RAROC-Based Contingent Claim Valuation

King Ming Chan Wayne

Supervisor: Professor Erik Schlögl

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School of Finance and Economics

University of Technology, Sydney

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Certificate of Original Authorship

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ABSTRACT

The present dissertation investigates the valuation of a contingent claim based on the criterion RAROC, an abbreviation of Risk-Adjusted Return on Capital. RAROC is defined as the ratio of expected return to risk, and may therefore be regarded as a performance measure. RAROC-based pricing theory can indeed be considered as a subclass of the broader 'good-deal' pricing theory, developed by Bernardo and Ledoit (2000) and Cochrane and Saá-Requejo (2000). By fixing some specific target value of RAROC, a RAROC-based good-deal price for a contingent claim is determined as follows: upon charging the counterparty with this price and using available funds, we are able to construct a hedging portfolio such that the maximum achievable RAROC of our hedged position meets the target RAROC.

As a first step, we consider the standard Black-Scholes model, but allow only static hedging strategies. Assuming that the contingent claim in question is a call option, we examine the behavior of maximum value of RAROC as a function of initial call price, as well as the corresponding optimal static hedging strategy. In this analysis we consider two specifications for the risk component of RAROC, namely Value-at-Risk and Expected Shortfall.

Subsequently, we allow continuous-time trading strategies, while remaining in the Black-Scholes framework. In this case we suppose that the initial price of the call option is limited to be below the Black-Scholes price. Perfect hedging is thus impossible, and the position must contain some residual risk. For ease of analysis, we restrict our attention to a specific class of hedging strategies and examine the maximum RAROC for each strategy in this class. In the interest of tractability, the version of RAROC adopted risk is measured simply as expected loss.

While the previous approach only permits us to examine the constrained maximum RAROC over a specific class of hedging strategies, we would like to employ a more general method in order to study the global maximum RAROC over all hedging strategies. To do so, we introduce the notion of dynamic RAROC-based good-deal prices. In particular, with reference to the dynamic good-deal pricing theory of Becherer (2009), such prices are required to satisfy certain dynamic conditions, so that inconsistent decision-making between different times can be avoided. This task is accomplished by constructing prices that behave like time-consistent dynamic coherent risk measures. As soon as the construction process is finished, we set up a discrete time incomplete market, and demonstrate how to determine the dynamic RAROC-based good-deal price for a call option. Furthermore, by following Becherer (2009), we derive the dynamics of RAROC-based good-deal prices as solutions for discrete-time backward stochastic difference equations. Finally, we introduce RAROC-based good-deal hedging strategies, and examine their representation in terms of discrete-time backward stochastic difference equations.

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1. INTRODUCTION

Mathematical finance emerged from the important papers of Harrison and Kreps (1979) and Black and Scholes (1973). The first showed the relationship between arbitrage and martingales in asset pricing, while the second demonstrated the replication of a call option using two assets, a stock and a bond, derived the governing partial differential equation, and the Black-Scholes formula. Contingent claim valuation in *complete markets* has developed rapidly and there exists a rich literature on the pricing and hedging of exotic options and optimal portfolio selection. In a complete market every contingent claim can be replicated with a hedging portfolio constructed from assets in the market. The procedure for determining the price of a contingent claim in a complete market is basically as follows: Cashflows of the contingent claim and the hedging portfolio are first made to be identical, once an initial price differential between them is found, one can adopt the investment strategy of 'buy low, sell high' (buy the cheap and sell the expensive) to secure a riskless profit. So, in order to prevent such a scenario and be fair to both parties, the fair price of the claim should be determined as the value of this hedging portfolio. In the language of mathematical finance, this riskless trading profit is called an *arbitrage opportunity*[1]. Though the theory is interesting and apparently sound, the reality is unfortunately more involved. One cannot completely replicate a contingent claim most of the time with a suitable hedging portfolio that can produce its cashflows. Hence the previous valuation framework is not feasible in reality. Therefore, attention has been directed to contingent claim valuation in *incomplete markets*.

An incomplete market can be treated as a generalization of a complete market which removes the strong binding assumption of complete replication of any contingent claim. In this market, one is able to consider more sources of randomness to better describe and model a financial market in order to make the theoretical results more consistent with the observations. Examples of random factors that can be incorporated in an incomplete market are stochastic volatility of asset prices and the occurrence of jumps in price processes. With all these complications, it is not hard to understand that *not all* contingent claims can be hedged *perfectly*, hence we are facing the presence of non-zero residual risk whenever a contingent claim is traded. As a result, there is a requirement for a *risk management* policy in which people are concerned about the *right*

[1] A contingent claim V is called an arbitrage if (i) $P(V_0 = 0) = 1$; (ii) $P(V_T \geq 0) = 1$; (iii) $P(V_T > 0) > 0$.

measurement of risk exposure in a position. To sum up, we perceive a close relationship between incomplete markets and risk measurement.

As a rule of thumb, risk management is all about examining the *worst* event that would trigger, under some given time horizon, a loss more than some threshold at some confidence level. Any loss due to the occurrence of this event is commonly regarded as *unexpected loss*. In the event of an unexpected loss, the financial health of an institution can be seriously damaged, or it can be driven to bankruptcy. This brings us to the topic of *economic capital*, which is defined as the threshold stated previously, or, the *possible* amount of unexpected loss. It serves as a *loss buffer*, which prevents bankruptcy in the event of unexpected loss. A careful determination of the appropriate amount of economic capital is important. However there is no consistency or consensus in the approach/standard one should follow. This is because its calculation depends on *internal* assumptions and models, subject to a financial institution's own assessment. If we suppose that an optimal amount of economic capital is decided, then it will be reserved through the issuance of equity, i.e. the economic capital are obtained from new equity holders.

According to this flow of arguments, we apparently observe that there should be some interaction between contingent claim valuation and economic capital in an incomplete market. Therefore it motivates us to investigate and examine pricing of a contingent claim under the presence of economic capital, so that we might develop a pricing theory which is better in a practical sense.

1.1 Motivations and Objectives

Economic capital acts as a buffer against loss and plays a crucial role in the prevention of institutional failure. An institution can be so risk-averse that it reserves extraordinarily high levels of economic capital. Although this decreases the likelihood of bankruptcy or financial distress, it comes at a cost, since raising capital is costly. For instance, equity holders demand an appropriate return on the capital they supply. Secondly, liquidity in the market is not always sufficient enough to meet demand, thus limiting the amount of capital one can build up. Lastly, the return of an investment opportunity should be of finite magnitude and stochastic in nature, which creates the chance that, at maturity, the institution may not have adequate funds for providing the expected return on capital required by its financing body. All of these points discourage the accumulation of an unreasonably large amount of economic capital. However, regarding the opposite situation, a too low level of economic capital exposes a financial institution to bankruptcy. As a result, we expect that there is some *optimal* amount of economic capital for a contingent claim.

It is not hard to understand that the risk exposure after selling a contingent claim depends on the price that was charged. This is because the corresponding amount of cash is available for constructing a hedging portfolio. For the case of a high initial price, we may be able to superhedge a contingent claim, while, for a low initial price, only a partial hedge is formed, resulting in some residual risk exposure by the seller. This line of argument establishes a linkage: economic capital depends on risk and risk depends on initial price, so pricing of a contingent claim and economic capital should be coupled together.

Of course, we must also take account of compensation for capital funding costs when pricing and hedging a contingent claim. In other words, simply reserving economic capital for loss absorption is not completely satisfactory. Rather, *how to use economic capital in a more efficient manner* should be treated. Risk always exists no matter how one hedges against a contingent claim, and so building up economic capital is inevitable. However, some profit from the hedged position should also be possible to compensate the suppliers of economic capital.

To incorporate both the return and the risk of a contingent claim, we may make use of some performance measure to summarize the interaction between return and risk. For example, the ratio called Risk-Adjusted Return On Capital[2] ($RAROC$) can be chosen, which can be understood as reward per unit risk. All of these motivate us to investigate the following pricing problem:

> Suppose an arbitrary target value of $RAROC$, say $\overline{R}$, is fixed, how do we determine the price $C_0^{\overline{R}}$ for a contingent claim C such that there would exist a hedging strategy ϑ, and a hedging portfolio X^{ϑ}, under which the $RAROC$ that can be achieved by the hedged position $X^{\vartheta} - C$ is $\overline{R}$. Moreover, the price $C_0^{\overline{R}}$ should be optimal in the sense that, for any price below $C_0^{\overline{R}}$, there does not exist any hedging strategy that can lead to the desired value of $RAROC$ in the hedged position.

We shall regard the price $C_0^{\overline{R}}$ described above as the $RAROC$-based price for the contingent claim C.

1.2 Literature Review

In this section we will review relevant literature regarding some of the aspects discussed before, as well as the evolution of contingent claim pricing theory. Mathematical de-

[2]It is defined as the ratio of expected profit-and-loss to risk.

tails will be provided whenever the corresponding literature is relevant to later chapters.

Black and Scholes (1973) showed that, in case that profit from arbitrage is not allowed, one should evaluate a contingent claim with some *basis assets* (such as stocks and bonds) along with some replicating strategy. This would result in a single price for the contingent claim under consideration in the context of a complete market. In Harrison and Kreps (1979), Harrison and Pliska (1981) and Harrison and Pliska (1983), the authors developed the foundation work of martingale pricing theory, and in particular, it was shown that the existence of martingale measures is equivalent to absence of arbitrage opportunities under the assumption of completeness in the market. These papers initiated the modern theory of continuous-time contingent claim valuation using the mathematical machinery of stochastic calculus and stochastic integration. Unfortunately, there exist some contingent claims for which no replicating strategy can be found in the context of an incomplete market. In order to address valuation problems in this context, while maintaining the hypothesis of absence of arbitrage, El Karoui and Quenez (1995) derived the so-called no-arbitrage price bounds for contingent claims using stochastic control methods.

Sometimes it is not useful to value a contingent claim in terms of a price interval, since the interval can be unacceptably wide, see Merton (1973) and Britten-Jones and Neuberger (1996). Hence, one may ask for some other pricing methodology which could produce a 'better' price from a practical point of view, for instance, a single number instead of an interval. So, together with the replication-based approach, one might adopt the quadratic hedging technique for a contingent claim in an incomplete market. The desired hedging strategy in this theory is determined in such a way that the *residual risk* of a position is at *minimum*, where the magnitude of the residual risk is measured by a certain criterion. Specifically, the 'mean-variance' criterion was studied in Föllmer and Sondermann (1986), Duffie and Richardson (1991), Schweizer (2001) and Lim (2005), while the hedging strategy corresponding to 'local risk-minimization' was studied in Föllmer and Schweizer (1991), Chan (1999) and Schweizer (2001). Comparisons between these two hedging strategies were provided by Health and Platen (2001).

The price obtained from a replication or quadratic hedging argument is entirely independent of one's preference. It might be interesting to equip an investor with a *utility function*, and determine an optimal hedging strategy that way. Under the imposition of utility functions, contingent claim prices are *not preference-free*. Within the utility-indifference framework, one determines the price of a contingent claim by asking oneself at what price the maximum utility that can be achieved when an individual does not trade the contingent claim is the same as that after he takes a position in the claim, see

Henderson and Hobson (2009). The corresponding price is called the *utility-indifference price*. Hodges and Neuberger (1989) is the first paper that uses utility functions in contingent claim valuation. subsequently, a rich literature on utility-based pricing developed, for example, Hodges and Neuberger (1989), Musiela and Zariphopoulou (2004) and Grasselli and Hurd (2007), etc.

A more recent advance is the notion of quantile hedging, introduced in Föllmer and Leukert (1999), whose main idea was to construct the best hedging strategy in terms of probability of 'success' when perfect replication is not available. Here 'success' is defined as the event at maturity that the payoff of a hedging portfolio is equal to or greater than that of a contingent claim. The best hedging strategy is the one under which the corresponding hedging portfolio maximizes the probability of success. Since this concept is relevant to our later studies, we pause to describe its mathematical formulation.

Denote H_0 and ϑ as the initial funds at hand and a hedging strategy. The resulting value process of a self-financing hedging portfolio H, with initial price H_0, is defined by:

$$H_t = H_0 + \int_0^t \vartheta_u dS_u, \qquad \forall t \in [0, T].$$

Denote V as a contingent claim and call the set $\{H_T \geq V_T\}$ the 'success set', i.e. the scenario in which the hedging portfolio H can offer sufficient cashflows for meeting those from V. The task, as in quantile hedging, is about solving the following problem:

$$\max_{\vartheta} P(H_T \geq V_T) \quad \text{subject to} \quad H_0 \leq \tilde{V}_0$$

where $\tilde{V}_0$ represents the perfect-replication price in a complete market and the superhedging price in an incomplete market. Föllmer and Leukert (1999) showed how to solve this problem in complete and incomplete markets, by an application of Neyman-Pearson Lemma. In a similar fashion, Spivak and Cvitanić (1999) solved the same probability maximization problem using a duality approach.

Artzner et al. (1999) proposed the concept of 'coherence' that should be possessed by any 'reasonable' measure of risk. This leads to the notion of coherent risk measures. Föllmer and Leukert (2000) and Rudloff (2009) developed a valuation framework based on coherent risk, where some appropriate loss function or coherent risk measure is chosen as an objective function and the goal for an investor is to identify a hedging strategy that minimizes residual risk under that measure, subject to the constraint that initial funds are insufficient to set up a superhedging portfolio. This is obviously an optimization problem, and the Neyman-Pearson Lemma is again the key device for handling it, together with the use of randomized tests.

One might also resort to a risk measure in place of a utility function in the implementation of utility-based pricing theory discussed in the previous paragraph. This results in so-called risk-indifference pricing, which is analogous to utility-indifference pricing. It was studied by Xu (2006) and Øksendal and Sulem (2009). Loosely speaking, an investor tries to price a contingent claim such that the minimal risk exposure as measured by some chosen risk measure, is invariant with respect to buying or selling the claim. That is to say, the minimal risk for the agent without the contingent claim is the same as that with the contingent claim. Other relevant work in this regard are Scheemaekere (2008) and Scheemaekere (2009). Mathematically, by following Øksendal and Sulem (2009), we start with a given *convex risk measure* ρ and a set of portfolios $\mathcal{P}$. Denote $X_x^{\vartheta}(T)$ as the terminal wealth of an agent whose initial wealth is x, and who employs a strategy ϑ, then the *minimal risk* of the agent without a contingent claim V is

$$\Phi_0(x) = \inf_{\vartheta \in \mathcal{P}} \rho\big(X_x^{\vartheta}(T)\big).$$

If the agent receives an initial payment p for selling the claim V, then the minimal risk in this case is

$$\Phi_V(x+p) = \inf_{\vartheta \in \mathcal{P}} \rho\big(X_{x+p}^{\vartheta}(T) - V_T\big).$$

Now the agent's selling price V_0^{ask} for the claim V is the solution p of the following equation

$$\Phi_V(x+p) = \Phi_0(x).$$

In a similar fashion, one can derive the agent's buying price V_0^{bid} for the claim V.

We shall now briefly discuss research on coherent risk measures as well as their dynamic counterparts. By assuming a finite sample space, Artzner et al. (1999) first axiomized the so-called 'acceptance set', which takes account of terminal net worths agreed by regulators and then showed the correspondence between acceptance sets and measures of risk. These axioms, together with the definition of coherence, serve as the key features of risk measures used nowadays. While initial results assumed finite dimensional probability spaces, Delbaen (2002) extended to infinite probability spaces. Some well-defined coherent risk measures and their properties were introduced by Acerbi and Tasche (2002b), Acerbi and Tasche (2002a), Bertsimasa et al. (2004), Martin and Tasche (2007) and Acerbi et al. (2008), while Jorion (2007) approached the subject from a practical point of view. We should emphasize the fact the class of coherent risk measures discussed so far is static by nature. They have not been integrated with the evolution of information through time. In other words, if one attempts to apply these coherent risk measures recursively at different times, with a multi-period time horizon, one may run into inconsistent risk measurement and counter-intuitive conclusions. For example, a trade may be riskier tomorrow than today, which generates inconsistency in decision-making. For concrete examples of this problem, refer to Artzner et al. (2007). As a remedy, Artzner et al. (2007) suggested the use of Bellman's principle for the construction of a proper risk measure used dynamically. That is, if $\rho_t(X)$ denotes the risk of X, as measured by a risk measure ρ at time t, then for any $s < t$, the property of $\rho_s(X) = \rho_s(-\rho_t(X))$ should be satisfied. This leads to the definition of time-consistent dynamic coherent risk measures, see Delbaen (2006), Cheridito and Stadje (2009) and Cheridito and Kupper (2011) for more detail.

As soon as the magnitude of risk is suitably measured, one can allocate economic capital for protection against it. Optimal allocation of economic capital among portfolios or conglomerates or financial institutions is studied by Tasche (2004) and Mierzejewski (2008). For details on practical implementation and the calculation of economic capital, one may consult van Lelyveld (2006). However, the concept of economic capital rarely appears in the asset pricing literature before the recent introduction of the theory of 'good-deal pricing'. We shall discuss good-deal pricing at length now, because of its importance to our analysis.

The concept of no-arbitrage pricing is widely accepted. However, no-arbitrage bid-ask spreads are too wide when compared to those observed in the market. Consider pricing with a single-period time horizon. Becherer (2009) generated his results to a multi-period setting by introducing dynamic good-deal pricing theory. In this theory one can produce good-deal bid prices and ask prices for a contingent claim at different times, based on the available information at that moment. Moreover, when viewed from a time perspective, the good-deal bid and ask prices behave as time-consistent dynamic coherent risk measures. Other than good-deal prices, the corresponding good-deal hedging strategies are explored. It is shown that both the prices and hedging strategies are intimately related to backward stochastic differential equations. The theory of backward stochastic differential equations was initiated by Pardoux and Peng (1990), and has many applications in mathematical finance, see Karoui et al. (1997). Since good-deal pricing based on *RAROC* is discussed in Cherny (2008), we hope to use the work by Becherer (2009) to develop a theory of dynamic *RAROC*-based good-deal pricing.

1.3 Structure of the Thesis

The thesis consists of four chapters. The major problems we consider are, under a prescribed model,

Q1: At a given price charged for a contingent claim, what is the maximum *RAROC* of the hedged position, and how does it behave as a function of price?

Q2: What hedging strategy/portfolio allows the *RAROC* of the hedged position to attain its maximum?

Q3: For a given target *RAROC*, how should we determine the price of a claim dynamically?

In order to answer the above questions, the theme of each chapter is developed as follows:

1. In Chapter 2, we study the maximization of *RAROC* in a very simple setting. We assume that the market consists of a money market account offering a constant riskfree return and one tradeable risky asset, which follows a geometric Brownian motion. This is indeed the Black-Scholes model. However, in order to generate incompleteness, we additionally impose a restriction on the trading strategies one can use. That is, the market only allows for static trading strategies. Under these assumptions, we consider a European call option. We investigate the problem of maximizing *RAROC* from both the buyer's and the seller's perspectives. The maximum *RAROC* is also obtained as a function of the price of the claim.

2. In Chapter 3, we remove the limitation on trading strategies and permit continuous-time trading. As in Chapter 2, the Black-Scholes model is used for modeling the market and the contingent claim is a Europrean call option. In this market, if the option price is constrained to be less than the Black-Scholes price, the investor is not able to hedge perfectly, because of insufficient funds for setting up a perfect hedge. So, residual risk exists at maturity. We specifically choose a class of hedging strategies that can potentially lead to a maximum *RAROC*, and study the corresponding behavior as a function of option price.

3. In Chapter 4, we extend the notion of conditional expected shortfall. This is used in Chapter 5 to derive dynamic *RAROC*-based good-deal prices.

4. In Chapter 5, based on the results in Chapter 4, dynamic *RAROC*-based good-deal prices are derived, and are shown to behave as time-consistent dynamic coherent risk measures. Furthermore dynamic *RAROC*-based good-deal hedging strategies are introduced. Examples of how to determine such good-deal prices and good-deal hedging strategies in a one-period incomplete market model are demonstrated. Finally, the connections to backward stochastic difference equations[3] are established.

[3]More precisely, we mean a discrete-time backward stochastic differential equation.

2. STATIC RAROC MAXIMIZATION

In a complete financial market, one can hedge a contingent claim without the presence of residual risk. The standard Black-Scholes model provides an example of a complete market, in the sense that any claim written on the risky asset can be hedged perfectly, so long as continuous-time trading is possible. In reality, however, only discrete-time trading is possible. This introduces incompleteness into the Black-Scholes model, and means that contingent claims can no longer be hedged without incurring residual risk. This residual risk must be managed somehow. One possible approach is to reserve a certain amount of economic capital designed to reduce the impact of unhedgeable losses, and to prevent bankruptcy when it occurs. However, the providers of such capital will demand an acceptable return on their investment. To provide this return we require a hedging portfolio that can generate a profit, in addition to its hedging properties. The natural question then is how should we evaluate different hedging strategies, in order to choose the best one? To answer this question we require some measure of risk-adjusted return for hedging portfolios. The measure we shall use is risk-adjusted return on capital (RAROC).

This chapter considers the standard Black-Scholes model under the constraint that only static hedging strategies are permitted. Perfect hedging is therefore impossible, and only super-replicating strategies can eliminate all risk. Given an initial price for a European call option, we study the problem of determining the maximum RAROC that can be achieved by a static hedging portfolio. We seek also to identify the corresponding strategy. To make the problem interesting, we suppose that the initial price of the contract is less than the superhedging price, so that residual risk cannot be eliminated. We consider RAROC-maximizing static hedges for the call option, where risk is measured using value-at-risk and expected shortfall.

2.1 Background

Consider a filtered probability space $\big(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, P\big)$ satisfying the usual hypothesis, and let W be a standard Brownian motion defined on the space. We consider a financial market $\mathcal{M}$ containing two assets S and B, their prices are determined by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad \text{and} \quad \frac{dB_t}{B_t} = r dt.$$

We may regard S as the price of a risky stock, while B may be thought of as the value of a riskfree money-market account. Under the usual assumptions of continuous trading, no transaction costs, etc, this is a complete arbitrage-free market in continuous-time so long as $\mu \neq r$. Any contingent claim written on S can therefore be perfectly replicated by some self-financing trading strategy, provided that one can rebalance the portfolio dynamically, see Karatzas and Shreve (1998). In order to generate incompleteness, we shall allow only static hedging strategies for all market participants.

Let m_t be the number of units of S held in a self-financing portfolio at time t. The value of the corresponding portfolio is

$$H_t = m_t S_t + (H_t - m_t S_t) = m_t S_t + \frac{H_t - m_t S_t}{B_t} B_t,$$

where $\frac{H_t - m_t S_t}{B_t}$ is the number of units of the money-market account in the portfolio. If no rebalancing is carried out over $(t, t + \Delta t)$, the value of the portfolio at $t + \Delta t$ will be

$$H_{t+\Delta t} = m_t S_{t+\Delta t} + \frac{H_t - m_t S_t}{B_t} B_{\Delta t} = m_t S_{t+\Delta t} + (H_t - m_t S_t) e^{r\Delta t}.$$

A single-period or static strategy implies that $m_t = m_0 = m$ for all $t \geq 0$. Consequently, we have

$$H_t = m S_t + (H_0 - m S_0) e^{rt}.$$

Suppose now that the static strategy m is used to hedge a European call with a strike price K and expiry date T. If the option was initially sold for C_0, then the value of the hedged position at the maturity of the contract is

$$V_T := V(m, C_0, S_T) = m S_T + (C_0 - m S_0) e^{rT} - (S_T - K)^+. \tag{2.1.1}$$

Subject to the restriction of a single-period hedge, we are not able to replicate the option perfectly and it is impossible to have $V_T = 0$ P-a.s. In other words, V_T is a non-degenerate random variable. We can therefore use its distribution to define a risk measure ρ_α that expresses the risk exposure of the hedged position at some confidence level $\alpha \in (0, 1)$. One of the many candidates for ρ_α is value-at-risk VaR_α, which is defined by

$$\text{VaR}_\alpha(V_T) := \inf \big\{ x \in \mathbb{R} \;\big|\; P(V_T + x \leq 0) \leq \alpha \big\}. \tag{2.1.2}$$

In some sense, ρ_α can help us judge the risk-mitigation quality of a chosen hedging strategy. Intuitively, the best strategy should be the one that generates the lowest value for ρ_α, for a given α. In a similar vein, instead of determining the best risk-reducing hedge, we may compare hedging strategies on the basis of their risk-adjusted performance. We therefore require an appropriate measure of risk-adjusted performance. For this purpose we introduce risk-adjusted return on capital (RAROC), which we define as follows:

$$\text{RAROC}(V_T) := R(m, C_0; \rho_\alpha) := \frac{E^P[V_T]}{\rho_\alpha(V_T)}. \tag{2.1.3}$$

The RAROC of a hedging portfolio evidently measures its expected return per unit of risk. Using this criterion, the objective of a hedger who is faced with unhedgeable risk is to maximize the RAROC of his position. The advantage of RAROC maximization over simple risk minimization is the possibility of more efficient use of capital, since RAROC takes into account the return generated from a hedging strategy along with its associated risk. From now on we shall focus on the problem of achieving the most effective use of economic capital when pricing and hedging a contingent claim, rather than simply risk minimization. More precisely, we wish to determine

$$R^* := \sup_m R(m, C_0; \rho_\alpha) \qquad \text{and} \qquad m^* := \arg\max_m R(m, C_0; \rho_\alpha),$$

for a given initial price C_0 for the European call option under consideration.

2.2 RAROC Maximization with Value-at-Risk

Suppose the risk measure ρ_α used in the definition of RAROC is value-at-risk VaR_α. Let us assume that the hedger may not buy or sell more than one unit of the risky asset. In other words, we suppose that $m \in [0, 1]$.

2.2.1 The Seller's Position

For fixed m, C_0, S_0, examples of the terminal value V_T, as a function of S_T, are illustrated in Figure 2.1.

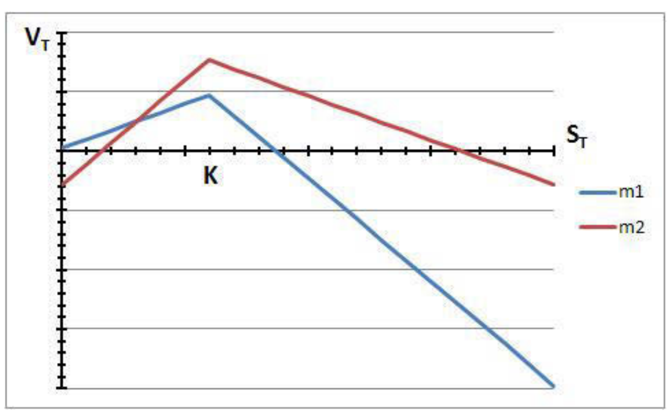


Fig. 2.1: Examples of V_T against S_T $(0 < m_1 < m_2 < 1)$

As seen in Figure 2.1, subject to the value of m, negative values of V_T are not restricted to the high asset price region, but can also occur in the low asset price region. More precisely, V_T starts to be negative as soon as S_T reaches some level $\overline{S}_T > K$ or some level $\underline{S}_T < K$. Here, $\overline{S}_T > K$ refers to the situation when the call option is in-the-money (ITM), while $\underline{S}_T < K$ refers to that when the call option is out-of-the-money (OTM). The values of $\overline{S}_T$ and $\underline{S}_T$ can be easily determined by solving $V(m, \overline{S}_T) = 0$ and $V(m, \underline{S}_T) = 0$, yielding

$$\begin{aligned} V(m, \overline{S}_T) &= m\overline{S}_T + (C_0 - mS_0)e^{rT} - (\overline{S}_T - K)^+ \\ &= m\overline{S}_T + (C_0 - mS_0)e^{rT} - (\overline{S}_T - K) = 0 \\ \Longrightarrow \quad \overline{S}_T = \overline{S}_T(m, C_0) &= \frac{(C_0 - mS_0)e^{rT} + K}{1-m}, \end{aligned}$$

and

$$\begin{aligned} V(m, \underline{S}_T) &= m\underline{S}_T + (C_0 - mS_0)e^{rT} - (\underline{S}_T - K)^+ \\ &= m\underline{S}_T + (C_0 - mS_0)e^{rT} = 0 \\ \Longrightarrow \quad \underline{S}_T = \underline{S}_T(m, C_0) &= \frac{(mS_0 - C_0)e^{rT}}{m}. \end{aligned}$$

Finally, for any $m \in (0, 1)$, $C_0 \in (0, S_0)$ and $S_0 \in \mathbb{R}_+$, we have

$$V(m, S_T) < 0 \qquad \text{if and only if} \qquad S_T < \underline{S}_T \text{ or } S_T > \overline{S}_T. \tag{2.2.1}$$

In words, a loss in the hedged position occurs when the asset price breaches these two thresholds. Moreover, this situation of loss remains valid as long as the inequality $\underline{S}_T < K < \overline{S}_T$ is satisfied. Moreover, for a fixed S_0, if m and C_0 are chosen such that $\underline{S}_T < K$, then

$$\overline{S}_T = \frac{(C_0 - mS_0)e^{rT} + K}{1-m} = \frac{-m\underline{S}_T + K}{1-m} = K + \frac{m}{1-m}(K - \underline{S}_T),$$

which guarantees that $K < \overline{S}_T$. So the condition $\underline{S}_T < K < \overline{S}_T$ is equivalent to either $\underline{S}_T < K$ or $K < \overline{S}_T$.

Given $\alpha \in (0, 1)$, we define the lower and upper α-quantiles of the risky asset price as follows, see Föllmer and Schied (2002b).

$$\underline{S}_{T,\alpha} := \sup \{ x \in \mathbb{R} \mid P(S_T < x) < \alpha \} \quad \text{and} \tag{2.2.2}$$

$$\overline{S}_{T,\alpha} := \inf \{ x \in \mathbb{R} \mid P(S_T \leq x) > 1 - \alpha \} \tag{2.2.3}$$

Typically, for $\alpha \ll 0.5$, we have $\underline{S}_{T,\alpha} < S_0 e^{\mu T} < \overline{S}_{T,\alpha}$.

Due to the simplicity of the structure of the hedged position, we can express VaR_α as a function of the α-quantile of the asset price. That is to say, for a given α, VaR_α can be expressed as $\mathrm{VaR}_\alpha = V(m, \underline{S}_{T,\alpha})$ or $\mathrm{VaR}_\alpha = V(m, \overline{S}_{T,\alpha})$ depending on whether we are analyzing the situation from the seller's or buyer's perspective. Note that it may not be possible to do so under multi-period hedging. Also note that, due to (2.2.1), if $\underline{S}_{T,\alpha} > \underline{S}_T$ and $\overline{S}_{T,\alpha} < \overline{S}_T$ then $\mathrm{VaR}_\alpha^A \leq 0$, where the VaR_α^A denotes VaR_α of the seller's position. It is because in this situation the probability of loss is smaller than α, hence, by the definition of VaR_α in (2.1.2), a negative value of VaR_α results. Moreover, if the strike of the European call option satisfies $K \geq \overline{S}_{T,\alpha}$, then $\overline{S}_T > \overline{S}_{T,\alpha}$, for some values of m, whence $\mathrm{VaR}_\alpha^A \leq 0$. We shall neglect situations that produce negative values for VaR_α^A.

Assume $\overline{S}_{T,\alpha} \geq \overline{S}_T$, leading to $\mathrm{VaR}_\alpha^A > 0$. First we obtain the maximum value of m such that the loss happens in the high price region, which we denote by $\overline{m}$:

$$\begin{aligned} & \overline{m}\overline{S}_{T,\alpha} - (\overline{S}_{T,\alpha} - K)^+ + (C_0 - \overline{m}S_0)e^{rT} = (C_0 - \overline{m}S_0)e^{rT} \\ \Longrightarrow \quad & \overline{m} := \frac{\overline{S}_{T,\alpha} - K}{\overline{S}_{T,\alpha}} = 1 - \frac{K}{\overline{S}_{T,\alpha}}. \end{aligned}$$

Then, for $m \in [0, \overline{m}]$, VaR_α^A can be calculated as follows:

$$\mathrm{VaR}_\alpha^A(V_T) = V(m, \overline{S}_{T,\alpha}) = -\big(m\overline{S}_{T,\alpha} - (\overline{S}_{T,\alpha} - K) + (C_0 - mS_0)e^{rT}\big). \tag{2.2.4}$$

In the case where $m \in (\overline{m}, 1)$, VaR_α^A should be obtained from its definition. Setting $x := \mathrm{VaR}_\alpha^A$, we must solve

$$\begin{aligned} & mS_T - (S_T - K)^+ + (C_0 - mS_0)e^{rT} + x \leq 0 \\ \Longleftrightarrow \quad & \begin{cases} mS_T + (C_0 - mS_0)e^{rT} + x \leq 0 & \text{for} \quad S_T \leq K \\ mS_T - (S_T - K) + (C_0 - mS_0)e^{rT} + x \leq 0 & \text{for} \quad S_T > K \end{cases} \\ \Longleftrightarrow \quad & \begin{cases} S_T \leq \frac{(mS_0 - C_0)e^{rT}}{m} - \frac{x}{m} & \text{for} \quad S_T \leq K \\ S_T \geq \frac{(C_0 - mS_0)e^{rT} + K}{1-m} + \frac{x}{1-m} & \text{for} \quad S_T > K \end{cases} \\ \Longleftrightarrow \quad & S_T \leq \left(\underline{S}_T - \frac{x}{m}\right) \wedge K \quad \text{or} \quad S_T \geq \left(\overline{S}_T + \frac{x}{1-m}\right) \vee K. \end{aligned}$$

Since we have assumed that $\mathrm{VaR}_\alpha^A = x > 0$, $m \in (\overline{m}, 1)$ and $\underline{S}_T < K < \overline{S}_T$, it follows from the inequalities above that

$$S_T \leq \underline{S}_T - \frac{x}{m} \quad \text{or} \quad S_T \geq \overline{S}_T + \frac{x}{1-m}.$$

Since these two events are mutually exclusive, we obtain

$$\begin{aligned}
&P\Big(mS_T - (S_T - K)^+ + (C_0 - mS_0)e^{rT} + x \leq 0\Big) \\
&= P\Big(S_T \leq \underline{S}_T - \frac{x}{m}\Big) + P\Big(S_T \geq \overline{S}_T + \frac{x}{1-m}\Big) \\
&= \Phi\left(\frac{\ln \frac{\underline{S}_T - \frac{x}{m}}{S_0} - \big(\mu - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right) + 1 - \Phi\left(\frac{\ln \frac{\overline{S}_T + \frac{x}{1-m}}{S_0} - \big(\mu - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable. Computing VaR_α^A therefore amounts to solving for x in

$$\Phi\left(\frac{\ln \frac{\underline{S}_T - \frac{x}{m}}{S_0} - \big(\mu - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right) + 1 - \Phi\left(\frac{\ln \frac{\overline{S}_T + \frac{x}{1-m}}{S_0} - \big(\mu - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right) = \alpha. \tag{2.2.5}$$

Instead of finding the analytical solution, we shall describe an approximation in the next section.

2.2.2 Approximating VaR_α^A

We begin by introducing the following quantities:

$$\overline{\phi} := \phi\left(\frac{\ln \frac{\overline{S}_T + \frac{x}{1-m}}{S_0} - \big(\mu - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right) \quad \text{and} \quad \underline{\phi} := \phi\left(\frac{\ln \frac{\underline{S}_T - \frac{x}{m}}{S_0} - \big(\mu - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right)$$

where $\phi(\cdot)$ is the probability density function of a standard normal random variable. Noting that $x \equiv x(m, C_0)$, we apply the Implicit Function Theorem and differentiate

(2.2.5) with respect to m. This gives

$$\frac{\underline{\phi}}{m\underline{S}_T - x}\left(\frac{C_0e^{rT}+x}{m} - \frac{\partial x}{\partial m}\right) - \frac{\overline{\phi}}{(1-m)\overline{S}_T + x}\left(\frac{K+(C_0-S_0)e^{rT}+x}{1-m} + \frac{\partial x}{\partial m}\right) = 0$$

$$\Longrightarrow \quad \frac{\partial x}{\partial m} = \frac{\dfrac{C_0e^{rT}+x}{m}\dfrac{\underline{\phi}}{m\underline{S}_T - x} - \dfrac{K+(C_0-S_0)e^{rT}+x}{1-m}\dfrac{\overline{\phi}}{(1-m)\overline{S}_T+x}}{\dfrac{\underline{\phi}}{m\underline{S}_T - x} + \dfrac{\overline{\phi}}{(1-m)\overline{S}_T+x}}$$

$$= \frac{\underline{\phi}(C_0e^{rT}+x)(1-m)\big((1-m)\overline{S}_T + x\big) - \overline{\phi}m\big(K+(C_0-S_0)e^{rT}+x\big)(m\underline{S}_T - x)}{\underline{\phi}m(1-m)\big((1-m)\overline{S}_T+x\big) + \overline{\phi}m(1-m)(m\underline{S}_T - x)}. \tag{2.2.6}$$

Similarly, applying the Implicit Function Theorem and differentiating (2.2.5) with respect to C_0, we get

$$\frac{\partial x}{\partial C_0} = -e^{rT}. \tag{2.2.7}$$

We therefore obtain the following expression for the desired Value-at-Risk:

$$x(m, C_0) = f(m) - C_0e^{rT},$$

where f is some (complicated) function. Seeking a suitable f that satisfies both (2.2.6) and (2.2.7) is not trivial. However, we can obtain a numerical solution for (2.2.6) in order to gain some insights into the approximation $\widehat{x}$ of the true solution x. The reason for us to approximate the true Value-at-Risk is because in later stages we have to determine the optimal hedge ratio m such that the RAROC of the hedged position would meet the target RAROC. Without explicit expression of VaR, it is very challenging to perform the corresponding optimization when determining the optimal hedge ratio. Hence, it is better to obtain a nice approximation of VaR to avoid this difficulty.

Using finite difference methods, the numerical solution of $\frac{\partial x}{\partial m}$ in (2.2.6) is shown in the figure below, where the real-world probability P is assumed to be the risk-neutral probability Q, hence, the drift μ being the same as the riskfree rate r:

Parameters: $r = 0.02, \mu = 0.02, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

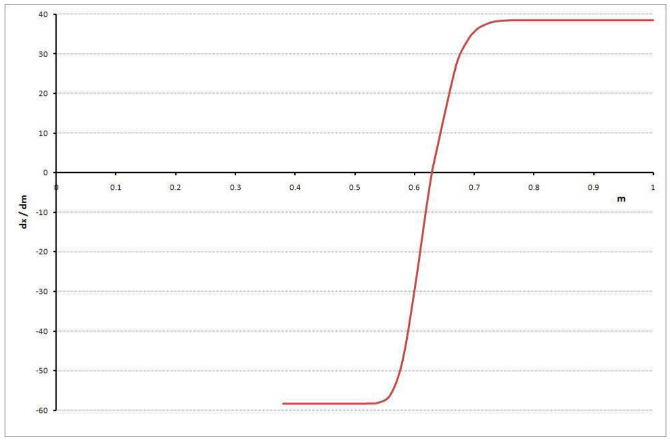


Fig. 2.2: Numerical solution of $\frac{\partial x}{\partial m}$ in (2.2.6) under $P = Q$ and $\mu = r$

The numerical solution of $\frac{\partial x}{\partial m}$ resembles the function tanh. So we propose the following transformation of tanh to serve as the approximation of the LHS in (2.2.6):

$$\frac{\partial \widehat{x}}{\partial m} = \theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4.$$

The approximation $\widehat{\mathrm{VaR}}_\alpha^A$ can be obtained easily by integrating the above, yielding

$$\widehat{\mathrm{VaR}}_\alpha^A = \widehat{x} = \frac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT} \tag{2.2.8}$$

where $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ are independent of m.

It remains to determine appropriate values for $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$. To meet this goal, we impose the following boundary conditions

$$\begin{aligned}
\mathrm{VaR}_\alpha^A\big|_{m=0} &= \lim_{m\downarrow 0}\Big(\frac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}\Big) \\
\mathrm{VaR}_\alpha^A\big|_{m=\overline{m}} &= \lim_{m\downarrow \overline{m}}\Big(\frac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}\Big) \\
\mathrm{VaR}_\alpha^A\big|_{m=1} &= \lim_{m\uparrow 1}\Big(\frac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}\Big) \\
\frac{\partial x}{\partial m}\Big|_{m=0} &= \lim_{m\downarrow 0}\Big(\theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4\Big) \\
\frac{\partial x}{\partial m}\Big|_{m=\overline{m}} &= \lim_{m\downarrow \overline{m}}\Big(\theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4\Big) \\
\frac{\partial x}{\partial m}\Big|_{m=1} &= \lim_{m\uparrow 1}\Big(\theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4\Big),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{VaR}_\alpha^A\big|_{m=0} &= -(K - \overline{S}_{T,\alpha} + C_0 e^{rT}) \\
\mathrm{VaR}_\alpha^A\big|_{m=\overline{m}} &= -(C_0 - \overline{m} S_0) e^{rT} \\
\mathrm{VaR}_\alpha^A\big|_{m=1} &= -(\underline{S}_{T,\alpha} + (C_0 - S_0) e^{rT}) \\
\frac{\partial x}{\partial m}\Big|_{m=0} &= -(\overline{S}_{T,\alpha} - S_0 e^{rT}) < 0 \\
\frac{\partial x}{\partial m}\Big|_{m=\overline{m}} &= -(\overline{S}_{T,\alpha} - S_0 e^{rT}) < 0 \\
\frac{\partial x}{\partial m}\Big|_{m=1} &= -(\underline{S}_{T,\alpha} - S_0 e^{rT}) > 0.
\end{aligned}$$

Clearly there are six boundary conditions, and it suffices to select five for the determination of $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$. It is strongly advised to include the four boundary conditions[1] at $m = \overline{m}$ and $m = 1$ in order to get a more consistent $\widehat{\mathrm{VaR}}_\alpha^A$.

The comparison between $\widehat{\mathrm{VaR}}_\alpha^A$ and VaR_α^A in the case when $\mu = r$ is illustrated in Figure 2.3 below. We have used the four boundary conditions at $m = \overline{m}$ and $m = 1$ and $\frac{\partial x}{\partial m}\big|_{m=1}$ for finding the values of $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$. The solid line represents $\widehat{\mathrm{VaR}}_\alpha^A$ while the dots represent VaR_α^A, whose values were obtained by solving (2.2.5) numerically.

Parameters: $r = 0.02, \mu = 0.02, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

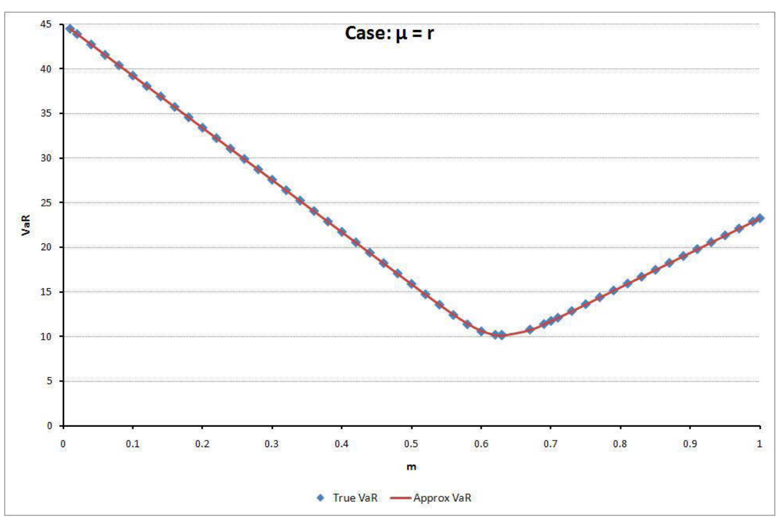


Fig. 2.3: Comparison of VaR_α^A and $\widehat{\mathrm{VaR}}_\alpha^A$ under $P = Q$ and $\mu = r$

In the case when $\mu \neq r$, analogous results are shown in Figure 2.4.

[1]Because the position and shape of $\widehat{\mathrm{VaR}}_\alpha^A$ are relatively more sensitive to these inputs.

Parameters: $r = 0.02, \mu = 0.08, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

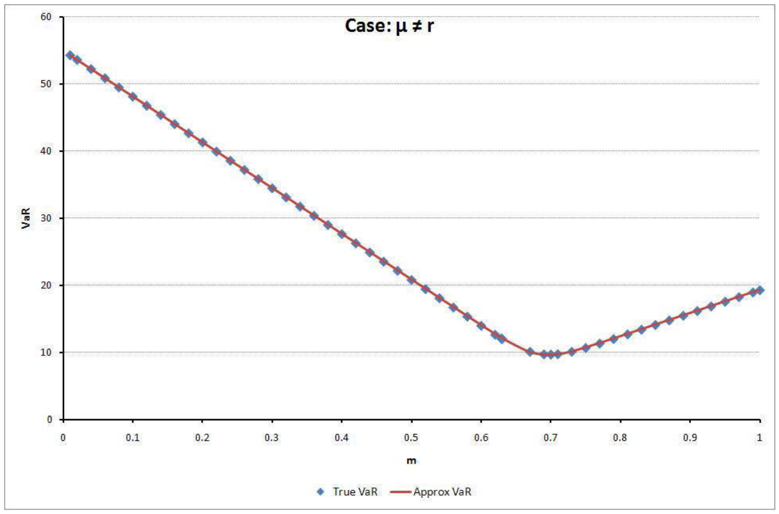


Fig. 2.4: Comparison of VaR_α^A and $\widehat{\mathrm{VaR}}_\alpha^A$ under $P \neq Q$ and $\mu \neq r$

Without further assumptions, the five parameters $\theta_1, \ldots, \theta_5$ allow too many degrees of freedom in specifying $\widehat{\mathrm{VaR}}_\alpha^A$. In order to facilitate later analysis, we shall focus on some specific properties of $\widehat{\mathrm{VaR}}_\alpha^A$. We suppose the values of $\theta_1, \ldots, \theta_5$ are determined such that

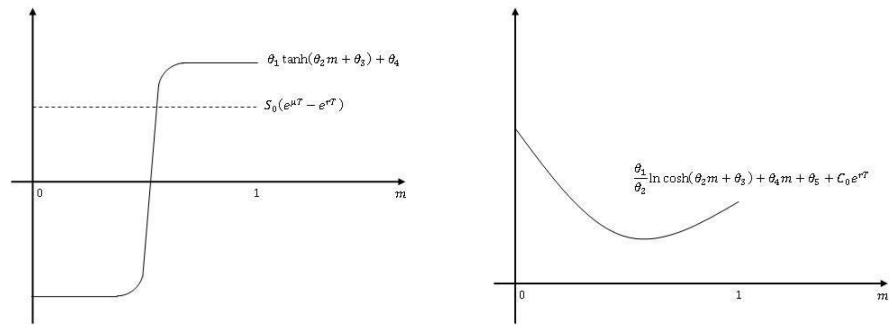
i. the function $\theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4$ resembles the function in Figure 2.2, in particular, $\theta_1 > 0, \theta_2 > 0, \theta_3 < 0, \theta_4 < 0$,

ii. there exists one unique root of $m \in [0, 1]$ for the equation $\theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4 = 0$, or, equivalently, $\left|\frac{\theta_4}{\theta_1}\right| < 1$,

iii. $S_0(e^{\mu T} - e^{rT}) \leq |\theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4|$ for all $m \in [0, 1]$,

iv. the function $\frac{\theta_1}{\theta_2} \ln \cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}$ is qualitatively similar to those in Figure 2.3 and Figure 2.4.

The following diagrams summarize the behavior described above:



(a) $\theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4$ (b) $\frac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}$

Fig. 2.5: Assumptions on VaR profile

2.2.3 Maximum RAROC under $P = Q$ and $\mu = r$

For the sake of simplicity, we assume that $P = Q$ and $\theta_1, \ldots, \theta_5$ in (2.2.8) are already determined. The expectation of V_T in (2.1.1) can then be written as

$$E^P[V_T] = E^Q[V_T] = \left(C_0 - C_0^{BS}\right)e^{rT}$$

where $C_0^{BS} := S_0\Phi(d_1) - Ke^{rT}\Phi(d_2)$ is the standard Black-Scholes formula for a call option. Combining with (2.2.8), the RAROC is expressed as

$$R \equiv R(m, C_0) = \frac{\left(C_0 - C_0^{BS}\right)e^{rT}}{\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}} \tag{2.2.9}$$

The initial price C_0 charged by the seller is called the ask-price. If we assume that the ask-price is fixed, then the seller aims to maximize RAROC, and the optimal static hedge m^* is defined as the maximizer of $R(m, C_0)$:

$$m^* = \arg\max_m R(m, C_0)$$

We introduce the following notation:

R^* : the maximum value of R, which is $R(m^*, C_0)$
$C_0^{A,-}$: the value of C_0 below which a negative and finite value of R^* would result
$C_0^{A,0}$: the value of C_0 above which a positive and finite value of R^* would result
$C_0^{A,\infty}$: the value of C_0 above which an infinite value of R^* would result
$m^*(C_0)$: the value of m such that a positive and finite value of R^* is achieved for a given C_0
$m^\infty(C_0)$: the value of m such that an infinite value of R^* is achieved for a given C_0

Proposition 2.2.1. *Given the assumptions on the VaR profile in Section 2.2.2, there exist values of* $C_0^{A,-}$,$C_0^{A,\infty}$ *and* $m^*(C_0)$, $m^\infty(C_0)$ *such that:*

i. When $C_0 < C_0^{A,-}$, $m^* = 0$ *and* $R^* = R(0, C_0) < 0$.

ii. When $C_0^{A,-} \leq C_0 < C_0^{A,\infty}$, $m^* = m^*(C_0) \in [0, 1]$ *and* $R^* = R(m^*(C_0), C_0) \geq 0$.

iii. When $C_0 \geq C_0^{A,\infty}$, $m^* = m^\infty(C_0) \in [0, 1]$ *and* $R^* = R(m^\infty(C_0), C_0) = +\infty$.

Proof. Refer to Appendix 2.5 for the justification as well as the characterization of $C_0^{A,-}$, $C_0^{A,\infty}$, $m^*(C_0)$, $m^\infty(C_0)$. As an illustration, one may refer to Figure 2.6 to understand the locations of $C_0^{A,-}, C_0^{A,0}, C_0^{A,\infty}$. □

Remark 2.2.3.1. When the ask-price C_0 is too low (less than $C_0^{A,-}$), the seller cannot attain a positive RAROC. In that case the maximum value R^* is attained by holding zero units of the underlying and thus leaving the position unhedged. At first sight, such an optimal hedging portfolio violates our intuition. The reason is that we permit R to take negative values. When C_0 is too low, expected profit-and-loss from the hedged position is negative. In such cases, maximizing R is tantamount to maximizing the denominator part, which is VaR. In other words, we are instructed to *maximize the risk* in order to improve RAROC. Of course one may exclude the prices which cannot lead to positive expected profits so that such a 'weird' observation is eliminated, for instance.

Remark 2.2.3.2. Indeed, in the context of $P = Q$, one can easily identify $C_0^{A,-}$ as C_0^{BS} because we are evaluating the profit of the hedged position by making use of the risk-neutral measure Q. In a risk-neutral world, the drift of the underlying asset S is the riskfree rate and the expected future price is the forward price $S_0 e^{rT}$, so if we borrow from the money market account to buy a unit of S, there should be no profit-and-loss *on average*[2] at maturity. On the one hand, upon selling for C_0, we can invest the whole premium in the money-market account and earn the riskfree rate. In a risk-neutral world, the expected future value of the European call is $C_0^{BS} e^{rT}$. Eventually the profit-and-loss at maturity is merely $(C_0 - C_0^{BS})e^{rT}$. So if $C_0 < C_0^{BS}$ the seller suffers a loss. The seller cannot hedge perfectly by means of a *static hedge*, which implies a positive risk as measured by VaR_α. A negative value for R follows from this.

Remark 2.2.3.3. For the case of $C_0 \in [C_0^{A,-}, C_0^{A,\infty})$, a profit on the hedged position can be expected because the difference $C_0 - C_0^{BS}$ is now positive. Since this is deterministic and independent of the underlying, the holding of the underlying only affects the risk as measured by VaR, and not the return. As a result, the seller only needs to minimize the risk in order to maximize RAROC. Without a sufficiently large ask-price, a non-zero risk exposure results, leading to a finite value of R^*.

[2] Of course this is definitely wrong in a pointwise manner, i.e. for each fixed $\omega \in \Omega$.

Remark 2.2.3.4. Explosion of R^* is observed even if the seller does not superhedge because he can hedge by holding m^∞ units of the underlying, so that the VaR_α of the hedged position is zero although the position is not completely riskfree. More precisely, when $C_0 \geq C_0^{A,\infty}$, holding m^∞ units of the underlying would guarantee $\overline{S}_{T,\alpha} = \overline{S}_T$, so according to (2.2.1), we would have a zero VaR_α for the hedged position. Together with a positive expected profit, we would end up with an infinite value of R^*.

Remark 2.2.3.5. As we compute the expected profit in *risk-neutral* manner, we ignore any information about the *real* drift of the underlying. This leads to the result that the expected profit-and-loss is not a function of the units of holding of the underlying. Consequently, the optimal static hedging strategy m^* for maximizing RAROC is determined as the minimizer of risk. This is equivalent to saying that the notion of the RAROC-maximizing hedge coincides with the risk-minimizing hedge as long as the expected profit is computed in a risk-neutral sense.

The diagram below demonstrates the behavior of the maximum RAROC for different values of the initial ask-price C_0.

Parameters: $r = 0.02, \mu = 0.02, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

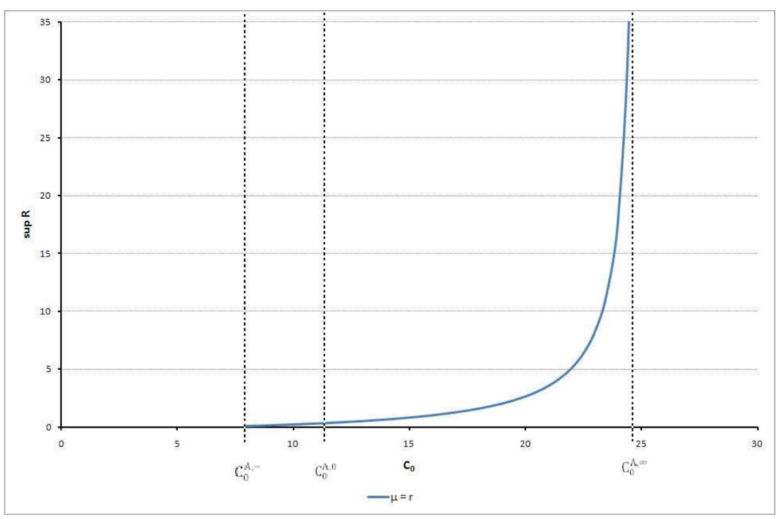


Fig. 2.6: R^* when $\mu = r$ and risk measurement in VaR_α

2.2.4 *Maximum RAROC under $P \neq Q$ and $\mu > r$*

A more realistic study involves the use of the real-world probability measure P. This also means that we no longer have the Black-Scholes pricing formula for the European call option because $E^P[C_T] \neq E^Q[C_T]$. In this case we adapt the formula for $E^P[C_T]$

in Rubinstein (1976), which gives a Black-Scholes-like pricing formula

$$E^P[C_T] = S_0 e^{\mu T}\Phi(d_1) - K\Phi(d_2) =: C_0^{bs},$$

where $d_1 := \frac{\ln\frac{S_0}{K}+\left(\mu+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_2 := d_1 - \sigma\sqrt{T}$. Then we have

$$R = R(m, C_0) = \frac{mS_0 e^{\mu T} - C_0^{bs} + (C_0 - mS_0)e^{rT}}{\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}}. \tag{2.2.10}$$

The associated maximum value and the optimal static hedging strategy are given by the following result.

Proposition 2.2.2. *Given the assumptions on the VaR profile in Section 2.2.2, there exist some values for $C_0^{A,-}$, $C_0^{A,0}$, $C_0^{A,\infty}$ and $m^*(C_0)$, $m^\infty(C_0)$, such that:*

i. *If $C_0 < C_0^{A,-}$, then $m^* = 1$ and $R^* = R(1, C_0) < 0$.*

ii. *If $C_0^{A,-} \le C_0 < C_0^{A,0}$, then $m^* = 1$ and $R^* = R(1, C_0) \ge 0$.*

iii. *If $C_0^{A,0} \le C_0 < C_0^{A,\infty}$, then $m^* = m^*(C_0) \in [0, 1)$ and $R^* = R(m^*(C_0), C_0) \ge 0$.*

iv. *If $C_0 \ge C_0^{A,\infty}$, then $m^* = m^\infty(C_0) \in [0, 1]$ and $R^* = R(m^\infty(C_0), C_0) = +\infty$.*

Proof. See Appendix 2.5 for the proof and the characterization of $C_0^{A,-}$, $C_0^{A,0}$, $C_0^{A,\infty}$, $m^*(C_0)$ and $m^\infty(C_0)$. As an illustration, one may refer to Figure 2.7 to understand the locations of $C_0^{A,-}, C_0^{A,0}, C_0^{A,\infty}$. □

Remark 2.2.4.1. As long as the real-world probability measure is used, the drift rate of the underlying is the real-world drift μ, which is usually larger than the riskfree rate r. The expected profit-and-loss is therefore a function of the units of the underlying. As a result, both the numerator and denominator of the RAROC are functions of m. Under the assumed VaR profile, we are suggested to take $m = 1$ in order to maximize RAROC. This means we should leverage by borrowing money to hold 1 unit of the underlying. In this case, we are not exposed to any risk from the European call since 1 unit of the underlying can superhedge it but we do experience risk due to borrowing. However, due to the small value of the initial ask-price C_0, high leverage is required and so the expected profit-and-loss is negative, which leads to a negative value of R^*.

Remark 2.2.4.2. When $C_0^{A,-} < C_0 \le C_0^{A,0}$, we do not need to borrow too much money in order to acquire 1 unit of the underlying, and we can expect a positive profit. This produces a positive value of R^*.

Remark 2.2.4.3. For the case of $C_0^{A,0} < C_0 \le C_0^{A,\infty}$, the maximum RAROC is attained without holding 1 unit of the underlying. This means it is not necessary to use aggressive leverage for improving RAROC, because even though the expected profit can be obtained, the marginal increase in profit cannot outweigh the risk from borrowing, hence, inducing a deteriorating effect on RAROC.

Remark 2.2.4.4. If the initial ask-price C_0 is sufficiently high, in addition to a positive expected profit, one would achieve zero risk as measured by VaR_α. This leads to an infinite value for R^*.

Remark 2.2.4.5. If the investor aims only at constructing the hedging strategy which minimizes VaR_α under $P \neq Q$, he does not fully utilize economic capital in the sense that the profit per unit risk under such a strategy is not optimal. If reserving capital is unavoidable, a better use of capital is to consider the possibility of a positive outcome from it. So the investor should determine the policy such that the profit/return per unit of economic capital is maximized, instead of minimizing the amount of capital he must put aside.

The graph below shows the behavior of R^* for different values of C_0 when $\mu \neq r$. Comparison to the situation when $\mu = r$ is also illustrated.

Parameters: $r = 0.02, \mu = 0.08/0.02, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

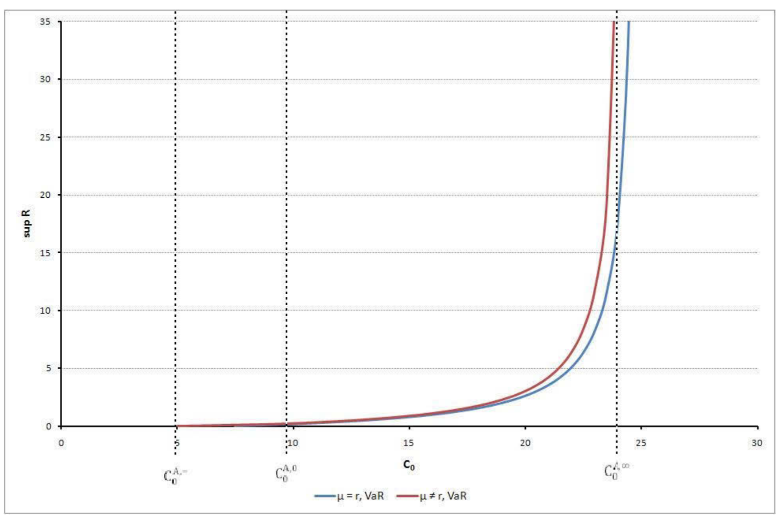


Fig. 2.7: R^* when $\mu \neq r$ and $\mu = r$

2.2.5 *The Buyer's Position*

We analyze the situation of the buyer. Upon purchase of a European call, the buyer should *short sell* a certain amount of the underlying to hedge. In the single-period case, the value of the buyer's hedged position is given by

$$V_T := V(m, C_0, S_T) = (S_T - K)^+ - mS_T - (C_0 - mS_0)e^{rT}. \tag{2.2.11}$$

Let VaR_α^B denote the Value-at-Risk of the buyer. We can determine VaR_α^B as follows, where $x = \mathrm{VaR}_\alpha^B$:

$$\begin{aligned} V_T + x \le 0 \quad &\iff \quad (S_T - K)^+ - mS_T - (C_0 - mS_0)e^{rT} + x \le 0 \\ &\iff \quad \begin{cases} -mS_T - (C_0 - mS_0)e^{rT} + x \le 0 & \text{for} \quad S_T \le K \\ (S_T - K) - mS_T - (C_0 - mS_0)e^{rT} + x \le 0 & \text{for} \quad S_T > K \end{cases} \\ &\iff \quad \underline{S}_T + \frac{x}{m} \le S_T \le \overline{S}_T - \frac{x}{1-m}. \end{aligned}$$

It then follows that VaR_α^B is the solution x of the following equation:

$$\Phi\left(\frac{\ln \frac{\overline{S}_T - \frac{x}{1-m}}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{\ln \frac{\underline{S}_T + \frac{x}{m}}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = \alpha. \tag{2.2.12}$$

By applying the Implicit Function Theorem, we obtain

$$\frac{\overline{\phi}}{(1-m)\overline{S}_T - x}\left(\frac{K + (C_0 - S_0)e^{rT} - x}{1-m} - \frac{\partial x}{\partial m}\right) - \frac{\underline{\phi}}{m\underline{S}_T + x}\left(\frac{C_0 e^{rT} - x}{m} + \frac{\partial x}{\partial m}\right) = 0 \tag{2.2.13}$$

$$\text{and} \qquad \frac{\partial x}{\partial C_0} = e^{rT}, \tag{2.2.14}$$

where,

$$\overline{\phi} := \phi\left(\frac{\ln \frac{\overline{S}_T - \frac{x}{1-m}}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right), \qquad \underline{\phi} := \phi\left(\frac{\ln \frac{\underline{S}_T + \frac{x}{m}}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

and $\phi(\cdot)$ is the probability density function of a standard normal random variable.

2.2.6 Approximating VaR_α^B

As in Section 2.2.2, we deduce an approximate solution for the system of PDEs (2.2.13) and (2.2.14). Note that VaR of a buyer's position should satisfy the following boundary conditions

$$\mathrm{VaR}_\alpha^B|_{m=0} = x(0, C_0) = C_0 e^{rT} \quad \text{and} \quad \mathrm{VaR}_\alpha^B|_{m=1} = x(1, C_0) = K - S_0 e^{rT} + C_0 e^{rT}.$$

When $m > 1$, we can solve analytically for VaR_α^B. Under this case, losses occurs in the region $\{S_T > \overline{S}_T - \frac{x}{1-m}\}$ of terminal price. This implies that VaR_α^B is obtained by solving

$$\Phi\left(\frac{\ln\frac{\overline{S}_T-\frac{x}{1-m}}{S_0}-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)=1-\alpha$$

$$\Longrightarrow \quad \mathrm{VaR}_\alpha^B = x(m,C_0) = m\big(\overline{S}_{T,\alpha}-S_0e^{rT}\big)+(K-\overline{S}_{T,\alpha})+C_0e^{rT}, \tag{2.2.15}$$

where $\overline{S}_{T,\alpha}$ is the $(1-\alpha)$-quantile of S_T, as defined in (2.2.3).

For the case of $0 \le m \le 1$, analytical expression for VaR_α^B is approximated. Refer to Figure 2.8 for an example of a numerical solution for $\frac{\partial x}{\partial m}$,

Parameters: $r = 0.02, \mu = 0.02, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

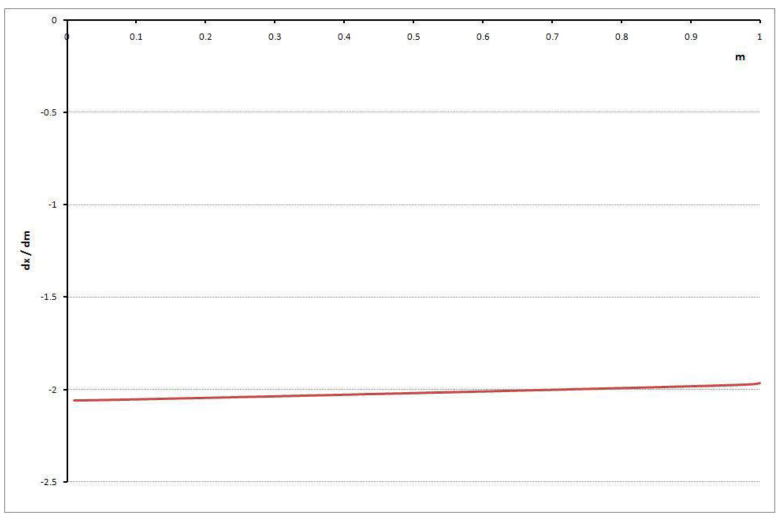


Fig. 2.8: Numerical Solution of $\frac{\partial x}{\partial m}$ under $P = Q$ and $\mu = r$

Figure 2.8 indicates a nearly-linear relationship between $\frac{\partial x}{\partial m}$ and m, in particular $\frac{\partial x}{\partial m}$ remains roughly the same throughout $m \in [0,1]$. Hence, for the sake of convenience, we assume a constant $\frac{\partial x}{\partial m}$, resulting in an approximation of (2.2.13),

$$\frac{\partial x}{\partial m} = \theta_1 \qquad \text{and} \qquad \frac{\partial x}{\partial C_0} = e^{rT}.$$

We are able obtain the corresponding solution of these PDEs, which is

$$\mathrm{VaR}_\alpha^B = \theta_1 m + C_0 e^{rT},\ \forall m \in [0,1].$$

This provides an approximation for VaR_α^B in (2.2.15). Contrary to the seller's case, the VaR_α^B is a linear function of m.

Remark 2.2.6.1. To understand the validity of assuming a constant $\frac{\partial x}{\partial m}$, or equivalently, that VaR_α^B is a linear function of m, we consider the payoff of the buyer's hedged position shown in Figure 2.9. For any $m \in (0,1)$, the potential loss suffered by the buyer is bounded, while it is unbounded for the case of the seller. As m is varied, the value of a buyer's hedged position is obtained by rotating about some point on the vertical axis $S_T = K$. The maximum loss occurs when $S_T = K$. This observation allows us to assume, for a sufficiently fixed small value of α, that $\text{VaR}_\alpha^B(m) \approx V_T(S_T^\alpha, m)$ at a fixed S_T^α. We see that varying m does not alter the value of $V_T(S_T^\alpha, m)$ in a nonlinear manner. This justifies the assumption that VaR_α^B is a linear function of m. Furthermore, as all losses are bounded, the value of VaR_α^B corresponding to $\alpha = 0$ is the maximum loss, which is $V_T(S_T)|_{S_T=K}$. For any sufficiently small value of α, VaR_α^B is close to $\text{VaR}_{\alpha=0}^B = V_T(K)$, so we observe that the function $\text{VaR}_\alpha^B(m)$, $m \in [0,1]$ for different small values of α would be close to each other. In other words, $\text{VaR}_\alpha^B(m)$ is insensitive to small values of α. See Figure 2.10 for a better illustration.

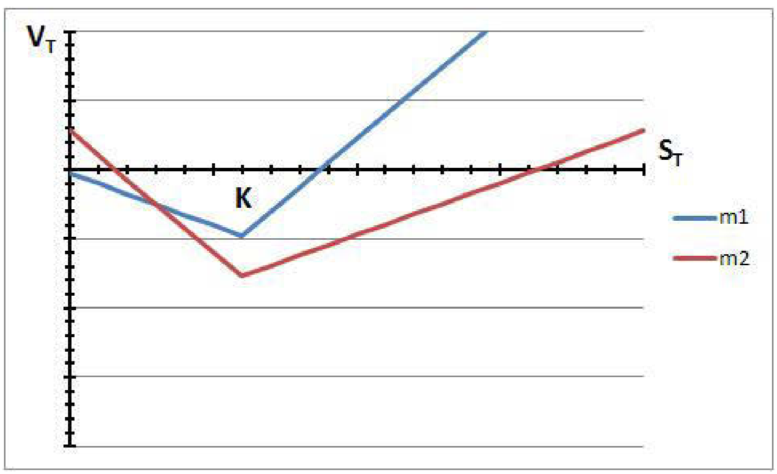


Fig. 2.9: Examples of V_T against S_T $(0 < m_1 < m_2 < 1)$

Parameters: $r = 0.02, \mu = 0.08, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

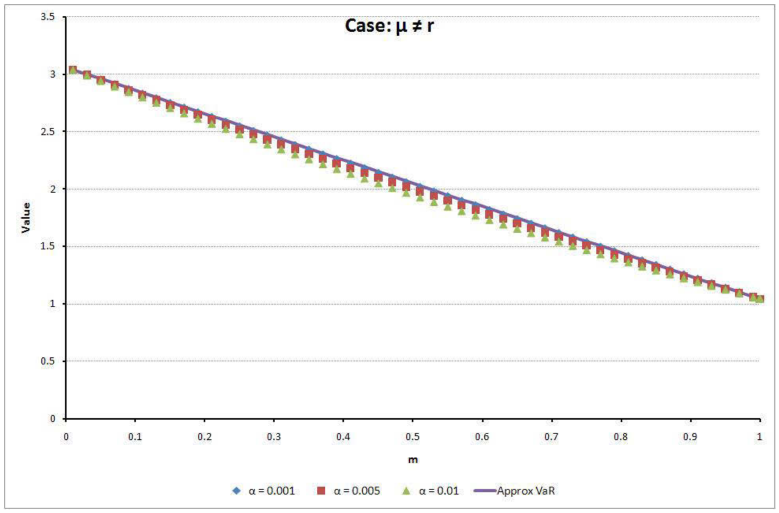


Fig. 2.10: Comparison of VaR^B_α and VaR^B under $P = Q$ and $\mu \neq r$

Remark 2.2.6.2. Since the approximation involves only one parameter θ_1, it is sufficient to select one boundary condition to determine its value. If we choose $\text{VaR}^B_\alpha|_{m=1} = K - S_0e^{rT} + C_0e^{rT}$ (by substituting $m = 1$ into (2.2.15)) as the boundary condition, we have $\theta_1 = K - S_0e^{rT}$, so VaR^B_α is given by $\text{VaR}^B_\alpha = (K - S_0e^{rT})m + C_0e^{rT}$. In this case, the approximation is independent of α, i.e. $\text{VaR}^B_\alpha = \text{VaR}^B$. In other words, the value of risk of the buyer is insensitive to the confidence level α.

2.2.7 Maximum RAROC when $P \neq Q$ and $\mu > r$

Given that the VaR of the buyer's position is

$$\text{VaR}^B_\alpha = x(m, C_0) = \theta_1 m + C_0e^{rT}, \tag{2.2.16}$$

the buyer's RAROC is expressed as

$$R = R(m, C_0) = \frac{C_0^{bs} - mS_0e^{\mu T} - (C_0 - mS_0)e^{rT}}{\theta_1 m + C_0e^{rT}}.$$

Similar to Section 2.2.3, the price C_0 will be known as a bid-price for the European call option. For each bid-price, we shall determine the optimal hedging strategy m^* such that RAROC is maximized, i.e.

$$m^* = \arg\max_m R(m, C_0).$$

Consequently we arrive at the following result.

Proposition 2.2.3. *Assume the VaR profile is defined by* (2.2.16)*. Then there exists values for* $C_0^{B,-}$*,* $C_0^{B,0}$*,* $C_0^{B,\infty}$*,* $m^*(C_0)$ *and* $m^\infty(C_0)$ *such that*

i. If $C_0 <_0^{B,\infty}$*, then* $m^* = m^\infty(C_0) \in [0, 1)$ *and* $R^* = R(m^\infty(C_0), C_0) = \infty$.

ii. If $C_0^{B,\infty} \leq C_0 < C_0^{B,0}$*, then* $m^* = 1$ *and* $R^* = R(1, C_0) > 0$.

iii. If $C_0^{B,0} \leq C_0 < C_0^{B,-}$*, then* $m^* = 0$ *and* $R^* = R(0, C_0) > 0$.

iv. If $C_0 \geq C_0^{B,-}$*, then* $m^* = 0$ *and* $R^* = R(0, C_0) < 0$.

Proof. See Appendix 2.5. As an illustration, one may refer to Figure 2.11 to understand the locations of $C_0^{B,-}, C_0^{B,0}, C_0^{B,\infty}$. □

Remark 2.2.7.1. When $C_0 < C_0^{B,\infty}$, we can regard the price of the European call option as cheap. Since purchasing the call is financed by borrowing money from the money market account, this implies that the buyer is exposed to a small liability at the option maturity. This leads to a low level of risk, since the long position of the buyer of the European call does not involve risk. Instead, it can potentially generate a large profit (unbounded upside payoff). On average, the profit from the call option can cover the relatively small liability at option maturity. As a result, the buyer can short sell a certain amount of the underlying such that the hedged position satisfies $\text{VaR}_\alpha^B = 0$, resulting in an infinite RAROC.

Remark 2.2.7.2. If the purchase price of the European call option increases, $C_0^{B,\infty} < C_0 \leq C_0^{B,0}$, then it is impossible to manage a hedged position to reach $\text{VaR}_\alpha^B = 0$. In this case the buyer must short sell one unit of the underlying to maximize RAROC.

Remark 2.2.7.3. When the initial bid-price becomes even more expensive, $C_0^{B,0} < C_0 \leq C_0^{B,-}$, the buyer is advised to short sell none of the underlying so that the only risk at option maturity is due to the liability. The loss from short selling when the asset price goes up does not occur in this case. Such a hedging strategy leads to a maximum RAROC.

Remark 2.2.7.4. In case the initial bid-price is very high, $C_0^{B,-} < C_0$, the future liability due to borrowing is so extreme that even when the European call is exercised at a moderately high asset price, the liability cannot be repaid. Only an extremely high asset price at maturity allows the buyer to cover the borrowing costs. Any short-sale strategy would induce additional loss to the buyer when the asset price at maturity is high. This prevents the buyer from short selling any amount of underlying, so that the optimal hedging strategy is to do nothing. The next figure shows the dependence between R^* and C_0, from a buyer's perspective.

Parameters: $r = 0.02, \mu = 0.08, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

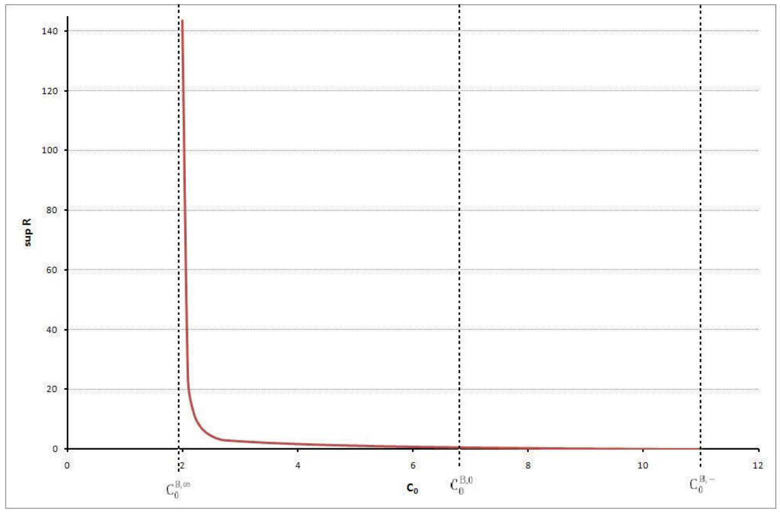


Fig. 2.11: R^* when $\mu \neq r$

2.3 RAROC Maximization with Expected Shortfall

2.3.1 Seller's Position

VaR is often criticized due to its lack of coherence as a risk measure. More precisely, the subadditivity property of a coherent risk measure is not observed, see Acerbi et al. (2008) for an excellent example. Hence, it might be worthwhile to study RAROC under the use of a coherent risk measure. Among all choices, we shall resort to *expected shortfall* which is defined as

$$ES_\alpha(V) := -\frac{1}{\alpha}\Big(E\big[V\mathbf{1}_{\{V\leq -\mathrm{VaR}_\alpha(V)\}}\big] + \mathrm{VaR}_\alpha(V)\big(P(V \leq -\mathrm{VaR}_\alpha(V)) - \alpha\big)\Big),$$

see Lütkebohmert (2009). When V is a continuous random variable, it coincides with the notion of *tail conditional expectation*, which is given by

$$TCE_\alpha(V) := -E\big[V \,\big|\, V \leq -\mathrm{VaR}_\alpha\big]. \tag{2.3.1}$$

This is true in present context due to the assumed dynamics of the underlying asset, which is a geometric Brownian motion. Generally ES and TCE are related by

$$ES_\alpha = TCE_\alpha + (\lambda - 1)\big(TCE_\alpha - \mathrm{VaR}_\alpha\big), \qquad \text{where} \quad \lambda := \frac{P(V \leq -\mathrm{VaR}_\alpha)}{\alpha} \geq 1.$$

One may refer to Acerbi and Tasche (2002a) for more details.

In order to determine expected shortfall of a seller's position, we denote VaR_α^A as x

and recognize

$$\begin{aligned} V_T \le -\mathrm{VaR}_\alpha^A \quad &\Longleftrightarrow \quad mS_T - (S_T - K)^+ + (C_0 - mS_0)e^{rT} + \mathrm{VaR}_\alpha^A \le 0 \\ &\Longleftrightarrow \quad S_T \le \underline{S}_T - \frac{x}{m} \quad \text{or} \quad S_T \ge \overline{S}_T + \frac{x}{1-m}, \end{aligned}$$

where $\left\{S_T \le \underline{S}_T - \frac{x}{m}\right\}$ and $\left\{S_T \ge \overline{S}_T + \frac{x}{1-m}\right\}$ are mutually disjoint. This enables us to derive expected shortfall from (2.3.1), which is

$$\begin{aligned} ES_\alpha^A(V_T;\mathrm{VaR}) &= TCE_\alpha(V) = -E\big[V \,\big|\, V \le -\mathrm{VaR}_\alpha\big] \\ &= -\frac{mS_0e^{\mu T}}{\alpha} + \frac{mE^P\Big[S_T\mathbf{1}_{\{S_T \ge \underline{S}_T - \frac{x}{m}\}}\Big]}{\alpha} - \frac{(m-1)E^P\Big[S_T\mathbf{1}_{\{S_T \ge \overline{S}_T + \frac{x}{1-m}\}}\Big]}{\alpha} \\ &\quad - \frac{K \cdot P\big(S_T \ge \overline{S}_T + \frac{x}{1-m}\big)}{\alpha} - (C_0 - mS_0)e^{rT}, \end{aligned} \tag{2.3.2}$$

where, by (2.5.6) in Appendix 2.5,

$$\begin{aligned} E^P\Big[S_T\mathbf{1}_{\{S_T \ge \underline{S}_T - \frac{x}{m}\}}\Big] &= E^P\Big[S_T \,\Big|\, S_T \ge \underline{S}_T - \frac{x}{m}\Big] \cdot P\Big(S_T > \underline{S}_T - \frac{x}{m}\Big) \\ &= S_0e^{\mu T}\left(1 - \Phi^\circ\left(\ln\left(\frac{\underline{S}_T - \frac{x}{m}}{S_0}\right)\right)\right) \\ &= S_0e^{\mu T}\left(1 - \Phi\left(\frac{\ln\frac{\underline{S}_T - \frac{x}{m}}{S_0} - (\mu + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right), \end{aligned}$$

and,

$$\begin{aligned} E^P\Big[S_T\mathbf{1}_{\{S_T \ge \overline{S}_T + \frac{x}{1-m}\}}\Big] &= E^P\Big[S_T \,\Big|\, S_T \ge \overline{S}_T + \frac{x}{1-m}\Big] \cdot P\Big(S_T > \overline{S}_T + \frac{x}{1-m}\Big) \\ &= S_0e^{\mu T}\left(1 - \Phi^\circ\left(\ln\left(\frac{\overline{S}_T + \frac{x}{1-m}}{S_0}\right)\right)\right) \\ &= S_0e^{\mu T}\left(1 - \Phi\left(\frac{\ln\frac{\overline{S}_T + \frac{x}{1-m}}{S_0} - (\mu + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right), \end{aligned}$$

where Φ° is the probability distribution function of a normal random variable with drift $\mu^\circ := (\mu - \frac{\sigma^2}{2})T + \sigma^2 T = (\mu + \frac{\sigma^2}{2})T$ and standard deviation $\sigma\sqrt{T}$.

The next figure compares VaR_α^A and ES_α^A at a given value of C_0, in which the value of VaR_α^A is calculated by solving (2.2.5) and that of ES_α^A is obtained by (2.3.2) with $x = \mathrm{VaR}_\alpha^A$. Both of them represent the true values, not the approximate ones.

Parameters: $r = 0.02, \mu = 0.08, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

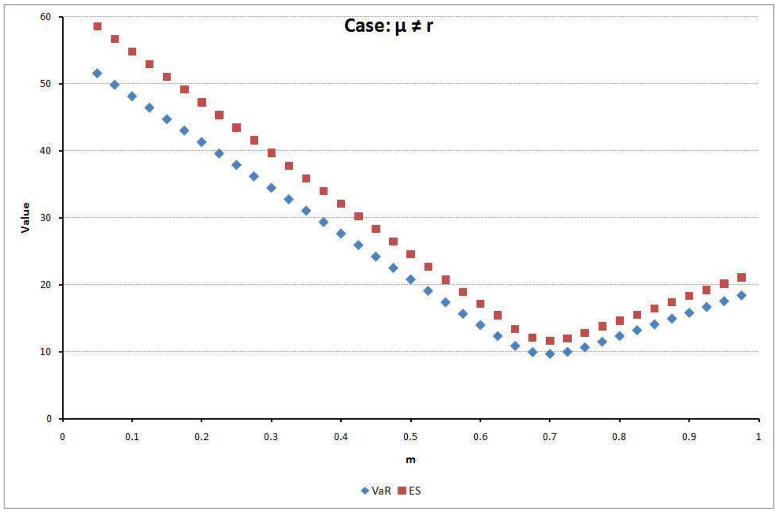


Fig. 2.12: Difference between VaR_α^A and ES_α^A

2.3.2 Approximation of ES_α^A

We adopt the same approximation of VaR_α^A , i.e. $\frac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m^{1,\varepsilon} + \theta_3) + \theta_4 m^{1,\varepsilon} + \theta_5 - C_0 e^{rT}$, for approximating ES_α^A for easing further analysis. Then we couple with the following boundary conditions in order to determine $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$.

$$\begin{cases} ES_\alpha^A(V_T; \widehat{\mathrm{VaR}}_\alpha^A)\big|_{m=m^{1,\varepsilon}} &= \dfrac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m^{1,\varepsilon} + \theta_3) + \theta_4 m^{1,\varepsilon} + \theta_5 - C_0 e^{rT} \\ ES_\alpha^A(V_T; \widehat{\mathrm{VaR}}_\alpha^A)\big|_{m=m^{2,\varepsilon}} &= \dfrac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m^{2,\varepsilon} + \theta_3) + \theta_4 m^{2,\varepsilon} + \theta_5 - C_0 e^{rT} \\ ES_\alpha^A(V_T; \widehat{\mathrm{VaR}}_\alpha^A\big|_{m=m^{3,\varepsilon}} &= \dfrac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m^{3,\varepsilon} + \theta_3) + \theta_4 m^{3,\varepsilon} + \theta_5 - C_0 e^{rT} \\ ES_\alpha^A(V_T; \widehat{\mathrm{VaR}}_\alpha^A)\big|_{m=m^{4,\varepsilon}} &= \dfrac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m^{4,\varepsilon} + \theta_3) + \theta_4 m^{4,\varepsilon} + \theta_5 - C_0 e^{rT} \\ ES_\alpha^A(V_T; \widehat{\mathrm{VaR}}_\alpha^A)\big|_{m=m^{5,\varepsilon}} &= \dfrac{\theta_1}{\theta_2} \ln\cosh(\theta_2 m^{5,\varepsilon} + \theta_3) + \theta_4 m^{5,\varepsilon} + \theta_5 - C_0 e^{rT} \end{cases}$$

where $m^{1,\varepsilon}, m^{2,\varepsilon}, m^{3,\varepsilon}, m^{4,\varepsilon}, m^{5,\varepsilon}$ are values of m arbitrarily chosen from $[0, \varepsilon] \cup [1-\varepsilon, 1)$ for sufficiently small $\varepsilon > 0$.

Remark 2.3.2.1. This set of boundary conditions is proposed because $\widehat{\mathrm{VaR}}_\alpha^A$ can well approximate the true VaR_α^A for both large and small value of m while certain (though small) error exists for intermediate value of m. It is obvious that the determination of $ES_\alpha^A = ES_\alpha^A(V_T; \mathrm{VaR}_\alpha^A)$ requires the input of VaR_α^A. It is inconvenient and time-consuming to evaluate VaR_α^A before one computes ES_α^A. As a result, in order to avoid such additional computational efforts, we may simply use the approximation $\widehat{\mathrm{VaR}}_\alpha^A$ to obtain $\widehat{ES}_\alpha^A = \widehat{ES}_\alpha^A(V_T; \widehat{\text{VaR}}_\alpha^A)$. However, we should carefully control the propagation of errors from $\widehat{\text{VaR}}_\alpha^A$ into $\widehat{ES}_\alpha^A$. This means we should use those values of $\widehat{\text{VaR}}_\alpha^A$ that are sufficiently close to the true value VaR_α^A, and they appear at extreme values of m. Lastly, we should note that the minimizers of $\widehat{\text{VaR}}_\alpha^A$ and $\widehat{ES}_\alpha^A$ are generally different.

The performance of the approximation $\widehat{ES}_\alpha^A = \widehat{ES}_\alpha^A(V_T; \widehat{\text{VaR}}_\alpha^A)$ is displayed in the following figure. The solid line represents the approximation while the dots corresponds to the true values, which are obtained by solving VaR_α^A from (2.2.5) and substituting $x = \text{VaR}_\alpha^A$ into (2.3.2) for computing expected shortfall.

Parameters: $r = 0.02, \mu = 0.08, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

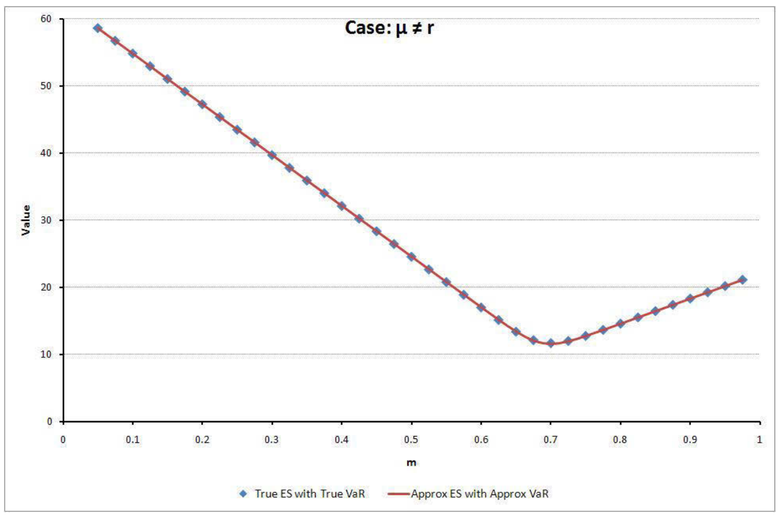


Fig. 2.13: Quality of $\widehat{ES}_\alpha^A$

2.3.3 Maximum RAROC under $P \neq Q$ and $\mu > r$

By making use of the approximation of expected shortfall, the resultant RAROC can be expressed as

$$R = R(m, C_0) = \frac{mS_0e^{\mu T} - C_0^{bs} + (C_0 - mS_0)e^{rT}}{\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}}.$$

It resembles the version of RAROC in (2.2.10), but we should emphasize that the values of $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ are different. We repeat the same procedures as in Section 2.2.4 to analyze the maximum RAROC. The major changes are the values of $C_0^{A,-}$,$C_0^{A,0}$,$C_0^{A,\infty}$ and $m^*(C_0)$,$m^\infty(C_0)$ while the assertions of Proposition 2.2.2 remain valid in present context.

The next figure presents the maximum RAROC as a function of C_0 as well as the comparison to that in Section 2.2.4. Note that approximations $\widehat{\mathrm{VaR}}_\alpha^A$ and $\widehat{ES}_\alpha^A$ are used in both cases.

Parameters: $r = 0.02, \mu = 0.08, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

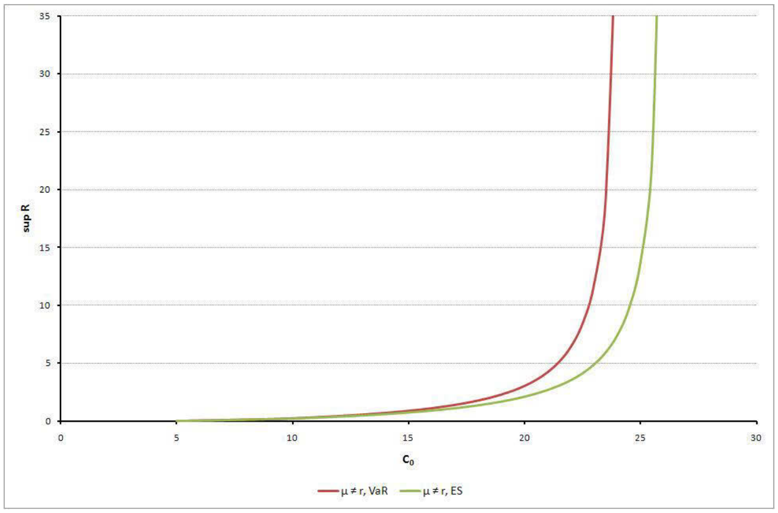


Fig. 2.14: R^* under VaR and ES when $\mu \neq r$

Remark 2.3.3.1. By definition, $ES \geq$ VaR holds, hence, one should expect that, to reach the same level of R^* that can be achieved under the use of value-at-risk, the initial price C_0 should be increased after switching from using value-at-risk to expected shortfall because the denominator (the risk) in R becomes larger under expected shortfall, which would diminish the value of RAROC. Equivalently, at a given value of C_0, R^* would be lower when using ES as risk measure.

Remark 2.3.3.2. If the readers are cautious, he/she might question after reading the graphs in Figure 2.14, that is, why R^* under VaR_α and ES_α can reach infinity at a price C_0 below the initial asset price S_0. More precisely, the no-arbitrage price of a European call option under a financial market in which only a static hedge is allowed is not unique. Rather, there is a no-arbitrage bid-ask price bound for the European call option. In the present situation, the no-arbitrage bid-ask price bound should be $C_0 \in [0, S_0]$. As a result, if we are a seller, we should not sell the European call option for more than S_0, otherwise, an arbitrage strategy (sell C for $C_0 > S_0$ and buy S at S_0) can be established. For any $C_0 < S_0$, any hedged position possesses a certain amount of risk. Returning to the definition of RAROC, an infinite value of RAROC can only be achieved when the denominator, which corresponds to the risk of the position, is zero. Based on the previous argument, we could comment that $R^* = \infty$ can only happen when $C_0 \geq S_0$. However the graphs in Figure 2.14 show a violation, namely, an infinite value of R_* is located at some value of $C_0 < S_0$. So what is the problem here? This is merely due to the *support* in the definitions of VaR_α and ES_α. Considering VaR_α first, in order to determine the value of VaR_α at a given level of α, we require a loss random variable L, search for the α-quantile L_α and define $\text{VaR}_\alpha = L_\alpha$. Now the support[3] of L plays a crucial role here. If the support is $\mathbb{R}^+$, the positive part of the real-line, then L is actually referring to loss, while if the support is $\mathbb{R}$, the whole real-line, then L is referring to profit-and-loss, in which case negative L means a profit. In view of this, we can readily see that, at a confidence level α, the α-quantile L_α of L is guaranteed to be positive if the support is $\mathbb{R}^+$. However, it is no longer true when the support is $\mathbb{R}$, L_α can be zero or even negative. In our study, we assume the later case, that is, the support of L is $\mathbb{R}$, because this can ease our analysis and allow us to relate VaR_α to the lower and upper α-quantile $\underline{S}_{T,\alpha}, \overline{S}_{T,\alpha}$. Consequently, it is possible to have zero risk even if $C_0 < S_0$ since the corresponding $\text{VaR}_\alpha = L_\alpha$ is zero as determined from the profit-and-loss distribution. Similarly for the case of ES_α, the reason of having $ES_\alpha = 0$ without $C_0 \geq S_0$ is that ES_α is about taking the weighted (with probabilities) average of all losses $L > L_\alpha$. If $L_\alpha < 0$ is satisfied, it may happen that the weighted average is zero, resulting in $ES_\alpha = 0$. In summary,

if the support of L, hence VaR_α, is $\mathbb{R}^+$, then $R^* = \infty \Longleftrightarrow C_0 = S_0$.
if the support of L, hence VaR_α, is $\mathbb{R}$, then $R^* = \infty$ is possible when $C_0 < S_0$.

2.3.4 Buyer's Position

Consider a buyer's RAROC under the use of expected shortfall. The expected shortfall for a buyer's position is given by

$$\begin{aligned} ES_\alpha^B(V_T; \text{VaR}_\alpha^B) = &-\frac{E^P\left[S_T \mathbf{1}_{\{S_T \geq K\}}\right] - E^P\left[S_T \mathbf{1}_{\{S_T \geq \overline{S}_T - \frac{x}{1-m}\}}\right]}{\alpha} \\ &+ K \cdot \frac{P(K \leq S_T \leq \overline{S}_T - \frac{x}{1-m})}{\alpha} \\ &+ \frac{m}{\alpha}\left(E^P\left[S_T \mathbf{1}_{\{S_T \geq \underline{S}_T + \frac{x}{m}\}}\right] - E^P\left[S_T \mathbf{1}_{\{S_T \geq \overline{S}_T - \frac{x}{1-m}\}}\right]\right) \\ &+ (C_0 - mS_0)e^{rT}, \end{aligned} \tag{2.3.3}$$

[3]This also means the range of values that $L(\omega)$ can take at each $\omega \in \Omega$.

where, again by (2.5.6),

$$E^P\Big[S_T\mathbf{1}_{\{S_T\geq K\}}\Big]=S_0e^{\mu T}\Bigg(1-\Phi\Bigg(\frac{\ln\frac{K}{S_0}-(\mu+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\Bigg)\Bigg), \tag{2.3.4}$$

$$E^P\Big[S_T\mathbf{1}_{\{S_T\geq \underline{S}_T+\frac{x}{m}\}}\Big]=S_0e^{\mu T}\Bigg(1-\Phi\Bigg(\frac{\ln\frac{\underline{S}_T+\frac{x}{m}}{S_0}-(\mu+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\Bigg)\Bigg), \tag{2.3.5}$$

$$E^P\Big[S_T\mathbf{1}_{\{S_T\geq \overline{S}_T-\frac{x}{1-m}\}}\Big]=S_0e^{\mu T}\Bigg(1-\Phi\Bigg(\frac{\ln\frac{\overline{S}_T-\frac{x}{1-m}}{S_0}-(\mu+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\Bigg)\Bigg), \tag{2.3.6}$$

$$\begin{aligned}P(K\leq S_T\leq \overline{S}_T-\frac{x}{1-m})&=P(S_T\leq\overline{S}_T-\frac{x}{1-m})-P(S_T\leq K)\\&=\Phi\Bigg(\frac{\ln\frac{\overline{S}_T-\frac{x}{1-m}}{S_0}-(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\Bigg)-\Phi\Bigg(\frac{\ln\frac{K}{S_0}-(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\Bigg).\end{aligned} \tag{2.3.7}$$

The figure below compares the values of value-at-risk and expected shortfall of a buyer's position at a fixed C_0 (Here $C_0\approx 6.8$ as an example).

Parameters: $r=0.02,\mu=0.08,T=1,\sigma=0.15,S_0=100,K=100,\alpha=0.001$

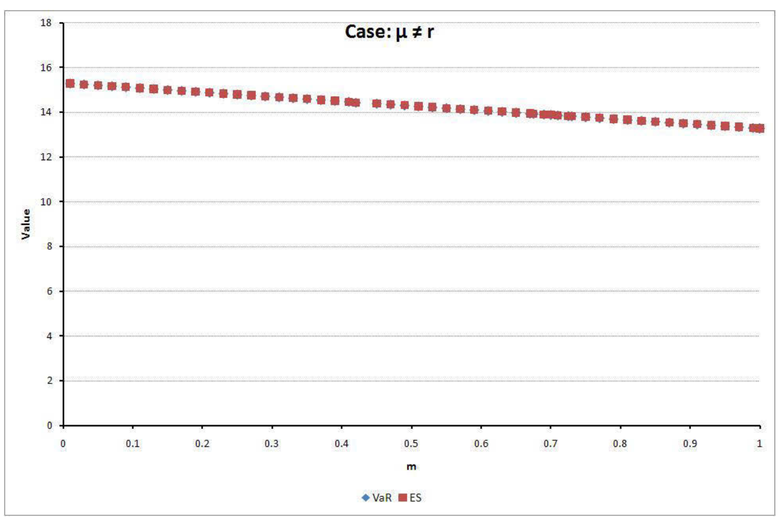


Fig. 2.15: Buyer's VaR and ES when $\mu\neq r$

Remark 2.3.4.1. Even though the two curves look the same in Fig. 2.15, they are different in values and the condition of $ES>$ VaR still holds at every m. Contrarily, there is a clear discrepancy in VaR and ES of a seller's position, see Fig. 2.12. The explanation is attributed to the possibility of unbounded loss for a seller. There are unlimited amounts of loss beyond VaR with extremely small probabilities (at most α), expected shortfall takes the weighted average of loss beyond VaR, which eventually leads to a value different from zero, resulting a distinguishing deviation from value-at-risk. For a buyer's position, the maximum loss is bounded and this corresponds to the case that no hedging is done and the option expires with out-of-moneyness. As a consequence, value-at-risk with sufficiently small confidence level will not be deviated too much from the maximum loss. Moreover, the weighted average of loss is done within value-at-risk and maximum loss, which would not make ES substantially far away from VaR.

Due to the previous remark, we would simply assume $ES = \text{VaR}$ for the sake of convenience. Under this assumption, the maximum RAROC under expected shortfall is identical to that under value-at-risk. So we reproduce the same results in Section 2.2.7, that is, Proposition 2.2.3, in this context.

2.4 Conclusions

After analyzing RAROC of both the buyer's and seller's position, we can form a bid-ask price bound for a European call option under the criterion of maximizing RAROC. An example of such a bid-ask price bound is shown in the following figure.

Parameters: $r = 0.02, \mu = 0.08, T = 1, \sigma = 0.15, S_0 = 100, K = 100, \alpha = 0.001$

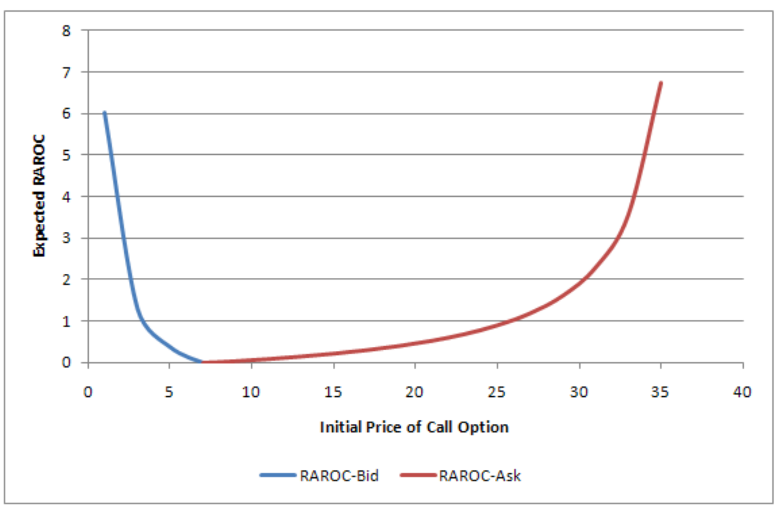


Fig. 2.16: R^* against C_0

At a given confidence level of α, suppose the investor sets the target RAROC to be $\overline{R}$, a bid-ask price bound for the European call option can be obtained in the following way: draw a horizontal line at $R^* = \overline{R}$ and locate the two intersection points in Figure 2.16. The C_0-coordinates of these two intersection points are the bid-price $C_0^{B,\overline{R}}$ and ask-price $C_0^{A,\overline{R}}$. The resultant bid-ask price bound is then $[C_0^{B,\overline{R}}, C_0^{A,\overline{R}}]$. If the investor is a buyer, he should at most pay $C_0^{B,\overline{R}}$ for the European call option in order to achieve at least $\overline{R}$ with some static hedging strategy. Otherwise, if he pays more than $C_0^{B,\overline{R}}$, he cannot yield his target RAROC in his hedged position.

Furthermore, in the present setup, the no-arbitrage price bound is $(0, S_0)$ where $S_0 = 100$. We can see that the bid-ask price bound $[C_0^{B,\overline{R}}, C_0^{A,\overline{R}}]$ in the above example is smaller than $[0, S_0]$. Most of the time the no-arbitrage price bounds can be too wide for completing a deal so a smaller bound is more favorable. By referring to this example, a tighter bid-ask price bound can be obtained, and so this underlines the possibility and practicality of making use of RAROC for pricing derivatives. Even though the price bound is smaller, absence of arbitrage remains valid. Of course, the reason for getting a relatively narrower bid-ask price bound is that we accept a risky deal which may induce losses after hedging. The level of risk tolerance is captured by the risk measure ρ at confidence level α.

Although the present financial market is not too realistic, it does provide some insight regarding the notion of RAROC-based pricing. In particular, an advantage of this pricing method is a tighter bid-ask price bound. We further explore the idea of RAROC-based pricing in a continuous-time setup in the next chapter.

2.5 Appendix

Since we frequently maximize a function in fractional form $f(m, C_0) = \frac{g(m,C_0)}{h(m,C_0)}$, the following notations are introduced for the sake of convenience:

$$\begin{aligned}
\mathcal{Z}^n(C_0) &:= \left\{ m \in [0,1] \,\middle|\, g(m, C_0) = 0 \right\}, \\
\mathcal{Z}^d(C_0) &:= \left\{ m \in [0,1] \,\middle|\, h(m, C_0) = 0 \right\}, \\
\mathcal{Z}^{n\cap d}(C_0) &:= \mathcal{Z}^n(C_0) \bigcap \mathcal{Z}^d(C_0).
\end{aligned}$$

At a fixed C_0, $\mathcal{Z}^n(C_0)$ collects the roots of $g(m, C_0) = 0$ while $\mathcal{Z}^d(C_0)$ collects the roots of $h(m, C_0) = 0$. $\mathcal{Z}^{n\cap d}(C_0)$ collects the roots that are common to both $\mathcal{Z}^n(C_0)$ and $\mathcal{Z}^d(C_0)$. Furthermore we define

$$\begin{aligned}
C_0^{A,-} &= \text{the value of } C_0 \text{ below which a negative and finite value of } R^* \text{ is obtained,} \\
C_0^{A,0} &= \text{the value of } C_0 \text{ above which a positive and finite value of } R^* \text{ is obtained,} \\
C_0^{A,\infty} &= \text{the value of } C_0 \text{ above which an infinite value of } R^* \text{ is obtained.}
\end{aligned}$$

Lastly, all assumptions stated in Section 2.2.2 are in force at all time.

2.5.1 Supplement to Section 2.2.3

Since R is given by

$$R = R(m, C_0) = \frac{\big(C_0 - C_0^{BS}\big)e^{rT}}{\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}}, \tag{2.5.1}$$

the corresponding first-order and second-order derivatives are

$$\begin{cases} \dfrac{\partial R}{\partial m} &= \dfrac{\big(C_0 - C_0^{BS}\big)e^{rT}}{\big(\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}\big)^2}\big(\theta_1\tanh(\theta_2 m + \theta_3) + \theta_4\big) \\ \dfrac{\partial^2 R}{\partial m^2} &= \dfrac{\big(C_0 - C_0^{BS}\big)e^{rT}\cdot\theta_1\theta_2\,\mathrm{sech}^2(\theta_2 m + \theta_3)}{\big(\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}\big)^2} \\ & \quad -\dfrac{\partial R}{\partial m}\dfrac{2\big(\theta_1\tanh(\theta_2 m + \theta_3) + \theta_4\big)}{\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}} \end{cases}.$$

Then, for a fixed C_0, $\mathcal{Z}^n(C_0)$, $\mathcal{Z}^d(C_0)$ and $\mathcal{Z}^{n\cap d}(C_0)$ are defined as

$$\begin{aligned} \mathcal{Z}^n(C_0) &:= \Big\{ m \in [0,1] \;\Big|\; \theta_1\tanh(\theta_2 m + \theta_3) + \theta_4 = 0 \Big\}, \\ \mathcal{Z}^d(C_0) &:= \Big\{ m \in [0,1] \;\Big|\; \frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT} = 0 \Big\}, \\ \mathcal{Z}^{n\cap d}(C_0) &:= \mathcal{Z}^n(C_0) \bigcap \mathcal{Z}^d(C_0). \end{aligned}$$

The following functions

$$\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT} \qquad \text{and} \qquad \theta_1\tanh(\theta_2 m + \theta_3) + \theta_4$$

play critical roles here. Under the assumptions in Section 2.2.2, $\mathcal{Z}^n(C_0)$ is a singleton and let the root be $m^{\mathcal{Z}^n}$, i.e. $\mathcal{Z}^n(C_0) = \{m^{\mathcal{Z}^n}\}$. Note that the function $\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}$ attains its minimum at $m = m^{\mathcal{Z}^n}$ because its derivative is $\theta_1\tanh(\theta_2 m + \theta_3) + \theta_4$. It is also apparent that the value of C_0 controls the y-intercept of $\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}$. Given the assumed graph of $\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5$ in Section 2.2.2, we could reach the conclusion that, for sufficiently small values of C_0, the equation

$$\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT} = 0$$

does not have any solutions $m \in [0,1]$. The corresponding value of C_0 for this situation is essentially $C_0^{A,\infty}$ and it can be determined by

$$\begin{cases} m^{z^n} &= \dfrac{\tanh^{-1}\left(-\frac{\theta_4}{\theta_1}\right) - \theta_3}{\theta_2} \qquad \text{where} \quad \tanh^{-1}(y) := \dfrac{1}{2}\ln\left(\dfrac{1+y}{1-y}\right) \\ C_0^{A,\infty} &= e^{-rT}\left(\dfrac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m^{z^n} + \theta_3) + \theta_4 m^{z^n} + \theta_5\right) \end{cases}. \tag{2.5.2}$$

More precisely, $C_0^{A,\infty}$ is the value of C_0 such that m^{z^n} is the unique root $m \in [0,1]$ of $\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT} = 0$ (as the minimum point touches the m-axis in this situation). For any $C_0 < C_0^{A,\infty}$, there exists no roots in $\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT} = 0$ because $\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT} > 0$ for all $m \in [0,1]$. The first-order condition $\frac{dR}{dm} = 0$ is satisfied at $m = m^{z^n}$ and a finite value of R^* is obtained which is calculated by $R^* = R(m^*, C_0) = R(m^{z^n}, C_0)$ and given as

$$R^* = R(m^*, C_0) = R(m^{z^n}, C_0) = \frac{\left(C_0 - C_0^{BS}\right)e^{rT}}{\left(C_0^{A,\infty} - C_0\right)e^{rT}} \in [0, +\infty).$$

Suppose $C_0^{BS} < C_0^{A,\infty}$ holds. Then one can readily observe that, when $C_0^{BS} < C_0 \leq C_0^{A,\infty}$, a finite and positive value of R^* can be attained. While, for the case of $C_0 \leq C_0^{BS}$, we would have $R(m) < 0$, $\forall m \in [0,1]$. As a consequence, $C_0^{A,-} = C_0^{BS}$ is valid. And the corresponding R^* is obtained by maximizing $\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}$ instead of minimizing, due to the negative value of $R(m)$ for all $m \in [0,1]$. Eventually, based on the assumed VaR profile, R^* is obtained at $m^* = 0$.

For the case of $C_0 \in \left(C_0^{A,\infty}, +\infty\right)$, the function $\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}$ intersects with the m-axis and so two roots, namely $m^{z^n,1}$ and $m^{z^n,2}$ can be found. We denote $m^* = \min(m^{z^n,1}, m^{z^n,2})$ as the smallest root in this case and define the corresponding value of R^* as

$$R^* = +\infty. \tag{2.5.3}$$

In summary, we conclude that

$$\begin{aligned} R^* &= \sup_m R(m, C_0) = R(m^*, C_0) \\ &= \begin{cases} R(m^*, C_0) = R(0, C_0) < 0 & \text{for} \quad C_0 \in \left[0, C_0^{A,-}\right) \\ R(m^*, C_0) = R(m^{z^n}, C_0) < +\infty & \text{for} \quad C_0 \in \left[C_0^{A,-}, C_0^{A,\infty}\right) \\ R(m^*, C_0) = R(\min(m^{z^n,1}, m^{z^n,2}), C_0) = +\infty & \text{for} \quad C_0 \in \left(C_0^{A,\infty}, +\infty\right) \end{cases}. \end{aligned}$$

2.5.2 Supplement to Section 2.2.4

Since R is given by

$$R = R(m, C_0) = \frac{mS_0 e^{\mu T} - C_0^{bs} + (C_0 - mS_0)e^{rT}}{\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}} =: \frac{g(m, C_0)}{h(m, C_0)}$$

and

$$\begin{cases} \dfrac{\partial R}{\partial m} &= \dfrac{g' \cdot h - g \cdot h'}{h^2} \qquad \text{where} \quad g' := \dfrac{\partial g}{\partial m}, h' := \dfrac{\partial h}{\partial m} \\ \dfrac{\partial^2 R}{\partial m^2} &= \dfrac{\frac{\partial}{\partial m}(g' \cdot h - g \cdot h') \cdot h^2 - (g' \cdot h - g \cdot h') \cdot 2hh'}{h^4} \\ &= \dfrac{g'' \cdot h - g \cdot h''}{h^2} - \dfrac{\partial R}{\partial m}\dfrac{2h'}{h} \\ &= -\dfrac{g \cdot h''}{h^2} - \dfrac{\partial R}{\partial m}\dfrac{2h'}{h} \qquad \because g'' = 0 \end{cases} \tag{2.5.4}$$

where

$$\begin{aligned} g' \cdot h - g \cdot h' = S_0\big(e^{\mu T} - e^{rT}\big) \cdot \Big(\frac{\theta_1}{\theta_2}\ln\cosh(\theta_2 m + \theta_3) + \theta_4 m + \theta_5 - C_0 e^{rT}\Big) \\ - \big(mS_0 e^{\mu T} - C_0^{bs} + (C_0 - mS_0)e^{rT}\big) \cdot \big(\theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4\big). \end{aligned}$$

There is a clear dependence of m in the numerator, contrary to the case of $P = Q$, see (2.5.1). Therefore the maximum value of R may not be attained at the value of m that corresponds to the minimizer of VaR as in the denominator.

Set

$$\begin{aligned} \mathcal{Z}^n(C_0) &= \Big\{ m \in [0,1] \,\Big|\, g' \cdot h - g \cdot h' = 0 \Big\}, \\ \mathcal{Z}^d(C_0) &= \Big\{ m \in [0,1] \,\Big|\, h = 0 \Big\}, \\ \mathcal{Z}^{n \cap d}(C_0) &= \mathcal{Z}^n(C_0) \bigcap \mathcal{Z}^d(C_0). \end{aligned}$$

To proceed further, we need to examine the following function:

$$f(m, C_0) = g' \cdot h - g \cdot h'.$$

Obviously it is a continuous function of (m, C_0). Since we are only interested in the compact set $[0,1]$ for m, at a fixed C_0, there exists a minimum point $f(\tilde{m}, C_0)$ for some $\tilde{m} \in [0,1]$. This is a well-known property of continuous functions (named Extreme Value Theorem). We suppose that there exists C_0 such that $f(\tilde{m}, C_0) \geq 0$, or equivalently, $g' \cdot h - g \cdot h' \geq 0$, $\forall m \in [0,1]$. If we compute

$$\frac{\partial}{\partial C_0}(g' \cdot h - g \cdot h') = -e^{rT}\left(S_0(e^{\mu T} - e^{rT}) + \theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4\right)$$

and under the assumption of $S_0(e^{\mu T} - e^{rT}) \leq |\theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4|$ in Section 2.2.2 and the behavior of the function $\theta_1 \tanh(\theta_2 m + \theta_3) + \theta_4$, see Figure 2.5, we can observe that there exists a $\hat{m} \in [0,1]$ such that $\frac{\partial}{\partial C_0}(g' \cdot h - g \cdot h')$ is positive on $[0, \hat{m}]$ and negative on $[\hat{m}, 1]$. Consequently, when C_0 is increased, this implies the values of $g' \cdot h - g \cdot h'$ on $[0, \hat{m}]$ are increased and those on $[\hat{m}, 1]$ are decreased. Assume there is a C_0 such that $g' \cdot h - g \cdot h' \geq 0,\ \forall m \in [0,1]$, increasing from this value of C_0 would eventually move the function $g' \cdot h - g \cdot h'$ (more specifically, the part on $[\hat{m}, 1]$) towards the m-axis and up to some point, there would exist a C_0 such that the function $g' \cdot h - g \cdot h'$ touches the m-axis. For each C_0, the function $g' \cdot h - g \cdot h'$ has its minimum at $\tilde{m}(C_0)$, we define

$$\begin{aligned} &C_0^{A,0} \\ &:= \sup\left\{ C_0 \in \mathbb{R}^+ \;\middle|\; \begin{array}{l} g'(\tilde{m}(C_0), C_0)h(\tilde{m}(C_0), C_0) - g(\tilde{m}(C_0), C_0)h'(\tilde{m}(C_0), C_0) \geq 0 \\ \text{and} \quad h(\tilde{m}(C_0), C_0) > 0 \end{array} \right\} \\ &= \left\{ C_0 \in \mathbb{R}^+ \;\middle|\; \begin{array}{l} g'(\tilde{m}(C_0), C_0) \cdot h(\tilde{m}(C_0), C_0) - g(\tilde{m}(C_0), C_0) \cdot h'(\tilde{m}(C_0), C_0) = 0 \\ \text{and} \quad h(\tilde{m}(C_0), C_0) > 0 \end{array} \right\}. \end{aligned} \tag{2.5.5}$$

As a result, for any $C_0 \leq C_0^{A,0}$, we have $g' \cdot h - g \cdot h' \geq 0,\ \forall m \in [0,1]$, which entails $\frac{\partial R}{\partial m} \geq 0,\ \forall m \in [0,1]$. In other words, $R(m)$ is an increasing function of m at this value of C_0 and in order to achieve the maximum value of R, we should take $m^* = 1$ to get

$$R^* = R(1, C_0) = \frac{S_0 e^{\mu T} - C_0^{bs} + (C_0 - S_0)e^{rT}}{\frac{\theta_1}{\theta_2} \ln\cosh(\theta_2 + \theta_3) + \theta_4 + \theta_5 - C_0 e^{rT}}.$$

From above, we can understand that the sign of R^* depends critically on $S_0 e^{\mu T} - C_0^{bs} + (C_0 - S_0)e^{rT}$, thus, we further define

$$C_0^{A,-} := e^{-rT}\big(C_0^{bs} - S_0(e^{\mu T} - e^{rT})\big),$$

under which $R^* \leq 0$ holds once $C_0 \leq C_0^{A,-}$. Here we assume that $C_0^{A,-} \leq C_0^{A,0}$ is valid.

For any $C_0 > C_0^{A,0}$, the maximum value of R is at $m^{z^n} \in [0,1)$ because at such C_0, the function $g' \cdot h - g \cdot h'$ would now intersect with m-axis, i.e. a solution of $g' \cdot h - g \cdot h' = 0$, i.e. $\frac{\partial R}{\partial m} = 0$, can be found. Moreover, evaluating $\frac{\partial^2 R}{\partial m^2}$ at this point (note that $g < 0$ and $h'' < 0$), which is negative, justifies a local maximum of R at this point. So, $m^* = m^{z^n}$ and $R^* < \infty$ provided that $h(m^{z^n}) > 0$.

We identify $C_0^{A,\infty}$ by using $R^* = \infty$. If C_0 is further increased, this would lead to $h(m^{z^n}) = 0$ because h is a decreasing function of C_0 at a given m, which can be verified from $\frac{\partial h}{\partial C_0} = -e^{rT}$. $h(m^{z^n}) = 0$ essentially indicates an infinite value of R^* since the risk as measured by h is zero. We shall determine $C_0^{A,\infty}$ as in (2.5.2) but $m^{\mathcal{Z}^n}$ cannot be determined explicitly in this case, rather, it is obtained by solving $g' \cdot h - g \cdot h' = 0$.

Assume $C_0^{A,-} \leq C_0^{A,0} \leq C_0^{A,\infty}$ holds and there exists a root $m^{\mathcal{Z}^n} \in \mathcal{Z}^n(C_0)$ such that $h(m^{\mathcal{Z}^n}) \neq 0$ for any $C_0 \in \big[C_0^{A,0}, C_0^{A,\infty}\big)$, then, we arrive at the following conclusion:

$$
\begin{aligned}
R^* &= \sup_m R(m, C_0) = R(m^*, C_0) \\
&= \begin{cases}
R(m^*, C_0) = R(1, C_0) < 0 & \text{for} \quad C_0 \in \big[0, C_0^{A,-}\big) \\
R(m^*, C_0) = R(1, C_0) \geq 0 & \text{for} \quad C_0 \in \big[C_0^{A,-}, C_0^{A,0}\big) \\
R(m^*, C_0) = R(m^{\mathcal{Z}^n}, C_0) < +\infty & \text{for} \quad C_0 \in \big[C_0^{A,0}, C_0^{A,\infty}\big) \\
R(m^*, C_0) = R(m^{\mathcal{Z}^n}, C_0) = +\infty & \text{for} \quad C_0 \in \big(C_0^{A,\infty}, +\infty\big)
\end{cases}.
\end{aligned}
$$

2.5.3 Supplement to Section 2.2.7

Since R is given by

$$
R = R(m, C_0) = \frac{C_0^{bs} - mS_0 e^{\mu T} - (C_0 - mS_0)e^{rT}}{\theta_1 m + C_0 e^{rT}},
$$

so,

$$
\begin{cases}
\dfrac{\partial R}{\partial m} &= \dfrac{-C_0^{bs}\theta_1 + C_0 e^{rT}(\theta_1 - S_0 e^{\mu T} + S_0 e^{rT})}{(\theta_1 m + C_0 e^{rT})^2} \\
\dfrac{\partial^2 R}{\partial m^2} &= \dfrac{-2\big(-C_0^{bs}\theta_1 + C_0 e^{rT}(\theta_1 - S_0 e^{\mu T} + S_0 e^{rT})\big)}{(\theta_1 m + C_0 e^{rT})^3} \\
&= -\dfrac{2}{\theta_1 m + C_0 e^{rT}} \dfrac{\partial R}{\partial m}
\end{cases}.
$$

Suppose $0 \leq -\theta_1 \leq S_0 e^{\mu T} - S_0 e^{rT} \leq -\theta_1 + S_0 e^{\mu T} - S_0 e^{rT} \leq C_0^{bs}$ holds, and, set

$$
\begin{aligned}
\mathcal{Z}^n(C_0) &= \Big\{ m \in [0,1] \,\Big|\, -C_0^{bs}\theta_1 + C_0 e^{rT}(\theta_1 - S_0 e^{\mu T} + S_0 e^{rT}) = 0 \Big\}, \\
\mathcal{Z}^d(C_0) &= \Big\{ m \in [0,1] \,\Big|\, \theta_1 m + C_0 e^{rT} = 0 \Big\}, \\
\mathcal{Z}^{n \cap d}(C_0) &= \mathcal{Z}^n(C_0) \bigcap \mathcal{Z}^d(C_0).
\end{aligned}
$$

This implies

$$
\begin{aligned}
\mathcal{Z}^n(C_0) &= \emptyset \quad \text{or} \quad [0,1], \\
\mathcal{Z}^d(C_0) &= \Big\{ m^{\mathcal{Z}^d} := -\frac{C_0 e^{rT}}{\theta_1} \Big\}, \\
\mathcal{Z}^{n \cap d}(C_0) &= \emptyset \quad \text{or} \quad \Big\{ m^{\mathcal{Z}^d} \Big\}.
\end{aligned}
$$

Consequently, when $0 \le C_0 \le -\theta_1 e^{-rT} =: C_0^{B,\infty}$, we have $0 \le m^{z^d} < 1$ and

$$R^* = R(m^{z^d}, C_0) = \frac{C_0^{bs} - m^{z^d} S_0 e^{\mu T} - (C_0 - m^{z^d} S_0) e^{rT}}{\theta_1 m^{z^d} + C_0 e^{rT}}$$
$$\ge \frac{C_0^{bs} - C_0 e^{rT}}{\theta_1 m^{z^d} + C_0 e^{rT}} = +\infty.$$

On the other hand, if $C_0^{B,\infty} \le C_0 \le e^{-rT} C_0^{bs} \left(\frac{-\theta_1}{-\theta_1 + S_0 e^{\mu T} - S_0 e^{rT}} \right) =: C_0^{B,0}$, we have

$$\frac{\partial R}{\partial m} \ge 0, \qquad \forall m \in [0,1].$$

This implies R is an increasing function of m, so the maximum is located at $m = 1$, which is

$$R^* = R(1, C_0) = \frac{C_0^{bs} - S_0 e^{\mu T} - (C_0 - S_0) e^{rT}}{\theta_1 + C_0 e^{rT}} \ge 0.$$

Otherwise, for $C_0 > C_0^{B,0}$, we have $\frac{\partial R}{\partial m} \le 0$ and so R is a decreasing function of m, leading to a maximum at $m = 0$,

$$R^* = R(0, C_0) = \frac{C_0^{bs} - C_0 e^{rT}}{C_0 e^{rT}}.$$

We denote

$$C_0^{B,-} = e^{-rT} C_0^{bs},$$

as soon as $C_0 \ge C_0^{B,-}$, we get

$$R^* = R(0, C_0) = \frac{C_0^{bs} - C_0 e^{rT}}{C_0 e^{rT}} \le 0.$$

In summary,

$$R^* = \sup_m R(m, C_0) = R(m^*, C_0)$$
$$= \begin{cases} R(m^{z^d}, C_0) = +\infty & \text{for} \quad C_0 \in \left[0, C_0^{B,\infty}\right] \\ R(1, C_0) = \dfrac{C_0^{bs} - S_0 e^{\mu T} - (C_0 - S_0) e^{rT}}{\theta_1 + C_0 e^{rT}} \ge 0 & \text{for} \quad C_0 \in \left(C_0^{B,\infty}, C_0^{B,0}\right] \\ R(0, C_0) = \dfrac{C_0^{bs} - C_0 e^{rT}}{C_0 e^{rT}} \ge 0 & \text{for} \quad C_0 \in \left(C_0^{B,0}, C_0^{B,-}\right] \\ R(0, C_0) = \dfrac{C_0^{bs} - C_0 e^{rT}}{C_0 e^{rT}} \le 0 & \text{for} \quad C_0 \in \left(C_0^{B,-}, +\infty\right) \end{cases}.$$

2.5.4 Supplement to Section 2.3.1

2.5.4.1 Derivation of (2.3.2)

Here we recall that, for a lognormal random variable Y, $Y \sim \mathcal{LN}(a,b)$, the conditional expectation $E[Y \mid Y > y]$ can be computed as

$$\begin{aligned} E[Y \mid Y > y] &= \frac{E[Y\mathbf{1}_{\{Y>y\}}]}{P(Y>y)} \\ &= \frac{e^{a+\frac{1}{2}b^2}}{P(Y>y)}\Big(1-\Phi^\circ(\ln y)\Big) = \frac{e^{a+\frac{1}{2}b^2}}{P(Y>y)}\Big(1-\Phi\Big(\frac{\ln y-(a+b^2)}{b}\Big)\Big) \end{aligned} \tag{2.5.6}$$

where Φ° is the probability distribution function of a normal random variable with drift $a^\circ := a+b^2$ and standard deviation b. Then,

$$\begin{aligned} &ES_\alpha^A(V_T;\mathrm{VaR}) \\ &= -E^P\Big[mS_T-(S_T-K)^+ + (C_0-mS_0)e^{rT} \;\Big|\; V_T \le -\mathrm{VaR}_\alpha^A\Big] \\ &= -E^P\Big[mS_T-(S_T-K)^+ \;\Big|\; V_T \le -\mathrm{VaR}_\alpha^A\Big] - (C_0-mS_0)e^{rT} \\ &= -\frac{E^P\Big[\big(mS_T-(S_T-K)^+\big)\mathbf{1}_{\{V_T\le -\mathrm{VaR}_\alpha^A\}}\Big]}{P(V_T \le -\mathrm{VaR}_\alpha^A)} - (C_0-mS_0)e^{rT} \\ &= -\frac{E^P\Big[\big(mS_T-(S_T-K)^+\big)\mathbf{1}_{\{S_T\le \underline{S}_T-\frac{x}{m}\}\cup\{S_T\ge \overline{S}_T+\frac{x}{1-m}\}}\Big]}{\alpha} - (C_0-mS_0)e^{rT} \\ &= -\frac{E^P\Big[\big(mS_T-(S_T-K)^+\big)\mathbf{1}_{\{S_T\le \underline{S}_T-\frac{x}{m}\}}\Big]}{\alpha} - \frac{E^P\Big[\big(mS_T-(S_T-K)^+\big)\mathbf{1}_{\{S_T\ge \overline{S}_T+\frac{x}{1-m}\}}\Big]}{\alpha} \\ &\quad - (C_0-mS_0)e^{rT} \\ &= -\frac{E^P\Big[mS_T\mathbf{1}_{\{S_T\le \underline{S}_T-\frac{x}{m}\}}\Big]}{\alpha} - \frac{E^P\Big[\big(mS_T-(S_T-K)\big)\mathbf{1}_{\{S_T\ge \overline{S}_T+\frac{x}{1-m}\}}\Big]}{\alpha} - (C_0-mS_0)e^{rT} \\ &= -\frac{E^P\big[mS_T\big]}{\alpha} + \frac{E^P\Big[mS_T\mathbf{1}_{\{S_T\ge \underline{S}_T-\frac{x}{m}\}}\Big]}{\alpha} - \frac{E^P\Big[\big((m-1)S_T+K\big)\mathbf{1}_{\{S_T\ge \overline{S}_T+\frac{x}{1-m}\}}\Big]}{\alpha} \\ &\quad - (C_0-mS_0)e^{rT} \\ &= -\frac{mS_0e^{\mu T}}{\alpha} + \frac{m\cdot E^P\Big[S_T\mathbf{1}_{\{S_T\ge \underline{S}_T-\frac{x}{m}\}}\Big]}{\alpha} - \frac{(m-1)\cdot E^P\Big[S_T\mathbf{1}_{\{S_T\ge \overline{S}_T+\frac{x}{1-m}\}}\Big]}{\alpha} \\ &\quad - \frac{K\cdot E^P\Big[\mathbf{1}_{\{S_T\ge \overline{S}_T+\frac{x}{1-m}\}}\Big]}{\alpha} - (C_0-mS_0)e^{rT} \\ &= -\frac{mS_0e^{\mu T}}{\alpha} + \frac{m\cdot E^P\Big[S_T\mathbf{1}_{\{S_T\ge \underline{S}_T-\frac{x}{m}\}}\Big]}{\alpha} - \frac{(m-1)\cdot E^P\Big[S_T\mathbf{1}_{\{S_T\ge \overline{S}_T+\frac{x}{1-m}\}}\Big]}{\alpha} \\ &\quad - \frac{K\cdot P\big(S_T\ge \overline{S}_T+\frac{x}{1-m}\big)}{\alpha} - (C_0-mS_0)e^{rT}. \end{aligned}$$

2.5.4.2 Derivation of (2.3.3)

Similar to the derivation of $ES^A_\alpha(V_T;\mathrm{VaR})$, we have

$$
\begin{aligned}
&ES^B_\alpha(V_T;\mathrm{VaR}^B_\alpha)\\
&= -E^P\Big[(S_T-K)^+ - mS_T - (C_0-mS_0)e^{rT} \,\Big|\, V_T \le -\mathrm{VaR}^B_\alpha\Big]\\
&= -E^P\Big[(S_T-K)^+ - mS_T - (C_0-mS_0)e^{rT} \,\Big|\, \underline{S}_T + \frac{x}{m} \le S_T \le \overline{S}_T - \frac{x}{1-m}\Big]\\
&= -E^P\Big[(S_T-K)^+ \,\Big|\, \underline{S}_T + \frac{x}{m} \le S_T \le \overline{S}_T - \frac{x}{1-m}\Big]\\
&\quad + mE^P\Big[S_T \,\Big|\, \underline{S}_T + \frac{x}{m} \le S_T \le \overline{S}_T - \frac{x}{1-m}\Big] + (C_0-mS_0)e^{rT}\\
&= -\frac{E^P\Big[(S_T-K)^+\mathbf{1}_{\{\underline{S}_T+\frac{x}{m}\le S_T\le\overline{S}_T-\frac{x}{1-m}\}}\Big]}{P(\underline{S}_T+\frac{x}{m}\le S_T\le\overline{S}_T-\frac{x}{1-m})} + \frac{mE^P\Big[S_T\mathbf{1}_{\{\underline{S}_T+\frac{x}{m}\le S_T\le\overline{S}_T-\frac{x}{1-m}\}}\Big]}{P(\underline{S}_T+\frac{x}{m}\le S_T\le\overline{S}_T-\frac{x}{1-m})}\\
&\quad + (C_0-mS_0)e^{rT}\\
&= -\frac{E^P\Big[(S_T-K)\mathbf{1}_{\{K\le S_T\le\overline{S}_T-\frac{x}{1-m}\}}\Big]}{\alpha} + \frac{m}{\alpha}E^P\Big[S_T\mathbf{1}_{\{\underline{S}_T+\frac{x}{m}\le S_T\le\overline{S}_T-\frac{x}{1-m}\}}\Big]\\
&\quad + (C_0-mS_0)e^{rT}\\
&= -\frac{E^P\Big[S_T\mathbf{1}_{\{K\le S_T\le\overline{S}_T-\frac{x}{1-m}\}}\Big]}{\alpha} + K\cdot\frac{P(K\le S_T\le\overline{S}_T-\frac{x}{1-m})}{\alpha}\\
&\quad + \frac{m}{\alpha}E^P\Big[S_T\mathbf{1}_{\{\underline{S}_T+\frac{x}{m}\le S_T\le\overline{S}_T-\frac{x}{1-m}\}}\Big] + (C_0-mS_0)e^{rT}\\
&= -\frac{E^P\Big[S_T\mathbf{1}_{\{S_T\ge K\}}\Big] - E^P\Big[S_T\mathbf{1}_{\{S_T\ge\overline{S}_T-\frac{x}{1-m}\}}\Big]}{\alpha} + K\cdot\frac{P(K\le S_T\le\overline{S}_T-\frac{x}{1-m})}{\alpha}\\
&\quad + \frac{m}{\alpha}\Big(E^P\Big[S_T\mathbf{1}_{\{S_T\ge\underline{S}_T+\frac{x}{m}\}}\Big] - E^P\Big[S_T\mathbf{1}_{\{S_T\ge\overline{S}_T-\frac{x}{1-m}\}}\Big]\Big) + (C_0-mS_0)e^{rT}.
\end{aligned}
$$

3. CONTINUOUS-TIME RAROC MAXIMIZATION

Static RAROC maximization has been studied in previous chapter. In particular, if we try to hedge a European call option with an initial endowment that is less than the Black-Scholes price, we are able to determine an optimal static hedge such that the RAROC of the hedged position is maximized. However, restriction of only a static hedge being allowed is not reasonable in a real financial market. As a result, we consider the case of continuous-time trading. Yet the market is still modeled by the standard Black-Scholes model. Furthermore, we consider the situation that one is already in a long/short position of a European call option but unfortunately can only devote an initial endowment less than the unique Black-Scholes price for a European call option (due to mismanagement for example). So, one cannot completely hedge and a residual risk exists in the hedged position. In this case, we investigate the use of the RAROC approach in setting up a hedging strategy for the call option.

Since a continuous-time trading strategy is used, we can no longer obtain approximations of value-at-risk and expected shortfall as in the previous chapter, and this would bring difficulties while maximizing RAROC. For the sake of analytical tractability, we would value-at-risk and expected shortfall for risk measurement. Instead, the magnitude of risk is determined as the expected value of losses. To further ease subsequent analysis, we specifically focus on a class of continuous-time hedging strategies for maximization of RAROC.

3.1 Background

Assume a complete probability space $(\Omega, \mathcal{F}, P)$ that is augmented with a right-continuous filtration $\{\mathcal{F}_t\}_{t \leq T}$, where $T \in \mathbb{R}^+$. In the context of a *complete* financial market with absence of arbitrage, a contingent claim C, which is a sufficiently smooth and integrable $\mathcal{F}_T$-measurable random variable, can be perfectly hedged. More precisely, by charging the counterparty with a suitable initial price x_0, one can seek a self-financing trading strategy π such that the corresponding portfolio $X^{x_0,\pi}$ replicates the contingent claim C at time $t = T$ *without the presence of any residual risk*, that is,

$$X_T^{x_0,\pi} = C_T \quad P\text{-a.s.} \quad \text{under a self-finaning trading strategy } \pi.$$

The price C_0 of the contingent claim C is naturally set to be x_0. Here, a trading strategy π of a portfolio $X^{x_0,\pi}$ with initial value $X_0 = x_0$ is called self-financing if the value of the portfolio $X^{x_0,\pi}$ satisfies

$$X_t^{x_0,\pi} = X_0 + \int_0^t \pi_u dS_u \qquad \forall t,$$

where S denotes the price process of a tradeable asset in the financial market.

In an incomplete market, *not all* contingent claims can be replicated and some residual risk might exist. The only way to get rid of this residual risk is through a *superhedging* portfolio, that is a portfolio $X^{x_0,\pi}$ with initial value x_0 and self-financing trading strategy π such that $X_T^{x_0,\pi} \geq C_T$ P-a.s.. Unfortunately, setting up this kind of portfolio often necessitates a huge amount of initial capital, see Cvitanić (2000). As a result, it would be worthwhile to study pricing and hedging of a contingent claim with the existence of residual risk when one does not superhedge. In particular, we consider an investor who does not have sufficient funding and so he can only allocate an amount of capital x_0 less than the perfect replication price C_0 for constructing a hedging portfolio $X_T^{x_0,\pi}$. This results in an imperfect hedge for the contingent claim, hence, inducing a residual risk $Y := X_T^{x_0,\pi} - C_T$. Indeed, the same situation is also investigated in Föllmer and Leukert (1999).

To give a proper assessment of residual risk, we might rely on a *coherent risk measure* which is introduced by Artzner et al. (1999). This measure of risk satisfies a set of axioms, under which it admits a general representation given by

$$\rho(Y) = \sup_{\nu \in \mathcal{P}} \mathbb{E}^\nu[-Y],$$

where $\mathcal{P}$ is a family of probability measures ν which are absolutely continuous with respect to P, i.e. $\nu \ll P$. For exposition of the axioms and the representation result, see Artzner et al. (1999). The measurement $\rho(Y)$ of the risk Y is done in such a way that Y is happened under the 'most adverse scenario[1]'. Under the worst-case scenario ν, the risk assessment is then computed as the expectation $\mathbb{E}^\nu[-Y]$. $\rho(Y)$ is essentially the greatest value of all these expectations. Apart from coherent risk measures, we could also make use of risk measures that are not coherent. A common example is value-at-risk (*VaR*) or *shortfall risk* $ES(Y)$ which is defined as

$$ES(Y) := \mathbb{E}[(-Y)^+] = \mathbb{E}[(-(X_T^{x_0,\pi} - C_T))^+] = \mathbb{E}[(C_T - X_T^{x_0,\pi})^+].$$

[1]Scenario here is not in pathwise sense $\omega \in \Omega$ but is described by a probability measure ν. That is why $\mathcal{P}$ is also regarded as a family of *probabilistic scenarios*.

This is not a coherent risk measure[2]. We would focus on shortfall risk throughout the whole chapter. It is also assumed in Cvitanić (2000) and Föllmer and Leukert (2000).

For the selection of an optimal hedging strategy, different criterion used lead to different answers, for instance, if one cares about the probability of successfully hedging, $P(X_T^{x_0,\pi} \geq C_T)$, the optimal hedging strategy would be the quantile-hedging strategy described in Föllmer and Leukert (1999). In our context, we choose risk-adjusted return on capital (RAROC) as the criterion and identify the optimal hedging strategy which maximizes RAROC. The RAROC (abbreviated as R) of a position X is defined as the ratio of expected profit-and-loss (PnL for short) to risk, which is mathematically expressed as

$$R(X) := \frac{\mathbb{E}^P[X]}{\rho(X)}. \tag{3.1.1}$$

Without loss of generality, we consider ourselves a seller in all subsequent analysis. Then the optimal hedging strategy that maximizes R of the seller's position is determined from the following optimization problem: for a given $x_0 < C_0$,

$$R^*(X_T^{x_0,\pi}) := \sup_{\substack{\pi \in \mathcal{A}(x_0) \\ X_T^{x_0,\pi} \in \mathcal{F}_T}} R(X_T^{x_0,\pi}) = \sup_{\substack{\pi \in \mathcal{A}(x_0) \\ X_T^{x_0,\pi} \in \mathcal{F}_T}} \frac{\mathbb{E}^P\left[X_T^{x_0,\pi} - C_T\right]}{\rho\left(X_T^{x_0,\pi} - C_T\right)} \tag{3.1.2}$$

where $Y = X_T^{x_0,\pi} - C_T$, $\mathcal{A}(x_0)$ is the set of self-financing trading strategies π with an initial endowment x_0 such that $X_t^{x_0,\pi} \geq 0$, $\forall t \in [0,T]$ and we have abused to use $X_T^{x_0,\pi} \in \mathcal{F}_T$ for meaning $\mathcal{F}_T$-measurability of $X_T^{x_0,\pi}$. We define $R^*(X_T^{x_0,\pi}) := +\infty$ whenever $x_0 \geq C_0$ since this corresponds to the circumstance that one can perfectly hedge the contingent claim, hence, zero risk. For convenience, we would simply use R and X to denote $R(X_T^{x_0,\pi})$ and $X^{x_0,\pi}$.

RAROC is a portfolio performance measure, so, the intuitive content behind (3.1.2) is that our characterization of hedging strategy leads to *maximum performance* but not simply *minimum risk*. For literature regarding risk-minimizing hedging strategy, one may consult Cvitanić (2000). Since we concern about performance, a 'good' portfolio should be able to offer some *positive* gain. Except in the context of superhedging, a positive profit from a hedging strategy is not discussed in the construction of ordinary hedging methodologies, in which they care about downside loss and put aside upside reward. This is a critical disparity between RAROC-maximzing hedging strategy and existing hedging techniques in literatures. The motivation for studying this problem is that residual risk is usually present due to imperfect hedging of a contingent claim in

[2]It violates the axiom of translation invariance, that is, $\rho(X+a) = \rho(X) - a$, $\forall a \in \mathbb{R}$, from the observation of $(X+a)^+ \neq X^+ - a$.

a realistic financial market and so reservation of economic capital is inevitable for risk management. Thus, in order to *utilize economic capital in a more effective way*, we should resort to a hedging portfolio which can possibly offer some positive profit apart from its hedging ability.

In Chapter 2, the problem is investigated preliminarily by restricting to the class of static hedging strategies. Most of the time the hedging strategy is not static, so it is a natural consequence to analyze under the scope of continuous-time trading. For the sake of simplicity, we would remain using the Black-Scholes model for the dynamics of the underlying asset. Dynamic programming is viable for solving the optimization problem in (3.1.2), however, thanks to the tractability of the Black-Scholes model, we can indeed tackle the dynamic stochastic control problem in (3.1.2) through two stages, namely,

i. solving a *static* optimization problem for an optimal $\mathcal{F}_T$-measurable random variable X_T^*, and then

ii. characterizing the corresponding optimal hedging strategy π^* to replicate X_T^*.

The success of this two-stage approach for solving the problem is due to the attainability of *any* $\mathcal{F}_T$-measurable random variable in a complete market. Breakdown of the single problem into two subproblems is pioneered by Pliska (1986). Adopting this method to the current problem in (3.1.2), at a given $x_0 < C_0$, we shall firstly work out the optimal solution X_T^* in the first subproblem defined by

$$\begin{aligned} &\underset{X_T \in \mathcal{F}_T}{\text{Maximize}} \quad R(X_T) = \frac{\mathbb{E}^P\big[X_T - C_T\big]}{\rho\big(X_T - C_T\big)} \\ &\text{subject to} \quad \mathbb{E}^Q[e^{-rT} X_T] \leq x_0, \end{aligned} \tag{3.1.3}$$

where Q is a pricing measure of the model under consideration, e.g. risk-neutral measures. After the optimal solution is found, the corresponding self-financing trading strategy π^* for the portfolio X_T^* can be identified, e.g. through techniques in martingale theory. However, we concentrate on the first subproblem of identifying the optimal hedging portfolio X_T^* in this chapter, leaving the second subproblem unattended.

3.2 RAROC as an Acceptability Index

In Cherny and Madan (2009), similar to the development of coherent risk measure in Artzner et al. (1999), they have proposed a set of eight axioms underlying measures of performance. If a measure satisfies the property of quasi-concavity, monotonicity, scale invariance and Fatou property, see Cherny and Madan (2009) for details, it is regarded as an acceptability index, which can be conceived intuitively as the likelihood of accepting a deal. Among the examples of acceptability indices they mentioned, RAROC in (3.1.1) has received in-depth discussion. We particularly prove the property of quasi-concavity in RAROC. We first have

Lemma 3.2.1. *For two positive functions f, g which map from $\mathbb{R}^+$ to $\mathbb{R}^+$, if f is concave and increasing and g is convex and decreasing, then $\dfrac{f}{g}$ is quasi-concave. Furthermore, if either f or g possess the above properties in the* strict *sense, for instance f is strictly concave, $\dfrac{f}{g}$ is strictly quasi-concave and strictly increasing.*

Proof. Recall that for a positive convex and decreasing function g, $\frac{1}{g}$ is an increasing concave function. It then suffices to prove that the product of two increasing concave functions is quasiconcave.

Let f, g be increasing concave functions. Without loss of generality, we assume $f(x)g(x) < f(y)g(y)$. Due to the fact that product of two increasing functions is also an increasing function, we have $f(x)g(x) < f(y)g(y) \Rightarrow x < y$. Hence, for any $\lambda \in (0, 1)$, we must have $x < \lambda x + (1 - \lambda)y < y$ and this leads to

$$\begin{aligned} f\Big(\lambda x + (1-\lambda)y\Big)g\Big(\lambda x + (1-\lambda)y\Big) &\geq \Big(\lambda f(x) + (1-\lambda)f(y)\Big)\Big(\lambda g(x) + (1-\lambda)g(y)\Big) \\ &\geq \Big(\lambda f(x) + (1-\lambda)f(x)\Big)\Big(\lambda g(x) + (1-\lambda)g(x)\Big) \\ &= f(x)g(x) \end{aligned}$$

in which the first and second inequality are respectively from the concavity and increasing property of f, g. As a result, the conclusion follows from $f\big(\lambda x + (1-\lambda)y\big)g\big(\lambda x + (1-\lambda)y\big) \geq \min\big(f(x)g(x), f(y)g(y)\big)$ for any $x, y \in \mathbb{R}^+$ and $\lambda \in (0, 1)$.

For the last claim, notice that all the inequalities are strict inequalities if the properties are taken in strict sense. □

Lemma 3.2.2. *$R(Y) := \dfrac{\mathbb{E}[Y]}{\rho(Y)}$ is quasi-concave and an increasing functional of Y. Moreover, if ρ satisfies the* Fatou *property, then R is strictly quasi-concave and strictly increasing.*

Proof. Note that the ordering and convergence for random variables here is in the almost-sure sense. Then we observe immediately that $\rho(Y)$ is a decreasing convex functional of Y by definition. Furthermore, a coherent risk measure is also a convex risk measure, see Föllmer and Schied (2002a), hence, monotonicity (more precisely, decreasing) and convexity of ρ are justified. On the other hand, $\mathbb{E}[Y]$ is obviously an increasing linear functional, so is increasing and concave. Consequently, by applying Lemma 3.2.1, it is readily concluded that $R(Y)$ is quasi-concave. Increasing property is obvious.

When the risk measure ρ is coherent and satisfies the Fatou property, this is equivalent to *continuity from above* in ρ, see Schied (2004), i.e.

$$Y_n \downarrow Y \text{ a.s.} \quad \Longrightarrow \quad \rho(Y_n) \uparrow \rho(Y).$$

For a fixed Y, if it is discontinuous at this point, owning to the increasing property of R, strict quasi-concavity trivially holds. On the other hand, if it is continuous, subject to Fatou property, strict quasi-concavity is again established. $\square$

Remark 3.2.0.1. Indeed the Fatou property of R is also justified in the proof of Lemma 3.2.2.

Quasi-concavity of an acceptability index refers to, for a given value of x,

$$\text{if } \alpha(X) \geq x \text{ and } \alpha(Y) \geq x, \text{ then } \alpha(\lambda X + (1-\lambda)Y) \geq x \text{ for any } \lambda \in [0,1].$$

This property can be heuristically understood as 'if both X and Y are acceptable, then any deal structured as a linear combination of X and Y should also be accepted'. Furthermore Cherny and Madan (2009) have obtained a representation result for the class of acceptability indices. In particular, for R under a coherent risk measure ρ, we have

$$R(X) := \frac{\mathbb{E}^P[X]}{\rho(X)} = \sup\left\{x \ : \ \beta_x(X) \geq 0\right\} := \sup\left\{x \ : \ \inf_{Q \in \mathcal{D}_x} \mathbb{E}^Q[X] \geq 0\right\},$$

where, at each fixed x, $-\beta_x(X) = -\inf_{Q\in\mathcal{D}_x} \mathbb{E}^Q[X]$ is a coherent risk measure and $\mathcal{D}_x$ is the support kernel for $\beta_x(X)$. $\mathcal{D}_x$ at each x is given by

$$\mathcal{D}_x = \frac{1}{1+x}\{P\} + \frac{x}{1+x}\mathcal{D},$$

in which $\mathcal{D}$ is the support kernel for ρ. Using this result, the optimization problem in (3.1.3) can be written as

$$\begin{aligned} &\underset{X_T \in \mathcal{F}_T}{\text{Maximize}} \quad R(X_T) = \sup\left\{x \ : \ \inf_{Q\in\mathcal{D}_x} \mathbb{E}^Q[X_T - C_T] \geq 0\right\} \\ &\text{subject to} \quad \mathbb{E}^Q[e^{-rT}X_T] \leq x_0 \end{aligned} \tag{3.2.1}$$

This formulation offers an advantage over the original version in (3.1.3). In (3.1.3), we need to optimize an objective function that is a fraction with both numerator and denominator depending on the random variable X and this poses challenges in solving the problem. However, in (3.2.1), the objective function is transformed into a non-fractional function, which is relatively easier to optimize.

Finally we should remark that the above result holds only for coherent risk measures ρ . Since we are interested in the shortfall risk ES, which is not coherent, we cannot arrive at the formulation in (3.2.1) but need to tackle the original version of problem in (3.1.3).

3.3 Maximization of RAROC under Shortfall Risk in Continuous-time

Föllmer and Leukert (2000) considered the shortfall risk of a hedged position. It is measured by $\mathbb{E}[L(X_T - C_T)]$ where $X_T - C_T$ is the value of the hedged position and $L(y) = l(y)\mathbf{1}_{(0,\infty)}(y)$ is some increasing convex loss function defined on $[0, \infty)$ with $L(0) = 0$. If this risk measure is used in R, the corresponding problem in (3.1.3) is

$$\begin{aligned} &\underset{X_T \in \mathcal{F}_T}{\text{Maximize}} \quad R := \frac{\mathbb{E}[X_T - C_T]}{\rho(X_T - C_T)} = \frac{\mathbb{E}[X_T - C_T]}{\mathbb{E}[L(C_T - X_T)]} \\ &\text{subject to} \quad \mathbb{E}^Q[e^{-rT} X_T] \le x_0. \end{aligned}$$

Loosely speaking, L represents the degree of investor's attention to the hedging error (or residual risk) $X_T - C_T$. The risk measurement $\rho(X)$ is conducted by 'penalizing' $X_T - C_T$ through the weighting function l. For simplicity we consider a linear function l, so that $L(y) = y\mathbf{1}_{(0,\infty)}(y)$, and $L(C_T - X_T) = (C_T - X_T)^+$ and $ES(C_T - X_T) = \mathbb{E}[(C_T - X_T)^+]$.

Due to $\mathbb{E}[(X_T - C_T)^-] = \mathbb{E}[(C_T - X_T)^+]$, the problem is written as

$$\begin{aligned} &\underset{X_T \in \mathcal{F}_T}{\text{Maximize}} \quad R = \frac{\mathbb{E}[(X_T - C_T)^+]}{\mathbb{E}[(X_T - C_T)^-]} - 1 \\ &\text{subject to} \quad \mathbb{E}^Q[e^{-rT} X_T] \le x_0 \end{aligned} \tag{3.3.1}$$

In this formulation, maximizing RAROC requires the inputs of the *profit* captured by $\mathbb{E}[(\cdot)^+]$, and, the *loss* captured by $\mathbb{E}[(\cdot)^-]$. Also, it becomes clear that several classes of risk-minimizing strategy, for example those described in Föllmer and Leukert (1999) and Spivak and Cvitanić (1999), would lead to $R = -1$ due to their features in common, that is, the desired hedging portfolios X_T can at most generate cashflow the same as C_T, meaning zero profit from the hedged position, i.e. $(X_T - C_T)^+ = 0$, with non-zero risk.

We rewrite the unconditional expectations in (3.3.1) using conditional expectations so that

$$R = \frac{\mathbb{E}[(X_T - C_T)^+]}{\mathbb{E}[(X_T - C_T)^-]} - 1 = \frac{\mathbb{E}[X - C | X \ge C]}{\mathbb{E}[C - X | X < C]} \cdot \frac{P(X \ge C)}{P(X < C)} - 1. \tag{3.3.2}$$

Then we are able to comment that if we wish to maximize R, we need to consider a suitable X that can maximize both $\frac{\mathbb{E}[X - C | X \ge C]}{\mathbb{E}[C - X | X < C]}$ and $\frac{P(X \ge C)}{P(X < C)}$. On the one hand, determining maximum $\frac{P(X \geq C)}{P(X<C)}$ resembles the problem of maximizing the probability of a 'success' event as in Föllmer and Leukert (1999). Let us call $\{X \geq C\}$ the 'success' event so that we may utilize the quantile-hedging technique in Föllmer and Leukert (1999) and characterize the 'success' event.

On the other hand, $\frac{\mathbb{E}[X-C|X \geq C]}{\mathbb{E}[C-X|X<C]}$ exhibits clearly that, apart from $X_T = C_T$ in the 'success' event, we should require the *strict inequality* $X_T > C_T$ in the 'success' event to get its value different from zero. Combining these arguments, we can derive the necessary condition for the optimal solution X^*.

Proposition 3.3.1. *If X^* is the optimal solution for the optimization problem in* (3.3.1)*, it is necessary and sufficient that X^* also maximizes* (3.3.2)*, or equivalently, the value of the product* $\dfrac{\mathbb{E}[X-C|X \geq C]}{\mathbb{E}[C-X|X<C]} \cdot \dfrac{P(X \geq C)}{P(X<C)}$.

This explains again why generic quantile-hedging cannot be the optimal solution. It is because the quantile-hedging technique aims at maximizing the probability of the 'success' event. The payoff of the hedged position under the quantile-hedging method is designed to be the same as the contingent claim at each state $\omega \in \Omega$, in this way one is able to 'be safe' at more states ω and so increases the probability of the 'success' event. More concisely, the computation of $P(X \geq C)$ does not involve the magnitude of $X(\omega) - C(\omega)$ at fixed ω but the calculation of $\mathbb{E}[X - C|X \geq C]$ takes this into account. In a quantile-hedging strategy, even though the chance of 'success' is maximized, the 'failure' event can still happen and once it happens, our position may not be hedged and a very serve loss can be incurred. This loss is not captured if one only aims at maximizing the probability of the 'success' event. As a result, *the risk-minimizing hedge is generally not a performance-maximizing hedge.*

As $\frac{P(X \geq C)}{P(X<C)}$ is non-negative[3], whether R is greater than -1 or positive depends on the value of $\frac{\mathbb{E}[X-C|X \geq C]}{\mathbb{E}[C-X|X<C]}$. We consider X in the form of

$$X = X(\varepsilon, A, B) = (C + \varepsilon B)\mathbf{1}_A$$

where B is a $\mathcal{F}_T$-measurable random variable such that $0 \leq B(\omega) < +\infty$ a.s., A is a $\mathcal{F}_T$-measurable event and $\varepsilon \in \mathbb{R}^+$. Under the assumption of a complete market, B can be replicated with some trading strategy, hence, we may understand B as a portfolio. Then the use of X can be regarded as *overhedging* C by holding ε units of B when event A happens so that extra profit is generated. When event A does not occur, we have a naked position, i.e. the hedge is ineffective.

[3]It is also finite unless $x_0 \geq C_0$.

3.4 Maximum RAROC under Candidate Hedging Portfolio

In this section we investigate the maximum value R^* of R that can be achieved by the aforementioned portfolio. Recall that $X = X(\varepsilon, A, B) = (C + \varepsilon B)\mathbf{1}_A$ and assume the contingent claim C is a European call option at strike K. We take $B = S$, where S is the underlying asset. We would perform an optimization with this X over both ε and A such that the hedge $X(\varepsilon, A)$ produces a surplus at maturity. The corresponding results and optimal values of ε and A will be unwounded step-by-step in this section.

Instead of treating $A \in \mathcal{F}_T$ as a free variable to be determined, it is indeed the 'success' event $\{X \geq C\}$, i.e. $A = \{X_T \geq C_T\}$. We express X as a function of ε given by

$$X = X(\varepsilon) = (C + \varepsilon S)\mathbf{1}_{\{X \geq C\}}. \tag{3.4.1}$$

Substituting this into (3.3.1),

$$\begin{aligned}
&\underset{X_T(\varepsilon)\in\mathcal{F}_T}{\text{Maximize}} \quad R = \frac{\mathbb{E}[(X_T(\varepsilon) - C_T)^+]}{\mathbb{E}[(X_T(\varepsilon) - C_T)^-]} - 1 \\
&\text{subject to} \quad \mathbb{E}^Q[e^{-rT} X_T(\varepsilon)] \leq x_0 \\
\Longrightarrow \quad &\underset{\varepsilon\in\mathbb{R}}{\text{Maximize}} \quad R = \frac{\mathbb{E}[\varepsilon S_T \mathbf{1}_{\{X_T(\varepsilon)\geq C_T\}}]}{\mathbb{E}[C_T \mathbf{1}_{\{X_T(\varepsilon)< C_T\}}]} - 1 \\
&\text{subject to} \quad \mathbb{E}^Q[e^{-rT}(C_T + \varepsilon S)\mathbf{1}_{\{X_T(\varepsilon)\geq C_T\}}] \leq x_0
\end{aligned}$$

We shall determine the optimal value ε^* for characterizing $X^* = X(\varepsilon^*)$. From (3.3.2), it is also equivalent to

$$\begin{aligned}
&\underset{\varepsilon\in\mathbb{R}}{\text{Maximize}} \quad R = \frac{\mathbb{E}[\varepsilon S_T|\{X_T(\varepsilon) \geq C_T\}]}{\mathbb{E}[C_T|\{X_T(\varepsilon) < C_T\}]} \cdot \frac{P(X_T(\varepsilon) \geq C_T)}{P(X_T(\varepsilon) < C_T)} - 1 \\
&\text{subject to} \quad \mathbb{E}^Q[e^{-rT}(C_T + \varepsilon S_T)\mathbf{1}_{\{X_T(\varepsilon)\geq C_T\}}] \leq x_0
\end{aligned} \tag{3.4.2}$$

The value of ε affects both values of $\frac{\mathbb{E}[\varepsilon S_T|\{X_T(\varepsilon)\geq C_T\}]}{\mathbb{E}[C_T|\{X_T(\varepsilon)<C_T\}]}$ and $\frac{P(X_T(\varepsilon)\geq C_T)}{P(X_T(\varepsilon)<C_T)}$. We shall characterize $\{X_T(\varepsilon) \geq C_T\}$ as $\{S_T \in \mathcal{S}\}$ where $\mathcal{S} = \mathcal{S}(\varepsilon)$ is some subset of $\mathbb{R}$. So the question now is how we locate the subset $\mathcal{S}$ such that R is maximized. To do so, we recognize that maximizing $\frac{P(X_T(\varepsilon)\geq C_T)}{P(X_T(\varepsilon)<C_T)}$ is reminiscent of the quantile-hedging problem, in which the probability of the 'success' event is maximized. That means we may utilize the solution techniques described in Föllmer and Leukert (1999) for identifying the 'success' event $\{X_T \geq C_T\}$ in terms of $\{S_T \in \mathcal{S}\}$. As a result, for a fixed $\varepsilon \in [0, 1]$, we firstly work with the quantile-hedging problem for a fictitious contingent claim $\widetilde{C} = \widetilde{C}(\varepsilon) = C + \varepsilon S$.

The associated quantile-hedging portfolio $\widetilde{X}$ is determined as the solution of

$$\begin{aligned} \underset{\widetilde{X}}{\text{Maximize}} \quad & P(\widetilde{X}_T \geq \widetilde{C}_T) \\ \text{subject to} \quad & \mathbb{E}^Q[e^{-rT}\widetilde{X}_T] \leq x_0 \end{aligned}$$

This would produce an optimal 'success' event $\{\widetilde{X}_T \geq \widetilde{C}_T\}$ in form of $\{S_T \in \mathcal{S}^*\}$. By substituting $\{S_T \in \mathcal{S}^*\}$, we can compute easily both $\mathbb{E}[\varepsilon S_T|\{X_T(\varepsilon) \geq C_T\}]$ and $\mathbb{E}[C_T|\{X_T(\varepsilon) < C_T\}]$. As soon as this is done, R is a function of ε and so the remaining task is merely optimization of $R = R(\varepsilon)$ over ε for maximum value R^*.

To sum up, the problem in (3.4.2) is solved according to the following procedures:

i. Define $X(\varepsilon, A) = (C + \varepsilon S)\mathbf{1}_A$ and solve two subproblems subsequently.

ii. Assume a fixed ε, we solve a quantile-hedging problem:

$$\begin{aligned} \underset{\widetilde{X}}{\text{Maximize}} \quad & P(\widetilde{X}_T \geq \widetilde{C}_T) \\ \text{subject to} \quad & \mathbb{E}^Q[e^{-rT}\widetilde{X}_T] \leq x_0 \end{aligned}$$

which would lead to maximum value in $\frac{P(X_T(\varepsilon)\geq C_T)}{P(X_T(\varepsilon)<C_T)}$. The corresponding 'success' event $\{\widetilde{X}_T \geq \widetilde{C}_T\}$ is characterized as $\{S_T \in \mathcal{S}\}$ for some subset $\mathcal{S}$. Here, $\mathcal{S}$ depends on ε.

iii. Substituting $\{S_T \in \mathcal{S}\}$ into $\mathbb{E}[\varepsilon S_T|\{X_T(\varepsilon) \geq C_T\}]$ and $\mathbb{E}[C_T|\{X_T(\varepsilon) < C_T\}]$, these expectations then become functions of ε and so is R.

iv. Finally, R is maximized over ε.

The maximum R obtained from the above procedures *cannot* be guaranteed to be the true one. This is because we attempt to characterize $\{X_T \geq C_T\}$ as $\{S_T \in \mathcal{S}^*\}$ through the quantile-hedging techniques, with this subset, this can only assure $\frac{P(X_T\geq C_T)}{P(X_T<C_T)}$ achieves its maximum. Yet the value of $\frac{\mathbb{E}[\varepsilon S_T|\{X_T\geq C_T\}]}{\mathbb{E}[C_T|\{X_T<C_T\}]}$ is not necessarily maximized. Eventually, their product $\frac{\mathbb{E}[\varepsilon S_T|\{X_T\geq C_T\}]}{\mathbb{E}[C_T|\{X_T<C_T\}]} \cdot \frac{P(X_T\geq C_T)}{P(X_T<C_T)}$ may not be maximized. The reason for us to adopt this approach is because analysis of (3.4.2) is simplified.

3.5 *Maximum RAROC with Candidate Hedging Portfolio under Black-Scholes model*

We consider a financial market in which the price dynamics of the underlying asset follows the standard Black-Scholes model with constant drift and volatility,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

Assume a constant riskfree rate $r \in \mathbb{R}$ is also present. It is well-known that a complete market admits a unique equivalent martingale measure Q, see Delbaen and Schachermayer (2006). In particular, under the standard Black-Scholes model, the Radon-Nikodým derivative or density process $\frac{dQ}{dP}$ takes the following form

$$\frac{dQ}{dP} = \exp\left(-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T - \left(\frac{\mu - r}{\sigma}\right) W_T\right),$$

see Musiela and Rutkowski (2008). By invoking Girsanov's Theorem, a Q-Brownian motion $\widetilde{W}$ satisfying

$$d\widetilde{W}_t = dW_t + \left(\frac{\mu - r}{\sigma}\right) dt,$$

is obtained, see Musiela and Rutkowski (2008).

Recall the first subproblem,

$$\begin{aligned} &\underset{\widetilde{X}}{\text{Maximize}} \quad P(\widetilde{X}_T \geq \widetilde{C}_T) \\ &\text{subject to} \quad \mathbb{E}^Q[e^{-rT}\widetilde{X}_T] \leq x_0, \end{aligned} \tag{3.5.1}$$

Proposition 2.8 of Föllmer and Leukert (1999) states that the solution of the following problem

$$\begin{aligned} &\underset{A}{\text{Maximize}} \quad P(A) \\ &\text{subject to} \quad \mathbb{E}^Q[e^{-rT}\widetilde{C}_T\mathbf{1}_A] \leq x_0 \end{aligned} \tag{3.5.2}$$

can lead to a solution of the problem in (3.5.1) with $\widetilde{X}_T = \widetilde{C}_T\mathbf{1}_A$ and $A = \{\widetilde{X}_T \geq \widetilde{C}_T\}$ P-a.s. See Föllmer and Leukert (1999) for more details. Moreover, if we rewrite the constraint in (3.5.2)

$$\begin{aligned} \mathbb{E}^Q[e^{-rT}\widetilde{C}_T\mathbf{1}_A] \leq x_0 \quad &\Longrightarrow \quad \mathbb{E}^Q\left[\frac{\widetilde{C}_T}{\mathbb{E}^Q[\widetilde{C}_T]} \cdot \mathbf{1}_A\right] \leq \frac{x_0 e^{rT}}{\mathbb{E}^Q[\widetilde{C}_T]} \\ &\Longrightarrow \quad Q^*(A) \leq \widetilde{x}_0 \qquad \text{where} \quad \widetilde{x}_0 := \frac{x_0 e^{rT}}{\mathbb{E}^Q[\widetilde{C}_T]}, \end{aligned}$$

and introduce a new measure Q^* with Radon-Nikodým derivative given by

$$\frac{dQ^*}{dQ} = \frac{\widetilde{C}_T}{\mathbb{E}^Q[\widetilde{C}_T]},$$

we have

$$\begin{aligned} &\underset{A}{\text{Maximize}} \quad P(A) \\ &\text{subject to} \quad Q^*(A) \leq \widetilde{x}_0. \end{aligned} \tag{3.5.3}$$

In this formulation, the problem in (3.5.2) is analogous to a Neyman-Pearson problem. Theorem 2.22 in Föllmer and Leukert (1999) provides a way to obtain the corresponding solution A of the above Neyman-Pearson problem through the use of Radon-Nikodým derivative $\frac{dQ}{dP}$. More precisely, the solution is given by

$$A = \left\{ \frac{dP}{dQ} > a \cdot \widetilde{C} \right\}$$

where a is a constant such that $Q^*(A) = \widetilde{x}_0$ is satisfied. Consequently, we have the 'success' event determined by

$$\{\widetilde{X}_T \geq \widetilde{C}_T\} = A = \left\{ \frac{dP}{dQ} > a \cdot \widetilde{C} \right\} \; P\text{-a.s.}$$

Applying these results to the fictitious contingent claim $\widetilde{C} = C + \varepsilon S$, we get

$$A^* = \left\{ \frac{dP}{dQ} > k^* \cdot (C + \varepsilon S) \right\} = \left\{ \frac{dP}{dQ} > k^* \cdot \big((S - K)^+ + \varepsilon S \big) \right\}$$

for some constant $k^* \in \mathbb{R}$.[4] A^* is indeed the union of two mutually disjoint sets and can be expressed as $\{S_T \in \mathcal{S}\}$ for some subset $\mathcal{S}$ of $\mathbb{R}$.

Proposition 3.5.1. *Under the standard Black-Scholes model, the event A^* is given by*

$$A^* = A_g(\lambda^*) \cup A_b(\lambda^*)$$

where $A_g(\lambda^*), A_b(\lambda^*)$ are mutually disjoint and λ^* is a constant such that $\mathbb{E}^Q[e^{-rT}(C + \varepsilon S)\mathbf{1}_{A^*}] = x_0$ holds.

Proof. Since we represent A^* as $\{S_T \in \mathcal{S}\}$, it is more convenient to have $\frac{dQ}{dP}$ in terms of S

$$\begin{aligned}
\frac{dQ}{dP} &= \exp\left(-\frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 T - \left(\frac{\mu - r}{\sigma} \right) W_T \right) \\
&= \left[\exp\left(\frac{1}{2} (\mu - r) T + \sigma W_T \right) \right]^{-\frac{\mu - r}{\sigma^2}} \\
&= \left[\exp\left(\frac{1}{2} (\mu - r) T - \left(\mu - \frac{1}{2}\sigma^2 \right) T \right) \cdot \frac{S_T}{S_0} \right]^{-\frac{\mu - r}{\sigma^2}} \\
&= \left[\exp\left(\tfrac{1}{2} \left(\sigma^2 - \mu - r \right) T \right) \cdot \frac{S_T}{S_0} \right]^{-\frac{\mu - r}{\sigma^2}} \\
&= \left[\frac{\exp\left(\tfrac{1}{2} \left(\sigma^2 - \mu - r \right) T \right)}{S_0} \right]^{-\frac{\mu - r}{\sigma^2}} \cdot S_T^{-\frac{\mu - r}{\sigma^2}} := \lambda^{-1} \cdot S_T^{-\frac{\mu - r}{\sigma^2}}.
\end{aligned}$$

[4]Recall that the existence of such k^* is provided by the Neyman-Pearson Lemma.

Consequently,

$$\begin{aligned}
A^* &= \left\{ \frac{dP}{dQ} > k^* \cdot \big((S-K)^+ + \varepsilon S \big) \right\} \\
&= \left\{ \lambda \cdot S^{\frac{\mu - r}{\sigma^2}} > k^* \cdot \big((S-K)^+ + \varepsilon S \big) \right\} \\
&= \left\{ S^{\frac{\mu - r}{\sigma^2}} > \lambda^* \cdot \big((S-K)^+ + \varepsilon S \big) \right\} \quad \text{where} \quad \lambda^* := \frac{k^*}{\lambda} \\
&= \left\{ \{ S^{\frac{\mu - r}{\sigma^2}} > \lambda^*(1+\varepsilon)S - \lambda^* K \} \cap \{ S \geq K \} \right\} \bigcup \\
&\quad \left\{ \{ S^{\frac{\mu - r}{\sigma^2}} > \lambda^* \varepsilon S \} \cap \{ S < K \} \right\} \qquad (3.5.4) \\
&:= A_{\mathrm{g}}(\lambda^*) \cup A_{\mathrm{b}}(\lambda^*).
\end{aligned}$$

It is obvious that $A_{\mathrm{g}}(\lambda^*), A_{\mathrm{b}}(\lambda^*)$ are mutually disjoint.

Regarding the value of λ^*, based on the requirement of $Q^*(A^*) = \widetilde{x}_0$ stated in Theorem 2.22 in Föllmer and Leukert (1999), it can be readily observe that $\mathbb{E}^Q[e^{-rT}(C + \varepsilon S)\mathbf{1}_{A^*}] = x_0$ holds with λ^*. □

Accordingly, the solution of the first subproblem is obtained from

Proposition 3.5.2. *Under the standard Black-Scholes model, for a fixed* ε *and* $\widetilde{C} = C + \varepsilon S$*, the problem in* (3.5.1)

$$\begin{aligned}
&\underset{\widetilde{X}}{\textit{Maximize}} && P(\widetilde{X}_T \geq \widetilde{C}_T) \\
&\textit{subject to} && \mathbb{E}^Q[e^{-rT}\widetilde{X}_T] \leq x_0
\end{aligned}$$

is solved by

$$\widetilde{X} = \widetilde{C}\mathbf{1}_{A^*}$$

where A^* *is obtained from Proposition 3.5.1 and solves the problem in* (3.5.3).

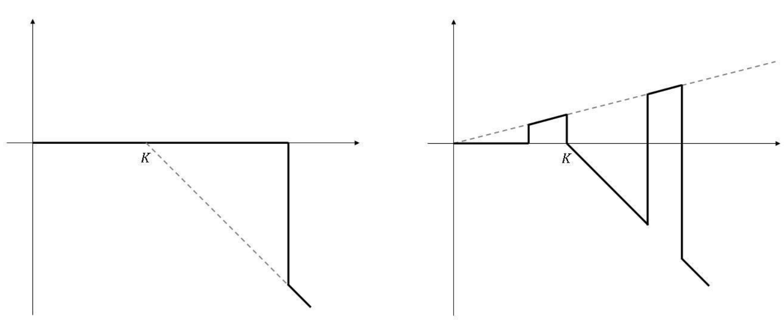
Proof. It is an immediate consequence of previous discussions. Note that the solution of the problem in (3.5.2) is also the solution of that in (3.5.1), moreover, the problems in (3.5.2) and (3.5.3) are equivalent, so we can apply Proposition 3.5.1 to obtain the solution of the problem in (3.5.3). □

Remark 3.5.0.2. Due to the decomposition of $A^* = A^*_{\mathrm{g}} \cup A^*_{\mathrm{b}}$ as union of two disjoint sets $A^*_{\mathrm{g}}, A^*_{\mathrm{b}}$, we may interpret the quantile-hedging portfolio $\widetilde{X} = \widetilde{X}(\varepsilon, A^*)$ as consisting of three options

$$X(\varepsilon, A^*) = C\mathbf{1}_{A^*_g} + \varepsilon S\mathbf{1}_{A^*_g} + \varepsilon S\mathbf{1}_{A^*_b}.$$

By the definitions of A^*_g, A^*_b, A^*_g refers to underlying prices at which C is in-the-money while A^*_b refers to underlying prices at which C is out-the-money. We then make several comments regarding this sum of three options:

- $C\mathbf{1}_{A^*_g}$, which can be regarded as a variant of a barrier call option, serves as a protection against the short position in C over the price region A^*_g, resulting in zero risk. However, we have a 'naked' position outside the price region A^*_g.
- We invest in ε units of two digital options $S\mathbf{1}_{A^*_g}$ and $S\mathbf{1}_{A^*_b}$ in order to acquire potential profit. The profit can only be earned when S_T is realized over the two price regions A^*_g and A^*_b. Specifically, over A^*_g, we are able to cover the short position of C_T and generate additional cashflow εS_T as profit. Over A^*_b, C is not exercised and so no risk in the short position. Moreover, we have an inflow of cash εS_T from holding ε unit of $S\mathbf{1}_{A^*_b}$.
- Compared to the typical quantile-hedging portfolio X^{qh}, it comprises only the first option $C\mathbf{1}_{A^*}$ so that protection against the short position of C is up to price region A^*. Outside this price region, the position is unhedged. See Föllmer and Leukert (1999) for details. Even though the probability of success is maximized and so the chance of unhedged position is minimized, once it is realized, a very serious loss can happen. In other words, the hedged position $X - C$ from using the quantile-hedging portfolio X^{qh} is non-positive (greater than or equal to zero) for all prices of S_T and negative for prices beyond a critical level $\overline{S}_T$. For any prices beyond $\overline{S}_T$, we have a naked position. However, by using $X(\varepsilon, A^*)$ it is possible to have both negative and positive values. The possible difference is illustrated in the following figure.



(a) Hedged position using quantile-hedging portfolio X^{qh}

(b) Hedged position using portfolio $X(\varepsilon, A^*)$

Fig. 3.1: Difference in hedged position $X - C$ at maturity

As seen from the above figure, we have certain probability of realizing a gain from the hedged position. Of course the gain comes from an extra amount of loss in the hedged position, thus increasing the expected shortfall/VaR of the position. Whether the RAROC hedge is worse than the quantile hedge cannot be answered explicitly. It should depend on the preference of the hedger, i.e. if his/her primary preference is the protection against the loss only, then he/she may consider the quantile hedge as more appropriate. However, if he/she wishes to maximize the risk-and-return performance, the RAROC hedge may be employed because it provides a reasonable comparison between risk and reward.

After acquiring the solution to the first subproblem from Proposition 3.5.2 and characterizing the corresponding event $A^* = A^*(\varepsilon)$, we substitute A^* for $\{X_T(\varepsilon) \geq C_T\}$ and derive

$$
\begin{aligned}
R(\varepsilon, A^*) &= \frac{\mathbb{E}[\varepsilon S_T | \{X_T(\varepsilon) \geq C_T\}]}{\mathbb{E}[C_T | \{X_T(\varepsilon) < C_T\}]} \cdot \frac{P(X_T(\varepsilon) \geq C_T)}{P(X_T(\varepsilon) < C_T)} - 1 \\
&= \frac{\mathbb{E}[\varepsilon S_T | A^*_{\mathrm{g}} \cup A^*_{\mathrm{b}}]}{\mathbb{E}[C_T | (A^*_{\mathrm{g}} \cup A^*_{\mathrm{b}})^c]} \cdot \frac{P(A^*_{\mathrm{g}} \cup A^*_{\mathrm{b}})}{P((A^*_{\mathrm{g}} \cup A^*_{\mathrm{g}})^c)} - 1 \\
&= \frac{\varepsilon \mathbb{E}[S_T \mathbf{1}_{A^*_{\mathrm{g}}}] + \varepsilon \mathbb{E}[S_T \mathbf{1}_{A^*_{\mathrm{b}}}]}{\mathbb{E}[C_T] - \mathbb{E}[C_T \mathbf{1}_{A^*_{\mathrm{g}} \cup A^*_{\mathrm{b}}}]} - 1 \\
&= \frac{\varepsilon \mathbb{E}[S_T \mathbf{1}_{A^*_{\mathrm{g}}}] + \varepsilon \mathbb{E}[S_T \mathbf{1}_{A^*_{\mathrm{b}}}]}{\mathbb{E}[C_T] - \mathbb{E}[C_T \mathbf{1}_{A^*_{\mathrm{g}}}]} - 1
\end{aligned}
$$

As a result, the second subproblem of maximizing R is merely a univariate optimization problem over $\varepsilon \in [0, 1]$. The maximum RAROC is thus defined as $R^* = \sup_{\varepsilon \in [0,1]} R(\varepsilon) = \sup_{\varepsilon \in [0,1]} R(\varepsilon, A^*(\varepsilon)) = R(\varepsilon^*, A^*(\varepsilon^*))$ where $\varepsilon^* = \arg\max_{\varepsilon \in [0,1]} R(\varepsilon)$. At a given initial endowment (less than the Black-Scholes price C_0) $x_0 < C_0$ and any target RAROC $\overline{R}$, the optimal hedging portfolio is identified as $X^* = X(\varepsilon^*, A^*(\varepsilon^*))$ such that $R^* = \overline{R}$, i.e. under such hedging portfolio, the maximum RAROC is the same as the target RAROC.

In next section, we would maximum RAROC under a given initial endowment for the call option. Maximum RAROC is different for the case of $\frac{\mu - r}{\sigma^2} - 1 \geq 0$ and $\frac{\mu - r}{\sigma^2} - 1 < 0$ because the corresponding event $A^* = A^*_{\mathrm{g}} \cup A^*_{\mathrm{b}}$, see (3.5.4), depends on the sign of $\frac{\mu - r}{\sigma^2} - 1$.

3.5.1 Maximum RAROC and Optimal Hedging Portfolio under $\frac{\mu - r}{\sigma^2} - 1 \geq 0$

Suppose $\frac{\mu - r}{\sigma^2} - 1 \geq 0$ holds and recall

$$
\begin{aligned}
A^*_{\mathrm{g}} &= \left\{ S^{\frac{\mu - r}{\sigma^2}} > \lambda^*(1 + \varepsilon) S - \lambda^* K \right\} \cap \{S \geq K\} \qquad \text{and} \\
A^*_{\mathrm{b}} &= \left\{ S^{\frac{\mu - r}{\sigma^2}} > \lambda^* \varepsilon S \right\} \cap \{S < K\}.
\end{aligned}
$$

It is readily seen that the call option is in-the-money on A^*_{g} and out-of-money on A^*_{b}, so

$$\mathbb{E}^Q[C\mathbf{1}_{A^*_{\mathrm{g}}}] = \mathbb{E}^Q[(S-K)^+\mathbf{1}_{A^*_{\mathrm{g}}}] = \mathbb{E}^Q[(S-K)\mathbf{1}_{A^*_{\mathrm{g}}}] = \mathbb{E}^Q[S\mathbf{1}_{A^*_{\mathrm{g}}}] - \mathbb{E}^Q[K\mathbf{1}_{A^*_{\mathrm{g}}}].$$

Meanwhile, both A^*_{g} and A^*_{b} can be geometrically understood as follows: they consist of the x-coordinates at which a straight line is below a curve described in the corresponding set A^*_{g} and A^*_{b}. In fact, A^*_{g} considers x-coordinates inside $S \geq K$ such that $y = \lambda^*(1+\varepsilon)x - \lambda^*K$ is $y = x^{\frac{\mu-r}{\sigma^2}}$ while A^*_{b} consider x-coordinates inside $S < K$ such that $y = \lambda^*\varepsilon x$ is under $y = x^{\frac{\mu-r}{\sigma^2}}$. One may refer to the following figure for a better understanding:

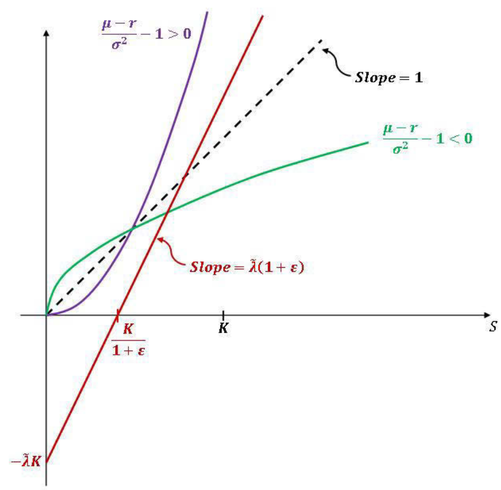


Fig. 3.2: Determination of A^*_{g} and A^*_{b} under two cases of $\frac{\mu-r}{\sigma^2} - 1$

We shall analyze the intersection points between the associated curve and straight line before the determination of A^*_{g} and A^*_{b}. Under the assumption of $\frac{\mu-r}{\sigma^2} - 1 \geq 0$, there are at most two intersection points with $x > 0$ between $y = x^{\frac{\mu-r}{\sigma^2}}$ and $y = \lambda^*(1+\varepsilon)x - \lambda^*K$ as in A^*_{g} whilst there is at most one intersection point with $x > 0$ between $y = x^{\frac{\mu-r}{\sigma^2}}$ and $y = \lambda^*\varepsilon x$ as in A^*_{b}. Based on these observations, at a given λ^*, we introduce the following quantities

$$S_{\underline{\mathrm{g}}} = S_{\underline{\mathrm{g}}}(\lambda^*) = \inf\{S \in \mathbb{R}^+ \mid S^{\frac{\mu-r}{\sigma^2}} = \lambda^*(1+\varepsilon)S - \lambda^*K\} \tag{3.5.5}$$

$$S_{\overline{\mathrm{g}}} = S_{\overline{\mathrm{g}}}(\lambda^*) = \sup\{S \in \mathbb{R}^+ \mid S^{\frac{\mu-r}{\sigma^2}} = \lambda^*(1+\varepsilon)S - \lambda^*K\} \tag{3.5.6}$$

$$S_{\mathrm{b}} = S_{\mathrm{b}}(\lambda^*) = \sup\{S \in \mathbb{R}^+ \mid S^{\frac{\mu-r}{\sigma^2}} = \lambda^*\varepsilon S\} \tag{3.5.7}$$

which represent the corresponding intersection points. We adapt the convention of $\sup\emptyset = -\infty$. A^*_{g} and A^*_{b} can be regarded as set-valued functions of these quantities.

However, since $S_{\underline{\mathrm{g}}}$, $S_{\overline{\mathrm{g}}}$ and S_{b} are functions of λ^*, we could also treat A^*_{g} and A^*_{b} as set-valued functions of λ^*.

We establish explicit structures of A^*_{g} and A^*_{b} through

Proposition 3.5.3. *Under the hypothesis of* $\frac{\mu-r}{\sigma^2}-1\geq 0$, A^*_g *and* A^*_b *are given as*

$$A^*_g = \begin{cases} \{S \geq K\} & \text{if} \quad \lambda^* < \widehat{\lambda}^*_g \\ \{K \leq S \leq S_{\underline{g}} \vee K\} \cup \{S \geq S_{\overline{g}} \vee K\} & \text{if} \quad \lambda^* \geq \widehat{\lambda}^*_g \end{cases}$$

$$A^*_b = \begin{cases} \{S_b < S < K\} & \text{if} \quad \lambda^* < \widehat{\lambda}^*_b \\ \emptyset & \text{if} \quad \lambda^* \geq \widehat{\lambda}^*_b \end{cases}$$

for some value of $\widehat{\lambda}^*_g$ *and* $\widehat{\lambda}^*_b$.

Proof. For fixed values of K and ε, hence $\frac{K}{1+\varepsilon}$, $y=\lambda^*(1+\varepsilon)x-\lambda^*K$ must pass through the point $(x,y)=(\frac{K}{1+\varepsilon},0)$. If there exists a tangent point (x',y') between $y=\lambda^*(1+\varepsilon)x-\lambda^*K$ and $y=x^{\frac{\mu-r}{\sigma^2}}$, it must satisfy $(x',y')=(x',x'^{\frac{\mu-r}{\sigma^2}})$ and so x' can be obtained by equating the slope between $(\frac{K}{1+\varepsilon},0)$ and (x',y') to that of $y=x^{\frac{\mu-r}{\sigma^2}}$ at (x',y'),

$$\begin{aligned} \frac{y'-0}{x'-\frac{K}{1+\varepsilon}} &= \frac{d}{dx}\left(x^{\frac{\mu-r}{\sigma^2}}\right)\Big|_{(x',y')} \\ \Longrightarrow \quad \frac{x'^{\frac{\mu-r}{\sigma^2}}}{x'-\frac{K}{1+\varepsilon}} &= \frac{\mu-r}{\sigma^2}x'^{\frac{\mu-r}{\sigma^2}-1} \\ \Longrightarrow \quad x' &= \frac{\frac{\mu-r}{\sigma^2}}{(\frac{\mu-r}{\sigma^2}-1)(1+\varepsilon)}K \end{aligned}$$

Then we define $\widehat{\lambda}^*_{\mathrm{g}}$ by associating the slope of $y=x^{\frac{\mu-r}{\sigma^2}}$ at (x',y') to that of $y=\lambda^*(1+\varepsilon)x-\lambda^*K$, resulting

$$\begin{aligned} \frac{\mu-r}{\sigma^2}\left(\frac{\frac{\mu-r}{\sigma^2}K}{(\frac{\mu-r}{\sigma^2}-1)(1+\varepsilon)}\right)^{\frac{\mu-r}{\sigma^2}-1} &= \widehat{\lambda}^*_{\mathrm{g}}(1+\varepsilon) \\ \Longrightarrow \quad \widehat{\lambda}^*_{\mathrm{g}} &= \frac{\frac{\mu-r}{\sigma^2}}{1+\varepsilon}\left(\frac{\frac{\mu-r}{\sigma^2}K}{(\frac{\mu-r}{\sigma^2}-1)(1+\varepsilon)}\right)^{\frac{\mu-r}{\sigma^2}-1}. \end{aligned}$$

$\widehat{\lambda}^*_{\mathrm{g}}$ can be geometrically interpreted as follows: if $\lambda^*<\widehat{\lambda}^*_{\mathrm{g}}$, $y=\lambda^*(1+\varepsilon)x-\lambda^*K$ does not intersect with $y=x^{\frac{\mu-r}{\sigma^2}}$, while, if $\lambda^*\geq\widehat{\lambda}^*_{\mathrm{g}}$, there exists (one for '=' and two for '>') intersection points. Analogously, we analyze and define $\widehat{\lambda}^*_{\mathrm{b}}$ as

$$\widehat{\lambda}^*_{\mathrm{b}} = \frac{K^{\frac{\mu-r}{\sigma^2}-1}}{\varepsilon} \tag{3.5.8}$$

to distinguish the number of intersection points between $y = \lambda^* \varepsilon x$ and $y = x^{\frac{\mu - r}{\sigma^2}}$. It is then a trivial consequence to derive

$$A_{\mathrm{g}}^* = \begin{cases} \{S \geq K\} & \text{if} \quad \lambda^* < \widehat{\lambda}_{\mathrm{g}}^* \\ \{K \leq S \leq S_{\underline{\mathrm{g}}} \vee K\} \cup \{S \geq S_{\overline{\mathrm{g}}} \vee K\} & \text{if} \quad \lambda^* \geq \widehat{\lambda}_{\mathrm{g}}^* \end{cases}$$

$$A_{\mathrm{b}}^* = \begin{cases} \{S_{\mathrm{b}} < S < K\} & \text{if} \quad \lambda^* < \widehat{\lambda}_{\mathrm{b}}^* \\ \emptyset & \text{if} \quad \lambda^* \geq \widehat{\lambda}_{\mathrm{b}}^* \end{cases}$$

□

3.5.1.1 Pricing of $X(\varepsilon, A^*(\lambda^*))$ and Determination of λ^*

Upon the characterizations of both A_{g}^* and A_{b}^*, the optimal hedging portfolio $X(\varepsilon, A^*(\lambda^*))$ is a three-option portfolio

$$X(\varepsilon, A^*(\lambda^*)) = C\mathbf{1}_{A_{\mathrm{g}}^*} + \varepsilon S\mathbf{1}_{A_{\mathrm{g}}^*} + \varepsilon S\mathbf{1}_{A_{\mathrm{b}}^*}.$$

We shall derive the closed-form pricing formula for $X(\varepsilon, A^*(\lambda^*))$ because it is necessary when we determine λ^* from $\mathbb{E}^Q[e^{-rT}X] = x_0$ in Proposition 3.5.1. Since we assume a complete market, there exists a unique martingale measure and any contingent claim is priced under this measure, see Karatzas and Shreve (1998), resulting in a unique price. The price at time 0 is computed by

$$C_0 = \mathbb{E}^Q[e^{-rT}C_T].$$

Applying to $X(\varepsilon, A^*(\lambda^*))$, the price X_0 is

$$\begin{aligned} X_0 &= \mathbb{E}^Q[e^{-rT}X(\varepsilon, A^*(\lambda^*))] \\ &= \mathbb{E}^Q[e^{-rT}(C\mathbf{1}_{A_{\mathrm{g}}^*} + \varepsilon S\mathbf{1}_{A_{\mathrm{g}}^*} + \varepsilon S\mathbf{1}_{A_{\mathrm{b}}^*})] \\ &= \mathbb{E}^Q[e^{-rT}C\mathbf{1}_{A_{\mathrm{g}}^*}] + \varepsilon\mathbb{E}^Q[e^{-rT}S\mathbf{1}_{A_{\mathrm{g}}^*}] + \mathbb{E}^Q[e^{-rT}S\mathbf{1}_{A_{\mathrm{b}}^*}]. \end{aligned}$$

In view of this, it suffices to determine $\mathbb{E}^Q[C\mathbf{1}_{A^*_{\mathrm{g}}}]$, $\mathbb{E}^Q[S\mathbf{1}_{A^*_{\mathrm{g}}}]$ and $\mathbb{E}^Q[S\mathbf{1}_{A^*_{\mathrm{b}}}]$. We introduce the following quantities

$$\Phi_{\underline{\mathrm{g}},+} = \Phi\left(\frac{\ln\frac{S_{\underline{\mathrm{g}}}\vee K}{S_0} - (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \qquad \Phi_{\underline{\mathrm{g}},-} = \Phi\left(\frac{\ln\frac{S_{\underline{\mathrm{g}}}\vee K}{S_0} - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right),$$

$$\Phi_{\overline{\mathrm{g}},+} = \Phi\left(\frac{\ln\frac{S_{\overline{\mathrm{g}}}\vee K}{S_0} - (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \qquad \Phi_{\overline{\mathrm{g}},-} = \Phi\left(\frac{\ln\frac{S_{\overline{\mathrm{g}}}\vee K}{S_0} - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right),$$

$$\Phi_{\mathrm{K},+} = \Phi\left(\frac{\ln\frac{K}{S_0} - (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \qquad \Phi_{\mathrm{K},-} = \Phi\left(\frac{\ln\frac{K}{S_0} - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right),$$

$$\Phi_{\mathrm{b}} = \Phi\left(\frac{\ln\frac{S_{\mathrm{b}}}{S_0} - (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

where Φ is the cumulative distribution function of normal distribution.

Proposition 3.5.4. *Assume a constant interest rate, we have*

$$Q(A^*_g) = \begin{cases} Q(S \geq K) = 1 - \Phi_{K,-} & \text{if} \quad \lambda^* < \widehat{\lambda}^*_g \\ Q(K \leq S \leq S_{\underline{g}} \vee K) + Q(S \geq S_{\overline{g}} \vee K) \\ \quad = \Phi_{\underline{g},-} - \Phi_{K,-} + 1 - \Phi_{\overline{g},-} & \text{if} \quad \lambda^* \geq \widehat{\lambda}^*_g \end{cases}$$

$$\mathbb{E}^Q[S\mathbf{1}_{A^*_g}] = \begin{cases} S_0 e^{rT}(1 - \Phi_{K,+}) & \text{if} \quad \lambda^* < \widehat{\lambda}^*_g \\ S_0 e^{rT}(\Phi_{\underline{g},+} - \Phi_{K,+} + 1 - \Phi_{\overline{g},+}) & \text{if} \quad \lambda^* \geq \widehat{\lambda}^*_g \end{cases}$$

$$\mathbb{E}^Q[S\mathbf{1}_{A^*_b}] = \begin{cases} S_0 e^{rT}(\Phi_{K,+} - \Phi_b) & \text{if} \quad \lambda^* < \widehat{\lambda}^*_b \\ 0 & \text{if} \quad \lambda^* \geq \widehat{\lambda}^*_b \end{cases}$$

If $\widehat{\lambda}^*_b \leq \widehat{\lambda}^*_g$, *the pricing formula for* $X(\varepsilon, A^*(\lambda^*))$ *is* $e^{-rT}\mathbb{E}^Q[X(\varepsilon, A^*(\lambda^*))]$, *where*

$$\begin{aligned} & \mathbb{E}^Q[X(\varepsilon, A^*(\lambda^*))] \\ = {} & \mathbb{E}^Q[C\mathbf{1}_{A^*_g} + \varepsilon S\mathbf{1}_{A^*_g} + \varepsilon S\mathbf{1}_{A^*_b}] \\ = {} & (1+\varepsilon)\mathbb{E}^Q[S\mathbf{1}_{A^*_g}] - KQ(A^*_g) + \varepsilon\mathbb{E}^Q[S\mathbf{1}_{A^*_b}] \\ = {} & \begin{cases} (1+\varepsilon)S_0 e^{rT}(1-\Phi_{K,+}) - K(1-\Phi_{K,-}) + \varepsilon S_0 e^{rT}(\Phi_{K,+} - \Phi_b) & \text{if} \quad \lambda^* < \widehat{\lambda}^*_b \\ (1+\varepsilon)S_0 e^{rT}(1-\Phi_{K,+}) - K(1-\Phi_{K,-}) & \text{if} \quad \widehat{\lambda}^*_b \leq \lambda^* < \widehat{\lambda}^*_g \\ (1+\varepsilon)S_0 e^{rT}(1-\Phi_{K,+} + \Phi_{\underline{g},+} - \Phi_{\overline{g},+}) \\ \quad - K(1-\Phi_{K,-} + \Phi_{\underline{g},-} - \Phi_{\overline{g},-}) & \text{if} \quad \lambda^* \geq \widehat{\lambda}^*_g \end{cases} \end{aligned}$$

Otherwise, when $\widehat{\lambda}_g^* \leq \widehat{\lambda}_b^*$*, it is given by*

$$
\begin{aligned}
&\mathbb{E}^Q[X(\varepsilon, A^*(\lambda^*))] \\
&= \mathbb{E}^Q[C\mathbf{1}_{A_g^*} + \varepsilon S\mathbf{1}_{A_g^*} + \varepsilon S\mathbf{1}_{A_b^*}] \\
&= (1+\varepsilon)\mathbb{E}^Q[S\mathbf{1}_{A_g^*}] - KQ(A_g^*) + \varepsilon\mathbb{E}^Q[S\mathbf{1}_{A_b^*}] \\
&= \begin{cases}
(1+\varepsilon)S_0 e^{rT}(1-\Phi_{K,+}) - K(1-\Phi_{K,-}) + \varepsilon S_0 e^{rT}(\Phi_{K,+} - \Phi_b) & \textit{if} \quad \lambda^* < \widehat{\lambda}_g^* \\
(1+\varepsilon)S_0 e^{rT}(1-\Phi_{K,+} + \Phi_{\underline{g},+} - \Phi_{\overline{g},+}) & \\
\quad - K(1-\Phi_{K,-} + \Phi_{\underline{g},-} - \Phi_{\overline{g},-}) + \varepsilon S_0 e^{rT}(\Phi_{K,+} - \Phi_b) & \textit{if} \quad \widehat{\lambda}_g^* \leq \lambda^* < \widehat{\lambda}_b^* \\
(1+\varepsilon)S_0 e^{rT}(1-\Phi_{K,+} + \Phi_{\underline{g},+} - \Phi_{\overline{g},+}) & \\
\quad - K(1-\Phi_{K,-} + \Phi_{\underline{g},-} - \Phi_{\overline{g},-}) & \textit{if} \quad \lambda^* \geq \widehat{\lambda}_b^*
\end{cases}
\end{aligned}
$$

Proof. Due to the fact that S is a geometric Brownian motion under Q satisfying

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t,$$

S_T is given by

$$S_T = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right).$$

Together with the characterizations of A_{g}^* and A_{b}^* in Proposition 3.5.3, one can obtain all the desired results by direct calculations. □

Remark 3.5.1.1. In particular, when $\lambda^* \to 0$, $\Phi_{\mathrm{b}} \to 0$ due to $S_{\mathrm{b}} \to 0$. Then we have $\mathbb{E}^Q[X(\varepsilon, A^*(\lambda^*))] \to (1+\varepsilon)S_0 e^{rT} - S_0 e^{rT}\Phi_{\mathrm{K},+} - K(1-\Phi_{\mathrm{K},-})$ where $S_0 e^{rT}(1-\Phi_{\mathrm{K},+}) - K(1-\Phi_{\mathrm{K},-})$ is $C_0^{\mathrm{BS}} e^{rT}$ with C_0^{BS} being the Black-Scholes call option price. On the other hand, as $\lambda^* \to +\infty$, $\mathbb{E}^Q[X(\varepsilon, A^*(\lambda^*))] \to 0$.

After we have obtained the (unique risk-neutral) price of $X(\varepsilon, A^*(\lambda^*))$ under fixed ε and λ^*, the price is implicitly a function of ε and λ^*. So, in order to solve the first subproblem in (3.5.1), we keep ε fixed and solve for an appropriate value, denoted by $\hat{\lambda}^* = \hat{\lambda}^*(\varepsilon)$, of λ^* from the budget constraint appeared $\mathbb{E}^Q[e^{-rT}X(\varepsilon, A^*(\lambda^*))] = x_0$ in order to obtain the solution. By then, based on Proposition 3.5.2, the first subproblem in (3.5.1) will be solved and the optimal 'success' event is $\{\widetilde{X}_T \geq \widetilde{C}_T\} = A^*(\hat{\lambda}^*)$.

Below is to demonstrate the risk-neutral price $e^{-rT}\mathbb{E}^Q[X(\varepsilon, A^*(\lambda^*))]$ as a function of λ^*.

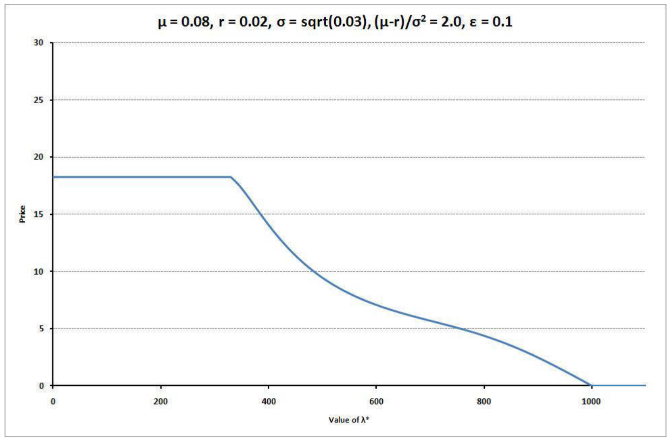


Fig. 3.3: Example of $\mathbb{E}^Q[X(\varepsilon, A(\lambda^*))]$ against λ^* under $\widehat{\lambda}^*_{\text{g}} < \widehat{\lambda}^*_{\text{b}}$, $\frac{\mu-r}{\sigma^2} = 2.0$, $\varepsilon = 0.1$

3.5.1.2 Determination of Maximum RAROC

We decompose the RAROC-maximization problem in (3.4.2) into two subproblems. The first subproblem, under a fixed ε,

$$\begin{aligned} &\underset{A\in\mathcal{F}_T}{\text{Maximize}} \quad P(A) \\ &\text{subject to} \quad \mathbb{E}^Q[e^{-rT}(C_T + \varepsilon S_T)\mathbf{1}_A] \leq x_0 \end{aligned}$$

is solved in the previous section and the solution is $\hat{A} = A^*(\hat{\lambda}^*) = A^*_{\text{g}}(\hat{\lambda}^*) \cup A^*_{\text{b}}(\hat{\lambda}^*)$ where A^*_{g} and A^*_{b} are described in Proposition 3.5.3, and, $\hat{\lambda}^*$ is obtained from satisfying the budget constraint $\mathbb{E}^Q[e^{-rT}X(\varepsilon, A^*(\lambda^*))] = x_0$. Note that $\hat{A}$ is regarded as the 'success' event $\{X_T \geq C_T\}$, we substitute and compute $\mathbb{E}[\varepsilon S_T|\{X_T \geq C_T\}]$ and $\mathbb{E}[C_T|\{X_T < C_T\}]$. Both

$$\frac{\mathbb{E}[\varepsilon S_T|\{X_T \geq C_T\}]}{\mathbb{E}[C_T|\{X_T < C_T\}]} := f(\varepsilon) \qquad \text{and} \qquad \frac{P(X_T \geq C_T)}{P(X_T < C_T)} := g(\varepsilon)$$

are now functions of ε, so is R since

$$R = R(\varepsilon) = \frac{\mathbb{E}[\varepsilon S_T|\{X_T \geq C_T\}]}{\mathbb{E}[C_T|\{X_T < C_T\}]} \cdot \frac{P(X_T \geq C_T)}{P(X_T < C_T)} - 1 = f(\varepsilon)g(\varepsilon) - 1.$$

Determining the maximum RAROC is the second subproblem

$$\underset{\varepsilon\in[0,1]}{\text{Maximize}} \quad R(\varepsilon)$$

which is a univariate optimization problem. After the optimal value of ε is found, we can conclude that the solution for the problem in (3.4.2) is $X(\varepsilon^*, \hat{A}(\varepsilon^*))$. This means the hedging portfolio $X(\varepsilon^*, \hat{A}(\varepsilon^*))$ would produce maximum RAROC at a fixed initial endowment for the call option (up to the reservation made in remark 3.4.0.2).

The next figure illustrates $R(\varepsilon)$ as a function of ε when x_0 is fixed and $\frac{\mu-r}{\sigma^2} - 1 \geq 0$ is satisfied.

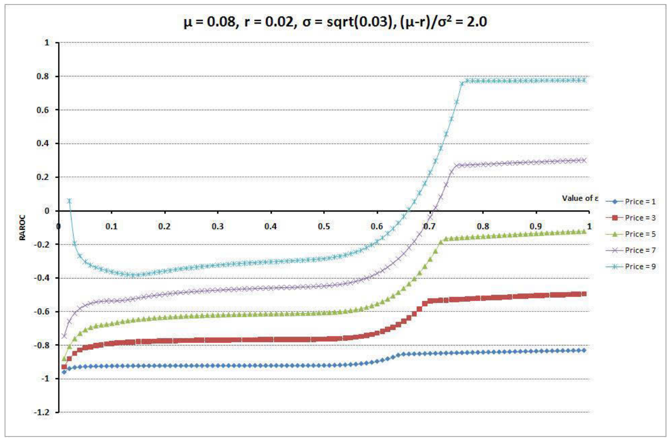


Fig. 3.4: RAROC against ε under different fixed initial endowment x_0 ($\widehat{\lambda}^*_{\mathrm{g}} < \widehat{\lambda}^*_{\mathrm{b}}$ and $\frac{\mu-r}{\sigma^2} = 2.0$)

According to Figure 3.4, we make several comments:

Firstly, as the initial endowment x_0 is increased, R at every ε is also increased. This is intuitively sound because we can improve the hedging portfolio and gain more profit when we sell the call option and prepare a higher initial endowment, resulting a better RAROC.

Secondly, based on present parameters of the Black-Scholes model, the Black-Scholes call option price is approximately 7.87. When the initial endowment x_0 is greater than 7.87, we can observe a singularity of R in a neighborhood of $\varepsilon = 0$. This takes place because we sell the call option at a price greater than the Black-Scholes price C_0 so that we have sufficient amount of money to set up a hedging portfolio for perfect replication of the call option. Consequently we have zero shortfall risk in the hedged position and are led to an infinite RAROC.

Suppose we sell the call option and hedge with an initial endowment less than 7.87, it is apparent from Figure 3.4 that no singularity is observed and R is a monotonic increasing function of ε. This means maximum RAROC is located at $\varepsilon = 1$. In words, we should hold exactly one unit of digital option $S\mathbf{1}_{A^*_{\mathrm{g}}}$ and $S\mathbf{1}_{A^*_{\mathrm{b}}}$. With these, we can overhedge the payoff of call option and acquire a profit equivalent to one unit of underlying asset upon the realization of events A^*_{g} and $\mathbf{1}_{A^*_{\mathrm{b}}}$. Moreover, maximum RAROC is not guaranteed to be positive. In particular, it is positive when the initial endowment is 'slightly' less than the Black-Scholes price, e.g. $x_0 = 7 < 7.87$ as seen from Figure 3.4.

At any initial endowment, we always have $R > -1$ over $\varepsilon \in (0, 1]$. This again indicates that the quantile-hedging portfolio is a 'bad' choice when one cares about RAROC, because we can readily find a 'better-performing' hedging portfolio in the sense that it offers a higher RAROC. In the quantile-hedging portfolio, the associated RAROC is -1 because we cannot gain any profit from the quantile-hedging portfolio together with non-zero risk. Indeed, we can construct a 'better' hedging strategy as measured by return per unit risk such that there is a chance of gaining some profit.

From Figure 3.4, there is seemingly a bound on maximum RAROC when the initial endowment is less than the Black-Scholes price. If this is true, the consequence is that, if we arbitrarily fix a target RAROC, and if the target is set too high, we may not be able to determine an initial endowment less than Black-Scholes price and a hedging portfolio to meet the target RAROC. So this pricing methodology is not practical. We should emphasize that this problem comes from our choice of hedging portfolio given by (3.4.1), which then limits the range of RAROC we can considered. So we cannot eliminate the existence of a hedging portfolio that can provide the desired target RAROC while the initial endowment of this hedging portfolio remains less than Black-Scholes price.

3.5.2 Maximum RAROC and Optimal Hedging Portfolio under $\frac{\mu-r}{\sigma^2} - 1 < 0$

Similar to the case of $\frac{\mu-r}{\sigma^2} - 1 \geq 0$, the 'success' event is still characterized as

$$\begin{aligned} A^* &= \left\{ \{S^{\frac{\mu-r}{\sigma^2}} > \lambda^*(1+\varepsilon)S - \lambda^* K\} \cap \{S \geq K\} \right\} \bigcup \left\{ \{S^{\frac{\mu-r}{\sigma^2}} > \lambda^* \varepsilon S\} \cap \{S < K\} \right\} \\ &:= A^*_{\mathrm{g}} \cup A^*_{\mathrm{b}} \qquad \text{where } A^*_{\mathrm{g}}, A^*_{\mathrm{b}} \text{ are disjoint.} \end{aligned}$$

However, owing to $\frac{\mu-r}{\sigma^2} - 1 < 0$, A^*_{g} and A^*_{b} would admit different representations.

Proposition 3.5.5. *Under the hypothesis of $\frac{\mu-r}{\sigma^2} - 1 < 0$, A^*_g and A^*_b are given by*

$$A^*_g = \begin{cases} \{K < S < S_g\} & \text{if} \quad \lambda^* < \widehat{\lambda}^* \\ \emptyset & \text{if} \quad \lambda^* \geq \widehat{\lambda}^* \end{cases}$$

$$A^*_b = \begin{cases} \{S < K\} & \text{if} \quad \lambda^* < \widehat{\lambda}^* \\ \{S < S_b\} & \text{if} \quad \lambda^* \geq \widehat{\lambda}^* \end{cases}$$

In particular, we have

$$A^* = \begin{cases} \{S < S_g\} & \textit{if} \quad \lambda^* < \widehat{\lambda}^* \\ \{S < S_b\} & \textit{if} \quad \lambda^* \geq \widehat{\lambda}^* \end{cases}$$

Proof. We start with A_{b}^*. By performing similar analysis, A_{b}^* is found to be

$$A_{\mathrm{b}}^* = \begin{cases} \{S < K\} & \text{if} \quad \lambda^* < \widehat{\lambda}_{\mathrm{b}}^* \\ \{S < S_{\mathrm{b}}\} & \text{if} \quad \lambda^* \geq \widehat{\lambda}_{\mathrm{b}}^* \end{cases}$$

where S_{b} and $\widehat{\lambda}_{\mathrm{b}}^*$ are defined in (3.5.7) and (3.5.8).

As for A_{g}^*, observing from Figure 3.2, we should realize that there is at most one intersection point between the straight line and the curve underlying A_{g}^* and we denote it as S_{g} if it exists. Furthermore, in order to retrieve $\widehat{\lambda}_{\mathrm{g}}^*$, we substitute $S = K$ into $S^{\frac{\mu-r}{\sigma^2}} = \lambda^*(1+\varepsilon)S - \lambda^* K$ and get

$$K^{\frac{\mu-r}{\sigma^2}} = \widehat{\lambda}_{\mathrm{g}}^*(1+\varepsilon)K - \widehat{\lambda}_{\mathrm{g}}^* K \quad \Longrightarrow \quad \widehat{\lambda}_{\mathrm{g}}^* := \frac{K^{\frac{\mu-r}{\sigma^2}-1}}{\varepsilon}.$$

This would allow us to express

$$A_{\mathrm{g}}^* = \begin{cases} \{K < S < S_{\mathrm{g}}\} & \text{if} \quad \lambda^* < \widehat{\lambda}_{\mathrm{g}}^* \\ \emptyset & \text{if} \quad \lambda^* \geq \widehat{\lambda}_{\mathrm{g}}^* \end{cases}.$$

If we compare this to (3.5.8), it is readily observed that $\widehat{\lambda}_{\mathrm{g}}^*$ and $\widehat{\lambda}_{\mathrm{b}}^*$ coincide, so we write $\widehat{\lambda}_{\mathrm{g}}^* = \widehat{\lambda}_{\mathrm{b}}^* = \widehat{\lambda}^*$. This fact enables us to combine A_{g}^*,A_{b}^* and obtain the following characterization of A^*

$$A^* = A_{\mathrm{g}}^* \cup A_{\mathrm{b}}^* = \begin{cases} \{S < S_{\mathrm{g}}\} & \text{if} \quad \lambda^* < \widehat{\lambda}^* \\ \{S < S_{\mathrm{b}}\} & \text{if} \quad \lambda^* \geq \widehat{\lambda}^* \end{cases}$$

□

3.5.2.1 Price of $X(\varepsilon, A^*(\lambda^*))$

The price of $X(\varepsilon, A(\lambda^*))$ is again computed by taking expectation under the risk-neutral measure Q, that is,

$$\mathbb{E}^Q[X(\varepsilon, A^*(\lambda^*))].$$

We obtain the following

Proposition 3.5.6. *Assume a constant interest rate, the pricing formula for* $X(\varepsilon, A^*(\lambda^*))$ *is* $e^{-rT}\mathbb{E}^Q[X(\varepsilon, A(\lambda^*))]$ *where* $\mathbb{E}^Q[X(\varepsilon, A(\lambda^*))]$ *is given by*

$$\mathbb{E}^Q[X(\varepsilon, A(\lambda^*))] = \mathbb{E}^Q[(C+\varepsilon S)\mathbf{1}_{A^*}] = \mathbb{E}^Q[C\mathbf{1}_{A_g^*}] + \mathbb{E}^Q[\varepsilon S\mathbf{1}_{A^*}]$$
$$= \begin{cases} \mathbb{E}^Q[(S-K)\mathbf{1}_{\{K<S<S_g\}}] + \mathbb{E}^Q[\varepsilon S\mathbf{1}_{\{S<S_g\}}] & \textit{if} \quad \lambda^* < \widehat{\lambda}^* \\ \mathbb{E}^Q[\varepsilon S\mathbf{1}_{S<S_b}] & \textit{if} \quad \lambda^* \geq \widehat{\lambda}^* \end{cases}$$
$$= \begin{cases} S_0e^{rT}(\Phi_{g,+} - \Phi_{K,+}) - K(\Phi_{g,-} - \Phi_{K,-}) + \varepsilon S_0e^{rT}\Phi_{g,+} & \textit{if} \quad \lambda^* < \widehat{\lambda}^* \\ \varepsilon S_0e^{rT}\Phi_b & \textit{if} \quad \lambda^* \geq \widehat{\lambda}^* \end{cases}$$

Proof. The proof resembles that of Proposition 3.5.4. So it is omitted here. □

Remark 3.5.2.1. $\lambda^* \to 0$ implies $S_\mathrm{g} \to +\infty$, hence, $\Phi_{\mathrm{g},+} \to 1$ and $\Phi_{\mathrm{g},-} \to 1$. Consequently, $\mathbb{E}^Q[X(\varepsilon, A(\lambda^*))] \to S_0e^{rT}(1-\Phi_{\mathrm{K},+}) - K(1-\Phi_{\mathrm{K},-}) + \varepsilon S_0 e^{rT}$, which reproduces the same result in Remark 3.5.1.1.

The figure below shows $\mathbb{E}^Q[X(\varepsilon, A^*(\lambda^*))]$ as a function of λ^*.

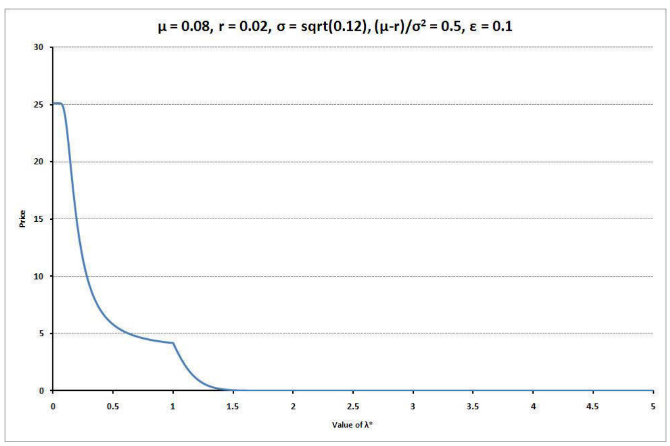


Fig. 3.5: Example of $\mathbb{E}^Q[X(\varepsilon, A(\lambda^*))]$ against λ^* under $\frac{\mu-r}{\sigma^2} = 0.5$, $\varepsilon = 0.1$

3.5.2.2 Determination of Maximum RAROC

After acquiring the pricing formula for $X(\varepsilon, A^*(\lambda^*))$, we determine an appropriate λ^* such that the budget constraint $\mathbb{E}^Q[e^{-rT}X(\varepsilon, A^*(\lambda^*))] = x_0$ is satisfied. Then we can study the corresponding maximum value of RAROC under the case of $\frac{\mu-r}{\sigma^2} - 1 < 0$. The following figure compares RAROC at different ε assuming a given initial endowment x_0 and $\frac{\mu-r}{\sigma^2} - 1 < 0$.

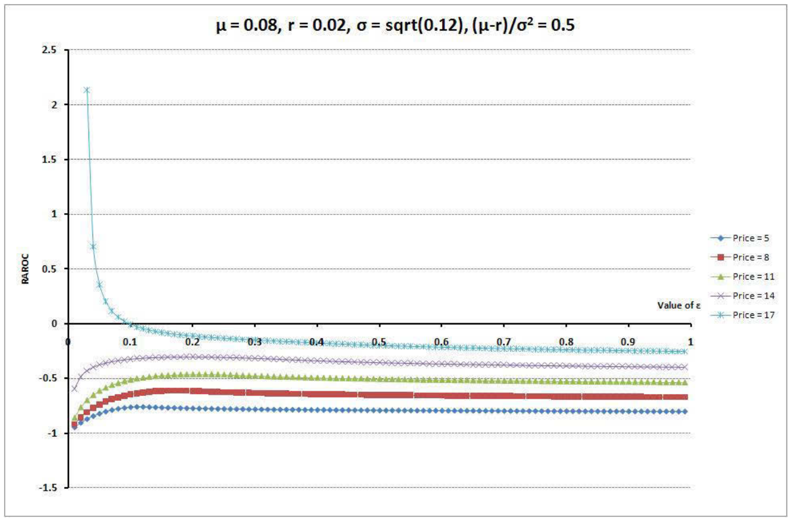


Fig. 3.6: RAROC against ε under different fixed initial capital ($\frac{\mu-r}{\sigma^2} = 0.5$)

Comparisons and comments can be made according to Figure 3.4 and Figure 3.6.

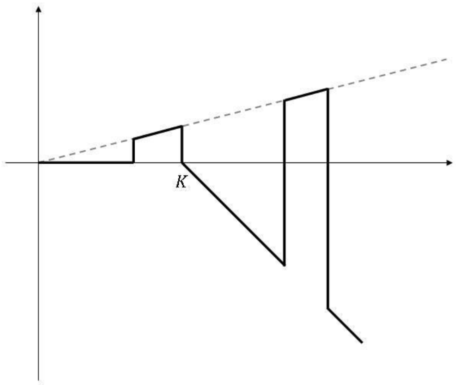
In Figure 3.6, the Black-Scholes call price is $C_0 \approx 14.63$. And we can conclude that if the initial endowment x_0 is less than C_0, we shall have a non-positive RAROC for all $\varepsilon \in (0, 1]$. This is different from the observations in Figure 3.4, in which positive RAROC is possible despite $x_0 < C_0$ provided that x_0 is not too much smaller than C_0. Moreover, from Figure 3.6, we can see that positivity as well as singularity can happen only when $x_0 \geq C_0$ is satisfied.

At a fixed initial endowment, even though RAROC is negative for all $\varepsilon \in (0, 1]$, it is not a monotonic increasing function of ε and there apparently exists a maximum RAROC for some $\varepsilon^* \in (0, 1)$, see RAROC under $x_0 = 17$ for example. This observation cannot be found when $x_0 < C_0$ and $\frac{\mu-r}{\sigma^2} - 1 \geq 0$ hold, in which case RAROC is a monotonic increasing function of ε and so a maximum value occurs at $\varepsilon^* = 1$.

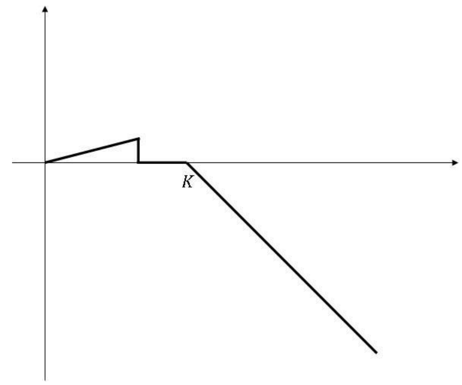
Note that the corresponding structure of the hedging portfolio under $\frac{\mu-r}{\sigma^2} - 1 < 0$ is completely different from that under $\frac{\mu-r}{\sigma^2} - 1 \geq 0$ due to the distinct characterization of A^*. Below is an example of the difference in the hedging portfolios under the two cases.

3.6 Conclusions

Based on the previous studies, when we use $X(\varepsilon, A) = (C + \varepsilon S)\mathbf{1}_A$ as a hedging portfolio and try to generate profit, positive RAROC cannot be taken for granted. Instead, the sign of RAROC depends heavily on the ratio $\frac{\mu-r}{\sigma^2}$. Particularly, positive RAROC



(a) Using $X(\varepsilon, A^*)$ under $\frac{\mu-r}{\sigma^2} - 1 \geq 0$



(b) Using $X(\varepsilon, A^*)$ under $\frac{\mu-r}{\sigma^2} - 1 < 0$

Fig. 3.7: Difference in payoff of hedged position $X - C$ at maturity

can only be achieved when $\frac{\mu-r}{\sigma^2} > 1$ is true, for instance, a model in which underlying asset has a large drift and small volatility. In other words, one is 'less uncertain' about the future realization of high asset price. However, when $\frac{\mu-r}{\sigma^2} < 1$ holds, the hedging portfolio cannot generate profit such that the RAROC is positive.

We focused on a specific hedging portfolio and studied the associated RAROC. All results and discussions are only valid for this hedging portfolio. Since we have only used one type of hedging portfolios, this means that we cannot exclude the existence of a hedging portfolio which can offer a target RAROC if we set an initial endowment for the call option to be x_0 less than the Black-Scholes price. There might exist such a hedging portfolio and so it is crucial to investigate what is the hedging portfolio that would lead to target RAROC with this initial endowment x_0 because we can sell the option at x_0 and achieve a target RAROC at our hedged position with this hedging portfolio. If we use $X(\varepsilon, A) = (C + \varepsilon S)\mathbf{1}_A$, we can only price the call option in this manner when some 'reasonable' target RAROC is set. For other values of RAROC, we fail to attain with this hedging portfolio. This problem is obviously due to the fact that we have defined explicit structure of hedging portfolio to be used and it is just a small subset of al admissible hedging portfolios. Furthermore, we should determine the *minimum* price x_0 for a contingent claim C such that we are able to obtain a hedging portfolio with RAROC of the hedged position being the target RAROC. This should be the *true* price for the contingent claim C under the framework of RAROC-based pricing. Even though we consider a complete market and set up a hedging strategy based on the RAROC approach, if one assumes an incomplete market, one may also make use of the RAROC approach to determine the corresponding bid price and ask price for the call option without being arbitraged because, by definition, the prices obtained from the RAROC approach are within the no-arbitrage price interval.

As a result, we should avoid analyzing from a restricted class of hedging strategy and studying how RAROC can be performed within this hedging strategy. Rather, we should work from a 'top-down' perspective. More precisely, we start by formulating what is RAROC-based price. Once this notion of price is determined, we would characterize the desired hedging strategy. Details are discussed in the next two chapters.

4. CONSTRUCTION OF DYNAMIC RAROC-BASED GOOD-DEAL PRICES

Let us recall our attempt at hedging a contingent claim such that the RAROC of the hedged position is maximized. The corresponding setup cost of the hedging portfolio will be termed as RAROC-based price of the contingent claim. In Chapter 1, we performed the analysis under a standard Black-Scholes model with a restriction on admissible trading strategies, namely, only static trading strategy is allowed. In Chapter 2, we continued the investigation and permitted continuous-time trading strategies. There we hedged a contingent claim with less than its unique perfect-replication price under the Black-Scholes model. Then we fixed a specific class of hedging strategies and studied the associated range of RAROC that can be achieved with this class. We found that the results are seriously dependent of model parameters.

The observations and conclusions in Chapter 3 are not completely satisfactory. Firstly, the values of RAROC obtained depended on the class of hedging strategies. It is possible that other classes can provide a different range of RAROC. Secondly, and most importantly, the cost of the hedging portfolio is not guaranteed to be a minimum for a fixed value of RAROC. It is not known whether there exists some other hedging portfolio such that the cost is lower and the target RAROC can be met. Consequently, we shall avoid selecting a particular class of trading strategies and defining RAROC-based prices with respect to this class of hedging strategies. Instead a more general approach should be adopted. More concisely, we shall start by formulating a 'good' definition of a RAROC-based price as well as some 'reasonable' properties it should possess. This will provide us a 'strategy-independent' RAROC-based price. Once this is accomplished, the hedging strategy is characterized. Through this approach, we can obtain a more robust RAROC-based pricing method.

4.1 Background

In this section, we shall review literature on the theory of good-deal pricing in order to formulate properly the notion of RAROC-based good-deal prices. Typically a 'good-deal' refers to an investment that possesses an 'attractive reward'. Here 'attractive reward' is quantified by some suitable criterion. For instance, in the context of arbitrage pricing theory, an arbitrage opportunity is undoubtedly a 'good-deal' since the 'attractive reward' is a positive return with zero chance of loss. One may also evaluate an investment opportunity with a performance measure and identify a ‘good-deal’ to be one with a corresponding to extremely large value of the measure. For example, if the Sharpe ratio is used as a performance measure, then an investment with a high Sharpe ratio is regarded as a ‘good-deal’ because intuitively a high Sharpe ratio signifies a large profit can be made per unit risk (more precisely, standard deviation). Adopting this approach, our primary performance measure is RAROC and an investment with high value of RAROC is termed a ‘good-deal’.

As soon as a ‘good-deal’ is defined, the idea of good-deal prices can be formulated. The development of good-deal pricing theory can be traced back to the work by Bernardo and Ledoit (2000) and Cochrane and Saá-Requejo (2000). This pricing method features its success in tightening the price bound of a contingent claim in an incomplete market, when compared to the one determined by arbitrage pricing. Due to this fact, there is an increasing number of researches focusing on the development of this pricing theory. Among all, Cherny (2008) is the main reference used.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote $\mathcal{A}$ as the set of all portfolios in the financial market that can be attained by self-financing trading strategies. In Cherny (2008), RAROC of a portfolio $X \in \mathcal{A} \subseteq L^0$, where L^0 is the space of all measurable random variables, is calculated by

$$RAROC(X) := \frac{\inf_{Q \in \mathcal{PD}} \mathbb{E}^Q[X]}{-\inf_{Q \in \mathcal{RD}} \mathbb{E}^Q[X]}$$

where $\mathcal{PD}$ and $\mathcal{RD}$ are two fixed collections of probability measures, called the *profit-determining set* and the *risk-determining set* respectively. As hinted at the names, one evaluates the profit and risk of the portfolio by making use of those probability measures defined in each of these sets. Thus the profit and risk of a position that yields a P&L X are $\inf_{Q \in \mathcal{PD}} \mathbb{E}^Q[X]$ and $-\inf_{Q \in \mathcal{RD}} \mathbb{E}^Q[X]$ respectively. It is akin to the probabilistic scenarios in coherent risk measure discussed in Artzner et al. (1999). Obviously it is not necessary that $\mathcal{PD} = \mathcal{RD}$. For example, one may be extremely optimistic about the profit in the future while seriously cautious about the risk as well, indicating the use of an aggressive profit-determining set and a conservative risk-determining set. We fix a positive number R meaning the upper limit on a possible RAROC and recall from Cherny (2008) the

Definition 4.1.1. A model satisfies the RAROC-based no-good-deal (NGD) condition if there exists no $X \in \mathcal{A}$ such that $RAROC(X) > R$, for a fixed $R > 0$.

As a usual practice, we wish to preclude the existence of ‘good-deals’ under a model, which means both $\mathcal{PD}$ and $\mathcal{RD}$ cannot be arbitrary. Instead, they should be chosen such that the following relationship is satisfied, see Cherny (2008).

Corollary 4.1.1. *A model satisfies the RAROC-based NGD condition if and only if*

$$\left(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\right) \bigcap \mathcal{R} \neq \emptyset$$

where $\mathcal{R}$ is the set of risk-neutral measures under the model.

Here, $\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}$ is an abbreviation for the set of probability measures given by

$$\begin{aligned} &\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD} \\ &\coloneqq \left\{ Q \in \mathcal{P} \;\middle|\; Q = \frac{1}{1+R}Q_1 + \frac{R}{1+R}Q_2 \quad \text{where} \quad Q_1 \in \mathcal{PD}, Q_2 \in \mathcal{RD} \right\} \end{aligned}$$

where $\mathcal{P}$ is the set of probability measures. By using this corollary, we can define the RAROC-based NGD price of a contingent claim F as follows:

Definition 4.1.2. A RAROC-based NGD price of a contingent claim F is a real number x such that the extended model $(\Omega, \mathcal{F}, P, \mathcal{PD}, \mathcal{RD}, A + \{h(F-x) : h \in \mathbb{R}\})$ satisfies the NGD condition.

More specifically, assume $R > 0$ and $0 < \alpha < 1$ are fixed, we consider the following $\mathcal{PD}$ and $\mathcal{RD}$

$$\mathcal{PD} \coloneqq \{P\} \qquad \text{and} \qquad \mathcal{RD} \coloneqq \left\{ Q \in \mathcal{P}^a \;\middle|\; 0 \leq \frac{dQ}{dP} \leq \frac{1}{\alpha} \quad P\text{-a.s.} \right\}. \tag{4.1.1}$$

where $\frac{dQ}{dP}$ is the corresponding Radon-Nikodým derivative or density of a measure $Q \in \mathcal{P}^a$. Here, $\mathcal{P}^a$ is the set of absolutely continuous[1] probability measures Q with respect to P, $Q \ll P$. Unless stated otherwise, we use $\frac{dQ}{dP}$ to identify Q and vice versa. One can realize immediately that $\mathcal{RD}$ is essentially used in the definition of expected shortfall ES^α, see Artzner et al. (1999). In other words, we consider the ratio of real-world profit to expected shortfall and so RAROC of a portfolio X can be expressed as

$$RAROC(X) \coloneqq \frac{\mathbb{E}^P[X]}{\mathrm{ES}^\alpha(X)}$$

and we shall dwell on this version of RAROC without any further mention.

It should be noted that the theory of *static* good-deal pricing is explored in Cherny (2008), where the theory is developed assuming that one is concerned with a good-deal price at a fixed time point. There is no trivial relationship between good-deal prices computed at different time points. The *dynamic* relationship is left unattended in Cherny

[1] A probability measure Q is absolutely continuous with respect to another probability measure P if any P-null set is also a Q-null set, i.e. any $A \in \mathcal{F}$ such that $P(A) = 0$ implies $Q(A) = 0$.

(2008). However this dynamic relationship is critical if one wishes to develop a more robust good-deal pricing theory. Briefly speaking, without any dynamic properties in the good-deal price process, when one performs decision-making it may happen that a 'good-deal as seen tomorrow would not be assessed as good-deal today', hence, leading to an inconsistent approach. This line of argument can also be found in the evolution of coherent risk measures, in which a static coherent risk measure is further developed to a dynamic coherent risk measure. Particularly, a dynamic coherent risk measure is considered in order to circumvent the *time-inconsistency problem* arisen from applying static coherent risk measure dynamically. One may refer to Riedel (2004) and Cheridito and Stadje (2009) for more discussions on this issue.

After the *RAROC*-based NGD condition is set up, Cherny (2008) derived a fair price interval for a contingent claim F which is given by

$$I_{NGF}(F) = \left\{ \mathbb{E}^Q[F] \,\middle|\, Q \in \left(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD} \right) \bigcap \mathcal{R} \right\}$$

Consequently, it is naturally to define the bid price and ask price of F to be the infimum and supremum of $I_{NGF}(F)$ respectively, i.e.

$$F^{\text{BID}} := \inf I_{NGF}(F) \qquad \text{and} \qquad F^{\text{ASK}} := \sup I_{NGF}(F).$$

For any prices outside the fair price interval, it is possible to set up a portfolio with the corresponding RAROC greater than or equal to R by definition.

To reflect the dynamic aspect of good-deal pricing theory, we may refer to Becherer (2009). They study both good-deal pricing and hedging of a contingent claim via a different approach, and more importantly, the theory of dynamic good-deal pricing is emphasized. They begin with a prescribed set $\mathcal{Q}^{\text{ngd}}$ of probability measures and define respectively upper and lower good-deal valuation bounds/good-deal prices π_t^u, π_t^l for a contingent claim X as

$$\pi_t^u(X; \mathcal{Q}^{\text{ngd}}) := \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^{\text{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t] \quad \text{and}$$
$$\pi_t^l(X; \mathcal{Q}^{\text{ngd}}) := \operatorname*{ess\,inf}_{Q \in \mathcal{Q}^{\text{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t].$$

These definitions are motivated by the bid price and ask price derived from $I_{NGF}(F)$ discussed above. It is shown in Becherer (2009) that the set $\mathcal{Q}^{\text{ngd}}$ is *m-stable*[2], then by invoking Theorem 12 in Delbaen (2006) or by referring to Theorem 3.7 in Klöppel and

[2]m-stability of a set of probability measures is firstly introduced in Delbaen (2006), we direct readers to Delbaen (2006) for more details.

Schweizer (2007),

Definition 4.1.3. Let $\mathcal{S}$ be a closed convex set of probability measures, d be a bounded random variable for each stopping time $T \leq \infty$, and

$$\Phi_T(f) = \operatorname*{ess\,inf}_{Q\in\mathcal{S}} \mathbb{E}^Q[X|\mathcal{F}_T],$$

then the following are equivalent:

i. The set $\mathcal{S}$ is m-stable.

ii. For every bounded random variable f, the family $\Phi_T(f)$ satisfies: for every two stopping times $\sigma \leq \tau$ we have $\Phi_\sigma(f) = \Phi_\sigma(\Phi_\tau(f))$.

it is an immediate consequence that both valuation bounds π_t^u and π_t^l inherit the properties of time-consistent dynamic coherent risk measures. A brief introduction on time-consistent dynamic coherent risk measures will be given later and one may see Cheridito and Kupper (2011) for more comprehensive discussions on time-consistent dynamic coherent risk measures. With the feature of time-consistency, the time-inconsistency problem in decision-making mentioned before can be eliminated.

As soon as dynamic good-deal prices, i.e. $\{\pi_t^u\}_{0\leq t\leq T}$ and $\{\pi_t^l\}_{0\leq t\leq T}$,are established, the dynamics of π_t^u and π_t^l are of interest. Instead of forward stochastic differential equations, Becherer (2009) characterizes the dynamics of π_t^u and π_t^l through backward stochastic differential equations (BSDEs), so π_t^u and π_t^l are represented as the solutions of the associated BSDEs. This creates the opportunity to study the properties of π_t^u and π_t^l from a dynamic perspective. After dynamic good-deal pricing is examined, dynamic good-deal hedging is also investigated in a similar vein, in which another set $\mathcal{P}^{\text{ngd}}$ of probability measures is prescribed. We postpone the discussion on this until the next chapter.

Inspired by the importance of the set of probability measures $\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}$ in Corollary 4.1.1 as well as the critical input of $\mathcal{Q}^{\text{ngd}}$ in defining the valuation bounds π_t^u and π_t^l, the specific choices of $\mathcal{PD}$ and $\mathcal{RD}$ in (4.1.1) suggest that $\bar{\mathcal{Q}}^{\text{ngd}}$ should be defined as

$$\begin{aligned}\bar{\mathcal{Q}}^{\text{ngd}} &:= \frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD} \\ &= \frac{1}{1+R}\left\{Q \;\middle|\; \frac{dQ}{dP} = 1 \quad P\text{-a.s.}\right\} + \frac{R}{1+R}\left\{Q \;\middle|\; 0 \leq \frac{dQ}{dP} \leq \frac{1}{\alpha} \quad P\text{-a.s.}\right\}.\end{aligned}$$

Based on this, we are able to follow an analogous analysis to that in Becherer (2009), and we shall focus on the upper good-deal valuation bound or upper good-deal price π_t^u

because of the relationship $\pi_t^u(X) = -\pi_t^l(-X)$. π_t^u under $\bar{\mathcal{Q}}^{\text{ngd}}$ is given by

$$\pi_t^u(X; \bar{\mathcal{Q}}^{\text{ngd}}) := \operatorname*{ess\,sup}_{Q \in \bar{\mathcal{Q}}^{\text{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t].$$

$\bar{\mathcal{Q}}^{\text{ngd}}$ is essentially the convex combination of $\mathcal{PD}$ and $\mathcal{RD}$. Besides, it can also be represented more succinctly as

$$\bar{\mathcal{Q}}^{\text{ngd}} = \left\{ Q \in \mathcal{P}^a \;\middle|\; 0 < \frac{1}{1+R} \leq \frac{dQ}{dP} \leq \frac{\alpha + R}{\alpha(1+R)} \quad P\text{-a.s.} \right\}.$$

To see this, for any $Q \in \frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}$, it is trivial to have $0 < \frac{1}{1+R} \leq \frac{dQ}{dP} \leq \frac{\alpha+R}{\alpha(1+R)}$. Conversely, for any $0 < \frac{1}{1+R} \leq \frac{dQ}{dP} \leq \frac{\alpha+R}{\alpha(1+R)}$, it can be decomposed into

$$\frac{dQ}{dP} = \frac{1}{1+R} + \frac{R}{1+R}\frac{dQ'}{dP} \quad \text{where} \quad \frac{dQ'}{dP} := \frac{1+R}{R}\left(\frac{dQ}{dP} - \frac{1}{1+R}\right).$$

Obviously $\frac{dQ'}{dP}$ satisfies $0 \leq \frac{dQ'}{dP} \leq \frac{1}{\alpha}$ P-a.s., hence, $\frac{dQ'}{dP} \in \mathcal{RD}$. This proves the equivalent representation of $\bar{\mathcal{Q}}^{\text{ngd}}$. In addition, from this representation, we know that any $Q \in \bar{\mathcal{Q}}^{\text{ngd}}$ is not only absolutely continuous but also equivalent[3] to P, $Q \sim P$, i.e.

$$\bar{\mathcal{Q}}^{\text{ngd}} = \left\{ Q \in \mathcal{P}^e \;\middle|\; 0 < \frac{1}{1+R} \leq \frac{dQ}{dP} \leq \frac{\alpha + R}{\alpha(1+R)} \quad P\text{-a.s.} \right\}.$$

As mentioned, if π_t^u behaves as a time-consistent dynamic coherent risk measure, we will not encounter a time-inconsistency problem when we evaluate a contingent claim dynamically. The fact that π_t^u is a time-consistent dynamic coherent risk measure plays a critical role in the analysis of Becherer (2009) and this validity is guaranteed due to m-stability of their choice of $\mathcal{Q}^{\text{ngd}}$. According to Delbaen (2006), m-stability is a sufficient condition on a set $\mathcal{S}$ of probability measures such that $\pi_t^u(X; \mathcal{S}) = \operatorname{ess\,sup}_{Q \in \mathcal{S}} \mathbb{E}^Q[X|\mathcal{F}_t]$ becomes a time-consistent dynamic coherent risk measure. For the sake of clarity, we recall the definition of m-stability from Delbaen (2006).

Definition 4.1.4. A set $\mathcal{S}$ of probability measures, all elements of which are equivalent to P, is called multiplicativity stable (m-stable) if, for all elements $Q^1, Q^2 \in \mathcal{S}$ with density processes Z^1, Z^2 and for all stopping times $\tau \leq T$, it holds that $Z_T := Z_\tau^1 \dfrac{Z_T^2}{Z_\tau^2}$ is the density of some $Q \in \mathcal{S}$.

To check whether π_t^u under $\bar{\mathcal{Q}}^{\text{ngd}}$ is a time-consistent dynamic coherent risk measure or not, an immediate thought is to verify if $\bar{\mathcal{Q}}^{\text{ngd}}$ obeys the property of m-stability. Unfortunately $\bar{\mathcal{Q}}^{\text{ngd}}$ does not possess m-stability for arbitrarily fixed values of α and R. To see this, it suffices to check for deterministic stopping time $\tau(\omega) = t$, $\forall \omega \in \Omega$.

[3] A probability measure Q is equivalent to another probability measure P if both of them are absolutely continuous with respect to each other, i.e. $Q \ll P$ and $P \ll Q$ hold.

Since any $Q \in \bar{\mathcal{Q}}^{\text{ngd}}$ should satisfy $\frac{1}{1+R} \le Z_T \le \frac{\alpha+R}{\alpha(1+R)}$ P-a.s., this implies $\frac{1}{1+R} \le \mathbb{E}[Z_T|\mathcal{F}_t] \le \frac{\alpha+R}{\alpha(1+R)}$ P-a.s., hence for any $\tau(\omega) = t$, $\frac{Z_T}{Z_t} = \frac{Z_T}{\mathbb{E}[Z_T|\mathcal{F}_t]}$ satisfies

$$\frac{\frac{1}{1+R}}{\frac{\alpha+R}{\alpha(1+R)}} = \frac{\alpha}{\alpha+R} \le \frac{Z_T}{Z_t} \le \frac{\frac{\alpha+R}{\alpha(1+R)}}{\frac{1}{1+R}} = \frac{\alpha+R}{\alpha} \quad P\text{-a.s.}$$

$$\Longrightarrow \quad \frac{1}{1+R} \cdot \frac{\alpha}{\alpha+R} \le Z_t^1 \frac{Z_T^2}{Z_t^2} \le \frac{\alpha+R}{\alpha(1+R)} \cdot \frac{\alpha+R}{\alpha} \quad P\text{-a.s.}$$

As a result, for any positive values of α and R, it is obvious that $\frac{\alpha}{\alpha+R} < 1$ holds and the density $Z_t^1 \frac{Z_T^2}{Z_t^2}$ generally does not produce a Q that belongs to $\bar{\mathcal{Q}}^{\text{ngd}}$.

In other words, it is not possible to obtain a time-consistent dynamic coherent risk measure π_t^u in the present context if we define π_t^u as $\pi_t^u(X; \bar{\mathcal{Q}}^{\text{ngd}})$, owing to the lack of m-stability. However, we *do* wish to represent π_t^u as $\pi_t^u = \operatorname{ess\,sup}_{Q \in \mathcal{Q}^{\text{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t]$ with some suitable $\mathcal{Q}^{\text{ngd}}$ such that π_t^u behaves as a time-consistent dynamic coherent risk measure. This can be accomplished by modifying $\bar{\mathcal{Q}}^{\text{ngd}}$ appropriately to obtain a desired $\mathcal{Q}^{\text{ngd}}$. Meanwhile, this new $\mathcal{Q}^{\text{ngd}}$ should resemble the original $\bar{\mathcal{Q}}^{\text{ngd}}$ the greatest extent possible. How this is done is described in the next section.

4.2 *Construction of Time-Consistent Dynamic Valuation Bounds in Discrete-Time*

It would be a challenging task to define a proper set $\mathcal{Q}^{\text{ngd}}$ such that a time-consistent dynamic risk measure π_t^u is obtained. To avoid such difficulty, we may determine $\mathcal{Q}^{\text{ngd}}$ by starting with some specifications of the partial structure of $\mathcal{Q}^{\text{ngd}}$. When we design the specifications, we should remember the desire of turning π_t^u into a time-consistent dynamic coherent risk measure. This can be regarded as a 'bottom-up' approach in the construction of a time-consistent dynamic coherent risk measure. A particularly useful paper on this aspect of constructing a time-consistent dynamic coherent risk measure is Cheridito and Kupper (2011), in which their method of construction is versatile from our point of view. However, since only a discrete-time formulation of a time-consistent dynamic coherent risk measure is studied in Cheridito and Kupper (2011), we assume a discrete-time setup throughout the whole chapter. Moreover, the construction can only be applied to a contingent claim $X \in L^\infty$, the space of essentially bounded $\mathcal{F}_T$ random variables.

We relate a coherent risk measure ρ to $\rho = -\phi$ where ϕ is a monetary utility function, see e.g. Artzner et al. (1999). This means the primary object to be studied is a monetary utility function ϕ. The definition of a dynamic monetary utility function as well as properties of monetary utility functions are reviewed here for convenience, see

Cheridito and Kupper (2011),

Definition 4.2.1. A dynamic monetary utility function $(\phi_t)_{t=0,1,\ldots,T}$ is a family of monetary utility functions indexed by t. For each index $t \in \{0, 1, \ldots, T\}$, the mapping $\phi_t : L^\infty(\mathcal{F}_T) \mapsto L^\infty(\mathcal{F}_t)$ is called a monetary utility function if it satisfies the following properties:

i. Normalization: $\phi_t(0) = 0$

ii. Monotonicity: $\phi_t(X) \geq \phi_t(Y)$ for all $X, Y \in L^\infty(\mathcal{F}_T)$ such that $X \geq Y$

iii. Translation property: $\phi_t(X + m) = \phi_t(X) + m$ for all $X \in L^\infty(\mathcal{F}_T)$ and $m \in L^\infty(\mathcal{F}_t)$

If, in addition, it possesses the property of

i. $\mathcal{F}_t$-concavity: $\phi_t(\lambda X + (1-\lambda)Y) \geq \lambda\phi_t(X) + (1-\lambda)\phi_t(Y)$ for all $X, Y \in L^\infty(\mathcal{F}_T)$, and, $\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$,

we shall call it a concave monetary utility function.

Time-consistency of a dynamic monetary utility function is defined as

Definition 4.2.2. A dynamic monetary utility function $(\phi_t)_{t=0,1,\ldots,T}$ is time-consistent if

$$\phi_{t+1}(X) \geq \phi_{t+1}(Y) \quad \Longrightarrow \quad \phi_t(X) \geq \phi_t(Y)$$

for all $X, Y \in L^\infty(\mathcal{F}_T)$ and $t = 0, 1, \ldots, T$.

Furthermore, due to the properties of a dynamic monetary utility function, time-consistency can be equivalently characterized as the fulfilment of a dynamic programming principle in ϕ_t, namely,

$$\phi_t(X) = \phi_t(\phi_{t+1}(X)) \quad \text{for all } X \in L^\infty(\mathcal{F}_T) \text{ and } t = 0, 1, \ldots, T-1.$$

Remark 4.2.0.2. In light of this, we can understand that the amount of risk is consistently measured across time. Intuitively, risk at maturity as measured today is the same as tomorrow's risk as measured today.

Taking this dynamic programming principle as primitive, the first step of the construction described in Cheridito and Kupper (2011) is to prepare ourselves with an arbitrary family of monetary utility functions

$$\varphi_t : L^\infty(\mathcal{F}_{t+1}) \mapsto L^\infty(\mathcal{F}_t), \qquad t = 0, 1, \ldots, T-1,$$

and then we define implicitly a time-consistent dynamic monetary utility function $(\phi_t)_{t=0,1,\dots,T}$ by means of backward induction:

$$\phi_T(X) = X \qquad \text{and} \qquad \phi_t(X) = \varphi_t(\phi_{t+1}(X)), \qquad t = 0, 1, \dots, T-1. \tag{4.2.1}$$

We shall call the family $(\varphi_t)_{t=0,1,\dots,T}$ the generator of $(\phi_t)_{t=0,1,\dots,T}$. The freedom in choosing the family $(\varphi_t)_{t=0,1,\dots,T}$ is a great feature in this construction method. In this framework, it is highly convenient and flexible to create a time-consistent dynamic monetary utility function, hence, a time-consistent dynamic coherent risk measure, and they can enjoy certain desired properties by definition. After a time-consistent dynamic coherent risk measure, we can then determine the corresponding structure of $\mathcal{Q}^{\text{ngd}}$.

Remark 4.2.0.3. In essence, a time-consistent dynamic monetary utility function ϕ_t is constructed through its 'intertemporal atoms' φ_t, thus, this can be regarded as a 'bottom-up' construction.

Here we provide some conventions used throughout the chapter and establish the connection between an absolutely-continuous probability measure Q and its density $Z = \frac{dQ}{dP}$. One may also refer to Cheridito and Kupper (2011). Define the following sets of one-step transition densities

$$\mathcal{D}_t := \left\{ Z \in L^1_+(\mathcal{F}_t) \;\middle|\; \mathbb{E}^P[Z|\mathcal{F}_{t-1}] = 1 \quad P\text{-a.s.} \right\}, \qquad t = 1, \dots, T.$$

On one hand, every sequence $z = (Z_1, \dots, Z_T) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_T$ defines a probability measure Q^Z in $\mathcal{P}^a$ with density

$$\frac{dQ}{dP} = Z_1 \cdots Z_T.$$

On the other hand, since every probability measure $Q \in \mathcal{P}^a$ induces a non-negative martingale

$$M_t^Q := \mathbb{E}^P\left[\frac{dQ}{dP} \middle| \mathcal{F}_t \right], \qquad t = 0, \dots, T$$

and

$$\left\{ \omega \;\middle|\; M_{t-1}^Q = 0 \right\} \subseteq \left\{ \omega \mid M_t^Q = 0 \right\}, \qquad t = 1, \dots, T,$$

so we define a sequence $\left\{Z_t^Q\right\}$ given by

$$Z_t^Q := \begin{cases} \frac{M_t^Q}{M_{t-1}^Q} & \text{on} \quad \left\{\omega \,\middle|\, M_{t-1}^Q > 0\right\} \\ 1 & \text{on} \quad \left\{\omega \,\middle|\, M_{t-1}^Q = 0\right\} \end{cases}, \qquad t = 1, \dots, T, \tag{4.2.2}$$

which is an element in $\mathcal{D}_1 \times \cdots \times \mathcal{D}_T$ with the property that

$$\frac{dQ}{dP} = Z_1^Q \cdots Z_T^Q.$$

As a result, we are able to identify any $Q \in \mathcal{P}^a$ with density $Z = \frac{dQ}{dP} \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_T$ and vice versa. Moreover, we shall understand

$$\mathbb{E}^Q[X|\mathcal{F}_t] := \mathbb{E}^P\left[Z_{t+1}^Q \cdots Z_T^Q X \middle| \mathcal{F}_t\right], \qquad t = 0, \dots, T$$

for a fixed $Q \in \mathcal{P}^a$, unless stated otherwise.

To start with, we prepare ourselves with the following family $(\varphi_t)_{t=0,1,\dots,T}$

$$\begin{aligned} \varphi_t(X) &= \operatorname*{ess\,inf}_{Z_{t+1} \in \mathcal{D}_{t+1} \cap \mathcal{Q}_{t+1}^{\text{ngd}}} \mathbb{E}^P[Z_{t+1}X|\mathcal{F}_t], \qquad t = 0, \dots, T-1, \quad \text{and} \\ \varphi_T(X) &= X \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_t &:= \{Z \in L_+^1(\mathcal{F}_t) \mid \mathbb{E}^P[Z|\mathcal{F}_{t-1}] = 1 \quad P\text{-a.s.}\} \qquad \text{and} \\ \mathcal{Q}_t^{\text{ngd}} &:= \left\{Z \in L_+^1(\mathcal{F}_t) \,\middle|\, 0 < \frac{1}{r_1} \le Z \le \frac{1}{r_2} \quad P\text{-a.s.}\right\}, \quad t = 0, 1, \dots, T-1, \end{aligned}$$

for some fixed $r_1 > 1 > r_2 > 0$. Here, $\mathcal{D}_t$ is the set of one-step (between t and $t+1$) transition densities and $\mathcal{Q}_t^{\text{ngd}}$ is the set of one-step transition densities which are bounded above by $\frac{1}{r_2}$ and below by $\frac{1}{r_1}$.

Remark 4.2.0.4. By the property of conditional expectation,

$$\mathbb{E}^P[Z_{t+1}X|\mathcal{F}_t] = \mathbb{E}^P\left[\mathbb{E}^P[Z_{t+1}X|\mathcal{F}_{t+1}]\middle|\mathcal{F}_t\right] = \mathbb{E}^P\left[Z_{t+1}\mathbb{E}^P[X|\mathcal{F}_{t+1}]\middle|\mathcal{F}_t\right],$$

we have

$$\varphi_t(X) = \varphi_t(\mathbb{E}^P[X|\mathcal{F}_{t+1}]).$$

This can be understood as "the amount of risk for a contingent claim X with maturity T is $\varphi_t(X)$ at time t and it is the same as that for a 'fictitious' contingent claim $\mathbb{E}^P[X|\mathcal{F}_{t+1}]$

with maturity $t+1$."

By the properties of essential infimum, see Karatzas and Shreve (1998), we can observe that each φ_t is a concave monetary utility function and so the family $(\varphi_t)_{t=0,1,\dots,T}$ can be used as the generator of a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0,1,\dots,T}$.

Remark 4.2.0.5. It is indeed quite straightforward to show it is a concave monetary utility function. Proofs regarding normalization, monotonicity and translation property are elementary and so omitted for brevity. To show concavity, we use the following fact

$$\operatorname*{ess\,inf}_t (X_t + Y_t) \geq \operatorname*{ess\,inf}_t X_t + \operatorname*{ess\,inf}_t Y_t.$$

It can also be verified that $\varphi_t(X)$ is continuous from above. As a matter of fact, suppose $\{X^n\}$ is decreasing to X P-a.s. , and ξ, X^n, X are all bounded random variables, by the Monotone Convergence Theorem for conditional expectation, we have

$$\mathbb{E}^P[\xi_{t+1}X|\mathcal{F}_t] = \lim_n \mathbb{E}^P[\xi_{t+1}X^n|\mathcal{F}_t] \qquad P\text{-a.s.}$$

Moreover, by the definition of an essential infimum, we have

$$\begin{aligned}
& \mathbb{E}^P[\xi_{t+1}X^n|\mathcal{F}_t] \geq \operatorname*{ess\,inf}_{\xi_{t+1}} \mathbb{E}^P[\xi_{t+1}X^n|\mathcal{F}_t] \qquad P\text{-a.s.} \\
\Longrightarrow\quad & \lim_n \mathbb{E}^P[\xi_{t+1}X^n|\mathcal{F}_t] \geq \overline{\lim_n} \operatorname*{ess\,inf}_{\xi_{t+1}} \mathbb{E}^P[\xi_{t+1}X^n|\mathcal{F}_t] \qquad P\text{-a.s.} \\
\Longrightarrow\quad & \mathbb{E}^P[\xi_{t+1}X|\mathcal{F}_t] \geq \overline{\lim_n} \operatorname*{ess\,inf}_{\xi_{t+1}} \mathbb{E}^P[\xi_{t+1}X^n|\mathcal{F}_t] \qquad P\text{-a.s.} \\
\Longrightarrow\quad & \varphi_t(X) \geq \overline{\lim_n} \varphi_t(X^n) \qquad P\text{-a.s.}
\end{aligned}$$

However, due to the monotonicity of φ, we have

$$X^n \geq X \quad \Longrightarrow \quad \varphi_t(X^n) \geq \varphi_t(X).$$

This altogether implies

$$\varphi_t(X) = \lim_n \varphi_t(X^n).$$

The representation of ϕ_t is yet to be made explicit even though it is well-defined by $(\varphi_t)_{t=0,1,\dots,T}$ and backward inductions in (4.2.1). Moreover, the main object we are interested in is ϕ_t which is implicity defined. To achieve this, the representation result in Cheridito and Kupper (2011) shows that ϕ_t can indeed be characterized as $\operatorname{ess\,inf}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t]$ for some $\mathcal{Q}^{\mathrm{ngd}}$. The remaining work of present chapter is devoted to the determination of $\mathcal{Q}^{\text{ngd}}$.

The analysis is close to that in Vogelpoth (2006). In the meantime, the concepts of dynamic expected shortfall, or equivalently, dynamic average value-at risk in Cheridito and Kupper (2011), and its representation are studied. ϕ_t can be regarded as a variant of dynamic expected shortfall. This point will become transparent as one approaches the end.

4.2.1 Explicit Representation of ϕ_t

In this section, we derive the representation of ϕ_t implicity defined by $(\varphi_t)_{t=0,1,\ldots,T}$ and backward inductions in (4.2.1). By following Cheridito and Kupper (2011), we first define the mapping $\varphi_t^{min} : \mathcal{D}_{t+1} \mapsto \overline{L}_+(\mathcal{F}_t)$[4]

$$\varphi_t^{min}(Z_{t+1}) \coloneqq \operatorname*{ess\,sup}_{X \in L^\infty(\mathcal{F}_{t+1})} \left\{ \varphi_t(X) - \mathbb{E}^P[Z_{t+1}X|\mathcal{F}_t] \right\}.$$

Since φ_t is continuous from above, we conclude from Lemma 2.1 in Cheridito and Kupper (2011) that φ_t^{min} is the smallest dynamic penalty function such that

$$\varphi_t(X) = \operatorname*{ess\,inf}_{Q \in \mathcal{P}_a} \{\mathbb{E}^Q[X|\mathcal{F}_t] + \varphi_t^{min}(Q)\}$$

[5]and from Corollary 2.1 in Cheridito and Kupper (2011) that we can represent ϕ_t explicitly as

$$\begin{aligned}\phi_t(X) = \operatorname*{ess\,inf}_{Q \in \mathcal{P}^a} \mathbb{E}^Q\left[X + \sum_{i=t+1}^{T} \varphi_{i-1}^{min}(Q) \middle| \mathcal{F}_t \right] &= \operatorname*{ess\,inf}_{Q \in \mathcal{P}^a} \mathbb{E}^Q[X + \phi^{min}(Q)|\mathcal{F}_t] \\ &\coloneqq \operatorname*{ess\,inf}_{Q \in \mathcal{Q}^{\text{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t]\end{aligned}$$

where $\phi^{min}(Q) \coloneqq \sum_{i=1}^{T} \varphi_{i-1}^{min}(Q)$, for all $t = 0, 1, \ldots, T-1$ and $X \in L^\infty(\mathcal{F}_{t+1})$. $\mathcal{Q}^{\text{ngd}}$ is a subset of $\mathcal{P}^a$ resultant from the presence of $\phi^{min}(Q)$, of which the function ϕ^{min} would penalize inappropriate probability measures Q by assigning them with $\phi^{min}(Q) = +\infty$ (so that the essential infimum cannot be located at such Q). This $\mathcal{Q}^{\text{ngd}}$ is what we hope for. The specific content of $\mathcal{Q}^{\text{ngd}}$ is acquired based on similar analysis in Vogelpoth (2006).

In particular, we shall make use of regular conditional probabilities and regular conditional distributions[6]. Hereafter, we assume a probability space $(\Omega, \mathcal{G}, P)$ is given

[4] $\overline{L}_+(\mathcal{F}_t)$ is the space of $L_+(\mathcal{F}_t)$ augmented with $X = +\infty$ P-a.s.

[5] Here we mean $\varphi_t^{min}(Q) = \varphi_t^{min}(\frac{dQ}{dP})$

[6] They are introduced because of ease in analysis. Indeed results derived under the presence of regular conditional probability and regular conditional distribution are sufficient to meet our purpose,

and it is equipped with an auxiliary σ-algebra $\mathcal{F}$ which is a sub-σ-algebra of $\mathcal{G}$, i.e. $\mathcal{F} \subseteq \mathcal{G}$. Let us recall the definition of regular conditional probability from Gray (2009).

Definition 4.2.3. Given a probability space $(\Omega, \mathcal{G}, P)$ with auxiliary sub-σ-algebra $\mathcal{F} \subseteq \mathcal{G}$, if a mapping $P^{\mathcal{F}} : \Omega \times \mathcal{G} \mapsto [0,1]$ satisfies the following properties:

i. For each $\omega \in \Omega$, $P^{\mathcal{F}}(\omega, \cdot) : \mathcal{G} \mapsto [0,1]$ is a probability measure;

ii. For each $A \in \mathcal{G}$, $P^{\mathcal{F}}(\cdot, A) : \Omega, \mapsto [0,1]$ is $\mathcal{F}$-measurable;

iii. For each $A \in \mathcal{G}$ and $B \in \mathcal{F}$, $\int_B P^{\mathcal{F}}(\omega, A)P(d\omega) = P(B \cap A)$;

then it is called a regular conditional probability with respect to $\mathcal{F}$.

Remark 4.2.1.1. In view of condition (ii) and (iii), it is also equivalent to saying that, for a fixed $A \in \mathcal{G}$, $P^{\mathcal{F}}(\cdot, A) : \Omega, \mapsto [0,1]$, as a function of ω, is a version of conditional probability of A with respect to $\mathcal{F}$, that is,

$$P^{\mathcal{F}}(\omega, A) = P(A|\mathcal{F})(\omega) = \mathbb{E}[\mathbf{1}_A|\mathcal{F}](\omega) \quad P\text{-a.s.}$$

Remark 4.2.1.2. It should be emphasized that $P^{\mathcal{F}}$ is defined over the whole sample space Ω whilst the usual conditional probability of an event A given $\mathcal{F}$ is defined in the P-almost sure sense over Ω.

The existence of a regular conditional probability with respect to a sub-σ-algebra $\mathcal{F}$ under a given probability space $(\Omega, \mathcal{G}, P)$ is justified when the sub-σ-algebra $\mathcal{F}$ is 'sufficiently simple'. In particular, we have the following theorem from Gray (2009), see also Vogelpoth (2006),

Theorem 4.2.1. *Given a probability space $(\Omega, \mathcal{G}, P)$ and a discrete sub-σ-algebra $\mathcal{F} \subseteq \mathcal{G}$ (that is, $\mathcal{F}$ has a finite or countable number of members), then there exists a regular conditional probability measure with respect to $\mathcal{F}$.*

Remark 4.2.1.3. An example of such a probability space and sub-σ-algebra, which will also be encountered in next chapter, is a finite space[7] S equipped with the discrete topology[8] $\mathcal{T}_d$ and discrete metric[9] d. If we consider $(S, \mathcal{S} := \sigma(\mathcal{T}_d) = \mathcal{T}_d, P)$, then, since

particularly in the next chapter where we consider a probability space that provides the existence of regular conditional probabilities and regular conditional distributions.

[7]That is, a space with only finitely many elements.

[8]Discrete topology is also referring to the power set.

[9]Discrete metric d is given as

$$d(x,y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}.$$

Such a metric is compatible with the discrete topology, hence, making a finite space readily a completely separable metric space.

there are finitely many members in $\mathcal{T}_d$, any sub-σ-algebra $\mathcal{F} \subseteq \mathcal{T}_d$ must also contain finitely many members, hence, by the above theorem, a regular conditional probability with respect to any sub-σ-algebra $\mathcal{F} \subseteq \mathcal{T}_d$ exists.

Assume a given $(\Omega, \mathcal{G}, P)$ augmented with a sub-σ-algebra $\mathcal{F} \subseteq \mathcal{G}$ and a fixed random variable X with its state space in $(\mathcal{S}, \mathcal{B})$ where $\mathcal{B}$ is Borel σ-algebra. Moreover, suppose the current probability space admits the presence of a regular conditional probability respect to $\mathcal{F}$. We define two mappings $P_X^{\mathcal{F}}$ and $F_X^{\mathcal{F}}$.

Definition 4.2.4. For a random variable X defined on $(\Omega, \mathcal{G}, P)$, the mapping

$$P_X^{\mathcal{F}} : \Omega \times \mathcal{B} \mapsto [0,1], \qquad (\omega, B) \mapsto P_X^{\mathcal{F}}(\omega, B) := P^{\mathcal{F}}(\omega, \{X \in B\}),$$

is called a regular conditional distribution[10] of X given $\mathcal{F}$. The mapping

$$F_X^{\mathcal{F}} : \Omega \times \mathbb{R} \mapsto [0,1], \qquad (\omega, x) \mapsto F_X^{\mathcal{F}}(\omega, x) := P^{\mathcal{F}}(\omega, \{X \leq x\}),$$

is called a regular conditional distribution function of X given $\mathcal{F}$.

Remark 4.2.1.4. Note that $P_X^{\mathcal{F}}$ inherits properties of $P^{\mathcal{F}}$, hence, $P_X^{\mathcal{F}}(\cdot, B)$ is a version of $\mathbb{E}[\mathbf{1}_{\{X \in B\}} | \mathcal{F}]$, i.e. for a fixed $B \in \mathcal{B}$, $P_X^{\mathcal{F}}(\omega, B) = \mathbb{E}[\mathbf{1}_{\{X \in B\}} | \mathcal{F}](\omega) \quad P$-a.s..

Upon the introduction of the mapping $F_X^{\mathcal{F}}$, we also establish the notion of conditional quantile according to Vogelpoth (2006).

Definition 4.2.5. For a random variable X defined on $(\Omega, \mathcal{G}, P)$, the conditional r-quantile of X given $\mathcal{F}$, denoted by $q_X^{\mathcal{F}}$, is a mapping

$$q_X^{\mathcal{F}} : \Omega \times (0,1) \mapsto \mathbb{R}, \quad (\omega, r) \mapsto q_X^{\mathcal{F}}(\omega, r),$$

such that for each fixed $\omega \in \Omega$, $q_X^{\mathcal{F}}(\omega, \cdot)$ is a function of r satisfying

$$F_X^{\mathcal{F}}\big(\omega, q_X^{\mathcal{F}}(\omega, r) - \big) \leq r \leq F_X^{\mathcal{F}}\big(\omega, q_X^{\mathcal{F}}(\omega, r)\big)$$

where $q_X^{\mathcal{F}}(\omega, r) - := \lim_{y \uparrow r} q_X^{\mathcal{F}}(\omega, y)$. In other words, $q_X^{\mathcal{F}}(\omega, \cdot)$ is an inverse function of $F_X^{\mathcal{F}}(\omega, \cdot)$ for each fixed $\omega \in \Omega$.

Henceforward, we omit the subscript X and use $P^{\mathcal{F}}$ to mean $P_X^{\mathcal{F}}$ for brevity. Moreover, $q_X^{\mathcal{F}}(r)$ will always refer to $q_X^{\mathcal{F}}(\omega, r)$ without emphasizing the dependence of ω. We recall from Vogelpoth (2006) the

[10] For more details regarding regular conditional distributions, see, for example, Ikeda and Watanabe (1989) and Shiryaev (1995).

Lemma 4.2.1. *For any fixed $r \in (0,1)$, consider the associated conditional r-quantile $q_X^{\mathcal{F}}(r)$ of a P-almost surely bounded random variable X defined on $(\Omega, \mathcal{G}, P)$, the ω-sets $\{X \le q_X^{\mathcal{F}}(r)\}$ and $\{X < q_X^{\mathcal{F}}(r)\}$ are $\mathcal{G}$-measurable. Then, for the $\mathcal{F}$-measurable mappings defined by*

$$
\begin{aligned}
&P^{\mathcal{F}}\big(\cdot, \{X \le q_X^{\mathcal{F}}(r)\}\big) : (\Omega, \mathcal{F}) \mapsto [0,1], \qquad \omega \mapsto P^{\mathcal{F}}\big(\omega, \{X \le q_X^{\mathcal{F}}(r)\}\big), \quad \textit{and} \\
&P^{\mathcal{F}}\big(\cdot, \{X < q_X^{\mathcal{F}}(r)\}\big) : (\Omega, \mathcal{F}) \mapsto [0,1], \qquad \omega \mapsto P^{\mathcal{F}}\big(\omega, \{X < q_X^{\mathcal{F}}(r)\}\big),
\end{aligned}
$$

we have

$$
P^{\mathcal{F}}\big(\cdot, \{X < q_X^{\mathcal{F}}(r)\}\big) \le r \le P^{\mathcal{F}}\big(\cdot, \{X \le q_X^{\mathcal{F}}(r)\}\big) \quad P\text{-a.s.}.
$$

Proof. One may refer to Vogelpoth (2006) for the proof. □

Remark 4.2.1.5. Specifically, on the ω-set $\{P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(r)\}\big) = 0\}$, we shall have the above equality hold, i.e. $P^{\mathcal{F}}\big(\omega, \{X < q_X^{\mathcal{F}}(r)\}\big) = P^{\mathcal{F}}\big(\omega, \{X \le q_X^{\mathcal{F}}(r)\}\big) = r$. To see this, on the ω-set $\{P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(r)\}\big) = 0\}$,

$$
\begin{aligned}
& P^{\mathcal{F}}\big(\omega, \{X < q_X^{\mathcal{F}}(r)\}\big) \le r \le P^{\mathcal{F}}\big(\omega, \{X \le q_X^{\mathcal{F}}(r)\}\big) \\
\Longrightarrow\quad & P^{\mathcal{F}}\big(\omega, \{X < q_X^{\mathcal{F}}(r)\}\big) + P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(r)\}\big) \le r \le P^{\mathcal{F}}\big(\omega, \{X \le q_X^{\mathcal{F}}(r)\}\big) \\
\Longrightarrow\quad & P^{\mathcal{F}}\big(\omega, \{X \le q_X^{\mathcal{F}}(r)\}\big) \le r \le P^{\mathcal{F}}\big(\omega, \{X \le q_X^{\mathcal{F}}(r)\}\big) \\
\Longrightarrow\quad & P^{\mathcal{F}}\big(\omega, \{X \le q_X^{\mathcal{F}}(r)\}\big) = r.
\end{aligned}
$$

The following defines two mappings $I_X^{\mathcal{F}}(r_1, r_2)(\omega)$ and $\kappa_X^{\mathcal{F}}(r_1, r_2)(\omega)$ on Ω.

Definition 4.2.6. Fix $r_1 > 1 > r_2 > 0$ and $1 > \bar{r} := \frac{1-\frac{1}{r_1}}{\frac{1}{r_2}-\frac{1}{r_1}} > 0$. Denote $q_X^{\mathcal{F}}(r)(\omega)$ as the conditional r-quantile of X given $\mathcal{F}$. Define the $\mathcal{G}$-measurable mapping $I_X^{\mathcal{F}}(r_1, r_2)(\omega)$: $\Omega \mapsto \mathbb{R}$ by

$$
\omega \mapsto I_X^{\mathcal{F}}(r_1, r_2)(\omega) := \frac{1}{r_2}\mathbf{1}_{\{X < q_X^{\mathcal{F}}(\bar{r})\}} + \frac{1}{r_1}\mathbf{1}_{\{X > q_X^{\mathcal{F}}(\bar{r})\}} + \kappa_X^{\mathcal{F}}(r_1, r_2)(\omega)\mathbf{1}_{\{X = q_X^{\mathcal{F}}(\bar{r})\}} \tag{4.2.3}
$$

and the $\mathcal{F}$-measurable mapping $\kappa_X^{\mathcal{F}}(r_1, r_2)(\omega) : \Omega \mapsto \mathbb{R}$ by

$$
\omega \mapsto \kappa_X^{\mathcal{F}}(r_1, r_2)(\omega) := \begin{cases} 1 & \text{on} \quad \{P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\}\big) = 0\} \\[2ex] \dfrac{1 - \dfrac{1}{r_2}P^{\mathcal{F}}(\omega, \{X < q_X^{\mathcal{F}}(\bar{r})\}) - \dfrac{1}{r_1}P^{\mathcal{F}}(\omega, \{X > q_X^{\mathcal{F}}(\bar{r})\})}{P^{\mathcal{F}}(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\})} & \\ & \text{on} \quad \{P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\}\big) > 0\} \end{cases} \tag{4.2.4}
$$

We then have the following

Lemma 4.2.2. *For fixed* $r_1 > 1 > r_2 > 0$*, the* $\mathcal{F}$*-measurable mapping* $\kappa_X^{\mathcal{F}}(r_1, r_2)$ *takes values on* $\left[\frac{1}{r_1}, \frac{1}{r_2}\right]$ P*-almost surely. Moreover, for the* $\mathcal{G}$*-measurable mapping* $I_X^{\mathcal{F}}(r_1, r_2)(\omega)$*, we have* $\mathbb{E}\left[I_X^{\mathcal{F}}(r_1, r_2) \middle| \mathcal{F}\right] = 1, \qquad P\text{-a.s.}$

Proof. Since the set $\left\{\omega \mid P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\}\big) \geq 0\right\}$ is $\mathcal{F}$-measurable and of probability measure one by definition, it is sufficient to consider this set to establish the assertion.

For $\omega \in \left\{P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\}\big) = 0\right\}$, we have $\kappa_X^{\mathcal{F}}(r_1, r_2) = 1 \in \left(\frac{1}{r_1}, \frac{1}{r_2}\right]$ by definition, thus, only the situation on $\left\{\omega \mid P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\}\big) > 0\right\}$ remains to be proven. On one hand, consider

$$
\begin{aligned}
&1 - \frac{1}{r_2} P^{\mathcal{F}}\big(\omega, \{X < q_X^{\mathcal{F}}(\bar{r})\}\big) - \frac{1}{r_1} P^{\mathcal{F}}\big(\omega, \{X > q_X^{\mathcal{F}}(\bar{r})\}\big) - \frac{1}{r_2} P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\}\big) \\
&= 1 - \frac{1}{r_2} P^{\mathcal{F}}\big(\omega, \{X < q_X^{\mathcal{F}}(\bar{r})\}\big) - \frac{1}{r_1} + \frac{1}{r_1} P^{\mathcal{F}}\big(\omega, \{X \leq q_X^{\mathcal{F}}(\bar{r})\}\big) \\
&\quad - \frac{1}{r_2} P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\}\big) \\
&= 1 - \frac{1}{r_1} - \left(\frac{1}{r_2} - \frac{1}{r_1}\right) P^{\mathcal{F}}\big(\omega, \{X \leq q_X^{\mathcal{F}}(\bar{r})\}\big) \\
&\leq 1 - \frac{1}{r_1} - \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \bar{r} \\
&= 0
\end{aligned}
$$

where we have invoked Lemma 4.2.1 for arriving at the second last inequality. This leads us to

$$
\begin{aligned}
&1 - \frac{1}{r_2} P^{\mathcal{F}}\big(\omega, \{X < q_X^{\mathcal{F}}(\bar{r})\}\big) - \frac{1}{r_1} P^{\mathcal{F}}\big(\omega, \{X > q_X^{\mathcal{F}}(\bar{r})\}\big) \leq \frac{1}{r_2} P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\}\big) \\
\Longrightarrow \quad & \kappa_X^{\mathcal{F}}(\bar{r}) \leq \frac{1}{r_2}.
\end{aligned}
$$

On the other hand, consider

$$
\begin{aligned}
&1 - \frac{1}{r_2} P^{\mathcal{F}}\big(\omega, \{X < q_X^{\mathcal{F}}(\bar{r})\}\big) - \frac{1}{r_1} P^{\mathcal{F}}\big(\omega, \{X > q_X^{\mathcal{F}}(\bar{r})\}\big) - \frac{1}{r_1} P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\}\big) \\
&= 1 - \frac{1}{r_2} P^{\mathcal{F}}\big(\omega, \{X < q_X^{\mathcal{F}}(\bar{r})\}\big) - \frac{1}{r_1} + \frac{1}{r_1} P^{\mathcal{F}}\big(\omega, \{X \leq q_X^{\mathcal{F}}(\bar{r})\}\big) \\
&\quad - \frac{1}{r_1} P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar{r})\}\big) \\
&= 1 - \frac{1}{r_1} - \left(\frac{1}{r_2} - \frac{1}{r_1}\right) P^{\mathcal{F}}\big(\omega, \{X < q_X^{\mathcal{F}}(\bar{r})\}\big) \\
&\geq 1 - \frac{1}{r_1} - \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \bar{r} \\
&= 0.
\end{aligned}
$$

in which Lemma 4.2.1 is used again to get the second last inequality. This implies

$$\begin{aligned}
& 1-\frac{1}{r_2}P^{\mathcal{F}}\big(\omega,\{X<q_X^{\mathcal{F}}(\bar{r})\}\big)-\frac{1}{r_1}P^{\mathcal{F}}\big(\omega,\{X>q_X^{\mathcal{F}}(\bar{r})\}\big)\geq\frac{1}{r_1}P^{\mathcal{F}}\big(\omega,\{X=q_X^{\mathcal{F}}(\bar{r})\}\big)\\
\Longrightarrow\quad & \kappa_X^{\mathcal{F}}(\bar{r})\geq\frac{1}{r_1}.
\end{aligned}$$

So the first assertion follows.

To verify the second assertion, we perform again analysis on the set $\{\omega \mid P^{\mathcal{F}}(\omega,\{X = q_X^{\mathcal{F}}(\bar{r})\}) \geq 0\}$. For $\omega \in \big\{\omega \mid P^{\mathcal{F}}\big(\omega,\{X=q_X^{\mathcal{F}}(\bar{r})\}\big)=0\big\}$, since $\kappa_X^{\mathcal{F}}(r_1,r_2)(\omega)=1$, so

$$\begin{aligned}
\mathbb{E}\big[I_X^{\mathcal{F}}(r_1,r_2)\big|\mathcal{F}\big] &= \frac{1}{r_2}\mathbb{E}\big[\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar{r})\}}\big|\mathcal{F}\big]+\frac{1}{r_1}\mathbb{E}\big[\mathbf{1}_{\{X>q_X^{\mathcal{F}}(\bar{r})\}}\big|\mathcal{F}\big]+\mathbb{E}\big[\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar{r})\}}\big|\mathcal{F}\big]\\
&= \frac{1}{r_2}P^{\mathcal{F}}\big(\omega,\{X<q_X^{\mathcal{F}}(\bar{r})\}\big)\\
&\quad +\frac{1}{r_1}P^{\mathcal{F}}\big(\omega,\{X>q_X^{\mathcal{F}}(\bar{r})\}\big)+P^{\mathcal{F}}\big(\omega,\{X=q_X^{\mathcal{F}}(\bar{r})\}\big)\\
&= \frac{1}{r_2}P^{\mathcal{F}}\big(\omega,\{X\leq q_X^{\mathcal{F}}(\bar{r})\}\big)+\frac{1}{r_1}P^{\mathcal{F}}\big(\omega,\{X>q_X^{\mathcal{F}}(\bar{r})\}\big)\\
&= \frac{1}{r_1}+\left(\frac{1}{r_2}-\frac{1}{r_1}\right)P^{\mathcal{F}}\big(\omega,\{X\leq q_X^{\mathcal{F}}(\bar{r})\}\big)\\
&= \frac{1}{r_1}+\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\bar{r}\\
&= 1,
\end{aligned}$$

owing to the comment in Remark 4.2.1.5.

For $\omega \in \big\{\omega \mid P^{\mathcal{F}}\big(\omega,\{X=q_X^{\mathcal{F}}(\bar{r})\}\big)>0\big\}$, $\kappa_X^{\mathcal{F}}(r_1,r_2)(\omega)$ is given by

$$\kappa_X^{\mathcal{F}}(r_1,r_2)(\omega)=\frac{1-\frac{1}{r_2}P^{\mathcal{F}}\big(\omega,\{X<q_X^{\mathcal{F}}(\bar{r})\}\big)-\frac{1}{r_1}P^{\mathcal{F}}\big(\omega,\{X>q_X^{\mathcal{F}}(\bar{r})\}\big)}{P^{\mathcal{F}}\big(\omega,\{X=q_X^{\mathcal{F}}(\bar{r})\}\big)},$$

together with its $\mathcal{F}$-measurability, this implies

$$\begin{aligned}
\mathbb{E}\big[I_X^{\mathcal{F}}(r_1,r_2)\big|\mathcal{F}\big] &= \frac{1}{r_2}\mathbb{E}\big[\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar{r})\}}\big|\mathcal{F}\big]+\frac{1}{r_1}\mathbb{E}\big[\mathbf{1}_{\{X>q_X^{\mathcal{F}}(\bar{r})\}}\big|\mathcal{F}\big]\\
&\quad +\kappa_X^{\mathcal{F}}(r_1,r_2)\mathbb{E}\big[\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar{r})\}}\big|\mathcal{F}\big]\\
&= \frac{1}{r_2}P^{\mathcal{F}}\big(\omega,\{X<q_X^{\mathcal{F}}(\bar{r})\}\big)+\frac{1}{r_1}P^{\mathcal{F}}\big(\omega,\{X>q_X^{\mathcal{F}}(\bar{r})\}\big)\\
&\quad +\kappa_X^{\mathcal{F}}(r_1,r_2)P^{\mathcal{F}}\big(\omega,\{X=q_X^{\mathcal{F}}(\bar{r})\}\big)\\
&= 1.
\end{aligned}$$

Hence the second assertion is justified. □

Remark 4.2.1.6. As an immediate consequence of Lemma 4.2.2, we are able to conclude that, P-a.s., $\frac{1}{r_1} \leq I_X^{\mathcal{F}}(r_1, r_2) \leq \frac{1}{r_2}$ and $\frac{1}{r_1} \leq \frac{I_X^{\mathcal{F}}(r_1,r_2)}{\mathbb{E}[I_X^{\mathcal{F}}(r_1,r_2)|\mathcal{F}]} \leq \frac{1}{r_2}$. Moreover, we can see that $\mathbb{E}\big[I_X^{\mathcal{F}}(r_1, r_2)\big] = \mathbb{E}\big[\mathbb{E}\big[I_X^{\mathcal{F}}(r_1, r_2)\big|\mathcal{F}\big]\big] = 1$.

Before we define the notion of an extended conditional expected shortfall, we introduce the following convention, for any $Z \in \mathcal{P}^a$,

$$\frac{Z}{\mathbb{E}[Z|\mathcal{F}]} := \begin{cases} \frac{Z}{\mathbb{E}[Z|\mathcal{F}]} & \text{on} \quad \{\mathbb{E}[Z|\mathcal{F}] > 0\} \\ 1 & \text{on} \quad \{\mathbb{E}[Z|\mathcal{F}] = 0\}\,. \end{cases}.$$

Definition 4.2.7. For fixed $r_1 > 1 > r_2 > 0$ and a random variable X defined on $(\Omega, \mathcal{G}, P)$, the mapping

$$\widetilde{\mathrm{ES}}^{\mathcal{F}}_{r_1,r_2} : L^\infty(\mathcal{G}) \mapsto L^\infty(\mathcal{F}), \qquad X \mapsto \widetilde{\mathrm{ES}}^{\mathcal{F}}_{r_1,r_2}(X) := \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^{\mathcal{F}}(r_1,r_2)} \mathbb{E}^Q[-X|\mathcal{F}]$$

is called an extended conditional expected shortfall at level (r_1, r_2) given $\mathcal{F}$, where $\mathcal{Q}^{\mathcal{F}}(r_1, r_2) \subseteq \mathcal{P}^a$ is the set of absolutely continuous probability measures $Q \ll P$ such that for each $Q \in \mathcal{Q}^{\mathcal{F}}(r_1, r_2)$, its corresponding density $Z = \frac{dQ}{dP}$ satisfies $\frac{1}{r_1} \leq \frac{Z}{\mathbb{E}[Z|\mathcal{F}]} \leq \frac{1}{r_2}$ P-a.s., i.e.

$$\mathcal{Q}^{\mathcal{F}}(r_1, r_2) := \left\{ Q \ll P \;\middle|\; Z = \frac{dQ}{dP} \quad \text{and} \quad \frac{1}{r_1} \leq \frac{Z}{\mathbb{E}[Z|\mathcal{F}]} \leq \frac{1}{r_2} \quad P\text{-a.s.} \right\}.$$

Remark 4.2.1.7. Following from Lemma 4.2.2 and Remark 4.2.1.6, we can readily observe $I_X^{\mathcal{F}}(r_1, r_2) \in \mathcal{Q}^{\mathcal{F}}(r_1, r_2)$.

The introduction of $I_X^{\mathcal{F}}(r_1, r_2)$ is to facilitate us in determining explicitly the extended conditional expected shortfall $\widetilde{\mathrm{ES}}^{\mathcal{F}}_{r_1,r_2}$ according to

Theorem 4.2.2. *For any fixed $r_1 > 1 > r_2 > 0$, a probability measure Q^* is defined by density $\frac{dQ^*}{dP} = I_X^{\mathcal{F}}(r_1, r_2)$. Then, for a random variable $X \in L^\infty(\mathcal{G})$, we have*

$$\operatorname*{ess\,sup}_{Q \in \mathcal{Q}^{\mathcal{F}}(r_1,r_2)} \mathbb{E}^Q[-X|\mathcal{F}] = \mathbb{E}^P\big[- X I_X^{\mathcal{F}}(r_1, r_2)\big|\mathcal{F}\big] = \mathbb{E}^{Q^*}[-X|\mathcal{F}], \quad P\text{-a.s.} \tag{4.2.5}$$

Hence, $\widetilde{ES}^{\mathcal{F}}_{r_1,r_2}$ can be expressed as

$$\widetilde{ES}^{\mathcal{F}}_{r_1,r_2}(X) := \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^{\mathcal{F}}(r_1,r_2)} \mathbb{E}^Q[-X|\mathcal{F}] = \mathbb{E}^P\big[- X I_X^{\mathcal{F}}(r_1, r_2)\big|\mathcal{F}\big], \quad P\text{-a.s.}$$

Proof. By abstract Bayes's rule, see Platen and Health (2006) for instance, $\frac{I_X^{\mathcal{F}}(r_1,r_2)}{\mathbb{E}[I_X^{\mathcal{F}}(r_1,r_2)|\mathcal{F}]} = I_X^{\mathcal{F}}(r_1, r_2)$ defines a probability measure Q^*on $(\Omega, \mathcal{F})$. Hence, by Lemma 4.2.2 and Remark 4.2.1.6, we have

$$\mathbb{E}^{Q^*}[X|\mathcal{F}] = \mathbb{E}^P\left[X\frac{I_X^{\mathcal{F}}(r_1,r_2)}{\mathbb{E}[I_X^{\mathcal{F}}(r_1,r_2)|\mathcal{F}]}\middle|\mathcal{F}\right] = \mathbb{E}^P\left[XI_X^{\mathcal{F}}(r_1,r_2)\middle|\mathcal{F}\right].$$

This establishes the last equality in (4.2.5).

For the second equality in (4.2.5), we first note that, by the definition of an essential supremum, it is readily seen that $\operatorname{ess\,sup}_{Q\in\mathcal{Q}^{\mathcal{F}}(r_1,r_2)}\mathbb{E}^Q[-X|\mathcal{F}] \geq \mathbb{E}^{Q^*}[-X|\mathcal{F}]$ holds because of $Q^*\in\mathcal{Q}^{\mathcal{F}}(r_1,r_2)$.

To establish the reverse inequality, $\operatorname{ess\,sup}_{Q\in\mathcal{Q}^{\mathcal{F}}(r_1,r_2)}\mathbb{E}^Q[-X|\mathcal{F}] \leq \mathbb{E}^{Q^*}[-X|\mathcal{F}]$, suppose $Q\in\mathcal{Q}^{\mathcal{F}}(r_1,r_2)$ is chosen, we analyze on the $\mathcal{F}$-measurable set $\left\{\omega \,\middle|\, \mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]\geq 0\right\}$ which is of probability measure one.

On $\left\{\omega \,\middle|\, \mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right] > 0\right\}$,

$$\mathbb{E}^P\left[\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]} - I_X^{\mathcal{F}}(r_1,r_2)\middle|\mathcal{F}\right] = 0 \quad P\text{-a.s.} \tag{4.2.6}$$

Consequently, for each ω,

$$\begin{aligned}
&\mathbb{E}^{Q^*}[-X|\mathcal{F}] - \mathbb{E}^Q[-X|\mathcal{F}]\\
&= \mathbb{E}^Q[X|\mathcal{F}] - \mathbb{E}^{Q^*}[X|\mathcal{F}]\\
&= \mathbb{E}^P\left[X\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]}\middle|\mathcal{F}\right] - \mathbb{E}^P\left[XI_X^{\mathcal{F}}(r_1,r_2)\middle|\mathcal{F}\right]\\
&= \mathbb{E}^P\left[X\left(\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]} - I_X^{\mathcal{F}}(r_1,r_2)\right)\middle|\mathcal{F}\right]\\
&= \mathbb{E}^P\left[X\left(\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]} - I_X^{\mathcal{F}}(r_1,r_2)\right)\middle|\mathcal{F}\right] - q_X^{\mathcal{F}}(\bar{r})\cdot\mathbb{E}^P\left[\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]} - I_X^{\mathcal{F}}(r_1,r_2)\middle|\mathcal{F}\right]\\
&= \mathbb{E}^P\left[\left(X - q_X^{\mathcal{F}}(\bar{r})\right)\left(\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]} - I_X^{\mathcal{F}}(r_1,r_2)\right)\middle|\mathcal{F}\right]
\end{aligned}$$

where $\mathcal{F}$-measurability of $q_X^{\mathcal{F}}(\bar{r})$ and (4.2.6) are used. Next we further distinguish three cases of $\left\{\omega \,\middle|\, \mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right] > 0\right\}$ according to the sign of $X - q_X^{\mathcal{F}}(\bar{r})$:

<u>Case 1:</u> On $\left\{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right] > 0\right\}\cap\left\{X - q_X^{\mathcal{F}}(\bar{r}) < 0\right\}$

From (4.2.3), we have $I_X^{\mathcal{F}}(r_1,r_2) = \frac{1}{r_2}$. Since $Q\in\mathcal{Q}^{\mathcal{F}}(r_1,r_2)$, this implies $\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]} - I_X^{\mathcal{F}}(r_1,r_2)\leq 0$, hence, $\left(\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]} - I_X^{\mathcal{F}}(r_1,r_2)\right)\left(X - q_X^{\mathcal{F}}(\bar{r})\right)\geq 0$.

Case 2: On $\left\{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]>0\right\}\cap\{X-q_X^{\mathcal{F}}(\bar{r})=0\}$

It is trivial that $\left(\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]}-I_X^{\mathcal{F}}(r_1,r_2)\right)\left(X-q_X^{\mathcal{F}}(\bar{r})\right)=0$.

Case 3: On $\left\{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]>0\right\}\cap\{X-q_X^{\mathcal{F}}(\bar{r})>0\}$

From (4.2.3), we have $I_X^{\mathcal{F}}(r_1,r_2)=\frac{1}{r_1}$. Since $Q\in\mathcal{Q}^{\mathcal{F}}(r_1,r_2)$, this implies $\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]}-I_X^{\mathcal{F}}(r_1,r_2)\geq 0$, hence, $\left(\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]}-I_X^{\mathcal{F}}(r_1,r_2)\right)\left(X-q_X^{\mathcal{F}}(\bar{r})\right)\geq 0$.

Combining these results, we conclude that $\left(\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]}-I_X^{\mathcal{F}}(r_1,r_2)\right)\left(X-q_X^{\mathcal{F}}(r_2)\right)\geq 0$ on $\left\{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]>0\right\}$, hence, $\mathbb{E}^{Q^*}[-X|\mathcal{F}]-\mathbb{E}^{Q}[-X|\mathcal{F}]\geq 0$ for all $Q\in\mathcal{Q}^{\mathcal{F}}(r_1,r_2)$ on $\left\{\omega\,\middle|\,\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]>0\right\}$.

For the situation on $\left\{\omega\,\middle|\,\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]=0\right\}$, by convention,

$$\frac{\frac{dQ}{dP}}{\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]}:=1\quad\text{on}\quad\left\{\omega\,\middle|\,\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]=0\right\}.$$

Furthermore, because of $\frac{1}{r_1}\leq 1\leq\frac{1}{r_2}$, by performing the same analysis as above, we can deduce that $\mathbb{E}^{Q^*}[-X|\mathcal{F}]-\mathbb{E}^{Q}[-X|\mathcal{F}]\geq 0$ for all $Q\in\mathcal{Q}^{\mathcal{F}}(r_1,r_2)$ is also true on $\left\{\omega\,\middle|\,\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]=0\right\}$.

From the conclusions on $\left\{\omega\,\middle|\,\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]>0\right\}$ and $\left\{\omega\,\middle|\,\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}\right]=0\right\}$, we have proven that $\mathbb{E}^{Q^*}[-X|\mathcal{F}]-\mathbb{E}^{Q}[-X|\mathcal{F}]\geq 0$ for all $Q\in\mathcal{Q}^{\mathcal{F}}(r_1,r_2)$ holds P-a.s.

As a final step, since an essential supremum is the least upper bound, the inequality of $\widetilde{\mathrm{ES}}^{\mathcal{F}}_{r_1,r_2}(X)\leq\mathbb{E}^{Q^*}[-X|\mathcal{F}]$ is justified and so we have completed the proof for all the equalities in (4.2.5). □

We seek the analytical representation of $\mathbb{E}^P[-XI_X^{\mathcal{F}}(r_1,r_2)|\mathcal{F}]$ and relate it to the notion of conditional Value-at-Risk mentioned in Vogelpoth (2006).

Corollary 4.2.1. *For any fixed* $r_1>1>r_2>0$*, denote* $1>\bar{r}:=\frac{1-\frac{1}{r_1}}{\frac{1}{r_2}-\frac{1}{r_1}}>0$*,* $\mathbb{E}^P\big[-XI_X^{\mathcal{F}}(r_1,r_2)|\mathcal{F}\big]$ *can be expressed as*

$$\mathbb{E}^P\big[-XI_X^{\mathcal{F}}(r_1,r_2)|\mathcal{F}\big](\cdot)=\frac{1}{r_1}\mathbb{E}^P[-X|\mathcal{F}](\cdot)+\left(1-\frac{1}{r_1}\right)ES_X^{\mathcal{F}}(\bar{r})(\cdot)\qquad(4.2.7)$$

where $ES_X^{\mathcal{F}}(r)$ is the conditional Value-at-Risk of X given $\mathcal{F}$ at level r, see Vogelpoth (2006), represented by

$$\begin{aligned} ES_X^{\mathcal{F}}(r)(\cdot) &= \mathbb{E}^P\left[-X \cdot \frac{1}{r}\left(\mathbf{1}_{\{X<q_X^{\mathcal{F}}(r)\}} + \frac{r - P^{\mathcal{F}}(\omega, \{X < q_X^{\mathcal{F}}(r)\})}{P^{\mathcal{F}}(\omega, \{X = q_X^{\mathcal{F}}(r)\})} \mathbf{1}_{\{X=q_X^{\mathcal{F}}(r)\}} \right) \middle| \mathcal{F} \right](\cdot) \\ &= \frac{1}{r}\mathbb{E}^P\left[\big(q_X^{\mathcal{F}}(r) - X\big)^+ \middle| \mathcal{F}\right](\cdot) - q_X^{\mathcal{F}}(r)(\cdot). \end{aligned}$$

Proof. We start with the left hand side of (4.2.7). To alleviate notations, we simply denote $\kappa_X^{\mathcal{F}}$ as $\kappa_X^{\mathcal{F}}(r_1, r_2)$. By the definition of $I_X^{\mathcal{F}}(r_1, r_2)$ in (4.2.3), we have

$$\begin{aligned} &\mathbb{E}^P\left[-X I_X^{\mathcal{F}}(r_1, r_2) \middle| \mathcal{F}\right] \\ &= \mathbb{E}^P\left[-X \left(\frac{1}{r_2}\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar r)\}} + \frac{1}{r_1}\mathbf{1}_{\{X>q_X^{\mathcal{F}}(\bar r)\}} + \kappa_X^{\mathcal{F}}\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar r)\}} \right) \middle| \mathcal{F}\right] \\ &= \frac{1}{r_1}\mathbb{E}^P[-X\mathbf{1}_{\{X>q_X^{\mathcal{F}}(\bar r)\}}|\mathcal{F}] + \mathbb{E}^P\left[-X \left(\frac{1}{r_2}\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar r)\}} + \kappa_X^{\mathcal{F}}\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar r)\}} \right) \middle| \mathcal{F}\right] \\ &= \frac{1}{r_1}\mathbb{E}^P[-X|\mathcal{F}] + \mathbb{E}^P\left[-X \left(-\frac{1}{r_1}\mathbf{1}_{\{X\le q_X^{\mathcal{F}}(\bar r)\}} + \frac{1}{r_2}\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar r)\}} + \kappa_X^{\mathcal{F}}\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar r)\}} \right) \middle| \mathcal{F}\right] \\ &= \frac{1}{r_1}\mathbb{E}^P[-X|\mathcal{F}] + \mathbb{E}^P\left[-X \left(\Big(\frac{1}{r_2} - \frac{1}{r_1}\Big)\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar r)\}} + \Big(\kappa_X^{\mathcal{F}} - \frac{1}{r_1}\Big)\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar r)\}} \right) \middle| \mathcal{F}\right] \\ &= \frac{1}{r_1}\mathbb{E}^P[-X|\mathcal{F}] + \Big(1 - \frac{1}{r_1}\Big)\mathbb{E}^P\left[-X \left(\frac{\frac{1}{r_2} - \frac{1}{r_1}}{1 - \frac{1}{r_1}}\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar r)\}} + \frac{\kappa_X^{\mathcal{F}} - \frac{1}{r_1}}{1 - \frac{1}{r_1}}\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar r)\}} \right) \middle| \mathcal{F}\right]. \end{aligned}$$

Consequently it suffices to establish the following equality

$$\mathbb{E}^P\left[-X \left(\frac{\frac{1}{r_2} - \frac{1}{r_1}}{1 - \frac{1}{r_1}}\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar r)\}} + \frac{\kappa_X^{\mathcal{F}} - \frac{1}{r_1}}{1 - \frac{1}{r_1}}\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar r)\}} \right) \middle| \mathcal{F}\right] = ES_X^{\mathcal{F}}(\bar r). \tag{4.2.8}$$

Note that the left hand side of (4.2.8) can be further written as

$$\begin{aligned} &\mathbb{E}^P\left[-X \left(\frac{\frac{1}{r_2} - \frac{1}{r_1}}{1 - \frac{1}{r_1}}\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar r)\}} + \frac{\kappa_X^{\mathcal{F}} - \frac{1}{r_1}}{1 - \frac{1}{r_1}}\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar r)\}} \right) \middle| \mathcal{F}\right] \\ &= \mathbb{E}^P\left[-X \left(\frac{1}{\bar r}\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar r)\}} + \frac{\kappa_X^{\mathcal{F}} - \frac{1}{r_1}}{1 - \frac{1}{r_1}}\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar r)\}} \right) \middle| \mathcal{F}\right] \\ &= \mathbb{E}^P\left[-X \cdot \frac{1}{\bar r}\left(\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar r)\}} + \frac{\kappa_X^{\mathcal{F}} - \frac{1}{r_1}}{\frac{1}{r_2} - \frac{1}{r_1}}\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar r)\}} \right) \middle| \mathcal{F}\right]. \end{aligned}$$

Recall the definition of $\kappa_X^{\mathcal{F}}$ in (4.2.4), we have

$$\frac{\kappa_X^{\mathcal{F}} - \frac{1}{r_1}}{\frac{1}{r_2} - \frac{1}{r_1}} = \begin{cases} \bar r & \text{on} \quad \big\{\omega \mid P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar r)\}\big) = 0\big\} \\ \dfrac{\bar r - P^{\mathcal{F}}(\omega, \{X < q_X^{\mathcal{F}}(\bar r)\})}{P^{\mathcal{F}}(\omega, \{X = q_X^{\mathcal{F}}(\bar r)\})} & \text{on} \quad \big\{\omega \mid P^{\mathcal{F}}\big(\omega, \{X = q_X^{\mathcal{F}}(\bar r)\}\big) > 0\big\} \end{cases}.$$

In view of this, we can conclude that

$$\mathbb{E}^P\left[-X\left(\frac{\frac{1}{r_2}-\frac{1}{r_1}}{1-\frac{1}{r_1}}\mathbf{1}_{\{X<q_X^{\mathcal{F}}(\bar{r})\}}+\frac{\kappa_X^{\mathcal{F}}-\frac{1}{r_1}}{1-\frac{1}{r_1}}\mathbf{1}_{\{X=q_X^{\mathcal{F}}(\bar{r})\}}\right)\Big|\mathcal{F}\right]=ES_X^{\mathcal{F}}(\bar{r}).$$

and so the proof is completed. □

By Theorem 4.2.2 and Corollary 4.2.1, a more direct expression of extended conditional expected shortfall $\widetilde{ES}^{\mathcal{F}}_{r_1,r_2}$ can be acquired: for a fixed random variable X,

$$\widetilde{ES}^{\mathcal{F}}_{r_1,r_2}(X)=\frac{1}{r_1}\mathbb{E}^P[-X|\mathcal{F}]+\left(1-\frac{1}{r_1}\right)ES_X^{\mathcal{F}}(\bar{r}).$$

With this result, we are now in a position to unveil the structure of $\mathcal{Q}^{\text{ngd}}$.

4.2.2 Determination of $\mathcal{Q}^{ngd}$

The idea of resolving the structure of $\mathcal{Q}^{\text{ngd}}$ is through the general representation result

$$\phi_t(X)=\operatorname*{ess\,inf}_{Q\in\mathcal{P}^a}\mathbb{E}^Q[X+\phi^{min}(Q)|\mathcal{F}_t]:=\operatorname*{ess\,inf}_{Q\in\mathcal{Q}^{\text{ngd}}}\mathbb{E}^Q[X|\mathcal{F}_t],$$

in which $\phi^{min}(Q):=\sum_{i=1}^T\varphi_{i-1}^{min}(Q)$ is characterized explicitly. For each $Q\in\mathcal{P}^a$, we compute $\varphi_{i-1}^{min}(Q):=\varphi_{i-1}^{min}(Z_i)$ directly by making use of the associated density $Z=\frac{dQ}{dP}$. Moreover, we are going to show that $\varphi_{i-1}^{min}(Z_{t+1})$ is expressed as

$$\begin{aligned}\varphi_{i-1}^{min}(Z_i)&:=\operatorname*{ess\,sup}_{X\in L^\infty(\mathcal{F}_i)}\left\{\varphi_{i-1}(X)-\mathbb{E}^P[Z_iX|\mathcal{F}_{i-1}]\right\}\\&=\begin{cases}0 & \text{on}\quad\left\{P^{\mathcal{F}_{i-1}}\left(\omega,\left\{Z_i<\frac{1}{1+R}\right\}\right)=0\right\}\bigcap\left\{P^{\mathcal{F}_{i-1}}\left(\omega,\left\{Z_i>\frac{\alpha+R}{\alpha(1+R)}\right\}\right)=0\right\}\\ \infty & \text{on}\quad\left\{P^{\mathcal{F}_{i-1}}\left(\omega,\left\{Z_i<\frac{1}{1+R}\right\}\right)>0\right\}\bigcup\left\{P^{\mathcal{F}_{i-1}}\left(\omega,\left\{Z_i>\frac{\alpha+R}{\alpha(1+R)}\right\}\right)>0\right\}\end{cases}\end{aligned}\tag{4.2.9}$$

which then enables us to derive $\phi^{min}(Q)$.

Since $\left\{Z_i<\frac{1}{1+R}\right\}$ and $\left\{Z_i>\frac{\alpha+R}{\alpha(1+R)}\right\}$ are disjoint, so are $\left\{P^{\mathcal{F}_{i-1}}(\omega,\{Z_i<\frac{1}{1+R}\})>0\right\}$ and $\left\{P^{\mathcal{F}_{i-1}}\left(\omega,\left\{Z_i>\frac{\alpha+R}{\alpha(1+R)}\right\}\right)>0\right\}$, we can rewrite above into

$$\begin{aligned}&\varphi_{i-1}^{min}(Z_i)\\&=\begin{cases}0 & \text{on}\quad\left\{P^{\mathcal{F}_{i-1}}\left(\omega,\left\{Z_i<\frac{1}{1+R}\right\}\right)=0\right\}\bigcap\left\{P^{\mathcal{F}_{i-1}}\left(\omega,\left\{Z_i>\frac{\alpha+R}{\alpha(1+R)}\right\}\right)=0\right\}\\ \infty & \text{on}\quad\left\{P^{\mathcal{F}_{i-1}}\left(\omega,\left\{Z_i<\frac{1}{1+R}\right\}\right)>0\right\}\\ \infty & \text{on}\quad\left\{P^{\mathcal{F}_{i-1}}\left(\omega,\left\{Z_i>\frac{\alpha+R}{\alpha(1+R)}\right\}\right)>0\right\}\end{cases}.\end{aligned}$$

As a result, $\phi^{min}(Q)$ is given by

$$\phi^{min}(Q) = \begin{cases} 0 & \text{on} \quad \left\{P^{\mathcal{F}_t}\left(\omega, \left\{Z_{t+1} < \frac{1}{1+R}\right\}\right) = 0\right\} \bigcap \left\{P^{\mathcal{F}_t}\left(\omega, \left\{Z_{t+1} > \frac{\alpha+R}{\alpha(1+R)}\right\}\right) = 0\right\} \\ & \quad \text{for all } t = 0, 1, \ldots, T-1 \\ \infty & \text{on} \quad \text{otherwise} \end{cases}.$$

We conclude that

$$\phi_t(X) = \operatorname*{ess\,inf}_{Q \in \mathcal{P}^a} \mathbb{E}^Q[X + \phi^{min}(Q)|\mathcal{F}_t] \coloneqq \operatorname*{ess\,inf}_{Q \in \mathcal{Q}^{\text{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t],$$

where $\mathcal{Q}^{\text{ngd}}$ corresponds to

$$\mathcal{Q}^{\text{ngd}} \coloneqq \left\{ Q \in \mathcal{P}^a \;\middle|\; \frac{1}{1+R} \le Z_{t+1} \le \frac{\alpha+R}{\alpha(1+R)} \text{ for all } t = 0, 1, \ldots, T-1 \text{ } P\text{-a.s.} \right\}.$$

Hence, it is enough to establish the assertion of $\varphi_t^{min}(Z_{t+1})$ being given by (4.2.9). We first prepare ourselves with

Lemma 4.2.3. *Given $\mathcal{F}_{t-1} \subseteq \mathcal{F}_t$ and any $A \in \mathcal{F}_{t-1}$, we have*

$$\begin{aligned} \mathbf{1}_A \operatorname*{ess\,inf}_{\xi \in \mathcal{S}(\Omega)} \mathbb{E}[\xi X|\mathcal{F}_{t-1}] &= \operatorname*{ess\,inf}_{\xi \in \mathcal{S}(\Omega)} \mathbf{1}_A \mathbb{E}[\xi X|\mathcal{F}_{t-1}] \\ &= \operatorname*{ess\,inf}_{\xi \in \mathcal{S}(\Omega)} \mathbb{E}[\xi \mathbf{1}_A X|\mathcal{F}_{t-1}] \\ &= \operatorname*{ess\,inf}_{\xi \in \mathcal{S}(A)} \mathbb{E}[\xi \mathbf{1}_A X|\mathcal{F}_{t-1}] \end{aligned}$$

where $0 \le a < 1 < b$ and

$$\mathcal{S}(B) \coloneqq \left\{ \xi \;\middle|\; \begin{matrix} \mathbb{E}[\xi|\mathcal{F}_{t-1}]=1 & P\text{-a.s. on } B \\ a \le \xi \le b & P\text{-a.s. on } B \end{matrix} \right\}.$$

Proof. The second equality follows trivially because of $A \in \mathcal{F}_{t-1}$ while the first equality is established by using Lemma 4 in Detlefsen and Scandolo (2005). Thus the last equality remains to be justified.

Due to the fact of $\mathcal{S}(\Omega) \subseteq \mathcal{S}(A)$,

$$\operatorname*{ess\,inf}_{\xi \in \mathcal{S}(\Omega)} \mathbb{E}[\xi \mathbf{1}_A X|\mathcal{F}_{t-1}] \ge \operatorname*{ess\,inf}_{\xi \in \mathcal{S}(A)} \mathbb{E}[\xi \mathbf{1}_A X|\mathcal{F}_{t-1}]$$

holds readily.

For the reverse inequality, suppose X and $\xi \in \mathcal{S}(A)$ are given, we define

$$\xi' := \xi \mathbf{1}_A + \mathbf{1}_{A^c}.$$

It is easy to observe that $\xi' \in \mathcal{S}(\Omega)$ holds. Moreover, since $\xi \mathbf{1}_A X \geq \xi' \mathbf{1}_A X$, we have

$$\mathbb{E}[\xi \mathbf{1}_A X | \mathcal{F}_{t-1}] \geq \mathbb{E}[\xi' \mathbf{1}_A X | \mathcal{F}_{t-1}]$$

by the property of conditional expectation. This allows us to conclude that for any X and $\xi \in \mathcal{S}(A)$, there exists $\xi' \in \mathcal{S}(\Omega)$ such that

$$\mathbb{E}[\xi' \mathbf{1}_A X | \mathcal{F}_{t-1}] \leq \mathbb{E}[\xi \mathbf{1}_A X | \mathcal{F}_{t-1}],$$

which further implies that

$$\operatorname*{ess\,inf}_{\xi \in \mathcal{S}(\Omega)} \mathbb{E}[\xi \mathbf{1}_A X | \mathcal{F}_{t-1}] \leq \operatorname*{ess\,inf}_{\xi \in \mathcal{S}(A)} \mathbb{E}[\xi \mathbf{1}_A X | \mathcal{F}_{t-1}].$$

So the assertion follows. □

Proposition 4.2.1. *Let Z be a non-negative random variable,*

i. if $P(Z < \infty) = 1$, then, for any $z, r > 0$ such that $P(Z > z) > r$, there exists $z' > z$ such that $P(Z > z') \leq r$.

ii. if $P(Z \leq K) = 1$ for some constant $K > 0$, then, for any $z < K, r > 0$ such that $P(z < Z \leq K) < r$, there exists $z' < z$ such that $P(z' < Z \leq K) \geq r$.

Proof. For (i), note that $P(Z = \infty) = 0$. Since P is a probability measure, continuity of probability measure justifies

$$P(Z = +\infty) = \lim_{z' \uparrow \infty} P(Z \geq z') = 0$$

as well as

$$P(Z > z) = \lim_{z' \downarrow z} P(Z \geq z') > r.$$

We also observe that $P(Z \geq z')$ increases as z' decreases, which implies that there exists some $z' > z$ such that $0 < P(Z \geq z') \leq r$.

For (ii), note that $P(0 \leq Z \leq K) = 1$. Again by continuity of a probability measure, we have

$$\lim_{z' \downarrow 0} P(z' \leq Z \leq K) = P(0 \leq Z \leq K) = 1,$$

Obviously $P(z' \leq Z \leq K)$ increases to 1 as z' decreases to 0, which subsequently entails the existence of some $z' < z$ such that $r \leq P(z' \leq Z \leq K)$ is satisfied. □

We shall prove below that $\varphi_t^{min}(Z_{t+1})$ admits the representation in (4.2.9). Let us recall its definition

$$\varphi_t^{min}(Z_{t+1}) \coloneqq \operatorname*{ess\,sup}_{X\in L^\infty(\mathcal{F}_{t+1})} \left\{\varphi_t(X) - \mathbb{E}^P[Z_{t+1}X|\mathcal{F}_t]\right\}.$$

Theorem 4.2.3. *For given i and $Z_i \in \mathcal{D}_i$, we have*

$$\begin{aligned}\varphi_{i-1}^{min}(Z_i) &\coloneqq \operatorname*{ess\,sup}_{X\in L^\infty(\mathcal{F}_i)} \left\{\varphi_{i-1}(X) - \mathbb{E}^P[Z_iX|\mathcal{F}_{i-1}]\right\}\\ &= \begin{cases} 0 & on \quad \left\{P^{\mathcal{F}_{i-1}}\left(\omega, \left\{Z_i < \frac{1}{1+R}\right\}\right) = 0\right\} \bigcap \left\{P^{\mathcal{F}_{i-1}}\left(\omega, \left\{Z_i > \frac{\alpha+R}{\alpha(1+R)}\right\}\right) = 0\right\} \\ \infty & on \quad \left\{P^{\mathcal{F}_{i-1}}\left(\omega, \left\{Z_i < \frac{1}{1+R}\right\}\right) > 0\right\} \bigcup \left\{P^{\mathcal{F}_{i-1}}\left(\omega, \left\{Z_i > \frac{\alpha+R}{\alpha(1+R)}\right\}\right) > 0\right\} \end{cases}\end{aligned}$$

and,

$$\begin{aligned}&\phi^{min}(Q) \coloneqq \sum_{i=1}^{T} \varphi_{i-1}^{min}(Z_i) < \infty\\ \iff\quad &\frac{1}{1+R} \leq Z_{t+1} \leq \frac{\alpha+R}{\alpha(1+R)} \quad \text{for all } t = 0,1,\ldots,T-1 \; P\text{-a.s.}\end{aligned}$$

Hence,

$$\phi_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{P}^a} \mathbb{E}^Q[X + \phi^{min}(Q)|\mathcal{F}_t] \coloneqq \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^{ngd}} \mathbb{E}^Q[X|\mathcal{F}_t]$$

where

$$\mathcal{Q}^{ngd} \coloneqq \left\{Q \in \mathcal{P}^a \;\middle|\; \frac{1}{1+R} \leq Z_{t+1} \leq \frac{\alpha+R}{\alpha(1+R)} \quad \text{for all } t = 0,1,\ldots,T-1 \; P\text{-a.s.}\right\}.$$

Proof. We abbreviate $1+R$ and $\frac{\alpha(1+R)}{\alpha+R}$ as r_1 and r_2 respectively. Then we should note that, according to the definition of $P^{\mathcal{F}}(\cdot, B)$ where B is an event (i.e $\in [0,1]\ \forall\omega \in \Omega$), both $\left\{\omega \mid P^{\mathcal{F}}\big(\omega, \{Z_{t+1} < \frac{1}{r_1}\}\big) \geq 0\right\}$ and $\left\{\omega \mid P^{\mathcal{F}}\big(\omega, \{Z_{t+1} > \frac{1}{r_2}\}\big) \geq 0\right\}$ have probability measure one. Along with their $\mathcal{F}$-measurability, we shall analyze the results on these two sets in order to arrive at the desired conclusions.

Begin with $\left\{P^{\mathcal{F}}\big(\omega, \{Z_{t+1} < \frac{1}{1+R}\}\big) = 0\right\} \bigcap \left\{P^{\mathcal{F}}\big(\omega, \{Z_{t+1} > \frac{\alpha+R}{\alpha(1+R)}\}\big) = 0\right\}$. On one hand, since $0 \in L^\infty(\mathcal{F}_{t+1})$, so, substituting $X = 0$ gives $\varphi_t(0) - \mathbb{E}^P[Z \cdot 0|\mathcal{F}] = 0$. Then, by definition of an essential supremum, we obtain

$$\varphi_t^{min}(Z_{t+1}) \geq 0.$$

On the other hand, due to Lemma 4.2.3 and definition of $\varphi_t(X)$,

$$\varphi_t(X) = \operatorname*{ess\,inf}_{Z_{t+1}\in\mathcal{D}_{t+1}\cap\mathcal{Q}^{\mathrm{ngd}}_{t+1}} \mathbb{E}^P[Z_{t+1}X|\mathcal{F}_t],$$

we observe that, for a given $X \in L^\infty$,

$$\varphi_t(X) - \mathbb{E}^P[Z_{t+1}X|\mathcal{F}] \leq 0 \qquad \text{for all} \quad Z_{t+1} \in \mathcal{D}_{t+1}.$$

Indeed this is because, by Lemma 4.2.3, for all $Z_{t+1} \in \mathcal{D}_{t+1}$,

$$\varphi_t(X) = \operatorname*{ess\,inf}_{Z_{t+1}\in\mathcal{D}_{t+1}\cap\mathcal{Q}^{\mathrm{ngd}}_{t+1}} \mathbb{E}^P[Z_{t+1}X|\mathcal{F}_t] = \operatorname*{ess\,inf}_{Z_{t+1}\in\mathcal{D}_{t+1}} \mathbb{E}^P[Z_{t+1}X|\mathcal{F}_t] \leq \mathbb{E}^P[Z_{t+1}X|\mathcal{F}_t]$$

which leads us to conclude

$$\varphi_t^{min}(Z_{t+1}) \leq 0$$

due to the least upper bound property of an essential supremum. So, we deduce that $\varphi_t^{min}(Z_{t+1}) = 0$ on $\Big\{P^{\mathcal{F}}(\omega, \{Z < \frac{1}{r_1}\}) = 0\Big\} \bigcap \Big\{P^{\mathcal{F}}(\omega, \{Z > \frac{1}{r_2}\}) = 0\Big\}$.

The remaining set $\Big\{P^{\mathcal{F}}(\omega, \{Z < \frac{1}{r_1}\}) > 0\Big\} \bigcup \Big\{P^{\mathcal{F}}(\omega, \{Z > \frac{1}{r_2}\}) > 0\Big\}$ is analyzed similarly. According to Lemma 4.2.3, this union of events can be treated individually.

For a fixed $\omega \in \Big\{P^{\mathcal{F}}(\omega, \{Z > \frac{1}{r_2}\}) > 0\Big\}$, if, furthermore, $P^{\mathcal{F}}\big(\omega, \big\{Z > \frac{1}{r_2}\big\}\big) \leq \bar{r}$ holds, take $\frac{1}{r_2'} = \frac{1}{r_2}$ and so $P^{\mathcal{F}}\big(\omega, \big\{Z > \frac{1}{r_2'}\big\}\big) \leq \bar{r}$ is valid. If, otherwise, $P^{\mathcal{F}}\big(\omega, \big\{Z > \frac{1}{r_2}\big\}\big) > \bar{r}$ holds, according to Proposition 4.2.1, we conclude that, at any fixed $\omega \in \Big\{P^{\mathcal{F}}(\omega, \{Z > \frac{1}{r_2}\}) > 0\Big\}$, there exists some $\frac{1}{r_2'} \geq \frac{1}{r_2}$ such that $0 < P^{\mathcal{F}}(\omega, \{Z \geq \frac{1}{r_2'}\}) \leq \bar{r}$ is satisfied. In addition to $\frac{1}{r_2'}$, we take some arbitrary $c_2, k_2 > 0$ and construct the following random variable X^{2,c_2}

$$X^{2,c_2} = -c_2(Z \wedge k_2)\mathbf{1}_{\left\{Z \geq \frac{1}{r_2'}\right\}} \in L^\infty.$$

On one hand, owing to $Z \geq 0$, we have $\{X^{2,c_2} < 0\} = \{Z \geq \frac{1}{r_2'}\}$ and this results in

$$P^{\mathcal{F}}(\omega, \{X^{2,c_2} < 0\}) = P^{\mathcal{F}}\left(\omega, \left\{Z \geq \frac{1}{r_2'}\right\}\right) \leq \bar{r}.$$

On the other hand, the conditional $\bar{r}$-quantile $q^{\mathcal{F}}_{X^{2,c_2}}(\bar{r})$ of X^{2,c_2} is given by $q^{\mathcal{F}}_{X^{2,c_2}}(\bar{r}) = 0$.

This can be verified from the observations

$$P^{\mathcal{F}}(\omega, \{X^{2,c_2} < 0\}) = P^{\mathcal{F}}\left(\omega, \left\{Z \geq \frac{1}{r_2'}\right\}\right) \leq \bar{r} \quad \text{and}$$
$$P^{\mathcal{F}}(\omega, \{X^{2,c_2} \leq 0\}) = 1,$$

thus,

$$P^{\mathcal{F}}(\omega, \{X^{2,c_2} < 0\}) \leq \bar{r} \leq P^{\mathcal{F}}(\omega, \{X^{2,c_2} \leq 0\}) \quad \Longrightarrow \quad q^{\mathcal{F}}_{X^{2,c_2}}(\bar{r}) = 0.$$

As a result, by Corollary 4.2.1, we can determine

$$\begin{aligned}
&\widetilde{\mathrm{ES}}^{\mathcal{F}}_{r_1,r_2}(X^{2,c_2}) \\
&= \frac{1}{r_1}\mathbb{E}^P[-X^{2,c_2}|\mathcal{F}] + \left(1 - \frac{1}{r_1}\right)\left(\frac{1}{\bar{r}}\mathbb{E}^P\left[(q^{\mathcal{F}}_{X^{2,c_2}}(\bar{r}) - X^{2,c_2})^+\middle|\mathcal{F}\right] - q^{\mathcal{F}}_{X^{2,c_2}}(\bar{r})\right) \\
&= \frac{1}{r_1}\mathbb{E}^P\left[c_2(Z \wedge k_2)\mathbf{1}_{\left\{Z \geq \frac{1}{r_2'}\right\}}\middle|\mathcal{F}\right] + \left(1 - \frac{1}{r_1}\right)\left(\frac{1}{\bar{r}}\mathbb{E}^P\left[c_2(Z \wedge k_2)\mathbf{1}_{\left\{Z \geq \frac{1}{r_2'}\right\}}\middle|\mathcal{F}\right]\right) \\
&= \frac{1}{r_2}\mathbb{E}^P\left[c_2(Z \wedge k_2)\mathbf{1}_{\left\{Z \geq \frac{1}{r_2'}\right\}}\middle|\mathcal{F}\right]
\end{aligned}$$

and so, evaluated at X^{2,c_2}, we have

$$\begin{aligned}
&\varphi_t(X^{2,c_2}) - \mathbb{E}^Q[X^{2,c_2}|\mathcal{F}] \\
&= \operatorname*{ess\,inf}_{Z_{t+1} \in \mathcal{D}_{t+1} \cap \mathcal{Q}^{\mathrm{ngd}}_{t+1}} \mathbb{E}^P[ZX^{2,c_2}|\mathcal{F}] - \mathbb{E}^Q[X^{2,c_2}|\mathcal{F}] \\
&= -\operatorname*{ess\,sup}_{Z_{t+1} \in \mathcal{D}_{t+1} \cap \mathcal{Q}^{\mathrm{ngd}}_{t+1}} \mathbb{E}^P[-ZX^{2,c_2}|\mathcal{F}] - \mathbb{E}^Q[X^{2,c_2}|\mathcal{F}] \\
&= -\widetilde{\mathrm{ES}}^{\mathcal{F}}_{r_1,r_2}(X^{2,c_2}) - \mathbb{E}^Q[X^{2,c_2}|\mathcal{F}] \\
&= -\frac{1}{r_2}\mathbb{E}^P\left[c_2(Z \wedge k_2)\mathbf{1}_{\left\{Z \geq \frac{1}{r_2'}\right\}}\middle|\mathcal{F}\right] + \mathbb{E}^P\left[Z \cdot c_2(Z \wedge k_2)\mathbf{1}_{\left\{Z \geq \frac{1}{r_2'}\right\}}\middle|\mathcal{F}\right] \\
&\geq \mathbb{E}^P\left[c_2(Z \wedge k_2)\mathbf{1}_{\left\{Z \geq \frac{1}{r_2'}\right\}}\middle|\mathcal{F}\right]\left(\frac{1}{r_2'} - \frac{1}{r_2}\right) \\
&> 0
\end{aligned}$$

under the condition of $\frac{1}{r_2'} \geq \frac{1}{r_2}$. Finally, since c_2 is arbitrary, we reach the conclusion

$$\begin{aligned}
\varphi^{min}_t(Z_{t+1}) &:= \operatorname*{ess\,sup}_{X \in L^\infty(\mathcal{F}_{t+1})} \left\{\varphi_t(X) - \mathbb{E}^P[Z_{t+1}X|\mathcal{F}_t]\right\} \\
&\geq \operatorname*{ess\,sup}_{c_2} \left\{\varphi_t(X^{2,c_2}) - \mathbb{E}^P[Z_{t+1}X^{2,c_2}|\mathcal{F}_t]\right\} \\
&= +\infty
\end{aligned}$$

on $\Big\{P^{\mathcal{F}}(\omega, \{Z > \frac{1}{r_2}\}) > 0\Big\}$.

Subsequently, in light of the results just shown, there is no loss of generality even if we assume $\Big\{P^{\mathcal{F}}(\omega, \{Z \leq \frac{1}{r_2}\}) = 1\Big\}$ on $\Big\{P^{\mathcal{F}}(\omega, \{Z < \frac{1}{r_1}\}) > 0\Big\}$. We prove $\varphi_t^{min}(Z_{t+1}) = +\infty$ in a similar vein as before. For a fixed $\omega \in \Big\{P^{\mathcal{F}}(\omega, \{Z < \frac{1}{r_1}\}) > 0\Big\}$, this means $P^{\mathcal{F}}(\omega, \{\frac{1}{r_1} \leq Z \leq \frac{1}{r_2}\}) < 1$. For a fixed $\bar{r} < 1$, if $P^{\mathcal{F}}(\omega, \{\frac{1}{r_1} \leq Z \leq \frac{1}{r_2}\}) \geq \bar{r}$ is true, we shall take $\frac{1}{r_1'} = \frac{1}{r_1}$ and $P^{\mathcal{F}}(\omega, \{\frac{1}{r_1'} \leq Z \leq \frac{1}{r_2}\}) \geq \bar{r}$ is trivial. If $P^{\mathcal{F}}(\omega, \{\frac{1}{r_1} \leq Z \leq \frac{1}{r_2}\}) < \bar{r}$ holds, by Proposition 4.2.1, there exists some $\frac{1}{r_1'} < \frac{1}{r_1}$ such that $\bar{r} \leq P^{\mathcal{F}}(\omega, \{\frac{1}{r_1'} \leq Z \leq \frac{1}{r_2}\}) < 1$ is satisfied. We analogously define the following random variable

$$X^{1,c_1,c_2} = c_1 \mathbf{1}_{\{Z < \frac{1}{r_1'}\}} + c_2 \mathbf{1}_{\{\frac{1}{r_1'} \leq Z \leq \frac{1}{r_2}\}}$$

for some arbitrary c_1 and c_2 satisfying $c_1 > c_2 > 0$. We then proceed to compute the quantity

$$\begin{aligned}\widetilde{\mathrm{ES}}^{\mathcal{F}}_{r_1,r_2}(X^{1,c_1,c_2}) = \frac{1}{r_1}\mathbb{E}^P[-X^{1,c_1,c_2}|\mathcal{F}] + \left(1 - \frac{1}{r_1}\right)\Bigg(&\frac{1}{\bar{r}}\mathbb{E}^P\big[(q^{\mathcal{F}}_{X^{1,c_1,c_2}}(\bar{r}) - X^{1,c_1,c_2})^+\big|\mathcal{F}\big] \\ &- q^{\mathcal{F}}_{X^{1,c_1,c_2}}(\bar{r})\Bigg).\end{aligned}$$

Noting the facts $\{X^{1,c_1,c_2} < c_2\} = \emptyset$ and $\{X^{1,c_1,c_2} \leq c_2\} = \{\frac{1}{r_1'} \leq Z \leq \frac{1}{r_2}\}$, we can determine $q^{\mathcal{F}}_{X^{1,c_1,c_2}}(\bar{r}) = c_2$ because of

$$\begin{aligned}&P^{\mathcal{F}}(\omega, \{X^{1,c_1,c_2} < c_2\}) = 0 \quad \text{and} \\ &P^{\mathcal{F}}(\omega, \{X^{1,c_1,c_2} \leq c_2\}) = P^{\mathcal{F}}\left(\omega, \left\{\frac{1}{r_1'} \leq Z \leq \frac{1}{r_2}\right\}\right) \geq \bar{r},\end{aligned}$$

by the existence of $\frac{1}{r_1'}$, this implies

$$P^{\mathcal{F}}(\omega, \{X^{1,c_1,c_2} < c_2\}) \leq \bar{r} \leq P^{\mathcal{F}}(\omega, \{X^{1,c_1,c_2} \leq c_2\})$$

and so justifying $q^{\mathcal{F}}_{X^{1,c_1,c_2}}(\bar{r}) = c_2$. Substituting we obtain

$$\begin{aligned}\widetilde{\mathrm{ES}}^{\mathcal{F}}_{r_1,r_2}(X^{1,c_1,c_2}) &= \frac{1}{r_1}\mathbb{E}^P\left[-c_1\mathbf{1}_{\{Z<\frac{1}{r_1'}\}} - c_2\mathbf{1}_{\{\frac{1}{r_1'}\leq Z\leq\frac{1}{r_2}\}}\middle|\mathcal{F}\right] \\ &\quad + \left(1 - \frac{1}{r_1}\right)\left(\frac{1}{\bar{r}}\mathbb{E}^P\left[\left(c_2 - c_1\mathbf{1}_{\{Z<\frac{1}{r_1'}\}} - c_2\mathbf{1}_{\{\frac{1}{r_1'}\leq Z\leq\frac{1}{r_2}\}}\right)^+\middle|\mathcal{F}\right] - c_2\right) \\ &= -\frac{c_1}{r_1}\mathbb{E}^P\left[\mathbf{1}_{\{Z<\frac{1}{r_1'}\}}\middle|\mathcal{F}\right] - \frac{c_2}{r_1}\mathbb{E}^P\left[\mathbf{1}_{\{\frac{1}{r_1'}\leq Z\leq\frac{1}{r_2}\}}\middle|\mathcal{F}\right] - c_2\left(1 - \frac{1}{r_1}\right).\end{aligned}$$

As a consequence, we arrive at

$$
\begin{aligned}
&\varphi_t(X^{1,c_1,c_2}) - \mathbb{E}^Q[X^{1,c_1,c_2}|\mathcal{F}] \\
=& \operatorname*{ess\,inf}_{Z_{t+1}\in\mathcal{D}_{t+1}\cap\mathcal{Q}^{\text{ngd}}_{t+1}} \mathbb{E}^P[ZX^{1,c_1,c_2}|\mathcal{F}] - \mathbb{E}^Q[X^{1,c_1,c_2}|\mathcal{F}] \\
=& -\operatorname*{ess\,sup}_{Z_{t+1}\in\mathcal{D}_{t+1}\cap\mathcal{Q}^{\text{ngd}}_{t+1}} \mathbb{E}^P[-X^{1,c_1,c_2}Z|\mathcal{F}] - \mathbb{E}^Q[X^{1,c_1,c_2}|\mathcal{F}] \\
=& -\widetilde{\mathrm{ES}}^{\mathcal{F}}_{r_1,r_2}(X^{1,c_1,c_2}) - \mathbb{E}^Q[X^{1,c_1,c_2}|\mathcal{F}] \\
=& \frac{c_1}{r_1}\mathbb{E}^P\left[\mathbf{1}_{\{Z<\frac{1}{r_1'}\}}\middle|\mathcal{F}\right] + \frac{c_2}{r_1}\mathbb{E}^P\left[\mathbf{1}_{\{\frac{1}{r_1'}\le Z\le\frac{1}{r_2}\}}\middle|\mathcal{F}\right] + c_2\left(1-\frac{1}{r_1}\right) - c_1\mathbb{E}^P\left[Z\mathbf{1}_{\{Z<\frac{1}{r_1'}\}}\middle|\mathcal{F}\right] \\
&- c_2\mathbb{E}^P\left[Z\mathbf{1}_{\{\frac{1}{r_1'}\le Z\le\frac{1}{r_2}\}}\middle|\mathcal{F}\right] \\
=& c_1\mathbb{E}^P\left[\left(\frac{1}{r_1}-Z\right)\mathbf{1}_{\{Z<\frac{1}{r_1'}\}}\middle|\mathcal{F}\right] + c_2\mathbb{E}^P\left[\left(\frac{1}{r_1}-Z\right)\mathbf{1}_{\{\frac{1}{r_1'}\le Z\le\frac{1}{r_2}\}}\middle|\mathcal{F}\right] + c_2\left(1-\frac{1}{r_1}\right) \\
=& (c_1-c_2)\mathbb{E}^P\left[\left(\frac{1}{r_1}-Z\right)\mathbf{1}_{\{Z<\frac{1}{r_1'}\}}\middle|\mathcal{F}\right] + c_2\mathbb{E}^P\left[\left(\frac{1}{r_1}-Z\right)\mathbf{1}_{\{Z<\frac{1}{r_1'}\}}\middle|\mathcal{F}\right] \\
&+ c_2\mathbb{E}^P\left[\left(\frac{1}{r_1}-Z\right)\mathbf{1}_{\{\frac{1}{r_1'}\le Z\le\frac{1}{r_2}\}}\middle|\mathcal{F}\right] + c_2\left(1-\frac{1}{r_1}\right) \\
=& (c_1-c_2)\mathbb{E}^P\left[\left(\frac{1}{r_1}-Z\right)\mathbf{1}_{\{Z<\frac{1}{r_1'}\}}\middle|\mathcal{F}\right] + c_2\mathbb{E}^P\left[\left(\frac{1}{r_1}-Z\right)\mathbf{1}_{\{0\le Z\le\frac{1}{r_2}\}}\middle|\mathcal{F}\right] + c_2\left(1-\frac{1}{r_1}\right) \\
=& (c_1-c_2)\mathbb{E}^P\left[\left(\frac{1}{r_1}-Z\right)\mathbf{1}_{\{Z<\frac{1}{r_1'}\}}\middle|\mathcal{F}\right] + c_2\left(\frac{1}{r_1}-1\right) + c_2\left(1-\frac{1}{r_1}\right) \\
=& (c_1-c_2)\mathbb{E}^P\left[\left(\frac{1}{r_1}-Z\right)\mathbf{1}_{\{Z<\frac{1}{r_1'}\}}\middle|\mathcal{F}\right] > 0
\end{aligned}
$$

for fixed c_2 and sufficiently large $c_1 > c_2$, and the second last equality is obtained because of the assumption of $\left\{P^{\mathcal{F}}(\omega, \{Z \le \frac{1}{r_2}\}) = 1\right\}$ and $\mathbb{E}^P[Z|\mathcal{F}] = 1$ from $Z \in \mathcal{D}$. Eventually, we arrive at the assertion of

$$
\begin{aligned}
\varphi_t^{min}(Z_{t+1}) &:= \operatorname*{ess\,sup}_{X\in L^\infty(\mathcal{F}_{t+1})} \left\{\varphi_t(X) - \mathbb{E}^P[Z_{t+1}X|\mathcal{F}_t]\right\} \\
&\ge \operatorname*{ess\,sup}_{(c_1,c_2)} \left\{\varphi_t(X^{1,c_1,c_2}) - \mathbb{E}^Q[X^{1,c_1,c_2}|\mathcal{F}]\right\} \\
&= +\infty.
\end{aligned}
$$

This justifies $\operatorname{ess\,sup}\varphi_t(X) - \mathbb{E}^Q[X|\mathcal{F}] = +\infty$ on $\omega \in \left\{P^{\mathcal{F}}(\omega, \{Z < \frac{1}{r_1}\}) > 0\right\}$. □

4.3 Conclusions

In this chapter, I have made two contributions. Firstly, in Vogelpoth (2006) and Cherdito and Kupper (2011), they considered the dynamic expected shortfall, which is a dynamic coherent risk measure with the property of time-consistency. Expected shortfall is well-known to most readers. I introduced the notion of extended conditional expected shortfall which is a more general version of expected shortfall and derived the corresponding results under the framework of Vogelpoth (2006) and Cherdito and Kupper(2011). Finally I established the dynamic extended conditional expected shortfall possessing the time-consistency feature. More precisely, I have handled the situation of $\frac{1}{r_1} > 0$ while the results of Vogelpoth (2006) and Cherdito and Kupper(2011) are corresponding to the special case of $\frac{1}{r_1} = 0$. My results are more general than theirs, which can be reproduced by letting $\frac{1}{r_1} \to 0$. Furthermore, from my results, we can know how the risks calculated under the extended conditional expected shortfall and the usual conditional expected shortfall are different. In essence, the presence of $\frac{1}{r_1} > 0$ has an impact on the risk calculation according to

$$\widetilde{ES}^{\mathcal{F}}_{r_1,r_2}(X) = \frac{1}{r_1}\mathbb{E}^P[-X|\mathcal{F}] + \left(1 - \frac{1}{r_1}\right) ES^{\mathcal{F}}_X(\bar{r})$$

where $\widetilde{ES}^{\mathcal{F}}_{r_1,r_2}(X)$ is the extended conditional expected shortfall and $ES^{\mathcal{F}}_X(\bar{r})$ is the usual conditional expected shortfall. In the extended conditional expected shortfall, it is the weighted average of the expected loss $\mathbb{E}^P[-X|\mathcal{F}]$ and $ES^{\mathcal{F}}_X(\bar{r})$ the usual conditional expected shortfall by using the weight $\frac{1}{r_1}$.

Secondly, in order to establish dynamic valuation bounds π^u_t and π^l_t for pricing a contingent claim X, we develop π^u_t and π^l_t in such a way that they behave as time-consistent dynamic coherent risk measures. In particular, they are constructed as the essential supremum and essential infimum of conditional expectations with respect to a collection of absolutely continuous probability measures $\mathcal{Q}^{\mathrm{ngd}}$, namely,

$$\pi^u_t(X;\mathcal{Q}^{\mathrm{ngd}}) := \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t] \quad \text{and}$$
$$\pi^l_t(X;\mathcal{Q}^{\mathrm{ngd}}) := \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t].$$

In view of these, $\mathcal{Q}^{\mathrm{ngd}}$ is a critical constituent and should be chosen appropriately so that the resultant dynamic valuation bounds π^u_t and π^l_t can inherit the dynamic characteristics of a time-consistent dynamic coherent risk measure. A sufficient condition for $\mathcal{Q}^{\mathrm{ngd}}$ to achieve this goal is the property of m-stability in $\mathcal{Q}^{\mathrm{ngd}}$. However, it is in general a challenging task to search for a family of probability measures with such a special feature. So, instead of moving along this direction, we attempt to meet our ambition through a 'bottom-up' approach. More precisely, we construct a time-consistent dynamic risk measure ϕ_t by making use of generators φ_t which describe the one-step (between time t and time $t+1$) behavior of ϕ_t and a one-step version $\mathcal{Q}^{\mathrm{ngd}}_t$. Then we infer the family $\mathcal{Q}^{\text{ngd}}$ as a final step. Specifically, if $\mathcal{Q}^{\text{ngd}}$ is defined by

$$\mathcal{Q}^{\text{ngd}} := \left\{ Q \in \mathcal{P}^a \;\middle|\; \frac{1}{1+R} \leq Z_{t+1} \leq \frac{\alpha + R}{\alpha(1+R)} \text{ for all } t = 0, 1, \ldots, T-1 \; P\text{-a.s.} \right\},$$

then $\pi_t^u(X; \mathcal{Q}^{\text{ngd}})$ and $\pi_t^l(X; \mathcal{Q}^{\text{ngd}})$ possess the properties of a time-consistent dynamic coherent risk measure and can be used for pricing purposes. We regard the respective prices as computed by $\pi_t^u(X; \mathcal{Q}^{\text{ngd}})$ and $\pi_t^l(X; \mathcal{Q}^{\text{ngd}})$ the RAROC-based (ask- and bid-) price of a contingent claim X, which are the primary objects to be studied in the next chapter. Moreover, these prices are 'optimal' due to their definitions. Take $\pi_t^u(X; \mathcal{Q}^{\text{ngd}})$ as an example, it is the *minimum* price that a seller charges in order to have a hedging portfolio such that the RAROC of the hedged position can meet a target value of the RAROC.

5. DYNAMIC RAROC-BASED GOOD-DEAL PRICING AND HEDGING

In the previous chapter, we introduced a pair of dynamic valuation bounds, $\pi_t^u(X;\mathcal{Q}^{\mathrm{ngd}})$ and $\pi_t^l(X;\mathcal{Q}^{\mathrm{ngd}})$, for a contingent claim X. Their construction is carried out in such a way that they behave as time-consistent dynamic coherent risk measures and so posses certain nice dynamic properties, see Proposition 2.0.6 in Becherer (2009) for enumeration of these properties, among which the 'time-consistency' feature deserves particular emphasis here. It refers to the following

for any stopping time $s \le t$, if $\pi_t(X^1) \ge \pi_t(X^2)$ holds, this implies $\pi_s(X^1) \ge \pi_s(X^2)$.

To see how this characteristic aligns with intuition, we interpret $\pi_t(X)$ as the risk of X measured at time t. Imagine that we make an investment decision based on $\pi_t(X)$ in the following way: under some fixed level of 'risk tolerance' x, if $\pi_t(X) \le x$ holds, we would trade X at time t. And now if we are at an earlier time $s < t$, according to the property of time-consistency in π, we would have the implication of $\pi_s(X) \le x$, hence, we are also encouraged to trade X at an earlier time s. In view of this, it is now readily seen that the property of time-consistency prevents us from 'regretting in the future any decision made at earlier times' or arriving at inconsistent decisions as time goes by. According to Wang (1999), if the objective function in our optimization problem is not time-consistent and we have chosen a strategy in earlier stages, subject to future scenarios, we may abandon the chosen one in favor of another one, leading to inconsistencies in choices over time. Consequently, one may attempt to produce prices for the contingent claim X in an incomplete market by using the mentioned dynamic valuation bounds π_t^u and π_t^l. In doing so, one is assured that 'time-inconsistency' problems never arise because π_t^u and π_t^l are constructed to be time-consistent. Moreover, since both π_t^u and π_t^l involve the collection of probability measures $\mathcal{Q}^{\mathrm{ngd}}$ as a constituent, and $\mathcal{Q}^{\mathrm{ngd}}$ is defined in terms of the performance measure $RAROC$, they are known as $RAROC$-based no-good-deal prices ($RAROC$-based NGD prices for short), or more precisely, the upper dynamic valuation bound π_t^u is the $RAROC$-based NGD ask price and the lower dynamic valuation bound π_t^l is the $RAROC$-based NGD bid price.

We wish to highlight again the importance of time-consistency in valuation of a contingent claim X in an incomplete market. Under the environment of an incomplete market, not all contingent claims can be replicated and so the presence of residual risk remains. However, along with the residual risk, the contingent claim X may offer an attractive profit in the meantime. The decision to trade the claim X or not should be backed by taking both profit and risk simultaneously into account. Specifically the return-to-risk ratio can be used to compare profit and risk in a fair way under this circumstance and a potential candidate of return-to-risk ratio is the performance measure $RAROC$. Suppose $RAROC$ is chosen for assessing a contingent claim, if the $RAROC$ of a contingent claim X shows an extraordinarily high value, this signals that a relatively large profit can be generated per unit residual risk that one exposed to. Investors may regard this as a 'good-deal', thus being tempted to undertake the trade. Of course a trade that is identified as a 'good-deal' today does not necessarily remain a 'good-deal' in the future. Rational investors would normally retain their position in the contingent claim X as time goes by only when a 'good-deal' is still a 'good-deal' on another day. So this motivates one to encapsulate the ingredient of 'good deal as viewed today remains good deal as viewed on another day' in the generation of prices. Formally such sound statements can be captured by mathematical means through the time-consistency property of a dynamic coherent risk measure. As a consequence it is reasonable to define prices of a contingent claim X in an incomplete market by using the pair of dynamic valuation bounds $\pi_t^u(X; \mathcal{Q}^{\text{ngd}})$ and $\pi_t^l(X; \mathcal{Q}^{\text{ngd}})$.

For the sake of obtaining more constructive results regarding the dynamic valuation bounds π_t^u and π_t^l, we shall situate ourselves in a specific model, in which explicit computations of π_t^u and π_t^l can be illustrated. For the choice of model, we recall that the constructions of π_t^u and π_t^l are made under the assumption of a finite time horizon, hence, we are confined to those discrete-time models. Furthermore, in order to follow the approach by Becherer (2009), particularly on the characterization of π_t^u and π_t^l as solutions of backward stochastic differential equations (BSDEs), this indicates us that a discrete-time model supporting the analysis of BSDEs should be selected. A crucial element in the study of the theory of BSDEs is definitely the existence of solutions and most of the time such existence is provided by a specific feature of a probability space, that is, the predictable representation property (also known as the martingale representation property[1]). With this property, the existence of solutions of BSDEs can be shown constructively. More elaboration on the construction will be given in later sections. By and large, we will select a discrete-time model that has the predictable representation property, and so π_t^u and π_t^l can be characterized as solutions of discrete-time BSDEs (BSΔEs, also named as backward stochastic difference equations).

[1]A local martingale M has the predictable representation if, for any $\mathcal{F}^M$-local martingale X, there is a predictable process H such that $X_t = X_0 + \int_0^t H_s dM_s$, where $\mathbb{F}^M := \{\mathcal{F}_t^M\}$ is the natural filtration of M. See Klebaner (2005) for instance.

5.1 Background

RAROC-based good-deal pricing is particularly interesting in an incomplete market. As a result, we shall study in detail the *RAROC*-based NGD ask price π_t^u in a relatively simple and tractable incomplete market model. As we also require the presence of predictable representation property embedded in the model, a potential model for the financial market would be a discrete-time 2-dimensional Bernoulli market with two tradeable asset $S, \overline{S}$. More precisely, we suppose a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_{t_n}\}, P)$ with a finite time set $\mathcal{T} = \{t_0, t_1, \ldots, t_N\}$ satisfying the usual conditions. Randomness here is driven by a 2-dimensional random walk process $R \in \mathbb{R}^2$ defined by

$$R_{t_n} = \begin{pmatrix} R^1_{t_n} \\ R^2_{t_n} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n \Delta R^1_{t_n} \\ \sum_{i=1}^n \Delta R^2_{t_n} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n \sqrt{\Delta t_n} B^1_{t_n} \\ \sum_{i=1}^n \sqrt{\Delta t_n} B^2_{t_n} \end{pmatrix},$$

$$\Delta R_{t_n} = \begin{pmatrix} \Delta R^1_{t_n} \\ \Delta R^2_{t_n} \end{pmatrix} = \begin{pmatrix} \sqrt{\Delta t_n} B^1_{t_n} \\ \sqrt{\Delta t_n} B^2_{t_n} \end{pmatrix}$$

where $B^1_{t_n}$ and $B^2_{t_n}$ are i.i.d. Bernoulli random variables such that $P(B^1_{t_n} = \pm 1) = P(B^2_{t_n} = \pm 1) = \frac{1}{2}$, $\forall t_n \in \mathcal{T}$. As a result, without loss of generality, we assume that $\Omega = \{\omega_1, \omega_2, \ldots, \omega_{4^n}\}$. The filtration $\mathcal{F}_{t_n}$ is assumed to be the natural filtration of R, i.e. the filtration generated by $R \in \mathbb{R}^2$. The price processes of the underlying assets $S, \overline{S}$ available in the market are assumed to be governed by

$$\begin{aligned} S_{t_n} - S_{t_{n-1}} &= S_{t_{n-1}} \left(\mu_{t_{n-1}} \Delta t_n + \sigma^1_{t_{n-1}} \Delta R^1_{t_n} + \sigma^2_{t_{n-1}} \Delta R^2_{t_n} \right) \\ &= S_{t_{n-1}} \left(\mu_{t_{n-1}} \Delta t_n + \sigma_{t_{n-1}} \Delta R_{t_n} \right) \end{aligned}$$

$$\begin{aligned} \overline{S}_{t_n} - \overline{S}_{t_{n-1}} &= \overline{S}_{t_{n-1}} \left(\overline{\mu}_{t_{n-1}} \Delta t_n + \overline{\sigma}^1_{t_{n-1}} \Delta R^1_{t_n} + \overline{\sigma}^2_{t_{n-1}} \Delta R^2_{t_n} \right) \\ &= \overline{S}_{t_{n-1}} \left(\overline{\mu}_{t_{n-1}} \Delta t_n + \overline{\sigma}_{t_{n-1}} \Delta R_{t_n} \right) \end{aligned}$$

where $\sigma_{t_n} = \begin{pmatrix} \sigma^1_{t_n} & \sigma^2_{t_n} \end{pmatrix}$ and $\overline{\sigma}_{t_n} = \begin{pmatrix} \overline{\sigma}^1_{t_n} & \overline{\sigma}^2_{t_n} \end{pmatrix}$. However, in order to generate incompleteness in this market model, we further assume that one can only trade $\overline{S}$ and S is not tradeable. For any payoffs contingent on S, the market is incomplete and so one cannot completely hedge the corresponding payoffs.

As mentioned in the introduction, the predictable representation property plays a critical role if we wish to establish connections to BSΔE for *RAROC*-based NGD prices π_t^u and π_t^l. The prescribed model does provide us with this property, which means that, for any $\mathcal{F}_{t_{n+1}}$-measurable random variable Y, there exists some $\mathcal{F}_{t_n}$-measurable processes

α_{t_n}, β_{t_n} and $\hat{\beta}_{t_n}$ such that

$$\begin{aligned} Y &= \alpha_{t_n} + \beta_{t_n}^{\mathsf{T}} \cdot \Delta R_{t_{n+1}} + \hat{\beta}_{t_n} \Delta \hat{R}_{t_{n+1}} \\ &= \alpha_{t_n} + (\beta_{t_n}^1 \quad \beta_{t_n}^2) \cdot \Delta R_{t_{n+1}} + \hat{\beta}_{t_n} \Delta \hat{R}_{t_{n+1}} \\ &= \alpha_{t_n} + \beta_{t_n}^1 \Delta R_{t_{n+1}}^1 + \beta_{t_n}^2 \Delta R_{t_{n+1}}^2 + \hat{\beta}_{t_n} \Delta \hat{R}_{t_{n+1}} \\ &= \alpha_{t_n} + \beta_{t_n}^1 \sqrt{\Delta t_{n+1}} B_{t_{n+1}}^1 + \beta_{t_n}^2 \sqrt{\Delta t_{n+1}} B_{t_{n+1}}^2 + \hat{\beta}_{t_n} \sqrt{\Delta t_{n+1}} \hat{B}_{t_{n+1}} \end{aligned}$$

where $\hat{B}_{t_n} := B_{t_n}^1 B_{t_n}^2$ is an additional 1-dimensional Bernoulli random variable with $P(\hat{B}_{t_n} = \pm 1) = \frac{1}{2}$, and, $\hat{R}_{t_{n+1}} := \sqrt{\Delta t_{n+1}} \hat{B}_{t_{n+1}}$. Moreover $\hat{B}_{t_n}$ is orthogonal[2] to both $B_{t_n}^1$ and $B_{t_n}^2$. Note that enlargement of the filtration $\mathcal{F}_{t_n}$ is not needed for supporting the measurability of $\hat{R}_{t_n}$ if we demand the presence of a supplementary random variable $\hat{R}_{t_n}$ since it is constructed by making use of the two $\mathcal{F}_{t_n}$-adapted processes $R_{t_n}^1, R_{t_n}^2$, hence, $\hat{R}_{t_n}$ is automatically adapted to $\mathcal{F}_{t_n}$ by definition. We should emphasize that it is necessary to have the existence of $\hat{B}_{t_n}$ in order to endow the existing probability space with the predictable representation property. For proof of this, we refer readers to Lemma 6.1 in Cheridito et al. (2011) as well as the justification of the predictable representation property in the current model setup.

Remark 5.1.0.1. As a consequence of the predictable representation property, we can further obtain a representation result for the density process Z of some absolutely continuous probability measure $Q \in \mathcal{P}^a$ where $\mathcal{P}^a$ denotes the set of absolutely continuous probability measures. Namely, for any $Q \in \mathcal{P}^a$ in the defined probability space, the corresponding density process $Z \in \mathcal{F}_{t_N}$ can be represented as

$$Z = \prod_{0 \le n \le N-1} \left(1 + q_{t_n}^1 \Delta R_{t_{n+1}}^1 + q_{t_n}^2 \Delta R_{t_{n+1}}^2 + \hat{q}_{t_n} \Delta \hat{R}_{t_{n+1}}\right) = \prod_{0 \le n \le N-1} \left(1 + q_{t_n}^{\mathsf{T}} \Delta R_{t_{n+1}}\right)$$

for some $\mathcal{F}_{t_n}$-adapted (not necessarily deterministic[3]) process $q_{t_n} := \begin{pmatrix} q_{t_n}^1 & q_{t_n}^2 & \hat{q}_{t_n} \end{pmatrix}^{\mathsf{T}} \in \mathbb{R}^3$. In fact, for any $Z \in \mathcal{F}_{t_N}$, we can express it as

$$Z = \prod_{0 \le n \le N-1} \frac{\mathbb{E}[Z|\mathcal{F}_{t_{n+1}}]}{\mathbb{E}[Z|\mathcal{F}_{t_n}]}$$

in which we define

$$\frac{\mathbb{E}[Z|\mathcal{F}_{t_{n+1}}]}{\mathbb{E}[Z|\mathcal{F}_{t_n}]} := 1 \qquad \text{on} \quad \{\omega \mid \mathbb{E}[Z|\mathcal{F}_{t_n}] = 0\}.$$

In this way it suffices to observe that, for each $Z \in \mathcal{F}_{t_{n+1}}$, $Z = 1 + q_{t_n}^1 \Delta R_{t_{n+1}}^1 + q_{t_n}^2 \Delta R_{t_{n+1}}^2 + \hat{q}_{t_n} \Delta \hat{R}_{t_{n+1}}$ for some $q_{t_n} \in \mathcal{F}_{t_n}$ holds. However such a decomposition is readily available due to the predictable representation property, this then allows us to

[2]Two random variables X and Y are said to be orthogonal if $\mathbb{E}[XY] = 0$ is satisfied.

[3]Deterministic process x means a collection of $\mathbb{R}$-valued number x_t, $x = (x_t)_{t \in \mathcal{T}}$, indexed by t.

conclude the validity of the stated representation for Z.

Remark 5.1.0.2. If, in addition, we have $Z > 0$ held P-almost surely, the associated probability measure Q will be an equivalent probability measure, $Q \in \mathcal{P}^e$, where $\mathcal{P}^e$ denotes the set of equivalent probability measures. It can be shown, see Stadje (2010), that the following 3-dimensional process

$$R^Q_{t_n} = \begin{pmatrix} R^{Q,1}_{t_n} \\ R^{Q,2}_{t_n} \\ \hat{R}^Q_{t_n} \end{pmatrix} = \begin{pmatrix} R^1_{t_n} - \sum_{i=0}^{n-1} q^1_{t_i} \Delta t_{i+1} \\ R^2_{t_n} - \sum_{i=0}^{n-1} q^2_{t_i} \Delta t_{i+1} \\ \hat{R}_{t_n} - \sum_{i=0}^{n-1} \hat{q}_{t_i} \Delta t_{i+1} \end{pmatrix}$$

is a martingale under Q. To verify this assertion, it is sufficient to show $\mathbb{E}^Q[\Delta R_{t_n} - q_{t_{n-1}} \Delta t_n | \mathcal{F}_{t_{n-1}}] = 0$ holds for all $n = 1, \ldots, N$ because of the fact $R^Q_{t_n} = R^Q_{t_{n-1}} + \Delta R_{t_n} - q_{t_{n-1}} \Delta t_n$. This is indeed true because

$$\begin{aligned} \mathbb{E}^Q[\Delta R_{t_n} - q_{t_{n-1}} \Delta t_n | \mathcal{F}_{t_{n-1}}] &= \mathbb{E}\left[\frac{Z_{t_n}}{Z_{t_{n-1}}} \Delta R_{t_n} \middle| \mathcal{F}_{t_{n-1}} \right] - q_{t_{n-1}} \Delta t_n \\ &= \mathbb{E}[(1 + q_{t_{n-1}} \Delta R_{t_n}) \Delta R_{t_n} | \mathcal{F}_{t_{n-1}}] - q_{t_{n-1}} \Delta t_n \\ &= 0 + q_{t_{n-1}} \Delta t_n - q_{t_{n-1}} \Delta t_n = 0 \end{aligned}$$

As discussed in the remark above, the random variable $\Delta \hat{R}_{t_{n+1}}$ is essential to have the predictable representation property valid in this model. Moreover, even though the price process S solely depends on $\Delta R^1_{t_{i+1}}$ and $\Delta R^2_{t_{i+1}}$, the density process Z of a probability measure Q is generally dependent of $R^1_{t_{i+1}}$, $R^2_{t_{i+1}}$ and $\hat{R}_{t_{n+1}}$, as seen from previous remarks, and so too for those pricing measures and martingale measures, which come into play when one computes prices. This suggests we retain $\hat{R}_{t_{n+1}}$ in the price process of S, and we simply assign $\hat{\sigma}_{t_n} = 0$ to be the volatility coefficient $\hat{R}_{t_{n+1}}$. More specifically, we augment the original process $R = (R^1 \quad R^2)$ to $R = (R^1 \quad R^2 \quad \hat{R})$ and so the price process of S is rewritten as

$$\begin{aligned} S_{t_n} - S_{t_{n-1}} &= S_{t_{n-1}} \left(\mu_{t_{n-1}} \Delta t_n + \sigma_{t_{n-1}} \Delta R_{t_n} \right) \\ &= S_{t_{n-1}} \sigma_{t_{n-1}} \left(\xi_{t_{n-1}} \Delta t_n + \Delta R_{t_n} \right), \end{aligned}$$

in which $\sigma_{t_n} = \begin{pmatrix} \sigma^1_{t_n} & \sigma^2_{t_n} & 0 \end{pmatrix}$. A similar price process can also be obtained for the other underlying asset $\overline{S}$. Furthermore, in line with Becherer (2009), we define the market price of risk process ξ as

$$\xi_{t_n} := \sigma^\intercal_{t_n} (\sigma_{t_n} \sigma^\intercal_{t_n})^{-1} \mu_{t_n}.$$

We denote $\mathcal{M}^e = \mathcal{M}^e(S)$ as the set of martingale measures for S. Note that the present setup is also a Markovian model so that any conditional expectation with respect to the filtration is essentially that with respect to the state of S_t realized at t. In other words, $\mathbb{E}[\cdot|\mathcal{F}_t] = \mathbb{E}[\cdot|S_t]$.

5.2 RAROC-Based NGD Prices

Observing the relationship between the *RAROC*-based NGD bid price and the ask price given by $\pi_t^u(X) = -\pi_t^l(-X)$, we shall restrict all analysis to the *RAROC*-based NGD ask price $\pi_t^u(X)$ in the sequel. The definition of the upper dynamic valuation bound $\pi_t^u(X)$ in Becherer (2009) is

$$\pi_t^u := \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^{\text{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t]$$

$$\text{where} \quad \mathcal{Q}^{\text{ngd}} := \left\{ Q \in \mathcal{M}^e \;\middle|\; \begin{array}{l} \mathbb{E}\left[-\log \frac{Z_{\tau_2}}{Z_{\tau_1}} \middle| \mathcal{F}_{\tau_1} \right] \leq \frac{1}{2}\mathbb{E}\left[-\int_{\tau_1}^{\tau_2} h_u^2 du \middle| \mathcal{F}_{\tau_1} \right] \\ \text{for all stopping times } \tau_1, \tau_2 \text{such that } \tau_1 \leq \tau_2 \leq T \end{array} \right\}.$$

Here h represents some upper limit on the growth rate of all portfolios in the market. See Becherer (2009) for details. It is also proven in Becherer (2009) that this version of $\mathcal{Q}^{\text{ngd}}$ satisfies the property of being multiplicatively stable (m-stable), a concept introduced by Delbaen (2006). Equipped with a set of probability measures that is m-stable, the resultant upper dynamic valuation bound π_t^u is a time-consistent dynamic coherent risk measure, which is a direct consequence of results in Delbaen (2006).

Returning to our definition of $\pi_t^u(X)$, let us define r_1 and r_2 as

$$r_1 := 1 + R \qquad \text{and} \qquad r_2 := \frac{\alpha(1+R)}{\alpha + R}$$

where $0 < \alpha < 1$ is the confidence level and R is the target *RAROC* that we require. Obviously $r_1 > 1 > r_2$ is satisfied and it was shown in the previous chapter that

$$\pi_t^u := \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^{\text{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t] \tag{5.2.1}$$

$$\text{where} \quad \mathcal{Q}^{\text{ngd}} := \left\{ Q \in \mathcal{P}^a \;\middle|\; \frac{1}{r_1} \leq \frac{\mathbb{E}^P[Z|\mathcal{F}_{t_{n+1}}]}{\mathbb{E}^P[Z|\mathcal{F}_{t_n}]} \leq \frac{1}{r_2},\ P\text{-a.s.} \quad \forall t_n \in \mathcal{T} \right\} \tag{5.2.2}$$

is a time-consistent dynamic coherent risk measure. We recall that $\mathcal{Q}^{\text{ngd}}$ is constructed such that π_t^u possesses the property of time-consistency. In other words, we have not made use of any m-stability of $\mathcal{Q}^{\text{ngd}}$ to derive a time-consistent dynamic coherent risk measure, whether it is m-stable or not is unspecified (Indeed it is m-stable and this will be proven in later stages). In view of our version of $\mathcal{Q}^{\text{ngd}}$, it consists of those $Q \in \mathcal{P}^a$ where $\mathcal{P}^a$ is the set of probability measures absolutely continuous with P. As we are interested in conducting the valuation of a contingent claim X with π_t^u, to exclude the possibility of arbitrage opportunity, not all pricing measures Q from $\mathcal{P}^a$ can meet this purpose. Instead, we should restrict the attention to a smaller set $\mathcal{M}^e$ in $\mathcal{P}^a$ that are equivalent martingale measures. If Q is chosen from $\mathcal{M}^e \subseteq \mathcal{P}^a$, absence of arbitrage is guaranteed. In particular, if we wish to have a valid *RAROC*-based NGD ask price, we have to modify π^u to

$$\pi_t^u := \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^{\mathrm{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t]$$

$$\begin{aligned}
\text{where} \quad \mathcal{Q}^{\mathrm{ngd}} &:= \left\{ Q \in \mathcal{M}^e \;\middle|\; \frac{1}{r_1} \leq \frac{\mathbb{E}^P[Z|\mathcal{F}_{t_{n+1}}]}{\mathbb{E}^P[Z|\mathcal{F}_{t_n}]} \leq \frac{1}{r_2},\ P\text{-a.s.} \quad \forall t_n \in \mathcal{T} \right\} \\
&:= \left\{ Q \in \mathcal{P}^a \;\middle|\; \frac{1}{r_1} \leq \frac{\mathbb{E}^P[Z|\mathcal{F}_{t_{n+1}}]}{\mathbb{E}^P[Z|\mathcal{F}_{t_n}]} \leq \frac{1}{r_2},\ P\text{-a.s.} \quad \forall t_n \in \mathcal{T} \right\} \bigcap \mathcal{M}^e \\
&= \mathcal{P}^{\mathrm{ngd}} \cap \mathcal{M}^e
\end{aligned}$$

in which $\mathcal{P}^{\mathrm{ngd}}$ refers to previous $\mathcal{Q}^{\mathrm{ngd}}$ in (5.2.2). However, it should be stressed that any $Q \in \mathcal{P}^{\mathrm{ngd}}$ is not guaranteed to be an element of $\mathcal{M}^e$ too, so $\mathcal{Q}^{\mathrm{ngd}}$ may be a proper subset of $\mathcal{P}^{\mathrm{ngd}}$. With the replacement of a new $\mathcal{Q}^{\mathrm{ngd}}$, the 'nice' dynamic behavior and properties that π^u originally possesses when it is equipped with $\mathcal{P}^{\mathrm{ngd}}$ might not be inherited. In other words, π^u under the new $\mathcal{Q}^{\mathrm{ngd}}$ may depart from being a time-consistent dynamic coherent risk measure and we may be hindered from defining a *RAROC*-based NGD ask price of a contingent claim. Fortunately such a departure can be prevented because the property of m-stability can be identified in this new $\mathcal{Q}^{\mathrm{ngd}}$, and then by the results in Delbaen (2006), the associated π^u retains all the behavior and properties of a time-consistent dynamic coherent risk measure. So it justifies the definition of a *RAROC*-based NGD ask price by using this version of π^u which also precludes arbitrage opportunities. The assertion that$\mathcal{Q}^{\mathrm{ngd}}$ is m-stable is now shown.

To begin with, we prepare ourselves with the following

Lemma 5.2.1. *For any m-stable sets of probability measures $\mathcal{S}^1$ and $\mathcal{S}^2$, their intersection, $\mathcal{S}^1 \cap \mathcal{S}^2$, is also a m-stable set of probability measure.*

Proof. Indeed this is immediate. For any $Q^1, Q^2 \in \mathcal{S}^1 \cap \mathcal{S}^2$ with density process Z^1, Z^2 respectively, for any sopping time $\tau \leq T$, the process

$$Z_T = Z_\tau^1 \frac{Z_T^2}{Z_\tau^2}$$

is ensured to be the density process of some $Q \in \mathcal{S}^1$ because $Q^1, Q^2 \in \mathcal{S}^1$. At the same time, it is also the density process of some $Q \in \mathcal{S}^2$ because $Q^1, Q^2 \in \mathcal{S}^2$. A unique probability measure Q is defined by the same density process Z. Hence, we can assert that it is a density process for some $Q \in \mathcal{S}^1 \cap \mathcal{S}^2$. This proves that $\mathcal{S}^1 \cap \mathcal{S}^2$ is m-stable. □

Proposition 5.2.1. *The sets $\mathcal{M}^e$, $\mathcal{P}^{ngd}$ and $\mathcal{Q}^{ngd}$ are m-stable.*

Proof. Regarding $\mathcal{M}^e$, its m-stability is proven in Delbaen (2006). Since $\mathcal{Q}^{\mathrm{ngd}} = \mathcal{P}^{\mathrm{ngd}} \cap \mathcal{M}^e$ holds, because of Lemma 5.2.1, it suffices to prove that $\mathcal{P}^{\mathrm{ngd}}$ is m-stable.

For any $Q^1, Q^2 \in \mathcal{P}^{\mathrm{ngd}}$ with density process Z^1, Z^2 respectively, and for any stopping times $\tau \leq T$, consider

$$Z_T = Z^1_\tau \frac{Z^2_T}{Z^2_\tau}$$

Due to the explicit representation of Z in the current setup, we also have

$$Z_T = Z^1_\tau \frac{Z^2_T}{Z^2_\tau} = \prod_{0 \leq n \leq M(\omega)} (1 + (q^1)^\intercal \Delta R) \cdot \prod_{M(\omega)+1 \leq n \leq N} (1 + (q^2)^\intercal \Delta R)$$

because the stopping time τ satisfies $\tau(\omega) = t_{M(\omega)}$ and $M(\omega)$ can only take values from $\{0, 1, \ldots, N\}$.

To prove the claim, we need to verify that, for any t_n,

$$\frac{1}{r_1} \leq \frac{\mathbb{E}[Z_T | \mathcal{F}_{t_{n+1}}]}{\mathbb{E}[Z_T | \mathcal{F}_{t_n}]} \leq \frac{1}{r_2}.$$

Observe that, for any fixed t_n, on $\{\tau(\omega) \leq t_n\}$,

$$\mathbb{E}[Z_T | \mathcal{F}_{t_n}] = \mathbb{E}\left[Z^1_\tau \frac{Z^2_T}{Z^2_\tau} \middle| \mathcal{F}_{t_n} \right] = Z^1_\tau \frac{1}{Z^2_\tau} \mathbb{E}[Z^2_T | \mathcal{F}_{t_n}] = Z^1_\tau \frac{Z^2_{t_n}}{Z^2_\tau}.$$

On $\{\tau(\omega) > t_n\}$,

$$\begin{aligned}
\mathbb{E}[Z_T | \mathcal{F}_{t_n}] = \mathbb{E}\left[Z^1_\tau \frac{Z^2_T}{Z^2_\tau} \middle| \mathcal{F}_{t_n} \right] &= \mathbb{E}\left[\mathbb{E}\left[Z^1_\tau \frac{Z^2_T}{Z^2_\tau} \middle| \mathcal{F}_\tau \right] \middle| \mathcal{F}_{t_n} \right] \\
&= \mathbb{E}\left[Z^1_\tau \frac{1}{Z^2_\tau} \mathbb{E}\left[Z^2_T \middle| \mathcal{F}_\tau \right] \middle| \mathcal{F}_{t_n} \right] \\
&= \mathbb{E}\left[Z^1_\tau \frac{1}{Z^2_\tau} Z^2_\tau \middle| \mathcal{F}_{t_n} \right] \\
&= \mathbb{E}[Z^1_\tau | \mathcal{F}_{t_n}] = Z^1_{t_n}.
\end{aligned}$$

Then, by using these facts, we have, for any t_n,

$$\frac{\mathbb{E}[Z_T|\mathcal{F}_{t_{n+1}}]}{\mathbb{E}[Z_T|\mathcal{F}_{t_n}]} = \begin{cases} \dfrac{Z^1_\tau \frac{Z^2_{t_{n+1}}}{Z^2_\tau}}{Z^1_\tau \frac{Z^2_{t_n}}{Z^2_\tau}} = \dfrac{Z^2_{t_{n+1}}}{Z^2_{t_n}} & \text{on} \quad \{\tau(\omega) \le t_n < t_{n+1}\} \\ \dfrac{Z^1_\tau \frac{Z^2_{t_{n+1}}}{Z^2_\tau}}{Z^1_{t_n}} = \dfrac{Z^1_{t_{n+1}} \frac{Z^2_{t_{n+1}}}{Z^2_{t_{n+1}}}}{Z^1_{t_n}} = \dfrac{Z^1_{t_{n+1}}}{Z^1_{t_n}} & \text{on} \quad \{t_n < \tau(\omega) \le t_{n+1}\} = \{\tau(\omega) = t_{n+1}\} \\ \dfrac{Z^1_{t_{n+1}}}{Z^1_{t_n}} & \text{on} \quad \{t_n < t_{n+1} < \tau(\omega)\} \end{cases}.$$

In view of this, we can conclude that $\frac{1}{r_1} \le \frac{\mathbb{E}[Z_T|\mathcal{F}_{t_{n+1}}]}{\mathbb{E}[Z_T|\mathcal{F}_{t_n}]} \le \frac{1}{r_2}$ holds because $Q^1, Q^2 \in \mathcal{P}^{\mathrm{ngd}}$ are assumed. So the proof is completed. □

5.3 Computation of RAROC-Based NGD Ask Price

We recall here that, under the present model setup, the density process Z for a probability measure $Q \in \mathcal{P}^a$ would admit the following representation

$$Z = \prod_{0 \le n \le N-1} \Big(1 + q^1_{t_n} \Delta R^1_{t_{n+1}} + q^2_{t_n} \Delta R^2_{t_{n+1}} + \hat{q}_{t_n} \Delta \hat{R}_{t_{n+1}}\Big) = \prod_{0 \le n \le N-1} \Big(1 + q^\intercal_{t_n} \Delta R_{t_{n+1}}\Big)$$

$$\text{where} \quad q_{t_n} = \begin{pmatrix} q^1_{t_n} \\ q^2_{t_n} \\ \hat{q}_{t_n} \end{pmatrix} \quad \text{and} \quad \Delta R_{t_n} = \begin{pmatrix} \Delta R^1_{t_n} \\ \Delta R^2_{t_n} \\ \Delta \hat{R}_{t_n} \end{pmatrix}.$$

This fact would make the computation of *RAROC*-based NGD ask price π^u tantamount to solving a stochastic optimization problem. To see this, *RAROC*-based NGD ask price π^u is defined as

$$\pi^u_{t_n} := \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^{\mathrm{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_{t_n}]$$

$$\text{where} \quad \mathcal{Q}^{\mathrm{ngd}} := \left\{ Q \in \mathcal{M}^e \;\middle|\; \frac{1}{r_1} \le \frac{\mathbb{E}^P[Z|\mathcal{F}_{t_{n+1}}]}{\mathbb{E}^P[Z|\mathcal{F}_{t_n}]} \le \frac{1}{r_2}, \quad \forall t_n \in \mathcal{T} \right\}.$$

By substituting the above expression for Z into $\mathcal{Q}^{\text{ngd}}$, we have

$$\mathcal{Q}^{\text{ngd}} := \left\{ Q \in \mathcal{M}^e \,\middle|\, \frac{1}{r_1} \le \frac{\mathbb{E}^P[Z|\mathcal{F}_{t_{n+1}}]}{\mathbb{E}^P[Z|\mathcal{F}_{t_n}]} \le \frac{1}{r_2}, \quad \forall t \in \mathcal{T} \right\}$$

$$= \left\{ Q \in \mathcal{M}^e \,\middle|\, \frac{1}{r_1} \le \left(1 + q^1_{t_n}\Delta R^1_{t_{n+1}} + q^2_{t_n}\Delta R^2_{t_{n+1}} + \hat{q}_{t_n}\Delta \hat{R}_{t_{n+1}}\right) \le \frac{1}{r_2}, \quad \forall n \le N-1 \right\}$$

$$\text{since} \quad \frac{\mathbb{E}^P[Z|\mathcal{F}_{t_{n+1}}]}{\mathbb{E}^P[Z|\mathcal{F}_{t_n}]} = 1 + q^1_{t_n}\Delta R^1_{t_{n+1}} + q^2_{t_n}\Delta R^2_{t_{n+1}} + \hat{q}_{t_n}\Delta \hat{R}_{t_{n+1}}$$

$$= \bigcap_{n=0}^{N-1} \mathcal{Q}^{\text{ngd}}_{t_n}$$

$$\text{where} \quad \mathcal{Q}^{\text{ngd}}_{t_n} := \left\{ Q \in \mathcal{M}^e \,\middle|\, \frac{1}{r_1} \le \left(1 + q^1_{t_n}\Delta R^1_{t_{n+1}} + q^2_{t_n}\Delta R^2_{t_{n+1}} + \hat{q}_{t_n}\Delta \hat{R}_{t_{n+1}}\right) \le \frac{1}{r_2} \right\}.$$

Furthermore,

$$\mathbb{E}^Q[X|\mathcal{F}_{t_n}] = \mathbb{E}\left[\frac{Z_T}{Z_{t_n}} X \,\middle|\, \mathcal{F}_{t_n}\right] = \mathbb{E}\left[\prod_{n \le i \le N-1} \left(1 + q^{\intercal}_{t_i}\Delta R_{t_{i+1}}\right) X \,\middle|\, \mathcal{F}_{t_n}\right].$$

In view of these, we can conclude that to determine π^u is merely about solving a stochastic optimization problem with respect to variable q_{t_n}. Before we proceed further, we should recognize that, since there are N time steps, we ought to tackle a high-dimensional optimization problem in which we aim at seeking a maximum over all candidates of density process Z of Q belonging to $\mathcal{Q}^{\text{ngd}}$. Each of these density process is completely parameterized by $3 \times N$ parameters, namely, $q_{t_n} = \{(q^1_{t_n} \quad q^2_{t_n} \quad \hat{q}_{t_n})\}_{0 \le n \le N-1}$. At the first sight, it might seem cumbersome and computationally intensive to solve the problem. Yet, fortunately, this complexity can be significantly reduced due to the formulation of π^u_t.

Recall that π^u_t is constructed in such a way that it behaves as a time-consistent dynamic risk measure, the corresponding structure of $\mathcal{Q}^{\text{ngd}}$ is acquired upon the stated behavior of π^u_t. In other words, if there exists an optimal element $Q^* \in \mathcal{Q}^{\text{ngd}}$ with density process Z^* in the above optimization problem, at such Z^*, the associated π^u_t *must* retain the desired behavior. In particular, we require π^u satisfies the property of recursiveness, or dynamic programming principle, described by

$$\phi_{t_n}(X) = \phi_{t_n}(\phi_{t_{n+1}}(X))$$

where $\phi := \pi^u$. To achieve this goal, we make use of the generators φ introduced in Chapter 4 because the property of recursiveness is readily available in a time-consistent dynamic coherent risk measure ϕ once we derive ϕ from φ. More specifically, φ and ϕ

should obey the following relationship

$$\phi_{t_n}(X) = \varphi_{t_n}(\phi_{t_{n+1}}(X)).$$

It should be emphasized that the generator φ_{t_n} refers to a one-period (between t_n and t_{n+1}) optimization problem while ϕ_{t_n} is generally a $(N-n)$-period (between t_n and t_N) optimization problem. If Z^*, comprised of $\{(q^{*,1}_{t_n} \quad q^{*,2}_{t_n} \quad \hat{q}^*_{t_n})\}_{0\leq n\leq N-1}$, is the optimal element for yielding $\phi_{t_n}(X)$, due to the above equality, it should also be an optimal element for solving the one-period optimization problem $\varphi_{t_n}(\phi_{t_{n+1}}(X))$. However, since φ_{t_n} stands for optimizing between t_n and t_{n+1}, only the parameter $\{(q^{*,1}_{t_n} \quad q^{*,2}_{t_n} \quad \hat{q}^*_{t_n})\}$ of Z^* matters for determining the optimal value. As a result, we comment that, under the presence of φ_{t_n}, at each t_n, we can confront the $(N-n)$-period optimization problem defined by ϕ_{t_n} by breaking it down into two subproblems: the first one corresponds to the determination of $\phi_{t_{n+1}}(X)$ while the second one corresponds to a one-period optimization problem set by $\varphi_{t_n}(\cdot)$. In other words, this highlights that the solution of the original maximization problem $\phi_{t_n}(X)$ at t_n can be approached by means of backward induction, in which the original problem is decomposed into $N-n$ stand-alone one-period optimization problems. Then we can perform $(N-n)$ one-period maximizations rather than a single $(N-n)$-period maximization. At t_0, the desired maximizing density process Z^* can be obtained by concatenating/pasting together all N one-period maximizing density processes $\{Z^*_{t_n}\}_{n=0,1,\ldots,N-1}$. It should be remarked that, generally speaking, without the construction of a time-consistent dynamic coherent risk measure ϕ by the means of generators φ, we cannot attempt to construct the solution Z^* by making use of such a concatenation technique with each one-period maximizing the density process $\{Z^*_{t_n}\}_{n=0,1,\ldots,N-1}$ because any fixed density process Z_{t_0} at t_0 would produce a non-vanishing effect on $\phi_{t_n}(X)$ at other times $t_n > t_0$, which cannot be isolated/untangled in a trivial manner, hence, it would 'intervene' and complicate the calculations of each one-step maximization problems at subsequent times. This forces us to determine Z^* as a whole but not as a 'package' of $\{Z^*_{t_n}\}_{n=0,1,\ldots,N-1}$.

In summary, we solve the maximization problem for $\pi^u_{t_n}(X)$ with the following procedures:

i. Start with t_{N-1}, we solve for $\pi^u_{t_{N-1}}(X)$ which is a one-period optimization problem.

ii. Denote $Z^*_{t_{N-1}}$ as the maximizing density process for $\pi^u_{t_{N-1}}(X)$ at t_{N-1}. Suppose $\pi^u_{t_{N-1}}(X)$ is now solved and known.

iii. To solve the maximization problem in $\pi^u_{N-2}(X)$ at t_{N-2}, we make use of the relationship $\pi^u_{t_{N-2}}(X) = \varphi_{t_{N-2}}(\pi^u_{t_{N-1}}(X))$ to replace the generic $\pi^u_{t_{N-2}}(X)$. Owing to the property of recursiveness and the definition of generators φ, we can treat $\pi^u_{t_{N-2}}(X)$ as a one-period optimization problem defined by φ and the solution is guaranteed to be optimal and identical to $\pi^u_{t_{N-2}}(X)$. Assume it is solved and denote the corresponding maximizing density process as $Z^*_{t_{N-2}}$.

iv. Repeat the procedures above until we reach t_n. We then conclude that the corresponding maximizing density $Z^*_{t_n}$ for solving the original problem $\phi_{t_n}(X)$ is obtained by concatenation, which gives

$$Z^*_{t_n} = Z^*_{t_n} \times Z^*_{t_{n+1}} \times \cdots \times Z^*_{t_{N-2}} \times Z^*_{t_{N-1}}.$$

In light of this routine, our primary focus for the rest of the chapter will be placed on the determination of $\pi^u_{N-1}(X)$, which has a resemblance to other one-period maximization problems at different times $\{t_n\}_{n=0,1,\ldots,N-2}$. Once the methodology for determining $\pi^u_{N-1}(X)$ is elaborated, analogous methods can be immediately adopted to yield the respective solutions at different times.

5.3.1 Computation of $\pi^u_{N-1}(X)$

The density process related to solving for $\pi^u_{N-1}(X)$ is given by $\frac{Z}{\mathbb{E}^P[Z|\mathcal{F}_{t_{N-1}}]}$, which admits the following representation

$$\begin{aligned}\frac{Z}{\mathbb{E}^P[Z|\mathcal{F}_{t_{N-1}}]} &= \frac{\prod_{0\le i\le N-1}\left(1+q^1_{t_i}\Delta R^1_{t_{i+1}}+q^2_{t_i}\Delta R^2_{t_{i+1}}+\hat{q}_{t_i}\Delta\hat{R}_{t_{i+1}}\right)}{\prod_{0\le i\le N-2}\left(1+q^1_{t_i}\Delta R^1_{t_{i+1}}+q^2_{t_i}\Delta R^2_{t_{i+1}}+\hat{q}_{t_i}\Delta\hat{R}_{t_{i+1}}\right)}\\ &= 1+q^1_{t_{N-1}}\Delta R^1_{t_N}+q^2_{t_{N-1}}\Delta R^2_{t_N}+\hat{q}_{t_{N-1}}\Delta\hat{R}_{t_N}.\end{aligned}$$

It can be easily seen that such a density process is essentially parameterized by three variables, $q^1_{t_{N-1}}$, $q^2_{t_{N-1}}$ and $\hat{q}_{t_{N-1}}$. Moreover, in order to satisfy the associated $Q\in\mathcal{Q}^{\text{ngd}}_{t_{N-1}}$, there should be some constraints on the choices of $q^1_{t_{N-1}}$, $q^2_{t_{N-1}}$ and $\hat{q}_{t_{N-1}}$. First of all, we should establish the following with $q^1_{t_{N-1}}$, $q^2_{t_{N-1}}$ and $\hat{q}_{t_{N-1}}$

$$\frac{1}{r_1}\le 1+q^1_{t_{N-1}}\Delta R^1_{t_N}+q^2_{t_{N-1}}\Delta R^2_{t_N}+\hat{q}_{t_{N-1}}\Delta\hat{R}_{t_N}\le\frac{1}{r_2}\quad P\text{-a.s.}$$

The associated Q, however, can only be assured to be an element of $\mathcal{P}^{\text{ngd}}_{t_{N-1}}$. Nevertheless, we see that each Z is completely determined by the 3-dimensional coefficient $q=(q^1\quad q^2\quad \hat{q})^\intercal$ satisfying the above inequality and so is the variable we should optimize through for $\pi^u_{N-1}(X)$. Hereafter we shall omit t_{N-1} for the sake of brevity. Note that each Q is equivalent to P since $\frac{1}{r_1}>0$, hence, we are permitted to define a new process $R^Q_{t_n}$ which is a martingale under Q. To preclude arbitrage opportunities when pricing a contingent claim, we shall also require $Q\in\mathcal{M}^e$ but not only $Q\in\mathcal{P}^{\text{ngd}}_{t_{N-1}}$. Taking this additional factor into account, we derive the corresponding constraint imposed on the choice of $q=(q^1\quad q^2\quad \hat{q})^\intercal$ as demonstrated below.

Denote the image I of σ^{T} and the kernel K of σ respectively as

$$I_{t_n} = \operatorname{Im} \sigma_{t_n}^{\mathsf{T}} \qquad \text{and} \qquad K_{t_n} = \operatorname{Ker} \sigma_{t_n},$$

since, for any $Q \in \mathcal{P}^e$,

$$\sigma_{t_{n-1}} \left(\xi_{t_{n-1}} \Delta t_n + \Delta R_{t_n} \right) = \sigma_{t_{n-1}} \left(\left(\xi_{t_{n-1}} + q_{t_{n-1}} \right) \Delta t_n + \Delta R_{t_n}^Q \right)$$

holds, so, $Q \in \mathcal{M}^e(S)$ is satisfied if and only if the drift of the above process is eliminated, that is, $\sigma_{t_{n-1}} \left(\xi_{t_{n-1}} + q_{t_{n-1}} \right) = 0$, hence, this implies $q_{t_{n-1}} = -\xi_{t_{n-1}} + \eta_{t_{n-1}}$ for some $\eta_{t_{n-1}} \in K_{t_n}$. Equivalently, $\sigma_{t_{n-1}} \left(\xi_{t_{n-1}} + q_{t_{n-1}} \right) = 0$ is true if and only if $\sigma_{t_{n-1}} q_{t_{n-1}} = -\mu_{t_{n-1}}$, due to the definition of $\xi_{t_{n-1}}$. We then arrive at the appropriate constraint that is placed on $q = (q^1 \quad q^2 \quad \hat{q})^{\mathsf{T}}$ for absence of arbitrage, i.e. $\sigma_{t_{n-1}} q_{t_{n-1}} = -\mu_{t_{n-1}}$.

By aggregating all the constraints explored above, to ensure the associated Q belongs to $\mathcal{Q}_{t_{N-1}}^{\mathrm{ngd}}$ under a density process Z, we conclude that the corresponding $q = (q^1 \quad q^2 \quad \hat{q})^{\mathsf{T}}$ in Z should be chosen in such a way that it simultaneously satisfies

$$\frac{1}{r_1} \leq 1 + q_{t_{N-1}}^1 \Delta R_{t_N}^1 + q_{t_{N-1}}^2 \Delta R_{t_N}^2 + \hat{q}_{t_{N-1}} \Delta \hat{R}_{t_N} \leq \frac{1}{r_2} \; P\text{-a.s.} \quad \text{and}$$
$$\sigma_{t_{N-1}}^1 q_{t_{N-1}}^1 + \sigma_{t_{N-1}}^2 q_{t_{N-1}}^2 = -\mu_{t_{N-1}}.$$

The first constraint involves a P-almost sure statement under random variables $\Delta R_{t_N}^1$, $\Delta R_{t_N}^2$, $\Delta \hat{R}_{t_N}$ in which the 'P-almost sure' part can be omitted. This is because we are now in a discrete probability space governed by $(\Delta R_{t_N}^1, \Delta R_{t_N}^2, \Delta \hat{R}_{t_N})$, and each state $\omega \in \Omega$ is assumed with non-zero probability of occurrence, the constraint can be essentially imposed in a pointwise manner. In other words, one should consider one constraint for each scenario $\omega \in \Omega$ realized by $(\Delta R_{t_N}^1(\omega), \Delta R_{t_N}^2(\omega), \Delta \hat{R}_{t_N}(\omega))$. With reference to the current market configuration, even though there are three Bernoulli random variables $(\Delta R_{t_N}^1, \Delta R_{t_N}^2, \Delta \hat{R}_{t_N})$, one should not trivially conclude that a total of 8 scenarios, hence, 8 constraints, are in action. Actually it should be a total of 4 scenarios only, rather than 8 scenarios as in independent case, because the random variable $\Delta \hat{R}_{t_N}$ is a function of the other two random variables $\Delta R_{t_N}^1$ and $\Delta R_{t_N}^2$ (hence depending on the values of $\Delta R_{t_N}^1$ and $\Delta R_{t_N}^2$). In other words, $\Delta \hat{R}_{t_N}$ is orthogonal to but not independent of $\Delta R_{t_N}^1$, $\Delta R_{t_N}^2$. So, one only need to consider the number of scenarios as realized by $\left(\Delta R_{t_N}^1, \Delta R_{t_N}^2\right)$. Due to the assumed independence between them, we are led to study 4 scenarios and so an imposition of 4 inequalities (each of which corresponds to a realization of $(\Delta R_{t_N}^1, \Delta R_{t_N}^2, \Delta \hat{R}_{t_N})$) and 1 equality, due to the no-arbitrage condition, on the 3-dimensional variable $q = (q^1 \quad q^2 \quad \hat{q})^{\mathsf{T}} \in \mathbb{R}^3$. More precisely, the set of constraints is given by

$$\begin{gathered}
\frac{1}{r_1} \leq 1 + q^1_{t_{N-1}}\sqrt{\Delta t_N} + q^2_{t_{N-1}}\sqrt{\Delta t_N} + \hat{q}_{t_{N-1}}\sqrt{\Delta t_N} \leq \frac{1}{r_2} \\
\frac{1}{r_1} \leq 1 + q^1_{t_{N-1}}\sqrt{\Delta t_N} - q^2_{t_{N-1}}\sqrt{\Delta t_N} - \hat{q}_{t_{N-1}}\sqrt{\Delta t_N} \leq \frac{1}{r_2} \\
\frac{1}{r_1} \leq 1 - q^1_{t_{N-1}}\sqrt{\Delta t_N} + q^2_{t_{N-1}}\sqrt{\Delta t_N} - \hat{q}_{t_{N-1}}\sqrt{\Delta t_N} \leq \frac{1}{r_2} \\
\frac{1}{r_1} \leq 1 - q^1_{t_{N-1}}\sqrt{\Delta t_N} - q^2_{t_{N-1}}\sqrt{\Delta t_N} + \hat{q}_{t_{N-1}}\sqrt{\Delta t_N} \leq \frac{1}{r_2} \\
\sigma^1_{t_{N-1}} q^1_{t_{N-1}} + \sigma^2_{t_{N-1}} q^2_{t_{N-1}} = -\mu_{t_{N-1}}
\end{gathered}$$

After the discussion on the associated set of constraints on $q = (q^1 \quad q^2 \quad \hat{q})^\mathsf{T}$, we now return to the problem of determining a $RAROC$-based NGD ask price $\pi^u_{t_{N-1}}$ at $t = t_{N-1}$. Under the existing setup of a Bernoulli market, along with the representation of the density process Z, the problem of computing a $RAROC$-based NGD ask price $\pi^u_{t_{N-1}}$ can be concisely expressed as

$$\begin{aligned}
&\underset{q^1_{t_{N-1}}, q^2_{t_{N-1}}, \hat{q}_{t_{N-1}}}{\text{Maximize}} \quad \mathbb{E}[(1 + q^1_{t_{N-1}}\Delta R^1_{t_N} + q^2_{t_{N-1}}\Delta R^2_{t_N} + \hat{q}_{t_{N-1}}\Delta \hat{R}_{t_{N-1}})X|\mathcal{F}_{t_{N-1}}] \\
&\text{subject to} \quad \begin{cases} \frac{1}{r_1} \leq 1 + q^1_{t_{N-1}}\Delta R^1_{t_N} + q^2_{t_{N-1}}\Delta R^2_{t_N} + \hat{q}_{t_{N-1}}\Delta \hat{R}_{t_N} \leq \frac{1}{r_2} \\ \sigma_{t_{N-1}} q_{t_{N-1}} = -\mu_{t_{N-1}} \end{cases}
\end{aligned}$$

Furthermore, by applying the predictable representation property on X, we have

$$X = \alpha^X_{t_{N-1}} + \beta^{X,1}_{t_{N-1}}\Delta R^1_{t_N} + \beta^{X,2}_{t_{N-1}}\Delta R^2_{t_N} + \hat{\beta}^X_{t_{N-1}}\Delta \hat{R}_{t_N}$$

for some $\alpha^X_{t_{N-1}}, \beta^{X,1}_{t_{N-1}}, \beta^{X,2}_{t_{N-1}}, \hat{\beta}^X_{t_{N-1}} \in \mathcal{F}_{t_{N-1}}$. Through substituting this representation of X, we can rewrite the objective function of the above optimization problem as

$$\begin{aligned}
&\mathbb{E}[(1 + q^1_{t_{N-1}}\Delta R^1_{t_N} + q^2_{t_{N-1}}\Delta R^2_{t_N} + \hat{q}_{t_{N-1}}\Delta \hat{R}_{t_N})X|\mathcal{F}_{t_{N-1}}] \\
&= \mathbb{E}[X|\mathcal{F}_{t_{N-1}}] + q^1_{t_{N-1}}\mathbb{E}[\Delta R^1_{t_N} X|\mathcal{F}_{t_{N-1}}] + q^2_{t_{N-1}}\mathbb{E}[\Delta R^2_{t_N} X|\mathcal{F}_{t_{N-1}}] + \hat{q}_{t_{N-1}}\mathbb{E}[\hat{R}_{t_N} X|\mathcal{F}_{t_{N-1}}] \\
&= \alpha^X_{t_{N-1}} + q^1_{t_{N-1}}\Delta t_N \beta^{X,1}_{t_{N-1}} + q^2_{t_{N-1}}\Delta t_N \beta^{X,2}_{t_{N-1}} + \hat{q}_{t_{N-1}}\Delta t_N \hat{\beta}^X_{t_{N-1}}
\end{aligned}$$

where we have made use of the fact that $\Delta R^1_{t_N}$, $\Delta R^2_{t_N}$ and $\Delta \hat{R}_{t_N}$ are orthogonal to each other. Consequently, in order to solve for the $RAROC$-based NGD ask price $\pi^u_{t_{N-1}}$, we need to solve the following equivalent problem

$$\begin{aligned}
&\underset{q^1_{t_{N-1}}, q^2_{t_{N-1}}, \hat{q}_{t_{N-1}}}{\text{Maximize}} \quad \alpha^X_{t_{N-1}} + q^1_{t_{N-1}}\Delta t_N \beta^{X,1}_{t_{N-1}} + q^2_{t_{N-1}}\Delta t_N \beta^{X,2}_{t_{N-1}} + \hat{q}_{t_{N-1}}\Delta t_N \hat{\beta}^X_{t_{N-1}} \\
&\text{subject to} \quad \begin{cases} \frac{1}{r_1} \leq 1 + q^1_{t_{N-1}}\Delta R^1_{t_N} + q^2_{t_{N-1}}\Delta R^2_{t_N} + \hat{q}_{t_{N-1}}\Delta \hat{R}_{t_N} \leq \frac{1}{r_2} \\ \sigma_{t_{N-1}} q_{t_{N-1}} = -\mu_{t_{N-1}} \end{cases}
\end{aligned}$$

under some given $\alpha^X_{t_{N-1}}, \beta^{X,1}_{t_{N-1}}, \beta^{X,2}_{t_{N-1}}, \hat{\beta}^X_{t_{N-1}} \in \mathcal{F}_{t_{N-1}}$. At this stage, we have described how to obtain a $RAROC$-based NGD ask price π^u in a general setting under a Bernoulli market. We have also derived the associated configuration of the optimization problem set by $\pi^u_{t_{N-1}}$.

In order to provide a better understanding of $\pi^u_{t_{N-1}}$ and show a more concrete way to solve the optimization problem, we are going to compute explicitly the numerical values of $\pi^u_{t_{N-1}}$ for illustrations. We will consider a Bernoulli market set-up with specific parameter values and determine the $RAROC$-based NGD ask price π^u step-by-step. Furthermore, we have mentioned the approach of 'working backward' for obtaining $\pi^u_{t_n}$. In the theory of BSDE, the existence of a solution of a BSDE can be proven via a constructive approach, which is a method to obtain the desired BSDE solution through the backward induction. We notice that the $RAROC$-based NGD ask price $\pi^u_{t_n}$ and the solution of BSDEs share some similarities and so naturally pose the question of whether there is any intimate relationship between dynamic good-deal pricing and BSDE. As a matter of fact, the answer is positive as shown in Becherer (2009). Establishing and investigating their connection is also one of the objectives in the upcoming sections.

5.4 Example - One-Period Model

Let us consider the following set of parameters with a one-period investment horizon, $N = 1$,

$$\begin{aligned}
&S_{t_0} = 1.0, \quad r = 0, \quad \mu_{t_0} = 0.1, \quad \Delta t = 1.0, \quad \sigma^1_{t_0} = 0.3, \sigma^2_{t_0} = 0.5,\\
&\overline{S}_{t_0} = 1.0, \quad \overline{\mu}_{t_0} = 0.1, \quad \overline{\sigma}^1_{t_0} = 0.4, \overline{\sigma}^2_{t_0} = 0.5,
\end{aligned}$$

Recall that we assume only $\overline{S}$ is tradeable. So the call option $C_{t_1} = (S_{t_1} - K)^+$ cannot be completely hedged since S cannot be traded. Then, as time elapses from t_0 to t_1, the price S_{t_1} at t_1 is

$$\begin{aligned}
S_{t_1} &= S_{t_0}\left(1 + \mu_{t_0}\Delta t + \sigma^1_{t_0}\Delta R^1_{t_1} + \sigma^2_{t_0}\Delta R^2_{t_1}\right)\\
&= S_{t_0}\left(1 + \mu_{t_0} + \sigma^1_{t_0}\Delta R^1_{t_1} + \sigma^2_{t_0}\Delta R^2_{t_1} + \hat{\sigma}_{t_0}\Delta \hat{R}_{t_1}\right),
\end{aligned}$$

where $(\Delta R^1 \quad \Delta R^2 \quad \Delta\hat{R}) = (\Delta B^1 \quad \Delta B^2 \quad \Delta B^1 \cdot \Delta B^2)$ and $\sigma_{t_0} = (\sigma^1_{t_0} \quad \sigma^2_{t_0} \quad \hat{\sigma}_{t_0}) = (0.3 \quad 0.5 \quad 0)$. More precisely, it could take value from

$$S_{t_1} = \begin{cases} 1.0 \times 1.9 = 1.9 & \text{when} \quad (\Delta B^1_{t_1}, \Delta B^2_{t_1}) = (1,1) \\ 1.0 \times 0.9 = 0.9 & \text{when} \quad (\Delta B^1_{t_1}, \Delta B^2_{t_1}) = (1,-1) \\ 1.0 \times 1.3 = 1.3 & \text{when} \quad (\Delta B^1_{t_1}, \Delta B^2_{t_1}) = (-1,1) \\ 1.0 \times 0.3 = 0.3 & \text{when} \quad (\Delta B^1_{t_1}, \Delta B^2_{t_1}) = (-1,-1) \end{cases}$$

The condition of absence of arbitrage is satisfied in this model since the set of martingale measures for S is non-empty, for instance, one may consider $(p_1 \quad p_2 \quad p_3 \quad p_4) = (\frac{5}{30} \quad \frac{9}{30} \quad \frac{6}{30} \quad \frac{10}{30})$. Apart from the existence of the underlying asset, we further assume that there is a call option $C_{t_1} = (S_{t_1} - K)^+$ with strike $K = 1.2$, which is the contingent claim to be priced. Then, its corresponding random payoff at t_1 is

$$C_{t_1} = \begin{cases} (1.7 - 1.2)^+ = 0.7 & \text{when} \quad (\Delta B^1_{t_1}, \Delta B^2_{t_1}) = (1,1) \\ (1.1 - 1.2)^+ = 0 & \text{when} \quad (\Delta B^1_{t_1}, \Delta B^2_{t_1}) = (1,-1) \\ (1.1 - 1.2)^+ = 0.1 & \text{when} \quad (\Delta B^1_{t_1}, \Delta B^2_{t_1}) = (-1,1) \\ (0.5 - 1.2)^+ = 0 & \text{when} \quad (\Delta B^1_{t_1}, \Delta B^2_{t_1}) = (-1,-1) \end{cases}$$

contingent on the realization of S_{t_1}. As discussed previously, this model configuration admits the predictable representation property. This implies that we can also express the random payoff C_{t_1} in a pointwise manner as

$$C_{t_1} = \alpha^C_{t_0} + \beta^{C,1}_{t_0}\Delta R^1_{t_1} + \beta^{C,2}_{t_0}\Delta R^2_{t_1} + \hat{\beta}^C_{t_0}\Delta\hat{R}_{t_1}$$

for some $\mathcal{F}_{t_0}$-measurable processes $\alpha^C_{t_0}, \beta^{C,1}_{t_0}, \beta^{C,2}_{t_0}, \hat{\beta}^C_{t_0}$.[4] In order to determine $\alpha^C_{t_0}$,$\beta^{C,1}_{t_0}$,$\beta^{C,2}_{t_0}$, $\hat{\beta}^C_{t_0}$, we have to solve the following system of equations

$$\left\{ \begin{array}{rcl} \alpha^C_{t_0} + \beta^{C,1}_{t_0} + \beta^{C,2}_{t_0} + \hat{\beta}^C_{t_0} & = & 0.7 \\ \alpha^C_{t_0} + \beta^{C,1}_{t_0} - \beta^{C,2}_{t_0} - \hat{\beta}^C_{t_0} & = & 0 \\ \alpha^C_{t_0} - \beta^{C,1}_{t_0} + \beta^{C,2}_{t_0} - \hat{\beta}^C_{t_0} & = & 0.1 \\ \alpha^C_{t_0} - \beta^{C,1}_{t_0} - \beta^{C,2}_{t_0} + \hat{\beta}^C_{t_0} & = & 0 \end{array} \right. \quad \Longrightarrow \quad \left\{ \begin{array}{rcl} \alpha^C_{t_0} & = & 0.2 \\ \beta^{C,1}_{t_0} & = & 0.15 \\ \beta^{C,2}_{t_0} & = & 0.2 \\ \hat{\beta}^C_{t_0} & = & 0.15 \end{array} \right.$$

in which each of the equations is derived in conformity to each realization of states of $(\Delta R^1_{t_1}, \Delta R^2_{t_1})$ at t_1. Moreover, we should note that the present setup is an incomplete market (because one cannot eliminate all the risks associated to the call option, which is driven by two Bernoulli processes, with one underlying asset). For that reason, a unique no-arbitrage price for the call option C is no longer available. Instead, a no-arbitrage bid-ask price bound for the call option C will emerge accordingly. In the remaining

[4] Actually $\alpha^C_{t_0}, \beta^{C,1}_{t_0}, \beta^{C,2}_{t_0}, \hat{\beta}^C_{t_0}$ are simply constants.

sections, we mainly focus on the analysis and computation of the no-arbitrage ask price $C_{t_0}^{\mathrm{NA}}$ out of the no-arbitrage bid-ask price bound as well as *RAROC*-based NGD ask price $C_{t_0}^{\mathrm{NGD}}$ of the call option C in accordance with the theory discussed in earlier sections.

5.4.1 Determination of No-Arbitrage Ask Price

Our first task is to acquire the no-arbitrage ask price $C_{t_0}^{\mathrm{NA}}$. In order to compute such a price for C, we merely solve

$$C_{t_0}^{\mathrm{NA}} = \sup_{Q \in \mathcal{M}^e} \mathbb{E}^Q[C_{t_1} | \mathcal{F}_{t_0}] = \sup_{Q \in \mathcal{M}^e} \mathbb{E}[Z_{t_1} C_{t_1} | \mathcal{F}_{t_0}]$$

by definition. Indeed this can be handled with ease.

Thanks to the representation of the density process Z and the predictable representation property on C, together with the orthogonality between ΔR^1, ΔR^2 and $\Delta \hat{R}$, we can rewrite the stated optimization problem on the right hand side into an equivalent form

$$\begin{aligned} &\underset{q_{t_0}^1, q_{t_0}^2, \hat{q}_{t_0}}{\text{Maximize}} \quad \alpha_{t_0}^C + q_{t_0}^1 \beta_{t_0}^{C,1} + q_{t_0}^2 \beta_{t_0}^{C,2} + \hat{q}_{t_0} \hat{\beta}_{t_0}^C \\ &\text{subject to} \quad \begin{cases} 1 + q_{t_0}^1 \Delta R_{t_1}^1 + q_{t_0}^2 \Delta R_{t_1}^2 + \hat{q}_{t_0} \Delta \hat{R}_{t_1} > 0 \\ \sigma_{t_0} q_{t_0} = -\mu_{t_0} \end{cases} \end{aligned}$$

where the first constraint is due to the requirement that Q is equivalent to P while the second constraint is to ensure S is a martingale under Q. Under the specific set of parameters we assumed, we have to solve explicitly

$$\begin{aligned} &\underset{q_{t_0}^1, q_{t_0}^2, \hat{q}_{t_0}}{\text{Maximize}} \quad 0.2 + 0.15 \cdot q_{t_0}^1 + 0.2 \cdot q_{t_0}^2 + 0.15 \cdot \hat{q}_{t_0} \\ &\text{subject to} \quad \begin{cases} 1 + q_{t_0}^1 \Delta R_{t_1}^1 + q_{t_0}^2 \Delta R_{t_1}^2 + \hat{q}_{t_0} \Delta \hat{R}_{t_1} > 0 \\ 0.3 \cdot q_{t_0}^1 + 0.5 \cdot q_{t_0}^2 = -0.1 \end{cases} \end{aligned}$$

$$\Longrightarrow \quad \begin{aligned} &\underset{\hat{q}_{t_0}}{\text{Maximize}} \quad 0.2 + 0.15 \cdot q_{t_0}^1 + 0.2 \cdot q_{t_0}^2 + 0.15 \cdot \hat{q}_{t_0} \\ &\text{subject to} \quad \begin{cases} 1 + q_{t_0}^1 \Delta R_{t_1}^1 + q_{t_0}^2 \Delta R_{t_1}^2 + \hat{q}_{t_0} \Delta \hat{R}_{t_1} > 0 \\ 0.3 \cdot q_{t_0}^1 + 0.5 \cdot q_{t_0}^2 = -0.1 \end{cases} \end{aligned}$$

Before tackling this problem, we recall that $\hat{R}_{t_1} = R_{t_1}^1 \cdot R_{t_1}^2$. Since it is a finite probability space in the present context, we only need to consider all possible (finite) scenarios actualized by $(\Delta R_{t_1}^1, \Delta R_{t_1}^2, \Delta \hat{R}_{t_1})$, each of which in turn derives an inequality and so the constraint set can be equally well represented as a system of equations. This leads us to the following version of the problem

$$\begin{aligned} &\underset{\hat{q}_{t_0}}{\text{Maximize}} \quad 0.2 + 0.15 \cdot q^1_{t_0} + 0.2 \cdot q^2_{t_0} + 0.15 \cdot \hat{q}_{t_0} \\ &\text{subject to} \quad \begin{cases} 1 + q^1_{t_0} + q^2_{t_0} + \hat{q}_{t_0} > 0 \\ 1 + q^1_{t_0} - q^2_{t_0} - \hat{q}_{t_0} > 0 \\ 1 - q^1_{t_0} + q^2_{t_0} - \hat{q}_{t_0} > 0 \\ 1 - q^1_{t_0} - q^2_{t_0} + \hat{q}_{t_0} > 0 \\ 0.3 \cdot q^1_{t_0} + 0.5 \cdot q^2_{t_0} = -0.1 \end{cases} \end{aligned}$$

The objective function should be maximized over all possible values of $\hat{q}_{t_0}$ belonging to the constraint set in which both $q^1_{t_0}$ and $q^2_{t_0}$ are under the control of $\hat{q}_{t_0}$. After optimizing the objective function over the constrain set, the optimal solution for the maximization problem is determined by $q^*_{t_0} = (-0.125 \quad -0.125 \quad 1)$, which *does not* belong to the constraint set, hence, the optimal value of the objective function is essentially a supremum but not a maximum. Besides, this specification of $q^*_{t_0}$ dictates that the optimal measure $q^*_{t_0}$ is indeed unique! Finally, we can conclude that the no-arbitrage ask price $C^{\text{NA}}_{t_0}$ for C, which is simply the optimal value of the objective function, should be

$$C^{\text{NA}}_{t_0} \approx 0.30625.$$

5.4.2 Determination of RAROC-Based NGD Ask Price

Our next task is to determine the *RAROC*-based NGD ask price. In order to compute the *RAROC*-based NGD ask price $C^{\text{NGD}}_{t_0}$ for C, we shall define values of α and R in advance. In particular, we study two different pairs of their values, which are $(\alpha = 0.01, R = 0.5)$ and $(\alpha = 0.4, R = 0.5)$.

For the case of $(\alpha = 0.01, R = 0.5)$, the corresponding values of r_1 and r_2 are respectively 1.5 and $\frac{1}{34}$. Under general values of r_1 and r_2, we can obtain the RAROC-based ask price $C^{\text{NGD}}_{t_0}$ of C through solving the problem below

$$\begin{aligned} &\underset{q^1_{t_0}, q^2_{t_0}, \hat{q}_{t_0}}{\text{Maximize}} \quad \alpha^C_{t_0} + q^1_{t_0}\beta^{C,1}_{t_0} + q^2_{t_0}\beta^{C,2}_{t_0} + \hat{q}_{t_0}\hat{\beta}^C_{t_0} \\ &\text{subject to} \quad \begin{cases} \frac{1}{r_1} \le 1 + q^1_{t_0}\Delta R^1_{t_1} + q^2_{t_0}\Delta R^2_{t_1} + \hat{q}_{t_0}\Delta \hat{R}_{t_1} \le \frac{1}{r_2} \\ \sigma_{t_0} q_{t_0} = -\mu_{t_0} \end{cases} . \end{aligned}$$

All the upcoming procedures are basically similar to the situation of determining the no-arbitrage ask price $C^{\text{NA}}_{t_0}$. The only difference in computing the two prices is the two slightly dissimilar sets of constraints. Firstly, we aim to reexpress the general problem into the one with $(\alpha = 0.01, R = 0.5)$

$$\underset{\hat{q}_{t_0}}{\text{Maximize}} \quad 0.2 + 0.15 \cdot q_{t_0}^1 + 0.2 \cdot q_{t_0}^2 + 0.15 \cdot \hat{q}_{t_0}$$

$$\text{subject to} \quad \begin{cases} \frac{1}{1.5} \leq 1 + q_{t_0}^1 + q_{t_0}^2 + \hat{q}_{t_0} \leq 34 \\ \frac{1}{1.5} \leq 1 + q_{t_0}^1 - q_{t_0}^2 - \hat{q}_{t_0} \leq 34 \\ \frac{1}{1.5} \leq 1 - q_{t_0}^1 + q_{t_0}^2 - \hat{q}_{t_0} \leq 34 \\ \frac{1}{1.5} \leq 1 - q_{t_0}^1 - q_{t_0}^2 + \hat{q}_{t_0} \leq 34 \\ 0.3 \cdot q_{t_0}^1 + 0.5 \cdot q_{t_0}^2 = -0.1 \end{cases} . \tag{5.4.1}$$

The optimal density process $q_{t_0}^*$ is found to be $q_{t_0}^* = (-0.125 \quad -0.125 \quad 0.3333)$, under the probability measure Q^* is now attainable from $\mathcal{Q}^{\text{ngd}}$. We can also see that uniqueness of the optimal pricing measure Q^* remains valid here. As a result, contrary to the previous case, the *RAROC*-based NGD ask price $C_{t_0}^{\text{NGD}}$ is not only a supremum but also a maximum in the present situation and is calculated as

$$C_{t_0}^{\text{NGD}}(0.01, 0.5) \approx 0.2062.$$

Now if we further increase the value of α while keeping R constant, for instance, $(\alpha = 0.4, R = 0.5)$, we can achieve a reduction in the *RAROC*-based NGD ask price. To understand this in a heuristic manner, we may perceive the value of α as the degree of risk we can tolerate. When the value of α is increased, we essentially reduce our risk limit, i.e. allowing for more risk in the position, thus inducing a decrease in the price. We shall compute analogously the new *RAROC*-based NGD ask price as previous case in order to provide rigorous judgement to this intuition. Here, we make use of Appendix 5.11 for an alternate characterization of the constraint set and reach the resultant optimization problem

$$\underset{\hat{q}_{t_0}}{\text{Maximize}} \quad 0.2 + 0.15 \cdot q_{t_0}^1 + 0.2 \cdot q_{t_0}^2 + 0.15 \cdot \hat{q}_{t_0}$$

$$\text{subject to} \quad \begin{cases} \frac{1}{1.5} \leq 1 + q_{t_0}^1 + q_{t_0}^2 + \hat{q}_{t_0} \leq \frac{3}{2} \\ \frac{1}{1.5} \leq 1 + q_{t_0}^1 - q_{t_0}^2 - \hat{q}_{t_0} \leq \frac{3}{2} \\ \frac{1}{1.5} \leq 1 - q_{t_0}^1 + q_{t_0}^2 - \hat{q}_{t_0} \leq \frac{3}{2} \\ \frac{1}{1.5} \leq 1 - q_{t_0}^1 - q_{t_0}^2 + \hat{q}_{t_0} \leq \frac{3}{2} \\ 0.3 \cdot q_{t_0}^1 + 0.5 \cdot q_{t_0}^2 = -0.1 \end{cases}$$

As a result, we deduce that the *RAROC*-based NGD ask price under ($\alpha = 0.4, R = 0.5$) is

$$C_{t_0}^{\mathrm{NGD}}(0.4, 0.5) \approx 0.1975.$$

This price is attained by using *any* optimal density process $q_{t_0}^*$ from the set given by

$$q_{t_0}^* = \begin{pmatrix} q_{t_0}^1 & -\frac{1}{3} - q_{t_0}^1 & \hat{q}_{t_0} \end{pmatrix} = \begin{pmatrix} -0.0833 & -0.15 & 0.2667 \end{pmatrix}.$$

Note that this set of q_{t_0} belongs to those defined by the constraint set.

5.4.3 Dynamic RAROC-Based Good-Deal Hedging

Apart from the pricing of a contingent claim, it is also crucial to obtain an appropriate hedging strategy for managing the risk of a position. After the demonstration of how to yield no-good-deal prices for a contingent claim, we proceed to the task of exploring the idea of dynamic good-deal hedging. Such a notion was introduced in Becherer (2009). In this approach, hedging strategy is determined in such a way that a certain dynamic coherent risk measure is minimized. Put in another way, this risk measure assesses the residual risk resulting from such a hedging strategy, which is essentially the hedging cost due to the discrepancy between contingent claim payoff and hedging portfolio value, and the value of this assessment minimized among all possible hedging strategies. Moreover it is shown that the value coincides with no-good-deal prices.

To begin with, we introduce the following set of probability measures $\mathcal{P}^{\mathrm{ngd}}$ in a similar vein as Becherer (2009),

$$\mathcal{P}^{\mathrm{ngd}} := \left\{ Q \in \mathcal{P}^a \;\middle|\; \frac{1}{r_1} \leq \frac{Z}{\mathbb{E}^P[Z|\mathcal{F}_t]} \leq \frac{1}{r_2},\ P\text{-a.s.} \quad \forall t \in \mathcal{T} \right\}. \tag{5.4.2}$$

It is reminiscent of $\mathcal{Q}^{\mathrm{ngd}}$ and the key difference is $\mathcal{Q}^{\mathrm{ngd}}$ is required to be a subset of $\mathcal{M}^e$. In other words, $\mathcal{Q}^{\mathrm{ngd}} \subseteq \mathcal{M}^e$ while $\mathcal{P}^{\mathrm{ngd}} \subseteq \mathcal{P}^a$, i.e. S is not necessarily a martingale under a probability measure in $\mathcal{P}^{\mathrm{ngd}}$. We further define the following time-consistent dynamic coherent risk measure ρ given by

$$\rho_t(X) := \operatorname*{ess\,sup}_{Q \in \mathcal{P}^{\mathrm{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t].$$

This is well-defined because $\mathcal{P}^{\mathrm{ngd}}$ is previously shown to be m-stable, see Proposition 5.2.1. With this risk measure, we are going to determine the good-deal hedging strategy

ϕ through minimizing

$$\rho_t\left(X - \int \phi dS\right) = \operatorname*{ess\,sup}_{Q\in\mathcal{P}^{\mathrm{ngd}}} \mathbb{E}^Q\left[X - \int \phi dS \middle| \mathcal{F}_t\right].$$

We again place ourselves in the previous market with concrete parameter values and take $t = t_0$. As it is a one-period model, the stochastic integral $\int \phi dS$ is equivalent to $\phi_{t_0}(S_{t_1} - S_{t_0})$. In order to find out the corresponding good-deal hedging strategy, we suppose ϕ is fixed and then characterize $X - \int \phi dS = C_{t_1} - \phi_{t_0}(S_{t_1} - S_{t_0})$ from the predictable representation property as before

$$\begin{cases} \alpha_{t_0}^{\phi} + \beta_{t_0}^{\phi,1} + \beta_{t_0}^{\phi,2} + \hat{\beta}_{t_0}^{\phi} &= 0.7 - 0.9\phi \\ \alpha_{t_0}^{\phi} + \beta_{t_0}^{\phi,1} - \beta_{t_0}^{\phi,2} - \hat{\beta}_{t_0}^{\phi} &= 0 + 0.1\phi \\ \alpha_{t_0}^{\phi} - \beta_{t_0}^{\phi,1} + \beta_{t_0}^{\phi,2} - \hat{\beta}_{t_0}^{\phi} &= 0.1 - 0.3\phi \\ \alpha_{t_0}^{\phi} - \beta_{t_0}^{\phi,1} - \beta_{t_0}^{\phi,2} + \hat{\beta}_{t_0}^{\phi} &= 0 + 0.7\phi \end{cases} \Longrightarrow \begin{cases} \alpha_{t_0}^{\phi} &= \frac{0.8-0.4\phi}{4} \\ \beta_{t_0}^{\phi,1} &= \frac{0.6-1.2\phi}{4} \\ \beta_{t_0}^{\phi,2} &= \frac{0.8-2\phi}{4} \\ \hat{\beta}_{t_0}^{\phi} &= \frac{0.6}{4} \end{cases}.$$

This yields

$$\begin{aligned} X - \int \phi dS &= C_{t_1} - \phi_{t_0}(S_{t_1} - S_{t_0}) \\ &= \alpha_{t_0}^{\phi} + \beta_{t_0}^{\phi,1}\Delta R_{t_1}^1 + \beta_{t_0}^{\phi,2}\Delta R_{t_1}^2 + \hat{\beta}_{t_0}^{\phi}\Delta \hat{R}_{t_1} \\ &= \frac{0.8 - 0.4\phi}{4} + \frac{0.6 - 1.2\phi}{4}\Delta R_{t_1}^1 + \frac{0.8 - 2\phi}{4}\Delta R_{t_1}^2 + \frac{0.6}{4}\Delta \hat{R}_{t_1}. \end{aligned}$$

Next, since we can represent the density process of any fixed $Q \in \mathcal{P}^{\mathrm{ngd}}$ as $Z = 1 + q_{t_0}^1 \Delta R_{t_1}^1 + q_{t_0}^2 \Delta R_{t_1}^2 + \hat{q}_{t_0}\Delta \hat{R}_{t_1}$, and recall the orthogonality of $\Delta R_{t_1}^1$, $\Delta R_{t_1}^2$, $\Delta \hat{R}_{t_1}$, this permits us to express $\mathbb{E}^Q[X - \int \phi dS | \mathcal{F}_{t_0}]$ in a more explicit fashion, namely,

$$\begin{aligned} &\mathbb{E}^Q\left[X - \int \phi dS \middle| \mathcal{F}_{t_0}\right] \\ &= \mathbb{E}^Q\left[X - \int \phi dS\right] \\ &= \mathbb{E}^P\left[(1 + q_{t_0}^1 \Delta R_{t_1}^1 + q_{t_0}^2 \Delta R_{t_1}^2 + \hat{q}_{t_0}\Delta \hat{R}_{t_1})(\alpha_{t_0}^{\phi} + \beta_{t_0}^{\phi,1}\Delta R_{t_1}^1 + \beta_{t_0}^{\phi,2}\Delta R_{t_1}^2 + \hat{\beta}_{t_0}^{\phi}\Delta \hat{R}_{t_1})\right] \\ &= \alpha_{t_0}^{\phi} + q_{t_0}^1 \cdot \beta_{t_0}^{\phi,1} + q_{t_0}^2 \cdot \beta_{t_0}^{\phi,2} + \hat{q}_{t_0} \cdot \hat{\beta}_{t_0}^{\phi} \\ &= \frac{0.8}{4} + \frac{0.6}{4}\hat{q}_{t_0} + \frac{0.6 - 1.2\phi}{4} q_{t_0}^1 + \frac{0.8 - 2\phi}{4} q_{t_0}^2 - \frac{0.4}{4}\phi. \end{aligned}$$

The good-deal hedging strategy ϕ^* is thus characterized as

$$\begin{aligned}
\phi^* &= \arg\min_{\phi} \rho_{t_0}\left(X - \int \phi dS\right) \\
&= \arg\min_{\phi} \operatorname*{ess\,sup}_{Q\in\mathcal{P}^{\text{ngd}}} \mathbb{E}^Q\left[X - \int \phi dS \middle| \mathcal{F}_{t_0}\right] \\
&= \arg\min_{\phi} \operatorname*{ess\,sup}_{\substack{q^1_{t_0}, q^2_{t_0}, \hat{q}_{t_0} \\ Q\in\mathcal{P}^{\text{ngd}}}} \left(\frac{0.8}{4} + \frac{0.6}{4}\hat{q}_{t_0} + \frac{0.6-1.2\phi}{4}q^1_{t_0} + \frac{0.8-2\phi}{4}q^2_{t_0} - \frac{0.4}{4}\phi\right)
\end{aligned}$$

In view of this, we shall approach the good-deal hedging strategy ϕ^* in two stages. As a first step, suppose ϕ is fixed, we solve the 'inner' maximization problem for the value of $\rho(X - \int \phi dS)$ and once the solution is found, it is a function of ϕ since ϕ is a priori fixed. The 'inner' maximization problem is given by

$$\operatorname*{ess\,sup}_{Q\in\mathcal{P}^{\text{ngd}}} \mathbb{E}^Q\left[X - \int \phi dS \middle| \mathcal{F}_{t_0}\right] = \operatorname*{ess\,sup}_{\substack{q^1_{t_0}, q^2_{t_0}, \hat{q}_{t_0} \\ Q\in\mathcal{P}^{\text{ngd}}}} \left(\frac{0.8}{4} + \frac{0.6}{4}\hat{q}_{t_0} + \frac{0.6-1.2\phi}{4}q^1_{t_0} + \frac{0.8-2\phi}{4}q^2_{t_0} - \frac{0.4}{4}\phi\right)$$

which, in the present set-up, is identical to the following constrained maximization problem

$$\begin{aligned}
&\operatorname*{Maximize}_{\hat{q}_{t_0}, q^1_{t_0}, q^2_{t_0}} \quad \frac{0.8}{4} + \frac{0.6}{4}\hat{q}_{t_0} + \frac{0.6-1.2\phi}{4}q^1_{t_0} + \frac{0.8-2\phi}{4}q^2_{t_0} - \frac{0.4}{4}\phi \\
&\text{subject to} \quad \begin{cases}
\frac{1}{r_1} \le 1 + q^1_{t_0} + q^2_{t_0} + \hat{q}_{t_0} \le \frac{1}{r_2} \\
\frac{1}{r_1} \le 1 + q^1_{t_0} - q^2_{t_0} - \hat{q}_{t_0} \le \frac{1}{r_2} \\
\frac{1}{r_1} \le 1 - q^1_{t_0} + q^2_{t_0} - \hat{q}_{t_0} \le \frac{1}{r_2} \\
\frac{1}{r_1} \le 1 - q^1_{t_0} - q^2_{t_0} + \hat{q}_{t_0} \le \frac{1}{r_2}
\end{cases}.
\end{aligned}$$

Once this problem is tackled, the optimal value ρ is a function of ϕ, $\rho = \rho^*(\phi)$. Then, the second step would be optimizing over all ϕ such that $\rho(\phi) = \rho(X - \int \phi dS)$ is minimized. The corresponding minimizer ϕ^* is defined as the good-deal hedging strategy and we will show that the optimal value $\rho(\phi^*) = \rho(X - \int \phi^* dS)$ coincides with the *RAROC*-based NGD ask price $\pi^u_{t_0}(X)$.

5.4.3.1 Case of $(\alpha = 0.01, R = 0.5)$

After we specify the values of $(\alpha = 0.01, R = 0.5)$, we are required to solve

$$\underset{\hat{q}_{t_0},q^1_{t_0},q^2_{t_0}}{\text{Maximize}} \quad \frac{0.8}{4} + \frac{0.6}{4}\hat{q}_{t_0} + \frac{0.6-1.2\phi}{4}q^1_{t_0} + \frac{0.8-2\phi}{4}q^2_{t_0} - \frac{0.4}{4}\phi$$

$$\text{subject to} \quad \begin{cases} \frac{2}{3} \le 1 + q^1_{t_0} + q^2_{t_0} + \hat{q}_{t_0} \le 34 \\ \frac{2}{3} \le 1 + q^1_{t_0} - q^2_{t_0} - \hat{q}_{t_0} \le 34 \\ \frac{2}{3} \le 1 - q^1_{t_0} + q^2_{t_0} - \hat{q}_{t_0} \le 34 \\ \frac{2}{3} \le 1 - q^1_{t_0} - q^2_{t_0} + \hat{q}_{t_0} \le 34 \end{cases}.$$

We can adopt the equivalent description of the constraint set and have the above problem rewritten as

$$\underset{\hat{q}_{t_0},q^1_{t_0},q^2_{t_0}}{\text{Maximize}} \quad \frac{0.8}{4} + \frac{0.6}{4}\hat{q}_{t_0} + \frac{0.6-1.2\phi}{4}q^1_{t_0} + \frac{0.8-2\phi}{4}q^2_{t_0} - \frac{0.4}{4}\phi$$

$$\text{subject to} \quad \begin{cases} -\frac{1}{3} - \hat{q}_{t_0} \le q^1_{t_0} + q^2_{t_0} \le 33 - \hat{q}_{t_0} \\ -\frac{1}{3} + \hat{q}_{t_0} \le q^1_{t_0} - q^2_{t_0} \le 33 + \hat{q}_{t_0} \\ -\frac{1}{3} + \hat{q}_{t_0} \le -q^1_{t_0} + q^2_{t_0} \le 33 + \hat{q}_{t_0} \\ -\frac{1}{3} - \hat{q}_{t_0} \le -q^1_{t_0} - q^2_{t_0} \le 33 - \hat{q}_{t_0} \\ -\frac{1}{3} \le \hat{q}_{t_0} \le \frac{1}{3} \end{cases}.$$

According to the objective function, we shall differentiate several cases of ϕ, resulting in different optimal values of $\hat{q}_{t_0}, q^1_{t_0} + q^2_{t_0}$. In particular, we consider the cases of $\phi \le 0$, $0 < \phi \le 0.4$, $0.4 < \phi \le 0.5$ and $\phi > 0.5$.

The above problem can be simplified into the following form:

$$\underset{\hat{q}_{t_0},q^1_{t_0},q^2_{t_0}}{\text{Maximize}} \quad \frac{0.8}{4} + \frac{0.6}{4}\hat{q}_{t_0} + \frac{0.6-1.2\phi}{4}q^1_{t_0} + \frac{0.8-2\phi}{4}q^2_{t_0} - \frac{0.4}{4}\phi$$

$$\text{subject to} \quad \begin{cases} -\frac{1}{3} - \hat{q}_{t_0} \le q^1_{t_0} + q^2_{t_0} \le \frac{1}{3} + \hat{q}_{t_0} \\ -\frac{1}{3} + \hat{q}_{t_0} \le q^1_{t_0} - q^2_{t_0} \le \frac{1}{3} - \hat{q}_{t_0} \\ -\frac{1}{3} \le \hat{q}_{t_0} \le \frac{1}{3} \end{cases}.$$

For a given value of ϕ, we can interpret the optimization problem in a geometric manner: for any fixed ϕ, a vector with coordinates $(\frac{0.6}{4}, \frac{0.6-1.2\phi}{4}, \frac{0.8-2\phi}{4})$ is given in the $\hat{q}_{t_0}$-$q^1_{t_0}$-$q^2_{t_0}$ 3-dimensional space. The optimization problem can be viewed as determining the vector over the constraint set such that it produces the maximum projection to the given vector.

The optimization problem is solved by means of numerical methods. We comment that the good-deal hedging strategy ϕ^* for such a market configuration is to hold $\phi^* \approx 0.44$ units of underlying asset S, resulting the amount of expected residual risk $\rho^* = \rho(\phi^*)$ of 0.206. Here we can also justify without difficulty that $\rho^* = \operatorname{ess\,inf}_\phi \rho(\phi) = \pi^u$, i.e. the amount of expected residual risk ρ^* coincides with $RAROC$-based NGD ask price $\pi^u_{t_0}(X)$.

5.4.3.2 *Case of* $(\alpha = 0.4, R = 0.5)$

Next, we shall study the good-deal hedging strategy ϕ^* under a larger value of α with fixed R. Assume the values of $(\alpha = 0.4, R = 0.5)$, we confront with the same form optimization problem under a different set of constraints

$$\underset{\hat{q}_{t_0}, q^1_{t_0}, q^2_{t_0}}{\text{Maximize}} \quad \frac{0.8}{4} + \frac{0.6}{4}\hat{q}_{t_0} + \frac{0.6 - 1.2\phi}{4} q^1_{t_0} + \frac{0.8 - 2\phi}{4} q^2_{t_0} - \frac{0.4}{4}\phi$$

$$\text{subject to} \quad \begin{cases} \frac{2}{3} \leq 1 + q^1_{t_0} + q^2_{t_0} + \hat{q}_{t_0} \leq \frac{3}{2} \\ \frac{2}{3} \leq 1 + q^1_{t_0} - q^2_{t_0} - \hat{q}_{t_0} \leq \frac{3}{2} \\ \frac{2}{3} \leq 1 - q^1_{t_0} + q^2_{t_0} - \hat{q}_{t_0} \leq \frac{3}{2} \\ \frac{2}{3} \leq 1 - q^1_{t_0} - q^2_{t_0} + \hat{q}_{t_0} \leq \frac{3}{2} \end{cases}.$$

By analyzing the constraint, we can further simplify the problem into the following version:

$$\underset{\hat{q}_{t_0}, q^1_{t_0}, q^2_{t_0}}{\text{Maximize}} \quad \frac{0.8}{4} + \frac{0.6}{4}\hat{q}_{t_0} + \frac{0.6 - 1.2\phi}{4} q^1_{t_0} + \frac{0.8 - 2\phi}{4} q^2_{t_0} - \frac{0.4}{4}\phi$$

$$\text{subject to} \quad \begin{cases} -\frac{1}{2} + \hat{q}_{t_0} \leq q^1_{t_0} + q^2_{t_0} \leq \frac{1}{2} - \hat{q}_{t_0} \\ -\frac{1}{3} + \hat{q}_{t_0} \leq q^1_{t_0} - q^2_{t_0} \leq \frac{1}{3} - \hat{q}_{t_0} \\ -\frac{1}{3} \leq \hat{q}_{t_0} \leq \frac{1}{3} \end{cases}.$$

Again we need to consider separately the situations of $\phi \leq 0$, $0 < \phi \leq 0.4$, $0.4 < \phi \leq 0.5$ and $\phi > 0.5$. Since the good-deal hedging strategy ϕ^* is obtained by optimizing $\rho(\phi)$ for a minimum value $\rho^* = \rho(\phi^*)$, by analyzing the respective minimum value of $\rho(\phi)$ over each case, we can arrive at the assertion of holding $\phi^* \approx 0.29$ unit of underlying asset S is the desired good-deal hedging strategy ϕ^* because of the fact that

$$\rho^* = \operatorname*{ess\,inf}_\phi \rho(\phi) = 0.1975 \qquad \text{when} \quad \phi = 0.29.$$

Here, we can again observe the equality of $\rho^* = \operatorname{ess\,inf}_\phi \rho(\phi) = \pi^u$. The amount of expected residual risk ρ^* and $RAROC$-based NGD ask price $\pi^u_{t_0}(X)$ are identical in values.

5.5 Good-Deal Price and Backward Stochastic Differential Equation

In the previous section, we demonstrated how a $RAROC$-based NGD ask price $\pi^u_{t_0}(X)$ is determined as the optimal value of the associated objective function by means of an optimization method. Even though the present example used is simply a one-period model, it has been mentioned earlier that one may break down multi-period time horizon into a number of one-period models and solve each one-period optimization problem, then the resultant measure obtained by pasting each one-period optimal measure $Z^*_{t_n}$ is guaranteed to be the optimal measure Z^* for the multi-period case by the principle of dynamic programming. Regardless of a one-period or multi-period framework, the optimization approach can only provide information about the value of $\pi^u_{t_0}(X)$ but none on the dynamics of $\pi^u_{t_0}(X)$. In other words, the question about how $\pi^u_{t_0}(X)$ evolves with time cannot be answered by simply optimizing over the associated objective function. If one can uncover the dynamics that the $RAROC$-based NGD ask price $\pi^u_{t_0}(X)$ exhibits, provided one exists, one can explore further properties that $\pi^u_{t_0}(X)$ shares. Moreover, whenever the respective optimization problem for $\pi^u_{t_0}(X)$ is too complicated to be solved, one may resort to Monte Carlo simulation methods with respect to its own dynamics for yielding $\pi^u_{t_0}(X)$. This thus underlines the importance of investigating the dynamics of $RAROC$-based NGD ask price $\pi^u_{t_0}(X)$. In a continuous-time setup, Becherer (2009) has obtained a positive answer to this question. He provides definite dynamics for a $RAROC$-based NGD ask price $\pi^u_{t_0}(X)$ under the framework of good-deal pricing. In particular, the dynamics is characterized by the means of a backward stochastic differential equation (BSDE).

There are a few key elements for achieving their results. First of all, they place themselves in a continuous-time Brownian market and it is well-known that the predictable representation property automatically exists under such setup, see Protter (2004) and Revuz and Yor (1999). Such a property is crucial in proving the existence and uniqueness of solution of a given BSDE. Then he derives the corresponding BSDE for $RAROC$-based NGD ask price using the well-known comparison theorem in the theory of BSDE, see Karoui et al. (1997). In view of this, since the predictable representation property is already shown to be at our disposal upon the choice of existing discrete-time model, in order to follow the same strategy, we need to utilize the existence and uniqueness theorem and comparison theorem in the theory of discrete-time BSDE to cope with present discrete-time framework. By surveying literature in this field of interest, some of the work are mainly about the approximation of a continuous-time BSDE, and, the associated numerical methods and convergence, see Ma et al. (2002) for example. Instead of treating the discrete-time BSDE from this perspective, we wish to extract those related to studying discrete-time BSDE, also called BSΔE, as an entity in its own right. Those are the sources for us to gather results regarding existence and uniqueness of solution of a given BSΔE and comparison theorem for BSΔE. Among these, Cohen and Elliott (2010) carry out such an investigation in this direction and so we will adopt their approach. We show that, under the present framework and definition of $\mathcal{Q}^{\text{ngd}}$, we cannot expect to derive the same results as Becherer (2009). Unless the 'geometry' of $\mathcal{Q}^{\text{ngd}}$ is 'nice' enough or the special case of a one-period model is taken, one cannot achieve the same as their results. Hence we will also explore under what situation one would yield identical conclusions as Becherer (2009) by making use of the comparison theorem offered by Cohen and Elliott (2010). Illustration with our one-period model is supplied when we approach the end.

5.6 Theory of Backward Stochastic Difference Equation

As discussed in Becherer (2009), the NGD ask price π_t^u is shown to be intimately related to BSDE. More precisely, π^u can be characterized as the solution of some appropriate BSDE. This thus unveils the dynamic structure of π^u. Since it is a continuous-time setting in Becherer (2009), the continuous-time dynamics of π^u is naturally mapped to a BSDE, while, the analogue object of BSDE in a discrete-time setup is called backward stochastic difference equation (BSΔE). See Cohen and Elliott (2010) for example. Before we follow closely Becherer (2009) to establish a BSΔE for describing the discrete-time dynamics of *RAROC*-based NGD ask price, we shall discuss a few of the theory of BSΔE.

Consider a discrete-time finite state process ΔR and define the process $M_t := \Delta R_t - \mathbb{E}[\Delta R_t|\mathcal{F}_{t-1}]$ which is a martingale, a solution (Y, Z) of a BSΔE means (Y, Z) satisfies an equation of the form

$$\begin{cases} \Delta Y_{t+1}(\omega) = Y_{t+1}(\omega) - Y_t(\omega) = -F(\omega, t, Y_t(\omega), Z_t(\omega))\Delta t + Z_t(\omega)M_{t+1}(\omega) \\ \quad Y_T(\omega) = X(\omega) \end{cases}$$

where F is a given adapted map and X is a given $\mathcal{F}_T$-measurable random variable, called the driver and terminal condition respectively. Most of the time, under a given F and X, the desired solution (Y, Z) is constructed by means of backward induction when the background probability space supports the predictable representation property.

For instance, if we introduce the following quantities

$$\begin{aligned} \xi_{t_n}(Z) &:= \sigma_{t_n}^{\intercal}(\sigma_{t_n}\sigma_{t_n}^{\intercal})^{-1}\mu_{t_n}Z \\ \Pi_{t_n}(Z) &:= \sigma_{t_n}^{\intercal}(\sigma_{t_n}\sigma_{t_n}^{\intercal})^{-1}\sigma_{t_n}Z \\ \Pi_{t_n}^{\perp}(Z) &:= (\mathbf{I} - \Pi_{t_n})(Z) = Z - \Pi_{t_n}(Z) \end{aligned}$$

and consider the following BSΔE:

$$\begin{cases} Y_{t_n} - Y_{t_{n+1}} = -\Delta Y_{t_{n+1}} \\ \qquad\qquad\quad = \Big(-\begin{pmatrix} q^1_{t_n} & q^2_{t_n} & 0 \end{pmatrix}^\intercal \Pi_{t_n}\left(Z_{t_n}\right) + \begin{pmatrix} 0 & 0 & \hat{q}_{t_n} \end{pmatrix}^\intercal \Pi^{\perp}_{t_n}\left(Z_{t_n}\right) \Big) \Delta t - Z_{t_n} \Delta R_{t_{n+1}} \\ \qquad\quad Y_{t_N} = X \end{cases}$$

Suppose there exists a solution, in order to construct the desired solution for this BSΔE, we approach it by making use of the following steps recursively:

i. Suppose $Y_{t_{n+1}}$ is already found, we proceed backward to obtain the appropriate Y_{t_n}.

ii. We consider $Y_{t_{n+1}} - \mathbb{E}^P[Y_{t_{n+1}}|\mathcal{F}_{t_n}]$. This is obviously a martingale and due to the predictable representation property, there exists $Z_{t_n} \in \mathcal{F}_{t_n}$ such that $Y_{t_{n+1}} - \mathbb{E}^P[Y_{t_{n+1}}|\mathcal{F}_{t_n}] = Z_{t_n} \Delta R_{t_{n+1}}$.

iii. With this Z_{t_n}, we substitute into the BSΔE and derive

$$\begin{aligned} Y_{t_n} - Y_{t_{n+1}} &= \Big(-\begin{pmatrix} q^1_{t_n} & q^2_{t_n} & 0 \end{pmatrix}^\intercal \Pi_{t_n}\left(Z_{t_n}\right) + \begin{pmatrix} 0 & 0 & \hat{q}_{t_n} \end{pmatrix}^\intercal \Pi^{\perp}_{t_n}\left(Z_{t_n}\right) \Big) \Delta t \\ &\quad - Z_{t_n} \Delta R_{t_{n+1}} \\ \implies \quad Y_{t_n} - Y_{t_{n+1}} &= \Big(-\begin{pmatrix} q^1_{t_n} & q^2_{t_n} & 0 \end{pmatrix}^\intercal \Pi_{t_n}\left(Z_{t_n}\right) + \begin{pmatrix} 0 & 0 & \hat{q}_{t_n} \end{pmatrix}^\intercal \Pi^{\perp}_{t_n}\left(Z_{t_n}\right) \Big) \Delta t \\ &\quad - Y_{t_{n+1}} + \mathbb{E}^P[Y_{t_{n+1}}|\mathcal{F}_{t_n}] \\ \implies \quad Y_{t_n} &= \Big(-\begin{pmatrix} q^1_{t_n} & q^2_{t_n} & 0 \end{pmatrix}^\intercal \Pi_{t_n}\left(Z_{t_n}\right) + \begin{pmatrix} 0 & 0 & \hat{q}_{t_n} \end{pmatrix}^\intercal \Pi^{\perp}_{t_n}\left(Z_{t_n}\right) \Big) \Delta t \\ &\quad + \mathbb{E}^P[Y_{t_{n+1}}|\mathcal{F}_{t_n}] \end{aligned}$$

The last equality then gives rise to an $\mathcal{F}_{t_n}$-measurable Y_{t_n}.

iv. Proceeding backward from t_N to t_0 and aggregating all $(Y_{t_n}, Z_{t_n})_{0 \le n \le N}$, we conclude that the BSΔE is solved and the solution is found (by making use of $(Y_{t_n}, Z_{t_n})_{0 \le n \le N}$).

Of course, it is not guaranteed that there always exists a solution for any given type of BSΔE. This thus leads to a theorem regarding the necessary and sufficient conditions for ensuring the existence and uniqueness of a solution of the BSΔE. We recall this important result from Cohen and Elliott (2010).

Definition 5.6.1. For two processes Z^1, Z^2, we shall write $Z^1 \sim_M Z^2$ if $Z^1_t M_{t+1} = Z^2_t M_{t+1}$ P-a.s. holds for all $t = 0, 1, \dots, T-1$.

Theorem 5.6.1. *Suppose F is such that the following two assumptions hold:*

i. For any Y, if $Z^1 \sim_M Z^2$, then $F(\omega, t, Y_t, Z^1_t) = F(\omega, t, Y_t, Z^2_t)$ P-a.s. *for all t.*

ii. For any $z \in \mathbb{R}^{K\times N}$*, for all* t*, for* P*-almost all* ω*, the map*

$$y \mapsto y - F(\omega, t, y, z)$$

is a bijection $\mathbb{R}^K \to \mathbb{R}^K$*.*

Then for any terminal condition X *essentially bounded,* $\mathcal{F}_T$*-measurable, and with values in* $\mathbb{R}^K$*, the BS*Δ*E has an adapted solution* (Y, Z)*. Moreover, this solution is unique up to indistinguishability for* Y *and equivalence* $\sim_M$ *for* Z*.*

Apart from the existence and uniqueness theorem above, there is another notable theorem underlying the theory of BSDEs and BSΔEs, which is the so-called Comparison Theorem. We adopt the comparison theorem for BSΔE in Cohen and Elliott (2010).

Theorem 5.6.2. *Consider two BS*Δ*Es corresponding to drivers* F^1, F^2 *and terminal conditions* X^1, X^2*. Suppose both BS*Δ*Es admit unique solutions and let* (Y^1, Z^1) *and* (Y^2, Z^2) *be the respective solutions. Suppose the following conditions hold:*

i. $X^1 \geq X^2$ P-a.s.

ii. P-a.s.*, for all* t*,*

$$F^1(\omega, t, Y_t^2, Z_t^2) \geq F^2(\omega, t, Y_t^2, Z_t^2).$$

iii. P-a.s.*, for all* t*, for all* i*, the* i*-th component of* F^1*, given by* $e_i^\top F^1$*, satisfies*

$$e_i^\top F^1(\omega, t, Y_t^2, Z_t^1) - e_i^\top F^1(\omega, t, Y_t^2, Z_t^2) \geq \min_{j\in\mathbb{J}_t} \left\{ e_i^\top \left(Z_t^1 - Z_t^2\right)\left(r_j - \mathbb{E}[\Delta R_{t+1}|\mathcal{F}_t]\right)\right\}.$$

iv. P-a.s.*, for all* t*, if*

$$Y_t^1 - F^1(\omega, t, Y_t^1, Z_t^1) \geq Y_t^2 - F^1(\omega, t, Y_t^2, Z_t^1)$$

then $Y_t^1 \geq Y_t^2$*.*

It is then true that $Y^1 \geq Y^2$ P-a.s.

Here, e_i is a n-dimensional vector with all zeros except the i-th component and $\mathbb{J}_t$ at each t is defined as $\mathbb{J}_t := \{i \,:\, P(\Delta R_{t+1} = r_i|\mathcal{F}_t) > 0\}$, i.e. the collection of all states of ΔR_{t+1} at $t+1$ that have positive probability of occurrence conditional on $\mathcal{F}_t$ at t.

5.7 Relate the NGD Price to Backward Stochastic Differential Equations

We recall some notations introduced in the previous section

$$\begin{aligned} \xi_{t_n} &\coloneqq \sigma_{t_n}^\top(\sigma_{t_n}\sigma_{t_n}^\top)^{-1}\mu_{t_n}, \\ \Pi_{t_n}(Z) &\coloneqq \sigma_{t_n}^\top(\sigma_{t_n}\sigma_{t_n}^\top)^{-1}\sigma_{t_n}Z, \\ \Pi_{t_n}^\perp(Z) &\coloneqq (\mathbf{I}-\Pi_{t_n})(Z) = Z - \Pi_{t_n}(Z). \end{aligned}$$

In fact, ξ_{t_n} is the market price of risk process, $\Pi_{t_n}(Z)$ denotes the orthogonal projection of Z onto $I_{t_n} = \operatorname{Im}\sigma_{t_n}^\top$ and $\Pi_{t_n}^\perp(Z)$ represents the orthogonal projection of Z onto $K_{t_n} = \operatorname{Ker}\sigma_{t_n}$. As a result, it is readily seen that $Z = \Pi_{t_n}(Z) + \Pi_{t_n}^\perp(Z)$ holds.

In Becherer (2009), they firstly show that, for any given $Q \sim P$ with density process $\frac{dQ}{dP} = \mathcal{E}(\int \lambda dW)$ where λ is a predictable and bounded process, the dynamics of $Y_t = \mathbb{E}^Q[X|\mathcal{F}_t]$ satisfies the following BSDE

$$\begin{cases} -dY_t = \lambda_t^\top Z_t dt - Z_t dW_t \\ Y_T = X \end{cases}.$$

Moreover, by using the orthogonal projection Π_t, λ can be further expressed as $\lambda = -\xi + \eta$ where $-\xi = \Pi(\lambda)$ and $\eta \coloneqq \Pi^\perp(\lambda)$, that is, the sum of the market price of risk $-\xi$ and an orthogonal component η. Together with the orthogonal decomposition of $Z = \Pi_t(Z) + \Pi_t^\perp(Z)$, the 'dot product' $\lambda_t^\top Z_t$ is essentially the same as $\lambda_t^\top Z_t = -\xi_t^\top \Pi_t(Z) + \eta_t^\top \Pi_t^\perp(Z)$ due to the orthogonality of $\Pi_t(\cdot)$ and $\Pi_t^\perp(\cdot)$. This gives rise to an equivalent form of BSDE

$$\begin{cases} -dY_t = \left(-\xi_t^\top \Pi_t(Z) + \eta_t^\top \Pi_t^\perp(Z)\right) dt - Z_t dW_t \\ Y_T = X \end{cases}. \tag{5.7.1}$$

For any $Q \in \mathcal{P}^e$, we may identify Q with its density process $\frac{dQ}{dP}$, since $\frac{dQ}{dP} = \mathcal{E}(\int \lambda dW)$ in the setup, Q can also be identified by λ. Furthermore, as ξ is fixed, we can further see from $\lambda = -\xi + \eta$ that Q can indeed be parameterized by η. So, from here onwards, we will speak of $Q/\frac{dQ}{dP}/\lambda/\eta$ interchangeably. As a result, in order to have $Q \in \mathcal{Q}^{\text{ngd}}$, there should be some conditions $\mathcal{E}^{\text{ngd}}$ imposed on η, see Proposition 4.0.11 in Becherer (2009). The same interchangeability is applied to the discrete-time model we considered, in which q is the analogue instance for λ and $Q/\frac{dQ}{dP}/q$ are describing the same object.

The BSDE in (5.7.1) can be seen as a system of BSDEs parameterized by η. Denote (Y^η, Z^η) as the solution of BSDE in 5.7.1 under a fixed η. Since the good-deal ask price π_t^u is defined as $\pi_t^u := \operatorname{ess\,sup}_{Q\in\mathcal{Q}^{\text{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t]$, where $\operatorname{ess\,sup}_{Q\in\mathcal{Q}^{\text{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t]$ is equivalent to $\operatorname{ess\,sup}_{\eta\in\mathcal{E}^{\text{ngd}}} Y^\eta$, so once $Y_t = \operatorname{ess\,sup}_{\eta\in\mathcal{E}^{\text{ngd}}} Y^\eta$ is found, we can obtain $\pi_t^u = Y_t$. In order to yield the desired Y_t under the class of BSDE solution Y^η, since the dynamics of Y^η is known, it is natural to employ the comparison theorem. Namely, if some η^* is found such that it maximizes the driver $-\xi_t^\intercal \Pi_t(Z) + \eta_t^\intercal \Pi_t^\perp(Z)$ at Z over all other η in a P-a.s. manner, i.e.

$$-\xi_t^\intercal \Pi_t(Z) + {\eta^*}_t^\intercal \Pi_t^\perp(Z) \geq -\xi_t^\intercal \Pi_t(Z) + \eta_t^\intercal \Pi_t^\perp(Z) \qquad \text{for any} \quad \eta \in \mathcal{E}^{\text{ngd}}$$

then $Y_t = \operatorname{ess\,sup}_{\eta\in\mathcal{E}^{\text{ngd}}} Y^\eta$ holds where (Y, Z) is the solution of BSDE in (5.7.1) under $\eta = \eta^*$, see Proposition 4.0.13 in Becherer (2009). Based on this, we are led to their conclusion of

$$\eta^* = \frac{\sqrt{h_t^2 - \|\xi_t\|^2}}{\|\Pi_t^\perp(Z)\|} \Pi_t^\perp(Z)$$

with dynamics of $Y_t = \operatorname{ess\,sup}_{\eta\in\mathcal{E}^{\text{ngd}}} Y^\eta$ given by

$$\begin{cases} -dY_t = \left(-\xi_t^\intercal \Pi_t(Z) + \sqrt{h_t^2 - \|\xi_t\|^2} \|\Pi_t^\perp(Z)\| \right) dt - Z_t dW_t \\ \quad Y_T = X \end{cases}. \tag{5.7.2}$$

We shall now begin the investigation of whether the same results are found in our context of good-deal pricing. As a first step, we prove that

Proposition 5.7.1. *Let $Q \sim P$ with density*

$$\frac{dQ}{dP} =: D = \prod_{0\leq i\leq N-1} \left(1 + q_{t_i}^1 \Delta R_{t_{i+1}}^1 + q_{t_i}^2 \Delta R_{t_{i+1}}^2 + \hat{q}_{t_i} \Delta \hat{R}_{t_{i+1}}\right).$$

Then there exists a unique solution (Y, Z) to the BSΔE

$$\begin{cases} \Delta Y_{t+1} = Y_{t+1} - Y_t = -q_t^\intercal Z_t \Delta t + Z_t \Delta R_t \\ \quad Y_T = X \end{cases}$$

and Y_t is given by

$$Y_t = \mathbb{E}^Q[X|\mathcal{F}_t].$$

Proof. In accordance with the notation defined in Theorem 5.6.1, we have $M_t := \Delta R_t - \mathbb{E}[\Delta R_t|\mathcal{F}_{t-1}] = \Delta R_t$ because of $\mathbb{E}[\Delta R_t|\mathcal{F}_{t-1}] = 0$ in present model. We then take $F(\omega, t, Y_t, Z_t^1) = F(\omega, t, Z_t^1) = q_t^\intercal Z_t$. To verify condition (i) in Theorem 5.6.1, assume $Z^1 \sim_M Z^2$ holds, i.e. $Z^1 M = Z^2 M$ P-a.s., as in present probability space, each state of M is of positive probability, we have $Z^1 M = Z^2 M$ P-a.s. equivalent to $Z^1(\omega)M(\omega) =$

$Z^2(\omega)M(\omega)$ for each realization of M. So instead of treating $Z^1(\omega)M(\omega) = Z^2(\omega)M(\omega)$ and M as random variables, we may regard them as matrices. Particularly we have

$$M = \Delta t \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

in which each column represents each of its realizations. This matrix has full row rank, hence, we can define the right pseudo-inverse M_R^{-1}, see Laub (2005), as

$$M_R^{-1} = M^\intercal (MM^\intercal)^{-1}.$$

Similarly, we also regard q and Z^1, Z^2 as matrices in which each column represents each of its realizations. By the property of right pseudo-inverses, $MM_R^{-1} = \mathbf{1}$, we have, in matrix sense,

$$\begin{aligned} q_t^\intercal (Z_t^1 - Z_t^2) &= q_t^\intercal (Z_t^1 - Z_t^2) M M_R^{-1} \\ &= q_t^\intercal \underbrace{(Z_t^1 - Z_t^2)M}_{=0} M_R^{-1} = 0 \end{aligned}$$

due to the assumption of $Z^1 \sim_M Z^2$. This implies $q_t^\intercal Z_t^1 = q_t^\intercal Z_t^2$, hence, $F(\omega, t, Y_t, Z_t^1) = F(\omega, t, Y_t, Z_t^2)$ P-a.s. is justified. For the second condition in Theorem 5.6.1, it is trivial that the map $y \mapsto y - q^\intercal z$ is a bijection with respect to variable y because $q^\intercal z$ is independent of y. Consequently we conclude the existence of a solution of the stated BSΔE by Theorem 5.6.1.

To prove the second claim, we apply Ito's Lemma to the process DY

$$\begin{aligned} \Delta(D_t Y_t) &= D_{t-1}\Delta Y_t + Y_{t-1}\Delta D_t + \Delta D_t \Delta Y_t \\ &= D_{t-1}(-q_t^\intercal Z_t \Delta t + Z_t \Delta R_t) + Y_{t-1}(D_{t-1} q_t^\intercal \Delta R_t) + D_{t-1} q_t^\intercal Z_t (\Delta R_t)^2 \\ &= D_{t-1}(-q_t^\intercal Z_t \Delta t + Z_t \Delta R_t) + Y_{t-1}(D_{t-1} q_t^\intercal \Delta R_t) + D_{t-1} q_t^\intercal Z_t \Delta t \\ &= D_{t-1} Z_t \Delta R_t + Y_{t-1} D_{t-1} q_t^\intercal \Delta R_t \end{aligned}$$

where we have used $D_t - D_{t-1} = D_{t-1} q_t^\intercal \Delta R_t$. In view of this, we conclude that DY is a P-martingale. Hence,

$$D_t Y_t = \mathbb{E}^P[D_T Y_T | \mathcal{F}_t] \quad \Longrightarrow \quad Y_t = \mathbb{E}^P\left[\frac{D_T}{D_t} X \middle| \mathcal{F}_t\right] = \mathbb{E}^Q[X|\mathcal{F}_t].$$

□

Furthermore, for any $Q \in \mathcal{P}^e$, so is $Q \in \mathcal{Q}^{\text{ngd}}$, in our model, we still have its density decomposed as $q = \xi + \eta$. As a result, it may be possible that we obtain the same result as Becherer (2009) on the dynamics of good-deal prices. Unfortunately this is generally not true because of the potential failure in applying the comparison theorem for obtaining the dynamics of $Y_t = \operatorname{ess\,sup}_{\eta\in\mathcal{E}^{\text{ngd}}} Y^\eta$. Firstly we should note that for any $Q \in \mathcal{Q}^{\text{ngd}}$, it is regardless of any model that we can write the dynamics of $Y_t^\eta = \mathbb{E}^Q[X|\mathcal{F}_t]$ from

$$\begin{cases} \Delta Y_{t+1} = Y_{t+1} - Y_t = -q_t^\mathsf{T} Z_t \Delta t + Z_t \Delta R_t \\ \quad Y_T = X \end{cases}$$

to

$$\begin{cases} \Delta Y_{t+1} = Y_{t+1} - Y_t = -\left(-\xi_t^\mathsf{T}\Pi_t(Z_t) + \eta_t^\mathsf{T} \cdot \Pi_t^\perp(Z_t)\right)\Delta t + Z_t \Delta R_t \\ \quad Y_T = X \end{cases}$$

due to the orthogonal decomposition under the action of Π. So in order to arrive at $Y_t = \operatorname{ess\,sup}_\eta Y^\eta$ by applying the comparison theorem, the remaining task is to determine the optimal η^* such that it maximizes, for fixed Z_t, $\eta_t^\mathsf{T} \cdot \Pi_t^\perp(Z_t)$ over all η, according to earlier discussion just before Proposition 5.7.1. If a continuous-time setting is used, this will be sufficient when one applies the comparison theorem for continuous-time BSDEs. However, it is not the case when one applies the comparison theorem, Theorem 5.6.2, for discrete-time BSDE. A counterexample is shown in Cohen and Elliott (2010) (Example 2) to demonstrate that the conclusion of Theorem 5.6.2, $Y^1 \geq Y^2$, can be invalid if only Assumption (i), (ii) and (iv) are met. As a result, further requirements on η^* should be imposed for successful application of Theorem 5.6.2 and desired comparison results. We shall discuss each assumption defined in Theorem 5.6.2 in order to apply the theorem to derive the correct η^*.

For a fixed $\eta^* \in \mathcal{Q}^{\text{ngd}}$, we denote (Y^*, Z^*) as the solution of the BSΔE with driver $F^{\eta^*}(\omega, t, Y(\omega), Z(\omega)) = F^{\eta^*}(\omega, t, Z(\omega)) = -\xi_t^\mathsf{T}\Pi_t(Z_t) + \eta_t^{*\mathsf{T}} \cdot \Pi_t^\perp(Z_t)$ and terminal condition $Y^{\eta^*} = X$, i.e. satisfying

$$\begin{cases} \Delta Y_{t+1} = Y_{t+1} - Y_t = -\left(-\xi_t^\mathsf{T}\Pi_t(Z_t) + \eta_t^{*\mathsf{T}} \cdot \Pi_t^\perp(Z_t)\right)\Delta t + Z_t \Delta R_t \\ \quad Y_T = X \end{cases}$$

and (Y^η, Z^η) as that of the BSΔE with driver $F^\eta(\omega, t, Y(\omega), Z(\omega)) = F^\eta(\omega, t, Z(\omega)) = -\xi_t^\mathsf{T}\Pi_t(Z_t) + \eta_t^\mathsf{T} \cdot \Pi_t^\perp(Z_t)$ and terminal condition $Y^\eta = X$, i.e. satisfying

$$\begin{cases} \Delta Y_{t+1} = Y_{t+1} - Y_t = -\left(-\xi_t^\mathsf{T}\Pi_t(Z_t) + \eta_t^\mathsf{T} \cdot \Pi_t^\perp(Z_t)\right)\Delta t + Z_t \Delta R_t \\ \quad Y_T = X \end{cases}$$

In addition, we suppose that, at such $\eta^* \in \mathcal{Q}^{\mathrm{ngd}}$, it maximizes $\eta_t^\intercal \cdot \Pi_t^\perp(Z_t^\eta)$ evaluated at fixed Z_t^η over all η. We shall justify Assumption (i) to (iv) in Theorem 5.6.2 below to understand the requirements of η^* such that we are permitted to apply Theorem 5.6.2 and conclude that $Y^* = \operatorname{ess\,sup}_\eta Y^\eta$ holds.

Assumption (i) is trivially satisfied because $Q^{\eta^*} = X = Q^\eta$.

The validity of Assumption (iv) is immediate once we recognize the fact that both drivers F^{η^*} and F^η are independent of Y, hence

$$\begin{aligned} & Y^{\eta^*} - F^{\eta^*}\left(\omega, t, Y^{\eta^*}(\omega), Z^{\eta^*}(\omega)\right) \geq Y^\eta - F^{\eta^*}\left(\omega, t, Y^\eta(\omega), Z^{\eta^*}(\omega)\right) \\ \Longrightarrow\quad & Y^{\eta^*} - F^{\eta^*}\left(\omega, t, Z^{\eta^*}(\omega)\right) \geq Y^\eta - F^{\eta^*}\left(\omega, t, Z^{\eta^*}(\omega)\right) \\ \Longrightarrow\quad & Y^{\eta^*} \geq Y^\eta. \end{aligned}$$

For Assumption (ii), by the definition of η^*, the optimizer of the function $\eta_t^\intercal \cdot \Pi_t^\perp(Z_t^\eta)$ at fixed Z_t^η over all η, along with the independence of Y in both F^{η^*} and F^η, ensures that

$$\begin{aligned} F^{\eta^*}(\omega, t, Y^\eta(\omega), Z^\eta(\omega)) &= F^{\eta^*}(\omega, t, Z^\eta(\omega)) \\ &\geq F^\eta(\omega, t, Z^\eta(\omega)) = F^\eta(\omega, t, Y^\eta(\omega), Z^\eta(\omega)). \end{aligned}$$

Assumption (iii) is crucial for the success in applying the Comparison Theorem. This assumption imposes further structure on η^* apart from its maximization of the product $\eta_t^\intercal \cdot \Pi_t^\perp(Z_t^\eta)$. Recall the details of this assumption

$$e_i^\intercal F^1(\omega, t, Y_t^2, Z_t^1) - e_i^\intercal F^1(\omega, t, Y_t^2, Z_t^2) \geq \min_{j \in \mathbb{J}_t} \left\{ e_i^\intercal \left(Z_t^1 - Z_t^2\right)\left(r_j - \mathbb{E}[\Delta R_{t+1} | \mathcal{F}_t]\right) \right\}.$$

Here, $r_j - \mathbb{E}[\Delta R_{t+1}|\mathcal{F}_t]$ refers to the realization of M_{t+1} at $t+1$ and we have $r_j - \mathbb{E}[\Delta R_{t+1}|\mathcal{F}_t] = r_j$ because of $\mathbb{E}[\Delta R_{t+1}|\mathcal{F}_t] = 0$ under the present setup (in other words, $M_{t+1} = \Delta R_{t+1}$ is valid, hence the realization of M_{t+1} also means that of ΔR_{t+1}). Moreover, $\mathbb{J}_t$ at each t is defined as $\mathbb{J}_t := \{i : P(\Delta R_{t+1} = r_i|\mathcal{F}_t) > 0\}$, i.e. the collection of all states of ΔR_{t+1} at $t+1$ that have positive probability of occurrence conditional on $\mathcal{F}_t$. Returning to Section 5.1, we can observe that it is now a Markovian model with transition probability homogenous in time and independent of states. According to this, since there are four possible states in ΔR_{t+1}, $\mathbb{J}_t$ should include all these four states at every time t. Together with F^{η^*}, F^η being one-dimensional, the above can essentially be written as

$$F^{\eta^*}(\omega, t, Y_t^\eta, Z_t^{\eta^*}) - F^{\eta^*}(\omega, t, Y_t^\eta, Z_t^\eta) \geq \min_{j \in \mathbb{J}_t} \left\{ \left(Z_t^{\eta^*} - Z_t^\eta\right) r_j \right\}$$

or

$$F^{\eta^*}(\omega,t,Y_t^\eta,Z_t^{\eta^*}) - F^{\eta^*}(\omega,t,Y_t^\eta,Z_t^\eta) \geq \left\{\left(Z_t^{\eta^*} - Z_t^\eta\right)\Delta R_{t+1}\right\} \qquad P\text{-a.s.}$$

Equivalently, at each $\omega \in \Omega$, the above can be expressed as

$$(\eta_t^*)^\mathsf{T} \cdot \left(\Pi_t^\perp(Z_t^{\eta^*}) - \Pi_t^\perp(Z_t^\eta)\right) \geq \min\left\{\begin{array}{c} (z_1^{\eta^*}-z_1^\eta)+(z_2^{\eta^*}-z_2^\eta)+(z_3^{\eta^*}-z_3^\eta), \\ (z_1^{\eta^*}-z_1^\eta)-(z_2^{\eta^*}-z_2^\eta)-(z_3^{\eta^*}-z_3^\eta), \\ -(z_1^{\eta^*}-z_1^\eta)+(z_2^{\eta^*}-z_2^\eta)-(z_3^{\eta^*}-z_3^\eta), \\ -(z_1^{\eta^*}-z_1^\eta)-(z_2^{\eta^*}-z_2^\eta)+(z_3^{\eta^*}-z_3^\eta) \end{array}\right\}$$

$$\Longrightarrow \qquad (\eta_t^*)^\mathsf{T} \cdot \left(\Pi_t^\perp(Z_t^{\eta^*} - Z_t^\eta)\right) \geq \min\left\{\begin{array}{c} (z_1^{\eta^*}-z_1^\eta)+(z_2^{\eta^*}-z_2^\eta)+(z_3^{\eta^*}-z_3^\eta), \\ (z_1^{\eta^*}-z_1^\eta)-(z_2^{\eta^*}-z_2^\eta)-(z_3^{\eta^*}-z_3^\eta), \\ -(z_1^{\eta^*}-z_1^\eta)+(z_2^{\eta^*}-z_2^\eta)-(z_3^{\eta^*}-z_3^\eta), \\ -(z_1^{\eta^*}-z_1^\eta)-(z_2^{\eta^*}-z_2^\eta)+(z_3^{\eta^*}-z_3^\eta) \end{array}\right\} \tag{5.7.3}$$

where $(z_1^{\eta^*} \quad z_2^{\eta^*} \quad z_3^{\eta^*})$ and $(z_1^\eta \quad z_2^\eta \quad z_3^\eta)$ are the respective values of $Z_t^{\eta^*}$ and Z_t^η realized at ω.

Consequently when we are able to find η^* that maximizes the product $\eta_t^\mathsf{T} \cdot \Pi_t^\perp(Z_t^\eta)$ at fixed Z_t^η over all η and satisfies (5.7.3) at the same time, we may apply Theorem 5.6.2 to conclude that $\pi_t^u = Y^* = \operatorname{ess\,sup}_\eta Y^\eta$ with dynamics described by

$$\begin{cases} \Delta Y_{t+1} = Y_{t+1} - Y_t = -\left(-\xi_t^\mathsf{T}\Pi_t(Z_t) + \eta_t^{*\mathsf{T}} \cdot \Pi_t^\perp(Z_t)\right)\Delta t + Z_t \Delta R_t \\ \quad Y_T = X \end{cases}.$$

This is not yet the same result as Becherer (2009) (c.f. (5.7.2)). We have to proceed one step further to determine that the dynamics of Y^* should be

$$\begin{cases} \Delta Y_{t+1} = Y_{t+1} - Y_t = -\left(-\xi_t^\mathsf{T}\Pi_t(Z_t) + \|\eta_t^*\|\|\Pi_t^\perp(Z_t)\|\right)\Delta t + Z_t \Delta R_t \\ \quad Y_T = X \end{cases}.$$

Whether one can make this assertion or not depends critically on the 'geometry' of η. Generally speaking, the product $\eta_t^\mathsf{T} \cdot \Pi_t^\perp(Z_t)$ equals to $\|\eta_t\|\|\Pi_t^\perp(Z_t^\eta)\|\cos\theta$ where θ is the angle between η_t and $\Pi_t^\perp(Z_t^\eta)$. Maximizing $\eta_t^\mathsf{T} \cdot \Pi_t^\perp(Z_t^\eta)$ at fixed Z_t^η can therefore be regarded as an optimization problem over (l^η, θ^η) where $l^\eta := \|\eta_t\|$ and θ^η is the angle between η_t and $\Pi_t^\perp(Z_t^\eta)$. This two-dimensional optimization problem can be reduced to a one-dimensional problem when the 'geometry' of η is 'nice' enough. For instance, if it is a solid circle centered at origin, then we can focus on those η along the boundary of the circle because they have the maximum length l^η. Moreover, because of the circular geometry, there always exists an element of η which lies in the same direction of any given vector, hence we are guaranteed to locate some η^* such that the maximum value of $\eta_t^\intercal \cdot \Pi_t^\perp(Z_t^\eta)$ is exactly $\|\eta_t^*\|\|\Pi_t^\perp(Z_t^\eta)\|$. As a matter of fact, this is also the observation that permits one to locate the corresponding η_t^* and arrive at the BSDE of good-deal price found in Becherer (2009). More precisely, η derived from the set $\mathcal{Q}^{\mathrm{ngd}}$ defined in Becherer (2009) is characterized in terms of norm, see Proposition 4.0.11 in Becherer (2009) which states that $\|\eta\|^2 \leq h^2 - \|\xi\|^2$ for some given h where ξ is the market price of risk. In light of this, it is obvious that the 'geometry' of η is a circle centered at the origin.

However, in our context, the characterization of $\mathcal{Q}^{\mathrm{ngd}}$ is described in a pointwise manner. This usually prevents the geometry of the derived η from being a 'circular' one as before. This thus makes the optimization of the product $\eta_t^\intercal \cdot \Pi_t^\perp(Z_t^\eta)$ at fixed Z_t^η non-trivial in the sense that we cannot ensure that there exists a η^p lying in the same direction as $\Pi_t^\perp(Z_t^\eta)$. Even it exists, the maximum value of the product $\eta_t^\intercal \cdot \Pi_t^\perp(Z_t^\eta)$ cannot be guaranteed at η^*. There might exist some non-parallel η^{np} that further maximizes the product, for example, the value of the product at this non-parallel η^{np} can be greater than that from a parallel η^p as long as the value of $\|\eta^{np}\|\cos\theta$ is larger than $\|\eta^p\|$. In other words, the maximization problem of $\eta_t^\intercal \cdot \Pi_t^\perp(Z_t^\eta)$ over all (l^η, θ^η) from $\mathcal{Q}^{\mathrm{ngd}}$ cannot reproduce the dynamics of Y^* as in Becherer (2009). Of course, there may still exist circumstances under which the same dynamics of Y^* can be obtained. Such a circumstance is related to a 'simple' geometry of η_t as shown in next section.

5.7.1 Investigation on the One-Period Model

If we are content with the existing one-period model, we shall be able to derive dynamics of π_t^u as (5.7.2). Firstly we compute explicitly the following quantities

$$\xi_{t_n} := \sigma_{t_n}^\intercal(\sigma_{t_n}\sigma_{t_n}^\intercal)^{-1}\mu_{t_n} = \frac{\mu}{\sigma_1^2+\sigma_2^2}\begin{pmatrix}\sigma_1\\ \sigma_2\\ 0\end{pmatrix}$$

$$\Pi_{t_n} := \sigma_{t_n}^\intercal(\sigma_{t_n}\sigma_{t_n}^\intercal)^{-1}\sigma_{t_n} = \frac{1}{\sigma_1^2+\sigma_2^2}\begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2 & 0\\ \sigma_1\sigma_2 & \sigma_2^2 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \text{and} \quad \Pi_{t_n}^\perp := \mathbf{I} - \Pi_{t_n} = \begin{pmatrix}\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} & -\frac{\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2} & 0\\ -\frac{\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2} & \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Hence, for any $Z = (z_1 \quad z_2 \quad z_3)$, the corresponding orthogonal projections $\Pi_{t_n}(Z)$ and $\Pi^{\perp}_{t_n}(Z)$ are

$$\Pi_{t_n}(Z) = \begin{pmatrix} \frac{\sigma_1^2 z_1 + \sigma_1\sigma_2 z_2}{\sigma_1^2+\sigma_2^2} \\ \frac{\sigma_1\sigma_2 z_1 + \sigma_2^2 z_2}{\sigma_1^2+\sigma_2^2} \\ 0 \end{pmatrix} \qquad \text{and} \qquad \Pi^{\perp}_{t_n}(Z) = \begin{pmatrix} \frac{\sigma_2^2 z_1 - \sigma_1\sigma_2 z_2}{\sigma_1^2+\sigma_2^2} \\ \frac{\sigma_1^2 z_2 - \sigma_1\sigma_2 z_1}{\sigma_1^2+\sigma_2^2} \\ z_3 \end{pmatrix}.$$

Furthermore, for any $Q \in \mathcal{M}^e$, its corresponding density process q can be decomposed as $q = -\xi + \eta$, which gives

$$\begin{pmatrix} q^1 \\ q^2 \\ \hat{q} \end{pmatrix} = -\frac{\mu}{\sigma_1^2+\sigma_2^2} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \eta^1 \\ \eta^2 = -\frac{\sigma_1}{\sigma_2}\eta^1 \\ \hat{\eta} \end{pmatrix} \qquad \eta^1, \hat{\eta} \in \mathbb{R}$$

where the first component is the minimal martingale measure and the second component is obtained due to $\sigma_1 q_1 + \sigma_2 q_2 = -\mu$ and $\xi^{\mathsf{T}}\eta = 0$ where $\eta = (\eta^1 \quad \eta^2 \quad \hat{\eta})^{\mathsf{T}}$.

Consider the one-period model we assumed in Section 5.4, we may then use the Comparison Theorem because Assumption (iii) discussed in (5.7.3) is no longer a constraint to us. It can be satisfied readily. To see this, observe that for any η, Z^{η} is determined through predictable representation property, i.e.

$$Y_{t_{n+1}} - \mathbb{E}^P[Y_{t_{n+1}}|\mathcal{F}_{t_n}] = Z^{\eta}_{t_n}\Delta R_{t_{n+1}}.$$

As it is only a one-period model, we thus have

$$Y_{t_1} - \mathbb{E}^P[Y_{t_1}|\mathcal{F}_{t_0}] = Z^{\eta}_{t_0}\Delta R_{t_1}.$$

Since $Y_{t_1} = X$, by the uniqueness of predictable representation property, we conclude that, for any η^*, η

$$Z^{\eta^*}_{t_0}\Delta R_{t_1} = X - \mathbb{E}^P[X|\mathcal{F}_{t_0}] = Z^{\eta}_{t_0}\Delta R_{t_1} \quad \Longrightarrow \quad Z^{\eta^*}_{t_0} = Z^{\eta}_{t_0}$$

Consequently, Assumption (iii) is trivially true because $0 \geq 0$ holds. We can then focus on seeking η^* such that Assumption (ii) is met. For this purpose, we compute the numerical quantities of $\xi, \Pi, \Pi^{\perp}$ under the configuration of

$$S_{t_0} = 1.0, \quad r = 0, \quad \mu_{t_0} = 0.1, \quad \Delta t = 1.0, \quad \sigma^1_{t_0} = 0.3, \sigma^2_{t_0} = 0.5, \quad \alpha = 0.01, \quad R = 0.5.$$

Straightforward calculations give

$$\begin{aligned}
\xi_{t_n} &:= \sigma_{t_n}^{\mathsf{T}}(\sigma_{t_n}\sigma_{t_n}^{\mathsf{T}})^{-1}\mu_{t_n} \\
&= (0.3 \quad 0.5 \quad 0)^{\mathsf{T}}\left((0.3 \quad 0.5 \quad 0)\,(0.3 \quad 0.5 \quad 0)^{\mathsf{T}}\right)^{-1}\cdot 0.1 \\
&= \begin{pmatrix} \frac{3}{34} \\ \frac{5}{34} \\ 0 \end{pmatrix} \\
\Pi_{t_n} &:= \sigma_{t_n}^{\mathsf{T}}(\sigma_{t_n}\sigma_{t_n}^{\mathsf{T}})^{-1}\sigma_{t_n} \\
&= (0.3 \quad 0.5 \quad 0)^{\mathsf{T}}\left((0.3 \quad 0.5 \quad 0)\,(0.3 \quad 0.5 \quad 0)^{\mathsf{T}}\right)^{-1}(0.3 \quad 0.5 \quad 0) \\
&= \begin{pmatrix} \frac{9}{34} & \frac{15}{34} & 0 \\ \frac{15}{34} & \frac{25}{34} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\Pi_{t_n}^{\perp} &:= \mathbf{I} - \Pi_{t_n} \\
&= \begin{pmatrix} \frac{25}{34} & -\frac{15}{34} & 0 \\ -\frac{15}{34} & \frac{9}{34} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Regarding η and its associated constraints such that it defines a Q belonging to $\mathcal{Q}^{\mathrm{ngd}}$, the decomposition of q in this configuration is

$$q = \begin{pmatrix} q^1 \\ q^2 \\ \hat{q} \end{pmatrix} = -\xi + \eta = \begin{pmatrix} -\frac{3}{34} \\ -\frac{5}{34} \\ 0 \end{pmatrix} + \begin{pmatrix} \eta^1 \\ \eta^2 = -\frac{0.3}{0.5}\eta^1 \\ \hat{\eta} \end{pmatrix}$$

and to have $Q \in \mathcal{Q}^{\mathrm{ngd}}$ requires the components of q to satisfy

$$\begin{cases} -\frac{1}{3} - \hat{q}_{t_0} \leq q_{t_0}^1 + q_{t_0}^2 \leq \frac{1}{3} + \hat{q}_{t_0} \\ -\frac{1}{3} + \hat{q}_{t_0} \leq q_{t_0}^1 - q_{t_0}^2 \leq \frac{1}{3} - \hat{q}_{t_0} \\ -\frac{1}{3} \leq \hat{q}_{t_0} \leq \frac{1}{3} \end{cases}.$$

This implies that

$$\begin{cases} -\frac{1}{3} \leq \hat{\eta}_{t_0} \leq \frac{1}{3} \\ -\frac{1}{3} - \hat{\eta}_{t_0} \leq -\frac{3}{34} + \eta^1_{t_0} - \frac{5}{34} - \frac{0.3}{0.5}\eta^1 \leq \frac{1}{3} + \hat{\eta}_{t_0} \\ -\frac{1}{3} + \hat{\eta}_{t_0} \leq -\frac{3}{34} + \eta^1_{t_0} + \frac{5}{34} + \frac{0.3}{0.5}\eta^1 \leq \frac{1}{3} - \hat{\eta}_{t_0} \\ \eta^2_{t_0} = -\frac{0.3}{0.5}\eta^1_{t_0} \end{cases}$$

$$\Longrightarrow \begin{cases} -\frac{1}{3} \leq \hat{\eta}_{t_0} \leq \frac{1}{3} \\ -\frac{10}{102} - \hat{\eta}_{t_0} \leq \frac{2}{5}\eta^1 \leq \frac{58}{102} + \hat{\eta}_{t_0} \\ -\frac{40}{102} + \hat{\eta}_{t_0} \leq \frac{8}{5}\eta^1 \leq \frac{28}{102} - \hat{\eta}_{t_0} \\ \eta^2_{t_0} = -\frac{3}{5}\eta^1_{t_0} \end{cases}$$

$$\Longrightarrow \begin{cases} -\frac{1}{3} \leq \hat{\eta}_{t_0} \leq \frac{1}{3} \\ \max\left(\frac{5}{2}\left(-\frac{10}{102} - \hat{\eta}_{t_0}\right), \frac{5}{8}\left(-\frac{40}{102} + \hat{\eta}_{t_0}\right)\right) \leq \eta^1 \leq \min\left(\frac{5}{2}\left(\frac{58}{102} + \hat{\eta}_{t_0}\right), \frac{5}{8}\left(\frac{28}{102} - \hat{\eta}_{t_0}\right)\right) \\ \eta^2_{t_0} = -\frac{3}{5}\eta^1_{t_0} \end{cases} .$$

We consider the value of $\eta_{t_n}^{\mathsf{T}} \Pi_{t_n}^{\perp}(Z_{t_n})$ for arbitrary fixed $Z_{t_n} = \begin{pmatrix} Z^1_{t_n} & Z^2_{t_n} & \hat{Z}_{t_n} \end{pmatrix}$ with $\Pi_{t_n}^{\perp}(Z_{t_n}) = \begin{pmatrix} Z^{\Pi^\perp,1}_{t_n} & Z^{\Pi^\perp,2}_{t_n} & \hat{Z}^{\Pi^\perp}_{t_n} \end{pmatrix}$, that is

$$\begin{aligned} \eta^{\mathsf{T}} \Pi_{t_n}^{\perp}(Z_{t_n}) &= \eta^1_{t_n} \cdot \frac{1}{34}(25 Z^{\Pi^\perp,1}_{t_n} - 15 Z^{\Pi^\perp,2}_{t_n}) + \eta^2_{t_n} \cdot \frac{1}{34}(-15 Z^{\Pi^\perp,1}_{t_n} + 9 Z^{\Pi^\perp,2}_{t_n}) + \hat{\eta}_{t_n} \cdot \hat{Z}^{\Pi^\perp}_{t_n} \\ &= \eta^1_{t_n} \cdot \frac{1}{34}(10 Z^{\Pi^\perp,1}_{t_n} - 6 Z^{\Pi^\perp,2}_{t_n}) + \hat{\eta}_{t_n} \cdot \hat{Z}^{\Pi^\perp}_{t_n}. \end{aligned}$$

Obviously the maximum value depends critically on the signs of $10 Z^{\Pi^\perp,1}_{t_n} - 6 Z^{\Pi^\perp,2}_{t_n}$ and $\hat{Z}^{\Pi^\perp}_{t_n}$. Individual treatment on each possible combination of the signs of them allows determination of the corresponding optimal $\eta^*_{t_n}$. For instance, if $10 Z^{\Pi^\perp,1}_{t_n} - 6 Z^{\Pi^\perp,2}_{t_n} \geq 0$, this implies we should choose $\eta^1_{t_n} = \min\left(\frac{5}{2}\left(\frac{58}{102} + \hat{\eta}_{t_0}\right), \frac{5}{8}\left(\frac{28}{102} - \hat{\eta}_{t_0}\right)\right)$. Then, under this substitution, we need further suppose two cases, depending on the relative magnitudes between $\hat{Z}^{\Pi^\perp}_{t_n}$ and $10 Z^{\Pi^\perp,1}_{t_n} - 6 Z^{\Pi^\perp,2}_{t_n}$, in order to conclude the choice of $\hat{\eta}_{t_n}$ for maximization. Similar analysis should be done in the case of $10 Z^{\Pi^\perp,1}_{t_n} - 6 Z^{\Pi^\perp,2}_{t_n} < 0$ in order to derive the BSΔE for *RAROC*-based NGD ask price π^u, under current setup and choice of $(\alpha = 0.01, R = 0.5)$ and one-period time horizon, is represented by

$$\begin{aligned} Y_{t_0} - Y_{t_1} = -\Delta Y_{t_1} &= \left(-\xi_{t_0}^{\mathsf{T}} \Pi_{t_0}(Z_{t_0}) + \sqrt{h^2 - \|\xi_{t_0}\|^2} \cdot \left|\Pi_{t_0}^{\perp}(Z_{t_0})\right|\right) \Delta t - Z_{t_0} \Delta R_{t_1} \\ Y_{t_1} &= C \end{aligned}$$

with some appropriate value of h, the length of q. In other words, it resembles the continuous-time version of the BSDE found in Becherer (2009), i.e. (5.7.2).

5.8 Relating NGD Hedging to Backward Stochastic Differential Equations

In Section 5.4.3, we discussed the idea of good-deal hedging and explicitly solved the corresponding good-deal hedging strategy in our one-period model. In particular, the good-deal hedging strategy φ is determined in such a way that the coherent dynamic risk measure ρ associated with $\mathcal{P}^{\mathrm{ngd}}$ in (5.4.2), namely,

$$\rho_t\left(X - \int_t^T \varphi dS\right) := \operatorname*{ess\,sup}_{Q\in\mathcal{P}^{\mathrm{ngd}}} \mathbb{E}^Q\left[X - \int_t^T \varphi dS \middle| \mathcal{F}_t\right],$$

is minimized. Denote ρ^* as $\rho^* = \operatorname{ess\,inf}_\phi \rho_t(X - \int \varphi dS)$ and φ^* as the corresponding optimizer. We have demonstrated the direct computation of φ^* by optimizing on the value function $\rho_t\left(X - \int_t^T \varphi dS\right)$. Through this approach we cannot gather any information regarding the dynamics of $\rho_t\left(X - \int_t^T \varphi dS\right)$ as well as ρ^*. In a continuous-time framework, other than good-deal price, Becherer (2009) also derives the governing BSDE for both dynamics, which are respectively

$$\begin{aligned} -dY_t &= (-\xi_t^\mathsf{T}\phi_t + h_t\|\phi_t - Z_t\|) - Z_t dW_t \\ Y_T &= X \end{aligned}$$

and

$$\begin{aligned} -dY_t &= (-\xi_t^\mathsf{T}\phi_t + h_t\|\phi_t^* - Z_t\|) - Z_t dW_t \\ Y_T &= X \end{aligned}$$

where $\phi = \sigma^\mathsf{T}\varphi$ and $\phi^* = \sigma^\mathsf{T}\varphi^*$.

In order to retrieve the corresponding BSΔE dynamics for $\rho_t\left(X - \int_t^T \varphi dS\right)$, we derive the BS$\Delta$E that characterizes $\mathbb{E}^Q\left[X - \int_t^T \varphi dS \middle| \mathcal{F}_t\right]$, under fixed $Q \in \mathcal{P}^{\mathrm{ngd}}$ and fixed φ. Recall the notation $\phi = \sigma^\mathsf{T}\varphi$ and consider the following BSΔE

$$\begin{aligned} \tilde{Y}_{t_n} - \tilde{Y}_{t_{n+1}} = -\Delta\tilde{Y}_{t_{n+1}} &= q_{t_n}^\mathsf{T} \cdot Z_{t_n}\Delta t_n - Z_{t_n}\Delta R_{t_{n+1}} \\ \tilde{Y}_{t_N} &= X \end{aligned}$$

Existence and uniqueness of solution of this BSΔE is already justified in Proposition 5.7.1. Then, we observe that the dynamics of $\int_t^T \varphi dS = \sum_{i=n}^{N-1} \varphi_{t_i} \Delta S_{t_{i+1}}$ is

$$
\begin{aligned}
\Delta\left(\sum_{i=n}^{N-1} \varphi_{t_i} \Delta S_{t_{i+1}}\right) = \phi_{t_n} \Delta S_{t_{n+1}} &= \xi_{t_n} \cdot \phi_{t_n} \Delta t_{n+1} + \phi_{t_n} \Delta R_{t_{n+1}} \\
&= \xi_{t_n} \cdot \phi_{t_n} \Delta t_{n+1} + q_{t_n}^{\mathsf{T}} \cdot \phi_{t_n} \Delta t_{n+1} \\
&\quad + \phi_{t_n} (\Delta R_{t_{n+1}} - q_{t_n}^{\mathsf{T}} \Delta t_{n+1}) \\
&= \xi_{t_n} \cdot \phi_{t_n} \Delta t_{n+1} + q_{t_n}^{\mathsf{T}} \cdot \phi_{t_n} \Delta t_{n+1} + \phi_{t_n} \Delta R_{t_{n+1}}^Q .
\end{aligned}
$$

This implies

$$
\begin{aligned}
\bar{Y}_{t_n} = \mathbb{E}^Q\left[\int_{t_n}^T \phi dS \middle| \mathcal{F}_{t_n} \right] &= \mathbb{E}^Q\left[\sum_{i=n}^{N-1} \phi_{t_i} \Delta S_{t_{i+1}} \middle| \mathcal{F}_{t_n} \right] \\
&= \sum_{i=n}^{N-1} \mathbb{E}^Q[\phi_{t_i} \Delta S_{t_{i+1}} | \mathcal{F}_{t_n}] \\
&= \sum_{i=n}^{N-1} \Big(\xi_{t_i} \cdot \phi_{t_i} \Delta t_{i+1} + q_{t_i}^{\mathsf{T}} \cdot \phi_{t_i} \Delta t_{i+1} \Big)
\end{aligned}
$$

and so the corresponding BSΔE for $\bar{Y}$ is

$$
\begin{aligned}
\bar{Y}_{t_n} - \bar{Y}_{t_{n+1}} &= \xi_{t_i} \cdot \phi_{t_i} \Delta t_{i+1} + q_{t_i}^{\mathsf{T}} \cdot \phi_{t_i} \Delta t_{i+1} \\
\bar{Y}_{t_N} &= 0
\end{aligned}
$$

With $\tilde{Y}$ and $\bar{Y}$, $Y_{t_n} = \mathbb{E}^Q\big[X - \int_t^T \phi dS \big| \mathcal{F}_t\big]$ can be related by

$$
Y_{t_n} = \mathbb{E}^Q\left[X - \int_{t_n}^T \phi dS \middle| \mathcal{F}_{t_n} \right] = \mathbb{E}^Q[X | \mathcal{F}_t] - \mathbb{E}^Q\left[X - \int_{t_n}^T \phi dS \middle| \mathcal{F}_{t_n} \right] = \tilde{Y}_{t_n} - \bar{Y}_{t_n}.
$$

By combining all the results, it can be readily seen that the BSΔE for $Y_t = \mathbb{E}^Q\big[X - \int_t^T \phi dS \big| \mathcal{F}_t\big]$ should satisfy

$$
\begin{aligned}
Y_{t_n} - Y_{t_{n+1}} &= -\Delta Y_{t_{n+1}} = -\Delta \tilde{Y}_{t_{n+1}} + \Delta \bar{Y}_{t_{n+1}} \\
&= q_{t_n}^{\mathsf{T}} \cdot Z_{t_n} \Delta t_n - Z_{t_n} \Delta R_{t_{n+1}} - \xi_{t_n} \cdot \phi_{t_n} \Delta t_{n+1} - q_{t_n}^{\mathsf{T}} \cdot \phi_{t_n} \Delta t_{n+1} \\
&= \Big(-\xi_{t_n} \cdot \phi_{t_n} + q_{t_n}^{\mathsf{T}} \cdot (Z_{t_n} - \phi_{t_n}) \Big) \Delta t_n - Z_{t_n} \Delta R_{t_{n+1}} \\
Y_{t_N} &= X.
\end{aligned}
$$

Again, as the driver is also linked to the value of the product $q_{t_n}^{\mathsf{T}} \cdot (Z_{t_n} - \phi_{t_n})$, if we wish to proceed one step further to yield the BSΔE for $\rho_t(X - \int_t^T \varphi dS) = \operatorname{ess\,sup}_{Q \in \mathcal{P}^{\mathrm{ngd}}} \mathbb{E}^Q\big[X - \int_t^T \varphi dS \big| \mathcal{F}_t\big]$, we may need to investigate whether there exists an optimal element q^* such that the Comparison Theorem can be employed and it is really in the same direction as $Z_{t_n} - \phi_{t_n}$. If the answer is positive, then we would have a BSΔE for $Y_t = \mathbb{E}^Q\big[X - \int_t^T \varphi dS \big| \mathcal{F}_t\big]$ in parallel to that in Becherer (2009),

$$\begin{aligned} -dY_t &= (-\xi_t^{\intercal}\phi_t + h_t\|\phi_t - Z_t\|) - Z_t dW_t \\ Y_T &= X \end{aligned}$$

for some adapted process h_t. Of course, to explore the existence of such a q^* requires analysis comparable to those in previous section for the case of good-deal price processes. We defer this study as well as the question of whether it is possible to determine ρ^* with BSΔE given by

$$\begin{aligned} -dY_t &= (-\xi_t^{\intercal}\phi_t + h_t\|\phi_t^* - Z_t\|) - Z_t dW_t \\ Y_T &= X \end{aligned}$$

to the future.

We end this section by discussing some intuitive understanding about $\rho_t\left(X - \int_t^T \varphi dS\right)$. As discussed before, equipped with $\mathcal{P}^{\text{ngd}}$, $\rho_t\left(X - \int_t^T \varphi dS\right)$ behaves as a time-consistent dynamic coherent risk measure, and so one can make use of it to measure the risk of an investment dynamically. Since one normally performs hedging to reduce the risk after taking position in some contingent claim X, the resultant hedged position can thus be represented by the random variable $X - \int_t^T \varphi dS$ where $\int_t^T \varphi dS$ is the value of the hedging portfolio based on hedging strategy φ. If we are at time t, the risk associated with the hedged position at maturity time T is assessed at time t by using $\rho_t(X - \int_t^T \varphi dS)$. As a result, a rational choice of the 'best' hedging strategy φ^* should be the one which can minimize the future risk exposure, i.e. $\varphi^* = \arg\min_\phi \rho_t\left(X - \int_t^T \varphi dS\right)$. Apart from this, if the aforementioned q^* exists, then $Y_t = \rho_t\left(X - \int_t^T \varphi dS\right)$ possesses the dynamics

$$\begin{aligned} -dY_t &= \left(-\xi_{t_n} \cdot \phi_{t_n} + {q_{t_n}^*}^{\intercal} \cdot (Z_{t_n} - \phi_{t_n})\right) \Delta t_n - Z_{t_n}\Delta R_{t_{n+1}} \\ Y_T &= X \end{aligned}$$

in which case the solution Y_t can also be expressed as

$$Y_t = \mathbb{E}^P[Y_{t_{n+1}}|\mathcal{F}_{t_n}] - \xi_{t_n} \cdot \phi_{t_n}\Delta t_{n+1} + q_{t_n}^* \cdot (Z_{t_n} - \varphi_{t_n})\Delta t_{n+1}.$$

We can understand how the risk Y_t at each time is appraised by the time-consistent dynamic coherent risk measure ρ from this form of representation. Suppose the risk at t_{n+1} is already allocated, i.e. $Y_{t_{n+1}}$ is well-defined, the risk Y_{t_n} at t_n is determined from three components:

i. The real-world expected risk $\mathbb{E}^P[Y_{t_{n+1}}|\mathcal{F}_{t_n}]$ at t_n because $Y_{t_{n+1}}$ depends on the realization of the future state or economy at t_{n+1} which is uncertain at t_n, a forecast of $Y_{t_{n+1}}$ at t_n is computed to estimate the future risk level at t_n.

ii. The future risk $Y_{t_{n+1}}$ is driven by the movement of the underlying asset, in particular, even if there is no randomness involved, the drift part of the asset may lead to an increase in value in the contingent claim X hence higher levels of risk would be exposed as time evolves. Based on this argument, the future risk $Y_{t_{n+1}}$ should somehow contain a 'drift' segment. Consequently one can hold φ_{t_n} units of the underlying asset to reduce this effect, resulting a deduction of $\mu_{t_n} \cdot \varphi_{t_n} \Delta t$ from the future risk level $Y_{t_{n+1}}$.

iii. In general, the cancelation of randomness due to holding of φ_{t_n} units of the underlying asset is not complete in the sense that there remains certain residual randomness in the position, that is, $\int (Z - \varphi) dR$ is non-zero. As a result, one is subject to a certain amount of randomness. Recall the fact that a covariation process between two random processes can generate an extra drift component. This is well-known when the driving process is a Brownian motion. In the present context, the covariation process between $\int Z dR$ and $-\int \varphi dR$ is given by $(Z_{t_n} - \varphi_{t_n})\Delta t_{n+1}$ and would induce an upward pressure in the future risk level $Y_{t_{n+1}}$. Moreover, the probabilistic scenarios considered in risk-management should prepare one for the 'worst' cases happening in the future. The impact on future risk level $Y_{t_{n+1}}$ under these 'worst-case' scenarios can be taken into account by using a penalty term $q^*_{t_n} \cdot (Z_{t_n} - \varphi_{t_n})\Delta t$, which increases, with a factor of $q^*_{t_n}$, the influence of covariation.

The net value between them is then the resultant Y_{t_n}.

5.9 Numerical Results and Sensitivity Analysis under a Multi-Period Model

In this section, we conduct some numerical studies and sensitivity analysis under a multi-period model, in which all parameters are invariant throughout time, i.e. constant for any time. Firstly, we illustrate the convergence of a *RAROC*-based NGD ask-price π^u_t as the size of time step Δt goes to 0 in the following figure. For each size of time step Δt, suppose there is a sequence of times $t = 0, 1, \ldots, N$, we start with the one-period optimization problem at each state ω_i at $t = N - 1$ which is similar to that in (5.4.1). After we have computed the prices at all states ω_i at $t = N - 1$, we can move back to $t = N - 2$ and repeat similar one-period optimization problem at each state ω_i at $t = N - 2$ until the price at $t = 0$ is calculated.

Parameters: $r = 0, \mu = 0.1, T = 1, \sigma^1 = \sigma^2 = 0.3, S_0 = 1.0, K = 1.2, \alpha = 0.01, R = 0.5$

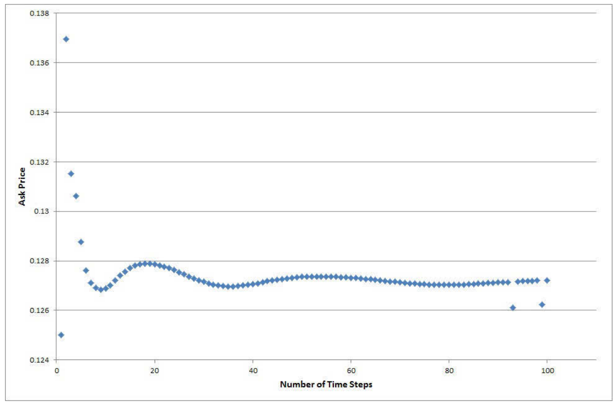


Fig. 5.1: Convergence of $RAROC$-based NGD ask-price π_t^u as $\Delta t \to 0$

Here are some discussions on Figure 5.1. We can observe that the $RAROC$-based NGD ask-price is converging as the size of time steps goes to zero. However, there are some outliers at some sizes of time steps. The presence of them is indeed related to the issue of instability in numerical schemes. When we implement a numerical scheme, there can be some spurious oscillations in the numerical solutions. See Kwok (2008) for example. The occurrence of spurious oscillations can be due to the roundoff errors or truncation errors during the numerical computation. The errors can be amplified and propagated when computing numerical solutions and eventually erode the solutions. The method here used to compute the option prices can be regarded as the lattice tree method. In order to avoid the problem of instability, one has to choose carefully the parameters σ, μ and the probabilities p_i of the tree. See Avellaneda and Laurence (1999) for more discussions. The study of numerical stability is important but it is out of the scope of this chapter. So we leave this unattempted and reserve for future research.

In view of Figure 5.1, we can conjecture that, as Δt goes to 0, $\pi_t^{u,\Delta t}$ converges to a limit π_t^u and π_t^u can be regarded as the price of the call option under the dynamics of underlying asset S driven by a *continuous-time* 2-dimensional Brownian motion, that is,

$$\frac{dS_t}{S_t} = \mu dt + \sigma^{\mathsf{T}} dW_t \qquad \text{where} \qquad \sigma = \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \quad \text{and} \quad W_t = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}.$$

This is because the multi-period model under the configuration described in Section 5.1 can be regarded as a discrete-time approximation of the above dynamics. More precisely, the multi-period model given by

$$S_{t_n} - S_{t_{n-1}} = S_{t_{n-1}} \left(\mu_{t_{n-1}} \Delta t_n + \sigma^1_{t_{n-1}} \Delta R^1_{t_n} + \sigma^2_{t_{n-1}} \Delta R^2_{t_n} \right)$$

will converge to

$$dS_t = S_t \left(\mu_t dt + \sigma^1_t dW^1_t + \sigma^2_t dW^2_t \right)$$

as we decrease the size of time step Δt. Such observed convergence also entails that the convergence of the discrete-time density process Z^N to its continuous-time analog. In other words, as $N \to \infty$, we have

$$Z^N = \prod_{0 \le n \le N-1} \left(1 + q^{\intercal}_{t_n} \Delta R_{t_{n+1}} \right) \quad \longrightarrow \quad Z = 1 + \int_0^T \xi_t dW_t$$

for some ξ such that $\frac{1}{r_1} \le \xi_t \le \frac{1}{r_2}$ P-a.s.[5]. Apart from the convergence, we also demonstrate in the figure below the sensitivity of $RAROC$-based NGD ask-price π^u_t with respect to the value of α.

Parameters: $r = 0, \mu = 0.1, T = 1, \sigma^1 = \sigma^2 = 0.3, S_0 = 1.0, K = 1.2, R = 0.1, \Delta t = 0.05$

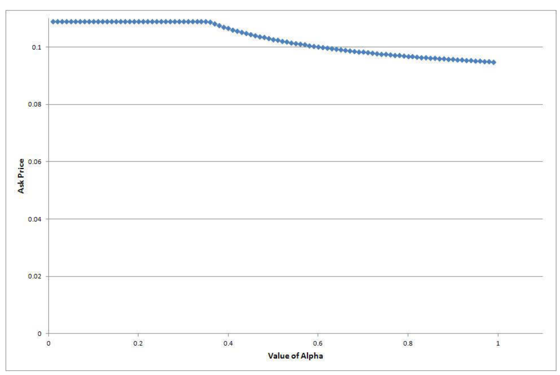


Fig. 5.2: $RAROC$-based NGD ask-price π^u_t under different values of α

We can see that the $RAROC$-based NGD ask-price π^u_t is a decreasing function of α for a fixed value of R. This phenomenon can be explained as follows: if we intend to use a larger value of α, this means we are less risk-averse and so tolerate more losses for achieving a fixed return R, as a result, we permit ourselves to spend less of the initial funds thus acquiring a riskier hedging portfolio that can potentially generate enough profit to meet the target R.

[5] Of course, it remains to justify what is the mode of convergence in this situation but we are not going to ponder this question in this chapter.

We also display the sensitivity of $RAROC$-based NGD ask-price π_t^u with respect to the value of R.

Parameters: $r = 0, \mu = 0.1, T = 1, \sigma^1 = \sigma^2 = 0.3, S_0 = 1.0, K = 1.2, \alpha = 0.01, \Delta t = 0.05$

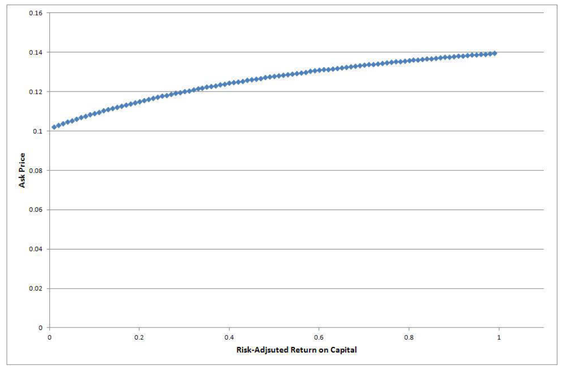


Fig. 5.3: $RAROC$-based NGD ask-price π_t^u under different values of R

From Figure 5.3, we can see that the $RAROC$-based NGD ask-price π_t^u is an increasing function of R for a fixed α. This is indeed very intuitive because, as soon as we demand more reward as given by R for the same level of risk as given by α, we should ask for a higher initial price so that we have more available funds for setting up a hedging portfolio that offers better profit while possessing at least the same amount of risk.

Finally, we show below the $RAROC$-based NGD ask-price π_t^u of a call option C as a function of the price of underlying asset S_0 and its corresponding 'delta' $\Delta^{\text{ngd}} = \frac{\partial \pi_t^u}{\partial S}$ when we price it with π_t^u. The Δ^{ngd} measures the sensitivity of the price with respect to the change of the underlying price. Whenever there is a change in the underlying price, we have to rebalance the amount of the underlying so that we can maintain the RAROC of the hedged position and meet the target RAROC at maturity. Regarding the delta, it can recover the general pattern as in usual delta Δ^{BS} of Black-Scholes call option price (see Figure 5.8) but, interestingly, it exhibits 'erratic behavior'. The implication is then a fluctuating behavior in 'gamma' $\Gamma^{\text{ngd}} = \frac{\partial^2 \pi_t^u}{\partial S^2}$ of a call option. The Γ^{ngd} is the change of Δ^{ngd} due to the change in the underlying price. If Γ^{ngd} is sufficiently small, this indicates Δ^{ngd} will not change too much when the underlying price moves, so we do not need to rebalance the hedged position too often. For very large Γ^{ngd}, we have to rebalance frequently.

Parameters: $r = 0, \mu = 0.1, T = 1, \sigma^1 = \sigma^2 = 0.3, K = 1.2, \alpha = 0.01, R = 0.1, \Delta t = 0.05$

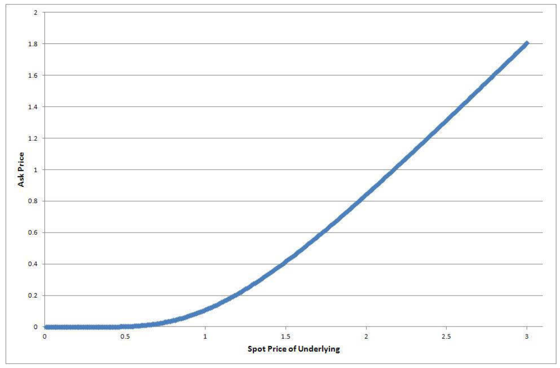


Fig. 5.4: $RAROC$-based NGD ask-price π_t^u of call option as a function of spot price S_0

Parameters: $r = 0, \mu = 0.1, T = 1, \sigma^1 = \sigma^2 = 0.3, K = 1.2, \alpha = 0.01, R = 0.1, \Delta t = 0.05$

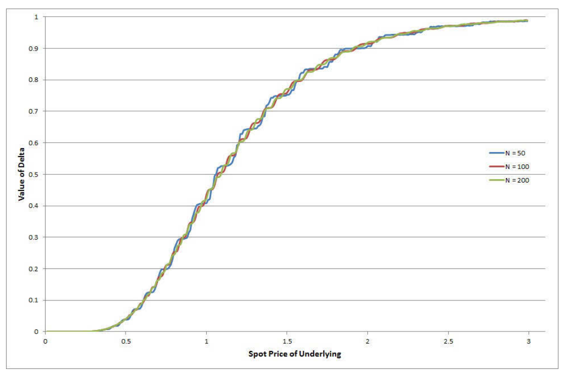


Fig. 5.5: Delta Δ^{ngd} of call option under $RAROC$-based NGD ask-price π_t^u

Parameters: $r = 0, T = 1, \sigma = 0.3, K = 1.2$

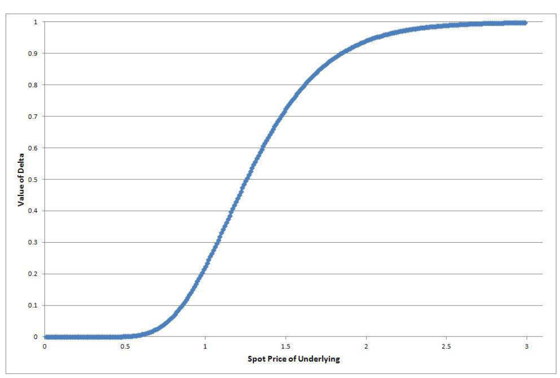


Fig. 5.6: Black-Scholes Delta Δ^{BS} of call option

Parameters: $r = 0, \mu = 0.1, T = 1, \sigma^1 = \sigma^2 = 0.3, K = 1.2, \alpha = 0.01, R = 0.1, \Delta t = 0.05$

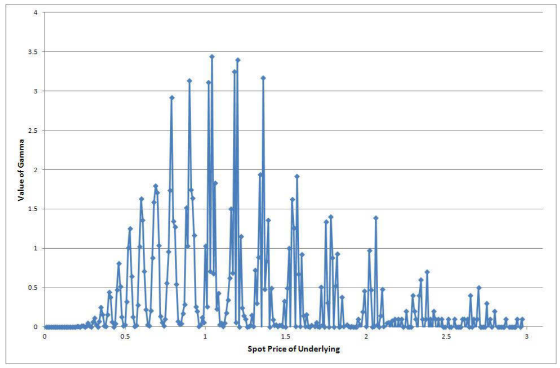


Fig. 5.7: Gamma Γ^{ngd} of call option under $RAROC$-based NGD ask-price π_t^u

Parameters: $r = 0, T = 1, \sigma = 0.3, K = 1.2$

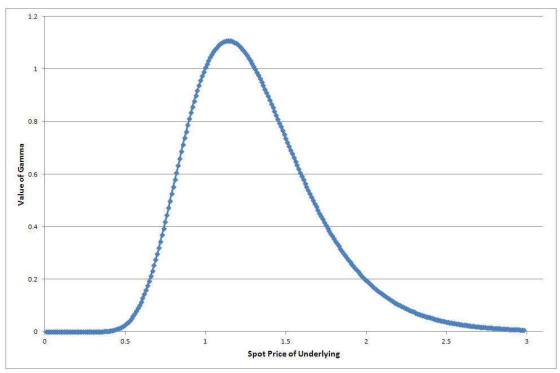


Fig. 5.8: Black-Scholes Gamma Γ^{BS} of call option

The results in Figures 5.9 and 5.11 exhibit strange behaviors and we shall explain as follows: the fluctuating behaviors of both figures are not the feature of the RAROC approach, instead, they are primarily related to the numerical matter. We have shown the convergence of the prices as the size of time steps decreases in Figure 5.5. In Figure 5.9, I have shown the results obtained at different number of time steps for comparison. As seen from the figure, we can indeed observe the convergence of the graphs into a much smoother graph. Since, for each fixed price, when we increase the number of time steps, Figure 5.5 indicates that the option price converges in a fluctuating manner but not a monotonic one. So, assume we fix the number of time steps/the size of time steps, and consider a small range $[S - \varepsilon, S + \varepsilon]$ at each fixed price S, the numerical solutions for each price within this price range do not converge at a sufficiently fast rate and their rates of convergence are different from each other. Consequently, the numerical option prices within this price range have different degrees of accuracy which lead to the fluctuating behavior in Figure 5.9. However, if we consider a sufficiently large number of time steps, the graph in Figure 5.5 should become free of fluctuations. After this explanation, we can also understand the cause of the erratic behavior of Γ in Figure 5.11: it is due to the unstable behavior of Δ in Figure 5.5 since Γ is the first order derivative of Δ, rapidly changing Δ implies fluctuating Γ.

5.10 Conclusions

With reference to Becherer (2009), we can arrive at a well-defined notion of dynamic *RAROC*-based good-deal bid and ask prices π_t^l, π_t^u for a contingent claim X which are respectively given by

$$\pi_t^u(X;\mathcal{Q}^{\mathrm{ngd}}) := \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t] \quad \text{and}$$
$$\pi_t^l(X;\mathcal{Q}^{\mathrm{ngd}}) := \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} \mathbb{E}^Q[X|\mathcal{F}_t]$$

through some suitable set of probability measures $\mathcal{Q}^{\mathrm{ngd}}$. They are constructed in such a way that their behaviors resemble those of time-consistent dynamic coherent risk measures. This is sufficient and particularly helpful in one's decision making process, in the sense that, if one makes an investment decision through such prices, one would not run into the time-inconsistency problems, i.e. a 'good-deal' today may not be 'good-deal' on another day, because such doubtful decision is prohibited by the property of time-consistency in the dynamic *RAROC*-based good-deal prices. In addition to this benefit, under certain special cases, for example the previously mentioned one-period model, we are able to derive the dynamics of π_t^l, π_t^u via backward stochastic difference equations, which opens up the opportunity of obtaining the prices by means of Monte Carlo simulation. Within the present theoretical framework, we have also discussed the notion of dynamic *RAROC*-based good-deal hedging, which is determined through minimizing certain dynamic coherent risk measures. All of these thus provide a solid support to the theory of valuating a contingent claim based on the *RAROC* criterion.

5.11 Appendix

5.11.1 Derivation of Constraints on $q^1, q^2, \hat{q}$ under General Values of σ^1, σ^2

Without loss of generality, we assume $\sigma^2 > \sigma^1$. Under this configuration, the constraint set takes the following form:

$$\begin{cases} \frac{1}{r_1} \leq 1 + q^1 + q^2 + \hat{q} \leq \frac{1}{r_2} \\ \frac{1}{r_1} \leq 1 + q^1 - q^2 - \hat{q} \leq \frac{1}{r_2} \\ \frac{1}{r_1} \leq 1 - q^1 + q^2 - \hat{q} \leq \frac{1}{r_2} \\ \frac{1}{r_1} \leq 1 - q^1 - q^2 + \hat{q} \leq \frac{1}{r_2} \\ \sigma^1 q^1 + \sigma^2 q^2 = -\mu \end{cases}$$

$$\Longrightarrow \begin{cases} \frac{1}{r_1} \leq 1 + q^1 + \frac{-\mu-\sigma^1 q^1}{\sigma^2} + \hat{q} \leq \frac{1}{r_2} & (1) \\ \frac{1}{r_1} \leq 1 + q^1 + \frac{\mu+\sigma^1 q^1}{\sigma^2} - \hat{q} \leq \frac{1}{r_2} & (2) \\ \frac{1}{r_1} \leq 1 - q^1 + \frac{-\mu-\sigma^1 q^1}{\sigma^2} - \hat{q} \leq \frac{1}{r_2} & (3) \\ \frac{1}{r_1} \leq 1 - q^1 + \frac{\mu+\sigma^1 q^1}{\sigma^2} + \hat{q} \leq \frac{1}{r_2} & (4) \\ q^2 = \frac{-\mu-\sigma^1 q^1}{\sigma^2} & (5) \end{cases}.$$

Suppose $\hat{q}$ exists and is fixed in such a way that the above inequalities hold, we arrive at

$$\begin{cases} \frac{1}{r_1} - 1 - \hat{q} \leq q^1 + \frac{-\mu-\sigma^1 q^1}{\sigma^2} \leq \frac{1}{r_2} - 1 - \hat{q} \\ \frac{1}{r_1} - 1 + \hat{q} \leq q^1 + \frac{\mu+\sigma^1 q^1}{\sigma^2} \leq \frac{1}{r_2} - 1 + \hat{q} \\ \frac{1}{r_1} - 1 + \hat{q} \leq -q^1 + \frac{-\mu-\sigma^1 q^1}{\sigma^2} \leq \frac{1}{r_2} - 1 + \hat{q} \\ \frac{1}{r_1} - 1 - \hat{q} \leq -q^1 + \frac{\mu+\sigma^1 q^1}{\sigma^2} \leq \frac{1}{r_2} - 1 - \hat{q} \\ q^2 = \frac{-\mu-\sigma^1 q^1}{\sigma^2} \end{cases}$$

$$\Longrightarrow \begin{cases} \frac{1}{r_1} - 1 - \hat{q} \leq q^1 + \frac{-\mu-\sigma^1 q^1}{\sigma^2} \leq \frac{1}{r_2} - 1 - \hat{q} & (6) \\ \frac{1}{r_1} - 1 + \hat{q} \leq q^1 + \frac{\mu+\sigma^1 q^1}{\sigma^2} \leq \frac{1}{r_2} - 1 + \hat{q} & (7) \\ -\frac{1}{r_2} + 1 - \hat{q} \leq q^1 + \frac{\mu+\sigma^1 q^1}{\sigma^2} \leq -\frac{1}{r_1} + 1 - \hat{q} & (8) \\ -\frac{1}{r_2} + 1 + \hat{q} \leq q^1 + \frac{-\mu-\sigma^1 q^1}{\sigma^2} \leq -\frac{1}{r_1} + 1 + \hat{q} & (9) \\ q^2 = \frac{-\mu-\sigma^1 q^1}{\sigma^2} & (10) \end{cases}.$$

From this, we can deduce that

$$\begin{cases} \max\left(\frac{1}{r_1} - 1 - \hat{q}, -\frac{1}{r_2} + 1 + \hat{q}\right) \leq q^1 + \frac{-\mu-\sigma^1 q^1}{\sigma^2} \leq -\max\left(\frac{1}{r_1} - 1 - \hat{q}, -\frac{1}{r_2} + 1 + \hat{q}\right) \\ \max\left(\frac{1}{r_1} - 1 + \hat{q}, -\frac{1}{r_2} + 1 - \hat{q}\right) \leq q^1 + \frac{\mu+\sigma^1 q^1}{\sigma^2} \leq -\max\left(\frac{1}{r_1} - 1 + \hat{q}, -\frac{1}{r_2} + 1 - \hat{q}\right) \\ q^2 = \frac{-\mu-\sigma^1 q^1}{\sigma^2} \end{cases}.$$

Let $K_1 = \max\left(\frac{1}{r_1} - 1 - \hat{q}, -\frac{1}{r_2} + 1 + \hat{q}\right)$ and $K_2 = \max\left(\frac{1}{r_1} - 1 + \hat{q}, -\frac{1}{r_2} + 1 - \hat{q}\right)$, we further derive that

$$\begin{cases} \frac{\sigma^2 K_1 + \mu}{\sigma^2 - \sigma^1} \leq q^1 \leq \frac{-\sigma^2 K_1 + \mu}{\sigma^2 - \sigma^1} \\ \frac{\sigma^2 K_2 - \mu}{\sigma^2 + \sigma^1} \leq q^1 \leq \frac{-\sigma^2 K_2 - \mu}{\sigma^2 + \sigma^1} \\ q^2 = \frac{-\mu - \sigma^1 q^1}{\sigma^2} \end{cases}$$

$$\Longrightarrow \quad \begin{cases} \max\left(\frac{\sigma^2 K_1 + \mu}{\sigma^2 - \sigma^1}, \frac{\sigma^2 K_2 - \mu}{\sigma^2 + \sigma^1}\right) \leq q^1 \leq \min\left(\frac{-\sigma^2 K_1 + \mu}{\sigma^2 - \sigma^1}, \frac{-\sigma^2 K_2 - \mu}{\sigma^2 + \sigma^1}\right) \\ q^2 = \frac{-\mu - \sigma^1 q^1}{\sigma^2} \end{cases}.$$

As a result, this provides the values of q^1 and q^2 as defined by the constraint set. Moreover, we should observe that the range of $\hat{q}$ should be chosen such that

$$\max\left(\frac{\sigma^2 K_1 + \mu}{\sigma^2 - \sigma^1}, \frac{\sigma^2 K_2 - \mu}{\sigma^2 + \sigma^1}\right) \leq \min\left(\frac{-\sigma^2 K_1 + \mu}{\sigma^2 - \sigma^1}, \frac{-\sigma^2 K_2 - \mu}{\sigma^2 + \sigma^1}\right)$$

is satisfied. This is thus the general framework for solving the desired optimization problem for arbitrary values of σ^1, σ^2, R and α.

5.11.2 Derivation of Constraints on $q^1, q^2, \hat{q}$ for NA Ask Price

We assume $(q^1, q^2, \hat{q})$ exists, we then have

$$\begin{cases} 1 + q^1 + q^2 + \hat{q} > 0 \\ 1 + q^1 - q^2 - \hat{q} > 0 \\ 1 - q^1 + q^2 - \hat{q} > 0 \\ 1 - q^1 - q^2 + \hat{q} > 0 \\ q^1 + q^2 = -\frac{0.1}{0.3} \end{cases}$$

$$\Longrightarrow \quad \begin{cases} \hat{q} > -\frac{2}{3} & (1) \\ 2q^1 - \hat{q} > -\frac{4}{3} & (2) \\ -2q^1 - \hat{q} > -\frac{2}{3} & (3) \\ \hat{q} > -\frac{4}{3} & (4) \\ q^2 = -\frac{1}{3} - q^1 & (5) \end{cases}.$$

Adding Eqn (2) to Eqn (3), we have $\hat{q} < 1$. Together with Eqn (1) and (4), we have

$$-\frac{2}{3} < \hat{q} < 1.$$

On the other hand, by rearranging Eqn (2) and (3),

$$\begin{cases} 2q^1 - \hat{q} & > & -\frac{4}{3} \\ -2q^1 - \hat{q} & > & -\frac{2}{3} \end{cases} \qquad \Longrightarrow \qquad \begin{cases} q^1 & > & \frac{\hat{q}}{2} - \frac{2}{3} \\ q^1 & < & -\frac{\hat{q}}{2} + \frac{1}{3} \end{cases}.$$

As $-\frac{2}{3} < \hat{q} < 1$, we obtain

$$\frac{\hat{q}}{2} - \frac{2}{3} \; < \; q^1 \; < \; -\frac{\hat{q}}{2} + \frac{1}{3}.$$

In order to have q^1 existed, this implicitly places requirements on $\hat{q}$ which is

$$\frac{\hat{q}}{2} - \frac{2}{3} < -\frac{\hat{q}}{2} + \frac{1}{3} \qquad \Longrightarrow \qquad \hat{q} < 1.$$

Combining this with $-\frac{2}{3} < \hat{q} < 1$, we have $-\frac{2}{3} < \hat{q} < 1$. Consequently, we can conclude that the constraint set is essentially given by

$$\begin{cases} -\frac{2}{3} \; < \; \hat{q} \; < \; 1 \\ \frac{\hat{q}}{2} - \frac{2}{3} \; < \; q^1 \; < \; -\frac{\hat{q}}{2} + \frac{1}{3} \\ q^2 \; = \; -\frac{1}{3} - q^1 \end{cases}.$$

5.11.3 Derivation of Constraints on $q^1, q^2, \hat{q}$ for NGD Ask Price under $(\alpha = 0.01, R = 0.5)$

We assume $(q^1, q^2, \hat{q})$ exists, we then have

$$\begin{cases} \frac{1}{1.5} & \leq & 1 + q^1 + q^2 + \hat{q} & \leq & 34 \\ \frac{1}{1.5} & \leq & 1 + q^1 - q^2 - \hat{q} & \leq & 34 \\ \frac{1}{1.5} & \leq & 1 - q^1 + q^2 - \hat{q} & \leq & 34 \\ \frac{1}{1.5} & \leq & 1 - q^1 - q^2 + \hat{q} & \leq & 34 \\ & & q^1 + q^2 = -\frac{0.1}{0.3} & & \end{cases}$$

$$\Longrightarrow \begin{cases} 0 & \leq & \hat{q} & \leq & \frac{100}{3} & (1) \\ -\frac{2}{3} & \leq & 2q^1 - \hat{q} & \leq & \frac{98}{3} & (2) \\ 0 & \leq & -2q^1 - \hat{q} & \leq & \frac{100}{3} & (3) \\ -\frac{2}{3} & \leq & \hat{q} & \leq & \frac{98}{3} & (4) \\ & & q^2 = -\frac{1}{3} - q^1 & & & (5) \end{cases}.$$

Adding Eqn (2) to Eqn (3), we have $-33 \leq \hat{q} \leq \frac{1}{3}$. Together with Eqn (1) and (4), we have

$$0 \; \leq \; \hat{q} \; \leq \; \frac{1}{3}.$$

On the other hand, by rearranging Eqn (2) and (3),

$$\begin{cases} -\frac{2}{3}+\hat{q} & \leq & 2q^1 & \leq & \frac{98}{3}+\hat{q} \\ \hat{q} & \leq & -2q^1 & \leq & \frac{100}{3}+\hat{q} \end{cases} \quad \Longrightarrow \quad \begin{cases} \frac{\hat{q}}{2}-\frac{1}{3} & \leq & q^1 & \leq & \frac{\hat{q}}{2}+\frac{47}{3} \\ -\frac{\hat{q}}{2}-\frac{50}{3} & \leq & q^1 & \leq & -\frac{\hat{q}}{2} \end{cases}.$$

As $0 \leq \hat{q} \leq \frac{1}{3}$, we obtain

$$\frac{\hat{q}}{2}-\frac{1}{3} \leq q^1 \leq -\frac{\hat{q}}{2}.$$

In order to have the above inequality well-defined, this implicitly places requirements on $\hat{q}$ which is

$$\frac{\hat{q}}{2}-\frac{1}{3} \leq -\frac{\hat{q}}{2} \qquad \Longrightarrow \qquad \hat{q} \leq \frac{1}{3}.$$

Combining this with $0 \leq \hat{q} \leq \frac{1}{3}$, we have $0 \leq \hat{q} \leq \frac{1}{3}$. Consequently, we can conclude that the constraint set is essentially given by

$$\begin{cases} 0 \leq \hat{q} \leq \frac{1}{3} \\ \frac{\hat{q}}{2}-\frac{1}{3} \leq q^1 \leq -\frac{\hat{q}}{2} \\ q^2 = -\frac{1}{3}-q^1 \end{cases}.$$

5.11.4 Derivation of Constraints on $q^1, q^2, \hat{q}$ for NGD Ask Price under ($\alpha = 0.4, R = 0.5$)

We assume $(q^1, q^2, \hat{q})$ exists, we then have

$$\begin{cases} \frac{1}{1.5} & \leq & 1+q^1+q^2+\hat{q} & \leq & \frac{3}{2} \\ \frac{1}{1.5} & \leq & 1+q^1-q^2-\hat{q} & \leq & \frac{3}{2} \\ \frac{1}{1.5} & \leq & 1-q^1+q^2-\hat{q} & \leq & \frac{3}{2} \\ \frac{1}{1.5} & \leq & 1-q^1-q^2+\hat{q} & \leq & \frac{3}{2} \\ & & q^1+q^2 = -\frac{0.1}{0.3} & & \end{cases}$$

$$\Longrightarrow \begin{cases} 0 & \leq & \hat{q} & \leq & \frac{5}{6} & (1) \\ -\frac{2}{3} & \leq & 2q^1-\hat{q} & \leq & \frac{1}{6} & (2) \\ 0 & \leq & -2q^1-\hat{q} & \leq & \frac{5}{6} & (3) \\ -\frac{2}{3} & \leq & \hat{q} & \leq & \frac{1}{6} & (4) \\ & & q^2 = -\frac{1}{3}-q^1 & & & (5) \end{cases}.$$

Adding Eqn (2) to Eqn (3), we have $-\frac{1}{2} \leq \hat{q} \leq \frac{1}{3}$. Together with Eqn (1) and (4), we have

$$0 \leq \hat{q} \leq \frac{1}{6}.$$

On the other hand, by rearranging Eqn (2) and (3),

$$\begin{cases} -\frac{2}{3}+\hat{q} & \leq & 2q^1 & \leq & \frac{1}{6}+\hat{q} \\ \hat{q} & \leq & -2q^1 & \leq & \frac{5}{6}+\hat{q} \end{cases} \Longrightarrow \begin{cases} \frac{\hat{q}}{2}-\frac{1}{3} & \leq & q^1 & \leq & \frac{\hat{q}}{2}+\frac{1}{12} \\ -\frac{\hat{q}}{2}-\frac{5}{12} & \leq & q^1 & \leq & -\frac{\hat{q}}{2} \end{cases}.$$

As $0 \leq \hat{q} \leq \frac{1}{6}$, we obtain

$$\frac{\hat{q}}{2}-\frac{1}{3} \leq q^1 \leq -\frac{\hat{q}}{2}.$$

In order to have the above inequality well-defined, this implicitly places requirements on $\hat{q}$ which is

$$\frac{\hat{q}}{2}-\frac{1}{3} \leq -\frac{\hat{q}}{2} \qquad \Longrightarrow \qquad \hat{q} \leq \frac{1}{3}.$$

Combining this with $0 \leq \hat{q} \leq \frac{1}{6}$, we have $0 \leq \hat{q} \leq \frac{1}{6}$. Consequently, we can conclude that the constraint set is essentially given by

$$\begin{cases} 0 \leq \hat{q} \leq \frac{1}{6} \\ \frac{\hat{q}}{2}-\frac{1}{3} \leq q^1 \leq -\frac{\hat{q}}{2} \\ q^2 = -\frac{1}{3}-q^1 \end{cases}.$$

5.11.5 Derivation of Constraints on $q^1, q^2, \hat{q}$ for NGD Hedging under $(\alpha = 0.01, R = 0.5)$

We assume $(q^1, q^2, \hat{q})$ exists, we then have

$$\begin{cases} \frac{1}{1.5} & \leq & 1+q^1+q^2+\hat{q} & \leq & 34 \\ \frac{1}{1.5} & \leq & 1+q^1-q^2-\hat{q} & \leq & 34 \\ \frac{1}{1.5} & \leq & 1-q^1+q^2-\hat{q} & \leq & 34 \\ \frac{1}{1.5} & \leq & 1-q^1-q^2+\hat{q} & \leq & 34 \end{cases}$$

$$\Longrightarrow \begin{cases} -\frac{1}{3} & \leq & q^1+q^2+\hat{q} & \leq & 33 & (1) \\ -\frac{1}{3} & \leq & q^1-q^2-\hat{q} & \leq & 33 & (2) \\ -\frac{1}{3} & \leq & -q^1+q^2-\hat{q} & \leq & 33 & (3) \\ -\frac{1}{3} & \leq & -q^1-q^2+\hat{q} & \leq & 33 & (4) \end{cases}.$$

Hereafter we treat $(q^1+q^2, q^1-q^2, \hat{q})$ as the independent variables, instead of $(q^1, q^2, \hat{q})$. Let $\hat{q} \in \mathbb{R}$ be a fixed arbitrary number. Assume both q^1+q^2 and q^1-q^2 exist, we rearrange Eqn (1) to (4) to

$$\begin{cases} -\frac{1}{3}-\hat{q} & \leq & q^1+q^2 & \leq & 33-\hat{q} \\ -\frac{1}{3}+\hat{q} & \leq & q^1-q^2 & \leq & 33+\hat{q} \\ -\frac{1}{3}+\hat{q} & \leq & -(q^1-q^2) & \leq & 33+\hat{q} \\ -\frac{1}{3}-\hat{q} & \leq & -(q^1+q^2) & \leq & 33-\hat{q} \end{cases}$$

$$\Longrightarrow \quad \begin{cases} -\frac{1}{3}-\hat{q} & \leq & q^1+q^2 & \leq & 33-\hat{q} & (1) \\ -\frac{1}{3}+\hat{q} & \leq & q^1-q^2 & \leq & 33+\hat{q} & (2) \\ -33-\hat{q} & \leq & q^1-q^2 & \leq & \frac{1}{3}-\hat{q} & (3) \\ -33+\hat{q} & \leq & q^1+q^2 & \leq & \frac{1}{3}+\hat{q} & (4) \end{cases}.$$

In view of this, we can conclude that

$$\begin{cases} \max\left(-\frac{1}{3}-\hat{q},-33+\hat{q}\right) & \leq & q^1+q^2 & \leq & -\max\left(-\frac{1}{3}-\hat{q},-33+\hat{q}\right) \\ \max\left(-\frac{1}{3}+\hat{q},-33-\hat{q}\right) & \leq & q^1-q^2 & \leq & -\max\left(-\frac{1}{3}+\hat{q},-33-\hat{q}\right) \end{cases}.$$

Define $K_1 = \max\left(-\frac{1}{3}-\hat{q},-33+\hat{q}\right)$ and $K_2 = \max\left(-\frac{1}{3}+\hat{q},-33-\hat{q}\right)$. There are in total 4 cases for investigation. We shall also study the corresponding value of $\hat{q}$ for each of these cases to hold and ensure the existence of both q^1+q^2 and q^1-q^2.

Case 1: $\quad K_1 = -\frac{1}{3}-\hat{q}$ and $K_2 = -\frac{1}{3}+\hat{q}$

In order to have the stated value of K_1 and K_2, it is necessary that

$$\begin{cases} K_1 = -\frac{1}{3}-\hat{q} & \Rightarrow & -\frac{1}{3}-\hat{q} \geq -33+\hat{q} & \Rightarrow & \hat{q} \leq \frac{47}{3} \\ K_2 = -\frac{1}{3}+\hat{q} & \Rightarrow & -\frac{1}{3}+\hat{q} \geq -33-\hat{q} & \Rightarrow & \hat{q} \geq -\frac{47}{3} \end{cases}.$$

Moreover, to ensure the existence of q^1+q^2 and q^1-q^2, we require that

$$\begin{cases} -\frac{1}{3}-\hat{q} \leq \frac{1}{3}+\hat{q} & \Rightarrow & \hat{q} \geq -\frac{1}{3} \\ -\frac{1}{3}+\hat{q} \leq \frac{1}{3}-\hat{q} & \Rightarrow & \hat{q} \leq \frac{1}{3} \end{cases}.$$

Combining the above, we can conclude that, under this case, the value of $\hat{q}$ should be

$$-\frac{1}{3} \leq \hat{q} \leq \frac{1}{3}.$$

Case 2: $\quad K_1 = -\frac{1}{3}-\hat{q}$ and $K_2 = -33-\hat{q}$

In order to have the stated value of K_1 and K_2, it is necessary that

$$\begin{cases} K_1 = -\frac{1}{3} - \hat{q} \quad \Rightarrow \quad -\frac{1}{3} - \hat{q} \geq -33 + \hat{q} \quad \Rightarrow \quad \hat{q} \leq \frac{47}{3} \\ K_2 = -33 - \hat{q} \quad \Rightarrow \quad -\frac{1}{3} + \hat{q} \leq -33 - \hat{q} \quad \Rightarrow \quad \hat{q} \leq -\frac{47}{3} \end{cases}.$$

Moreover, to ensure the existence of $q^1 + q^2$ and $q^1 - q^2$, we require that

$$\begin{cases} -\frac{1}{3} - \hat{q} \leq \frac{1}{3} + \hat{q} \quad \Rightarrow \quad \hat{q} \geq -\frac{1}{3} \\ -33 - \hat{q} \leq 33 + \hat{q} \quad \Rightarrow \quad \hat{q} \geq -33 \end{cases}.$$

Combining the above, we can conclude that, under this case, there is no solution for the values of $\hat{q}$.

<u>Case 3:</u> $K_1 = -33 + \hat{q}$ and $K_2 = -\frac{1}{3} + \hat{q}$

In order to have the stated value of K_1 and K_2, it is necessary that

$$\begin{cases} K_1 = -33 + \hat{q} \quad \Rightarrow \quad -\frac{1}{3} - \hat{q} \leq -33 + \hat{q} \quad \Rightarrow \quad \hat{q} \geq \frac{47}{3} \\ K_2 = -\frac{1}{3} + \hat{q} \quad \Rightarrow \quad -\frac{1}{3} + \hat{q} \geq -33 - \hat{q} \quad \Rightarrow \quad \hat{q} \geq -\frac{47}{3} \end{cases}.$$

Moreover, to ensure the existence of $q^1 + q^2$ and $q^1 - q^2$, we require that

$$\begin{cases} -33 + \hat{q} \leq 33 - \hat{q} \quad \Rightarrow \quad \hat{q} \leq 33 \\ -\frac{1}{3} + \hat{q} \leq \frac{1}{3} - \hat{q} \quad \Rightarrow \quad \hat{q} \leq \frac{1}{3} \end{cases}.$$

Combining the above, we can conclude that, under this case, there is no solution for the values of $\hat{q}$.

<u>Case 4:</u> $K_1 = -33 + \hat{q}$ and $K_2 = -33 - \hat{q}$

In order to have the stated value of K_1 and K_2, it is necessary that

$$\begin{cases} K_1 = -33 + \hat{q} \quad \Rightarrow \quad -\frac{1}{3} - \hat{q} \leq -33 + \hat{q} \quad \Rightarrow \quad \hat{q} \geq \frac{47}{3} \\ K_2 = -33 - \hat{q} \quad \Rightarrow \quad -\frac{1}{3} + \hat{q} \leq -33 - \hat{q} \quad \Rightarrow \quad \hat{q} \leq -\frac{47}{3} \end{cases}.$$

Moreover, to ensure the existence of $q^1 + q^2$ and $q^1 - q^2$, we require that

$$\begin{cases} -33 + \hat{q} \leq 33 - \hat{q} \quad \Rightarrow \quad \hat{q} \leq 33 \\ -33 - \hat{q} \leq 33 + \hat{q} \quad \Rightarrow \quad \hat{q} \geq -33 \end{cases}.$$

Combining the above, we can conclude that, under this case, there is no solution for the values of $\hat{q}$.

Consequently, we summarize and obtain the equivalent constraint as follows

$$\begin{cases} -\frac{1}{3} - \hat{q} \le q^1 + q^2 \le \frac{1}{3} + \hat{q} \\ -\frac{1}{3} + \hat{q} \le q^1 - q^2 \le \frac{1}{3} - \hat{q} \\ -\frac{1}{3} \le \hat{q} \le \frac{1}{3} \end{cases}.$$

5.11.6 Derivation of Constraints on $q^1, q^2, \hat{q}$ for NGD Hedging under $(\alpha = 0.4, R = 0.5)$

We assume $(q^1, q^2, \hat{q})$ exists, we then have

$$\begin{cases} \frac{1}{1.5} & \le & 1 + q^1 + q^2 + \hat{q} & \le & \frac{3}{2} \\ \frac{1}{1.5} & \le & 1 + q^1 - q^2 - \hat{q} & \le & \frac{3}{2} \\ \frac{1}{1.5} & \le & 1 - q^1 + q^2 - \hat{q} & \le & \frac{3}{2} \\ \frac{1}{1.5} & \le & 1 - q^1 - q^2 + \hat{q} & \le & \frac{3}{2} \end{cases}$$

$$\Longrightarrow \begin{cases} -\frac{1}{3} & \le & q^1 + q^2 + \hat{q} & \le & \frac{1}{2} & \quad (1) \\ -\frac{1}{3} & \le & q^1 - q^2 - \hat{q} & \le & \frac{1}{2} & \quad (2) \\ -\frac{1}{3} & \le & -q^1 + q^2 - \hat{q} & \le & \frac{1}{2} & \quad (3) \\ -\frac{1}{3} & \le & -q^1 - q^2 + \hat{q} & \le & \frac{1}{2} & \quad (4) \end{cases}.$$

Hereafter we treat $(q^1 + q^2, q^1 - q^2, \hat{q})$ as the independent variables, instead of $(q^1, q^2, \hat{q})$. Let $\hat{q} \in \mathbb{R}$ be a fixed arbitrary number. Assume both $q^1 + q^2$ and $q^1 - q^2$ exist, we rearrange Eqn (1) to (4) to

$$\begin{cases} -\frac{1}{3} - \hat{q} & \le & q^1 + q^2 & \le & \frac{1}{2} - \hat{q} \\ -\frac{1}{3} + \hat{q} & \le & q^1 - q^2 & \le & \frac{1}{2} + \hat{q} \\ -\frac{1}{3} + \hat{q} & \le & -(q^1 - q^2) & \le & \frac{1}{2} + \hat{q} \\ -\frac{1}{3} - \hat{q} & \le & -(q^1 + q^2) & \le & \frac{1}{2} - \hat{q} \end{cases}$$

$$\Longrightarrow \begin{cases} -\frac{1}{3} - \hat{q} & \le & q^1 + q^2 & \le & \frac{1}{2} - \hat{q} & \quad (1) \\ -\frac{1}{3} + \hat{q} & \le & q^1 - q^2 & \le & \frac{1}{2} + \hat{q} & \quad (2) \\ -\frac{1}{2} - \hat{q} & \le & q^1 - q^2 & \le & \frac{1}{3} - \hat{q} & \quad (3) \\ -\frac{1}{2} + \hat{q} & \le & q^1 + q^2 & \le & \frac{1}{3} + \hat{q} & \quad (4) \end{cases}.$$

In view of this, we can conclude that

$$\begin{cases} \max\left(-\frac{1}{3} - \hat{q}, -\frac{1}{2} + \hat{q}\right) & \le & q^1 + q^2 & \le & -\max\left(-\frac{1}{3} - \hat{q}, -\frac{1}{2} + \hat{q}\right) \\ \max\left(-\frac{1}{3} + \hat{q}, -\frac{1}{2} - \hat{q}\right) & \le & q^1 - q^2 & \le & -\max\left(-\frac{1}{3} + \hat{q}, -\frac{1}{2} - \hat{q}\right) \end{cases}.$$

Define $K_1 = \max\left(-\frac{1}{3} - \hat{q}, -\frac{1}{2} + \hat{q}\right)$ and $K_2 = \max\left(-\frac{1}{3} + \hat{q}, -\frac{1}{2} - \hat{q}\right)$. There are in total 4 cases for investigation. We shall also study the corresponding value of $\hat{q}$ for each of these cases to hold and ensure the existence of both $q^1 + q^2$ and $q^1 - q^2$.

Case 1: $K_1 = -\frac{1}{3} - \hat{q}$ and $K_2 = -\frac{1}{3} + \hat{q}$
In order to have the stated value of K_1 and K_2, it is necessary that

$$\begin{cases} K_1 = -\frac{1}{3} - \hat{q} \quad \Rightarrow \quad -\frac{1}{3} - \hat{q} \geq -\frac{1}{2} + \hat{q} \quad \Rightarrow \quad \hat{q} \leq \frac{1}{12} \\ K_2 = -\frac{1}{3} + \hat{q} \quad \Rightarrow \quad -\frac{1}{3} + \hat{q} \geq -\frac{1}{2} - \hat{q} \quad \Rightarrow \quad \hat{q} \geq -\frac{1}{12} \end{cases}.$$

Moreover, to ensure the existence of $q^1 + q^2$ and $q^1 - q^2$, we require that

$$\begin{cases} -\frac{1}{3} - \hat{q} \leq \frac{1}{3} + \hat{q} \quad \Rightarrow \quad \hat{q} \geq -\frac{1}{3} \\ -\frac{1}{3} + \hat{q} \leq \frac{1}{3} - \hat{q} \quad \Rightarrow \quad \hat{q} \leq \frac{1}{3} \end{cases}.$$

Combining the above, we can conclude that, under this case, the value of $\hat{q}$ should be

$$-\frac{1}{12} \leq \hat{q} \leq \frac{1}{12}.$$

Case 2: $K_1 = -\frac{1}{3} - \hat{q}$ and $K_2 = -\frac{1}{2} - \hat{q}$
In order to have the stated value of K_1 and K_2, it is necessary that

$$\begin{cases} K_1 = -\frac{1}{3} - \hat{q} \quad \Rightarrow \quad -\frac{1}{3} - \hat{q} \geq -\frac{1}{2} + \hat{q} \quad \Rightarrow \quad \hat{q} \leq \frac{1}{12} \\ K_2 = -\frac{1}{2} - \hat{q} \quad \Rightarrow \quad -\frac{1}{3} + \hat{q} \leq -\frac{1}{2} - \hat{q} \quad \Rightarrow \quad \hat{q} \leq -\frac{1}{12} \end{cases}.$$

Moreover, to ensure the existence of $q^1 + q^2$ and $q^1 - q^2$, we require that

$$\begin{cases} -\frac{1}{3} - \hat{q} \leq \frac{1}{3} + \hat{q} \quad \Rightarrow \quad \hat{q} \geq -\frac{1}{3} \\ -\frac{1}{2} - \hat{q} \leq \frac{1}{2} + \hat{q} \quad \Rightarrow \quad \hat{q} \geq -\frac{1}{2} \end{cases}.$$

Combining the above, we can conclude that, under this case, the value of $\hat{q}$ should be

$$-\frac{1}{3} \leq \hat{q} \leq -\frac{1}{12}.$$

Case 3: $K_1 = -\frac{1}{2} + \hat{q}$ and $K_2 = -\frac{1}{3} + \hat{q}$

In order to have the stated value of K_1 and K_2, it is necessary that

$$\begin{cases} K_1 = -\frac{1}{2} + \hat{q} \;\Rightarrow\; -\frac{1}{3} - \hat{q} \leq -\frac{1}{2} + \hat{q} \;\Rightarrow\; \hat{q} \geq \frac{1}{12} \\ K_2 = -\frac{1}{3} + \hat{q} \;\Rightarrow\; -\frac{1}{3} + \hat{q} \geq -\frac{1}{2} - \hat{q} \;\Rightarrow\; \hat{q} \geq -\frac{1}{12} \end{cases}.$$

Moreover, to ensure the existence of $q^1 + q^2$ and $q^1 - q^2$, we require that

$$\begin{cases} -\frac{1}{2} + \hat{q} \leq \frac{1}{2} - \hat{q} \;\Rightarrow\; \hat{q} \leq \frac{1}{2} \\ -\frac{1}{3} + \hat{q} \leq \frac{1}{3} - \hat{q} \;\Rightarrow\; \hat{q} \leq \frac{1}{3} \end{cases}.$$

Combining the above, we can conclude that, under this case, the value of $\hat{q}$ should be

$$\frac{1}{12} \leq \hat{q} \leq \frac{1}{3}.$$

Case 4: $K_1 = -\frac{1}{2} + \hat{q}$ and $K_2 = -\frac{1}{2} - \hat{q}$

In order to have the stated value of K_1 and K_2, it is necessary that

$$\begin{cases} K_1 = -\frac{1}{2} + \hat{q} \;\Rightarrow\; -\frac{1}{3} - \hat{q} \leq -\frac{1}{2} + \hat{q} \;\Rightarrow\; \hat{q} \geq \frac{1}{12} \\ K_2 = -\frac{1}{2} - \hat{q} \;\Rightarrow\; -\frac{1}{3} + \hat{q} \leq -\frac{1}{2} - \hat{q} \;\Rightarrow\; \hat{q} \leq -\frac{1}{12} \end{cases}.$$

Moreover, to ensure the existence of $q^1 + q^2$ and $q^1 - q^2$, we require that

$$\begin{cases} -\frac{1}{2} + \hat{q} \leq \frac{1}{2} - \hat{q} \;\Rightarrow\; \hat{q} \leq \frac{1}{2} \\ -\frac{1}{2} - \hat{q} \leq \frac{1}{2} + \hat{q} \;\Rightarrow\; \hat{q} \geq -\frac{1}{2} \end{cases}.$$

Combining the above, we can conclude that, under this case, there is no solution for the values of $\hat{q}$.

Consequently, we summarize and obtain the equivalent constraint as follows

$$\left\{ \begin{array}{l} \begin{array}{c} -\frac{1}{3} - \hat{q} \leq q^1 + q^2 \leq \frac{1}{3} + \hat{q} \\ -\frac{1}{2} - \hat{q} \leq q^1 - q^2 \leq \frac{1}{2} + \hat{q} \\ -\frac{1}{3} \leq \hat{q} \leq \frac{1}{12} \end{array} \bigcup \begin{array}{c} -\frac{1}{3} - \hat{q} \leq q^1 + q^2 \leq \frac{1}{3} + \hat{q} \\ -\frac{1}{3} + \hat{q} \leq q^1 - q^2 \leq \frac{1}{3} - \hat{q} \\ -\frac{1}{12} \leq \hat{q} \leq \frac{1}{12} \end{array} \\ \\ \bigcup \begin{array}{c} -\frac{1}{2} + \hat{q} \leq q^1 + q^2 \leq \frac{1}{2} - \hat{q} \\ -\frac{1}{3} + \hat{q} \leq q^1 - q^2 \leq \frac{1}{3} - \hat{q} \\ \frac{1}{12} \leq \hat{q} \leq \frac{1}{3} \end{array} \end{array} \right. .$$

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