Relaxed Stability Conditions Based on Taylor Series Membership Functions for Polynomial Fuzzy-Model-Based Control Systems

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***Abstract*— In this paper, we investigate the stability of polynomial fuzzy-model-based (PFMB) control systems, aiming to relax stability conditions by considering the information of membership functions. To facilitate the stability analysis, we propose a general form of approximated membership functions, which is implemented by Taylor series expansion. Taylor series membership functions (TSMF) can be brought into stability conditions such that the relation between membership grades and system states is expressed. To further reduce the conservativeness, different types of information are taken into account: the boundary of membership functions, the property of membership functions, and the boundary of operating domain. Stability conditions are obtained from Lyapunov stability theory by sum of squares (SOS) approach. Simulation examples demonstrate the effect of each piece of information.**

I. Introduction

STABILITY analysis is a systematic process proving the feasibility of designed controllers for the stabilization of control systems. It is very challenging even though mathematical models of systems are known beforehand, especially for nonlinear systems. Takagi-Sugeno (T-S) fuzzy model [1], [2] represents nonlinear systems using local linear systems weighted by membership functions, which is in favor of stability analysis. Based on the T-S fuzzy model, Lyapunov stability theory [3] was employed to guarantee the stabilization by a set of conditions in terms of linear matrix inequalities (LMIs) [4], [5], which convex programming techniques can handle and numerically obtain solutions.

Following the basic framework of fuzzy-model-based stability analysis, two main research areas have been studied for decades. One is combining it with other control problems [6]–[9]. Another is relaxing stability conditions by considering the following three areas. First, the positivity of fuzzy summations was investigated [10], [11] by the concept of parallel distributed compensation (PDC) [3] and slack matrices through S-procedure [12], which were generalized by the Pólya's theory [13], [14] solving permutation problems of membership functions in the higher order of summation terms. Second, instead of the common quadratic Lyapunov function, various types of Lyapunov function candidates were utilized such as piecewise linear Lyapunov function [15], [16], fuzzy Lyapunov function [17], [18], and switching Lyapunov function [19], [20]. Third, the information of membership functions was brought into stability analysis to reduce the conservativeness [21], [22].

Since the polynomial fuzzy-model-based (PFMB) control system was proposed [23], [24] to generalize the T-S fuzzy-model-based control system, the above approaches were imitated for PFMB control systems. In modeling, the sector nonlinearity technique [5] was extended using Taylor series expansion [25] to construct progressively more precise polynomial fuzzy models. In stability analysis, stability conditions derived from polynomial Lyapunov function candidates are established by sum of squares (SOS) approach [26] instead of LMI approach. Certainly, the conservativeness of SOS-based conditions still exists and requires relaxation by the above mentioned three areas. The PDC design was applied [23], [24], as well as polynomial fuzzy Lyapunov function [27], multiple polynomial Lyapunov function [28] and switching polynomial Lyapunov function [29]. With regard to the information of membership functions, symbolic variables were employed to represent membership functions [25], [30], [31] such that they can remain in SOS-based conditions which is in favor of introducing more slack matrices through some Positivstellensatz multipliers. Moreover, approximated membership functions were exploited to directly bring the information into stability analysis [32], [33] such that the relation between membership grades and system states is expressed rather than the information independent of system states.

In this paper, we aim to relax stability conditions for PFMB control systems by considering the information of membership functions. Inspired by [33], we extend the piecewise linear membership functions to more systematic approximated membership functions with the consideration of approximation error. As an example, Taylor series expansion is chosen to implement the approximation. The advantage of Taylor series is that it yields polynomials which can be handled in SOS-based conditions and it provides the truncation order and expansion points to be determined by users. Based on Taylor series membership functions

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(TSMFs), stability conditions can be progressively relaxed as the truncation order increases and the interval of expansion points decreases. Meanwhile, we consider the following information: the boundary of membership functions, the property of membership functions, and the boundary of operating domain, which can further relax the SOS conditions.

This paper is organized as follows. In Section II, notations and the formulation of polynomial fuzzy model and controller are presented. In Section III, TSMFs and relaxed SOS-based conditions are proposed. In Section IV, simulation examples are offered to show the improvement of designed controllers. In Section V, a conclusion is drawn.

II. Preliminary

A. Notation

The following notation is employed throughout this paper [26]. A monomial in $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is a function of the form $x_1^{d_1}(t)x_2^{d_2}(t)\cdots x_n^{d_n}(t)$, where $d_i \geq 0, i = 1, 2, \ldots, n$, are integers. The degree of a monomial is $d = \sum_{i=1}^{n} d_i$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is a finite linear combination of monomials with real coefficients. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is an SOS if it can be written as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(t))^2$, where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and m is a nonnegative integer. It can be concluded that if $\mathbf{p}(\mathbf{x}(t))$ is an SOS, $\mathbf{p}(\mathbf{x}(t)) \geq 0$. The expressions of $\mathbf{M} > 0, \mathbf{M} \geq 0, \mathbf{M} < 0$, and $\mathbf{M} \leq 0$ denote the positive, semi-positive, negative, and semi-negative definite matrices $\mathbf{M}$, respectively.

B. Polynomial Fuzzy Model

The i^{th} rule of the polynomial fuzzy model for the nonlinear plant is presented as follows [23]:

$$\begin{aligned}\text{Rule } i : &\text{IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND} \cdots \text{AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i \\ &\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t),\end{aligned} \tag{1}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is the state vector, and n is the dimension of the nonlinear plant; $f_\alpha(\mathbf{x}(t))$ is the premise variable corresponding to its fuzzy term M_α^i in rule i, $\alpha = 1, 2, \ldots, \Psi$, and Ψ is a positive integer; $\mathbf{A}_i(\mathbf{x}(t)) \in \Re^{n\times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \Re^{n\times m}$ are the known polynomial system and input matrices, respectively; $\hat{\mathbf{x}}(t) = [\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_N(t)]^T$ is a vector of monomials in $\mathbf{x}(t)$, and it is assumed that $\hat{\mathbf{x}}(t) = \mathbf{0}$, iff $\mathbf{x}(t) = \mathbf{0}$; $\mathbf{u}(t) \in \Re^m$ is the control input vector. Thus, the dynamics of the nonlinear plant is given by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\big), \tag{2}$$

where p is the number of rules in the polynomial fuzzy model; $w_i(\mathbf{x}(t))$ is the normalized grade of membership, $w_i(\mathbf{x}(t)) = \dfrac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{x}(t)))}{\sum_{k=1}^{p}\prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(\mathbf{x}(t)))}$, $w_i(\mathbf{x}(t)) \geq 0, i = 1, 2, \ldots, p$, and $\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1$; $\mu_{M_\alpha^i}(f_\alpha(\mathbf{x}(t))), \alpha = 1, 2, \ldots, \Psi$, are grades of membership corresponding to the fuzzy term M_α^i.

C. Polynomial Fuzzy Controller

The j^{th} rule of the polynomial fuzzy controller is presented as follows:

$$\begin{aligned}\text{Rule } j : &\text{IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ AND} \cdots \text{AND } g_\Omega(\mathbf{x}(t)) \text{ is } N_\Omega^j \\ &\text{THEN } \mathbf{u}(t) = \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)),\end{aligned} \tag{3}$$

where $g_\beta(\mathbf{x}(t))$ is the premise variable corresponding to its fuzzy term N_β^j in rule j, $\beta = 1, 2, \ldots, \Omega$, and Ω is a positive integer; $\mathbf{G}_j(\mathbf{x}(t)) \in \Re^{m\times N}$ is the polynomial feedback gain in rule j. Thus, the following polynomial fuzzy controller is applied to the nonlinear plant represented by the polynomial fuzzy model (2):

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \tag{4}$$

where c is the number of rules in the polynomial fuzzy controller; $m_j(\mathbf{x}(t))$ is the normalized grade of membership, $m_j(\mathbf{x}(t)) = \dfrac{\prod_{l=1}^{\Omega} \mu_{N_l^j}(g_l(\mathbf{x}(t)))}{\sum_{k=1}^{c}\prod_{l=1}^{\Omega}(\mu_{N_l^k}(g_l(\mathbf{x}(t)))}$, $m_j(\mathbf{x}(t)) \geq 0, j = 1, 2, \ldots, c$, and $\sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1$; $\mu_{N_\beta^j}(g_\beta(\mathbf{x}(t))), \beta = 1, 2, \ldots, \Omega$, are grades of membership corresponding to the fuzzy term N_β^j.

The polynomial fuzzy model and controller in this paper do not share the same membership functions, meaning non-PDC design is employed which improves the design flexibility and reduces the complexity of the controller [34].

III. Stability Analysis

A. Taylor Series Membership Function

In this section, TSMFs are introduced to approximate the original membership functions such that they can be brought into stability conditions. In the following analysis, for brevity, $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(\mathbf{x}(t))$ are denoted as $\mathbf{x}$ and $\hat{\mathbf{x}}$ respectively. Without losing generality, we assume that membership functions depend on all system states $\mathbf{x}$.

Since the approximation is carried out in each substate space, the overall state space which is denoted as ψ is divided into s connected substate spaces (hypercubes) which are denoted as $\psi_l, l = 1, 2, \ldots, s$. Specifically, in each dimension of $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$, $x_r, r = 1, 2, \ldots, n$, are divided into s_r connected substate spaces. Hence, we have the relation that $\psi = \bigcup_{l=1}^{s} \psi_l$ and $s = \prod_{r=1}^{n} s_r$.

Sample points are exploited to implement the segmentation of state space. Therefore, in each substate space ψ_l, we have 2 sample points denoted as x_{r1l} (lower bound) and x_{r2l} (upper bound) in each dimension x_r, and 2^n sample points in all. In what follows, these sample points are exploited as expansion points for Taylor series. The approximation of original membership functions is achieved by fuzzy blending of the membership grades at samples points in each substate space.

Let us define $h_{ij}(\mathbf{x}) = w_i(\mathbf{x})m_j(\mathbf{x})$, and denote the approximation of $h_{ij}(\mathbf{x})$ as $\overline{h}_{ij}(\mathbf{x})$. Therefore, the approximated membership function is defined as

$$\overline{h}_{ij}(\mathbf{x}) = \sum_{l=1}^{s} \sigma_l(\mathbf{x}) \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{ri_rl}(x_r)\delta_{iji_1i_2\cdots i_nl}(\mathbf{x}) \quad \forall i, j, \tag{5}$$

where $\sigma_l(\mathbf{x})$ is a scalar index of substate spaces, satisfying $\sigma_l(\mathbf{x}) = 1, \mathbf{x} \in \psi_l$, $l = 1, 2, \ldots, s$; otherwise, $\sigma_l(\mathbf{x}) = 0$; $\delta_{iji_1i_2\cdots i_nl}(\mathbf{x})$ is a predefined scalar polynomial of $\mathbf{x}$ as grades of membership function $h_{ij}(\mathbf{x})$ at sample points $x_r = x_{ri_rl}, r = 1, 2, \ldots, n, i_r = 1, 2,$ in substate space ψ_l; $v_{ri_rl}(x_r)$ is the membership function corresponding to fuzzy term $\delta_{iji_1i_2\cdots i_nl}(\mathbf{x})$, exhibiting the following properties: $0 \leq v_{ri_rl}(x_r) \leq 1$, $v_{r1l}(x_r) + v_{r2l}(x_r) = 1$ for all $r, i_r, l, \mathbf{x} \in \psi_l$, and $\sum_{l=1}^{s} \sigma_l(\mathbf{x}) \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{ri_rl}(x_r) = 1$ (Readers may refer to [33] for further examples of obtaining (5), which are some special cases of (5)).

Remark 1: There are different approaches to define membership grades of sample points $\delta_{iji_1i_2\cdots i_nl}(\mathbf{x})$ in (5). In this paper, particularly, the method of Taylor series expansion is employed to define $\delta_{iji_1i_2\cdots i_nl}(\mathbf{x})$. The general form of multi-variable Taylor series expansion [35] is given by

$$f(\mathbf{x}) = \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\sum_{r=1}^{n} (x_r - x_{r0}) \frac{\partial}{\partial x_r} \Big)^k \times f(\mathbf{x})|_{(x_r = x_{r0}, r=1,2,\ldots,n)}, \tag{6}$$

where $f(\mathbf{x})$ is an arbitrary function of $\mathbf{x}$; $x_{r0}, r = 1, 2, \ldots, n$, are expansion points; $\frac{\partial}{\partial x_r} f(\mathbf{x})|_{(x_r = x_{r0}, r=1,2,\ldots,n)}$ is a constant calculated by taking the partial derivative of $f(\mathbf{x})$ and then substituting $\mathbf{x}$ by $x_r = x_{r0}$. From the Taylor series expansion (6), we substitute expansion points and $f(\mathbf{x})$ by sample points and $h_{ij}(\mathbf{x})$ such that $\delta_{iji_1i_2\cdots i_nl}(\mathbf{x})$ is obtained:

$$\delta_{iji_1i_2\cdots i_nl}(\mathbf{x}) = \sum_{k=0}^{\lambda-1} \frac{1}{k!} \Big(\sum_{r=1}^{n} (x_r - x_{ri_rl}) \frac{\partial}{\partial x_r} \Big)^k \times h_{ij}(\mathbf{x})|_{(x_r = x_{ri_rl}, r=1,2,\ldots,n)} \quad \forall i, j, i_1, i_2, \ldots, i_n, l, \mathbf{x} \in \psi_l, \tag{7}$$

where λ is the predefined truncation order, which means the polynomial with the order $\lambda - 1$ is applied for approximation. The TSMF is obtained by substituting (7) into (5). It is noted that the membership function $h_{ij}(\mathbf{x})$ is required to be differentiable if TSMFs are employed.

B. Polynomial Fuzzy-Model-Based Control Systems

In this section, the stability of the PFMB control system is analyzed. Without any ambiguity, $w_i(\mathbf{x}(t))$, $m_j(\mathbf{x}(t))$ $h_{ij}(\mathbf{x})$, and $\overline{h}_{ij}(\mathbf{x})$ are denoted as w_i , m_j, h_{ij}, and $\overline{h}_{ij}$, respectively. The PFMB control system formed by the polynomial fuzzy model (2) and the polynomial fuzzy controller (4) is

$$\dot{\mathbf{x}} = \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij} \big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}) \big) \hat{\mathbf{x}}. \tag{8}$$

The control objective is to make the PFMB control system (8) asymptotically stable i.e., $\mathbf{x}(t) \to 0$ as time $t \to \infty$, by determining the polynomial feedback gains $\mathbf{G}_j(\mathbf{x}(t))$.

To proceed with the stability analysis, from (8), we have

$$\dot{\hat{\mathbf{x}}} = \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \mathbf{T}(\mathbf{x})\dot{\mathbf{x}} = \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij} \big(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}) \big) \hat{\mathbf{x}}, \tag{9}$$

where $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x}), \tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$, $\mathbf{T}(\mathbf{x}) \in \Re^{N \times n}$ with its $(i,j)^{th}$ element defined as $T_{ij}(\mathbf{x}) = \partial \hat{x}_i(\mathbf{x}) / \partial x_j$. Due to the assumption that $\hat{\mathbf{x}}(t) = \mathbf{0}$, iff $\mathbf{x}(t) = \mathbf{0}$, the stability of control system (9) implies that of (8).

We investigate the stability of (9) by employing the following polynomial Lyapunov function candidate:

$$V(\mathbf{x}) = \hat{\mathbf{x}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}, \tag{10}$$

where $0 < \mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N \times N}$; $\tilde{\mathbf{x}}$ is defined in Remark 1. From (9) and (10), we have

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\hat{\mathbf{x}}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \dot{\hat{\mathbf{x}}} + \hat{\mathbf{x}}^T \frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} \hat{\mathbf{x}} \\ &= \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij} \hat{\mathbf{x}}^T \Big(\big(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}) \big)^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \\ &\quad + \mathbf{X}(\tilde{\mathbf{x}})^{-1} \big(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}) \big) \Big) \hat{\mathbf{x}} \\ &\quad + \hat{\mathbf{x}}^T \frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} \hat{\mathbf{x}}. \end{aligned} \tag{11}$$

Assumption 1 ([23], [26]): To deal with the term $\frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt}$ in (11), we define $\mathbf{K} = \{\zeta_1, \zeta_2, \ldots, \zeta_s\}$ as the set of row numbers that entries of the entire row of $\mathbf{B}_i(\mathbf{x})$ are all zeros for all i, and $\tilde{\mathbf{x}} = [x_{\zeta_1}, x_{\zeta_2}, \ldots, x_{\zeta_s}]^T$. Hence, we have $\frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} = \sum_{\zeta \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_\zeta} \sum_{i=1}^{p} w_i \mathbf{A}_i^{\zeta}(\mathbf{x}) \hat{\mathbf{x}}$, where $\mathbf{A}_i^{\zeta}(\mathbf{x}) \in \Re^N$ is the ζ^{th} row of $\mathbf{A}_i(\mathbf{x})$. Although this assumption is widely employed, it restricts the capability of polynomial Lyapunov function and further development can be achieved by removing this assumption [36].

Lemma 1 ([23], [26]): For any invertible polynomial matrix $\mathbf{X}(\mathbf{y})$ where $\mathbf{y} = [y_1, y_2, \ldots, y_n]^T$, the following equation is true.

$$\frac{\partial \mathbf{X}(\mathbf{y})^{-1}}{\partial y_j} = -\mathbf{X}(\mathbf{y})^{-1} \frac{\partial \mathbf{X}(\mathbf{y})}{\partial y_j} \mathbf{X}(\mathbf{y})^{-1} \quad \forall j$$

From Remark 1 and Lemma 1, we have

$$\frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} = -\mathbf{X}(\tilde{\mathbf{x}})^{-1} \Big(\sum_{i=1}^{p} \sum_{\zeta \in \mathbf{K}} w_i \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_\zeta} \mathbf{A}_i^{\zeta}(\mathbf{x}) \hat{\mathbf{x}} \Big) \mathbf{X}(\tilde{\mathbf{x}})^{-1}. \tag{12}$$

Let us denote $\mathbf{z} = \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}$ and $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}})^{-1}$, where $\mathbf{N}_j(\mathbf{x}) \in \Re^{m \times N}, j = 1, 2, \ldots, c$, are arbitrary polynomial matrices. From (11) and (12), we have

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij} \mathbf{z}^T \mathbf{Q}_{ij}(\mathbf{x}) \mathbf{z}, \tag{13}$$

where $\mathbf{Q}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T\tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{\zeta\in\mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_\zeta}\mathbf{A}_i^\zeta(\mathbf{x})\hat{\mathbf{x}}$ for $i = 1, 2, \ldots, p, j = 1, 2, \ldots, c$.

Remark 2: From the Lyapunov stability theory, the asymptotic stability of (9) is guaranteed by $V(\mathbf{x}) > 0$ and $\dot{V}(\mathbf{x}) < 0$ (excluding $\mathbf{x} = \mathbf{0}$), which can be implied by $\mathbf{Q}_{ij}(\mathbf{x}) < 0$ for all i, j. However, the information of membership functions w_i and m_j are not considered leading to very conservative stability conditions.

C. Relaxed SOS-based Stability Conditions

In the following, we firstly present a general approach for relaxing the non-PDC SOS-based stability conditions with approximated membership functions. Then detailed information to implement this approach is provided. For brevity, $\sigma_l(\mathbf{x})$ is denoted as σ_l.

In order to relax the stability conditions, we bring the approximated membership functions (5) into stability conditions by considering the boundary information of approximation error $\Delta h_{ijl} = h_{ij} - \overline{h}_{ij}$ for all $i, j, l, \mathbf{x} \in \psi_l$. The local lower and upper bounds of Δh_{ijl} are denoted as $\underline{\gamma}_{ijl}$ and $\overline{\gamma}_{ijl}$, respectively, which means $\underline{\gamma}_{ijl} \le \Delta h_{ijl} \le \overline{\gamma}_{ijl}$ for all $i, j, l, \mathbf{x} \in \psi_l$. Meanwhile, slack polynomial matrices $0 < \mathbf{Y}_{ijl}(\mathbf{x}) = \mathbf{Y}_{ijl}(\mathbf{x})^T \in \Re^{N\times N}$ for $\mathbf{x} \in \psi_l$ are introduced, which can be formulated by $\sum_{l=1}^{s} \sigma_l \mathbf{Y}_{ijl}(\mathbf{x}) > 0$. Moreover, it is required that $\sum_{l=1}^{s} \sigma_l \mathbf{Y}_{ijl}(\mathbf{x}) \ge \mathbf{Q}_{ij}(\mathbf{x})$ for all i, j. Recalling that $\sigma_l(\mathbf{x}) = 1$ for $\mathbf{x} \in \psi_l$ (otherwise $\sigma_l(\mathbf{x}) = 0$), the above inequalities can be implied by $\mathbf{Y}_{ijl}(\mathbf{x}) > 0$ and $\mathbf{Y}_{ijl}(\mathbf{x}) - \mathbf{Q}_{ij}(\mathbf{x}) \ge 0$ for all i, j, l. Based on the property that $(\sum_{l=1}^{s} \sigma_l \mathbf{M}_{1l}(\mathbf{x}))(\sum_{k=1}^{s} \sigma_k \mathbf{M}_{2k}(\mathbf{x})) = \sum_{l=1}^{s} \sigma_l \mathbf{M}_{1l}(\mathbf{x})\mathbf{M}_{2l}(\mathbf{x})$, (13) can be written as follows:

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= \sum_{i=1}^{p}\sum_{j=1}^{c} h_{ij}\mathbf{z}^T\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z} \\
&= \mathbf{z}^T \sum_{i=1}^{p}\sum_{j=1}^{c} \big(\overline{h}_{ij}\mathbf{Q}_{ij}(\mathbf{x}) + (h_{ij} - \overline{h}_{ij})\mathbf{Q}_{ij}(\mathbf{x})\big)\mathbf{z} \\
&= \mathbf{z}^T \sum_{i=1}^{p}\sum_{j=1}^{c} \big(\overline{h}_{ij}\mathbf{Q}_{ij}(\mathbf{x}) \\
&\quad + \sum_{l=1}^{s} \sigma_l(\Delta h_{ijl} - \underline{\gamma}_{ijl} + \underline{\gamma}_{ijl})\mathbf{Q}_{ij}(\mathbf{x})\big)\mathbf{z} \\
&\le \mathbf{z}^T \sum_{i=1}^{p}\sum_{j=1}^{c} \big((\overline{h}_{ij} + \sum_{l=1}^{s} \sigma_l\underline{\gamma}_{ijl})\mathbf{Q}_{ij}(\mathbf{x}) \\
&\quad + \sum_{l=1}^{s} \sigma_l(\Delta h_{ijl} - \underline{\gamma}_{ijl}) \sum_{k=1}^{s} \sigma_k \mathbf{Y}_{ijk}(\mathbf{x})\big)\mathbf{z} \\
&\le \mathbf{z}^T \sum_{l=1}^{s} \sigma_l \sum_{i=1}^{p}\sum_{j=1}^{c} \big((\overline{h}_{ij} + \underline{\gamma}_{ijl})\mathbf{Q}_{ij}(\mathbf{x}) \\
&\quad + (\overline{\gamma}_{ijl} - \underline{\gamma}_{ijl})\mathbf{Y}_{ijl}(\mathbf{x})\big)\mathbf{z}. \qquad (14)
\end{aligned}$$

To further relax stability conditions, we exploit the following information [31]: the boundary information of membership grades (with corresponding slack matrix $\mathbf{R}_{1\rho_1}(\mathbf{x})$) and the property of membership functions ($\mathbf{R}_{2\rho_2}(\mathbf{x})$)). Additionally, aiming at the information of substate spaces, we propose another type of information, namely the boundary of operating domain ($\mathbf{R}_{3\rho_3}(\mathbf{x})$). From (14), we have

$$\begin{aligned}
\dot{V}(\mathbf{x}) \le \mathbf{z}^T &\sum_{l=1}^{s} \sigma_l \sum_{i=1}^{p}\sum_{j=1}^{c} \big((\overline{h}_{ij} + \underline{\gamma}_{ijl})\mathbf{Q}_{ij}(\mathbf{x}) \\
&+ (\overline{\gamma}_{ijl} - \underline{\gamma}_{ijl})\mathbf{Y}_{ijl}(\mathbf{x})\big)\mathbf{z} \\
&+ \mathbf{z}^T\big(\sum_{k=1}^{3}\sum_{\rho_k=1}^{P_k} \phi_{k\rho_k}(\mathbf{x})\mathbf{R}_{k\rho_k}(\mathbf{x})\big)\mathbf{z}, \qquad (15)
\end{aligned}$$

where $\phi_{1\rho_1}(\mathbf{x}) \ge 0, \phi_{2\rho_2}(\mathbf{x}) = 0$, and $\phi_{3\rho_3}(\mathbf{x}) \ge 0$, $\rho_k = 1, 2, \ldots, P_k, k = 1, 2, \ldots, 3$, are predefined scalar polynomial functions; $\mathbf{R}_{2\rho_2}(\mathbf{x}) = \mathbf{R}_{2\rho_2}(\mathbf{x})^T \in \Re^{N\times N}$ is an arbitrary polynomial matrix; $0 < \mathbf{R}_{1\rho_1}(\mathbf{x}) = \mathbf{R}_{1\rho_1}(\mathbf{x})^T \in \Re^{N\times N}$ and $0 < \mathbf{R}_{3\rho_3}(\mathbf{x}) = \mathbf{R}_{3\rho_3}(\mathbf{x})^T \in \Re^{N\times N}$ are polynomial matrices.

In what follows, the details concerning the three pieces of information are discussed. Since the membership functions $v_{ri_rl}(x_r)$ in (5) can be either linear or nonlinear, and nonlinear functions cannot be solved in SOS-based stability conditions, we consider $v_{ri_rl}(x_r)$ not exist in final stability conditions for this paper.

Remark 3: If $v_{ri_rl}(x_r)$ is predefined as linear functions for $\mathbf{x} \in (-\infty, \infty)$, it can remain in SOS-based stability conditions. For this linear case, bringing $v_{ri_rl}(x_r)$ into stability conditions has potential to further relax the conditions. Future work can be done following this idea as a comparison with this paper.

1) Boundary Information of Membership Grades: Since the approximated membership functions $\overline{h}_{ij}$ have been brought into stability conditions, we can directly exploit their boundary information. We have $\underline{\eta}_{ijl} \le \overline{h}_{ij} \le \overline{\eta}_{ijl}$ for all i, j, $\mathbf{x} \in \psi_l$, where $\underline{\eta}_{ijl}$ and $\overline{\eta}_{ijl}$ are lower and upper bounds of approximated membership membership grades $\overline{h}_{ij}$ in substate space ψ_l, respectively. Then we have

$$\sum_{l=1}^{s} \sigma_l(\overline{h}_{ij} - \underline{\eta}_{ijl})\underline{\mathbf{W}}_{ijl}(\mathbf{x}) \ge 0 \quad \forall i, j, \qquad (16)$$

$$\sum_{l=1}^{s} \sigma_l(\overline{\eta}_{ijl} - \overline{h}_{ij})\overline{\mathbf{W}}_{ijl}(\mathbf{x}) \ge 0 \quad \forall i, j, \qquad (17)$$

where $0 < \underline{\mathbf{W}}_{ijl}(\mathbf{x}) = \underline{\mathbf{W}}_{ijl}(\mathbf{x})^T \in \Re^{N\times N}$ and $0 < \overline{\mathbf{W}}_{ijl}(\mathbf{x}) = \overline{\mathbf{W}}_{ijl}(\mathbf{x})^T \in \Re^{N\times N}$ for $\mathbf{x} \in \psi_l$ are polynomial matrices.

2) Property of Membership Functions: The membership function $v_{ri_rl}(x_r)$ owns the property that $\sum_{l=1}^{s} \sigma_l \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{ri_rl}(x_r) = 1$. Since we consider $v_{ri_rl}(x_r)$ not exist in final stability conditions, this information is lost. Therefore, we aim to bring such information into stability conditions. However, it is difficult to provide general equalities or inequalities representing such information due to different selection of function $v_{ri_rl}(x_r)$. In this paper, we provide an example by defining $v_{r1l}(x_r) = (x_{r2l} - x_r)/(x_{r2l} - x_{r1l})$ and

$v_{r2l}(x_r) = 1 - v_{r1l}(x_r)$ for all $r, l, \mathbf{x} \in \psi_l$, where x_{r1l} and x_{r2l} are lower and upper bounds of x_r in substate space ψ_l.

In this case, we have the following equality constraint [33]:

$$\sum_{l=1}^{s} \sigma_l \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{r i_r l}(x_r)(\chi(\mathbf{x}) - \overline{\chi}_{i_1 i_2 \cdots i_n l})\mathbf{K}_l(\mathbf{x}) = 0, \quad (18)$$

where we have the property that $\sum_{l=1}^{s} \sigma_l \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{r i_r l}(x_r)(\chi(\mathbf{x}) - \overline{\chi}_{i_1 i_2 \cdots i_n l}) = 0$; $\chi(\mathbf{x})$ is a monomials linear in $x_r, r = 1, 2, \ldots, n$; $\overline{\chi}_{i_1 i_2 \cdots i_n l} = \chi(\mathbf{x})|_{x_r = x_{r i_r l}}$ is the value of $\chi(\mathbf{x})$ at sample points $x_r = x_{r i_r l}$ in substate space ψ_l; $\mathbf{K}_l(\mathbf{x}) = \mathbf{K}_l(\mathbf{x})^T \in \Re^{N \times N}$ is an arbitrary polynomial matrix. It is noted that $\chi(\mathbf{x})$ is not necessarily a monomial in all system states x_r.

3) Boundary Information of Operating Domain: Once SOS-based stability conditions are satisfied, they hold for all $\mathbf{x} \in (-\infty, \infty)$. In practice, however, we usually only need to guarantee the satisfaction for a certain domain of $\mathbf{x}$, that is $x_k \in [x_{k1}, x_{k2}], k = 1, 2, \ldots, n$. In this paper, we only need to satisfy the local operating domain $x_k \in [x_{k1l}, x_{k2l}]$ for each substate space ψ_l. For this reason, we have the following constraint:

$$\sum_{l=1}^{s} \sigma_l \sum_{k=1}^{n} (x_k - x_{k1l})(x_{k2l} - x_k)\mathbf{L}_{kl}(\mathbf{x}) \geq 0, \quad (19)$$

where $0 < \mathbf{L}_{kl}(\mathbf{x}) = \mathbf{L}_{kl}(\mathbf{x})^T \in \Re^{N \times N}$ for $\mathbf{x} \in \psi_l$ is a polynomial matrix.

Now we substitute the approximated membership function $\overline{h}_{ij}$ in (15) by (5), and substitute general form of constraints by (16), (17), and (19). For the reason that $\sum_{l=1}^{s} \sigma_l \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{r i_r l}(x_r) = 1$ and $v_{r i_r l}(x_r)$ is independent of rule i, j, we have

$$\begin{aligned}\dot{V}(\mathbf{x}) \leq \mathbf{z}^T &\sum_{l=1}^{s} \sigma_l \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{r=1}^{n} v_{r i_r l}(x_r) \sum_{i=1}^{p} \sum_{j=1}^{c} \\ &\times \big((\delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}) + \underline{\gamma}_{ijl})\mathbf{Q}_{ij}(\mathbf{x}) \\ &\quad + (\overline{\gamma}_{ijl} - \underline{\gamma}_{ijl})\mathbf{Y}_{ijl}(\mathbf{x}) \\ &\quad + (\delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}) - \underline{\eta}_{ijl})\underline{\mathbf{W}}_{ijl}(\mathbf{x}) \\ &\quad + (\overline{\eta}_{ijl} - \delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}))\overline{\mathbf{W}}_{ijl}(\mathbf{x}) \\ &\quad + \sum_{k=1}^{n} (x_k - x_{k1l})(x_{k2l} - x_k)\mathbf{L}_{kl}(\mathbf{x})\big)\mathbf{z}. \quad (20)\end{aligned}$$

The satisfaction of $\dot{V}(\mathbf{x}) < 0$ can be guaranteed by $\sum_{i=1}^{p} \sum_{j=1}^{c} \big((\delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}) + \underline{\gamma}_{ijl})\mathbf{Q}_{ij}(\mathbf{x}) + (\overline{\gamma}_{ijl} - \underline{\gamma}_{ijl})\mathbf{Y}_{ijl}(\mathbf{x}) + (\delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}) - \underline{\eta}_{ijl})\underline{\mathbf{W}}_{ijl}(\mathbf{x}) + (\overline{\eta}_{ijl} - \delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}))\overline{\mathbf{W}}_{ijl}(\mathbf{x}) + \sum_{k=1}^{n} (x_k - x_{k1l})(x_{k2l} - x_k)\mathbf{L}_{kl}(\mathbf{x})\big) < 0$ for all $i_1, i_2, \ldots, i_n, l$. The above stability analysis result is summarized in the following theorem.

Theorem 1: The PFMB system (9), which is formed by the polynomial fuzzy model (2) and the polynomial fuzzy controller (4) connected in a closed loop, is guaranteed to be asymptotically stable if there exist polynomial matrices $\mathbf{Y}_{ijl}(\mathbf{x}) = \mathbf{Y}_{ijl}(\mathbf{x})^T \in \Re^{N \times N}$, $\underline{\mathbf{W}}_{ijl}(\mathbf{x}) = \underline{\mathbf{W}}_{ijl}(\mathbf{x})^T \in \Re^{N \times N}$, $\overline{\mathbf{W}}_{ijl}(\mathbf{x}) = \overline{\mathbf{W}}_{ijl}(\mathbf{x})^T \in \Re^{N \times N}$, $\mathbf{L}_{kl}(\mathbf{x}) = \mathbf{L}_{kl}(\mathbf{x})^T \in \Re^{N \times N}$, $\mathbf{N}_j(\mathbf{x}) \in \Re^{m \times N}$, $i = 1, 2, \ldots, p, j = 1, 2, \ldots, c, k = 1, 2, \ldots, n, l = 1, 2, \ldots, s$, and $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N \times N}$ such that the following SOS-based conditions are satisfied:

$$\begin{aligned}&\nu^T(\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\mathbf{I})\nu \text{ is SOS;} \\ &\nu^T(\mathbf{Y}_{ijl}(\mathbf{x}) - \varepsilon_2(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall i, j, l; \\ &\nu^T(\mathbf{Y}_{ijl}(\mathbf{x}) - \mathbf{Q}_{ij}(\mathbf{x}) - \varepsilon_3(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall i, j, l; \\ &\nu^T(\underline{\mathbf{W}}_{ijl}(\mathbf{x}) - \varepsilon_4(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall i, j, l; \\ &\nu^T(\overline{\mathbf{W}}_{ijl}(\mathbf{x}) - \varepsilon_5(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall i, j, l; \\ &\nu^T(\mathbf{L}_{kl}(\mathbf{x}) - \varepsilon_6(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall k, l; \\ &-\nu^T\Big(\sum_{i=1}^{p} \sum_{j=1}^{c} \big((\delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}) + \underline{\gamma}_{ijl})\mathbf{Q}_{ij}(\mathbf{x}) \\ &\quad + (\overline{\gamma}_{ijl} - \underline{\gamma}_{ijl})\mathbf{Y}_{ijl}(\mathbf{x}) \\ &\quad + (\delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}) - \underline{\eta}_{ijl})\underline{\mathbf{W}}_{ijl}(\mathbf{x}) \\ &\quad + (\overline{\eta}_{ijl} - \delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}))\overline{\mathbf{W}}_{ijl}(\mathbf{x}) \\ &\quad + \sum_{k=1}^{n} (x_k - x_{k1l})(x_{k2l} - x_k)\mathbf{L}_{kl}(\mathbf{x})\big) \\ &\quad + \varepsilon_7(\mathbf{x})\mathbf{I}\Big)\nu \text{ is SOS} \quad \forall i_1, i_2, \ldots, i_n, l; \quad (21)\end{aligned}$$

where $\nu \in \Re^N$ is an arbitrary vector independent of $\mathbf{x}$; $\delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x})$ is a predefined scalar polynomial of $\mathbf{x}$ in (5); $\underline{\gamma}_{ijl}, \overline{\gamma}_{ijl}, \underline{\eta}_{ijl}, \overline{\eta}_{ijl}, x_{k1l}$, and x_{k2l} are predefined constant scalars satisfying $\Delta h_{ij} = h_{ij} - \overline{h}_{ij}, \underline{\gamma}_{ijl} \leq \Delta h_{ij} \leq \overline{\gamma}_{ijl}, \underline{\eta}_{ijl} \leq \overline{h}_{ij} \leq \overline{\eta}_{ijl}$, and $x_{k1l} \leq x_k \leq x_{k2l}$ for all $i, j, k, l, \mathbf{x} \in \psi_l$; $\varepsilon_1(\tilde{\mathbf{x}}) > 0, \varepsilon_2(\mathbf{x}) > 0, \ldots, \varepsilon_7(\mathbf{x}) > 0$, are predefined scalar polynomials; the feedback gains are defined as $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}, j = 1, 2, \ldots, c$.

Remark 4: Referring to Theorem 1, the number of decision matrix variables is $1 + c + 3pcs + ns$, and the number of SOS conditions is $1 + 4pcs + ns + 2^n s$. When membership function $v_{r i_r l}(x_r)$ is defined as $v_{r1l}(x_r) = (x_{r2l} - x_r)/(x_{r2l} - x_{r1l})$ and $v_{r2l}(x_r) = 1 - v_{r1l}(x_r)$ for all $r, l, \mathbf{x} \in \psi_l$, where $x_{r1l} \leq x_r \leq x_{r2l}$, the information of the property of $v_{r i_r l}(x_r)$ can be brought into stability conditions. The SOS condition (21) in Theorem 1 is replaced by

$$\begin{aligned}&-\nu^T\Big(\sum_{i=1}^{p} \sum_{j=1}^{c} \big((\delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}) + \underline{\gamma}_{ijl})\mathbf{Q}_{ij}(\mathbf{x}) \\ &\quad + (\overline{\gamma}_{ijl} - \underline{\gamma}_{ijl})\mathbf{Y}_{ijl}(\mathbf{x}) \\ &\quad + (\delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}) - \underline{\eta}_{ijl})\underline{\mathbf{W}}_{ijl}(\mathbf{x}) \\ &\quad + (\overline{\eta}_{ijl} - \delta_{iji_1 i_2 \cdots i_n l}(\mathbf{x}))\overline{\mathbf{W}}_{ijl}(\mathbf{x}) \\ &\quad + (\chi(\mathbf{x}) - \overline{\chi}_{i_1 i_2 \cdots i_n l})\mathbf{K}_l(\mathbf{x}) \\ &\quad + \sum_{k=1}^{n} (x_k - x_{k1l})(x_{k2l} - x_k)\mathbf{L}_{kl}(\mathbf{x})\big) \\ &\quad + \varepsilon_7(\mathbf{x})\mathbf{I}\Big)\nu \text{ is SOS} \quad \forall i_1, i_2, \ldots, i_n, l; \quad (22)\end{aligned}$$

where $\chi(\mathbf{x})$ is a monomials linear in $x_r, r = 1, 2, \ldots, n$; $\overline{\chi}_{i_1 i_2 \cdots i_n l} = \chi(\mathbf{x})|_{x_r = x_{r i_r l}}$; $\mathbf{K}_l(\mathbf{x}) = \mathbf{K}_l(\mathbf{x})^T \in \Re^{N \times N}$ is an arbitrary polynomial matrix. In this case, the number of variables are $1 + c + 3pcs + ns + s$, and the number of SOS conditions remains the same.

IV. Simulation Examples

In the following, a 3-rule polynomial fuzzy model with the form of (2) is investigated to implement the designed controller. The system states are $\hat{\mathbf{x}}(t) = \mathbf{x}(t) = [x_1(t) \quad x_2(t)]^T$, and system matrices and input matrices are

$$\mathbf{A}_1(x_1) = \begin{bmatrix} 1.59 - 0.12x_1^2 & -7.29 - 0.25x_1 \\ 0.01 & -0.1 \end{bmatrix},$$
$$\mathbf{A}_2(x_1) = \begin{bmatrix} 0.02 - 0.63x_1^2 & -4.64 + 0.92x_1 \\ 0.35 & -0.21 \end{bmatrix},$$
$$\mathbf{A}_3(x_1) = \begin{bmatrix} -a + 0.31x_1 - 1.12x_1^2 & -4.33 \\ 0 & 0.05 \end{bmatrix},$$
$$\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \mathbf{B}_3 = \begin{bmatrix} -b + 6 \\ -1 \end{bmatrix},$$

where a and b are predefined constant parameters in the range of $0 \leq a \leq 10$ and $0 \leq b \leq 200$ at the interval of 1 and 20, respectively. The operating domain we consider for this model is $x_1 \in [-10, 10]$. The membership functions of this polynomial fuzzy model are selected as $w_1(x_1) = 1 - 1/(1 + e^{-(x_1+4)})$, $w_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$, and $w_3(x_1) = 1/(1 + e^{-(x_1-4)})$. To achieve the stabilization, a 2-rule polynomial fuzzy controller with the form of (4) is employed, with membership functions defined as $m_1(x_1) = e^{-x_1^2/12}$ and $m_2(x_1) = 1 - m_1(x_1)$.

Theorem 1 is applied to design the feedback gains of polynomial fuzzy controller. TSMFs (5) and (7) are exploited as approximated membership functions. In order to demonstrate the influence of different orders of TSMFs and intervals of expansion points, we make the comparison as shown in Table I. Without losing generality, we choose membership function

TABLE I
COMPARISON OF DIFFERENT ORDERS OF TSMFS AND INTERVALS OF EXPANSION POINTS

Case	Order λ	Interval	Expansion points
1	1	4	$x_1 = \{-10, -6, \ldots, 6, 10\}$
2	1	2	$x_1 = \{-10, -8, \ldots, 8, 10\}$
3	3	4	$x_1 = \{-10, -6, \ldots, 6, 10\}$
4	3	2	$x_1 = \{-10, -8, \ldots, 8, 10\}$

$v_{r i_r l}(x_r)$ in (5) as $v_{11l}(x_1) = (x_{12l} - x_1)/(x_{12l} - x_{11l})$ and $v_{12l}(x_1) = 1 - v_{11l}(x_1)$, for all $l, x_1 \in \psi_l$, where $x_{11l} \leq x_1 \leq x_{12l}$. It is noted that we remove the terms in Taylor series with the magnitude of coefficients less than 1×10^{-6} such that the computational efficiency is improved. Based on original membership functions and TSMFs, the predefined constant scalars $\underline{\gamma}_{ijl}, \overline{\gamma}_{ijl}, \underline{\eta}_{ijl}$, and $\overline{\eta}_{ijl}$ are obtained for Case 1-4.

Due the selection of membership function $v_{r i_r l}(x_r)$, the SOS condition (21) in Theorem 1 is replaced by (22) in Remark 4. To further reduce the computational burden, the number of slack matrices is decreased by $\mathbf{Y}_{ijl}(x_1) = \mathbf{Y}_{ij}(x_1), \underline{\mathbf{W}}_{ijl}(x_1) = \underline{\mathbf{W}}_{ij}(x_1), \overline{\mathbf{W}}_{ijl}(x_1) = \overline{\mathbf{W}}_{ij}(x_1), \mathbf{K}_l(x_1) = \mathbf{K}(x_1), \mathbf{L}_{kl}(x_1) = \mathbf{L}(x_1)$. Other parameters are chosen as follows: $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_7 = 1 \times 10^{-3}$, $\mathbf{X}$ of degree 0, $\mathbf{Y}_{ij}(x_1)$ of degree 8, $\underline{\mathbf{W}}_{ij}(x_1)$ and $\overline{\mathbf{W}}_{ij}(x_1)$ of degree 6, $\mathbf{K}(x_1)$ of degree 7, and $\mathbf{L}(x_1)$ of degree 6. The SOS-based stability conditions are solved numerically by the third-party MATLAB toolbox SOSTOOLS [37].

To demonstrate the effect of each type of slack matrices, the stabilization region obtained with only $\mathbf{Y}_{ij}(x_1)$, with only $\mathbf{Y}_{ij}(x_1)$, $\underline{\mathbf{W}}_{ij}(x_1)$, and $\overline{\mathbf{W}}_{ij}(x_1)$, with only $\mathbf{Y}_{ij}(x_1)$, $\underline{\mathbf{W}}_{ij}(x_1)$, $\overline{\mathbf{W}}_{ij}(x_1)$, and $\mathbf{L}(x_1)$, and with all slack matrices ($\mathbf{Y}_{ij}(x_1)$, $\underline{\mathbf{W}}_{ij}(x_1)$, $\overline{\mathbf{W}}_{ij}(x_1)$, $\mathbf{L}(x_1)$, and $\mathbf{K}(x_1)$) are shown in Fig. 1-4, respectively. The stabilization region is indicated by "×" for Case 1, "+" for Case 2, "□" for Case 3, and "○" for Case 4.

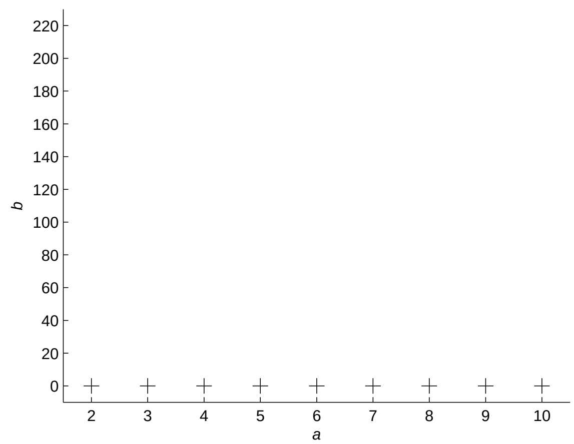

Fig. 1. Stabilization regions obtained from Theorem 1 with only $\mathbf{Y}_{ij}(x_1)$, indicated by "×" for Case 1, "+" for Case 2, "□" for Case 3, and "○" for Case 4.

From Fig. 1-4, it can be found that the stabilization region grows for all cases with the number of slack matrices increasing. It shows that these information of membership functions and operating domain as well as their corresponding slack matrices are effective for relaxing stability conditions. Moreover, by comparing Case 1 to Case 4, it is indicated that higher order and smaller interval lead to larger stabilization region. Additionally, when the interval is large, both the interval and the order play an important role; when the interval is small, they become less influential. It complies with what we expect because these SOS-based stability conditions are close to sufficient and necessary conditions as the interval is small. However, when higher order TSMFs are employed, corresponding higher order slack matrices are required simultaneously, which leads to unaffordable computational cost and makes sufficient and necessary conditions unattainable.

To verify the stabilization, we provide an example by choosing $a = 10$ and $b = 220$ in Case 4. The polynomial

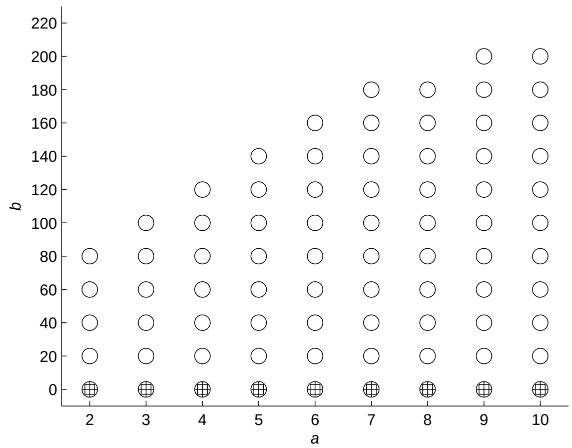

Fig. 2. Stabilization regions obtained from Theorem 1 with only $\mathbf{Y}_{ij}(x_1)$, $\underline{\mathbf{W}}_{ij}(x_1)$, and $\overline{\mathbf{W}}_{ij}(x_1)$, indicated by "×" for Case 1, "+" for Case 2, "□" for Case 3, and "○" for Case 4.

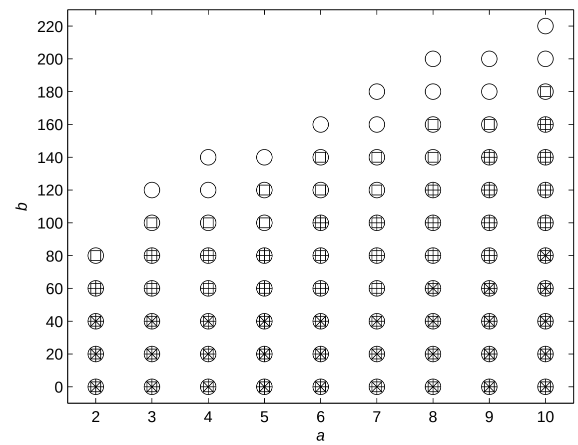

Fig. 4. Stabilization regions obtained from Theorem 1 with all slack matrices ($\mathbf{Y}_{ij}(x_1)$, $\underline{\mathbf{W}}_{ij}(x_1)$, $\overline{\mathbf{W}}_{ij}(x_1)$, $\mathbf{L}(x_1)$, and $\mathbf{K}(x_1)$), indicated by "×" for Case 1, "+" for Case 2, "□" for Case 3, and "○" for Case 4.

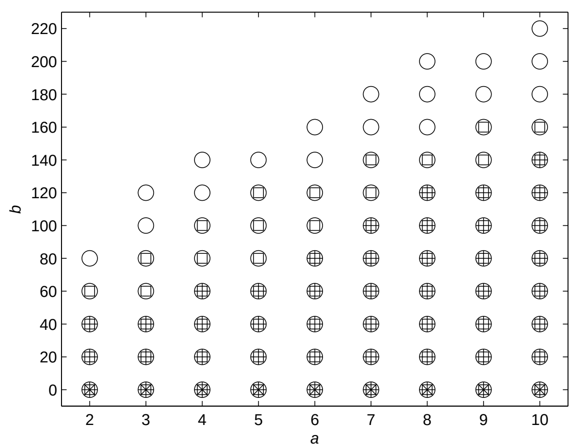

Fig. 3. Stabilization regions obtained from Theorem 1 with only $\mathbf{Y}_{ij}(x_1)$, $\underline{\mathbf{W}}_{ij}(x_1)$, $\overline{\mathbf{W}}_{ij}(x_1)$, and $\mathbf{L}(x_1)$, indicated by "×" for Case 1, "+" for Case 2, "□" for Case 3, and "○" for Case 4.

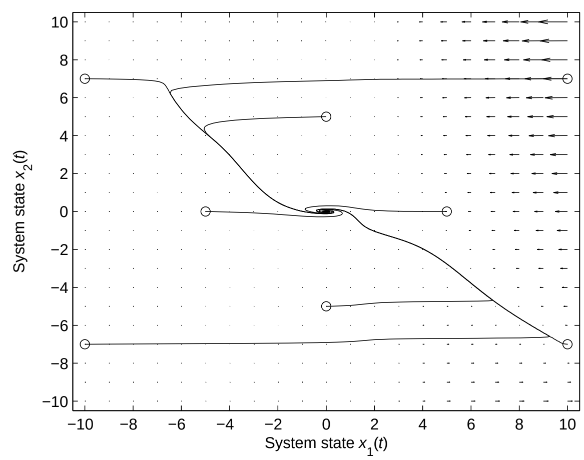

Fig. 5. Phase plot of $x_1(t)$ and $x_2(t)$ for $a = 10$ and $b = 220$ in Case 4.

feedback gains are obtained that $\mathbf{G}_1(x_1) = [0.0158x_1^2 + 0.0138x_1 - 0.1312 \quad 0.0720x_1^2 + 0.1453x_1 - 0.4839]$ and $\mathbf{G}_2(x_1) = [0.0058x_1^2 + 0.0000x_1 - 0.0459 \quad 0.0154x_1^2 - 0.0021x_1 - 0.0158]$. With initial conditions indicated by "○", the phase plot of $x_1(t)$ and $x_2(t)$ is shown in Fig. 5. With initial conditions $\mathbf{x}(0) = [10 \quad 10]^T$, the transient response of $\mathbf{x}(t)$ and control input $u(t)$ are shown in Fig. 6. It can been seen that the PFMB control system is guaranteed to be asymptotically stable in the domain $x_1 \in [-10, 10]$.

Compared with stability conditions in Remark 2 without any information of membership functions, there is no stabilization region within the same domain of parameters a and b. Since the polynomial fuzzy model and controller in this example do not share the same membership functions, PDC SOS-based stability conditions [23]–[25] in general cannot be applied. Accordingly, the relaxation and flexibility of the proposed method are exhibited from the comparison.

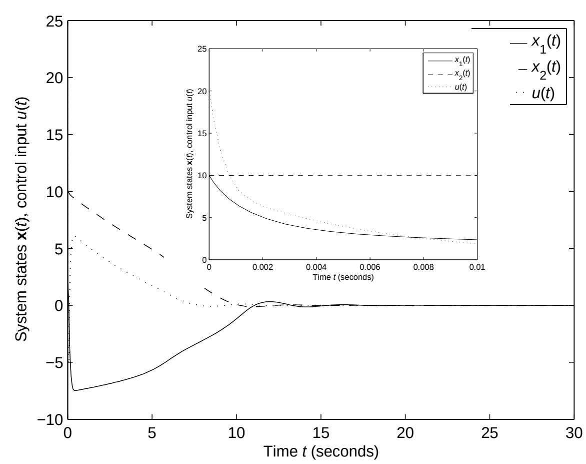

Fig. 6. Transient response of $\mathbf{x}(t)$ and control input $u(t)$ for $a = 10$ and $b = 220$ in Case 4.

V. Conclusion

The stability analysis of polynomial fuzzy-model-based control systems has been carried out. In favor of reducing the conservativeness, TSMFs have been proposed to approximate original membership functions. More information including the boundary of membership functions, the property of membership functions, and the boundary of operating domain, have been brought into stability conditions such that SOS-based conditions can be further relaxed. Future work can be done to bring specific membership function $v_{ri_rl}(x_r)$ into stability conditions, which has potential to further reduce the conservativeness.

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