, y), u(0, y, t)=u(x, 0, t)=0$.
(ii) The Sturm-Liouville problem: $L^{\prime \prime}+\left(-\cos ^{2} \theta+\lambda\right) L=0$, with $L(0)=L\left(\frac{\pi}{2}\right)=0$.
(iii) The eigenfunctions are $S\left(\lambda_{n}-\frac{1}{2}, \frac{1}{4}, \theta\right)$, the odd Mathieu functions.
(iv) The eigenvalues are $S\left(\lambda_{n}-\frac{1}{2}, \frac{1}{4}, \frac{\pi}{2}\right)=0$, or $\lambda_{1}=4.494793$, $\lambda_{2}=16.50208, \lambda_{3}=36.5009375, \ldots$
(v) The fundamental solution:

$$
\begin{aligned}
p(t, x, y, \xi, \eta) & =\frac{1}{2 t} e^{-\frac{x^{2}+y^{2}+\xi^{2}+\eta^{2}}{4 t}} \sum_{n=1}^{\infty} S\left(\lambda_{n}-\frac{1}{2}, \frac{1}{4}, \tan ^{-1}\left(\frac{y}{x}\right)\right) \\
& \times S\left(\lambda_{n}-\frac{1}{2}, \frac{1}{4}, \tan ^{-1}\left(\frac{\eta}{\xi}\right)\right) I_{\sqrt{\lambda_{n}}}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)
\end{aligned}
$$

where $c_{1}=0.797886, c_{2}=0.930257, c_{3}=1.12838$, etc.

More examples can be solved exactly but we will not attempt an exhaustive list.
4.0.13. Expansions in Higher Dimensions. Now we extend the results of the previous section to higher-dimensional problems. The results we obtain are not as strong as those for the two-dimensional problem, but are still very useful. We begin by studying the equation

$$
\begin{equation*}
u_{t}=\Delta u+\frac{1}{x^{2}} k\left(\frac{y}{x}, \frac{z}{x}\right) u . \tag{4.0.184}
\end{equation*}
$$

Introducing spherical coordinates,

$$
x=r \cos \theta \sin \phi, y=r \sin \theta \sin \phi, z=r \cos \phi
$$

with $r \geq 0, \theta \in[0,2 \pi], \phi \in[0, \pi]$ the equation becomes

$$
\begin{equation*}
u_{t}=u_{r r}+\frac{2}{r} u_{r}+\frac{1}{r^{2}}\left(\frac{1}{\sin ^{2} \phi} u_{\theta \theta}+\cot \phi u_{\phi}+u_{\phi \phi}+G(\theta, \phi) u\right) \tag{4.0.185}
\end{equation*}
$$

with $G(\theta, \phi)=\frac{k(\tan \theta, \cot \phi \sec \theta)}{\cos ^{2} \theta \sin ^{2} \phi}$. The PDE in spherical coordinates has a symmetry acting as

$$
\begin{equation*}
U_{\epsilon}(r, \theta, \phi, t)=\frac{1}{(1+4 \epsilon t)^{3 / 2}} e^{-\frac{\epsilon r^{2}}{1+4 \epsilon t}} u\left(\frac{r}{1+4 \epsilon t}, \theta, \phi, \frac{t}{1+4 \epsilon t}\right) \tag{4.0.186}
\end{equation*}
$$

valid for $\epsilon>-1 / 4 t$. As in the two-dimensional case we form a solution

$$
U(r, \theta, \phi, t)=\int_{0}^{\infty} \varphi(\epsilon) \frac{1}{(1+4 \epsilon t)^{3 / 2}} e^{-\frac{\epsilon r^{2}}{1+4 \epsilon t}} u\left(\frac{r}{1+4 \epsilon t}, \theta, \phi, \frac{t}{1+4 \epsilon t}\right) d \epsilon
$$

and we will let the solutions $u(r, \theta, \phi, t)$ be stationary solutions. We let $u(r, \theta, \phi)=R(r) \Psi(\theta, \phi)$ and we require

$$
\Psi\left(R^{\prime \prime}+\frac{2}{r} R^{\prime}\right)+\frac{R}{r^{2}} \frac{1}{r^{2}}\left(\frac{1}{\sin ^{2} \phi} \Psi_{\theta \theta}+\cot \phi \Psi_{\phi}+\Psi_{\phi \phi}+G(\theta, \phi) \Psi\right)=0 .
$$

So we have

$$
\begin{equation*}
\frac{1}{\sin ^{2} \phi} \Psi_{\theta \theta}+\cot \phi \Psi_{\phi}+\Psi_{\phi \phi}+(G(\theta, \phi)+\lambda) \Psi=0 \tag{4.0.187}
\end{equation*}
$$

with $r^{2} R^{\prime \prime}+2 r R^{\prime}-\lambda R=0$. This gives

$$
R(r)=c_{1} r^{\frac{1}{2}(-1+\sqrt{1+4 \lambda})}+c_{2} r^{\frac{1}{2}(-1-\sqrt{1+4 \lambda})} .
$$

We take $c_{2}=0$ and choose the eigenfunctions $L_{\lambda}^{n}(\theta, \phi)$ of (4.0.187) to form an orthonormal basis for $L^{2}\left(S^{2}\right)$ where $S^{2}$ is the two-dimensional unit sphere. Then we may expand arbitrary $f \in L^{2}\left(S^{2}\right)$ as

$$
\begin{equation*}
f(\theta, \phi)=\sum_{n} c_{n} L_{\lambda}^{n}(\theta, \phi), \tag{4.0.188}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{n}=\int_{0}^{2 \pi} \int_{0}^{\pi} f(\xi, \eta) L_{\lambda}^{n}(\xi, \eta) d \xi d \eta \tag{4.0.189}
\end{equation*}
$$

Now let

$$
\begin{equation*}
u(r, \theta, \phi, t)=\int_{0}^{\infty} \sum_{n} \varphi_{n}(\epsilon) \frac{e^{-\frac{\epsilon \epsilon^{2}}{1+4 \epsilon t}}}{(1+4 \epsilon t)^{3 / 2}} u_{n}\left(\frac{r}{1+4 \epsilon t}, \theta, \phi, t\right) d \epsilon \tag{4.0.190}
\end{equation*}
$$

so that we require

$$
\begin{equation*}
u(r, \theta, \phi, 0)=\sum_{n} \Phi_{n}\left(r^{2}\right) r^{\frac{1}{2}\left(-1+\sqrt{1+4 \lambda_{n}}\right)} L_{n}(\theta, \phi)=f(r, \theta, \phi) \tag{4.0.191}
\end{equation*}
$$

Comparing (4.0.191) to (4.0.188) implies

$$
\begin{equation*}
\Phi\left(r^{2}\right)=\frac{1}{r^{\frac{1}{2}\left(-1+\sqrt{1+4 \lambda_{n}}\right)}} \int_{0}^{2 \pi} \int_{0}^{\pi} f(r, \xi, \eta) \overline{L_{n}(\xi, \eta)} d \eta d \xi . \tag{4.0.192}
\end{equation*}
$$

If we let $l=\frac{1}{2}\left(-1+\sqrt{1+4 \lambda_{n}}\right)$, then our solution is

$$
\begin{aligned}
u(r, \theta, \phi, t) & =\sum_{n} \int_{0}^{\infty} \frac{\varphi_{n}(\epsilon) r^{l}}{(1+4 \epsilon t)^{\frac{3}{2}+l}} L_{n}(\theta, \phi) e^{-\frac{\epsilon r^{2}}{1+4 \epsilon t}} d \epsilon \\
& =\frac{1}{4 t} \sum_{n} \int_{\mathbb{R}_{+}^{2}} r^{l} e^{-\frac{r^{2}+z}{4 t}} r^{-l-\frac{1}{2}} z^{\frac{l}{2}+\frac{1}{4}} I_{l+\frac{1}{2}}\left(\frac{r \sqrt{z}}{2 t}\right) \varphi_{n}(\epsilon) e^{-\epsilon z} d \epsilon d z \\
& =\sum_{n} \int_{0}^{\infty} \Phi_{n}(z) e^{-\frac{r^{2}+z}{4 t}} r^{-\frac{1}{2}} z^{\frac{l}{2}+\frac{1}{4}} I_{l+\frac{1}{2}}\left(\frac{r \sqrt{z}}{2 t}\right) \varphi_{n}(\epsilon) e^{-\epsilon z} d z
\end{aligned}
$$

Convergence of the series and the interchange of sum and integrals follows as for Theorem 4.0.49 so that

$$
\begin{equation*}
u(r, \theta, \phi, t)=\int_{0}^{\infty} \int_{0}^{2 \pi} \int_{0}^{\pi} f(\rho, \xi, \eta) p(t, r, \theta, \phi, \xi, \eta) \rho d \eta d \xi d \rho, \tag{4.0.193}
\end{equation*}
$$

where

$$
p(t, r, \theta, \phi, \xi, \eta)=\frac{1}{2 t} \sqrt{\frac{\rho}{r}} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sin \eta \sum_{n} L_{\lambda_{n}}(\theta, \phi) \overline{L_{\lambda_{n}}(\xi, \eta)} I_{l+\frac{1}{2}}\left(\frac{r \rho}{2 t}\right)
$$

is a fundamental solution of (4.0.185). The sum is taken over all the eigenvalues. We can easily extend this calculation to the corresponding $n$-dimensional problem.

Theorem 4.0.52. Consider the equation

$$
\begin{aligned}
u_{t} & =u_{r r}+\frac{(n-1)}{r} u_{r}+\frac{1}{r^{2}}\left(\Delta_{S^{n-1}}+G(\boldsymbol{\Theta})\right) u \\
u(r, \boldsymbol{\Theta}, 0) & =f(r, \boldsymbol{\Theta})
\end{aligned}
$$

and $\alpha(\boldsymbol{\Theta}) \Psi(\boldsymbol{\Theta})+(1-\alpha(\boldsymbol{\Theta})) \frac{\partial \Psi}{\partial n}=0$, with $\alpha$ a continuous function and $\frac{\partial \Psi}{\partial n}$ the normal derivative on the surface of the unit sphere $S^{n-1}$ of dimension $n-1$. Here $\Psi$ is the restriction of $u$ to the sphere $S^{n-1}$, $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on $S^{n-1}$ and $f \in \mathcal{D}\left(\mathbb{R}^{n}\right)$. Let
$\boldsymbol{\Theta}=\left(\theta, \phi_{1}, \ldots, \phi_{m-2}\right)$. Then there is a solution of the form

$$
\begin{equation*}
U(r, \boldsymbol{\Theta}, t)=\int_{0}^{\infty} \int_{S^{n-1}} f(\rho, \xi) p(t, r, \boldsymbol{\Theta}, \rho, \xi) \rho d \xi d \rho \tag{4.0.194}
\end{equation*}
$$

where for $n \geq 2$,

$$
\begin{equation*}
p(t, r, \boldsymbol{\Theta}, \rho, \xi)=\frac{1}{2 t}\left(\frac{\rho}{r}\right)^{\frac{n}{2}-1} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{\lambda_{m}} L_{m}(\boldsymbol{\Theta}) \overline{L_{m}(\xi)} I_{\mu_{m}}\left(\frac{r \rho}{2 t}\right) \tag{4.0.195}
\end{equation*}
$$

Here $\mu_{m}=\frac{1}{2} \sqrt{4 \lambda_{m}+(n-2)^{2}}$ and $L_{m}(\boldsymbol{\Theta})$ are normalised eigenfunctions of the problem $\Delta_{S} L+(\lambda+G) L=0$ and $\lambda_{m}$ are the eigenvalues.

This result requires the solution of a PDE for the eigenfunctions. This can also be solved numerically. However a discussion of the numerical solution of an eigenvalue problem for elliptic PDEs is beyond the scope of the thesis. As before we may extend to larger classes of initial data by simply evaluating the given examples. We consider an application of the previous theorem.

Example 4.0.15. If $P_{l}(\boldsymbol{\Theta})$ denotes the $l$ th spherical harmonic on the sphere $S^{n-1}$, then $\Delta_{S^{n-1}} P_{l}=l(l+n-2) P_{l}$, see [36]. Applying Theorem 4.0.52, the fundamental solution for $u_{t}=\Delta u-\frac{A}{r^{2}} u, A \geq 0$ is

$$
p(t, r, \boldsymbol{\Theta}, \rho, \xi)=\frac{1}{2 t}\left(\frac{\rho}{r}\right)^{\frac{n}{2}-1} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{l=0}^{\infty} P_{l}(\boldsymbol{\Theta}) \overline{P_{l}(\xi)} I_{\mu_{l}}\left(\frac{r \rho}{2 t}\right)
$$

where $\mu_{l}=\sqrt{4 l(l+n-2)+(n-2)^{2}+4 A}$. Taking $A=0$ will give the expansion for the heat kernel on $\mathbb{R}^{n}$. On $S^{2}$,

$$
\Psi_{l}^{m}(\theta, \phi)=\sqrt{\frac{(2 l+1)(l-m)!}{4 \pi(l+m)!}} P_{l}^{m}(\cos \phi) e^{i m \theta}, l=0,1,2,3, \ldots
$$

$-l \leq m \leq l$, are the normalised spherical harmonics. Here $P_{l}^{m}(x)$ is the usual Legendre function, see [35]. Thus

$$
u_{t}=\Delta u-\frac{A}{x^{2}+y^{2}+z^{2}} u, A \geq 0
$$

has the fundamental solution

$$
\begin{align*}
p(t, r, \theta, \phi, \rho, \xi, \eta) & =\frac{1}{2 t} \sqrt{\frac{\rho}{r}} e^{-\frac{r^{2}+\rho^{2}}{4 t}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(2 l+1)(l-m)!}{4 \pi(l+m)!} P_{l}^{m}(\cos \phi) \\
& \times P_{l}^{m}(\cos \eta) e^{i m(\theta-\xi)} I_{\sqrt{l^{2}+l+A+\frac{1}{4}}}\left(\frac{r \rho}{2 t}\right) . \tag{4.0.196}
\end{align*}
$$

An interesting consequence of this result follows. We present the $n=3$ case, and the reader will easily see how it extends to arbitrary $n$.

Corollary 4.0.53. The following summation formula holds.

$$
\begin{gathered}
\sqrt{\frac{\rho}{r}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(2 l+1)(l-m)!}{(l+m)!} P_{l}^{m}(\cos \phi) P_{l}^{m}(\cos \eta) e^{i m(\theta-\xi)} I_{l+\frac{1}{2}}\left(\frac{r \rho}{2 t}\right) \\
\quad=\frac{\exp \left\{\frac{r \rho(\cos \eta \cos \phi+\cos \theta \cos \xi \sin \eta \sin \phi+\sin \eta \sin \theta \sin \xi \sin \phi)}{2 t}\right\}}{\sqrt{\pi t}}
\end{gathered}
$$

Proof. We equate (4.0.196) with $A=0$ and the three-dimensional heat kernel.

Now we give an example of a fundamental solution for an Itô process.

Example 4.0.16. Consider the $n$-dimensional Itô process

$$
\begin{aligned}
d X_{t}^{i} & =X_{t}^{i} \frac{c_{1} \alpha_{n}\left\|\mathbf{X}_{t}\right\| \sqrt{(n-2)^{2}+4 A}}{2\left\|\mathbf{X}_{t}\right\|^{2}\left(c_{2} \beta_{n}\left\|\mathbf{X}_{t}\right\| \sqrt{\sqrt{(n-2)^{2}+4 A}}+c_{2}\right)}+\sqrt{2} d W_{t}^{i}, A>0 \\
i & =1,2,3, \ldots, n
\end{aligned}
$$

where $\left\|\mathbf{X}_{t}\right\|^{2}=\left(X_{t}^{1}\right)^{2}+\cdots+\left(X_{t}^{n}\right)^{2}, \alpha_{n}=\left(2-n+\sqrt{(n-2)^{2}+4 A}\right)$ and $\beta_{n}=\left(n-2+\sqrt{(n-2)^{2}+4 A}\right)$. The transition probability density is

$$
\begin{aligned}
p(t, x, y) & =\frac{1}{2 t} \frac{\left(c_{1}+c_{2}\|y\| \sqrt{(n-2)^{2}+4 A}\right)}{\left(c_{1}+c_{2}\|x\| \sqrt{(n-2)^{2}+4 A}\right)}\left(\frac{\|y\|}{\|x\|}\right)^{\frac{n}{2}-1-\frac{1}{2} \beta_{n}} \\
& \times e^{-\frac{\|x\|^{2}+\|y\|^{2}}{4 t}} \sum_{l=0}^{\infty} P_{l}(x) \overline{P_{l}(y)} I_{\mu_{l}}\left(\frac{\|x\|\|y\|}{2 t}\right),
\end{aligned}
$$

$x=\left(x_{1}, \ldots, x_{n}\right), y=\left(y_{1}, \ldots, y_{n}\right)$ and $P_{l}$ is the $l$ th harmonic polynomial restricted to the unit sphere.

We finish this chapter with a final expansion result. The proof for this result proceeds along the lines Theorem 4.0.51.

Theorem 4.0.54. Consider the equation

$$
\begin{aligned}
u_{t} & =u_{r r}+\frac{(n-1)}{r} u_{r}+\frac{1}{r^{2}}\left(\Delta_{S^{n-1}}+G(\boldsymbol{\Theta})\right) u-\frac{1}{4} c r^{2} u, c>0 \\
u(r, \boldsymbol{\Theta}, 0) & =f(r, \boldsymbol{\Theta})
\end{aligned}
$$

and $\alpha(\boldsymbol{\Theta}) \Psi(\boldsymbol{\Theta})+(1-\alpha(\boldsymbol{\Theta})) \frac{\partial \Psi}{\partial n}=0$, where $\alpha$ is a continuous function and $\frac{\partial \Psi}{\partial n}$ is the normal derivative on the surface of the unit sphere $S^{n-1}$ of dimension $n-1$. Here $\Psi$ is the restriction of $u$ to the sphere $S^{n-1}$, $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on $S^{n-1}$ and $f \in \mathcal{D}\left(\mathbb{R}^{n}\right)$. Let $\Theta=\left(\theta, \phi_{1}, \ldots, \phi_{m-2}\right)$. Then there is a solution of the form

$$
\begin{equation*}
U(r, \boldsymbol{\Theta}, t)=\int_{0}^{\infty} \int_{S^{n-1}} f(\rho, \xi) p(t, r, \boldsymbol{\Theta}, \rho, \xi) \rho d \xi d \rho \tag{4.0.197}
\end{equation*}
$$

where for $n \geq 2$,

$$
\begin{aligned}
p(t, r, \boldsymbol{\Theta}, \rho, \xi)= & \frac{2 \sqrt{c}}{r^{(n-2) / 2}}(8(\cosh (2 \sqrt{c} t)-1))^{-\frac{n}{4}} \sum_{\lambda_{m}} \frac{1}{I_{\frac{\sqrt{(n-2)^{2}+4 \lambda_{m}}}{4}}\left(\frac{\sqrt{c} \rho^{2}}{4}\right)} \\
& \times e^{-\frac{\sqrt{c}\left(r^{2}+\rho^{2}\right) \sinh (2 \sqrt{c t})}{4(\cosh (2 \sqrt{c t})-1)}} \Gamma_{m}\left(r, \rho^{2}, t\right) \overline{L_{m}(\xi)} L_{m}(\boldsymbol{\Theta}), \\
\Gamma_{m}(r, \rho, t)= & \left.\mathcal{L}^{-1}\left(\frac{1}{\left(\varepsilon^{2}-\frac{c}{16}\right)^{\frac{n}{4}}} e^{\left(\frac{a \varepsilon r^{2}}{\varepsilon^{2}-\frac{c}{16}}\right.}\right) I_{\frac{\sqrt{(n-2)^{2}+4 \lambda_{m}}}{4}}\left(\frac{b r^{2}}{\varepsilon^{2}-\frac{c}{16}}\right)\right),
\end{aligned}
$$

and $a=\frac{c}{8(\cosh (2 \sqrt{c} t)-1)}, b=\frac{c^{3 / 2}}{32(\cosh (2 \sqrt{c} t)-1)}$, and $L_{m}(\boldsymbol{\Theta})$ are normalised eigenfunctions of the problem $\Delta_{S^{n-1}} L+(\lambda+G) L=0$ and $\lambda_{m}$ are the eigenvalues.

## Applications of the Theory

5.0.14. Applications to Representation Theory. Lie symmetries are typically only locally defined transformations. However if we can find an equivalence with a global representation then we can make the symmetry a global symmetry. Craddock has shown that the Lie symmetries of many important PDEs are in fact equivalent to global representations of the underlying symmetry groups; see $[\mathbf{2 4}]$ and $[\mathbf{2 5}]$. Recently Craddock and Dooley extended this work to some important classes of multidimensional PDEs, [18]. In this section we use the results proved earlier in the thesis to construct intertwining operators to make the symmetries in the $n$-dimensional case global. Connecting symmetries with representations allows for the application of representation theory to Lie symmetry analysis. Some consequences of this are in [18]. For simplicity, we consider the unitary case for equations in two space variables of the form

$$
\begin{equation*}
i u_{t}=\Delta u+\frac{1}{x^{2}} k\left(\frac{y}{x}\right) u \tag{5.0.198}
\end{equation*}
$$

The extension to the $n$-dimensional case is easy. In $[\mathbf{1 8}]$ the following irreducible projective representation of $S L(2, \mathbb{R})$ was introduced. See that paper for more details. $e_{S L(2, \mathbb{R})}$ and $e_{\mathbb{R}}$ are the identity elements of $S L(2, \mathbb{R})$ and $\mathbb{R}$ respectively.

Definition 5.0.55. For $\Re(\nu)>-2, \lambda \in \mathbb{R}^{*}$ and $f \in L^{2}\left(\mathbb{R}^{+}\right)$we define the modified Segal-Shale-Weil representation of $S L(2, \mathbb{R}) \times \mathbb{R}$ by

$$
\begin{align*}
& R_{\lambda}^{\nu}\left(\left(\begin{array}{cc}
1 & b \\
0 & 1
\end{array}\right), e_{\mathbb{R}}\right) f(z)=e^{-i \lambda b z^{2}} f(z)  \tag{5.0.199}\\
& R_{\lambda}^{\nu}\left(\left(\begin{array}{cc}
a & 0 \\
0 & a^{-1}
\end{array}\right), e_{\mathbb{R}}\right) f(z)=\sqrt{|a|} f(a z)  \tag{5.0.200}\\
& R_{\lambda}^{\nu}\left(\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right), e_{\mathbb{R}}\right) f(z)=\sqrt{|\lambda|} \tilde{f}_{\nu}(\lambda z) . \tag{5.0.201}
\end{align*}
$$

Here $\tilde{f}_{\nu}(y)=\int_{0}^{\infty} f(x) \sqrt{x y} J_{\nu}(x y) d y$, is the Hankel transform of $f$. The function $J_{\nu}$ is a Bessel function of the first kind. We also set $R_{\lambda}^{\nu}\left(e_{S L(2, \mathbb{R})}, s\right) f(z)=e^{i s} f(z)$.

As shown in [18], the projective representation $R_{\lambda}$ is irreducible and can be extended to the whole of $S L(2, \mathbb{R}) \times \mathbb{R}$ by the Bruhat decomposition of $S L(2, \mathbb{R})$.

For $f \in L^{2}\left(\mathbb{R}^{+}\right)$, we introduce the operator

$$
\begin{aligned}
(\mathcal{A} f)(r, \rho, \theta, t) & =\int_{0}^{\infty} \int_{0}^{2 \pi} \frac{\sqrt{\rho}}{4 \pi i t} f(\rho) L_{n}(\theta) \overline{L_{n}(\phi)} e^{-\frac{i\left(r^{2}+\rho^{2}\right)}{4 t}} J_{\nu}\left(\frac{r \rho}{2 t}\right) d \phi d \rho \\
& =\int_{\mathbb{R}^{2}} \frac{\sqrt{\rho}}{4 \pi i t} f(\rho) l_{n}(\theta, \phi) e^{-\frac{i\left(r^{2}+\rho^{2}\right)}{4 t}} J_{\nu}\left(\frac{r \rho}{2 t}\right) d \mu
\end{aligned}
$$

where $l_{n}(\theta, \phi)$ is a solution of (5.0.198) in polar coordinates, $d \mu=d \phi d \rho$, $\nu=\sqrt{\lambda_{n}}$ and $\lambda_{n}$ is the $n$th eigenvalue of the Sturm-Liouville problem in Theorem 4.0.49. The operator $\mathcal{A}$ is constructed by taking one term from the expansion for the fundamental solution.

Remark 5.0.56. Without loss of generality we assume that the domain of the angular variable here is $[0,2 \pi]$. More generally the domain
will be $[a, b]$ and in this case we simply replace the integral $\int_{0}^{2 \pi}$ with $\int_{a}^{b}$ and the arguments are unchanged.

The following result is an elementary consequence of Theorem 4.0.49.

Lemma 5.0.57. Let $u(r, \rho, t)=\mathcal{A} f$, for $f \in L^{2}\left(\mathbb{R}^{+}\right)$. Then $u$ is a solution of the equation $i u_{t}=u_{r r}+\frac{1}{r} u_{r}+\frac{1}{r^{2}} u_{\theta \theta}+\frac{K(\theta)}{r^{2}} u$, with $K(\theta)=$ $\frac{k(\tan \theta)}{\cos ^{2} \theta}$.

In previous work on this problem the intertwining operator has usually been constructed from a fundamental solution. In this case the fundamental solution is not required.

Theorem 5.0.58. The PDE $i u_{t}=\Delta u+\frac{1}{x^{2}} k\left(\frac{y}{x}\right) u$ has Lie symmetry group $S L(2, \mathbb{R}) \times \mathbb{R}$. Moreover if $\sigma$ represents the Lie symmetry operator and $R_{1}$ represents the modified Segal-Shale-Weil projective representation of $S L(2, \mathbb{R}) \times \mathbb{R}$, then for all $g \in S L(2, \mathbb{R}) \times \mathbb{R}$ and $f \in L^{2}\left(\mathbb{R}^{+}\right)$

$$
(\sigma(g) \mathcal{A} f)(x, y, t)=\left(\mathcal{A} R_{1}^{\nu}(g) f\right)(x, y, t)
$$

Moreover, the representations are irreducible.

Proof. It is sufficient to prove the result in polar coordinates. That is, we prove the equivalence for the $\operatorname{PDE} i u_{t}=u_{r r}+\frac{1}{r} u_{r}+$ $\frac{1}{r^{2}} u_{\theta \theta}+\frac{K(\theta)}{r^{2}} u$. The symmetries in polar coordinates are

$$
\begin{aligned}
\sigma\left(\exp \left(\epsilon \mathbf{v}_{1}\right)\right) u(r, \theta, t) & =u(r, \theta, t-\epsilon) \\
\sigma\left(\exp \left(\epsilon \mathbf{v}_{2}\right)\right) u(r, \theta, t) & =e^{-\epsilon} u\left(e^{\epsilon} r, \theta, e^{2 \epsilon} t\right) \\
\sigma\left(\exp \left(\epsilon \mathbf{v}_{3}\right)\right) u(r, \theta, t) & =\frac{1}{1+4 \epsilon t} \exp \left(-\frac{i \epsilon r^{2}}{1+4 \epsilon t}\right) u\left(\frac{r}{1+4 \epsilon t}, \theta, \frac{t}{1+4 \epsilon t}\right), \\
\sigma\left(\exp \left(\epsilon \mathbf{v}_{4}\right) u(r, \theta, t)\right. & =e^{i \epsilon} u(r, \theta, t) .
\end{aligned}
$$

Then we need to show that for $k=1,2,3$,

$$
\begin{equation*}
\left.\sigma\left(\exp \left(\epsilon \mathbf{v}_{k}\right)\right) \mathcal{A} f\right)(r, \theta, t)=\mathcal{A} R_{1}^{\nu}\left(\exp \left(\epsilon X_{k}\right) f\right)(r, \theta, t) \tag{5.0.202}
\end{equation*}
$$

where $X_{1}=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right) \quad X_{2}=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right), X_{3}=\left(\begin{array}{cc}0 & 0 \\ -1 & 0\end{array}\right)$, is a basis for the Lie algebra $\mathfrak{s l}_{2}$. The result for $\mathbf{v}_{4}$ is trivial. We can then use the fact that any element of $S L(2, \mathbb{R})$ can be written as a product of exponentials of these basis vectors. We will suppress the $\mathbb{R}$ component of $R_{1}^{\nu}$ for convenience. We exponentiate $X_{2}$ to get $\exp \left(\epsilon X_{2}\right)=\left(\begin{array}{cc}e^{\epsilon} & 0 \\ 0 & e^{-\epsilon}\end{array}\right)$. Thus we have

$$
R_{1}^{\nu}\left(\exp \left(\epsilon X_{2}\right) f\right)(\rho)=e^{\frac{1}{2} \epsilon} f\left(e^{\epsilon} \rho\right)
$$

Now for the equivalence calculation:

$$
\begin{aligned}
& \mathcal{A} R_{1}^{\nu}\left(\exp \left(\epsilon X_{2}\right) f\right)(r, \theta, t)=\int_{\mathbb{R}^{2}} e^{\frac{1}{2} \epsilon} f\left(e^{\epsilon} \rho\right) \frac{\sqrt{\rho}}{4 \pi t} l_{n}(\theta, \phi) e^{-\frac{i\left(r^{2}+\rho^{2}\right)}{4 t}} J_{\nu}\left(\frac{r \rho}{2 t}\right) d \mu \\
& =\int_{\mathbb{R}^{2}} e^{\frac{1}{2} \epsilon} f(\rho) \frac{\sqrt{e^{-\epsilon} \rho}}{4 \pi t} l_{n}(\theta, \phi) e^{-i\left(\frac{\left(r^{2}+e^{-2 \epsilon} \rho^{2}\right)}{4 t}\right.} J_{\nu}\left(\frac{r e^{-\epsilon} \rho}{2 t}\right) e^{-\epsilon} d \mu \\
& =\int_{\mathbb{R}^{2}} e^{-\epsilon} f(\rho) \frac{\sqrt{\rho}}{4 \pi t} l_{n}(\theta, \phi) e^{-\frac{i\left(\left(e^{\epsilon} r\right)^{2}+\rho^{2}\right)}{4 e^{2 \epsilon} t}} J_{\nu}\left(\frac{r e^{\epsilon} \rho}{2 e^{2 \epsilon} t}\right) d \mu \\
& =e^{-\epsilon} u\left(e^{\epsilon} r, \theta, e^{2 \epsilon} t\right)=\sigma\left(\exp \left(\epsilon \mathbf{v}_{2}\right) u\right)(r, \theta, t)
\end{aligned}
$$

For the $\mathbf{v}_{3}$ calculation, apply the symmetry

$$
\sigma\left(\exp \left(\epsilon \mathbf{v}_{3}\right)\right) u(x, y, t)=\frac{1}{1+4 \epsilon t} \exp \left(-\frac{i \epsilon r^{2}}{1+4 \epsilon t}\right) u\left(\frac{r}{1+4 \epsilon t}, \theta, \frac{t}{1+4 \epsilon t}\right)
$$

to

$$
p(r, \theta, \rho, \phi, t)=\frac{\sqrt{\rho}}{4 \pi i t} L_{n}(\theta) \overline{L_{n}(\phi)} e^{-\frac{i\left(r^{2}+\rho^{2}\right)}{4 t}} J_{\nu}\left(\frac{r \rho}{2 t}\right)
$$

Now the terms involving $J_{\nu}$ and $1 /(4 \pi i t)$ are unchanged by the symmetry. Clearly $\exp \left(-i \frac{\left.\frac{r^{2}}{1+4 \epsilon t}+\rho^{2}\right)}{\frac{\epsilon t}{1+4 \epsilon t}}\right)=e^{-i \epsilon \rho^{2}} \exp \left(\frac{-i\left(r^{2}+\rho^{2}\right)}{4 t}\right)$. Thus

$$
\begin{aligned}
& \left(\sigma\left(\exp \left(\epsilon \mathbf{v}_{3}\right)\right) \mathcal{A} f\right)(r, \theta, t)=\int_{\mathbb{R}^{2}} e^{-i \epsilon \rho^{2}} f(\rho) \frac{\sqrt{\rho}}{4 \pi i t} e^{-i \frac{r^{2}+\rho^{2}}{4 t}} l_{n}(\theta, \phi) J_{\nu}\left(\frac{r \rho}{2 t}\right) d \mu \\
& =\int_{\mathbb{R}^{2}} R_{1}^{\nu}\left(\exp \left(\epsilon X_{3}\right)\right) f(\rho) \frac{\sqrt{\rho}}{4 \pi i t} e^{-i \frac{r^{2}+\rho^{2}}{4 t}} l_{n}(\theta, \phi) J_{\nu}\left(\frac{r \rho}{2 t}\right) d \mu
\end{aligned}
$$

This establishes the third equivalence. Finally, $\mathfrak{H}_{\nu}\left(\frac{\sqrt{\rho}}{2 i t} e^{\frac{-i\left(r^{2}+\rho^{2}\right)}{4 t}} J_{\nu}\left(\frac{r \rho}{2 t}\right)\right)(z)=$ $e^{-\frac{i \nu \pi}{2}} \sqrt{z} e^{i t z^{2}} J_{\nu}(r z)$, (see [28]). An elementary calculation, detailed in [18], shows that

$$
\left.\mathcal{A} R_{1}^{\nu}\left(\exp \left(\epsilon X_{1}\right)\right) f\right)(\rho)=\mathfrak{H}_{\nu}\left(e^{i \epsilon \rho^{2}} \tilde{f}_{\nu}\right)(\rho)
$$

Now using $\int_{0}^{\infty} \tilde{f}_{\nu}(\rho) g(\rho) d \rho=\int_{0}^{\infty} f(\rho) \tilde{g}_{\nu}(\rho) d \rho$, we have

$$
\begin{aligned}
& \left(\mathcal{A} R_{1}^{\nu}\left(\exp \left(\epsilon X_{1}\right)\right) f\right)(r, \theta, t)=\int_{\mathbb{R}^{2}} \mathfrak{H}_{\nu}\left(e^{-i \epsilon \rho^{2}} \tilde{f}_{\nu}\right)(\rho) p(r, \theta, \phi, t) d \mu \\
& =\int_{\mathbb{R}^{2}} e^{-i \epsilon z^{2}} \tilde{f}_{\nu}(z) l_{n}(\theta, \phi) \frac{1}{2 \pi} e^{-\frac{i \nu \pi}{2}} \sqrt{z} e^{i t z^{2}} J_{\nu}(r z) d z d \phi \\
& =\int_{\mathbb{R}^{2}} \tilde{f}_{\nu}(z) l_{n}(\theta, \phi) \frac{1}{2 \pi} e^{-\frac{i \nu \pi}{2}} \sqrt{z} e^{i(t-\epsilon) z^{2}} J_{\nu}(r z) d z d \phi \\
& =\int_{\mathbb{R}^{2}} f(\rho) l_{n}(\theta, \phi) \frac{\sqrt{\rho}}{4 \pi i(t-\epsilon)} e^{-\frac{i\left(r^{2}+\rho^{2}\right)}{4(t-\epsilon)}} J_{\nu}\left(\frac{r \rho}{2(t-\epsilon)}\right) d \mu \\
& =u(x, t-\epsilon)
\end{aligned}
$$

Here $p(r, \theta, \phi, t)=l_{n}(\theta, \phi) \frac{\sqrt{\rho}}{4 \pi i t} e^{-\frac{i\left(r^{2}+\rho^{2}\right)}{4 t}} J_{\nu}\left(\frac{r \rho}{2 t}\right)$. This completes the proof.

This theorem may be extended to the PDE

$$
i u_{t}=\Delta u+\frac{1}{x_{1}^{2}} k\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right) u
$$

We work in the $n$-dimensional form of polar coordinates. From Theorem 4.0.52 in Chapter 4, it is straightforward to obtain an intertwining
operator. We take a single term from the series for the fundamental solution and let $t \rightarrow i t$. This leads to the intertwining operator

$$
\begin{equation*}
(\mathcal{A} f)(r, \boldsymbol{\Theta}, t)=\int_{0}^{\infty} \int_{S^{n-1}} f(\rho) \psi_{\lambda_{k}}(\boldsymbol{\Theta}, \xi) \frac{\rho^{\frac{1}{2}}}{2 t r^{\frac{n-2}{2}}} e^{-\frac{r^{2}+\rho^{2}}{4 t}} J_{l}\left(\frac{r \rho}{2 t}\right) d \xi d \rho \tag{5.0.203}
\end{equation*}
$$

where $l=\frac{1}{2} \sqrt{4 \lambda_{k}+(n-2)^{2}}$ and $\psi_{\lambda_{k}}=L_{k}(\boldsymbol{\Theta}) \overline{L_{k}(\xi)}$ and $L_{k}(\boldsymbol{\Theta})$ is the $k$ th eigenfunction of $\Delta_{S^{n-1}} u+(G+\lambda) u=0$, with $\lambda_{k}$ the corresponding eigenvalue. The calculations are essentially identical to the previous result. This leads to the following theorem.

Theorem 5.0.59. The PDE

$$
\begin{equation*}
i u_{t}=\Delta u+2 \nabla \phi \cdot \nabla u+B(x) u, x \in \mathbb{R}^{n} \tag{5.0.204}
\end{equation*}
$$

where $\Delta \phi+|\nabla \phi|^{2}+\frac{1}{x_{1}^{2}} k\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right)=B(x)$ has $S L(2, \mathbb{R}) \times \mathbb{R}$ as a global group of Lie point symmetries and if $\sigma$ represents the Lie symmetry operator and $R_{1}^{\nu}$ represents the modified Segal-Shale-Weil projective representation of $S L(2, \mathbb{R}) \times \mathbb{R}$, then for all $g \in S L(2, \mathbb{R}) \times \mathbb{R}$ and $f \in L^{2}\left(\mathbb{R}^{+}\right)$

$$
(\sigma(g) \overline{\mathcal{A}} f)(x, y, t)=\left(\overline{\mathcal{A}} R_{1}^{\nu}(g) f\right)(x, y, t)
$$

with $u=\overline{\mathcal{A}} f$ obtained from (5.0.203) by the change of variables $u=$ $e^{\phi} v$. The representations are irreducible.

Proof. Equation (5.0.204) is equivalent to $i v_{t}=\Delta v+\frac{1}{x_{1}^{2}} k\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right) v$, by letting $u=e^{\phi}$ and so they have isomorphic symmetry groups.

Analogous results can be established for PDEs of the form

$$
i u_{t}=\Delta u+\frac{1}{x_{1}^{2}} k\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right) u-\frac{1}{4} c\left(x_{1}^{2}+\cdots+x_{n}^{2}\right)
$$

but we do not consider this here.
5.0.15. The Nonunitary Case. Next we consider the nonunitary case for equations of the form

$$
\begin{equation*}
u_{t}=\Delta u+\frac{1}{x^{2}} k\left(\frac{y}{x}\right) u \tag{5.0.205}
\end{equation*}
$$

In order to analyse the nonunitary symmetries we need to introduce a new type of representation and a new representation space. This was done in $[\mathbf{2 4}]$. We also note that Rosinger has made an extensive study of the action of Lie groups on distributions, see [66].

Definition 5.0.60. Fix a nonnegative integer $n$. Then let

$$
\begin{equation*}
W^{\nu, n}=\left\{\sum_{k=1}^{N} \sum_{j=1}^{n} f_{j k}, N \in \mathbb{N}\right\} \tag{5.0.206}
\end{equation*}
$$

where $f_{j k}(x)=b_{j k} x^{\nu+\frac{1}{2}} e^{-\beta_{j k} x^{2}} L_{j}^{(\nu)}\left(a_{j k} x^{2}\right), b_{j k,}, a_{j k} \in \mathbb{R}, \operatorname{Im}\left(\beta_{j k}\right) \neq 0$, and $L_{j}^{(\nu)}(z)$ is a Laguerre polynomial. This splits as $W^{\nu, n}=W_{1}^{\nu, n} \oplus$ $W_{2}^{\nu, n} \oplus W_{3}^{\nu, n}$, with

$$
\begin{aligned}
& W_{1}^{\nu, n}=\left\{f \in W^{\nu, n}, \operatorname{Re}\left(\beta_{j k}\right)>0\right\} \\
& W_{2}^{\nu, n}=\left\{f \in W^{\nu, n}, \operatorname{Re}\left(\beta_{j k}\right)<0\right\} \\
& W_{3}^{\nu, n}=\left\{f \in W^{\nu, n}, \operatorname{Re}\left(\beta_{j k}\right)=0\right\}
\end{aligned}
$$

This is the space on which $R_{\lambda}$ acts. To define $R_{\lambda}$, let

$$
\begin{aligned}
\mathfrak{H}_{\nu}(f) & =\int_{0}^{\infty} \sqrt{x y} J_{\nu}(x y) f(x) d x, f \in W_{1}^{\nu, n} \\
\mathfrak{H}_{\nu}(f) & =\int_{0}^{i \infty} \sqrt{x y} J_{\nu}(x y) f(x) d x, f \in W_{2}^{\nu, n} \\
\mathfrak{H}_{\nu}(f) & =\lim _{R e(\beta) \rightarrow 0} \mathfrak{H}_{\nu}^{\prime}(F), f \in W_{3}^{\nu, n}, F \in W^{\nu, n}
\end{aligned}
$$

where $\lim _{\operatorname{Re}(\beta) \rightarrow 0} F=f$ and $\tilde{f}_{\nu}$ is the Hankel transform of $f$. The following proposition was given in [18].

Proposition 5.0.61. For $g \in S L(2, \mathbb{R})$ the action of $R_{\lambda}^{\nu}(g)$ defined by

$$
\begin{gather*}
R_{\lambda}^{\nu}\left(\left(\begin{array}{ll}
1 & b \\
0 & 1
\end{array}\right)\right) f(x)=e^{-\lambda i b x^{2}} f(x)  \tag{5.0.207}\\
R_{\lambda}^{\nu}\left(\left(\begin{array}{cc}
a & 0 \\
0 & a^{-1}
\end{array}\right)\right) f(x)=|a|^{1 / 2} f(a x), \tag{5.0.208}
\end{gather*}
$$

and

$$
R_{\lambda}^{\nu}\left(\left(\begin{array}{cc}
0 & -1  \tag{5.0.209}\\
1 & 0
\end{array}\right)\right) f(i x)=\lambda^{1 / 2} \tilde{f}(\lambda x)
$$

preserves $W^{\nu, n}$, for $\lambda \in \mathbb{C}-\{0\}$.

We consider the case where $\lambda=i$. Let $f=\sum_{j=1}^{3} f_{j} \in W^{\nu, n}$, $f_{j} \in W_{j}^{\nu, n}$. We introduce the intertwining operator

$$
(\mathcal{A} f)(r, \rho, \theta, t)=\sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma} \frac{\sqrt{\rho}}{4 \pi t} f_{j}(\rho) l_{m}(\theta, \phi) e^{-\frac{\left(r^{2}+\rho^{2}\right)}{4 t}} I_{\nu}\left(\frac{r \rho}{2 t}\right) d \rho d \phi
$$

where $l_{m}(\theta, \phi)=L_{m}(\theta) \overline{L_{m}(\phi)}, \nu=\sqrt{\lambda_{n}}$, with $\lambda_{m}$ the $m$ th eigenvalue of the Sturm-Liouville problem in Theorem 4.0.49 and $I_{\nu}$ is a modified Bessel function of the first kind. The contour of integration is chosen as either $\gamma(t)=t, t \geq 0$ for $f \in W_{1}^{\nu, n}$ or $\gamma(t)=i t, t \geq 0$ for $f \in W_{2,3}^{\nu, n}$. The next result is elementary.

Lemma 5.0.62. Let $u(r, \rho, t)=\mathcal{A} f$, for $f \in W^{\nu, n}$. Then $u$ is a solution of the equation $u_{t}=u_{r r}+\frac{1}{r} u_{r}+\frac{1}{r^{2}} u_{\theta \theta}+\frac{K(\theta)}{r^{2}} u$, with $K(\theta)=$ $\frac{k(\tan \theta)}{\cos ^{2} \theta}$.

We have the following result for the nonunitary case.

Theorem 5.0.63. The PDE $u_{t}=\Delta u+\frac{1}{x^{2}} k\left(\frac{y}{x}\right) u$ has Lie symmetry group $S L(2, \mathbb{R}) \times \mathbb{R}$. Moreover if $\sigma$ represents the Lie symmetry operator and $R_{i}^{\nu}$ represents the modified Segal-Shale-Weil projective representation, then for all $g \in S L(2, \mathbb{R})$ and $s \in \mathbb{R}$

$$
(\sigma(g, s) \mathcal{A} f)(x, y, t)=\left(\mathcal{A} R_{i}^{\nu}(g) \xi(s)\right) f(x, y, t), f \in W^{\nu, n}
$$

where $\xi(s)=e^{s}$.

Proof. As in the unitary case we work in polar coordinates. The symmetries in polar coordinates are

$$
\begin{aligned}
\sigma\left(\exp \left(\epsilon \mathbf{v}_{1}\right)\right) u(r, \theta, t) & =u(r, \theta, t+\epsilon) \\
\sigma\left(\exp \left(\epsilon \mathbf{v}_{2}\right)\right) u(r, \theta, t) & =e^{-\epsilon} u\left(e^{\epsilon} r, \theta, e^{2 \epsilon} t\right) \\
\sigma\left(\exp \left(\epsilon \mathbf{v}_{3}\right)\right) u(r, \theta, t) & =\frac{1}{1+4 \epsilon t} \exp \left(-\frac{\epsilon r^{2}}{1+4 \epsilon t}\right) u\left(\frac{r}{1+4 \epsilon t}, \theta, \frac{t}{1+4 \epsilon t}\right), \\
\sigma\left(\exp \left(\epsilon \mathbf{v}_{4}\right) u(r, \theta, t)\right. & =e^{\epsilon} u(r, \theta, t)
\end{aligned}
$$

The case $\sigma\left(\exp \left(\epsilon \mathbf{v}_{4}\right)\right)$ is trivial. We need to show that for $k=1,2,3$,

$$
\left.\sigma\left(\exp \left(\epsilon \mathbf{v}_{k}\right)\right) A f\right)(r, \theta, t)=\mathcal{A} R_{i}^{\nu}\left(\exp \left(\epsilon X_{k}\right) f\right)(r, \theta, t)
$$

where again

$$
X_{1}=\left(\begin{array}{cc}
0 & 0 \\
-1 & 0
\end{array}\right), \quad X_{2}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right), X_{3}=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right)
$$

is a basis for the Lie algebra $\mathfrak{s l}_{2}$. For the $\mathbf{v}_{3}$ case, apply the symmetry

$$
\sigma\left(\exp \left(\epsilon \mathbf{v}_{3}\right)\right) u(x, y, t)=\frac{1}{1+4 \epsilon t} \exp \left(-\frac{\epsilon r^{2}}{1+4 \epsilon t}\right) u\left(\frac{r}{1+4 \epsilon t}, \theta, \frac{t}{1+4 \epsilon t}\right)
$$

to

$$
p(r, \theta, \rho, \phi, t)=\frac{\sqrt{\rho}}{4 \pi t} l_{m}(\theta, \phi) e^{-\frac{\left(r^{2}+\rho^{2}\right)}{4 t}} I_{\nu}\left(\frac{r \rho}{2 t}\right) .
$$

Then we have

$$
\begin{gathered}
\left(\sigma\left(\exp \left(\epsilon \mathbf{v}_{3}\right)\right) A f\right)(r, \theta, t)=\sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma} e^{-\epsilon \rho^{2}} f_{j}(\rho) \frac{\sqrt{\rho}}{4 \pi t} e^{-\frac{\left(r^{2}+\rho^{2}\right)}{4 t}} l_{m}(\theta, \phi) \\
\times I_{\nu}\left(\frac{r \rho}{2 t}\right) d \rho d \phi \\
=\sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma} R_{i}^{\nu}\left(\exp \left(\epsilon X_{1}\right)\right) f(\rho) \frac{\sqrt{\rho}}{4 \pi t} e^{-\frac{\left(r^{2}+\rho^{2}\right)}{4 t}} l_{m}(\theta, \phi) I_{\nu}\left(\frac{r \rho}{2 t}\right) d \rho d \phi
\end{gathered}
$$

For the $\mathbf{v}_{2}$ calculation we have

$$
\begin{aligned}
& \mathcal{A} R_{i}^{\nu}\left(\exp \left(\epsilon X_{2}\right) f\right)(r, \theta, t)=\sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma} e^{\frac{1}{2} \epsilon} f_{j}\left(e^{\epsilon} \rho\right) \frac{\sqrt{\rho}}{4 \pi t} l_{m}(\theta, \phi) e^{-\frac{\left(r^{2}+\rho^{2}\right)}{4 t}} \\
& \times I_{\nu}\left(\frac{r \rho}{2 t}\right) d \rho d \phi \\
& =\sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma} e^{\frac{1}{2} \epsilon} f_{j}(\rho) \frac{\sqrt{e^{-\epsilon} \rho}}{4 \pi t} l_{m}(\theta, \phi) e^{-\frac{\left(r^{2}+e^{-2 \epsilon} \rho^{2}\right)}{4 t}} I_{\nu}\left(\frac{r e^{-\epsilon} \rho}{2 t}\right) e^{-\epsilon} d \rho d \phi \\
& =\sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma} e^{-\epsilon} f_{j}(\rho) \frac{\sqrt{\rho}}{4 \pi t} l_{m}(\theta, \phi) e^{-\frac{\left(\left(e^{\epsilon} r\right)^{2}+\rho^{2}\right)}{4 e^{2} t}} I_{\nu}\left(\frac{r e^{\epsilon} \rho}{2 e^{2 \epsilon} t}\right) d \rho d \phi \\
& =e^{-\epsilon} u\left(e^{\epsilon} r, \theta, e^{2 \epsilon} t\right)=\sigma\left(\exp \left(\epsilon \mathbf{v}_{2}\right) \mathcal{A} f\right)(r, \theta, t) .
\end{aligned}
$$

For the final equivalence calculation, we must determine the action $R_{i}^{\nu}\left(\left(\begin{array}{cc}1 & 0 \\ -\epsilon & 1\end{array}\right)\right) f$. First we need the following Hankel transform, valid for all $\Re(\beta)>0, \nu>1$. The Hankel transform is given by

$$
\mathfrak{H}_{\nu}\left(\frac{y^{1 / 2}}{2 \beta} e^{-\frac{a^{2}+y^{2}}{4 \beta}} I_{\nu}\left(\frac{a y}{2 \beta}\right)\right)(x)=x^{1 / 2} e^{-\beta x^{2}} J_{\nu}(a x),
$$

see formula 8.11 .23 on page 51 of [28]. Next we use a result from [18]. For all $f \in W^{\nu, n}$, we have

$$
\left(R_{i}^{\nu}\left(\begin{array}{cc}
1 & 0 \\
-\epsilon & 1
\end{array}\right) f\right)(y)=\mathfrak{H}_{\nu}\left(e^{-\epsilon y^{2}} \tilde{f}_{\nu}\right)(y)
$$

where $\tilde{f}_{\nu}=\mathfrak{H}_{\nu}(f)$. So we have

$$
\begin{aligned}
& \sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma}\left(R_{i}^{\nu}\left(\begin{array}{cc}
1 & 0 \\
-\epsilon & 1
\end{array}\right) f_{j}\right)(\rho) \frac{\sqrt{\rho}}{4 \pi t} e^{-\frac{\left(r^{2}+\rho^{2}\right)}{4 t}} l_{m}(\theta, \phi) I_{\nu}\left(\frac{r \rho}{2 t}\right) d \rho d \phi \\
& =\sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma} \mathfrak{H}_{\nu}\left(e^{-\epsilon \rho^{2}} \tilde{f}_{\nu}(\rho)\right) \frac{\sqrt{\rho}}{4 \pi t} e^{-\frac{\left(r^{2}+\rho^{2}\right)}{4 t}} l_{m}(\theta, \phi) I_{\nu}\left(\frac{r \rho}{2 t}\right) d \rho d \phi \\
& =\sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma} e^{-\epsilon \rho^{2}} \tilde{f}_{\nu}(\rho) \frac{\sqrt{\rho}}{2 \pi} e^{-t \rho^{2}} l_{m}(\theta, \phi) J_{\nu}(r \rho) d \rho d \phi \\
& =\sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma} \tilde{f}_{\nu}(\rho) \frac{\sqrt{\rho}}{2 \pi} e^{-(t+\epsilon) \rho^{2}} l_{m}(\theta, \phi) J_{\nu}(r \rho) d \rho d \phi \\
& =\sum_{j=1}^{3} \int_{0}^{2 \pi} \int_{\gamma} f_{j}(\rho) \frac{\sqrt{\rho}}{4 \pi(t+\epsilon)} e^{-\frac{\left(r^{2}+\rho^{2}\right)}{4(t+\epsilon)}} l_{m}(\theta, \phi) I_{\nu}\left(\frac{r \rho}{2(t+\epsilon)}\right) d \rho d \phi \\
& =u(r, \theta, t+e)
\end{aligned}
$$

This completes the proof for the nonunitary case.

The previous theorem can be extended to larger representation spaces.

Theorem 5.0.64. The representation $\left\{R_{i}^{\nu}, W^{\nu, n}\right\}$ and the intertwining operator $\mathcal{A}$ of Theorem 5.0.63 can be extended to $\mathcal{D}^{\prime}\left(\mathbb{R}^{+}\right)$and for all $g \in S L(2, \mathbb{R}), s \in \mathbb{R}$ and all $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{+}\right)$

$$
\begin{equation*}
(\sigma(g, s) \mathcal{A} f)(r, \theta, t)=\left(\mathcal{A} R_{i}^{\nu}(g) \xi(s) f\right)(r, \theta, t) \tag{5.0.210}
\end{equation*}
$$

Furthermore, the representations are irreducible.

The proof for this theorem is identical to that for Theorem 7.10 in [18]. We refer the reader there for the details. As for the unitary case, the extension to the general nonunitary case is straightforward. We let

$$
\begin{align*}
(\mathcal{A} f)(r, \rho, \theta, t) & =\sum_{j=1}^{3} \int_{S^{n}} \int_{\gamma} \frac{\sqrt{\rho}}{4 \pi t} f_{j}(\rho) L_{\lambda_{n}}(\theta) \overline{L_{\lambda_{n}}(\phi)} r^{-\frac{n}{2}+1} e^{-\frac{\left(r^{2}+\rho^{2}\right)}{4 t}} \\
& \times I_{\lambda_{n}}\left(\frac{r \rho}{2 t}\right) d \rho d \Omega \tag{5.0.211}
\end{align*}
$$

where $L_{\lambda_{n}}$ is the $n$th eigenfunction of $\Delta_{S}+(\lambda+G(\theta) L=0$ as in Chapter 4. Here $d \Omega$ is the surface measure on $S^{n}$.

Theorem 5.0.65. The PDE

$$
\begin{equation*}
u_{t}=\Delta u+2 \nabla \phi \cdot \nabla u+B(x) u, x \in \mathbb{R}^{n} \tag{5.0.212}
\end{equation*}
$$

where $\Delta \phi+|\nabla \phi|^{2}+\frac{1}{x_{1}^{2}} k\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right)=B(x)$ has $S L(2, \mathbb{R}) \times \mathbb{R}$ as a global group of Lie point symmetries and if $\sigma$ represents the Lie symmetry operator and $R_{i}$ represents the nonunitary modified Segal-Shale-Weil projective representation of $S L(2, \mathbb{R})$ given by Proposition 5.0.61, then for all $g \in S L(2, \mathbb{R}), s \in \mathbb{R}$ and $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{+}\right)$

$$
(\sigma(g, s) \overline{\mathcal{A}} f)(x, t)=\left(\overline{\mathcal{A}} R_{1}^{\nu}(g) \xi(s) f\right)(x, t)
$$

with $u=\overline{\mathcal{A}} f$ obtained from (5.0.211) by the change of variables $u=$ $e^{\phi} v$. Moreover, the representations are irreducible.

Equations of the form $u_{t}=\Delta u-\frac{1}{x_{1}^{2}} k\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right) u-c|x|^{2} u$ can be handled by the same methods, but we will not consider this here.
5.0.16. Pricing Derivatives. The results of the thesis can be used to solve problems in financial mathematics. The key is that the
pricing of derivative securities can be reduced to the solution of certain parabolic partial differential equations. See [52] for example for a reference on pricing derivatives.
5.0.16.1. Volatility Swaps in Platen's Benchmark Framework. Let us consider the price of a volatility swap. A volatility swap is a forward contract on the annualised volatility. In $[\mathbf{1 7}]$ the price of a volatility swap under Platen's benchmark approach is calculated. See [63] for an introduction to Platen's benchmark pricing methodology. The following result was proved in Chapter 2.

Proposition 5.0.66. Let $X=\left\{X_{t}: t \geq 0\right\}$ satisfy the $S D E$

$$
d X_{t}=\left(a-b X_{t}\right) d t+\sqrt{2 \sigma X_{t}} d W_{t} .
$$

Let $\beta=1+m-\alpha+\frac{\nu}{2}, m=\frac{1}{2}\left(\frac{a}{\sigma}-1\right)$ and $\nu=\frac{1}{\sigma} \sqrt{(a-\sigma)^{2}+4 \mu \sigma}$. Then if $m>\alpha-1$
$\mathbb{E}_{x}\left[\frac{\int_{0}^{t} \frac{d s}{X_{s}}}{X_{t}^{\alpha}}\right]=-x^{-m} e^{-\frac{b x}{\sigma\left(e^{b t}-1\right)}+b m t} \frac{d}{d \mu}\left(\left(\frac{b e^{b t}}{\left(e^{b t}-1\right) \sigma}\right)^{-m+\alpha-\frac{\nu}{2}} \times\right.$
$\left.\left(\frac{b^{2} x}{4 \sigma^{2} \sinh ^{2}\left(\frac{b t}{2}\right)}\right)^{\nu / 2} \frac{\Gamma\left(1+m-\alpha+\frac{\nu}{2}\right)}{\Gamma(1+\nu)}{ }_{1} F_{1}\left(\beta, 1+\nu, \frac{b x}{\sigma\left(e^{b t}-1\right)}\right)\right)\left.\right|_{\mu=0}$.
We let $L$ be the notional amount of the swap in dollars per annualised volatility point and $K$ the delivery price for annualised volatility. $\sigma_{0, T}^{2}=\frac{1}{T} \int_{0}^{T} \sigma_{u}^{2} d u$ where $\sigma_{u}$ is the volatility on an underlying $S$.

Proposition 5.0.67. The price of a volatility swap at $t=0$ in the Platen benchmark framework is given by

$$
V\left(0, S_{0}\right)=L \mathbb{E}\left[\frac{\sigma_{0, T}}{S_{T}}\right]-L K P_{T}\left(0, S_{0}\right)
$$

where

$$
P_{T}\left(0, S_{0}\right)=\mathbb{E}\left[\frac{1}{S_{T}}\right]
$$

Chan and Platen [17] take $S_{t}=A_{t} Y_{t}$ where $A_{t}=\alpha e^{\eta t}$ and

$$
d Y_{t}=\left(a-b Y_{t}\right) d t+\sqrt{2 \gamma Y_{t}} d W_{t}, Y_{0}>0
$$

Platen and Chan have assumed here that the interest rate $r=0$. More generally, $S_{t}=A_{t} Y_{t} e^{r t}$.

From the transition density for $Y_{t}, \mathbb{E}\left[\frac{1}{S_{T}}\right]$ can be calculated easily. Chan and Platen show that

$$
\begin{equation*}
\mathbb{E}\left[\frac{\sigma_{0, T}^{2}}{S_{T}}\right]=\frac{2 \gamma e^{-\eta T}}{\sqrt{T} \alpha} \mathbb{E}\left[\frac{\int_{0}^{T} \frac{d s}{Y_{s}}}{Y_{T}}\right] \tag{5.0.213}
\end{equation*}
$$

From this, Proposition 5.0.66 and the identity

$$
\begin{equation*}
\mathbb{E}[\sqrt{Y}]=\frac{1}{2 \sqrt{\pi}} \int_{0}^{\infty} \frac{1-E\left[e^{-Y \psi}\right]}{\psi^{3 / 2}} d \psi \tag{5.0.214}
\end{equation*}
$$

the price of a volatility swap can be computed. See [17] for a more detailed discussion of this problem.
5.0.17. Pricing General Derivative Securities. Numerous examples in the pricing of derivatives can be obtained using the results of the thesis, but an extensive exploration of the applications to finance is beyond the scope of the thesis. For a discussion of the notion of riskneutral measures, numeraires and option pricing, see [32]. We present below a simple application to the pricing of futures contracts.

Suppose that a derivative security with price $V(\mathbf{x}, t)$ depends on $n$ underlying assets $X_{t}^{1}, \ldots, X_{t}^{n}$ and that under the so-called risk-neutral
measure,

$$
\begin{equation*}
d X_{t}^{i}=b\left(\mathbf{X}_{t}\right) d t+\sum_{j=1}^{n} \bar{\sigma}_{i j}\left(\mathbf{X}_{t}\right) d W_{t}^{j}, X_{0}^{i}=x_{i} \tag{5.0.215}
\end{equation*}
$$

where $\mathbf{X}_{t}=\left(X_{t}^{1}, \ldots, X_{t}^{n}\right)$ and $W_{t}^{1}, \ldots, W_{t}^{n}$ are Wiener processes. If the rate of interest is $r(\mathbf{X}, t)$ and there are continuous payments $h(\mathbf{X}, t)$ then

$$
V_{t}+\frac{1}{2} \sum_{i, j} \sigma_{i j}(\mathbf{X}) V_{x_{i} x_{j}}+b(\mathbf{X}) \cdot \nabla V-r(\mathbf{x}, t) V+h(\mathbf{x}, t)=0
$$

where $\sigma_{i j}$ is the $i, j$ th component of $\bar{\sigma}(\mathbf{X}) \bar{\sigma}^{T}(\mathbf{X})$. See $[\mathbf{5 0}]$ for the derivation of this equation. We impose the terminal condition $V(\mathbf{x}, T)=$ $F(\mathbf{x}, T)$ and whatever boundary conditions are needed. Letting $t \rightarrow$ $T-t$ converts this to the forward equation

$$
\begin{align*}
V_{t} & =\frac{1}{2} \sum_{i, j} \sigma_{i j}(\mathbf{x}) V_{x_{i} x_{j}}+b(\mathbf{x}) \cdot \nabla V-r(\mathbf{x}, T-t) V+h(\mathbf{x}, T-t) \\
V(\mathbf{x}, 0) & =F(\mathbf{x}, 0) . \tag{5.0.216}
\end{align*}
$$

The multidimensional equations we have studied can be transformed into this form for many different cases of $\sigma$ and $b$. We may then price many different derivative securities depending on multiple underlying assets using the results of this thesis.

Example 5.0.17. We present a simple example of this methodology applied to a specific problem. If $h(\mathbf{x}, t)=r(\mathbf{x}, t) V$, then the resulting equation essentially prices a futures contract. Suppose that we have a set of assets which under the risk-neutral measure satisfy the SDEs

$$
d X_{t}^{i}=\frac{2 a_{i}}{a_{1} X_{t}^{1}+\cdots+a_{n} X_{t}^{n}} d t+\sqrt{2} d W_{t}^{i}, X_{0}^{i}=x^{i}, i=1, \ldots, n,
$$

with $a_{i}>0, i=1,2, \ldots$ Suppose that $V(\mathbf{x}, 0)=F(\mathbf{x})$ where $F$ can be represented as a generalised Laplace transform

$$
F(\mathbf{x})=\int_{\mathbb{R}_{+}^{n}} e^{-\sum_{i=1}^{n} \epsilon_{i} x_{i}^{2}} f\left(\epsilon_{1}, \ldots, \epsilon_{n}\right) d \epsilon_{1} \cdots d \epsilon_{n}
$$

It then follows that the price is given by

$$
\begin{equation*}
V(\mathbf{x}, t)=\int_{\mathbb{R}_{n}^{+}} p(t, \mathbf{x}, \mathbf{y}) F(\mathbf{y}) d \mathbf{y} \tag{5.0.217}
\end{equation*}
$$

where $p(t, \mathbf{x}, \mathbf{y})$ is the transition density for $\mathbf{X}_{t}$. See [16] and [29] for more on this idea. From Chapter Three we have the Laplace transform of this density. Specifically

$$
\begin{aligned}
U_{\epsilon_{1}, \ldots, \epsilon_{n}}\left(x_{1}, \ldots, x_{n}, t\right) & =\frac{1}{\sqrt{1+4 \epsilon_{1} t} \cdots \sqrt{1+4 \epsilon_{n} t}} \exp \left(-\sum_{i=1}^{n} \frac{\epsilon_{i} x_{i}^{2}}{1+4 \epsilon_{1} t}\right) \\
\times & \exp \left(\phi\left(\frac{x_{1}}{1+4 \epsilon_{1} t}, \ldots, \frac{x_{n}}{1+4 \epsilon_{n} t}\right)-\phi\left(x_{1}, \ldots, x_{n}\right)\right),
\end{aligned}
$$

satisfies

$$
\int_{\mathbb{R}_{+}^{n}} e^{-\sum_{i=1}^{n} \epsilon_{i} y_{i}^{2}} p(t, \mathbf{x}, \mathbf{y}) d \mathbf{y}=U_{\epsilon_{1}, \ldots, \epsilon_{n}}\left(x_{1}, \ldots, x_{n}, t\right)
$$

where $\phi\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\log \left(a_{1} x_{1}+\cdots+a_{n} x_{n}\right)$. We thus have

$$
\begin{aligned}
V(\mathbf{x}, t) & =\int_{\mathbb{R}_{n}^{+}} p(t, \mathbf{x}, \mathbf{y}) F(\mathbf{y}) d \mathbf{y} \\
& =\int_{\mathbb{R}_{n}^{+}} p(t, \mathbf{x}, \mathbf{y})\left(\int_{\mathbb{R}_{+}^{n}} e^{-\sum_{i=1}^{n} \epsilon_{i} y_{i}^{2}} f\left(\epsilon_{1}, \ldots, \epsilon_{n}\right) d \epsilon_{1} \cdots d \epsilon_{n}\right) d \mathbf{y} \\
& =\int_{\mathbb{R}_{n}^{+}} \int_{\mathbb{R}_{n}^{+}} f\left(\epsilon_{1}, \ldots, \epsilon_{n}\right) p(t, \mathbf{x}, \mathbf{y}) e^{-\sum_{i=1}^{n} \epsilon_{i} y_{i}^{2}} d \mathbf{y} d \epsilon_{1} \cdots d \epsilon_{n} \\
& =\int_{\mathbb{R}_{n}^{+}} f\left(\epsilon_{1}, \ldots, \epsilon_{n}\right) \frac{1}{\sqrt{1+4 \epsilon_{1} t} \cdots \sqrt{1+4 \epsilon_{n} t}} \exp \left(-\sum_{i=1}^{n} \frac{\epsilon_{i} x_{i}^{2}}{1+4 \epsilon_{1} t}\right) \\
& \times \exp \left(\phi\left(\frac{x_{1}}{1+4 \epsilon_{1} t}, \ldots, \frac{x_{n}}{1+4 \epsilon_{n} t}\right)-\phi\left(x_{1}, \ldots, x_{n}\right)\right) d \epsilon_{1} \cdots d \epsilon_{n}
\end{aligned}
$$

Thus, given any payoff in the relevant class, we can evaluate this integral to obtain the futures price.
5.0.18. Calculating Functionals. In Chapter 2 we calculated functionals for some one-dimensional processes. We can extend these results to higher-dimensional problems. We present one result here.

Consider the problem

$$
\begin{aligned}
v_{t} & =\Delta v+2 \nabla \Psi \cdot \nabla v+\left(\Delta \Psi+|\nabla \Psi|^{2}-K(x, y)\right) v, \\
v(x, y, 0) & =f(x, y), \quad(x, y) \in \Omega \subseteq \mathbb{R}^{2}, \quad B(\partial \Omega)=0,
\end{aligned}
$$

in which $\partial \Omega$ is the boundary of $\Omega$ and $B(\partial \Omega)=0$ denotes the boundary conditions. According to the Feynman-Kac formula, the solution can be written

$$
\begin{equation*}
v(x, y, t)=\mathbb{E}_{x, y}\left[f\left(X_{t}, Y_{t}\right) e^{-\int_{0}^{t} G\left(X_{s}, Y_{s}\right) d s}\right] \tag{5.0.218}
\end{equation*}
$$

where $G(x, y)=-\left(\Delta \Psi+|\nabla \Psi|^{2}-K(x, y)\right)$ and

$$
\begin{align*}
d X_{t} & =\sqrt{2} d W_{t}^{1}+2 \Psi_{x}\left(X_{t}, Y_{t}\right)  \tag{5.0.219}\\
d Y_{t} & =\sqrt{2} d W_{t}^{2}+2 \Psi_{y}\left(X_{t}, Y_{t}\right) \tag{5.0.220}
\end{align*}
$$

We define $\mathbb{E}_{x, y}\left[h\left(X_{t}, Y_{t}\right)\right]=\mathbb{E}\left[h\left(X_{t}, Y_{t}\right) \mid X_{0}=x, Y_{0}=y\right]$. The process $\left(X_{t}, Y_{t}\right)$ satisfies the boundary conditions implied by $B(\Omega)=0$.

Letting $v=e^{-\Psi} u$ leads to

$$
\begin{align*}
u_{t} & =\Delta u-K(x, y) u  \tag{5.0.221}\\
u(x, y, 0) & =e^{\Psi(x, y)} f(x, y), \bar{B}(\Omega)=0 \tag{5.0.222}
\end{align*}
$$

where $\bar{B}(\Omega)=0$ are the transformed boundary conditions. We solve this problem by integrating against the fundamental solution to obtain

$$
\begin{equation*}
u(x, y, t)=\int_{\Omega} e^{\Psi(\xi, \eta)} f(\xi, \eta) p(t, x, y, \xi, \eta) d \xi d \eta \tag{5.0.223}
\end{equation*}
$$

The solution of the original problem is thus

$$
\begin{align*}
v(x, y, t) & =\mathbb{E}_{x, y}\left[f\left(X_{t}, Y_{t}\right) e^{-\int_{0}^{t} G\left(X_{s}, Y_{s}\right) d s}\right] \\
& =\int_{\Omega} e^{\Psi(\xi, \eta)-\Psi(x, y)} f(\xi, \eta) p(t, x, y, \xi, \eta) d \xi d \eta \tag{5.0.224}
\end{align*}
$$

Now suppose that $K(x, y)=\frac{1}{x^{2}} k\left(\frac{y}{x}\right)$, where $k$ is a positive function. Then we have a representation of the fundamental solution as

$$
\begin{aligned}
& p(t, x, y, \xi, \eta)=\frac{1}{2 t} e^{-\frac{x^{2}+y^{2}+\xi^{2}+\eta^{2}}{4 t}} \sum_{n} l_{\lambda_{n}}(x, y) \overline{l_{\lambda_{n}}(\xi, \eta)} \\
& \times I_{\sqrt{\lambda_{n}}}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right)
\end{aligned}
$$

where $l_{\lambda_{n}}(x, y)=L_{\lambda_{n}}\left(\tan ^{-1}\left(\frac{y}{x}\right)\right)$, and $L_{\lambda_{n}}$ is the $n$th eigenfunction and $\lambda_{n}$ is the $n$th eigenvalue of the associated Sturm-Liouville problem, as defined in Chapter 4. Because the eigenfunctions are orthogonal we have the following easy result.

The key to calculating the functionals is a result proved in $[\mathbf{2 1}]$.

Proposition 5.0.68 (Craddock-Lennox). Let

$$
\begin{align*}
& d X_{t}=\sqrt{2} d W_{t}^{1}+2 \Psi_{x}\left(X_{t}, Y_{t}\right) d t  \tag{5.0.225}\\
& d Y_{t}=\sqrt{2} d W_{t}^{2}+2 \Psi_{y}\left(X_{t}, Y_{t}\right) d t \tag{5.0.226}
\end{align*}
$$

subject to the boundary conditions $B(\Omega)=0$, where $\Omega=[0, \infty) \times[a, b]$ in polar coordinates. Suppose that $l_{\lambda_{n}}(x, y) e^{-\Psi(x, y)}$ is bounded on $\Omega$,
where the eigenfunctions $l_{\lambda}$ and drift $\Psi$ are as defined above. Then if $f(x, y)=F\left(\sqrt{x^{2}+y^{2}}, \tan ^{-1}\left(\frac{y}{x}\right)\right)$, where $F(\rho, \phi)=e^{-\tilde{\Psi}(\rho, \phi)} L_{\lambda_{n}}(\phi)$, then $\mathbb{E}_{x, y}\left[f\left(X_{t}, Y_{t}\right) e^{-\int_{0}^{t} G\left(X_{s}, Y_{s}\right) d s}\right]=v(x, y, t)$ where

$$
\begin{aligned}
v(x, y, t)= & l_{\lambda_{n}}(x, y) e^{-\Psi(x, y)}\left(\frac{x^{2}+y^{2}}{4 t}\right)^{\frac{\mu_{n}}{2}} \frac{\Gamma\left(\frac{\mu_{n}}{2}+1\right)}{\Gamma\left(\mu_{n}+1\right)} \\
& \times e^{-\frac{x^{2}+y^{2}}{4 t}}{ }_{1} F_{1}\left(\frac{\mu_{n}}{2}+1, \mu_{n}+1, \frac{x^{2}+y^{2}}{4 t}\right), \mu_{n}=\sqrt{\lambda_{n}}
\end{aligned}
$$

Proof. By orthogonality of the eigenfunctions we have

$$
\begin{aligned}
& v(x, y, t)=L_{\lambda_{n}}(\theta) e^{-\tilde{\Psi}(r, \theta)} \int_{0}^{\infty} \frac{\rho}{2 t} e^{-\frac{r^{2}+\rho^{2}}{4 t}} I_{\mu_{n}}\left(\frac{r \rho}{2 t}\right) d \rho= \\
& 2^{-\mu_{n}} r^{\mu_{n}} t^{-\mu_{n} / 2} L_{\lambda_{n}}(\theta) e^{-\tilde{\Psi}(r, \theta)} \frac{\Gamma\left(\frac{\mu_{n}}{2}+1\right)}{\Gamma\left(\mu_{n}+1\right)}{ }_{1} F_{1}\left(\frac{\mu_{n}}{2} ; \mu_{n}+1 ;-\frac{r^{2}}{4 t}\right)= \\
& l_{\lambda_{n}}(x, y) e^{-\Psi(x, y)}\left(\frac{x^{2}+y^{2}}{4 t}\right)^{\frac{\mu_{n}}{2}} \frac{\Gamma\left(\frac{\mu_{n}}{2}+1\right)}{\Gamma\left(\mu_{n}+1\right)}{ }_{1} F_{1}\left(\frac{\mu_{n}}{2}, \mu_{n}+1,-\frac{x^{2}+y^{2}}{4 t}\right),
\end{aligned}
$$

in which $\mu_{n}=\sqrt{\lambda_{n}}$. We then use the well known relation for Kummer's confluent hypergeometric function ${ }_{1} F_{1}(a, b, z)=e^{z}{ }_{1} F_{1}(b-a, b,-z)$ in (5.0.227), see $[\mathbf{1}]$. That $v$ satisfies the $\operatorname{PDE} v_{t}=\Delta v+2 \nabla \Psi \cdot \nabla v-$ $G(x, y) v$ is clear. To show that this is the correct expectation, we observe that according to formula 13.1.4 of [1]

$$
\begin{equation*}
{ }_{1} F_{1}(a, b, z)=\frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{a-b}\left[1+O\left(|z|^{-1}\right)\right] \tag{5.0.227}
\end{equation*}
$$

as $z \rightarrow \infty$. From which it follows that $v$ is bounded. Specifically

$$
\begin{aligned}
& \left(\frac{x^{2}+y^{2}}{4 t}\right)^{\frac{\mu_{n}}{2}} \frac{\Gamma\left(\frac{\mu_{n}}{2}+1\right)}{\Gamma\left(\mu_{n}+1\right)} e^{-\frac{x^{2}+y^{2}}{4 t}}{ }_{1} F_{1}\left(\frac{\mu_{n}}{2}+1 ; \mu_{n}+1 ; \frac{x^{2}+y^{2}}{4 t}\right) \\
& \sim 1+O\left(|z|^{-1}\right),
\end{aligned}
$$

for $x, y$ large. Since $l_{\lambda_{n}}(x, y) e^{-\Psi(x, y)}$ is bounded, we conclude that $v$ is bounded and consequently the Feynman-Kac formula implies that $v$ is equal to the expectation.

We consider functionals of two Bessel processes.

Example 5.0.18. Suppose that $k$ is positive on the first quadrant. The change of variables $u(x, y, t)=e^{\Psi(x, y)} v(x, y, t)$ with $\Psi(x, y)=$ $\ln \left(x^{a} y^{b}\right)$ converts

$$
\begin{align*}
u_{t} & =\Delta u-\frac{1}{x^{2}} k\left(\frac{y}{x}\right) u  \tag{5.0.228}\\
u(x, y, 0) & =f(x, y)
\end{align*}
$$

to

$$
\begin{equation*}
v_{t}=\Delta v+\frac{2 a}{x} v_{x}+\frac{2 b}{y} v_{y}-\left(\frac{1}{x^{2}} k\left(\frac{y}{x}\right)+\frac{a-a^{2}}{x^{2}}+\frac{b-b^{2}}{y^{2}}\right) v \tag{5.0.229}
\end{equation*}
$$

$$
v(x, y, 0)=e^{-\Psi(x, y)} f(x, y)
$$

This allows us to compute functionals for the process $\left(X_{t}, Y_{t}\right)$ where

$$
\begin{equation*}
d X_{t}=\frac{2 a}{X_{t}} d t+\sqrt{2} d W_{t}^{1}, d Y_{t}=\frac{2 a}{Y_{t}} d t+\sqrt{2} d W_{t}^{2} \tag{5.0.230}
\end{equation*}
$$

with $X_{0}=x>0, Y_{0}=y>0$ These are essentially two independent Bessel processes. The change of variables $t \rightarrow 2 t$ in (5.0.229) will convert this to a problem for Bessel processes of dimension $a-1 / 2, b-$ $1 / 2$. For $a, b \leq 1, G(x, y)=\frac{1}{x^{2}} k\left(\frac{y}{x}\right)+\frac{a-a^{2}}{x^{2}}+\frac{b-b^{2}}{y^{2}}>0$. If $a, b>1$, then $G$ may become negative. Now suppose that $p(t, x, y, \xi, \eta)$ is the necessary fundamental solution for (5.0.228) subject to a given set of boundary conditions. Then we may represent a solution of (5.0.229)
on $\mathbb{R}_{+}^{2}$ as

$$
\begin{equation*}
v(x, y, t)=\int_{\mathbb{R}_{+}^{2}} f(\xi, \eta) \frac{\xi^{a} \eta^{b}}{x^{a} y^{b}} p(t, x, y, \xi, \eta) d \xi d \eta \tag{5.0.231}
\end{equation*}
$$

For example, suppose that we have absorbing boundary conditions and we take $\frac{1}{x^{2}} k\left(\frac{y}{x}\right)=\frac{A}{x^{2}+y^{2}}$ and $a, b<1, A>0$. Then we have

$$
\begin{align*}
& \mathbb{E}_{x, y}\left[\sin \left(2 n \tan ^{-1}\left(\frac{Y_{t}}{X_{t}}\right)\right) X_{t}^{-a} Y_{t}^{-b} e^{-\int_{0}^{t}\left(\frac{A}{X_{s}^{2}+Y_{s}^{2}}+\frac{a-a^{2}}{X_{s}^{2}}+\frac{b-b^{2}}{Y_{s}^{2}}\right) d s}\right] \\
& =x^{-a} y^{-b} \sin \left(2 n \tan ^{-1}\left(\frac{y}{x}\right)\right)\left(\frac{x^{2}+y^{2}}{4 t}\right)^{\frac{\mu_{n}}{2}} \frac{\Gamma\left(\frac{\mu_{n}}{2}+1\right)}{\Gamma\left(\mu_{n}+1\right)} e^{-\frac{x^{2}+y^{2}}{4 t}} \\
& \quad \times{ }_{1} F_{1}\left(\frac{\mu_{n}}{2}+1 ; \mu_{n}+1 ; \frac{x^{2}+y^{2}}{4 t}\right) \tag{5.0.232}
\end{align*}
$$

with $\mu_{n}=\sqrt{4 n^{2}+A}$. For example, if $n=1, \sin \left(2 \tan ^{-1}\left(\frac{y}{x}\right)\right)=\frac{2 x y}{x^{2}+y^{2}}$ and it is clear that if $x=0$, or $y=0$, then (5.0.232) is zero. If $a>1$ or $b>1$, then (5.0.232) is unbounded as $x, y \rightarrow 0$. But this is expected, since we will then have $a-a^{2}<0, b-b^{2}<0$. So if for example we take $x=0$ then the process will never leave zero and the quantity $\sin \left(2 n \tan ^{-1}\left(\frac{Y_{t}}{X_{t}}\right)\right) X_{t}^{-a} Y_{t}^{-b} e^{-\int_{0}^{t}\left(\frac{A}{X_{s}^{2}+Y_{s}^{2}}+\frac{a-a^{2}}{X_{s}^{2}}+\frac{b-b^{2}}{Y_{s}^{2}}\right) d s}$ will diverge, so the expectation (5.0.232) must be unbounded. If a Bessel process has dimension greater than 2 , it will never reach zero almost surely, so if $a, b>2.5$, and the process does not start on $(x, 0)$ or $(0, y)$, then (5.0.232) will remain finite.

EXAMPLE 5.0.19. Setting $u(x, y, t)=e^{\Psi(x, y)} v(x, y, t), \Psi(x, y)=$ $\frac{1}{2} \ln \left(x^{2}+y^{2}\right)$ converts

$$
\begin{equation*}
u_{t}=\Delta u-\frac{1}{x^{2}} \frac{x^{2}+\mu y^{2}}{x^{2}+y^{2}} u,(x, y) \in \mathbb{R}_{+}^{2} \tag{5.0.233}
\end{equation*}
$$

into

$$
v_{t}=\Delta v+\frac{2 x}{x^{2}+y^{2}} v_{x}+\frac{2 y}{x^{2}+y^{2}} v_{y}-\mu\left(\frac{y}{x}\right)^{2} \frac{1}{x^{2}+y^{2}} v
$$

A fundamental solution of (5.0.233) was given in the previous chapter, with absorbing boundary conditions. Thus if we consider

$$
\begin{equation*}
d X_{t}=\frac{2 X_{t}}{X_{t}^{2}+Y_{t}^{2}} d t+\sqrt{2} d W_{t}^{1}, d Y_{t}=\frac{2 Y_{t}}{X_{t}^{2}+Y_{t}^{2}} d t+\sqrt{2} d W_{t}^{2} \tag{5.0.234}
\end{equation*}
$$

with absorbing boundary conditions on $(x, 0)$ and $(y, 0)$, then

$$
\begin{align*}
& \mathbb{E}_{x, y}\left[\frac{1}{\sqrt{X_{t}^{2}+Y^{2}}} c_{n 2} F_{1}\left(-n-\frac{1}{2}, \beta ; \gamma ; \frac{X_{t}^{2}}{X_{t}^{2}+Y_{t}^{2}}\right) e^{-\int_{0}^{t} \mu\left(\frac{Y_{s}}{X_{s}}\right)^{2} \frac{1}{X_{s}^{2}+Y_{s}^{2}} d s}\right] \\
& =\frac{c_{n}}{x^{2}+y^{2}}{ }_{2} F_{1}\left(-n-\frac{1}{2}, \beta ; \gamma ; \frac{x^{2}}{x^{2}+y^{2}}\right)\left(\frac{x^{2}+y^{2}}{4 t}\right)^{\frac{\mu_{n}}{2}} \frac{\Gamma\left(\frac{\mu_{n}}{2}+1\right)}{\Gamma\left(\mu_{n}+1\right)} \\
& \quad \times \exp \left(-\frac{x^{2}+y^{2}}{4 t}\right)_{1} F_{1}\left(\frac{\mu_{n}}{2}+1 ; \mu_{n}+1 ; \frac{x^{2}+y^{2}}{4 t}\right),  \tag{5.0.235}\\
& \text { where } \mu_{n}=\sqrt{1-\mu+\left(2 n+\frac{3}{2}+\sqrt{\mu+1 / 4}\right)^{2}} .
\end{align*}
$$

More general initial conditions can be handled by expanding the initial condition in a series of eigenfunctions. We demonstrate this in the next example.

Example 5.0.20. Let $B_{t}^{1}$ and $B_{t}^{2}$ be two Brownian motions, with $B_{0}^{1}=\frac{x}{\sqrt{2}}=\bar{x}>0$ and $B_{0}^{2}=\frac{y}{\sqrt{2}}=\bar{y}>0$. We suppose that the process $\left(B_{t}^{1}, B_{t}^{2}\right)$ is absorbed on $(x, 0)$ and $(0, y)$. Define $r_{t}=\sqrt{\left(B_{t}^{2}\right)^{2}+\left(B_{t}^{2}\right)^{2}}$ and let $\theta_{t}$ be the angle the random vector $r_{t}$ makes with the $x$ axis at time $t$. We compute

$$
v(x, y, t)=\mathbb{E}_{\bar{x}, \bar{y}}\left[\exp \left(-\frac{1}{8} \int_{0}^{t}\left(\frac{\theta_{s}}{r_{s}}\right)^{2} d s\right)\right]
$$

We have to solve

$$
\begin{aligned}
u_{t} & =\Delta u-\frac{\tan ^{-1}\left(\frac{y}{x}\right)^{2}}{4\left(x^{2}+y^{2}\right)} u,(x, y) \in \mathbb{R}_{+}^{2} \\
u(x, y, 0) & =f(x, y), u(0, y, t)=u(x, 0, t)=0 .
\end{aligned}
$$

The eigenvalue problem is $L^{\prime \prime}+\left(\lambda-\frac{1}{4} \theta^{2}\right) L=0$ subject to $L(0)=$ $L\left(\frac{\pi}{2}\right)=0$. The solution of the differential equation is

$$
L(\theta, \lambda)=e^{-\frac{1}{4} \theta^{2}}\left(A_{1} F_{1}\left(\frac{1}{4}-\frac{1}{2} \lambda, \frac{1}{2}, \frac{1}{2} \theta^{2}\right)+B \theta_{1} F_{1}\left(\frac{3}{4}-\frac{1}{2} \lambda, \frac{3}{2}, \frac{1}{4} \theta^{2}\right)\right)
$$

The condition $L(0, \lambda)=0$ requires $A=0$. We then have the requirement that $L\left(\frac{\pi}{2}, \lambda\right)=0$. Using Mathematica, the eigenvalues are found to be approximately $\lambda_{1}=4.17, \lambda_{2}=16.2, \lambda_{3}=36.2, \lambda_{4}=64.2, \lambda_{5}=$ $100.2, \ldots$ We define the eigenfunctions as

$$
L_{i}\left(\theta, \lambda_{i}\right)=c_{i} \theta e^{-\frac{1}{4} \theta^{2}}{ }_{1} F_{1}\left(\frac{3}{4}-\frac{1}{2} \lambda_{i}, \frac{3}{2}, \frac{1}{4} \theta^{2}\right),
$$

where

$$
\int_{0}^{\frac{\pi}{2}} c_{i}^{2} \theta^{2} e^{-\frac{1}{2} \theta^{2}}{ }_{1} F_{1}\left(\frac{3}{4}-\frac{1}{2} \lambda_{i}, \frac{3}{2}, \frac{1}{4} \theta^{2}\right)^{2} d \theta=1
$$

So we have $c_{1}=2.29442, c_{2}=4.52946, c_{3}=6.77991, c_{4}=9.03401, c_{5}=$ 11.2893 etc. Thus

$$
\begin{aligned}
u(x, y, t)= & \int_{\mathbb{R}_{+}^{2}} f(\xi, \eta) \frac{1}{2 t} e^{-\frac{x^{2}+y^{2}+\xi^{2}+\eta^{2}}{4 t}} \sum_{i=1}^{\infty} l_{i}(x, y, \xi, \eta) \\
& \times I_{\sqrt{\lambda_{i}}}\left(\frac{\sqrt{x^{2}+y^{2}} \sqrt{\xi^{2}+\eta^{2}}}{2 t}\right) d \xi d \eta
\end{aligned}
$$

where $l_{i}(x, y, \xi, \eta)=c_{i}^{2} L_{i}\left(\tan ^{-1}\left(\frac{y}{x}\right), \lambda_{i}\right) L_{i}\left(\tan ^{-1}\left(\frac{\eta}{\xi}\right), \lambda_{i}\right)$. We set $f(\xi, \eta)=$

1. From this we can compute the expectation

$$
u(x, y, t)=\mathbb{E}_{x, y}\left[e^{-\int_{0}^{t} \frac{\tan ^{-1}\left(\frac{Y_{s}}{X_{s}}\right)^{2}}{4\left(X_{s}^{2}+Y_{s}^{2}\right)} d s}\right],
$$

where $X_{t}=\sqrt{2} B_{t}^{1}, Y_{t}=\sqrt{2} B_{t}^{2}$. The integration can be carried out numerically in Mathematica and we arrive at

$$
\begin{aligned}
& \mathbb{E}_{\bar{x}, \bar{y}}\left[e^{-\frac{1}{8} \int_{0}^{t}\left(\frac{\theta_{s}}{r_{s}}\right)^{2} d s}\right]=1.128 f_{1}(x, y, t)-0.00811 f_{2}(x, y, t)+0.376 f_{3}(x, y, t) \\
& \quad-0.001 f_{4}(x, y, t)+0.229 f_{5}(x, y, t)-0.0003 f_{6}(x, y, t) \\
& \quad+0.161 f_{7}(x, y, t)-\cdots,
\end{aligned}
$$

where

$$
\begin{aligned}
f_{i}(x, y, t)= & L_{i}\left(\tan ^{-1}\left(\frac{y}{x}\right), \lambda_{i}\right)\left(\frac{x^{2}+y^{2}}{4 t}\right)^{\frac{\mu_{i}}{2}} \frac{\Gamma\left(\frac{\mu_{i}}{2}+1\right)}{\Gamma\left(\mu_{i}+1\right)} \\
& \times e^{-\frac{x^{2}+y^{2}}{4 t}}{ }_{1} F_{1}\left(\frac{\mu_{i}}{2}+1, \mu_{i}+1, \frac{x^{2}+y^{2}}{4 t}\right), \mu_{i}=\sqrt{\lambda_{i}} .
\end{aligned}
$$

### 5.1. Conclusion

In this thesis we have sought closed-form expressions for fundamental solutions of higher-dimensional equations using symmetry group methods. Lie symmetry methods have been successfully applied to a range of problems and are especially effective for parabolic problems in one space dimension. However, they have been less successful for the higher-dimensional problems studied in this thesis, due to the fact that the dimension of the symmetry group generally does not grow as the dimension of the PDE increases. The standard techniques, such as group invariance or integral transform methods, have been extended to higher-dimensional problems only for some special cases of the classes of equations we consider in this thesis.

The major contribution of this thesis has been to overcome this problem for PDEs of the form

$$
\begin{equation*}
u_{t}=\Delta u+\left(\frac{1}{x_{1}^{2}} K\left(\frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right)+\sum_{i=1}^{n}\left(c_{i} x_{i}^{2}+a_{i} x_{i}\right)\right) u \tag{5.1.1}
\end{equation*}
$$

Six new results have been proved which provide explicit fundamental solutions for equations of this type.

In the case when there is a Heisenberg group of symmetries, discussed in Chapter 3, we have constructed Fourier transforms of fundamental solutions for some special cases of higher-dimensional PDEs. These have the the capacity to generate multiple fundamental solutions as well as transition densities for multidimensional diffusions. We presented several explicit examples and introduced two new multidimensional processes and their corresponding transition densities.

In general however this method cannot be applied to higher dimensional problems. For the classes of equations under study in this thesis,
typically we have only $S L(2, \mathbb{R})$ as the symmetry group. Here the dimension of the symmetry group remains unchanged as the dimension of the PDE grows, so we do not have enough one-parameter subgroups to construct integral transforms.

The major contribution of the thesis is presented in Chapter 4, where we show that it is possible to find fundamental solutions for higher-dimensional equations with only $S L(2, \mathbb{R})$ as the symmetry group. In combining the integration of an $S L(2, \mathbb{R})$ symmetry and the linearity of the equation, we can build solutions of initial value problems for arbitrary initial values. The expansion theorems we have established provide explicit fundamental solutions, in terms of series of Gaussians, Bessel functions and solutions of a Sturm-Liouville problem. Using these results we have computed many examples of explicit fundamental solutions and new transition densities for multidimensional problems.

The two-dimensional problem can be regarded as completely solved and we have also obtained some useful expressions for the $n$-dimensional case. Although these require solutions of an eigenfunction problem for a PDE, we are still able to give a series expansion of the fundamental solution. An explicit example was obtained and a new summation result for series of spherical harmonics was found as a corollary. The transition density for a third $n$-dimensional process was found.

A significant application of the results in Chapter 4 has been to show that these fundamental solutions provide the key to obtaining equivalences between Lie symmetries of higher-dimensional parabolic equations and group representations, extending the results obtained by Craddock and Dooley in connecting Lie symmetries of parabolic PDEs and representations of $S L(2, \mathbb{R})$.

We have also applied the results of Chapter 4 to the problem of computing functionals for multidimensional processes. Explicit formulae for two-dimensional functionals were obtained using the Feynman-Kac formula and the expansion theorems of Chapter 4. This led to the computation of new functionals of Bessel processes and planar Brownian motion.

A third application was in the pricing of derivative securities. The price of a volatility swap in the Platen framework was found as well as the price of a futures contract in the risk-neutral framework, when the dynamics follow a multidimensional process introduced in Chapter 3.

There are many possible future applications. A complete exploration of the new processes the results of the thesis provide should be investigated. There are possible applications in financial modelling to be explored. Since the complex transformation $t \rightarrow$ it converts (5.1.1) to the Schrodinger equation of quantum mechanics, there may be problems in physics which could be addressed by our methods.

The effective numerical and exact solution of the eigenvalue problems in higher dimensions needs be studied. The problem of efficiently summing the series giving the fundamental solution, and rates of convergence, when it is not possible to explicitly sum the series, should be addressed.

Finally, extending the results to different classes of parabolic equations of second order should be considered. A start would be to classify equations of the form $u_{t}=\Delta u+f(x, y) u_{x}+g(x, y) u_{y}$ with nontrivial symmetries in the case where $f_{y} \neq g_{x}$. The case where $\Delta u$ is replaced by another elliptic operator of second order is a further possible extension. The application of other types of symmetries such as non-local
symmetries to obtaining fundamental solutions in higher-dimensional problems could also be considered.

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[^0]:    ${ }^{1}$ There also exist symmetries which do not have group properties, as well as so called non-local symmetries. However we do not consider these in this thesis. See the book [ $\mathbf{7}]$ for more on this subject.

[^1]:    ${ }^{1}$ It has long been known that invariance under an $n$-dimensional Abelian group, allows an $n$-dimensional PDE to be solved by integral transform. In this chapter, we construct integral transforms when the symmetry group is non-Abelian.

