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MARTINGALES AND FIRST PASSAGE TIMES FOR ORNSTEIN–UHLENBECK PROCESSES WITH A JUMP COMPONENT*

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(Translated by the author)

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Abstract. Using martingale technique, we show that a distribution of the first-passage time over a level for the Ornstein–Uhlenbeck process with jumps is exponentially bounded. In the case of absence of positive jumps, the Laplace transform for this passage time is found. Further, the maximal inequalities are also given when the marginal distribution is stable.

Key words. exponential martingales, first-passage times, Ornstein–Uhlenbeck process, Laplace transform, moment Wald's identity, maximal inequalities, stable distribution

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1. Introduction. Let X_t, $t \geqq 0$, solve a linear stochastic equation

$$X_t = X_0 - \beta \int_0^t X_s \, ds + Y_t, \qquad t \geqq 0, \tag{1}$$

where $\{Y_t,\ t \geqq 0\}$ is a Lévy process (that is a process with independent homogeneous increments; see, e.g., [1], [2]), X_0 is a nonrandom initial value.

We consider here only the stable case, that is when $\beta > 0$. In the literature, this process is cited as an important example of a different class of random processes: shot noise processes (see, e.g., [3]), filtered Poisson processes [4], generalized Ornstein–Uhlenbeck (OU) processes ([5], [6], and [7]); etc.

One of the problems for models of that sort is to determine a distribution, or moments for the one-sided passage time

$$\tau_b = \inf\{t > 0\colon\ X_t > b\}, \qquad b > X_0.$$

Different approaches were used for studying this problem: integral equations (see, e.g., [1], [10], and [21]); martingale techniques ([7], [8], and [9]), etc. In this paper, we also apply the martingale technique, namely a special parametric family of martingales (Theorem 1). We find the Laplace transform of τ_b provided that the Lévy process Y_t possesses negative jumps only (Theorem 2; under somewhat less general conditions this result is known from [7] and [8]; see also Remark 3).

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When the process Y_t has positive jumps, the Laplace transform of τ_b, as well as its moments, are unknown, except for exponential distribution of positive jumps [10], or uniform distribution [22] of ones.

In this paper, we prove that the distribution of τ_b is exponentially bounded for all b under the assumption that the process Y_t has diffusion component or positive jumps (Theorem 3). This result is known from [9] under the additional assumption of finiteness of the mathematical expectation of Y_t. Moreover, with the help of the moment Wald identity (section 4), we derive a lower bound for the expectation $E(\tau_b)$. Notice that the moment Wald identity can also be used for deriving asymptotic expansions of $\mathbf{E}\tau_b$ as $b \to \infty$ (see [22]). In section 5, we use this moment identity for deriving the moment inequalities for $\sup_{t \leqq \tau}(X_t)$ even for an arbitrary stopping time τ provided that Y_t obeys one-sided stable distribution, i.e., in the absence of positive jumps (Theorem 5). The proof of this result uses techniques from [11], where the moment inequalities for $\sup_{t \leqq \tau} |X_t|$ are given for the Gaussian OU-process.

2. An exponential martingale family. We assume that the Lévy process Y_t and all other random objects are defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ supplied by a filtration (including the assumption of right-continuity, etc.).

Recall that any Lévy process

$$Y_t = mt + \sigma W_t + Z_t, \tag{2}$$

where m and σ are constants, W_t is a standard Brownian motion, and Z_t is a discontinuous process with independent homogeneous increments and paths from the Skorokhod space.

A (unique) solution of equation (1) has the following representation in terms of stochastic integrals with respect to W_t and Z_t:

$$\begin{aligned} X_t &= X_0 e^{-\beta t} + e^{-\beta t} \int_0^t e^{\beta s}\, dY_s \\ &= \frac{m}{\beta} + \left(X_0 - \frac{m}{\beta}\right) e^{-\beta t} + \sigma e^{-\beta t} \int_0^t e^{\beta s}\, dW_s + e^{-\beta t} \int_0^t e^{\beta s}\, dZ_s. \end{aligned} \tag{3}$$

It is well known that the jump component Z_t of the Lévy process can be represented in terms of integrals with respect to a Poisson random measure $p(dx, ds)$, (generated by jumps of Y_t), and a Lévy canonical measure of jumps $\Pi(dx)$:

$$Z_t = \int_0^t \int x I\{|x| < 1\} \left[p(dx, ds) - \Pi(dx)\, ds\right] + \int_0^t \int x I\big\{|x| \geqq 1\big\}\, p(dx, ds). \tag{4}$$

Here $I\{\cdot\}$ is an indicator function and the Lévy measure $\Pi(dx)$ must satisfy the following condition:

$$\int \min(x^2, 1)\, \Pi(dx) < \infty. \tag{5}$$

In what follows we define a class of martingales as a parametric family of "X_t" for the special case when the process Y_t obeys all exponential moments:

$$\mathbf{E} e^{uY_t} = \exp\big\{t\psi(u)\big\} < \infty \qquad \text{for all} \quad u \geqq 0, \tag{6}$$

where, as is well known [1] the cumulant function $\psi(u)$ has the following representation:

$$\psi(u) = mu + \frac{\sigma^2}{2} u^2 + \int \left(e^{ux} - ux I\{|x| < 1\} - 1\right) \Pi(dx).$$

Additionally, we suppose that

$$\mathbf{E}\log(1+Y_1^-)<\infty \tag{7}$$

(henceforth, $a^+ = \max(a,0)$, $a^- = (-a)^+$). We shall see that this condition is sufficient and necessary for finiteness of the following function $\varphi(u)$ which will be used in a definition of martingales below:

$$\varphi(u)=\frac{1}{\beta}\int_0^u v^{-1}\psi(v)\,dv=\frac{1}{\beta}\Big(m\,u+\frac{\sigma^2}{4}\,u^2+I_1(u)+I_2(u)\Big),\quad u\geqq 0, \tag{8}$$

where

$$I_1(u)=\int_0^u v^{-1}\Big[\int(e^{vx}-vxI\{|x|<1\}-1)I\{x>-1\}\,\Pi(dx)\Big]\,dv,$$
$$I_2(u)=\int_0^u v^{-1}\Big[\int(e^{vx}-1)I\{x\leqq -1\}\,\Pi(dx)\Big]\,dv.$$

By conditions (5) and (6) the integral $I_1(u)$ is well defined and finite. It is convenient to express the integrand in $I_2(u)$ as follows:

$$\begin{aligned}\int_0^u v^{-1}(e^{vx}-1)\,dv&=\int_0^{-xu}y^{-1}(e^{-y}-1)\,dy\\&=-\int_0^1 y^{-1}(1-e^{-y})\,dy-\int_1^{-xu}y^{-1}(1-e^{-y})\,dy\\&=-\mathit{EulerGamma}-\log(-xu)-\int_{-xu}^{\infty}y^{-1}e^{-y}\,dy,\end{aligned}$$

where *EulerGamma* is the Euler constant (in notation of the package Mathematica [23]):

$$\mathit{EulerGamma}=\int_0^1 y^{-1}(1-e^{-y})\,dy-\int_1^\infty y^{-1}e^{-y}\,dy$$

(see also [12, formula 8.367.12]). Therefore,

$$I_2(u)=D-\int\Big[\log(u)+\int_{-xu}^{\infty}y^{-1}e^{-y}\,dy\Big]\,I\{x\leqq -1\}\,\Pi(dx), \tag{9}$$

where $D=-\int[\mathit{EulerGamma}+\log(-x)]\,I\{x\leqq -1\}\,\Pi(dx)$. Obviously, (7) provides the existence of integral $\int\log(-x)I\{x\leqq -1\}\,\Pi(dx)$, so that, by (5), $I_2(u)$ is well defined and finite if and only if (7) holds true.

For definition of the above-mentioned parametric martingale family we introduce also the following "martingale" function

$$H(\mu,x)=\int_0^\infty e^{ux-\varphi(u)}u^{\mu-1}\,du,\qquad \mu>0.$$

The following simple estimate for an asymptotic limit of the function $\varphi(u)$ will allow us to find simple conditions for finiteness of the function $H(\mu,x)$.

LEMMA 1. *Let conditions* (6) *and* (7) *hold. If*

$$\sigma>0,\ \ or\ \Pi((0,\infty))>0,\ \ or\ \int|x|\,I\{-1<x<0\}\,\Pi(dx)=\infty, \tag{10}$$

then

$$\lim_{u\to\infty}\frac{\varphi(u)}{u}=\infty. \tag{11}$$

If

$$\sigma=0,\quad \Pi((0,\infty))=0,\quad \int|x|I\{-1<x<0\}\,\Pi(dx)<\infty, \tag{12}$$

then

$$\lim_{u\to\infty}\frac{\varphi(u)}{u}=\frac{1}{\beta}\left(m+\int|x|I\{-1<x<0\}\,\Pi(dx)\right). \tag{13}$$

Proof. From (9) it follows that

$$I_2(u)=-\log(u)\Pi\big((-\infty,-1]\big)+O(1),\qquad u\to\infty. \tag{14}$$

By the inequality $e^z-zI\{|z|<1\}-1\geqq z^2I\{z>0\}/2$, we find that

$$I_1(u)\geqq\frac{u^2}{4}\int x^2I\{x>0\}\,\Pi(dx),$$

and, hence,

$$\beta\varphi(u)\geqq mu+\frac{u^2}{4}\left(\sigma^2+\int x^2I\{x>0\}\,\Pi(dx)\right)+O(\log(u)).$$

So, if $\sigma>0$ or $\Pi((0,\infty))>0$, then by this lower bound we obtain (11).

Now assume (12). Then

$$\beta\,\varphi(u)=mu+\int\left[\int_0^u\frac{e^{v\,x}-v\,x-1}{v}\,dv\right]I\{-1<x<0\}\,\Pi(dx)+I_2(u).$$

By virtue of (14), the integral $I_2(u)$ is of the order $O(\log(u))$. Note that for $x<0$ and $v>0$ the following inequalities hold:

$$0\leqq\frac{e^{vx}-vx-1}{v}\leqq-x.$$

Taking into account the assumption $\int|x|I\{-1<x<0\}\,\Pi(dx)<\infty$, the dominated convergence theorem, and the l'Hospital rule, we find

$$\lim_{u\to\infty}\frac{1}{u}\int_0^u\int\frac{e^{v\,x}-v\,x-1}{v}\,dv\,I\{-1<x<0\}\,\Pi(dx)=\int|x|I\{-1<x<0\}\,\Pi(dx).$$

Thus, (13) holds.

If $\int|x|I\{-1<x<0\}\,\Pi(dx)=\infty$, the same arguments lead to the following estimate with any $\varepsilon>0$:

$$\lim_{u\to\infty}\frac{\varphi(u)}{u}\geqq\frac{1}{\beta}\left(m+\int|x|I\{-1<x<-\varepsilon\}\,\Pi(dx)\right).$$

Letting here $\varepsilon\to0$, we obtain (11). Lemma 1 is proved.

Notice that by Lemma 1 the function $H(\mu, x)$ is finite for any real x if condition (10) holds or if

$$\sigma = 0, \quad \Pi((0,\infty)) = 0, \quad m + \int |x| I\{-1 < x < 0\}\, \Pi(dx) > \beta x. \tag{15}$$

Remark 1. If

$$\sigma = 0, \quad \Pi((0,\infty)) = 0, \quad m + \int |x| I\{-1 < x < 0\}\, \Pi(dx) \leqq \beta X_0, \tag{16}$$

then for $b > X_0$ the stopping time $\tau_b = \infty$.

Indeed, the condition $\Pi((0,\infty)) = 0$ implies that the process X_t does not have any positive jumps at all. So, since the continuous part of X_t is not smaller than X_t itself, by (3), (4), and (16) we have the following deterministic upper bound:

$$\begin{aligned} X_t &\leqq \frac{m}{\beta} + \left(X_0 - \frac{m}{\beta}\right) e^{-\beta t} + e^{-\beta t} \int_0^t e^{\beta s} \left[\int |x|\, I\{-1 < x < 0\}\, \Pi(dx) \right] ds \\ &= \frac{1}{\beta}\left(m + \int |x| I\{-1 < x < 0\}\, \Pi(dx)\right) \\ &\quad + \left(X_0 - \frac{m}{\beta} - \frac{1}{\beta} \int |x| I\{-1 < x < 0\}\, \Pi(dx)\right) e^{-\beta t} \leqq X_0. \end{aligned}$$

Thus, under $b > X_0$ we have $\sup_{t>0} X_t < b$ and so $\tau_b = \infty$.

THEOREM 1. *Let conditions* (6) *and* (7) *hold. Further, assume* (10) *or* (15) *with* $x = X_0$ *hold. Then*

$$\left\{e^{-\beta\mu t} H(\mu, X_t),\ t \geqq 0\right\}, \qquad \mu > 0,$$

is the martingale.

Proof. Using standard tools of stochastic analysis (see, e.g., [1] or [13]) we obtain that, under conditions (6) and (7), the following process

$$\Sigma_t(u) = \exp\left\{ u e^{\beta t} X_t - \int_0^t \psi(u e^{\beta s})\, ds \right\} \tag{17}$$

is a martingale. The fact that this process is a *local martingale* can be checked using representation (3) and the generalized Itô formula. The uniform integrability (under assumption (6)), and, hence, the validity of the martingale property, is a consequence of the following exponential identity

$$\mathbf{E} \exp\left\{ \int_0^t f(s)\, dY_s - q_t \right\} = 1,$$

where

$$q_t = m \int_0^t f(s)\, ds + \frac{\sigma^2}{2} \int_0^t f^2(s)\, ds + \int_0^t \int \left(e^{f(s)x} - f(s)\, x I\{|x| < 1\} - 1\right) \Pi(dx)\, ds$$

and $f(s)$ is a bounded deterministic function.

Note that

$$\int_0^t \psi(u e^{\beta s})\, ds = \frac{1}{\beta} \int_u^{u e^{\beta t}} \frac{\psi(v)}{v}\, dv = \varphi(u e^{\beta t}) - \varphi(u).$$

Since $\mathbf{E}(\Sigma_t(u)) = \Sigma_t(0) = \exp\{uX_0\}$, from the latter formula it follows that

$$\mathbf{E}\exp\{uX_t\} = \exp\{uX_0e^{-\beta t} + \varphi(u) - \varphi(ue^{-\beta t})\}. \tag{18}$$

Applying the Fubini theorem and then introducing a new variable $z = ue^{-\beta t}$, we obtain

$$\begin{aligned}\mathbf{E}(H(\mu, X_t)) &= \int_0^\infty \mathbf{E}(e^{uX_t - \varphi(u)})\, u^{\mu-1}\, du \\ &= \int_0^\infty e^{uX_0e^{-\beta t} - \varphi(ue^{-\beta t})}\, u^{\mu-1}\, du = e^{\beta\mu t} H(\mu, X_0) < \infty.\end{aligned} \tag{19}$$

The finiteness of the function $H(\mu, X_0)$ is due to Lemma 1 and conditions (10) or (15) with $x = X_0$.

By (17), for all $s \leqq t$ we have

$$\mathbf{E}(\Sigma_t(u) \mid \mathcal{F}_s) = \Sigma_s(u) \qquad \text{a.s.}$$

Now, integrating both sides of the above equality with respect to $Q(du) = e^{-\varphi(u)}u^{\mu-1}\, du$ $(\mu > 0)$, with $\mu > 0$, over the interval $(0, \infty)$ we obtain

$$\begin{aligned}\int_0^\infty \mathbf{E}(\Sigma_t(u) \mid \mathcal{F}_s)\, e^{-\varphi(u)}u^{\mu-1}\, du &= \int_0^\infty \Sigma_s(u)\, e^{-\varphi(u)}u^{\mu-1}\, du \\ = e^{-\beta\mu s}\int_0^\infty e^{ue^{\beta s}X_s - \varphi(ue^{\beta s})}(ue^{\beta s})^{\mu-1}\, d(ue^{\beta s}) &= e^{-\beta\mu s} H(\mu, X_s).\end{aligned} \tag{20}$$

Hence, by the Fubini theorem, applied to the left-hand side of (20), we obtain the required martingale property

$$\mathbf{E}(e^{-\beta\mu t} H(\mu, X_t) \mid \mathcal{F}_s) = e^{-\beta\mu s} H(\mu, X_s) \qquad \text{a.s.}$$

Theorem 1 is proved.

Remark 2. The idea of constructing a special parametric martingale family is not new. A similar method was used in the papers [14] and [15] for boundary crossing problems related to a Brownian motion, and in the papers [7] and [16] for boundary crossing problems related to a stable Lévy process.

By the optional stopping theorem and Theorem 1 we have the following identity: for any stopping time τ and fixed $t < \infty$

$$\mathbf{E}\big[e^{-\beta\mu \min(\tau,t)}\, H(\mu, X_{\min(\tau,t)})\big] = H(\mu, X_0), \qquad \mu > 0, \tag{21}$$

We apply (21) to derive an explicit formula for the Laplace transform of τ_b provided that the process Y_t does not have positive jumps. Before proving this formula we have to introduce further notation. Set

$$\widetilde{H}(\mu, x) = \int_0^\infty e^{ux - \widetilde{\varphi}(u)}u^{\mu-1}\, du, \qquad \mu > 0,$$

with

$$\beta\widetilde{\varphi}(u) = m\,u + \frac{\sigma^2u^2}{4} + I_1(u) - \int\left[\log(u) + \int_{-xu}^\infty \frac{e^{-y}}{y}\, dy\right] I\{x \leqq -1\}\, \Pi(dx).$$

If (7) holds, then, by (9) and (8),

$$\widetilde{H}(\mu, x) = \exp\{D\beta^{-1}\} H(\mu, x),$$

where the constant D is defined above. Notice also that $\widetilde{H}(\mu, x)$ even if (7) fails.

THEOREM 2. *Let* $\Pi((0,\infty)) = 0$. *If*

$$\sigma > 0 \quad or \quad m + \int |x|\, I\{-1 < x < 0\}\, \Pi(dx) > \beta b, \tag{22}$$

then $P\{\tau_b < \infty\} = 1$ *and*

$$\mathbf{E}e^{-\beta\mu\tau_b} = \frac{\widetilde{H}(\mu, X_0)}{\widetilde{H}(\mu, b)}, \qquad \mu > 0.$$

Proof. First we assume that (7) holds and consider identity (21) with $\tau = \tau_b$. By Lemma 1 and (22), $|H(\mu, x)| < \infty$, $x \leqq b$. Under the absence of positive jumps, on the set $\{\tau_b < t\}$ we have $X_{\tau_b} = b$ and, on the set $\{\tau_b \geqq t\}$ we have $X_t \leqq b$ for all $t \geqq 0$. Hence, $H(\mu, X_{\min(\tau_b, t)}) \leqq H(\mu, b) < \infty$ and, by the Fatou lemma we can pass to the limit as $t \to \infty$ under the expectation in the left-hand side of (21). As the result, we get

$$\mathbf{E}\left[I\{\tau_b < \infty\} e^{-\beta\mu\tau_b}\right] = \frac{H(\mu, X_0)}{H(\mu, b)}, \qquad \mu > 0. \tag{23}$$

It is easy to verify, integrating by parts, that

$$\lim_{\mu \downarrow 0} \mu H(\mu, x) = 1. \tag{24}$$

Passing to the limit as $\mu \to 0$ in (23), by the Fatou lemma we get $\mathbf{P}\{\tau_b < \infty\} = 1$. Since

$$\frac{H(\mu, X_0)}{H(\mu, b)} = \frac{\widetilde{H}(\mu, X_0)}{\widetilde{H}(\mu, b)}, \qquad \mu > 0,$$

then, by (23) under (7), the statement of Theorem 1 is valid under the imposed assumption (7). If (7) fails, we may consider the OU-process X_t^N which solves (1) with Y_t replaced with truncated Lévy process

$$Y_t^N = mt + \sigma W_t + Z_t^N,$$

where

$$\begin{aligned} Z_t^N = &\int_0^t \int x I\{-1 < x < 0\} \left[p(dx, ds) - \Pi(dx)\, ds\right] \\ &+ \int_0^t \int x I\{-N \leqq x \leqq -1\}\, p(dx, ds). \end{aligned}$$

Denote by τ_b^N the corresponding crossing time of the level b and by $\widetilde{H}^N(\mu, x)$ the corresponding martingale function. Obviously, $\tau_b^N \to \tau_b$ a.s. as $N \nearrow \infty$. Notice that (7) holds for Y_t^N and we have

$$\mathbf{E}\, e^{-\beta\mu\tau_b^N} = \frac{\widetilde{H}^N(\mu, X_0)}{\widetilde{H}^N(\mu, b)}, \qquad \mu > 0,$$

where the function $\widetilde{H}^N(\mu, x)$ is defined above. Now it is easy to check that

$$\lim_{N\to\infty} \widetilde{H}^N(\mu, x) = \widetilde{H}(\mu, x) \qquad \text{for any} \quad x \leqq b.$$

Theorem 2 is proved.

Remark 3. In the case when $\Pi(dx) \equiv 0$ and $\sigma > 0$ the process X_t is Gaussian and, of course, the result of Theorem 2 for this case is well known (see, e.g., [17]). Note also that for this special case it is possible to derive an analytical inversion of the Laplace transform of τ_b based on the representation for the function $H(\mu, x)$ in terms of the parabolic cylinder function $D_{-\mu}(-x)$ which is well studied (see [12, formula 9.241.2.]).

In the case when $\Pi(-\infty, 0) > 0$ and $\sigma > 0$, Theorem 2 is proved by Hadjiev (see in [7, p. 85, Theorem 2]), where a slightly different parametric martingale family is used. The case $\Pi(-\infty, 0) > 0$ and $\sigma = 0$ is also discussed in [7] under an additional condition (see Hypothesis G in [7]).

3. Exponential boundedness of τ_b. In this section, we give sufficient conditions, weaker than in [9], for the exponential moments of τ_b to be finite. The existence of exponential moments for the first-passage times from interval (a, b):

$$\tau_{a,b} = \inf\{t > 0\colon\ X_t > b \text{ or } X_t < a\}, \qquad b > X_0 > a,$$

is established too.

THEOREM 3. *Let conditions* (10) *or* (15) *with* $x = b$ *hold. Assume also that*

$$\mathbf{E}(Y_1^-)^\delta < \infty \qquad \textit{for any} \quad \delta \in (0, 1). \tag{25}$$

Then there exists $\alpha > 0$ *such that*

$$\mathbf{E}\, e^{\alpha\tau_b} < \infty.$$

Proof. Assuming (6), we shall use an analytical continuation of the martingale family $e^{-\beta t} H(\mu, X_t)$, $\mu > 0$, to $\mu \in (-\delta, 0)$ with δ involved in (25).

For $\mu \in (-\delta, 0)$, set

$$H(\mu, x) = \int_0^\infty (e^{ux - \varphi(u)} - 1)\, u^{\mu-1}\, du \qquad \text{for} \quad \mu \in (-\delta, 0). \tag{26}$$

By (10) and Lemma 1, $|H(\mu, x)| < \infty$ if and only if

$$\int_0^1 |\varphi(u)|\, u^{\mu-1} du < \infty.$$

Owing to (5) and (8), for $\mu > -1$ we have

$$\int_0^1 |\varphi(u)|\, u^{\mu-1} du \leqq C + C \int_0^1 |I_2(u)|\, u^{\mu-1} du$$

(hereafter C is a positive generic constant).

The inequality

$$1 - e^{-z} \leqq C_\delta z^\delta, \qquad z > 0, \quad \delta > 0, \tag{27}$$

provides the following bound:

$$\int_0^1 |I_2(u)|\, u^{\mu-1}\, du = \int\int_0^1\int_0^u \frac{(1-e^{vx})}{v}\, dv\, I\{x \leqq -1\}\, \Pi(dx)\, u^{\mu-1}\, du$$
$$\leqq \frac{C_\delta}{\delta}\int |x|^\delta I\{x \leqq -1\}\, \Pi(dx) \int_0^1 u^{\mu-1+\delta}\, du$$

in which the latter integral in the right-hand side is finite for any $\mu \in (-\delta, 0)$, whereas (25) is equivalent to $\int |x|^\delta I\{x \leqq -1\}\, \Pi(dx) < \infty$.

The function $H(\mu, x)$, $\mu \in (-\delta, 0)$, defined in (26), can be considered as an analytical continuation of $H(\mu, x)$, $\mu > 0$. From (26) (see also (24)) it easily follows that

$$\lim_{\mu \uparrow 0} \mu H(\mu, x) = 1. \tag{28}$$

Just repeating the proof of Theorem 1 with $H(\mu, x)$, $\mu \in (-\delta, 0)$, we may prove that the process

$$e^{-\beta\mu t}\int_0^\infty (e^{uX_t} - e^{uX_0})\, u^{\mu-1}e^{-\varphi(u)}\, du + H(\mu, X_0)\, e^{-\beta\mu t}$$

is a martingale. Then, by the optional stopping theorem for martingales for any $t < \infty$, we have

$$\mathbf{E}e^{-\beta\mu\min(\tau_b, t)}\left[\int_0^\infty (e^{uX_{\min(\tau_b,t)}} - e^{uX_0})\, u^{\mu-1}e^{-\varphi(u)}\, du + H(\mu, X_0)\right] = H(\mu, X_0). \tag{29}$$

First consider the case $\Pi((0,\infty)) = 0$. Since $X_{\min(\tau_b, t)} \leqq b$ and $\mu < 0$, obviously,

$$\mathbf{E}e^{-\beta\mu\min(\tau_b, t)}\left[\mu\int_0^\infty (e^{ub} - e^{uX_0})\, u^{\mu-1}e^{-\varphi(u)}\, du + \mu H(\mu, X_0)\right] \leqq \mu H(\mu, X_0). \tag{30}$$

Further, whereas

$$0 < \int_0^\infty (e^{ub} - e^{uX_0})\, u^{\mu-1}e^{-\varphi(u)}\, du \longrightarrow \int_0^\infty (e^{ub} - e^{uX_0})\, u^{-1}e^{-\varphi(u)}\, du < \infty$$

as $\mu \to 0$ there exists $\mu_0 \in (-\delta, 0)$ such that for any $\mu \in (\mu_0, 0)$

$$\mu\int_0^\infty (e^{ub} - e^{uX_0})\, u^{\mu-1}e^{-\varphi(u)}\, du > -0.8.$$

On account of (28), there is $\mu_1 \in (\mu_0, 0)$ such that $0.9 < \mu_1 H(\mu_1, X_0) < 1.1$ So, (30) provides

$$\mathbf{E}e^{-\beta\mu_1\min(\tau_b, t)} \leqq \frac{\mu_1 H(\mu_1, X_0)}{\mu_1\int_0^\infty (e^{ub} - e^{uX_0})\, u^{\mu_1-1}e^{-\varphi(u)}\, du + \mu_1 H(\mu_1, X_0)} < \frac{1.1}{0.1}.$$

This bound is valid for any $t \geqq 0$. Hence, by the Fatou lemma, the statement of Theorem 3 holds true for $\alpha = -\mu_1\beta$.

If $\Pi((0,\infty)) > 0$, then, choosing a positive constant A with $\Pi((0, A]) > 0$, we introduce the OU-process X_t^A, generated by the Lévy process Y_t^A with positive jumps truncated by A, and the level crossing time τ_b^A and notice that $\tau_b^A \geqq \tau_b$. Applying identity (29) with τ_b replaced with τ_b^A and properly defined functions $\varphi_A(u)$ and $H_A(\mu, X_0)$), and taking into account $X^A_{\min(\tau_b^A,t)} \leqq b + A$ and $\mu < 0$, we get the following bound

$$\mathbf{E}e^{-\beta\mu\min(\tau_b^A,t)}\left[\mu\int_0^\infty (e^{u(A+b)} - e^{uX_0})\,u^{\mu-1}e^{-\varphi_A(u)}\,du + \mu H_A(\mu, X_0)\right]$$
$$\leqq \mu H_A(\mu, X_0).$$

The last part of the proof is similar to that for the case $\Pi((0,\infty)) = 0$.

Theorem 3 is proved.

COROLLARY 1. *Let* $\sigma > 0$ *or* $\Pi((-\infty,\infty)) > 0$. *Assume*

$$\mathbf{E}|Y_1|^\delta < \infty \qquad \text{for some} \quad \delta > 0.$$

Then there exists $\alpha > 0$ *such that*

$$\mathbf{E}e^{\alpha\tau_{a,b}} < \infty.$$

Proof. Denote $\gamma_a = \inf\{t > 0\colon X_t < a\}$ and note that $\tau_{a,b} = \min(\tau_b, \gamma_a)$ By Theorem 3, applied to τ_b and γ_a, the desired result holds.

4. The moment Wald identity. The theorem below generalizes Theorem 2 of [8].

THEOREM 4. *Denote* $T = \inf\{t \geqq 0\colon X_t \geqq f(t)\}$, $X_0 < f(0)$, *where* $f(t)$ *is a continuous deterministic function such that* $\sup_{t\geqq 0} f(t) = M < \infty$. *Let conditions* (6) *and* (25) *hold. Further, assume conditions* (10) *or* (15) *with* $x = M$ *hold.*

Then

$$\beta\,\mathbf{E}T = \mathbf{E}\int_0^\infty (e^{uX_T} - e^{uX_0})\,u^{-1}e^{-\varphi(u)}\,du < \infty. \tag{31}$$

Proof. First we shall show that under (10) or (15) with $x = M$ the process

$$\left\{\int_0^\infty (e^{uX_t} - e^{uX_0})\,u^{-1}e^{-\varphi(u)}\,du - \beta t,\ t \geqq 0\right\} \tag{32}$$

is a martingale. Indeed, since $\{e^{-\beta\mu t}H(\mu, X_t),\ t \geqq 0\}$ is the martingale for $\mu > 0$, we have

$$e^{-\beta\mu t}\mathbf{E}\Big(\big[H(\mu, X_t) - H(\mu, X_0)\big] \mid \mathcal{F}_s\Big) + (e^{-\beta\mu t} - 1)\,H(\mu, X_0)$$
$$= e^{-\beta\mu s}\big[H(\mu, X_s) - H(\mu, X_0)\big] + (e^{-\beta\mu s} - 1)\,H(\mu, X_0) \quad \text{a.s.} \tag{33}$$

Under the conditions of Theorem 4

$$\lim_{\mu\to 0}\big[H(\mu, z) - H(\mu, X_0)\big] = \int_0^\infty (e^{uz} - e^{uX_0})\,u^{-1}e^{-\varphi(u)}\,du$$

for any z, if (10) holds, or for any $z < M$, if (15) holds with $x = M$. Further, due to (24), we have

$$\lim_{\mu\to 0}(e^{-\beta\mu t} - 1)\,H(\mu, X_0) = -\beta t.$$

Applying now the dominated convergence theorem we can interchange the symbols of the limit as $\mu \to 0$ and the conditional expectation in the left side of (33). Thus, passing to the limit as $\mu \to 0$ in both parts of (33) we obtain the martingale property (32).

By the optional stopping theorem for martingales we find that

$$\beta\,\mathbf{E}\min(\tau,t) = \mathbf{E}\int_0^\infty (e^{uX_{\min(\tau,t)}} - e^{uX_0})\,u^{-1}e^{-\varphi(u)}\,du \tag{34}$$

is valid for any stopping time τ and fixed $t < \infty$.

To complete the proof, it remains only to verify that for computation of "$\lim_{t\to\infty}$" in the right-hand side of (34) with $\tau = T$ can be reduced to computation under the "expectation symbol." We verify that as follows. By Theorem 3,

$$\mathbf{E}T < \infty. \tag{35}$$

Further, since (34), with $\tau = T$, is equivalent to

$$\beta\,\mathbf{E}\min(T,t) = \mathbf{E}I\{T \leqq t\}\int_0^\infty (e^{uX_T} - e^{uX_0})\,u^{-1}e^{-\varphi(u)}\,du + e_t,$$

where $e_t = \mathbf{E}\,I\{T > t\}\int_0^\infty (e^{uX_t} - e^{uX_0})\,u^{-1}e^{-\varphi(u)}\,du$. Assume for a moment that

$$\lim_{t\to\infty} e_t = 0. \tag{36}$$

Then, by the dominated convergence theorem, (31) holds true.

For a verification of (36), we notice that $X_t \leqq M$ on the set $\{T > t\}$. Therefore, applying (27), we find that

$$\begin{aligned}
|e_t| &= \Big|\mathbf{E}I\{T > t,\ X_t \geqq X_0\}\int_0^\infty (e^{uX_t} - e^{uX_0})\,u^{-1}e^{-\varphi(u)}\,du \\
&\qquad - \mathbf{E}I\{T > t,\ X_t < X_0\}\int_0^\infty e^{uX_0}(1 - e^{u(X_t - X_0)})\,u^{-1}e^{-\varphi(u)}\,du\Big| \\
&\leqq \mathbf{P}\{T > t,\ X_t \geqq X_0\}\int_0^\infty (e^{uM} - e^{uX_0})\,u^{-1}e^{-\varphi(u)}\,du \\
&\qquad + \mathbf{E}\Big[I\{T > t,\ X_t < X_0\}\,C_\delta\int_0^\infty e^{uX_0}u^{-1+\delta}e^{-\varphi(u)}\,du\,\big((X_t - X_0)^-\big)^\delta\Big] \\
&\leqq C_1\mathbf{P}\{T > t\} + C_2\,\mathbf{E}\big[I\{T > t\}|X_t - X_0|^\delta\big],
\end{aligned} \tag{37}$$

where

$$C_1 = \int_0^\infty (e^{uM} - e^{uX_0})\,u^{-1}e^{-\varphi(u)}\,du, \qquad C_2 = C_\delta\int_0^\infty e^{uX_0}u^{-1+\delta}e^{-\varphi(u)}\,du.$$

Set

$$\begin{aligned}
\widehat{Y}_t &= \sigma W_t + \int_0^t\int xI\{|x| < 1\}\,\big[p(dx,ds) - \Pi(dx)\,ds\big], \\
\widetilde{Y}_t &= \int_0^t\int xI\{|x| \geqq 1\}\,p(dx,ds).
\end{aligned}$$

Since by (3) and (4),

$$X_t - X_0 = (\beta^{-1}m - X_0)(1 - e^{-\beta t}) + e^{-\beta t}\int_0^t e^{\beta s}\, d\widehat{Y}_s + e^{-\beta t}\int_0^t e^{\beta s}\, d\widetilde{Y}_s.$$

with the help of inequality $|a+b+c|^\delta \leqq |a|^\delta + |b|^\delta + |c|^\delta$, $\delta \in (0,1)$ we find that

$$|X_t - X_0|^\delta \leqq |\beta^{-1}m - X_0|^\delta + |\widehat{X}_t|^\delta + \int_0^t \int |x|^\delta I\{|x| \geqq 1\}\, p(dx, ds), \tag{38}$$

where $\widehat{X}_t$ is an OU-process with $\widehat{X}_0 = 0$, generated by the square-integrable martingale $\widehat{Y}_t$.

Below, we shall show that for any stopping time τ and $\delta \leqq 2$

$$\mathbf{E}\sup_{t\leqq\tau} |\widehat{X}_t|^\delta \leqq C_{\delta,\beta}\mathbf{E}\tau^{\delta/2}. \tag{39}$$

By the property of the stochastic integral (see, e.g., [1] or [13]), for any stopping time τ we have

$$\mathbf{E}\int_0^\tau \int |x|^\delta I\{|x| \geqq 1\}\, p(dx, ds) = \int |x|^\delta I\{|x| \geqq 1\}\, \Pi(dx)\, \mathbf{E}\tau,$$

where, by (6) and (25),

$$\int |x|^\delta I\{|x| \geqq 1\}\, \Pi(dx) < \infty. \tag{40}$$

Combining (35), (37), (38), (40), and (39) (inequality(39) will be proved in the sequel), we conclude that $\mathbf{E}\sup_{t\leqq T} |X_t - x| < \infty$ and in turn (36) holds.

To verify (39), we apply the Itô formula to $\widehat{X}_t^2 = e^{-2\beta t}M_t^2$ with

$$M_t = \int_0^t e^{\beta s}\, d\widehat{Y}_s,$$

and find that

$$\begin{aligned}\widehat{X}_t^2 = \int_0^t e^{-2\beta s}(2M_{s-}\, dM_s) + \int_0^t e^{-2\beta s}\, d\big([M_s, M_s] - \langle M_s, M_s\rangle\big) \\ + \int_0^t M_s^2\, de^{-2\beta s} + t\Big(\sigma^2 + \int x^2 I\{|x| < 1\}\, \Pi(dx)\Big).\end{aligned}$$

Here, the first and second integral terms are martingales, the third one is negative. For any bounded stopping time τ, these facts provide

$$\mathbf{E}\widehat{X}_\tau^2 \leqq \mathbf{E}(\tau)\Big(\sigma^2 + \int x^2 I\{|x| < 1\}\, \Pi(dx)\Big).$$

So, for $\delta \leqq 2$, (39) is provided by Lenglart's domination principle (see, e.g., [18, p. 156]).

Theorem 4 is proved.

Remark 4. Since $X_{\tau_b} \geqq b$, Theorem 4 provides

$$\beta\, \mathbf{E}\tau_b \geqq \int_0^\infty (e^{ub} - e^{uX_0})\, u^{-1} e^{-\varphi(u)}\, du.$$

Since the condition, given in (6), involves in Theorem 1 and 4, exponentially distributed, as well as others of such type, positive jumps are excluded from consideration. However, the truncation technique implementation (for large positive jumps) allows obtaining a lower bound in this case as well. So, instead of (6) we assume

$$K = \sup\Big\{u \geqq 0\colon\ \mathbf{E}e^{uY_t} = \exp\big\{t\psi(u)\big\}\ < \infty\Big\} < \infty$$

and define the function

$$\varphi(u) = \frac{1}{\beta}\int_0^u v^{-1}\psi(v)\,dv, \qquad u < K.$$

Then, repeating the steps of the proof of Theorem 4, first for the case with truncated jumps and then passing to the limit as the parameter of truncation increases to infinity, we obtain the following lower bound

$$\beta\,\mathbf{E}\tau_b \geqq \int_0^K (e^{ub} - e^{uX_0})\,u^{-1}e^{-\varphi(u)}\,du.$$

Remark 5. Identity (31) might also be used for creating corresponding bounds for two-sided stopping times $\tau_{a,b}$. If, for example, Y_t is the process with a symmetric distribution, $X_0 = 0$ and (6) hold, then (34) holds for X_t and $(-X_t)$ as well, that is, for any stopping time τ

$$\beta\,\mathbf{E}\min(\tau,t) = \mathbf{E}\int_0^\infty (e^{\pm uX_{\min(\tau,t)}} - 1)\,u^{-1}e^{-\varphi(u)}\,du$$

Hence,

$$\beta\,\mathbf{E}\min(\tau,t) = \mathbf{E}\int_0^\infty (\cosh(uX_{\min(\tau,t)}) - 1)\,u^{-1}e^{-\varphi(u)}\,du. \tag{41}$$

The same approach is used in [11] for the derivation of maximal inequalities for Gaussian OU-process. Since Gaussian OU-process is continuous, from (41), as $t \to \infty$, it follows that

$$\beta\,\mathbf{E}\tau_{-b,b} = \int_0^\infty \big(\cosh(ub) - 1\big)\,u^{-1}e^{-\varphi(u)}\,du < \infty, \qquad \varphi(u) = \frac{\sigma^2u^2}{4\beta}.$$

5. Maximal inequalities for stable OU processes. We consider now a spectral negative stable process Y_t (see [1] or [19]) with

$$\mathbf{E}e^{uY_1} = \exp\{\alpha^{-1}u^\alpha\}, \qquad u \geqq 0, \quad 1 < \alpha \leqq 2. \tag{42}$$

This process Y_t and, in turn, X_t do not have positive jumps. Moreover,

$$\varphi(u) = u^\alpha/(\alpha^2\beta).$$

By (18),

$$\mathbf{E}\exp\{uX_t\} = \exp\Big\{uX_0e^{-\beta t} + \frac{(1 - e^{-\alpha\beta t})\,u^\alpha}{\alpha^2\beta}\Big\}.$$

If $\alpha = 2, X_0 = 0$, the process X_t is Gaussian. Then by [11] the following remarkable inequality is valid: for any stopping time τ

$$C_1 \mathbf{E}\sqrt{\log(1+\beta\tau)} \leqq \sqrt{\beta}\mathbf{E}\Big(\max_{t\leqq\tau}|X_t|\Big) \leqq C_2\mathbf{E}\sqrt{\log(1+\beta\tau)}, \tag{43}$$

where $C_1 \geqq \frac{1}{3}$, $C_2 \leqq 3.3795$.

We prove here an analogue of (43).

THEOREM 5. *Let* (42) *hold and X_t solve* (1) *with $X_0 \geqq 0$. Then for any stopping time τ and all $p > 0$*

$$\begin{aligned} c_p\mathbf{E}\Big[\big(\log(1+\beta\tau)\big)^{p(1-1/\alpha)}\Big] &\leqq \mathbf{E}\Big[\Big(\sup_{t\leqq\tau}X_t\Big)^p\Big] \\ &\leqq a_p + C_p\mathbf{E}\Big[\big(\log(1+\beta\tau)\big)^{p(1-1/\alpha)}\Big], \end{aligned} \tag{44}$$

where positive constants a_p, c_p, and C_p do not depend on τ.

For the proof of inequality (43), Graversen and Peskir [11] apply Wald's moment identity with the formula for $\mathbf{E}\tau$ being similar to (41). Here, we apply identity (34) which is the one-sided analog of the above-mentioned formula from [11]. We also use the following simple consequence of Lenglart's domination principle.

LEMMA 2. *Let Q_t be a nonnegative, right continuous process and let A_t be an increasing continuous process, $A_0 = 0$. Assume that for all bounded stopping times τ*

$$\mathbf{E}Q_\tau \leqq \mathbf{E}A_\tau. \tag{45}$$

Then for all $p > 0$ and for all bounded stopping times τ there exist constants c_p and C_p such that

$$\mathbf{E}\Big(\Big[\log\Big(1+\sup_{t\leqq\tau}Q_t\Big)\Big]^p\Big) \leqq c_p + C_p\mathbf{E}([\log(1+A_\tau)]^p). \tag{46}$$

Proof. By Lenglart's principle, for any increasing continuous function $H(x)$ with $H(0) = 0$, (45) provides

$$\mathbf{E}\Big(\sup_{t\leqq\tau}H(Q_t)\Big) \leqq \mathbf{E}\big(\widetilde{H}(A_\tau)\big), \tag{47}$$

where

$$\widetilde{H}(x) = x\int_x^\infty \frac{1}{s}\,dH(s) + 2H(x).$$

Set $H(x) = (\log(1+x))^p$, $x \geqq 0$. By l'Hospital's rule, $\lim_{x\to 0}\widetilde{H}(x) = 0$ and

$$\lim_{x\to\infty}\frac{x}{H(x)}\int_x^\infty \frac{1}{s}\,dH(s) = 0.$$

Hence, $\lim_{x\to\infty}[\widetilde{H}(x)/H(x)] = 2$ and there are constants c_p and C_p such that $\widetilde{H}(x) \leqq c_p + C_pH(x)$. Therefore, (46) is implied by (47) with $H(x) = (\log(1+x))^p$.

Proof of Theorem 5. Denote $X_t^* = \sup_{s\leqq t}X_s$. In the absence of positive jumps for X_t, the process X_t^* is increasing and continuous. Then, (34) provides the following inequality as valid for any bounded stopping times τ

$$\beta\,\mathbf{E}\tau \leqq \mathbf{E}\int_0^\infty (e^{uX_\tau^*} - e^{uX_0})\,u^{-1}e^{-u^\alpha/(\alpha^2\beta)}\,du = \mathbf{E}\big(G(X_\tau^*) - G(X_0)\big),$$

where

$$G(y) = \int_0^\infty (e^{uy} - 1)\, u^{-1} e^{-u^\alpha/(\alpha^2\beta)}\, du.$$

Hence, (45) is valid for $Q_t = \beta t$ and $A_t = G(X_t^*) - G(x)$ the continuous increasing process, $A_0 = 0$. By Lemma 2,

$$\mathbf{E}\big[\big(\log(1+\beta\tau)\big)^{p(1-1/\alpha)}\big] \leqq c_p + C_p \mathbf{E}\Big[\big(\log\big[1 + G(X_\tau^*) - G(X_0)\big]\big)^{p(1-1/\alpha)}\Big].$$

Thus, the lower bound in (44) will be held, if the inequality $\log(1 + G(y)) \leqq C_1 + C_2 y^{\alpha/(\alpha-1)}$, $y > 0$ is valid. The latter bound readily follows from the well-known asymptotic relation:

$$G(y) = \exp\Big\{C_\alpha y^{\alpha/(\alpha-1)}\big(1 + o(1)\big)\Big\}, \qquad y \to \infty \tag{48}$$

(see, e.g., [20, Chap. 3, Exercise 7.3]). The boundedness requirement for stopping time τ is easily removed by applying the localization technique.

The upper bound (44) is derived with the help of (34) which, jointly with an obvious equality $e^x = e^{x^+} - 1 + e^{-x^-}$, for any bounded stopping time τ gives

$$\mathbf{E}\big(G(X_\tau^+)\big) \leqq G(X_0) + \beta \mathbf{E}\tau + \mathbf{E}\int_0^\infty (1 - e^{-uX_\tau^-})\, u^{-1} e^{-u^\alpha/(\alpha^2\beta)}\, du.$$

Since

$$\mathbf{E}\int_0^\infty (1 - e^{-uX_\tau^-})\, u^{-1} e^{-u^\alpha/(\alpha^2\beta)}\, du \leqq \mathbf{E}(X_\tau^-) \int_0^\infty e^{-u^\alpha/(\alpha^2\beta)}\, du$$

and $\mathbf{E}X_\tau^- \leqq |X_0|/\beta + C\mathbf{E}\tau$ (see, also the proof of Theorem 4), we find the following estimate:

$$\mathbf{E}\big(G(X_\tau^+)\big) \leqq c + C\mathbf{E}\tau.$$

Thus, (45) is valid with $A_t = c + Ct$ and $Q_t = G(X_t^+)$, where Q_t is a nonnegative right-continuous process. By Lemma 2,

$$\mathbf{E}\Big[\big(\log\big(1 + G(X_\tau^*)\big)\big)^{p(1-1/\alpha)}\Big] \leqq c_p + C_p \mathbf{E}\Big[\big(\log(1+\beta\tau)\big)^{p(1-1/\alpha)}\Big]$$

and it remains only to notice that (48) provides the following bound:

$$C + \log(1 + G(y)) \geqq C y^{\alpha/(\alpha-1)}, \qquad y > 0.$$

Theorem 5 is proved.

Remark 6. For $\alpha = 2$ and $X_0 = 0$, the application of (44) to $\max_{t \leqq \tau} X_t$ and $\max_{t \leqq \tau}(-X_t)$ leads to (43) without specification of the constants c_p and C_p.

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