Model and Extended Kuhn-Tucker Approach for Bilevel Multi-follower Decision Making in a Referential-Uncooperative Situation

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Abstract

When multiple followers are involved in a bilevel decision problem, the leader's decision will be affected, not only by the reactions of these followers, but also by the relationships among these followers. One of the popular situations within this framework is where these followers are uncooperatively making decisions while having cross reference of decision information. This situation is called a referential-uncooperative situation in this paper. The well-known Kuhn-Tucker approach has been successfully applied to a one-leader-and-one-follower linear bilevel decision problem. This paper extends this approach to deal with the above-mentioned linear referential-uncooperative bilevel multi-follower decision problem. The paper first presents a decision model for this problem. It then proposes an extended Kuhn-Tucker approach to solve this problem. Finally, a numeric example illustrates the application of the proposed Kuhn-Tucker approach.

***Keywords*:** bilevel programming, bilevel multi-follower decision, Kuhn-Tucker approach, optimization

1. Introduction

In a bilevel programming (BLP) problem, the leader cannot completely control his/her follower but is influenced by the reaction of his/her follower. Such a situation is appearing in decision making of many decentralized organizations. BLP was motivated by the game theory of Von Stackelberg [1] in the context of unbalanced economic

markets [2]. The majority of research on BLP has centered on the linear version of the problem in which only one follower is involved. There have been nearly two dozen algorithms, such as, the Kth best approach [3, 4], Kuhn-Tucker approach [5-7], complementarity pivot approach [8], penalty function approach [9-13], proposed for solving linear BLP problems since the field being caught the attention of researchers in the mid-1970s. Kuhn-Tucker approach has been proven to be a valuable analysis tool with a wide range of successful applications for linear BLP [2, 6, 7, 14-16].

Although much research has been carried out in the area, the existing bilevel technology has mainly limited to a specific situation comparing one leader and one follower. However, in a real-world bilevel decision problem, the lower level of a bilevel decision may involve multiple decision units. For example, the dean of a faculty is the leader, and all the heads of departments in the faculty are the followers in making a faculty annual budget. The leader (the dean, for example)'s decision will be affected, not only by the reactions of the multiple followers (these heads of departments in the faculty), but also by the relationships among these followers. Each of the leader's possible decisions is influenced by the various reactions of his/her followers who may have had a share in decision information, objectives and constraints. Hence, a bilevel multi-follower (BLMF) decision problem is a common case in any organizational decision practice, and involves various different decision situations.

Our previous work [17-20] presented overcame some fundamental deficiency of existing linear BLP theory. Based on that, we have recently generalized a framework for BLMF decision problems, and identified nine main kinds of relationships amongst these followers [21]. The uncooperative model is the most popular situation for BLMF decision problems. This model handles the case in which there is no shared decision variable among the followers. Under this uncooperative model, the most basic situation is that any follower also doesn't make any reference to other followers' decision. For a model and related approaches in finding an optimal solution for this particular decision situation, the reader is referred to [21-23]. Another such uncooperative situation is that though these followers are uncooperative (no sharing of decision variables) but have cross reference of information by considering other followers' decision results in each of their own decision

objective and constraint. We call this case as a referential-uncooperative situation, and this paper will particularly focus on this situation.

Following the introduction, this paper proposes a model for linear BLMF decision making in a referential-uncooperative situation in Section 2. An extended Kuhn-Tucker approach for solving this model is presented in Section 3. A numeric example for this approach is illustrated in Section 4. Further remarks are concluded in Section 5.

2. The linear BLMF decision model in a referential-uncooperative situation

Under the BLMF framework, if two followers don't have any shared decision variable, it is called an uncooperative relationship. But if one of them has a reference of another follower's decision information in his/her objective or constraint, the two followers are defined having a referential-uncooperative relationship. When there is a referential-uncooperative relationship in a BLMF model, this model is called a referential-uncooperative BLMF model. We present this model as follows.

For $x \in X \subset R^n$, $y_i \in Y_i \subset R^{m_i}$, $F : X \times Y_1 \times \ldots \times Y_K \to R^1$, and $f_i : X \times Y_1 \times \ldots \times Y_K \to R^1$, $i = 1,2,\ldots,K$, a linear BLMF decision problem where $K(\geq 2)$ followers are involved and there are not shared decision variables, but shared information in objective functions and constraint functions among the followers is defined as follows:

$$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s \tag{1a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leq b \tag{1b}$$

$$\min_{y_i \in Y_i} f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s \tag{1c}$$

$$\text{subject to } A_i x + \sum_{s=1}^{K} C_{is} y_s \leq b_i, \tag{1d}$$

where $c \in R^n$, $c_i \in R^n$, $d_i \in R^{m_i}$, $e_{is} \in R^{m_s}$, $b \in R^p$, $b_i \in R^{q_i}$, $A \in R^{p \times n}$, $B_i \in R^{p \times m_i}$, $A_i \in R^{q_i \times n}$, $C_{is} \in R^{q_i \times m_s}$, $i, s = 1,2,\ldots,K$.

Definition 1 A compact set is compact if every open cover of the entire space has a finite subcover. For example, $[a,b]$ is compact in R (the Heine-Borel theorem) [24].

Corresponding to (1), we give following basic definition.

Definition 2

(a) Constraint region:

$$S = \{(x, y_1, \ldots, y_K) \in X \times Y_1 \times \ldots \times Y_k, Ax + \sum_{s=1}^{K} B_s y_s \le b,$$

$$A_i x + \sum_{s=1}^{K} C_{is} y_s \le b_i, i = 1,2,\ldots,K\}.$$

The constraint region refers to all possible combinations of choices that the leader and followers may make.

(b) Projection of S onto the leader's decision space:

$$S(X) = \{x \in X : \exists y_i \in Y_i, Ax + \sum_{s=1}^{K} B_s y_s \le b, A_i x + \sum_{s=1}^{K} C_{is} y_s \le b_i, i = 1,2,\ldots,K\}.$$

(c) Feasible set for each follower $\forall x \in S(X)$:

$$S_i(x) = \{y_i \in Y_i : (x, y_1, \ldots, y_K) \in S\}.$$

The feasible region for each follower is affected by the leader's choice of x, and the allowable choices of each follower are the elements of S.

(d) Each follower's rational reaction set for $x \in S(X)$:

$$P_i(x) = \{y_i \in Y_i : y_i \in \arg\min[f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \ne i) : \hat{y}_i \in S_i(x)]\},$$

where $i = 1,2,\ldots,K$, $\arg\min[f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \ne i) : \hat{y}_i \in S_i(x)] =$ $\{y_i \in S_i(x) : f_i(x, y_1, \ldots, y_K) \le f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \ne i), \hat{y}_i \in S_i(x)\}$. The followers observe the leader's action and simultaneously react by selecting y_i from their feasible set to minimize their objective functions.

(e) Inducible region:

$$IR = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S, y_i \in P_i(x), i = 1,2,\ldots,K\}.$$

Thus in terms of the above notations, (1) can be written as

$$\min\{F(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in IR\} \tag{2}$$

We propose the following theorem to characterize the condition under which there is an optimal solution for (1).

Theorem 1 If S is nonempty and compact, there exists an optimal solution for a linear BLMFP problem.

Proof: Since S is nonempty, there exist a point $(x^*, y_1^*, \ldots, y_K^*) \in S$. Then, we have

$$x^* \in S(X) \neq \phi,$$

by Definition 2(b). Consequently, we have

$$S_i(x^*) \neq \phi,\ i = 1,2,\ldots,K,$$

by Definition 2(c). Because S is compact and Definition 2(d), we have

$$P_i(x^*) = \{y_i \in Y_i : y_i \in \arg\min[f_i(x^*, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \neq i) : \hat{y}_i \in S_i(x^*)]\}$$

$$= \{y_i \in Y_i : y_i \in \{y_i \in S_i(x^*) :$$

$$f_i(x^*, y_1, \ldots, y_K) \leq f_i(x^*, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \neq i), \hat{y}_i \in S_i(x^*)\}\} \neq \phi,$$

where $i = 1,2,\ldots,K$. Hence, there exists $y_i^0 \in P_i(x^*)$, $i = 1,2,\ldots,K$ such that $(x^*, y_1^0, \ldots, y_K^0) \in S$. Therefore, we have

$$IR = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S, y_i \in P_i(x), i = 1,2,\ldots,K\} \neq \phi,$$

by Definition 2(e). Because we are minimizing a linear function $\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s$ over IR, which is nonempty and bounded, an optimal solution to the linear BLMFP problem must exist. So the proof is completed.

3 An extended Kuhn-Tucker approach

Let write a linear programming (LP) as follows.

$$\min\ f(x) = cx$$

$$\text{subject to } Ax \geq b$$

$$x \geq 0,$$

where c is an n-dimensional row vector, b an m-dimensional column vector, A an $m \times n$ matrix with $m \leq n$, and $x \in R^n$.

Let $\lambda \in R^m$ and $\mu \in R^n$ be the dual variables associated with constraints $Ax \geq b$ and $x \geq 0$, respectively. Bard [2] gave the following proposition.

Proposition 1 A necessary and sufficient condition that (x^*) solves above LP is that there exist (row) vectors λ^*, μ^* such that (x^*, λ^*, μ^*) solves:

$$\lambda A - \mu = -c$$

$$Ax - b \geq 0$$

$$\lambda(Ax - b) = 0$$

$$\mu x = 0$$

$$x \geq 0, \lambda \geq 0, \mu \geq 0 .$$

Proof: (See reference [2] PP. 59-60)

Let $u_i \in R^p$, $v_i \in R^{q_1+q_2+\ldots+q_K}$ and $w_i \in R^{m_i}$ $(i = 1,2,\ldots,K)$ be the dual variables associated with constraints $(Ax + \sum_{s=1}^{K} B_s y_s \leq b)$, $(A^{'}x + \sum_{s=1}^{K} C_s^{'} y_s \leq b^{'})$ and $y_i \geq 0$ $(i = 1,\ldots,K)$, respectively, where $A^{'} = (A_1, A_2, \ldots, A_K)^T$, $C_i^{'} = (C_{i1}, C_{i2}, \ldots, C_{iK})^T$, $b^{'} = (b_1, b_2, \ldots, b_K)^T$. We have a following theorem.

Theorem 2 A necessary and sufficient condition that $(x^*, y_1^*, \ldots, y_K^*)$ solves the linear BLMFP problem (1) is that there exist (row) vectors $u_1^*, u_2^*, \ldots, u_K^*$, $v_1^*, v_2^*, \ldots, v_K^*$ and $w_1^*, w_2^*, \ldots, w_K^*$ such that $(x^*, y_1^*, \ldots, y_K^*, u_1^*, \ldots, u_K^*, v_1^*, \ldots, v_K^*, w_1^*, \ldots, w_K^*)$ solves:

$$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s \tag{3a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leq b \tag{3b}$$

$$A^{'}x + \sum_{s=1}^{K} C_s^{'} y_s \leq b^{'} \tag{3c}$$

$$u_i B_i + v_i C_i^{'} - w_i = -e_{ii} \tag{3d}$$

$$u_i(b - Ax - \sum_{s=1}^{K} B_s y_s) + v_i(b' - A'x - \sum_{s=1}^{K} C_s' y_s) + w_i y_i = 0 \tag{3e}$$

$$x \geq 0, y_j \geq 0, u_j \geq 0, v_j \geq 0, w_j \geq 0, j = 1,2,\ldots,K, \tag{3f}$$

where $i = 1,2\ldots,K$.

Proof:

(1) Let us get an explicit expression of (2).

Rewrite (2) as follows:

$$\min F(x, y_1,\ldots,y_K)$$

$$\text{subject to } (x, y_1,\ldots,y_K) \in IR.$$

We have

$$\min F(x, y_1,\ldots,y_K)$$

$$\text{subject to } (x, y_1,\ldots,y_K) \in S$$

$$y_i = P_i(x),$$

where $i = 1,2,\ldots,K$, by Definition 2(e). Then, we have

$$\min F(x, y_1,\ldots,y_K)$$

$$\text{subject to } (x, y_1,\ldots,y_K) \in S$$

$$y_i \in \arg\min[\, f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \neq i) : \hat{y}_i \in S_i(x)],$$

where $i = 1,2,\ldots,K$, by Definition 2(d). We rewrite it as:

$$\min F(x, y_1,\ldots,y_K)$$

$$\text{subject to } (x, y_1,\ldots,y_K) \in S$$

$$\min f_i(x, y_1,\ldots,y_K)$$

$$\text{subject to } y_i \in S_i(x),$$

where $i = 1,2,\ldots,K$. We have

$$\min F(x, y_1,\ldots,y_K)$$

$$\text{subject to } (x, y_1,\ldots,y_K) \in S$$

$$\min_{y_i \in Y_i} f_i(x, y_1,\ldots,y_K)$$

$$\text{subject to } (x, y_1,\ldots,y_K) \in S,$$

where $i = 1,2,\ldots,K$, by Definition 2(c). Consequently, we can have

$$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s \tag{4a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \le b \tag{4b}$$

$$A_j x + \sum_{s=1}^{K} C_{is} y_s \le b_i \text{ , } j = 1,2,\ldots,K \tag{4c}$$

$$\min_{y_i \in Y_i} f_i(x, y_i, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s \tag{4d}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \le b \tag{4e}$$

$$A_j x + \sum_{s=1}^{K} C_{is} y_s \le b_i \text{ , } j = 1,2,\ldots,K \text{ ,} \tag{4f}$$

where $i = 1,2,\ldots,K$, by Definition 2(a).

This simple transformation has shown that solving the linear BLMFP (1) is equivalent to solving (4a-f).

(2) Necessity is obvious from (4a-f).

(3) Sufficiency.

If $(x^*, y_1^*, \ldots, y_K^*)$ is the optimal solution of (1a-d), we need to show that there exist (row) vectors $u_1^*, u_2^*, \ldots, u_K^*$, $v_1^*, v_2^*, \ldots, v_K^*$ and $w_1^*, w_2^*, \ldots, w_K^*$ such that $(x^*, y_1^*, \ldots, y_K^*, u_1^*, \ldots, u_K^*, v_1^*, \ldots, v_K^*, w_1^*, \ldots, w_K^*)$ to solve (4a-f). Going one step farther, we only need to proof that there exist (row) vectors $u_1^*, u_2^*, \ldots, u_K^*$, $v_1^*, v_2^*, \ldots, v_K^*$ and $w_1^*, w_2^*, \ldots, w_K^*$ such that $(x^*, y_1^*, \ldots, y_K^*, u_1^*, \ldots, u_K^*, v_1^*, \ldots, v_K^*, w_1^*, \ldots, w_K^*)$ satisfies the follows

$$u_i B_i + v_i C_i^{'} - w_i = -e_{ii} \tag{5a}$$

$$u_i (b - Ax - \sum_{s=1}^{K} B_s y_s) = 0 \tag{5b}$$

$$v_i (b^{'} - A^{'} x - \sum_{s=1}^{K} C_s^{'} y_s) = 0 \tag{5c}$$

$$w_i y_i = 0 \tag{5d}$$

where $u_i \in R^p$, $v_i \in R^{q_1+q_2+\ldots+q_K}$, $w_i \in R^{m_i}$, $i = 1,2,\ldots,K$ and they are not negative variables. Because $(x^*, y_1^*, \ldots, y_K^*)$ is the optimal solution of (1a-d), we have

$$(x^*, y_1^*, \ldots, y_K^*) \in IR,$$

by (2). Thus we have

$$y_i^* \in P_i(x^*),$$

where $i = 1,2,\ldots,K$, by Definition 2(e). Consequently, $(y_1^*, y_2^*, \ldots, y_K^*)$ is the optimal solution to the following problem

$$\min(f_i(x^*, y_1, \ldots, y_K) : y_i \in S_i(x^*)),$$

where $i = 1,2,\ldots,K$, by Definition 2(d). Rewrite it as follows

$$\min f_i(x, y_1, \ldots, y_K)$$

$$\text{subject to } y_i \in S_i(x)$$

$$x = x^*$$

$$y_j = y_j^*, \; j = 1,2,\ldots,K, j \neq i,$$

where $i = 1,2,\ldots,K$. From Definition 2(c), we have

$$\min f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s \tag{6a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leq b \tag{6b}$$

$$A_j x + \sum_{s=1}^{K} C_{js} y_s \leq b_j, \; j = 1,2,\ldots,K \tag{6c}$$

$$x = x^* \tag{6d}$$

$$y_i \geq 0 \tag{6e}$$

$$y_j = y_j^*, j = 1,2,\ldots,K, j \neq i, \tag{6f}$$

where $i = 1,2,\ldots,K$. Let us define:

$A' = (A_1, A_2, \ldots, A_K)^{-1}$, $b' = (b_1, b_2, \ldots, b_K)^{-1}$, $C_i' = (C_{i1}, C_{i2}, \ldots, C_{iK})^{-1}$, $i = 1,2\ldots,K$. To simplify (6c), we can have

$$\min\ f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \le b$$

$$A' x + \sum_{s=1}^{K} C_s' y_s \le b'$$

$$x = x^*$$

$$y_i \ge 0,$$

$$y_j = y_j^*, j = 1,2,\ldots,K, j \ne i,$$

where $i = 1,2,\ldots,K$. Thus simplify it, we can have

$$\min\ f_i(y_i) = e_{ii} y_i \tag{7a}$$

$$\text{subject to } -\begin{pmatrix} B_i \\ C_i' \end{pmatrix} y_i \ge -\begin{pmatrix} b - Ax^* - \sum_{s=1, s\ne i}^{K} B_s y_s^* \\ b_i' - A' x^* - \sum_{s=1, s\ne i}^{K} C_s' y_s^* \end{pmatrix} \tag{7b}$$

$$y_i \ge 0 \tag{7c}$$

where $i = 1,2,\ldots,K$.

Now we see that y_i^* is the optimal solution of (7) which is a LP problem. By Proposition 1, there exists vector λ_i^*, μ_i^*, $i = 1,2,\ldots,K$ that satisfy a system below

$$\lambda_i \begin{pmatrix} B_i \\ C_i' \end{pmatrix} - \mu_i = -e_{ii}$$

$$\lambda_i (-\begin{pmatrix} B_i \\ C_i' \end{pmatrix} y_i + \begin{pmatrix} b - Ax^* - \sum_{s=1, s\ne i}^{K} B_s y_s^* \\ b' - A' x^* - \sum_{s=1, s\ne i}^{K} C_s' y_s^* \end{pmatrix}) = 0$$

$$\mu_i y_i = 0,$$

where $\lambda_i \in R^{p+q_1+\ldots+q_K}$, $\mu_i \in R^{m_i}$, $i = 1,2,\ldots,K$.

Let $u_i \in R^p$, $v_i \in R^{q_i+q_2+\ldots+q_K}$, $w_i \in R^{m_i}$ and define

$$\lambda_i = (u_i, v_i)$$

$$w_i = \mu_i,$$

where $i = 1,2,\ldots,K$. Thus we have $(x^*, y_1^*,\ldots,y_K^*, u_1^*,\ldots,u_K^*, v_1^*,\ldots,v_K^*, w_1^*,\ldots,w_K^*)$ that satisfy (8a-n). Our proof is completed.

Theorem 2 means that the most direct approach to solving (1a-d) is to solve the equivalent mathematical program given in (7a-c). One advantage that it offers is that it allows for a more robust model to be solved without introducing any new computational difficulties.

4. A numeric example

Example 1

Consider a following linear BLMF problem with $x_1, x_2 \in R^1$, $y_1, y_2 \in R^1$, $z \in R^1$ and $X = \{x_1 \geq 0, x_2 \geq 0\}$, $Y = \{y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\}$,

$$\min_{x_1 \in X, x_2 \in X} F(x_1, x_2, y_1, y_2, z) = -8x_1 - 4x_2 - 4y_1 + 40y_2 + 4y_3$$

subject to $2x_1 - y_1 + 2y_2 - 0.5y_3 \leq 1$

$$\min_{y_1 \in Y} f_1(x_1, x_2, y_1, y_2, z) = 2x_1 + x_2 + 2y_1 - y_2 - y_3$$

$$\min_{y_2 \in Y} f_2(x_1, x_2, y_1, y_2, z) = x_1 + 2x_2 - y_1 + 2y_2 - y_3$$

$$\min_{y_3 \in Y_3} f_3(x_1, x_2, y_1, y_2, z) = 3x_1 + 3x_2 + y_1 + y_2 - 2y_3$$

subject to $2x_2 + 2y_1 - y_2 - 0.5y_3 \leq 1$

$$-y_1 + y_2 + y_3 \leq 1.$$

Let us give Example 1 to show how the Kuhn-Tucker approach works. According to our approach, let us write all the inequalities but $x_1 \geq 0, x_2 \geq 0, y_3 \geq 0$ of the transferred form of Example 1 as follows:

$$g_{u,1}(x_1, x_2, y_1, y_2, y_3) = 1 - (2x_1 - y_1 + 2y_2 - 0.5y_3) \geq 0$$

$$g_{v,1}(x_1, x_2, y_1, y_2, y_3) = 1 - (2x_2 + 2y_1 - y_2 - 0.5y_3) \geq 0$$

$$g_{v,2}(x_1, x_2, y_1, y_2, y_3) = 1 - (-y_1 + y_2 + y_3) \geq 0$$

$$g_{w1,1}(x_1, x_2, y_1, y_2, y_3) = y_1 \geq 0$$

$$g_{w2,1}(x_1, x_2, y_1, y_2, y_3) = y_2 \geq 0$$

$$g_{w3,1}(x_1, x_2, y_1, y_2, y_3) = y_3 \geq 0 .$$

From (3a-f), we have

$$\min(-8x_1 - 4x_2 - 4y_1 + 40y_2 + 4y_3) \tag{8a}$$

$$\text{subject to } 2x_1 - y_1 + 2y_2 - 0.5y_3 \leq 1 \tag{8b}$$

$$2x_2 + 2y_1 - y_2 - 0.5y_3 \leq 1 \tag{8c}$$

$$-y_1 + y_2 + y_3 \leq 1 \tag{8d}$$

$$-u_{11} + 2v_{11} - v_{12} - w_{11} = -2 \tag{8e}$$

$$2u_{21} - v_{21} + v_{22} - w_{21} = -2 \tag{8f}$$

$$-0.5u_{31} - 0.5v_{31} + v_{32} - w_{31} = 2 \tag{8g}$$

$$g_{u,1}u_{11} + g_{v,1}v_{11} + g_{v,2}v_{12} + g_{w1,1}w_{11} = 0 \tag{8h}$$

$$g_{u,1}u_{21} + g_{v,1}v_{21} + g_{v,2}v_{22} + g_{w2,1}w_{21} = 0 \tag{8i}$$

$$g_{u,1}u_{31} + g_{v,1}v_{31} + g_{v,2}v_{32} + g_{w3,1}w_{31} = 0 \tag{8j}$$

$$x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0 \tag{8k}$$

$$u_{11} \geq 0, v_{11} \geq 0, v_{12} \geq 0, w_{11} \geq 0 \tag{8l}$$

$$u_{21} \geq 0, v_{21} \geq 0, v_{22} \geq 0, w_{21} \geq 0 \tag{8m}$$

$$u_{31} \geq 0, v_{31} \geq 0, v_{32} \geq 0, w_{31} \geq 0 . \tag{8n}$$

From (8e), (8f), (8g), (8l), (8m) and (8n), we can have following six possibilities.

Case 1: $(u_{11}, v_{11}, v_{12}, w_{11}, u_{21}, v_{21}, v_{22}, w_{21}, u_{31}, v_{31}, v_{32}, w_{31}) = (2,0,0,0,0,2,0,0,0,0,2,0)$

Case 2: $(u_{11}, v_{11}, v_{12}, w_{11}, u_{21}, v_{21}, v_{22}, w_{21}, u_{31}, v_{31}, v_{32}, w_{31}) = (0,0,2,0,0,2,0,0,0,0,2,0)$

Case 3: $(u_{11}, v_{11}, v_{12}, w_{11}, u_{21}, v_{21}, v_{22}, w_{21}, u_{31}, v_{31}, v_{32}, w_{31}) = (0,0,0,2,0,2,0,0,0,0,2,0)$

Case 4: $(u_{11}, v_{11}, v_{12}, w_{11}, u_{21}, v_{21}, v_{22}, w_{21}, u_{31}, v_{31}, v_{32}, w_{31}) = (2,0,0,0,0,0,0,2,0,0,2,0)$

Case 5: $(u_{11}, v_{11}, v_{12}, w_{11}, u_{21}, v_{21}, v_{22}, w_{21}, u_{31}, v_{31}, v_{32}, w_{31}) = (0,0,2,0,0,0,0,2,0,0,2,0)$

Case 6: $(u_{11}, v_{11}, v_{12}, w_{11}, u_{21}, v_{21}, v_{22}, w_{21}, u_{31}, v_{31}, v_{32}, w_{31}) = (0,0,0,2,0,0,0,2,0,0,2,0)$

From Case1, (8h), (8i), (8j) and (8k), we have

$$g_{u,1}(x_1, x_2, y_1, y_2, y_3) = 1 - (2x_1 - y_1 + 2y_2 - 0.5y_3) = 0$$

$$g_{v,1}(x_1, x_2, y_1, y_2, y_3) = 1 - (2x_2 + 2y_1 - y_2 - 0.5y_3) = 0$$

$$g_{v,2}(x_1, x_2, y_1, y_2, y_3) = 1 - (-y_1 + y_2 + y_3) = 0 .$$

Consequently, (8) can be rewritten as follows:

$$\min(-8x_1 - 4x_2 - 4y_1 + 40y_2 + 4y_3)$$

$$\text{subject to } 2x_1 - y_1 + 2y_2 - 0.5y_3 = 1$$

$$2x_2 + 2y_1 - y_2 - 0.5y_3 = 1$$

$$-y_1 + y_2 + y_3 = 1$$

$$x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0, z \geq 0 .$$

Using the simplex algorithm [2], we found that a solution occurs at the point $(x_1^1, x_2^1, y_1^1, y_2^1, y_3^1) = (1.5,0,1,0,2)$ with $F^1 = -8$, $f_1^1 = 3$, $f_2^1 = -1.5$ and $f_3^1 = 1.5$

By using the same way as that of Case 1, we have that a solution occurs at the point $(x_1^2, x_2^2, y_1^2, y_2^2, y_3^2) = (1.5,0,1,0,2)$ with $F^2 = -8$, $f_1^2 = 3$, $f_2^2 = -1.5$ and $f_3^2 = 1.5$ for Case 2; a solution occurs at the point $(x_1^3, x_2^3, y_1^3, y_2^3, y_3^3) = (0.75,0.75,0,0,1)$ with $F^3 = -5$, $f_1^3 = 1.25$, $f_2^3 = 1.25$ and $f_3^3 = 2.5$ for Case 3; a solution occurs at the point $(x_1^4, x_2^4, y_1^4, y_2^4, y_3^4) = (1.5,0,1,0,2)$ with $F^4 = -8$, $f_1^4 = 3$, $f_2^4 = -1.5$ and $f_3^4 = 1.5$ for Case 4; a solution occurs at the point $(x_1^5, x_2^5, y_1^5, y_2^5, y_3^5) = (1.5,0,1,0,2)$ with $F^5 = -8$, $f_1^5 = 3$, $f_2^5 = -1.5$ and $f_3^5 = 1.5$ for Case 5; a solution occurs at the point $(x_1^6, x_2^6, y_1^6, y_2^6, y_3^6) = (0.75,0.75,0,0,0)$ with $F^6 = -5$, $f_1^6 = 1.25$, $f_2^6 = 1.25$ and $f_3^6 = 2.5$ for Case 6.

By examining above procedure, we found that the solution occurs at the point $(x_1^*, x_1^*, y_1^*, y_2^*, z^*) = (1.5,0,1,0,2)$ with $F^* = -8$, $f_1^* = 3$, $f_2^* = -1.5$, $f_3^* = 1.5$ for this Example 1.

5. Further remark

Different relationships among followers in a BLMF decision problem could cause multiple different processes for deriving an optimal solution for the upper level's decision making. The referential-uncooperative situation is one of the popular cases in BLMF decision practices. For solving such a BLMF decision problem, this paper extended the Kuhn-Tucker approach from dealing with one leader and one follower to dealing with referential-uncooperative multiple followers. This paper further illustrated the details of the proposed approach by a numeric example. Initial experiment results showed this new extended approach more effectively for solving the proposed BLMF decision problem. Like most really powerful ideas, the basic notion of Nash equilibrium is very simple, even obvious. Its mathematical extensions and implications are not, however. The idea of this natural "sticking point" is that no single player can benefit from unilaterally changing his or her move - a non-cooperative best-response equilibrium [25]. As a future research, we are going to explore how use this concept into our BLMF research. Some practical use of this extended algorithm also will be considered as our future research task for BLMF decision making in the referential-uncooperative situation.

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