

ANALYTICAL AND NUMERICAL METHODS  
FOR DETECTION OF  
A CHANGE IN DISTRIBUTIONS

SAOWANIT SUKPARUNGSEE

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Principal Supervisor: Professor Alexander Novikov



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UNIVERSITY OF TECHNOLOGY, SYDNEY  
CERTIFICATE OF AUTHORSHIP/ORIGINALITY

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Author: Saowanit Sukparungsee  
Title: Analytical and Numerical Methods for Detection of a Change  
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Department: Mathematical Sciences  
Degree: Ph.D. Convocation: Mar Year: 2005

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# *Dedication*

*To Dad & Mom  
and Family*

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# List of Notation and Abbreviations

Notation and abbreviations are used in this thesis are:

<b>ARL</b>	Average Run Length
<b>AD</b>	Average Delay
<b>EWMA</b>	Exponentially Weighted Moving Average
<b>CUSUM</b>	Cumulative Sum
<b>MCA</b>	Markov Chain Approach
<b>IE</b>	Integral Equations
<b>MC</b>	Monte Carlo Simulations
<b>H</b>	control limit/boundary
$\theta$	moment of change-point
$\tau$	stopping time/alarm time
$\alpha_0$	in-control parameter
$\alpha$	out-of-control parameter
$\lambda$	weighted smoothing parameter in EWMA statistic, $\lambda \in (0, 1)$
$\chi$	overshoot
$\varphi$	characteristic function
$\psi$	cumulant generating function (logarithm of the moment generating function)

# List of Publications and Conference Presentations

Parts of the work presented in this thesis have been previously published as:

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3. Sukparungsee, Saowanit, & Novikov, Alexander. 2007. Analytical Approximations for Average Run Lengths in EWMA Charts in Case of Light-tailed Distributions. In *Proceedings of International Conference of Mathematical Sciences, 28-29 November, Bangi-Putrajaya, Malaysia (ICMS 2007)*.
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# Abstract

This thesis aims to derive analytical approximations and numerical algorithms for analysis and design of control charts used for detection of changes in distributions. In particular, we present a new analytical approach for evaluating of characteristics of the “Exponentially Weighted Moving Average (EWMA)” procedure in the case of Gaussian and non-Gaussian distributions with light-tails.

The main characteristics of a control chart are the mean of false alarm time or, Average Run Length (ARL), and the mean of delay for true alarm time or, Average Delay time (AD). ARL should be sufficiently large when the process is in-control and AD should be small when the process is out-of-control. Traditional methods for numerical evaluation ARL and AD are the Monte Carlo simulations (MC), Markov Chain Approach (MCA) and Integral Equations method (IE). These methods have the following essential drawbacks: the crude MC is very time consuming and difficult to use for finding optimal designs; MCA requires matrix inversions and, in general, is slowly convergent; IE requires intensive programming even for the case of Gaussian observations.

In this thesis we develop an approach based on a combination of the martingale technique and Monte Carlo simulations. With the use of a popular symbolic/numerical software Mathematica<sup>®</sup>, this new approach allows to obtain accurate procedures for finding the optimal weights, alarm boundaries and approximations ARL and AD for the case of Gaussian observations. Further, we show that our approach can be used also for non-Gaussian distributions with light-tails and, in particular, for Poisson and Bernoulli distributions.