Asset Pricing under Ambiguity and Heterogeneity

A Thesis Submitted for the Degree of
Doctor of Philosophy

by

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February 2015

Declaration of Authorship

I certify that this thesis has not previously been submitted for a degree nor has it been submitted as part of requirement for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Signed: ____

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Acknowledgements

The completion of this thesis would not be possible without the encouragement, guidance and support from my principal supervisor Prof Xue-Zhong (Tony) He. The discussions we had throughout my PhD candidature have proven to be most helpful. His positive attitude towards research and work will continue to be a motivation for my future professional career.

I would also like to thank my co supervisors Prof Carl Chiarella and Dr Corrado Di Guilmi for their continuous support and fellow PhD students including Dr Kai Li, Dr Swee Guan Yap, Dr Samuel Chege, Dr Yang Chang, Dr Wayne Chan and many more.

I appreciate the financial support from the University of Technology, Sydney (UTS), UTS Business School, Quantitative Finance Research Centre (QFRC) in contributing to cost of attending numerous national and international conferences and workshops which provided constant and timely feedback for my research. I am grateful to the University of Technology, Sydney for the UTS Business Faculty's Scholarship and the Quantitative Finance Research Centre's Scholarship, which are of great financial assistance.

Lastly, but not the least I am extremely grateful and fortunate for the constant support from my loving mother and father.

Abstract

Financial markets are becoming increasingly complex, volatile and uncertain in light of the recent financial crisis. Markets are characterised by a variety of anomalies and stylised facts that pose challenges to the traditional asset pricing theory, where market is represented by a single agent and investor is always perfectly aware of his (her) own preference forming rational expectation by maximising his (her) expected utility. However, empirical evidence suggests that instead, markets are populated with boundedly rational investors that are heterogeneous in beliefs and can often follow some heuristic trading rules. Further, the famous thought experiment known as the Ellsberg's Paradox reveals evidence that contradicts utility maximisation theory. In fact, it implies that investors are ambiguity-averse and prefer taking on risk in situations where they know specific odds rather than an alternate risk scenario in which the odds are completely ambiguous.

This thesis contributes to the development of the ambiguity literature by modelling uncertainty and incorporating boundedly rational behaviours to examine their joint impact on asset price dynamics as well as the various market anomalies. First, we provide a multi-asset setup to understand implication of ambiguity on correlated assets, and therefore market liquidity in time of uncertainty. Second, we propose two new dynamic ambiguity models and examine their impact on various market behaviours such as price deviations from the fundamental values, excess volatility, and long memories in return volatility. The main contributions are described below.

(i) Different from a single risky asset market, Chapter 2 adds to the ambiguity literature by exploring a multi-asset setup under ambiguity and heterogeneity, and studies the consequent implication on market illiquidity during a market downturn. We firstly explore how market illiquidity is impacted by ambiguity when risky assets are correlated. Second, we add on heterogeneity and study the implication

of heterogeneous beliefs on the first and second moments of a risky asset, and consequently the spillover effect among the correlated assets on equilibrium price, risk-free rate and market liquidity.

(ii) Although some researchers have discussed the relationship between ambiguity and volatility, most of these models remained in static setups and have not explicitly demonstrated models' capabilities to generate market anomalies and stylised facts in price and return series. Chapters 3 and 4 contribute to the literature by filling this gap. We develop dynamic ambiguity models that incorporate heuristic behaviours that investors exhibit in markets. By assuming that fundamental value of the risky assets are becoming increasingly ambiguous in times of market turmoils, we introduce models that incorporate three types of investors whose beliefs are updated through some heuristic strategies, namely fundamentalists, trend followers and noise investors. In particular, fundamentalists are assumed to be ambiguity-averse due to ambiguity about the fundamental value. The core difference between Chapters 3 and 4 lies in how we incorporate ambiguity in the fundamentalists' beliefs. In Chapter 3, we consider a simple and exogenous approach in structuring ambiguity, whereas Chapter 4 allows ambiguity to be endogenously embedded through an ambiguous signal received by fundamentalists and a Bayesian updating mechanism.

Overall, this thesis shows that asset pricing models under ambiguity and boundedly rational behaviour can help to characterise markets in time of turmoil and demonstrate models' capability to generate various financial market anomalies and stylised facts.

Contents

Chapter 1

Introduction

There is an increasing recognition that traditional asset-pricing models lack explanation power for many of the phenomenon observed in today's markets. A typical assumption of these models demands investors to be perfectly aware of their own preferences; they form rational expectations and maximise their expected utility. However, empirical evidence suggests that investors behaviour is not well described by this traditional paradigm. The famous Ellsbergs Paradox (see Ellsberg [1961]) points out that investors are not always able to derive a unique probability distribution over the reference state space; and they are referred to as *ambiguity aversion* or dislike for *uncertainty*. The distinct difference between pure *risk* and ambiguity is discussed in the seminal paper of Knight [2012]. Applications of ambiguity models received much attentions during and aftermath of the recent 2007/08 financial crisis. It is apparent that during a period of heightened ambiguity in the form of difficult-to-quantify information, the accountability of risk cannot sufficiently provide plausible explanations to some market phenomena, such as market freeze (lack of trading activities), non-participation, and excess volatility, discussed in Easley and O'Hara [2009], Easley and O'Hara [2010], and Illeditsch [2011] respectively.

Another growing body of literature that flourished due to inflexibility of those traditional asset models is heterogeneous agent-based models (HAMs). This strand belongs to a broader studies of "behavioural" models, and is loosely defined as involving some departure from the classical assumptions of single representative agent, strict rationality and unlimited computational power on that of the investors. These investors are often thought of as boundedly rational, where agent only uses some subset of the available public information. Defined in Simon [1957], investors are boundedly rational in the sense they are rational with the information they have and use certain heuristics to make decisions rather than a strict rigid rule of optimisation because of the complexity of the situation, and their inability to process and compute the expected utility of every alternative action noted in Keynes [2006]. HAMs are plausible alternative in understanding the mechanism to some of the well documented stylised facts, such as volatility clustering, lack of autocorrelation in returns and long memories in return volatility. Popular survey papers that study stylised facts and characterise financial markets under the HAMs framework include Hommes [2006], LeBaron [2006] and Chiarella et al. [2009].

Some of the commonly observed financial stylised facts are briefly mentioned here for the convenience of the discussion. Discussed in the seminal papers of LeRoy and Porter [1981] and Shiller [1981], *excess volatility* is referred to the increasing difficulty to justify the observed level of variability in asset returns by variations in fundamental economic variables. In particular, the occurrence of large (negative or positive) returns is not always explainable by the arrival of new information on the market. *Heavy tails* is the (unconditional) distribution of returns displays a heavy tail with positive excess kurtosis (>3) than a normal distributions. *Absence of autocorrelations in returns* is the case when (linear) autocorrelations of asset returns are insignificant, except for very small intra-day time scales where microstructure effects come into play. *Volatility clustering* noted by Mandelbrot

[1997], says large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes. A quantitative manifestation of this fact is that, while returns themselves are uncorrelated, absolute returns or their squares display a positive, significant and slowly decaying autocorrelations, the so called long memory or power law behaviour in volatility. These are studied in Bollerslev et al. [1992], Bouchaud and Potters [2001], and Guillaume et al. [1997]. In addition, *volume/volatility correlation* is positively correlated and that trading volume and volatility show the same type of long memory behaviour, see Lobato and Velasco [2000]. To motivate our research, we now provide some details on modelling heterogeneity and ambiguity in the literature.

1.1 Heterogeneous Agent Models

A distinctive feature of the Heterogeneous Agent Models (HAMs) is the ability of allowing different agents to co-exist in the financial market. The seminal paper of Brock and Hommes [1998] develops a simple standard discounted asset pricing model in a mean-variance framework by relaxing assumption of a representative agent, and assume agents update strategies based on past performance (i.e. realised prots) and they are boundedly rational. They show that such boundedly rational behaviour of agents can lead to market instability, and the resulting nonlinear dynamical system is capable of generating complex behaviour from local stability to high order cycles and chaos as the intensity of choice to switch predictor increases. Co-existence of multi-agents is firstly discussed in Zeeman [1974] as a mechanism to the rise and fall of bull and bear markets. The two most common agent types are fundamentalists and chartists. Quite often, fundamentalists are associated with a mean-reverting process where agents assume asset price will return to its fundamental. They are typically long-term investors over an investment of 3-5 years. Chartists, on the other hand, are often referred to agents whom believe in chasing trends (i.e. momentum or contrarian) as they extrapolate information from past prices over a shorter period.

Earlier paper of Beja and Goldman [1980] attempts to characterise how market maker adjusts prices according to aggregate excess demand. While, Day and Huang [1990] introduce a stylised market maker framework in discrete-time and their model exhibits complicated, chaotic price uctuations around a fundamental price with random switching bear and bull market episodes. Chiarella [1992] shows interaction of agents with heterogeneous expectations may lead to market instability in a continuous-time model. While, Lux [1995] models the herding behaviour through the master equation and shows that herding can give rise to realistic time series. Following from Brock and Hommes [1997] and Brock and Hommes [1998], Boswijk et al. [2007] study the behaviour of price to earnings ratio and offer an explanation for the recent stock prices run-up. Brock et al. [2009] incorporate derivatives and investigate the impact of hedging instruments on the market stability. In another direction, Chiarella and He [2003], and Chiarella et al. [2006] extend Brock and Hommes [1998] to study the impact of different time horizons on the price dynamics in discrete-time.

Within a multiple risky assets framework, the way agents form and update their beliefs about the covariance structure of asset returns also becomes an important factor in determining the dynamics of prices. A number of recent papers deal with the multiple risky assets decision problem within the HAM paradigm. Westerhoff [2004] considers a multi-asset model with fundamentalists who concentrate on only one market and chartists who invest in all markets, and offers reasons for the high degree of co-movements in stock prices observed empirically. Dieci and Westerhoff [2010, 2013] develop deterministic models to study two stock markets denominated in different currencies, which are linked via and with the related foreign exchange market, and explore potential spill-over effects between foreign exchange and stock markets. Also see Chiarella et al. [2005], Westerhoff and Dieci [2006], Chiarella et al. [2007], Chiarella, Dieci, He and Li [2013] and Chiarella, Dieci and He [2013] for the recent developments in multi-asset market dynamics.

The explanatory power of HAMs lies in their capability to generate time series of asset returns with properties similar to market stylised facts, but is also simple enough in structure so the origins of volatility clustering can be traced back to investor behaviours. The economic literature contains examples where switching of economic agents between two behavioural patterns leads to large aggregate fluctuations, see for example Kirman [1993]. In the context of financial markets, these behavioural patterns can be seen as trading rules and the resulting aggregate fluctuations as large movements in the market price i.e. heavy tails in returns. Lux and Marchesi [2000] show that heavy tails of asset returns and volatility clustering arise from behavioural switching of market participants between fundamentalist and chartist. Fundamentalists expect that the price follows the fundamental value in the long run. Chartists try to identify price trends, which results in a tendency to herding. Agents are allowed to switch between these two behaviours according to the performance of the various strategies. Chartists evaluate their performance according to realised gains, whereas for the fundamentalists, performance is measured according to the difference between the price and the fundamental value, which represents the anticipated gain of a convergence trade. An outbreak of volatility occurs when the fraction of agents using chartist techniques surpasses a certain threshold value, but such phases are brought to an end by stabilising tendencies. This behavioural switching is believed be the cause of volatility clustering, long memory and heavy tails in the Lux and Marchesi [2000]. Kirman and Teyssiere [2002] have proposed a variant of Kirman [1993] in which the proportion of fundamentalists in the market follows a Markov chain, of the type used in epidemiological models, describing herding of opinions. Simulations of this model exhibit autocorrelation patterns. More recently, He and Li [2007] study the source of power-law distributed fluctuations in volatility. Amilon [2008] and

Franke [2010] provide some empirical estimation of HAMs. Franke and Westerhoff [2011, 2012] characterise the structural stochastic volatility and estimate the models on daily returns by the method of simulated moments. Chen et al. [2012] summarise the ability of HAMs to generate stylised facts in econometrics approach. Overall, HAMs have shown great potentials to explain many market behaviours and stylised facts.

1.2 Ambiguity Models

The majority of ambiguity models have appeared in the asset pricing literature, particularly in the areas of optimal consumption and portfolio decisions. Motivated by the lack of explanation power of traditional models, the rise of ambiguity literature focuses on the relaxation of axioms that are strictly applicable to the standard expected utility maximisation. Traditionally, the theory of financial assets prices is mainly based on the expected utility (Von Neumann and Morgenstern [1947] and Savage [2012]) paradigm, which assumes that decision makers know, or act as if they know, the probabilities of all states of nature, typically referred to as risk. The issue is, in reality investor does not know the precise probabilities of events, highlighted in the sentimental paper of Ellsberg [1961]. This means that investors are exposed not only to risk but also to ambiguity, also known as *Knightian uncertainty*. Knight [2012] states "risk is randomness in which events have measurable probabilities". Uncertainty, however, describes events with unknown or objectively unmeasurable probabilities, Schmeidler [1989] has formally defined dislike for ambiguity as ambiguity aversion. More recently, portfolio choice experiments by Ahn et al. [2009] and Bossaerts et al. [2010] lead further support to the existence of ambiguity aversion in the behaviour of the investors.

The common approaches include subjective nonadditive probabilities of Gilboa [1989] and the Choquet expected utility (CEU) of Schmeidler [1989] where they state that beliefs can be represented by a single, but nonadditive, prior. The Multiple Prior (MP) or Maxmin Expected Utility (MEU) approach of Gilboa and Schmeidler [1989] however, does not assume that the investor's belief can be represented by a single additive prior. Instead, under ambiguity, individual has too little information to form a prior, and therefore considers a set of priors as possible. Being uncertainty averse, the individual takes into account the minimal expected utility over all priors in the set. In contrast to that, a different approach is described in Bewley [2002] in which he proposes that individual simply cannot rank all acts in times of uncertainty. He assumes that, due to ambiguity, some acts are simply not comparable. To complement this incomplete decision criterion, Bewley suggests an inertia assumption, which is essentially a form of status-quo bias. In other words, Bewley [2002] assumes that choices are made by inertia, and is only abandoned if a new choice yields higher expected utility for all possible probability distributions. For the purpose of this thesis, papers in discussion are limited by their relevancy to the topics.

1.3 Market Freeze and Limited-participation

In the past decade, ambiguity has offered compelling insights in phenomena observed during financial market crisis and most recently, the 2007/08 credit crisis. These characteristics include trading break-down and limited market participation that are often difficult to be understood under the traditional asset pricing framework. It is proposed in the seminal paper of Dow and Werlang [1992] that a market with one ambiguous asset and one riskless bond can induce a wide interval of prices in which the investor neither buys nor sells the risky asset. Epstein and Wang [1994] pursue a related line of argument and in Epstein and Wang [1995] they have shown a connection between ambiguity and markets extreme movements. More specifically, ambiguity aversion can generate equilibrium prices that are discontinuous processes of the state variables so that even small variations in the market fundamentals are responsible for sudden significant changes in security prices.

The result proposed in Dow and Werlang [1992] is important to understand the existence of *market freezes*, situations in which trading endogenously stops. Discussed in Easley and O'Hara [2010] against the rise of ambiguity, particular in that of the credit markets, they develop a model in which illiquidity (a non-trade spread) arises from ambiguity aversion with investors displaying Bewley [2002]'s preferences based on their heterogeneous beliefs about the future value of the risky asset. In other words, each of the investors trades away from his initial portfolio only if the trade is beneficial according to every belief he considers. As a result, no trade will occur and the market will "freeze". Therefore, the bid-ask price difference is an ambiguity spread, instead to that of the standard asymmetric information spread. The important difference between asymmetric information models and Easley and O'Hara [2010]'s model is the former spread reflects the informational advantage that some investor have with respect to knowledge of the risky assets value; while the spread of the latter is due to rise of ambiguity.

In another similar paper, Easley and O'Hara [2009] address the issue of market under participation in assets, empirical evidences discussed in Campbell [2006], by proposing a simple portfolio choice problem in a standard normal-CARA framework under ambiguity. Market is populated with heterogeneous investors, with some investors to exhibit Gilboa and Schmeidler [1989] preferences and form a limited participation equilibrium in presence of ambiguity. Guidolin and Rinaldi [2010] extend Easley and O'Hara [2009] and prove that a sufficient condition for ambiguity to induce market break-downs and limited participation equilibrium is that the spread between the highest and the lowest possible return of the idiosyncratic risk component is larger than the spread between the highest and the lowest possible return of the systematic component. The finding is consistent with the results in Mukerji and Tallon [2001] in which they show that ambiguity can endogenously generate an incomplete market with heterogeneous agents. In a similar vein, Ozsoylev and Werner [2011] propose an approach to connect ambiguity to liquidity and trading volume in a typical microstructure model that emphasises the interaction between uncertainty and private information. Again, in their model traders differ in the signals they receive. Arbitrageurs experience ambiguity, while the informed traders observe signals with certainty. They show that if the assets supply is non-deterministic and arbitragers are subjected to ambiguous signals, there exists a range of values that investors will choose not to trade. The results show limited participation also induces lower market depth, i.e. increases illiquidity. On the contrary, Routledge and Zin [2009] propose that uncertainty aversion does not always lower liquidity in a derivative market where market markers experience ambiguity and display multiple priors preferences. This occurs when ambiguity-averse market-maker posts a bid or ask that is more aggressive in order to "hedge ambiguity". Other related work include Caballero and Krishnamurthy [2008], Guidolin and Rinaldi [2010] and Routledge and Zin [2009], who study multi-priors models to capture the increase in uncertainty during financial crisis, where a shock to the economy suddenly increases ambiguity perceived by market participants can drive widespread withdrawal from that markets.

1.4 Equity Premium Puzzles, Risk-free Puzzles and Excess Volatility

The inability of a standard single-agent model, i.e. Lucas's [1978] is well known for its shortcomings in explaining puzzles such as equity premium, risk free rate and excess volatility. By adopting equilibrium setting Maenhout [2004] has shown how ambiguity aversion helps to shed light on both the equity premium and the risk-free rate puzzle. Abel [2002], Brandt et al. [2004], and Cecchetti and Lam [2010] model the agent's pessimism and doubt in specific ways and show that their modelling helps explain asset pricing puzzles. On a similar front and more recently, Ju and Miao [2012] show that their model can match the mean equity premium, the mean risk-free rate, and the volatility of the equity premium observed in the data when an ambiguity averse agent behaves pessimistically by attaching more weight to the pricing kernel in bad times when his continuation values are low. However, this result may be driven more by the separation between risk aversion and the elasticity of inter-temporal substitution than by ambiguity aversion. On that front, Collard et al. [2011] seem to provide a good match of the size and the volatility of assets returns of the market premium without Ju and Miao [2012]'s shortfall. Miao et al. [2012] show variance premium as a compounding effect of both belief distortion and variance differential about the uncertain economic regimes. They found the majority of the mean variance premium can be attributed to ambiguity aversion, and the model successfully captures recessions, and financial panics. Xu et al. [2014] attempt, in an equilibrium setting, to show in closed forms how ambiguity aversion helps to understand all three puzzles.

1.5 Bayesian Updating under Ambiguity

A standard result in Bayesian learning is discussed in the seminal paper of Blackwell et al. [1962] such that agents posterior beliefs converge to the true parameter of the dividend distribution. This implies that asset prices converge to agents valuations as time goes to infinity. However, learning and updating of beliefs can be significantly different under ambiguity than with no ambiguity. It is well known in Epstein and Schneider [2007] and the references therein, that learning with multiple priors may leave some ambiguity remaining in the long run. Such persistent ambiguity may lead to persistent speculative bubbles and give rise to persistent ambiguity that is unaffected by learning. Applying this learning model, Epstein and Schneider [2008] have extended the results on asset pricing under ambiguity to provide insights on the effects of the quality of information; while Leippold et al. [2008] embed this model in a continuous-time setting. Hansen [2007] surveys models of learning and robustness (see Hansen and Sargent [2008] for references within). A more recent paper from Illeditsch [2011] studies the behaviour of investor who is aversive to both risk and ambiguity and receives information with unknown precision with updating mechanism. The investor in his model acts according to a worst-case scenario. This leads to a situation where even small shocks can cause drastic changes in the worst-case scenario beliefs about the precision of the signal, i.e. induces uncertainty, which in turn leads to large price changes and excess volatility.

1.6 Motivation

The increasing complexity of modern financial markets means that much of the traditional models have faced great difficulty in providing satisfactory explanations to many of the characteristics and stylised facts observed in the financial markets today, e.g. excessive volatility.

Despite the idea of uncertainty (more specifically the aversion to ambiguity) is demonstrated earlier on in the famous thought experiment known as the Ellsberg [1961]'s Paradox; market implications of ambiguity are still poorly understood. Much of the works have been discussed in static setups, and to the best of our knowledge, whether ambiguity models are able to provide better explanatory power in explaining stylised facts than those of the traditional models has yet to be demonstrated. Therefore, the need to better understand market under ambiguity, not just risk alone, has become increasingly important against the backdrop of the most unprecedented financial crisis that the markets have experienced in decades. On the other hand, although many of the ambiguity models do consider heterogeneous belief or difference in opinions, they have assumed that investors interpret information differently because they agree to disagree. The lack of structures about their beliefs provides little intuitions. In fact, evidence in Zeeman [1974] points out that market is typically populated with investors who often follow some heuristic rules, which is an important feature of HAMs. These investors behaviours are often classified as fundamentalists, trend followers and noise investors. Specifically, the strength of HAMs lies in their ability to explain the origin of the observed behaviour of asset prices in terms of simple, behavioural rules of market participants. These models however, assume investors are utility maximisers and do not exhibit ambiguity. Since both ambiguity-aversion and heterogeneity are natural parts of investors' behaviours in financial markets, in this thesis, we add to the growing body of literature in ambiguity by proposing new models, in both static and dynamic setups, to study the coupling effect of ambiguity and heterogeneity on market dynamics and show our models are capable to characterise many market anomalies and stylised facts.

To demonstrate the real impact of ambiguity on the markets. We provide some market evidence of what the markets look like before and after a financial crisis. It is well acknowledged that ambiguity arises in times of uncertainty, and the recent 2007/08 financial crisis is an excellent example that can be observed and compared. Figures 1.1 and 1.2 below show returns series as well as autocorrelation of absolute returns of the pre and during the 2007/08 financial crisis. The separation point is selected just about a month before Lehman Brothers filed for bankruptcy on September 15, 2008. The two figures are based on three typical indices (daily returns) including 1)ASX 200; 2) Dow Jones All Industries; and 3) FTSE All Shares, and the period shown for pre-crisis is between 1/7/2003 - 30/7/2007 (pre) while 01/08/2007-30/8/2011 is used for the duration of the crisis. A period of four years is included to capture the aftermath, and for consistency a period of four years is selected for pre-crisis. The figures are separated into left and right panels for before and after the crisis. They show that markets hovered about ±2% with less volatility clustering prior to the credit crunch. On the contrary, all markets fluctuated violently with returns during the turmoil period, exceeding as much as ±10% in some cases with apparent volatility cluttering. Further, the autocorrelation of absolute returns has shown much stronger patterns during the crisis suggesting long memories. What is to note here that, the markets are populated with heterogeneous investors in both periods, yet the big contrast between before and during crisis highlights that ambiguity may indeed be an important factor that helps to shape the dynamics of the market and requires to be understood better.

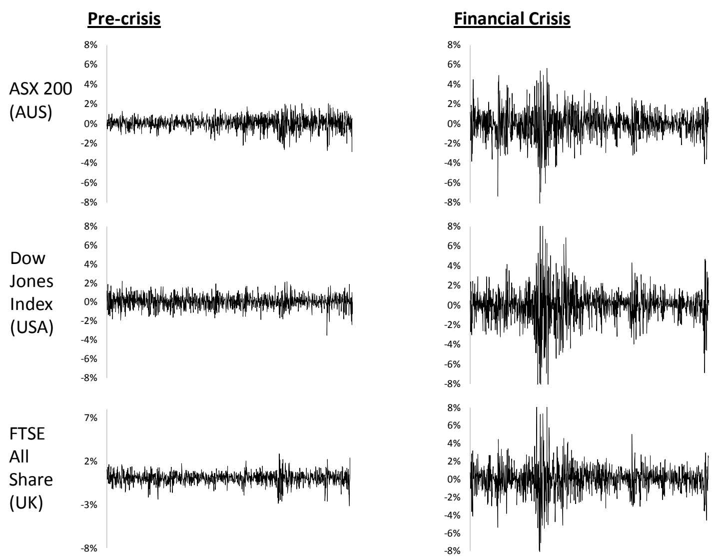


Figure 1.1: Returns series of pre and post financial crisis of three indices, 1) ASX 200; 2) Dow Jones; 3) FTSE All Shares, respectively, for the period between 1/7/2003 - 30/7/2007 (pre) and 01/08/2007-30/8/2011 (during).

This thesis seeks to move away from the traditional paradigms and proposes multi-assets and dynamic asset price models under ambiguity, with embedded features of HAMs. These proposed models shed light on the market mechanisms and implications as a result of the "hybrid" between ambiguity and heterogeneity. While works have been well underway to understand co-movements between different assets and market in the HAMs arena, we know little about the effect of ambiguity

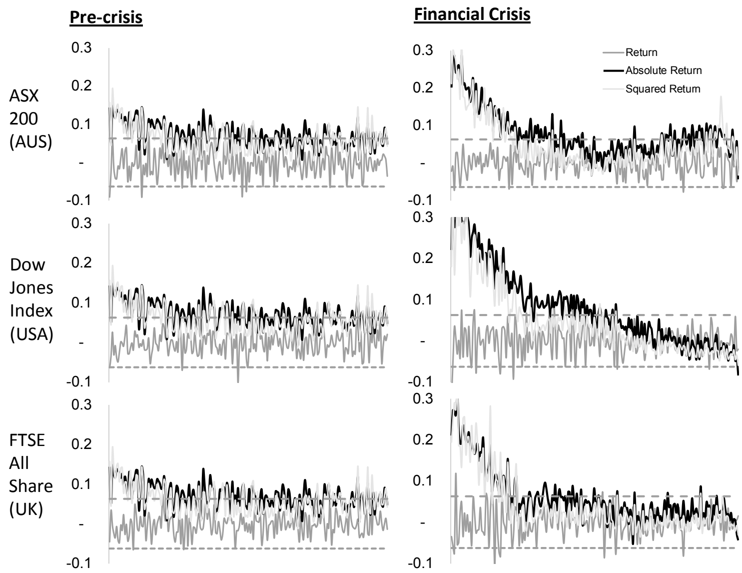


Figure 1.2: Autocorrelation of absolute returns of pre and post financial crisis of three indices, 1) ASX 200; 2) Dow Jones; 3) FTSE All Shares, respectively, for the period between 1/7/2003 - 30/7/2007 (pre) and 01/08/2007-30/8/2011 (during) over 200 lags.

in a multi-asset market. The second chapter of the thesis focuses on the coupling effect of ambiguity and heterogeneity in times of a market downturn. On a different front, the implication of ambiguity over longer periods of time is still widely untouched mainly due to the complexity to quantify and differentiate ambiguity in a multi-period model. Follow from this, the rest of the thesis introduces new models and focuses on the explanatory power of such models to generate realistic stylised facts including long memories in absolute returns, volatility clustering, and excess volatility under a simple mean-variance framework.

1.7 Structure of the Thesis

This thesis consists of three main components. Chapter 2 explores the coupling effect of ambiguity and heterogeneity on asset liquidity under uncertain times in a static setup. Chapter 3 establishes a dynamic equilibrium model of a repeated single period to explore the explanatory power of the model on stylised facts when the fundamentalists are ambiguous about the fundamental value of a risky asset. Chapter 4 steers in a similar direction as in Chapter 3, except the structure of ambiguity introduced here is endogenously applied with Bayesian updating mechanism, instead of exogenous applied as in Chapter 3. Chapter 5 summarises the main results; and discusses related future research. Finally, proofs and derivations of the models are provided in the Appendices unless otherwise stated.

In light of the "market freeze" which characterises the fact that the markets become illiquid and lack of trading activities observed in the recent financial crisis. Chapter 2 proposes a new model to study this illiquidity in a multi-risky asset setup as market experiences extreme downward pressure. The investors are assumed to be heterogeneous in beliefs and experience a systematic ambiguous shock in a two-period model. Further, risk-free rate can be endogenously derived to reflect demand for risky assets under uncertainty. In particular, we focus on the correlation effect on the demand for risky assets (and risk-free asset) when investors are heterogeneous in beliefs and subjected to ambiguity. In instance where a market is not influenced by ambiguity; market equilibrium will always prevail, i.e. trading activities can be observed. In presence of ambiguity however, market can no longer determine a singular price, rather a spread arises bounded by the lowest and highest possible prices, known as a no-trade spread and causes illiquidity. The results show that, when asset are more negatively correlated market illiquidity reduces due to asset diversification effect. However in general, ambiguity induces illiquidity on all assets having greater impact on riskier asset than a less risky asset despite the implication of correlations. Further, while heterogeneity does facilitate trades, it is not always true when the market experiences heightened uncertainty. Heterogeneity is more effective in reducing illiquidity when it is about the second moment of the risky asset, i.e. variance and covariance (a non-linear effect on the no-trade spread), than when it is about the first moment (a linear effect on the no-trade spread). Overall, market freeze is resilient in most cases under the influence of ambiguity, however, the higher the diversity between beliefs and correlations between risky assets the greater the reduction in market illiquidity.

Chapter 2 studies the coupling implication of heterogeneity and ambiguity in a multi-risky asset market. In particular, it emphasises on the lack of trading activities when investors are experiencing a systematic downward shock, resulting no trade between the investors by exhibiting Bewley [2002]'s preferences. Consequently, an equilibrium price is unable to be derived. While the static setup provides some insight in understanding the mechanisms, and the implications on market illiquidity; real markets are indeed dynamic, populated with investors that are often known to follow specific trading strategies (see Zeeman [1974] for evidence of co-existing investors in the markets) and of course complicated. Market evidences suggest that it is undoubtedly true that uncertainty has risen during the course of financial crisis (see Whaley [2000]; Williams [2009]; Bloom [2009]; and Bird and Yeung [2012]). Since fundamentalists typically depend their beliefs on the fundamental value of the risky asset that is not readily observable; when uncertainty increases, we assume that fundamentalists are ambiguous about the fundamental value. In Chapter 3, we introduce an ambiguity model that incorporates features of heterogeneity agent-based models, to analyse market dynamics in a multi-period setup and show that the model is able to generate market stylised facts due to ambiguity. The model allows for three different types of strategists, namely fundamentalists, trend chasers and noise traders. Each type follows a specific heuristic updating rule, in particular, fundamentalists are assumed to be ambiguous about the fundamental value of the risky asset such that they exhibit Gilboa and Schmeidler [1989] preferences (instead of Bewley [2002] in chapter 2); while allowing trend chasers to update based on past prices (in absence of ambiguity). Then, under min-max preference, the fundamentalists will choose not to participate in the market for a range of prices unless the movement in the price is large enough. Due to this nature of switching between participation and non-participation of the fundamentalists over a long period of time, we show that ambiguity leads to volatility clustering, long memories in absolute returns rather than attributed to heterogeneity.

Chapter 3 considers a simple and exogenous approach in structuring ambiguity. Different from Chapter 3, we introduce in Chapter 4 an ambiguous signal process that the fundamentalists receive. Further, the estimation about the signal is updated at every time period and is endogenously fed into the fundamentalists' conditional beliefs, and consequently the equilibrium price. Chapter 4 studies the implication of ambiguity and heterogeneous beliefs on asset price dynamics and shows that the model is capable of producing stylised facts, via an updating mechanism on the estimation of the signal about the future value of the risky asset received by the fundamentalists. Similar to the previous chapters, fundamentalists are ambiguity in the sense of Gilboa and Schmeidler [1989], and their preferences are governed by the worse-case scenario over a range of asset distributions. However, due to a newly imposed structure on ambiguity, the fundamentalists show two portfolio inertia in the model rather than one as in the Chapter 3. Consequently, the switching between the reliability of the signal over time as it updates at every period results different demands by the fundamentalists which consequently affects the market equilibrium. The switching mechanism is a result of the maxmin problem as fundamentalists expects the worst case scenario (depending on their positions) in times of uncertainty. We show that the interaction between different strategists under the influence of ambiguity gives rise to excessive volatility. Further, the excess volatility is comparatively larger under ambiguity than in absence of ambiguity. We also show that the model is capable of characterising insignificant autocorrelations (ACs) in the returns, but significant ACs in the absolute and squared returns, as well as volatility clustering.

Chapter 2

Liquidity and Valuation Under Ambiguity

2.1 Introduction

In traditional asset pricing models, investors act like utility maximisers and therefore trade with one another to reflect the arrival of market information, market clears and equilibrium always prevails. However, against the backdrop of the recent global financial crisis, one can observe particularly in the credit markets (see Easley and O'Hara [2010]), little to none trading activities were recorded for a period of time, and the market became illiquid. This is puzzling, one would expect investors to flee the markets by selling down the assets, and consequently generating trades. In light of this lack of understanding about the markets, a new strand of literature captured much of scholars' attention in the acknowledgement for ambiguity, or better known as ambiguity aversion - that is the dislike for uncertainty. The key difference between risk and ambiguity is documented in the seminal paper of Knight [2012] and discussed in Ellsberg [1961]. This chapter studies a market in a multi-asset setting to understand the joint impact of ambiguity and heterogeneity on market illiquidity. The results show that correlation, to some extend does mitigate market illiquidity. However ambiguity affects all risky assets nevertheless, with a greater impact on the riskier asset than a safer asset. Further, while heterogeneity does facilitate trades, it is not always true in market that experiences heightened uncertainty. Heterogeneity is more effective in reducing ambiguity when it is about the second moment of the risky asset, i.e. variance and covariance, than when it is applied to the first moment.

This chapter is most related to Easley and O'Hara [2010] and draws inspiration from its simple one-risky asset and one-risk-free asset economy under ambiguity. We extend the model to a multi-asset setting and offer insights to coupling effect between ambiguity and heterogeneity, and consequent implications on market dynamics. In their paper, investors are assumed to be heterogeneous in beliefs about the future payoff of the risky asset (first moment). Further, investors experience an unexpected adverse shock in the next period and become ambiguous about future value of the asset at that point in time. The type of preferences under ambiguity is described by Bewley [2002] "inertia assumption". In other words, investors will remain at status quo (its initial position) unless the alternative trade gives a greater expected utility in all other possible events. This behaviour of status quo is consistent with the empirical evidence in Samuelson and Zeckhauser [1988][1]. According to Bewley [2002]'s preference, when investors become ambiguous they no longer act as utility maximisers, and the market is unable to arrive at an equilibrium. The lack of equilibrium therefore forms an upper and a lower bound, forming a range of prices that investors are *not* willing to trade. Consequently, rather than arriving at an equilibrium price in a normal market condition (i.e. in

[1]Others that have adopted the same preference include Rigotti and Shannon [2005], and Illeditsch [2011]; while the other two common preferences in the ambiguity literature are the maximin expected utility studied in Schmeidler and Gilboa [1989] and Choquet expected utility studied in Schmeidler [1989]. The mentioned preferences relax one or another of the axioms made famous by Savage [2012] and is different to that of Bewley [2002]

absence of ambiguity), a no-trade spread is derived due to the fact that investors are ambiguous about the future asset value; given rise to a market freeze such that no trade occurs between this *equilibrium spread*. This chapter differs from Easley and O'Hara [2010] in two ways. First, we propose a multi-asset model governed by Bewley [2002]'s preference and study the correlation effect on market illiquidity through variance-covariance matrix. Secondly, the risk-free rate is endogenously derived to reflect investors' demand for risky assets (and risk-free asset) under the influence of ambiguity. The results show that ambiguity cannot be simply diversified through holdings of (negatively) correlated assets. Although correlation and heterogeneity reduce illiquidity, for large degree of ambiguity, no-trade spread is resilient.

Another distinctive feature in Easley and O'Hara [2010] is the arrival of no-trade spread. Typically, liquidity in absence of ambiguity is extensively researched in the context of market microstructure through the analysis of bid and ask spread of assets (see Hasbrouck [1997] for detailed discussion) that commonly arises from asymmetrical information, i.e. some investors have information advantage over others[2]. On the contrary, this particular ambiguity model derives bid-ask spread (or known as no-trade spread) due to presence of ambiguity. Although asymmetric information is not explicitly stated through the form of informed and uninformed investors, it is recognised through the difference of opinion that investors have about the future value of risky asset. This particular feature is important to demonstrate that market is illiquid even in presence of heterogeneity as long as investors are ambiguous about what is to come next.

On another front, study of implication of ambiguity has been proposed to characterise limited market participation. Under this scenario investors are typically assumed to exhibit Schmeidler and Gilboa [1989]'s preferences, where they move out of the market when ambiguity is large, and consequently generating trades

[2]Seminal papers on market microstructure includes Roll [1984] and Kyle [1985].

by closing out their positions. Among these literature, some have considered heterogeneity in addition to ambiguity, works like Cao et al. [2005] and Easley and O'Hara [2009]; while others have focused on scenario where ambiguity arises from individual risky asset rather than a systematic one, these are tackled in the works like Mukerji and Tallon [2001] and Guidolin and Rinaldi [2010]. However, due to assumed preferences, Bewley [2002]'s preferences (previously mentioned) results a complete non-participation of investors, whilst Schmeidler and Gilboa [1989]'s preferences shows limited-participation from certain type of investors (the latter type of preference is discussed in details in both Chapters 3 and 4).

An important feature that is present in ambiguity models is the assumption of heterogeneous beliefs among investors. Instead of assuming investors are homogeneous which collectively can be referred to as a representative agent, empirical evidences suggest markets in fact consist heterogeneous investors. Further, heterogeneity in absence of ambiguity is typically known for its power to facilitate greater trading activities, and has the ability to explain market volatility noted in Shiller [1981], Buraschi and Jiltsov [2006], Ziegler [2007], Buraschi et al. [2008],and Beber et al. [2010]. Much works have been done within a multi-risky asset framework, and the way investors form and update their beliefs about the covariance structure of asset returns becomes an important factor in determining the dynamics of prices. A number of recent papers that deal with the multiple risky assets decision problem within the heterogeneous agent-based paradigm include Westerhoff [2004], whom considers a multi-asset model with fundamentalists that are concentrated in a single market whilst chartists invested in all markets, he offers reasons for the high degree of co-movements in stock prices observed empirically[3]. However, studies of multi-asset within ambiguity arena is rather scarce, this chapter aims to fill in this gap.

[3]Readers are advised to read Dieci and Westerhoff [2010], Chiarella et al. [2005], Westerhoff and Dieci [2006] and Chiarella et al. [2007] for recent developments in multi-asset market dynamics under HAMs paradigm.

Evidently from above, ambiguity models have shown many advantages in capturing characteristics of markets in times of heightened uncertainty over traditional asset pricing models. This chapter introduces an economy that consists multi-risky asset and one-risk-free asset in an attempt to understand coupling effect of ambiguity and heterogeneity on market dynamics. Rather to assume risky assets are i.i.d.as in Easley and O'Hara [2010], Chapter 2 attempts to paint a more complete picture in a multi-asset market. To address this, a variance-covariance matrix is introduced, and independence between risky assets is relaxed. Further, their model assumes investors are only to differ in their opinion about the future payoff of a risky asset, the second moment is homogenous. In reality, Frijns et al. [2010] and reference therein found evidence in DAX30 that patterns observed in option prices are the result of heterogeneity in expectations about future volatility. To capture this, investors are allowed to differ in opinions about the first and second moment of the risky assets. Further, we explore the important role of assuming heterogeneity for both moments rather than just the first. The findings suggest that there exists a threshold such that, if the level of ambiguity is below it, trade occurs. Otherwise, risky asset becomes illiquid and no-trade spread prevails. It turns out, the threshold to trade depends on the level of the common shock, and the diversity of beliefs, but is independent of correlation; suggesting diversification hardly plays a role in presence of ambiguity. Also a reduction in spread is generally observed in an asset that is directly impacted by heterogeneity. However, the heterogeneity does not always help to reduce the spread of the correlated assets. Overall, in presence of ambiguity equilibrium no longer exists and induces a bid and ask spread; the effect of ambiguity is resilient in extreme market conditions and heterogeneity does not always help to facilitate trading activities and reduce market illiquidity.

The remainder of Chapter 2 is organised as follows. Section 2.2 outlines a two-period market with unexpected downward shock in period 1. Section 2.3 derives the equilibrium price at period 0 under heterogeneous beliefs. Investors enter period 1 with optimal portfolios from period 0; market equilibrium is studied in section 2.4 under two scenarios, where investors experience i) an unambiguous shock, and ii) an ambiguous shock to the future value of assets. The setups of the two scenarios are compared and discussed. Section 2.5 analyses model under special cases and provides numerical analysis on the impact of ambiguity and heterogeneity. Section 2.6 concludes. Appendix A provides technical derivations, proofs and case analysis.

2.2 The Model

In a mean-variance framework, consider an economy consisting I *(i=1,2,...,I)* investors; J *(j=1,2,...,J)* risky assets, and one risk-free asset with a rate of $R_f = 1 + r_f$. Rather than giving a fix rate, the risk free rate is endogenously derived in equilibrium. Investors may be heterogeneous about the first and second moments of the risky assets and the difference of opinion are due to the way investors interpret information. They may also differ in their risk appetite, as well as their belief about the correlation coefficient. To address the coupling effect of ambiguity and heterogeneity on the market, let $\rho_{i;j,k}$ be the belief of correlation between risky assets j and k of investor i. Finally, future payoff vector of J risky assets is denoted $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_J)^\top$. The payoffs of the risky assets are jointly normal. Investor i's expected payoff vector and variance-covariance matrix are $\mathbf{v_i}$ and $\mathbf{\Omega_i}$ respectively. The model considers two periods, $t = 0, 1$; and trades may occur at both $t = 0$ and $t = 1$.

At time $t = 0$, investors maximise their expected CARA utility given their beliefs, and hold their portfolios accordingly. At time $t = 1$, investors may experience one of two scenarios. As a base scenario, investors experience an unexpected systematic downward shock to their expected payoff vectors, and they know the size of shock (in absence of ambiguity). Investor i's expected payoff vector at $t = 1$ becomes $\mathbf{v_{i1}} = \alpha \mathbf{v_i}$, where the size of the downward shock is between $0 < \alpha < 1$ with certainty. In this scenario, inventors act like utility maximisers and portfolios are re-balanced with respect to their risk appetites and initial positions. In the second scenario however, investors no longer know the exact size of the downward shock. Each investor only knows that the size of shock is between a range, $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, such that $0 < \underline{\alpha} \leq \alpha \leq \bar{\alpha} \leq 1$. Therefore, the expected payoffs of investor i is $\alpha \bar{\mathbf{v}}_{\mathbf{i}} \in [\underline{\alpha} \bar{\mathbf{v}}_{\mathbf{i}}, \bar{\alpha} \bar{\mathbf{v}}_{\mathbf{i}}]$[4]. Under Bewley [2002]'s incomplete preferences, there are a range of prices that investors are not willing to trade, and the size of that price range varies depending on each investor's heterogeneous beliefs about the expected payoff, variance, covariance, and their appetite for risk. In the following, we examine the market equilibrium at periods 0 and 1 under two scenarios.

2.3 Market equilibrium at period 0

At the start of the period $t = 0$, investor i chooses a portfolio of assets through utility maximisation. Let $\mathbf{x_i} = (x_{i1}, x_{i2}, ..., x_{iJ})^\top$ be the position vector of J risky assets and m_i be the position of the risk-free asset. Then the future wealth of investor i's portfolio is $\tilde{w}_i = \mathbf{x_i}^\top \tilde{\mathbf{v}} + m_i R_f$, subjected to a budget constraint $m_i = w_{io} - \mathbf{x_i}^\top \mathbf{p_o}$, where w_{io} is the initial wealth and $\mathbf{p_o} = (p_{o1}, p_{o2,}..., p_{oJ})^\top$ is the equilibrium price vector of J risky assets to be determined and assume $\tilde{\mathbf{v}}$ is normally distributed. Assume investor i's expected utility is given by a constant absolute risk aversion (CARA) utility function $U_i(\tilde{w}_i) = -exp(-\frac{1}{\tau_i}\tilde{w}_i)$, and τ_i is the risk tolerance coefficient. Investor i's belief is in the form of $\mathbb{E}_i[\tilde{\mathbf{v}}] = \mathbf{v_i} = (v_{i1}, v_{i2}, \ldots, v_{iJ})^\top$ and $\mathbf{\Omega_i} = (\rho_{i;j,k}\sigma_{ij}\sigma_{ik})_{J\times J}$, where $\rho_{i;j,k}$ and σ_{ij}^2 are the correlation between risky assets j and k and the variance of risky asset j respectively. Under these assumptions, maximising investor i's expected utility is equivalent to maximising the certainty

[4] Here, α is a downward shock, to be consistent with the falling prices during the global financial crisis. For an absolute positive shock such that $\underline{\alpha} > 1$, a similar spread can be derived.

equivalent wealth

$$\max_{\mathbf{x_i}}\{\mathbf{x_i}^\top\bar{\mathbf{v}}_\mathbf{i}+(w_{io}-\mathbf{x_i}^\top\mathbf{p_o})R_f-\frac{1}{2\tau_i}\mathbf{x_i}^\top\mathbf{\Omega_i}\mathbf{x_i}\}. \tag{2.1}$$

By applying First Order Condition (FOC), the optimal portfolio composition for investor i becomes

$$\mathbf{x_i^*}=\tau_i\mathbf{\Omega_i^{-1}}(\mathbf{v_i}-\mathbf{p_o}R_f). \tag{2.2}$$

To find market equilibrium at $t=0$, we refer to Proposition 1 in Chiarella et al. [2010]. Let τ_a be the average risk tolerance coefficient across I investors, such that $\tau_a=\frac{1}{I}\sum_{i=1}^{I}\tau_i$, then we define the market variance-covariance matrix and expected payoff vector

$$\mathbf{\Omega_a^{-1}}=\frac{1}{I}\sum_{i=1}^{I}\frac{\tau_i}{\tau_a}\mathbf{\Omega_i^{-1}},\quad \textit{and}\quad \mathbf{v_a}=\frac{1}{I}\mathbf{\Omega_a}\sum_{i=1}^{I}\frac{\tau_i}{\tau_a}\mathbf{\Omega_i^{-1}}\mathbf{v_i}. \tag{2.3}$$

Applying market clearing condition $\frac{1}{I}\Sigma_{i=1}^{I}\mathbf{x_i^*}=\mathbf{x_m}$, where we denote $\mathbf{x_m}$ to be the market portfolio, the market equilibrium price $\mathbf{p_o}$ then satisfies

$$\mathbf{x_m}=\tau_a\mathbf{\Omega_a^{-1}}(\mathbf{v_a}-\mathbf{p_o}R_f); \tag{2.4}$$

and after re-arranging equation (2.4), the equilibrium price vector becomes

$$\mathbf{p_o^*}=\frac{1}{R_f}(\mathbf{v_a}-\frac{\mathbf{\Omega_a}\mathbf{x_m}}{\tau_a}). \tag{2.5}$$

One of the key difference between this model to that in Easley and O'Hara [2010] is endogenous derivation of the risk free rate. Intuitively, the rate should vary with the market. To derive the equilibrium risk-free rate R_f^*, let the risk-free asset be zero net supply $\sum_{i=1}^{I}(w_{io}-\mathbf{x_i}^\top\mathbf{p_o^*})=0$. Substitute equation (2.5) into $\mathbf{p_o^*}$ to re-express R_f^*

$$R_f^*=\frac{I}{w_{mo}}(\mathbf{v_a}-\frac{\mathbf{\Omega_a}\mathbf{x_m}}{\tau_a})^\top\mathbf{x_m}, \tag{2.6}$$

where $\sum_{i=1}^{I} w_{io} = w_{mo}$ is the total market wealth. Equation (2.5) shows that the equilibrium prices for J risky assets are determined by the difference between market expected payoff and a "compensation factor" for holding market risk, i.e. $\frac{\mathbf{\Omega_a x_m}}{\tau_a}$. It is also affected by the average market payoff and inversely related to the risk-free rate. The higher is the risk-free rate the lower is the risky asset price. Also, the rate is affected by correlation through market belief, shown in equation (2.6). Finally, by substituting equation (2.5) into equation (2.2), we obtain investor i's optimal portfolio composition at $t = 0$,

$$\mathbf{x_{io}^*} = \tau_i \mathbf{\Omega_i^{-1}}[(\mathbf{v_i} - \mathbf{v_a}) + \frac{\mathbf{\Omega_a x_m}}{\tau_a}]. \tag{2.7}$$

The composition of the portfolio depends on investor's relative belief to the market. Investor i is said to be optimistic (pessimistic) if the expected payoff $\mathbf{v_i}$ is higher (lower) than the market $\mathbf{v_a}$. Finally, we define status quo as the initial portfolio composition $\mathbf{x_{io}^*}$ at $t = 0$.

2.4 Market Equilibrium at period 1

After establishing an initial equilibrium at $t = 0$, each investor enters $t = 1$ with their respective portfolios. We look at two scenarios, scenario I is a benchmark case to set the scene; and scenario II incorporates ambiguity. To understand the implications of ambiguity on the market, we refer and compare the latter case (with ambiguity) to the benchmark (without ambiguity). Scenario I assumes heterogeneous investors experience an unanticipated downward shock to the expected payoff of J risky assets, and the size of the shock is known. Equilibrium price and position vector at $t = 1$ are then derived; investors trade to re-balance portfolio like (subjective) utility maximisers. Scenario II assumes investors experience the same downward shock but the size of the shock is unknown, i,e, ambiguous. In this case, we show that equilibrium does not always exist in this scenario, instead a bid-ask spread arises due to ambiguity.

2.4.1 Scenario I - shock with known size

At period 1, investors experience an unexpected downward shock, and they know the decrease in price is exactly by α. The expected payoff vector at the current period becomes $\mathbf{v_{i1}} = \alpha\mathbf{v_i}$, that is the payoff of assets is expected to decrease by α amount. We derive market equilibrium the same way as in the previous period, it then follows

$$\mathbf{p_1^*} = \frac{1}{R_f^*}(\alpha\mathbf{v_a} - \frac{\mathbf{\Omega_a x_m}}{\tau_a}), \tag{2.8}$$

$$\mathbf{x_{i1}^*} = \tau_i\mathbf{\Omega_i^{-1}}(\alpha(\mathbf{v_i} - \mathbf{v_a}) + \frac{\mathbf{\Omega_a x_m}}{\tau_a}). \tag{2.9}$$

This unexpected shock disturbs investors optimal holdings from $t = 0$. Because α is known with certainty, the expected payoff vector simply reduces proportionally by α, and the market adjusts to this shock with falling prices. In this market, investors hold less aggressive portfolios in comparison prior to the shock. The adjusted trading for investor i is given by the difference between the status quo and the current optimal portfolio holdings

$$\mathbf{t_i^*} = \mathbf{x_{io}^*} - \mathbf{x_{i1}^*} = \tau_i(1-\alpha)\mathbf{\Omega_i^{-1}}(\mathbf{v_a} - \mathbf{v_i}). \tag{2.10}$$

Despite an unexpected downward shock, the market still arrives at equilibrium. According to the above rule, the trading positions of investor i do not only depend on their relative beliefs to the market but the correlation between risky assets. The further the investor's expected payoff vector is from the market expectation, the more aggressive is the adjustment, holding all else constant. Investor's aggressiveness to trade is also directly linked to one's risk appetite; a smaller position is traded if the investor is more risk adverse; this is because the investor whom has a lower risk tolerance would have held a less aggressive portfolio from the previous period. Finally, if investor i has the same expectation as the market, than we expect trade to remain at status quo. The aggregated trading volume is the sum of all investors' trading activities in absolute value (divided by two to avoid double counting of buying and selling between two investors) and is given by

$$\mathbf{t_a^*} = \frac{1}{2}\sum_{i=1}^{I}|\mathbf{t_i}| = \frac{1}{2}\sum_{i=1}^{I}|\tau_i \mathbf{\Omega_i^{-1}}(1-\alpha)(\mathbf{v_a} - \mathbf{v_i})|. \tag{2.11}$$

2.4.2 Scenario II - shock with unknown size

Different from the former scenario, investors in this case experience an unexpected downward shock to the expected payoff of risky assets at $t = 1$, and the size of the shock is *unknown*. Investors are ambiguous in the sense that they do not have a single expectation of the payoff for risky asset j, rather a range of expected payoffs due to ambiguity, i.e. $[\underline{\alpha}\mathbf{v_i}, \bar{\alpha}\mathbf{v_i}]$[5]. According to Bewley [2002], investors have incomplete preferences that arise from uncertainty and thus cannot rank their choices like utility maximisers. In this model, investors will only trade to re-balance their portfolio if there is a set of trades that gives a higher expected utility than the status quo for every possible outcome. The future wealth at $t = 1$ is thus given by

$$\begin{aligned}\tilde{\mathbf{w}}_{i1} &= (\mathbf{x_{io}^*} + \mathbf{t_i})^\top \tilde{\mathbf{v}} + (m_{io}^* + m_i)R_f^* \\ &= \mathbf{x_{io}^*}^\top \tilde{\mathbf{v}} + \mathbf{t_i}^\top(\tilde{\mathbf{v}} - \mathbf{p_1}R_f^*)^\top + m_{io}^* R_f^*,\end{aligned} \tag{2.12}$$

[5]In this chapter, the range of possible events $[\underline{\alpha}, \bar{\alpha}]$ is assumed to be the same across all investors, where $\underline{\alpha}$ and $\bar{\alpha}$ are lower and upper bound respectively. In general, the range may vary from investor to investor, that is $[\underline{\alpha}_i, \bar{\alpha}_i]$, for investor i, $\underline{\alpha}_i$ is the worst case scenario and $\bar{\alpha}_i$ is the best case scenario.

after substituting the budget constraint equation $m_i + \mathbf{t_i}^\top \mathbf{p_1} = 0$. Here investor i's trading vector for J risky assets in the new period is $\mathbf{t_i} = (t_{i1}, t_{i2}, \ldots, t_{iJ})^\top$, m_i is the position for the risk-free asset; and $\mathbf{p_1} = (p_{11}, p_{12}, \ldots, p_{1J})^\top$ is a price vector at $t = 1$. The expected utility at period 1 is again equivalent to the certainty equivalent wealth

$$\max_{t_i}\{\mathbf{x_{io}^{*\top}}\mathbf{v_{i1}} + \mathbf{t_i}^\top(\mathbf{v_{i1}} - \mathbf{p_1}R_f^*) + m_{io}^* R_f^* - \frac{1}{2\tau_i}(\mathbf{x_{io}^*} + \mathbf{t_i})^\top \mathbf{\Omega_i}(\mathbf{x_{io}^*} + \mathbf{t_i})\}. \tag{2.13}$$

Because investors exhibits Bewley [2002]'s incomplete preferences, their decisions whether to remain at status quo or to trade depend on the following two conditions.

Condition i)

$$\begin{aligned} &\mathbf{x_{io}^{\top *}}\mathbf{v_{i1}} + \mathbf{t_i}^\top(\mathbf{v_{i1}} - \mathbf{p_1}R_f^*) + m_{io}^* R_f^* - \frac{1}{2\tau_i}(\mathbf{x_{io}^*} + \mathbf{t_i})^\top \mathbf{\Omega_i}(\mathbf{x_{io}^*} + \mathbf{t_i}) \\ &\geq \mathbf{x_{io}^{\top *}}\mathbf{v_{i1}} + m_{io}^* R_f^* - \frac{1}{2\tau_i}\mathbf{x_{io}^{\top *}}\mathbf{\Omega_i}\mathbf{x_{io}^*}; \end{aligned} \tag{2.14}$$

Condition ii)

$$\begin{aligned} &\mathbf{x_{io}^{\top *}}\mathbf{v_{i1}} + \mathbf{t}^{'\top}(\mathbf{v_{i1}} - \mathbf{p_1}R_f^*) + m_{io}^* R_f^* - \frac{1}{2\tau_i}(\mathbf{x_{io}^*} + \mathbf{t}')^\top \mathbf{\Omega_i}(\mathbf{x_{io}^*} + \mathbf{t}') \\ &\leq \mathbf{x_{io}^{\top *}}\mathbf{v_{i1}} + \mathbf{t_i}^\top(\mathbf{v_{i1}} - \mathbf{p_1}R_f^*) + m_{io}^* R_f^* - \frac{1}{2\tau_i}(\mathbf{x_{io}^*} + \mathbf{t_i})^\top \mathbf{\Omega_i}(\mathbf{x_{io}^*} + \mathbf{t_i}) \end{aligned} \tag{2.15}$$

for all $\mathbf{v_{i1}} \in [\underline{\alpha}\bar{\mathbf{v}}_\mathbf{i}, \bar{\alpha}\bar{\mathbf{v}}_\mathbf{i}]$ and $\mathbf{t}' \neq \mathbf{t_i}$. Condition (i) implies investor i will trade to re-balance portfolio if and only if this inequality is satisfied for all $\mathbf{v_{i1}} \in [\underline{\alpha}\bar{\mathbf{v}}_\mathbf{i}, \bar{\alpha}\bar{\mathbf{v}}_\mathbf{i}]$. Condition (ii) shows there is not another set of trade $\mathbf{t}'$ that gives a greater expected utility than $\mathbf{t_i}$. Due to this incomplete preference, unlike in scenario I, investors are no longer utility maximisers and the market does not always arrive at equilibrium. Instead, a bid-ask spread arises due to ambiguity. By maximising the certainty equivalent wealth function with respect to $\mathbf{t_i}$, the equality in equation

(2.14) becomes

$$(\mathbf{v_{i1}} - \mathbf{p_1} R_f^*) - \frac{1}{\tau_i}\mathbf{\Omega_i}(\mathbf{x_{io}^*} + \mathbf{t_i}) = 0. \tag{2.16}$$

Let $\mathbf{t_i} = \mathbf{0}$ and rearranging equation (2.16), the price vector becomes

$$\mathbf{p_1^i} = \frac{1}{R_f^*}(\mathbf{v_{i1}} - \frac{\mathbf{\Omega_i x_{io}^*}}{\tau_i}), \tag{2.17}$$

where $\mathbf{p_1^i}$ is a price vector that consists a range of prices that investor i will not trade[6]. Because the investor only knows the range of the expected payoff vector $\mathbf{v_{i1}} \in [\underline{\alpha}\mathbf{\bar{v}_i}, \bar{\alpha}\mathbf{\bar{v}_i}]$, investor i's spreads for risky assets become

$$\underline{\mathbf{p}}_1^i = \frac{1}{R_f^*}(\underline{\alpha}\mathbf{v_i} - \frac{1}{\tau_i}\mathbf{\Omega_i x_{io}^*}) \leq \mathbf{p_1^i} \leq \frac{1}{R_f^*}(\bar{\alpha}\mathbf{v_i} - \frac{1}{\tau_i}\mathbf{\Omega_i x_{io}^*}) = \bar{\mathbf{p}}_1^i. \tag{2.18}$$

To establish the bid-ask spread for risky assets, i.e. no-trade equilibrium spread; we apply the same condition as in Bewley [2002], and assume the interval between spreads is a non-empty intersection

$$\bigcap_{i=1}^{I}\Big[\frac{1}{R_f^*}(\underline{\alpha}\mathbf{v_i} - \frac{1}{\tau_i}\mathbf{\Omega_i x_{io}^*}), \frac{1}{R_f^*}(\bar{\alpha}\mathbf{v_i} - \frac{1}{\tau_i}\mathbf{\Omega_i x_{io}^*})\Big] \neq \emptyset. \tag{2.19}$$

Due to heterogeneity nature of investors, there are I spreads for any given risky asset. The bid price $\underline{\mathbf{p}}_1^i$ and ask price $\bar{\mathbf{p}}_1^i$ are the prices investor i is willing to sell and buy. Any prices between the upper and lower bounds results to no trade. The bid and ask spread of risky asset j is defined such that the bid price is the highest price an investor is willing to buy asset j given the largest possible drop on the expected payoff; and the ask price is the lowest price an investor is willing to sell given the smallest possible drop on the expected payoff. By using equations (2.5) and (2.7), the bid and ask spreads can be re-expressed in term of p_o^* (see Appendix

[6] It is important to note $\mathbf{p_1^i}$ is not an equilibrium price vector, but a price range for investor i.

A.1 for proof).

$$\mathbf{p_{bid}} = max_i\{\frac{1}{R_f^*}(\underline{\alpha}\mathbf{v_i} - \frac{1}{\tau_i}\mathbf{\Omega_i x_{io}^*})\} = \mathbf{p_o^*} - \frac{1}{R_f^*}min_i\{(1-\underline{\alpha})\mathbf{v_i}\} \tag{2.20}$$

and

$$\mathbf{p_{ask}} = min_i\{\frac{1}{R_f^*}(\bar{\alpha}\mathbf{v_i} - \frac{1}{\tau_i}\mathbf{\Omega_i x_{io}^*}\} = \mathbf{p_o^*} - \frac{1}{R_f^*}max_i\{(1-\bar{\alpha})\mathbf{v_i}\}. \tag{2.21}$$

Equations (2.20) and (2.21) give the bid and ask price vectors respectively. It is apparent from the above equations, that the bid and ask prices are affected by market belief - in particular covariance matrix, that is implicitly implied through $\mathbf{p_o^*}$ and R_f^*. Market belief, risk tolerance as well as the correlation coefficient determine the level of bid and ask prices. The coupling effect of ambiguity and heterogeneity about risky assets is carried through by the covariance matrix, which makes it the key feature of this model to better understand market dynamics in a multi-asset setting. The size of the spread is the difference of bid and ask price,

$$\mathbf{p_{ask}} - \mathbf{p_{bid}} = ((1-\underline{\alpha})min_i\{\mathbf{v_i}\} - (1-\bar{\alpha})max_i\{\mathbf{v_i}\}) > 0. \tag{2.22}$$

Note that even though the levels of the bid and ask prices are affected by the correlations; the spread is actually independent of them shown above. This is due to the fact that ambiguity is only specified to the first moment of the risky assets rather than to both moments. A natural question arises from above is then under what condition can trades be observed, i.,e. when the spread becomes zero. To find this threshold, we allow the lower and upper bounds to be

$$\underline{\alpha} = \alpha_o(1-\Delta) \quad and \quad \bar{\alpha} = \alpha_o(1+\Delta) \tag{2.23}$$

respectively, where $\alpha_o = \frac{\underline{\alpha}+\bar{\alpha}}{2}$ is the average of the lowest and highest possible drop in asset values; such that $\Delta \in (0,1)$ is a constant. The threshold for risky asset j

can then be expressed in term of Δ (refer to Appendix A.2 for proof),

$$\Delta_{oj} = \frac{[max_i\{\mathbf{v_{ij}}\} - min_i\{\mathbf{v_{ij}}\}](1-\alpha_o)}{[max_i\{\mathbf{v_{ij}}\} + min_i\{\mathbf{v_{ij}}\}]\alpha_o} < \Delta. \tag{2.24}$$

The above inequality describes the necessary condition for market to observe trading activities. For bid-ask spread to exists, Δ needs to be strictly greater than Δ_{oj}, otherwise trades prevail. The threshold depends on the ratio of the dispersion of investors' expectations. The less the investors are in an agreement with each other the more likely for trades to occur. Further, the ratio of the average size of drop α_o also impacts the threshold, the smaller the shock, the more likely the trades. Since the spread is independent of the correlation, the threshold is also unaffected by the correlation. At the threshold, there is a corresponding trading price. To derive this price for risky asset j, we substitute equation (2.24) into equations (2.20) and (2.21), and the trading price follows

$$\mathbf{p}'_{\mathbf{ask_j}} = \mathbf{p}'_{\mathbf{bid_j}} = \mathbf{p}^*_{\mathbf{o_j}} - \frac{2(1-\alpha_o)max_i\{\mathbf{v_{ij}}\}min_i\{\mathbf{v_{ij}}\}}{R^*_f(max_i\{\mathbf{v_{ij}}\} + min_i\{\mathbf{v_{ij}}\})} < \mathbf{p}^*_{\mathbf{o_j}}. \tag{2.25}$$

The trading price for risky asset j is always lower than the initial equilibrium price due to a downward shock to the future value of the asset, such that the second term is strictly positive, i.e. $\frac{2(1-\alpha_o)max_i\{\mathbf{v_{ij}}\}min_i\{\mathbf{v_{ij}}\}}{R^*_f(max_i\{\mathbf{v_{ij}}\}+min_i\{\mathbf{v_{ij}}\})} > 0$. However, the level of price depends on the diversity of investors' beliefs. Further, the price also depends on the correlation implicitly implied through R^*_f and $p^*_{o_j}$.

2.5 Numerical Analysis

To better understand the joint impact of ambiguity and heterogeneity on the pricing dynamics under extreme market condition, we dedicate this section to study some special cases of the proposed model. To simplify the analysis, we assume the market consists two investors, two risky assets and one risk-free asset.

Investors may differ in their risk preference. Risky asset 1 is considered to be safer than risky asset 2 with a lower market expected payoff and variance. Investors are allowed to be heterogeneous about the first and second moments of risky asset 2 only. Finally, we assume ρ is homogeneous in all cases. Here, we try to understand the role of heterogeneity that plays not just on the directly associated risky asset but its correlated asset in presence of ambiguity.

The following cases are considered. Benchmark case serves as a comparison to the other four cases by assuming investors are homogeneous in all aspects. Case 1 assumes investors to differ in their risk preference $\tau_1 = \tau_o(1-\gamma_\tau)$ and $\tau_2 = \tau_o(1+\gamma_\tau)$ where $\gamma_\tau \in (-1, 1)$, and they are also heterogeneous about the expected payoff of risky asset 2, $v_{12} = v_2(1-\gamma_v)$ and $v_{22} = v_2(1+\gamma_v)$ where $\gamma_v \in (-1, 1)$, all else being homogeneous. Case 2 assumes heterogeneity about the standard deviation of risky asset 2, $\sigma_{21} = \sigma_2(1-\gamma_\sigma)$ and $\sigma_{22} = \sigma_2(1+\gamma_\sigma)$ where $\gamma_\sigma \in (-1, 1)$, all else being homogeneous. Finally, Case 3 considers investors to differ about expected payoff and variance of risky asset 2, all else being homogeneous. Table 2.1 shows all the cases in a tabular form.

	Risky Pref.	Beliefs of Risky Asset 2
Benchmark case		
Investor 1	τ_o	v_2; σ_2
Investor 2	τ_o	v_2; σ_2
Case1		
Investor 1	$\tau_o(1-\gamma_\tau)$	$v_2(1-\gamma_v)$; σ_2
Investor 2	$\tau_o(1+\gamma_\tau)$	$v_2(1+\gamma_v)$; σ_2
Case 2		
Investor 1	τ_o	v_2; $\sigma_2(1-\gamma_\sigma)$
Investor 2	τ_o	v_2; $\sigma_2(1+\gamma_\sigma)$
Case 3		
Investor 1	τ_o	$v_2(1-\gamma_v)$; $\sigma_2(1-\gamma_\sigma)$
Investor 2	τ_o	$v_2(1+\gamma_v)$; $\sigma_2(1+\gamma_\sigma)$

TABLE 2.1: The parameters are $\tau_o = 1$, $v_1 = 0.52$, $v_2 = 1.64$, $\sigma_1 = 0.13$, $\sigma_2 = 0.2$, $\rho_o, \gamma_\tau, \gamma_v, \gamma_\sigma, \gamma_\rho \in (-1, 1)$. They are robustly chosen to avoid negative R_f^* for calculating realistic prices, demand and equilibrium bid-ask spreads.

2.5.1 Benchmark: Homogeneous belief

Under the Benchmark case, investors are homogeneous in the sense that they have identical belief about both risky assets, however, still subjected to ambiguity. It serves as a baseline to the rest of the cases. Since ρ is assumed homogeneous in all cases, we investigate the effect of correlation by allowing ρ to vary between (-1,1). With the above assumptions in mind, the market belief at $t = 0$ becomes $\mathbf{v_{a_{bm}}} = \mathbf{v_o} = (v_1, v_2)^\top$ and $\mathbf{\Omega_{a_{bm}}} = \mathbf{\Omega_o} = \begin{pmatrix} \sigma_{a,1}^2 & \rho\sigma_{a,1}\sigma_{a,2} \\ \rho\sigma_{a,1}\sigma_{a,2} & \sigma_{a,2}^2 \end{pmatrix}$, where $\mathbf{v_{a_{bm}}}$ and $\mathbf{\Omega_{a_{bm}}}$ are market expected payoff and variance-covariance matrix respectively[7]. Further, let $\rho\sigma_{a1}\sigma_{a2}$ be the covariance between two risky assets, the higher the ρ, the greater the portfolio is under-diversified (close to 1). For simplicity, assume investors have the same risk preference and the market portfolio is $x_m = (x_1, x_2) = (1, 1)$ henceforth unless otherwise stated. The equilibrium price vector at t=0 is thus

$$\begin{aligned} \mathbf{p^*_{o_{bm}}} &= (p^*_{o1_{bm}}, p^*_{o2_{bm}})^\top \\ &= \frac{1}{R_{f_{bm}}}\Big(v_1 - \frac{(\sigma_1^2 + \rho\sigma_1\sigma_2)}{\tau}, v_2 - \frac{(\sigma_2^2 + \rho\sigma_1\sigma_2)}{\tau}\Big)^\top; \end{aligned} \tag{2.26}$$

where $R_{f_{bm}}$ is endogenously derived

$$R_{f_{bm}} = \frac{2}{w_{mo}}[(v_1 + v_2) - \frac{1}{\tau}(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)]. \tag{2.27}$$

Equation (2.26) shows equilibrium price vector at $t = 0$. The price is the expected payoff of risky asset j reduced by a compensation term for holding market risk, discounted by $R_{f_{bm}}$. However, the exact prices depend on the relationship between risky assets, and $R_{f_{bm}}$ in equation (2.27) will always depends on the beliefs of

[7] Assume ρ is the same across all investors. But in general, it may differ from investor to investor.

both risky assets. There exists a negative relationship between correlation and $R_{f_{bm}}$, such that $\frac{\partial R_{f_{bm}}}{\partial \rho} < 0$ shown in Figure 2.1. The result shows homogeneous investors' demand to hold the risk-free asset increases as portfolio becomes under-diversified, i.e. diminishing diversification effect, and exhibits a linear relationship to the correlation coefficient.

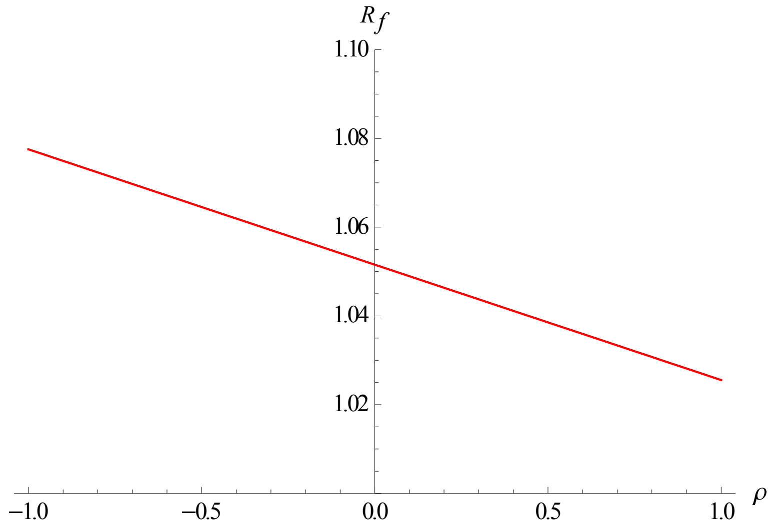


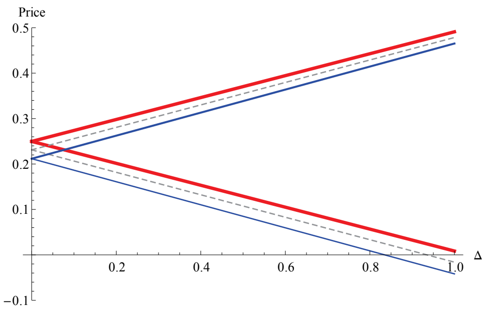
FIGURE 2.1: Equilibrium risk-free rate (y-axis) under the benchmark case with respect to the correlation coefficient (x-axis).

At $t = 1$, investors experience a systematic shock with unknown shock size. Using the same structure from equation (2.23) to describe ambiguity, the expected payoffs are $v_{i1} = v_1$ and $v_{i2} = v_2$, for both investors $i = 1, 2$. The bid and ask prices from equations (2.20) and (2.21) become

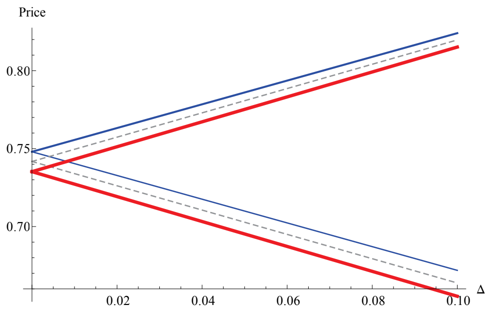
$$\mathbf{p_{bid_{bm}}} = \left(p^*_{o1_{bm}} - \frac{(1-\underline{\alpha})min\{v_{i1}\}}{R_{f_{bm}}}, p^*_{o2_{bm}} - \frac{(1-\underline{\alpha})min\{v_{i2}\}}{R_{f_{bm}}} \right)^\top \tag{2.28}$$
$$= \frac{1}{R_{fbm}} \left(\alpha_o(1-\Delta)v_1 - \frac{(\sigma_1^2 + \rho\sigma_1\sigma_2)}{\tau}, \alpha_o(1-\Delta)v_2 - \frac{(\sigma_2^2 + \rho\sigma_1\sigma_2)}{\tau} \right)^\top,$$

and

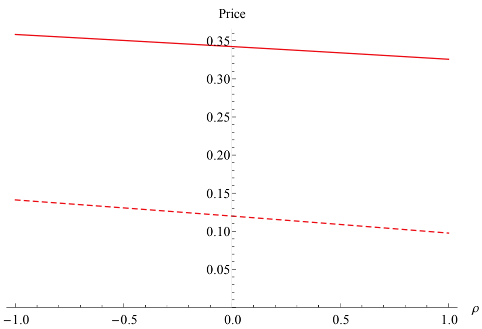
$$\mathbf{p_{ask_{bm}}} = \Big(p^*_{o1_{bm}} - \frac{(1-\bar{\alpha})max\{v_{i1}\}}{R_{f_{bm}}}, p^*_{o2_{bm}} - \frac{(1-\bar{\alpha})max\{v_{i2}\}}{R_{f_{bm}}}\Big)^\top \qquad (2.29)$$
$$= \frac{1}{R_{fbm}}\left(\alpha_o(1+\Delta)v_1 - \frac{(\sigma_1^2+\rho\sigma_1\sigma_2)}{\tau}, \alpha_o(1+\Delta)v_2 - \frac{(\sigma_2^2+\rho\sigma_1\sigma_2)}{\tau}\right)^\top.$$



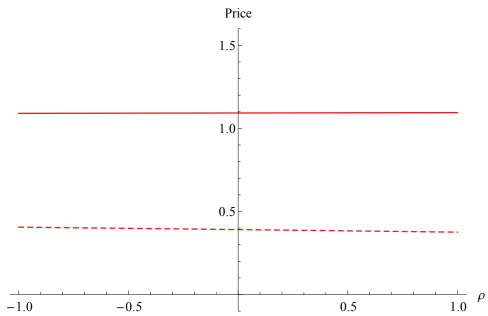
(a) Asset 1: $\rho > 0(thin), < 0(thick), = 0(dotted)$



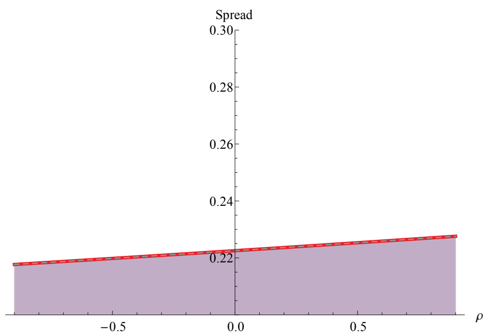
(b) Asset 2: $\rho > 0(thin), < 0(thick), = 0(dotted)$



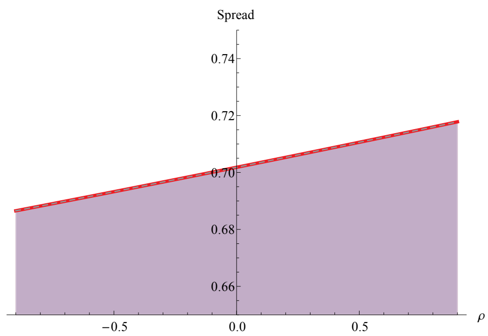
(c) Asset 1: $\Delta = 0.45$



(d) Asset 2: $\Delta = 0.45$



(e) Asset 1: $\Delta = 0.45$



(f) Asset 2: $\Delta = 0.45$

FIGURE 2.2: (a) and (b) shows the price level (y-axis) of risky asset 1 and 2 over different level of ambiguity (x-axis), given different correlation. (c) and (d) show the price level (y-axis) of the bid and ask prices over a various of correlation ρ (x-axis), given a particular level of ambiguity Δ. (e) and (f) show the level of spread (y-axis) over a various of correlation ρ (x-axis) given the same Δ.

Since homogeneity is assumed across board, the only difference between the bid and ask prices of risky asset j is the sign in front of Δ, shown in equations (2.28) and (2.29). The benchmark case demonstrates a clear picture that the bid-ask spreads arise from ambiguity. Visually, Figures 2.2(a) and 2.2(b) show spreads will always prevail as long as ambiguity is present in the market, irrespective of the signs to correlation coefficients. In particular, the upward shift in price of risky asset 2 suggests that in case of a highly positive correlation (thin line) investors prefers riskier asset as it is more likely to generate a higher payoff (at the cost of higher risk). In general, prices decrease with increasing correlation due to a diminishing diversification effect shown in Figures 2.2(c) and 2.2(d). Finally and most importantly, Figures 2.2(e) and 2.2(f) show the bid-ask spreads of both risky assets. Notably, the size of the spread increases with increasing correlation. Further, risky asset 2 shows a higher sensitivity (change in spread) to the correlation due to its riskier nature in comparison to the safer asset.

2.5.2 Case 1: Heterogeneity in the first moment of asset 2 and risk preferences

To understand the coupling effect of ambiguity and heterogeneity on bid-ask spreads, we study the heterogeneity impact of the first moment and the second moment separately in cases 1 and 2 before combining in case 3. The separate analysis will allow us to have a better grasp of how individual moment contributes in our model and to what extend the effects have on the overall price dynamics. Now, considers investors differ in risk preference and expected payoff of risky asset 2[8]. Investors' risk preference become $\tau_1 = \tau_o(1 - \gamma_\tau)$ and$\tau_2 = \tau_o(1 + \gamma_\tau)$, where $\gamma_\tau \in (-1, 1)$. Investor 1 is more (less) risk averse when $\gamma_\tau > 0 (< 0)$.

[8]Due to the nature of implied structure, two types of heterogeneity are assumed. Merely taking heterogeneity about expected payoff results a cancel-out resulting a reversion to the Benchmark case.

Further, investors' expected payoff vectors become $\mathbf{v_1} = (v_1, v_2(1-\gamma_v))^\top$ and $\mathbf{v_2} = (v_1, v_2(1+\gamma_v))^\top$, where $\gamma_v \in (-1,1)$. Investors are homogeneous about risky asset 1, this is purposely done so because we are interested if heterogeneity about a single asset translates to another, i.e. reduction in spread, through variance-covariance matrix and to what extend in presence of ambiguity.

The market belief at $t=0$ is then given by $\mathbf{v_{ac1}} = (v_1, v_2(1+\gamma_\tau\gamma_v))^\top$ and the variance-covariance is the same as the Benchmark. The market expected payoff of risky asset 2 differs to that of the benchmark case. By including heterogeneity, the sign of the additional linear term $\gamma_v\gamma_\tau$ determines the market expectation on asset 2, while asset 1 is implicitly affected through the risk-free rate. Equilibrium price vector $t=0$ can then be derived at

$$\begin{aligned}\mathbf{p^*_{o_{case1}}} &= (p^*_{o1_{case1}}, p^*_{o2_{case1}})^\top \\ &= \frac{1}{R_{f_{case1}}}\big(v_1 - \frac{(\sigma_1^2+\rho\sigma_1\sigma_2)}{\tau}, v_2(1+\gamma_\tau\gamma_v) - \frac{(\sigma_2^2+\rho\sigma_1\sigma_2)}{\tau}\big)^\top.\end{aligned} \tag{2.30}$$

where $R_{f_{case1}}$ is

$$R_{f_{case1}} = \frac{2}{w_{mo}}[(v_1 + v_2(1+\gamma_\tau\gamma_v)) - \frac{(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)}{\tau}]. \tag{2.31}$$

Equations (2.30) and (2.31) show equilibrium prices and risk free rate after introducing heterogeneity in Case 1 at t=0. In particular, $R_{f_{case1}}$ is endogenously derived, as such, it depends on the degree of correlation as well as $\gamma_\tau\gamma_v > 0 (< 0)$. Because γ_τ and γ_v are associated with the diversity of beliefs between the two investors, it is obviously that the larger the parameters, the greater the diversification between beliefs. When $\gamma_\tau\gamma_v > 0$, the market is dominated by an investor that is optimistic $\gamma_v > 0$, and high tolerant for risk, $\gamma_\tau > 0$. Therefore a higher risk free rate is required in order to attract investors' demand. On the other hand, when the market is pessimistic $\gamma_v < 0$ and else remains constant, one can expect the risk free rate to move in the opposite direction. The two scenarios are shown in Figure 2.3 by the upper solid and lower dotted lines respectively.

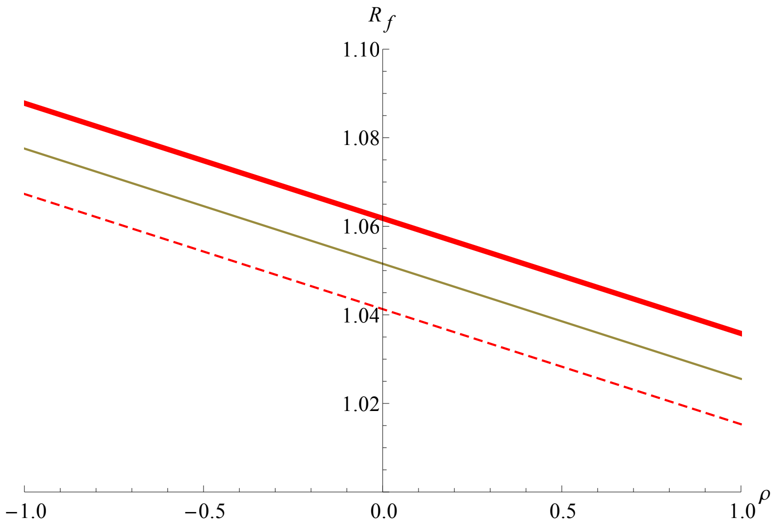


FIGURE 2.3: Equilibrium risk-free rate (y-axis) under the benchmark case with respect to the correlation coefficient (x-axis). An optimistic (thick), a benchmark (dotted), and a pessimistic (thin) are shown from top to bottom respectively. The parameters are $\gamma_\tau = \pm 0.05$ and $\gamma_v = \pm 0.25$.

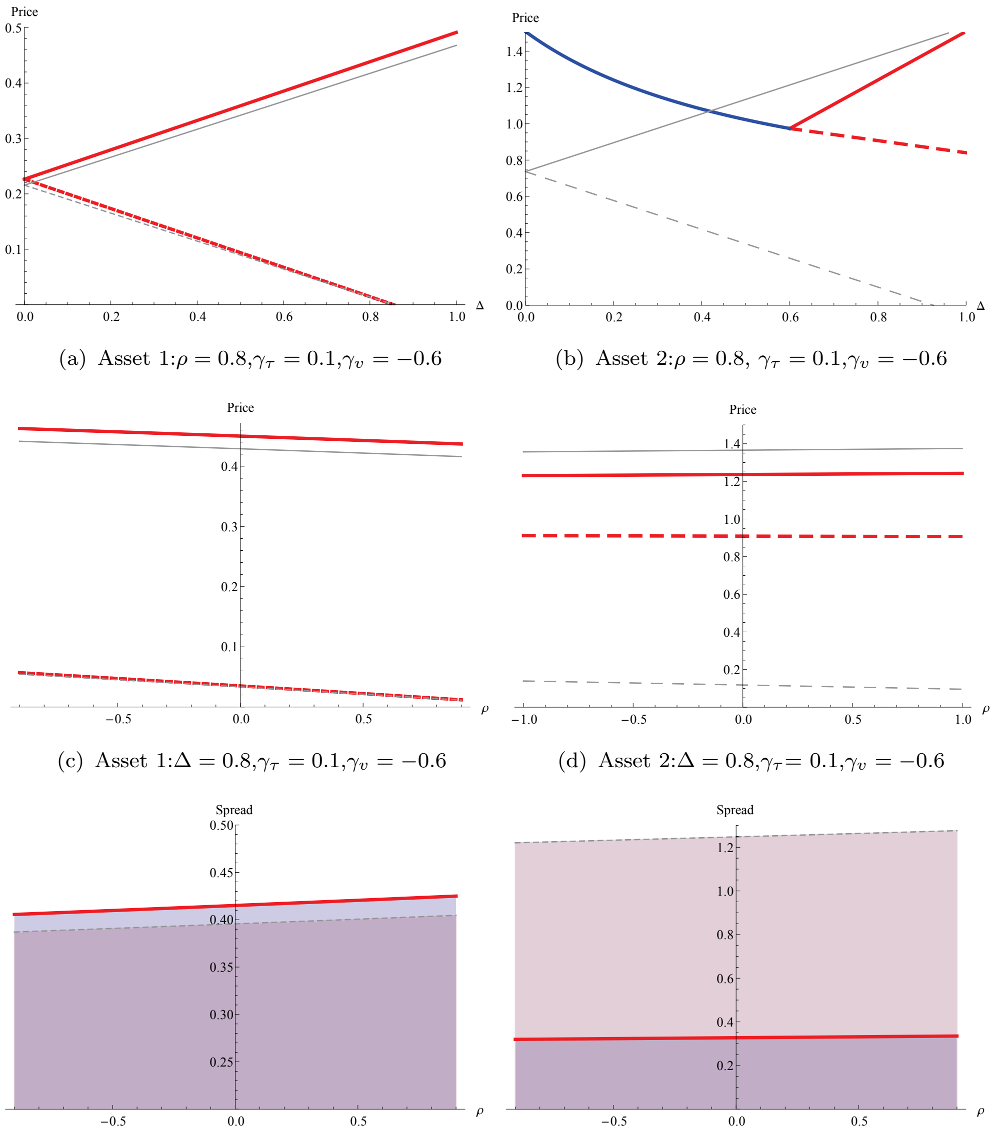
At $t = 1$, investors experience a systematic shock but do not know the magnitude of the shock. The bid and ask prices are derived respectively (see Appendix A.3.1),

$$\mathbf{p_{bid_{case1}}} = \big(p^*_{o1_{case1}} - \frac{(1-\underline{\alpha})min\{v_{i1}\}}{R_{f_{case1}}}, p^*_{o2_{case1}} - \frac{(1-\underline{\alpha})min\{v_{i2}\}}{R_{f_{case1}}}\big)^\top \tag{2.32}$$

$$\mathbf{p_{ask_{case1}}} = \big(p^*_{o1_{case1}} - \frac{(1-\bar{\alpha})max\{v_{i1}\}}{R_{f_{case1}}}, p^*_{o2_{case1}} - \frac{(1-\bar{\alpha})max\{v_{i2}\}}{R_{f_{case1}}}\big)^\top. \tag{2.33}$$

Different from the benchmark case, the added heterogeneity about the expected payoff and risk preference of risky asset 2 means the bid and ask price are determined by different investors, while the difference in opinion is implicitly flown to the safer asset through the equilibrium risk free rate.

Figure 2.4 shows the coupling effect of ambiguity and heterogeneity about the first moment (risky asset 2 only). Suppose we are in a pessimistic market, i.e. $\gamma_v\gamma_\tau <$



(a) Asset 1:$\rho = 0.8$,$\gamma_\tau = 0.1$,$\gamma_v = -0.6$

(b) Asset 2:$\rho = 0.8$, $\gamma_\tau = 0.1$,$\gamma_v = -0.6$

(c) Asset 1:$\Delta = 0.8$,$\gamma_\tau = 0.1$,$\gamma_v = -0.6$

(d) Asset 2:$\Delta = 0.8$,$\gamma_\tau = 0.1$,$\gamma_v = -0.6$

(e) Asset 1:$\Delta = 0.8$,$\gamma_\tau = 0.1$,$\gamma_v = -0.6$

(f) Asset 2:$\Delta = 0.8$,$\gamma_\tau = 0.1$,$\gamma_v = -0.6$

FIGURE 2.4: The figures are shown in a pessimistic market, i.e. $\gamma_\tau \gamma_v < 0$. The thick lines (both solid and dotted) represent case 1 and the thinner lines represent the benchmark case. (a) and (b) shows the price level (y-axis) of risky asset 1 and 2 over different level of ambiguity (x-axis), given different correlation. (c) and (d) show the price level (y-axis) of the bid and ask prices over a various of correlation ρ (x-axis), given a particular level of ambiguity Δ. (e) and (f) show the level of spread (y-axis) over a various of correlation ρ (x-axis) given the same Δ.

0 and assets are positively correlated, i.e. $\rho > 0$ to reflect an increasing asset correlation in an economic downturn. Figure 2.4(a) suggests investor's willingness to sell asset 1 decreases due to heterogeneity about risky asset 2. Recall the bid-ask price equation of risky asset 1 from above, the risk-free rate actually acts

as a medium to carry forward heterogeneity of the related asset via the variance-covariance matrix. On the other hand, it is clear in Figure 2.4(b) that direct impact of heterogeneity on the riskier asset helps to reduce market illiquidity immensely. Specifically, the single line leading up just before the separation of bid and ask prices shows an equilibrium price may be derived as long as ambiguity is small enough (and removing illiquidity); while asset 1 remains illiquid for all degree of ambiguity. This observation is very different to that of the market in absence of ambiguity. This is because under the influence of ambiguity, unless asset 1 is also subjected to some diversity in beliefs, investors cannot rank their preferences like utility maximisers, which induces no-trade spread regardless of the other risky asset. The threshold denoted Δ_o according to equation (2.22), is therefore given by $\Delta_o = \frac{1}{\alpha_o}\gamma_v(1-\alpha_o) < \Delta$. With ambiguity beyond the threshold $\Delta > \Delta_o$, the bid and ask spread is observed as in the benchmark case, however, the price levels certainly varied to reflect investors' heterogeneous beliefs about the first moment of asset 2. Figures 2.4(c) and 2.4(d) show an asymmetrical changes in the level of bid and ask prices due to heterogeneity about the riskier asset, even though the bid and ask prices have moved closer to each other in risky asset 2, it appears the sell price of the safer asset has increased instead. Finally, Figures 2.4(e) and 2.4(f) show the implication of the coupling effect has a drastic change on the risky asset 2 - that is the direct impact of heterogeneity reduces the consequent spread size, however the carried forward effect of heterogeneity in risky asset 1 shows the opposite, instead of a reduction, the spread has since increased. In summary, even though heterogeneity does reduce bid-ask spread or no-trade spread to some extend for asset 2, investors seem to be reluctant to sell risky asset 1 due to its safer nature in a pessimistic market.

2.5.3 Case 2: Heterogeneity of the second moment of Asset 2

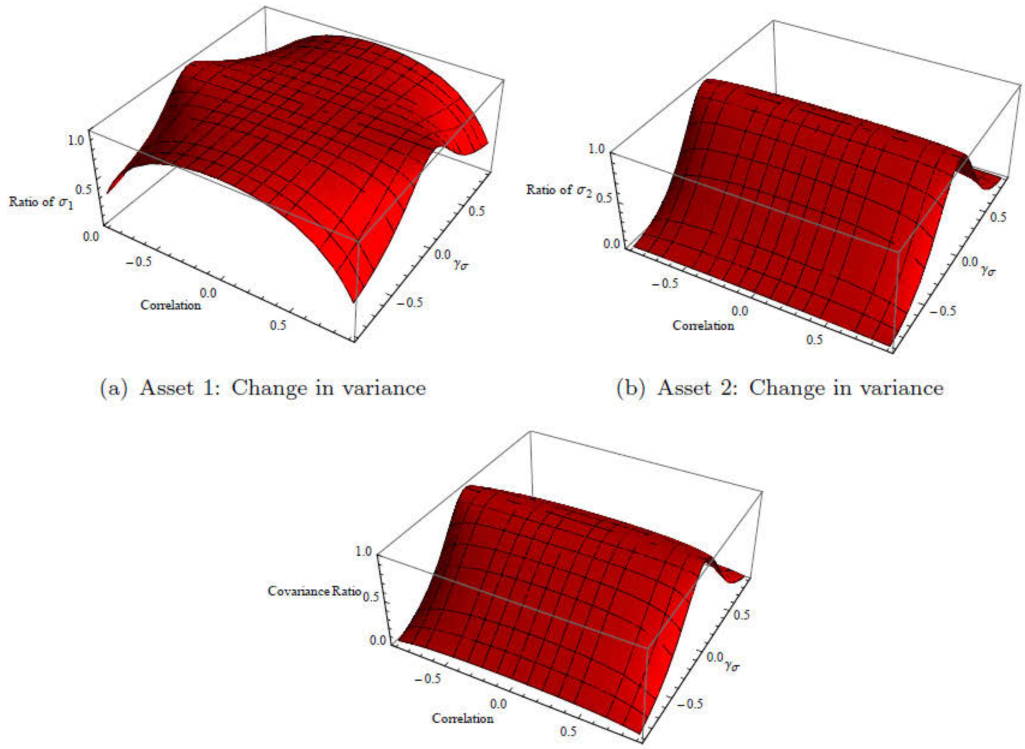
Now, let us consider investors are heterogeneous about the second moment of risky asset 2. That is, the standard deviations for risky asset 1 is $\sigma_{11} = \sigma_{21} = \sigma_1$; while the standard deviations of risky asset 2 become $\sigma_{12} = \sigma_2(1-\gamma_\sigma)$ and $\sigma_{22} = \sigma_2(1+\gamma_\sigma)$ where $\gamma_\sigma \in (-1,1)$ for investor 1 and 2 respectively. Investor 1 is said to be more (less) confident if $\gamma_\sigma > 0(\gamma_\sigma < 0)$. The market belief at $t=0$ becomes $\mathbf{v_{ac2}} = \mathbf{v_o} = (v_1, v_2)^\top$, and

$$\mathbf{\Omega_{ac2}} = \begin{pmatrix} (1-\frac{\gamma_\sigma^2\rho^2}{1+\gamma_\sigma^2-\rho^2})\sigma_1^2 & \rho\sigma_1\sigma_2(1-\frac{\gamma_\sigma^2(2-\rho^2)}{1+\gamma_\sigma^2-\rho^2}) \\ \rho\sigma_1\sigma_2(1-\frac{\gamma_\sigma^2(2-\rho^2)}{1+\gamma_\sigma^2-\rho^2}) & (\frac{(1-\gamma_\sigma^2)^2(1-\rho^2)}{1+\gamma_\sigma^2-\rho^2})\sigma_2^2 \end{pmatrix}.$$

The non-linear impact on the market variance-covariance matrix is quite apparent. Figure 2.5 shows coefficients of market variance of asset 1, defined by $(1-\frac{\gamma_\sigma^2\rho^2}{1+\gamma_\sigma^2-\rho^2})\sigma_1^2$ and asset 2, defined by $(\frac{(1-\gamma_\sigma^2)^2(1-\rho^2)}{1+\gamma_\sigma^2-\rho^2})\sigma_2^2$. As well as the coefficient of covariance, defined by $(1-\frac{\gamma_\sigma^2(2-\rho^2)}{1+\gamma_\sigma^2-\rho^2})\rho\sigma_1\sigma_2$. In particular, both risky assets exhibit a reduction in market risk due to heterogeneity about the second moment of the riskier asset (see Chiarella et al. [2011] for more discussion on this effect). The effect of heterogeneity is most prominent in risky asset 2, such that market risk is reduced due to diversified beliefs (as long as γ_σ is away from zero), while indirectly impacts the safer asset through variance-covariance matrix.

The equilibrium price vector at t=0 is derived as below

$$\begin{aligned} \mathbf{p^*_{o case2}} &= (p^*_{o1_{case2}}, p^*_{o2_{case2}})^\top \\ &= \frac{1}{R_{f_{case2}}}\Big(v_1 - \frac{(1-\rho^2)((1+\gamma_\sigma^2)\sigma_1^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}, \\ &\quad v_2 - \frac{(1-\rho^2)((1-\gamma_\sigma^2)^2\sigma_2^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2}\Big)^\top, \end{aligned} \tag{2.34}$$



(a) Asset 1: Change in variance

(b) Asset 2: Change in variance

(c) Covariance of Asset1 and Asset 2: Change in covariance

FIGURE 2.5: The changes in market confidence and covariance are demonstrated to be highly non-linear in both risky assets. All coefficients are strictly positive, i.e (0,1].

where $R_{f_{case2}}$ is

$$R_{f_{case2}} = \frac{2(v_1 + v_2)}{w_{mo}} - \frac{2(1-\rho^2)((1+\gamma_\sigma^2)\sigma_1^2 + 2(1-\gamma_\sigma^2)\rho\sigma_1\sigma_2 + (1-\gamma_\sigma^2)^2\sigma_2^2)}{(1+\gamma_\sigma^2-\rho^2)\tau_o w_{mo}}. \tag{2.35}$$

Equilibrium price vector shown in equation (2.34) is non-linearity and both prices depend on the dispersion in beliefs about the second moment of the risker asset. Further, the risk-free rate exhibits a similar non-linear relationship to correlation, which is illustrated in Figure 2.6. Compare to the benchmark case, case 2 always lies above the benchmark as it becomes increasingly insensitive to correlation to reflect a decreasing market risk due to heterogeneous beliefs among investors.

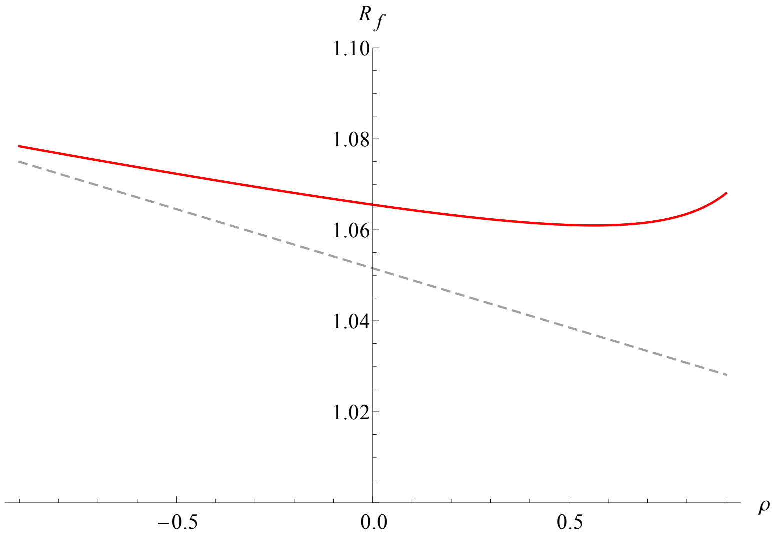


FIGURE 2.6: Equilibrium risk-free rate (y-axis) under the benchmark case with respect to the correlation coefficient (x-axis). Case 2 is shown above the benchmark. The parameter is $\gamma_\sigma = \pm 0.6$.

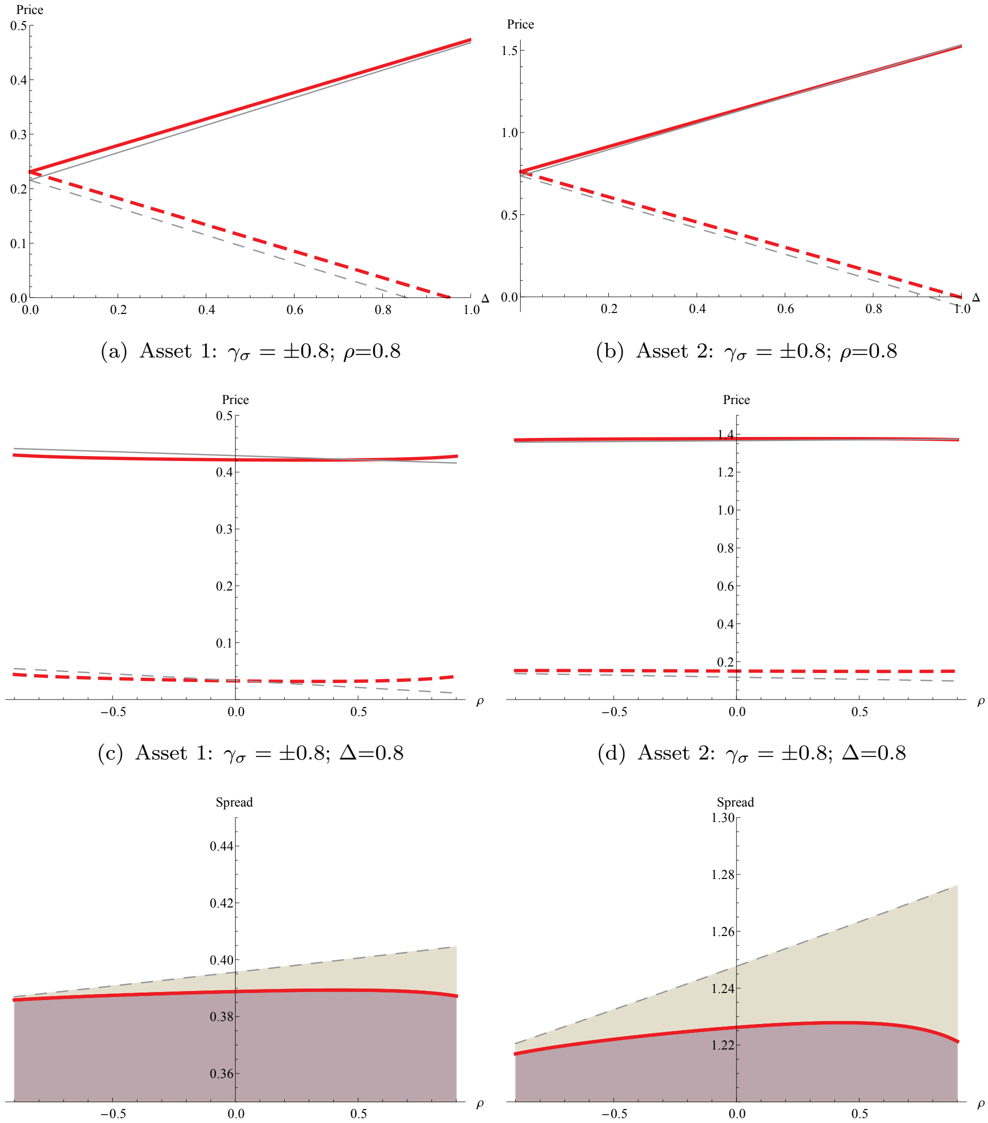
At $t = 1$, equilibrium bid and ask prices are derived respectively (refer to Appendix A.3.2)[9]

$$\mathbf{p_{bid_{case2}}} = \frac{1}{R_{f_{case2}}}\Big(\alpha_o(1-\Delta)v_1 - \frac{(1-\rho^2)((1+\gamma_\sigma^2)\sigma_1^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}, \\ \alpha_o(1-\Delta)v_2 - \frac{(1-\rho^2)((1-\gamma_\sigma^2)^2\sigma_2^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}\Big)^\top \tag{2.36}$$

$$\mathbf{p_{ask_{case2}}} = \frac{1}{R_{f_{case2}}}\Big(\alpha_o(1+\Delta)v_1 - \frac{(1-\rho^2)((1+\gamma_\sigma^2)\sigma_1^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}, \\ \alpha_o(1+\Delta)v_2 - \frac{(1-\rho^2)((1-\gamma_\sigma^2)^2\sigma_2^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}\Big)^\top. \tag{2.37}$$

Compared to the previous cases, we observe a systematic upward shift in price for both risky assets (regardless of the correlation coefficient) due to a decreased market risk, the results are shown in Figures 2.7(a) and 2.7(b). Because heterogeneity

[9] Due to the structure of the model in Case 2, prices and spreads are unaffected by the sign of γ_σ.



(a) Asset 1: $\gamma_\sigma = \pm 0.8$; ρ=0.8

(b) Asset 2: $\gamma_\sigma = \pm 0.8$; ρ=0.8

(c) Asset 1: $\gamma_\sigma = \pm 0.8$; Δ=0.8

(d) Asset 2: $\gamma_\sigma = \pm 0.8$; Δ=0.8

(e) Asset 1: $\gamma_\sigma = \pm 0.8$; Δ=0.8

(f) Asset 2: $\gamma_\sigma = \pm 0.8$; Δ=0.8

FIGURE 2.7: The thick lines (both solid and dotted) represent case 1 and the thinner lines represent the benchmark case. (a) and (b) shows the price level (y-axis) of risky asset 1 and 2 over different level of ambiguity (x-axis), given different correlation. (c) and (d) show the price level (y-axis) of the bid and ask prices over a various of correlation ρ (x-axis), given a particular level of ambiguity Δ. (e) and (f) show the level of spread (y-axis) over a various of correlation ρ (x-axis) given the same Δ.

is applied to the second moment of the riskier asset, the resulting reduction in market risk is translated to the safer asset through the variance-covariance matrix and is non-linear. Further, the non-linearity is also reflected in the bid and ask prices in Figures 2.7(c) and 2.7(d). More importantly, the change in spreads shown in

Figures 2.7(e) and 2.7(f) exhibits reduction in both assets as oppose to the previous case. In summary, case 3 demonstrates that when investors are heterogeneous about the second moment of the riskier asset, the implication does not only bring about non-linearity, the impact of ambiguity is also reduced in both assets, i.e. reduced market illiquidity. This suggests that in times of uncertainty, heterogeneity about the second moment alone can increase the possibility of trading activities as oppose to the first moment.

2.5.4 Case 3: Heterogeneity in the first and second moments of Asset 2

Finally, in this section we provide a complete picture by combining cases 1 and 2. Here, investors differ in opinions about the first and the second moments of risky asset 2, while risky asset 1 remains homogeneous. Investors' expected payoff vectors and standard deviations are then given by $\mathbf{v_2} = (v_1, v_2(1-\gamma_v))^\top$ and $\mathbf{v_2} = (v_1, v_2(1+\gamma_v))^\top$, where $\gamma_v \in (-1,1)$; and $\sigma_{11} = \sigma_{21} = \sigma_1$, and $\sigma_{12} = \sigma_2(1-\gamma_\sigma)$ and $\sigma_{22} = \sigma_2(1+\gamma_\sigma)$, where $\gamma_\sigma \in (-1,1)$, respectively.

At $t=0$, market belief is given by $\mathbf{v_{ac3}} = \mathbf{v_o} = (v_1 - \frac{\gamma_\sigma\gamma_v\rho\sigma_1}{\sigma_2(1+\gamma_\sigma^2-\rho^2)}v_2, v_2(1 - \frac{\gamma_\sigma\gamma_v(2-\rho^2)}{1+\gamma_\sigma^2-\rho^2})^\top$ and $\mathbf{\Omega_{ac3}} = \mathbf{\Omega_{ac2}}$. The same variance-covariance matrix is observed in Case 3 as in Case 2; however, market expected payoff vector differs. Due to variance-covariance matrix, market expected payoff of risky asset 1 is not only affected by risky asset 2 in both moments, the resulting impact is also non-linear.

The equilibrium price vector at $t=0$ becomes

$$\begin{aligned}\mathbf{p^*_{o_{case3}}} &= (p^*_{o1_{case3}}, p^*_{o2_{case3}})^\top \qquad (2.38)\\ &= \frac{1}{R_{f_{case3}}}\Big(v_1 - \frac{\gamma_\sigma\gamma_v\rho\sigma_1 v_2}{(1+\gamma_\sigma^2-\rho^2)\sigma_2} - (1-\rho^2)\frac{(1+\gamma_\sigma^2)\sigma_1^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2}{\tau_o(1+\gamma_\sigma^2-\rho^2}, \\ &\quad v_2 - \frac{\gamma_\sigma\gamma_v(2-\rho^2)v_2}{1+\gamma_\sigma^2-\rho^2} - (1-\rho^2)\frac{(1-\gamma_\sigma^2)^2\sigma_2^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2}{\tau_o(1+\gamma_\sigma^2-\rho^2)}\Big)^\top.\end{aligned}$$

where $R_{f_{case3}}$ is

$$R_{f_{case3}} = \frac{2((1+\gamma_\sigma^2-\rho^2)\sigma_2 v_1 - (\gamma_\sigma\gamma_v\rho\sigma_1 - (1+\gamma_\sigma^2-\rho^2-\gamma_\sigma\gamma_v(2-\rho^2)\sigma_2))v_2)}{(1+\gamma_\sigma^2-\rho^2)w_{mo}\sigma_2}$$
$$- \frac{2(1-\rho^2)((1+\gamma_\sigma^2)\sigma_1^2 + 2(1-\gamma_\sigma^2)\rho\sigma_1\sigma_2 + (1-\gamma_\sigma^2)^2\sigma_2^2)}{(1+\gamma_\sigma^2-\rho^2)\tau_o w_{mo}} \tag{2.39}$$

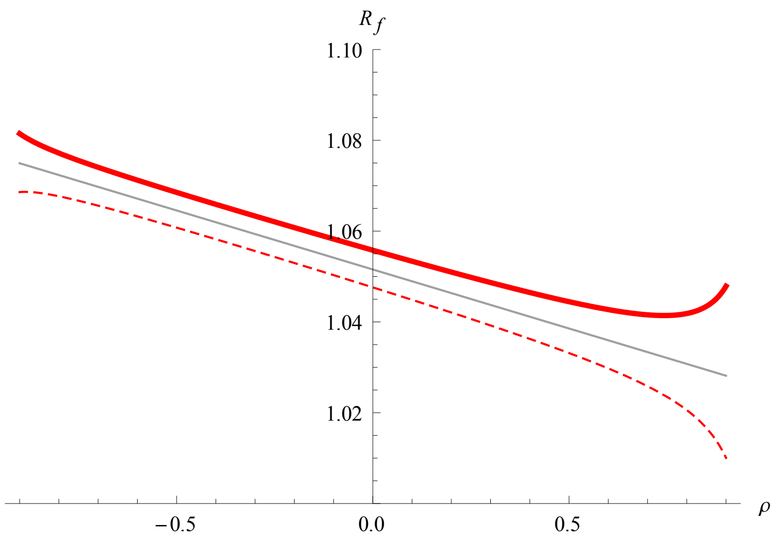
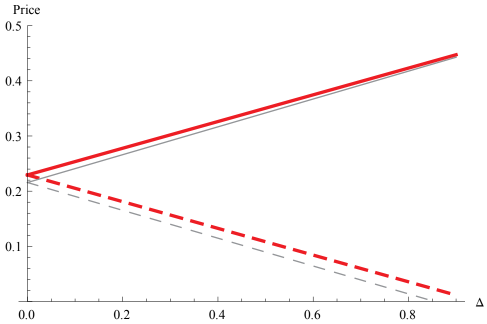
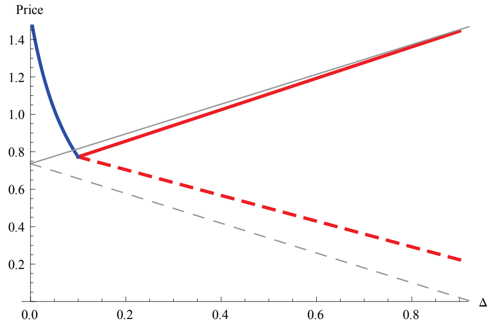


FIGURE 2.8: Equilibrium risk-free rate (y-axis) under the benchmark case with respect to the correlation coefficient (x-axis). The above risky free rate shows a optimistic market with high volatility (thick), a Benchmark (dotted), and a pessimistic market with the same volatility (thin) (from top to bottom). The parameter are $\gamma_v = \pm 0.05$ and $\gamma_\sigma = \pm 0.05$.

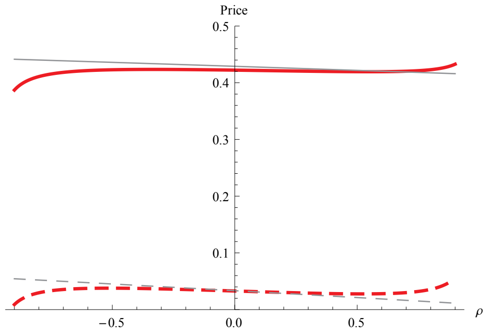
Figure 2.8 shows the risk-free rate for both $\gamma_v\gamma_\sigma > 0$ and $\gamma_v\gamma_\sigma < 0$ against the benchmark. Similarly, if the market is dominated by pessimistic investor ($\gamma_v < 0$), and higher volatility ($\gamma_\sigma > 0$) investors are less willing to invest in the risky asset and the risk-free rate will decrease due to increasing demand for risk-free asset. On the other hand, if the market is optimistic $\gamma_v > 0$ and the volatility remains the same, one can expect the risk free rate to move upwards due to increasing demand for risky assets. The two scenarios are shown in Figure 2.8 by the lower solid and upper dotted lines respectively.



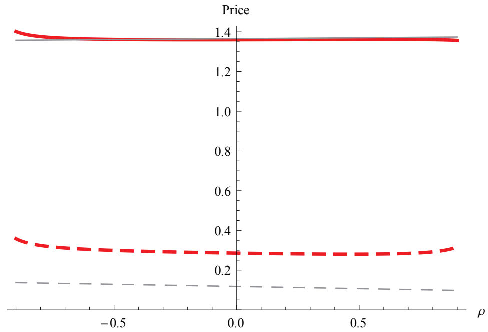
(a) Asset 1: $\gamma_v = -0.1;\gamma_\sigma = 0.1;\rho = 0.8$



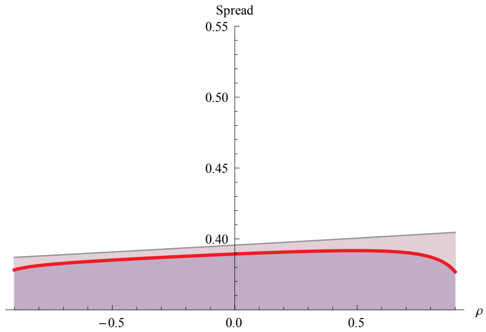
(b) Asset 2: $\gamma_v = -0.1;\gamma_\sigma = 0.1;\rho = 0.8$



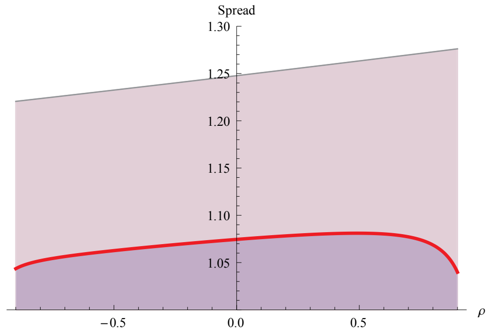
(c) Asset 1: $\gamma_v = -0.1;\gamma_\sigma = 0.1;\Delta = 0.8$



(d) Asset 2: $\gamma_v = -0.1;\gamma_\sigma = 0.1;\Delta = 0.8$



(e) Asset 1:$\gamma_v = -0.1;\gamma_\sigma = 0.1;\Delta = 0.8$



(f) Asset 2: $\gamma_v = -0.1;\gamma_\sigma = 0.1;\Delta = 0.8$

FIGURE 2.9: The figures are shown in a pessimistic market, i.e. $\gamma_\tau\gamma_v < 0$. The thick lines (both solid and dotted) represent case 1 and the thinner lines represent the benchmark case. (a) and (b) shows the price level (y-axis) of risky asset 1 and 2 over different level of ambiguity (x-axis), given different correlation. (c) and (d) show the price level (y-axis) of the bid and ask prices over a various of correlation ρ (x-axis), given a particular level of ambiguity Δ. (e) and (f) show the level of spread (y-axis) over a various of correlation ρ (x-axis) given the same Δ.

At period 1, bid and ask prices are again derived (refer to Appendix A.3.3). It is easy to identify that a single price only occurs in case of heterogeneity about the first moment show in Figure 2.9(b), while the non-linearity effect is due to heterogeneity about the second moment shown in Figures 2.9(c) to 2.9(f). In summary, heterogeneity does reduce market illiquidity in both directly and indirectly impacted assets. An important observation to note here is that although an amplified spread is observed in case 1 (heterogeneous about the first moment only), the diversified belief about the second moment dominates the overall effect, which reduces market illiquidity. In a market that experiences a downward shock, the uncertainty about the market can be reduced by heterogeneity, in particular, the market becomes more liquid when investors are heterogeneous about both mean and variance of the risk asset.

2.6 Conclusion

This chapter explores coupling effect of ambiguity and heterogeneity on market liquidity in a multi-risky asset market populated with investors who agree to disagree during market downturns. Heterogeneous investors become uncertain about the future asset values after experiencing a common downward shock. Market fails to arrive at an equilibrium and no-trade spread is showned in presence of ambiguity. The results show that despite the correlation coefficients between risky assets, ambiguity affects all assets with a greater impact on the riskier asset than a relative less risky asset. Further, while heterogeneity do facilitate trades, it is not always true when the market experiences heightened uncertainty, in fact, illiquidity worsen in some cases. Heterogeneity is more effective in reducing illiquidity when it is about the second moment of the risky asset, i.e. variance and covariance, than when it is about the first moment. In general, market freeze is observed in most case regardless of heterogeneity and correlation coefficients between the risky assets under the influence of ambiguity. Going forward, it should be fruitful to study ambiguity model in a dynamic setting allowing investors to update their beliefs according to their own set of rules. The interactions between investors under ambiguity in the market should generate interesting results that characterise common market anomalies and stylised facts. This is dealt with in Chapters 3 and 4.

Chapter 3

Price Dynamics and Excess Volatility under Heterogeneous Beliefs and Ambiguity

3.1 Introduction

The previous chapter studies the coupling implication of heterogeneity and ambiguity in a multi-risky asset market. In particular, it emphasises on the lack of trading activities when investors are experiencing a systematic downward shock and resulting no trade between the investors, exhibiting Bewley [2002]'s preferences. Consequently, an equilibrium price is unable to be derived. While the static setup provides some insight in understanding the mechanisms, and the implications on market illiquidity; real markets are indeed dynamic, populated with investors that are often known to follow specific trading strategies and of course complicated. Further, a static setup is unable to generate price and return series and therefore cannot discuss the implication of ambiguity in a dynamic market that typically exhibits a variety of stylised facts and anomalies. In this chapter, we introduce a dynamic model and show that it has the capability to generate volatility clustering and excess volatility, insignificant ACs in returns, and significant ACs in absolute and squared returns under the influence of ambiguity and heuristic behaviours of the investors. The important point to highlight here is that these stylised facts are the implications of ambiguity rather than heterogeneity embedded in the model.

Shiller [1992] and LeRoy and Porter [1981], among others, have argued that asset prices are more volatile than changes in the fundamentals would otherwise predict, they refer to this observation as *excess volatility*. Other evidences have shown volatility clustering, long memories in absolute and squared returns insignificant in ACs in returns[1]. The explanatory power of traditional asset pricing models for many of these market stylised facts is fairly limited, and much of the work has been done in the HAMs literature, where these models have shown great capabilities in producing market stylised facts and anomalies[2]. Although Amado and Laakkonen [2012] empirically address the linkage between ambiguity and excess volatility, little works have been done in the ambiguity literature to show that they have similar capability as the HAMs. This chapter and next chapter attempt to fill in this gap.

A distinctive feature of the heterogeneous agent models (HAMs) is the ability of allowing different agents to co-exist in the financial market. It is firstly discussed in Zeeman [1974] as a mechanism to the rise and fall of bull and bear markets. The two most common agent types are fundamentalists and chartists. This chapter incorporates these heuristic trading rules in a dynamic setting, while allow investors to update their belief from one period to the next in an intuitive manner. See He and Li [2007] for an updated version of the heuristic rule in which we have borrowed in this chapter, they show that agent heterogeneity, risk-adjusted

[1] See Cont [2001] and references within for empirical discussions.

[2] See recent paper include He and Li [2007] where they study the source of power-law distributed fluctuations in volatility. Others included Cont [2007], Amilon [2008], Franke [2010], Franke and Westerhoff [2011, 2012], Chen et al. [2012].

trend chasing through the geometric learning process, and the interplay of noisy fundamental and demand processes and the underlying deterministic dynamics can be the source of power-law distributed fluctuations. In this chapter however, we show that some of those commonly observed stylised facts arise from investor's uncertainty about the fundamental value of the risky asset. Therefore, rather than letting each investor to differ in opinion as in the previous chapter, we introduce three types of investors, i.e. fundamentalists, trend followers, and noise investors. Both fundamentalists and trend follows update their expectation about the future value of the risky asset to some specific rule.

Different from the previous chapter where the market experiences a systematic ambiguity shock, we discuss ambiguity only in fundamentalists in the sense of Gilboa [1989]. We motivate fundamentalists to experience ambiguity based on empirical evidences demonstrated in Whaley [2000]; Williams [2009]; Bloom [2009]; and Bird and Yeung [2012], where they show that the level of uncertainty about the true value of assets surged during the financial crises. Since the value of the fundamental is not readily observable, intuitively, investors are expected to become increasingly ambiguous about the true fundamental value of the asset as uncertainty heightens. Whereas trend followers on the other hand form expectations by extrapolating information from historical prices and therefore are assumed not been subjected to ambiguity. Similar idea is applicable to noise investors where they demand for other than investment reasons.

This chapter is most closely related to Easley and O'Hara [2009], who investigate the welfare effects of financial market policies when market is populated with both ambiguity-averse traders and utility maximisers under a static setup. They suggest there are limited-participation in the market when some traders choose to stay out of the market when they experience ambiguity. They relate this phenomenon to the lack of participation in equity markets from households. As a result, they suggest several government policies in attempt to reduce ambiguity and encourage market participation. We differ from Easley and O'Hara [2009] in two ways. First, the model is dynamical instead of static. Second, we assume three types of investor. In particular, the fundamentalists exhibit max-min utility preferences because they are uncertain about the exact fundamental value of the asset. Due to this type of preference, there will be time where the fundamentalists choose to stay out of the market unless price of the asset becomes more enticing to re-enter the market again. Trend followers are utility maximisers as they purely extrapolate information from past prices. While noise traders are liquidity traders whose investment decisions are made exogenously and are non-investment related. In the chapter, we borrow the setup of the two of investors from Easley and O'Hara [2009] but incorporate simple heuristic rules to reflect the common types of investors in the market and show that ambiguity and the interactions between the heuristic behaviours of the investors has explanatory power to stylised facts.

In this chapter, we contribute to the literature by employing heuristic behaviours from HAMs in an ambiguity model and show that it has the capability to generate a variety of stylised facts which have yet to be demonstrated by others (to the best of our knowledge). The model allows for three different types of strategists, where each type follows a specific heuristic updating rule. In particular, fundamentalists experience ambiguity in the fundamental value. Then, under min-max preference, they will choose not to participate in markets for a range of prices unless the movement in the price is large enough. Due to this nature of switching between participation and non-participation of the fundamentalists over time, coupled with the interaction between the three types of investors, we show through simulations that the model is able to produce stylised facts such as long memories in absolute returns, and excess volatility that are due to ambiguity rather than heterogeneity.

The remainder of the chapter is organised as follows. Section 3.2 establishes the general framework of the model. Section 3.3 analyses market dynamics under different market conditions, we show that our ***Model under Ambiguity (MuA)*** is able to generate a variety of market stylised facts including autocorrelation patterns in returns, absolute returns, and squared returns; as well as volatility clustering and excess volatility. Section 3.4 concludes. The proofs are provided in Appendix B.

3.2 The Model

In this section, we develop a standard asset pricing model with heterogeneous agents under ambiguity. Consider a market with one risky asset and one risk-free asset. The risk-free asset is perfectly elastically supplied at a gross return of $R_f = 1 + r_f/K$, where r_f is an annual constant risk-free rate[3]. The ex-dividend price and dividend per share of the risky asset is denoted p_t and d_t respectively at time t. The market consists of three types of investors; in particular, we study asset price implications of the interactions between ambiguity-averse *fundamentalists* $(i = f)$, utility maximising *trend followers* $(i = c)$ and *noise investors* $(i = n)$. Further, investors have CARA utility for wealth $U_{i,t}(W) = -e^{-a_i W_{i,t}}$. The future wealth of type i investor, $W_{i,t+1}$ is given by

$$W_{i,t+1} = x_{i,t}(p_{t+1} + d_{t+1}) + m_{i,t}R_f, \tag{3.1}$$

subjected to a budget constraint $m_{i,t} = W_{i,t} - x_{i,t}p_t$, where $m_{i,t}$ is the wealth invested in the risk-free asset, $x_{i,t}$ is the demand of the risky asset, and a_i is the risk aversion of investor i. Further p_{t+1} and d_{t+1} are the future price and dividend of the risky asset respectively. We assume the dividend process d_t follows a normal distribution $d_t \sim \mathcal{N}(\bar{d}, \sigma_d^2)$. We now introduce conditional expectation and variance, and derive the optimal demand of investor type i.

[3]To reflect risk-free rate in a yearly, monthly, or daily manner, we let K=1,12,52,250 respectively.

Fundamentalists believe that price may deviate away from the fundamental value of the asset in the short-term, however, the price always reverts back to the true fundamental value in the long-term. Typically, this fundamental value is assumed to follow a random walk process $p^*_{t+1} = p^*_t(1 + \sigma_{p^*}\epsilon_t)$, and is normally distributed with $\epsilon_t \sim \mathcal{N}(0, \sigma^2_\epsilon)$, $\sigma_\epsilon \geq 0$ and the initial price is $p^*_o = \bar{p} > 0$. The fundamentalists' conditional expectation and variance are thus

$$\mathbb{E}_{f,t}(p_{t+1}) = p_{t-1} + \gamma_f[\mathbb{E}(p^*_{t+1}) - p_{t-1}], \quad \textit{and} \quad \mathbb{V}_{f,t}(p_{t+1}) = \mathbb{V}_{f,t}(p^*_{t+1}), \tag{3.2}$$

where $\gamma_f > 0$ measures how quickly the fundamentalists believe the price would converge to the expected fundamental value. In extreme cases, the asset price converges instantaneously if $\gamma_f = 1$; and is independent of the fundamental value if $\gamma_f = 0$. Further, the conditional expectation depends on the past price and the expectation of the fundamental value, while variance is the volatility of the fundamental value.

From above, we know fundamentalists' beliefs are heavily affected by $\mathbb{E}(p^*_{t+1})$. Now, assume fundamentalists are ambiguous about the fundamental value

$$\mathbb{E}(p^*_{t+1}) = (1 + \mu)p^*_t, \quad \textit{and} \quad \mathbb{V}_{f,t}(p^*_{t+1}) = \sigma^2_f, \tag{3.3}$$

such that the true values of μ and σ^2_f are unknown. Rather, they are bounded by $\mu \in [\underline{\mu}, \overline{\mu}]$ with $-1 < \underline{\mu} < \overline{\mu} < 1$, and $\sigma^2_f \in (\underline{\sigma}^2_f, \overline{\sigma}^2_f)$. Therefore, the larger the range of μ and σ^2_f the greater the ambiguity. For example, fundamentalist is less ambiguous about the fundamental value if $\mu \in [-10\%, 10\%]$ than say if $\mu \in [-40\%, 40\%]$. While, in case $\mu = 0$, fundamentalists are not ambiguous. Consequently, the above conditional expectation and variance of the fundamentalists become

$$\mathbb{E}_{f,t}(p_{t+1}) = p_{t-1} + \gamma_f((1+\mu)p^*_{t+1} - p_{t-1}), \tag{3.4}$$
$$\mathbb{V}_{f,t}(p_{t+1}) = \sigma_f^2.$$

Due to the ambiguity of the fundamental value, the set of expectation and variance can be expressed

$$\mathbb{E}_{f,t}(p_{t+1}) \in (\underline{\mathbb{E}}_{f,t}(p_{t+1}), \overline{\mathbb{E}}_{f,t}(p_{t+1})), \quad \mathbb{V}_{f,t}(p_{t+1}) \in (\underline{\sigma}_f^2, \overline{\sigma}_f^2), \tag{3.5}$$

where $\underline{\mathbb{E}}_{f,t}(p_{t+1}) = p_{t-1} + \gamma_f[(1+\underline{\mu})\mathbb{E}(p^*_{t+1}) - p_{t-1}]$ and $\overline{\mathbb{E}}_{f,t}(p_{t+1}) = p_{t-1} + \gamma_f[(1+\overline{\mu})\mathbb{E}(p^*_{t+1}) - p_{t-1}]$. Therefore, γ_f affects the level of ambiguity. Fundamentalist chooses a portfolio to maximise the minimum expected utility over the set of possible distribution

$$\theta_t \in \{(\mathbb{E}_{f,t}(p_{t+1}), \mathbb{V}_{f,t}(p_{t+1}); \mathbb{E}_{f,t}(p_{t+1}) \in (\underline{\mathbb{E}}_{f,t}(p_{t+1}), \overline{\mathbb{E}}_{f,t}(p_{t+1})), \mathbb{V}_{f,t}(p_{t+1}) \in (\underline{\sigma}_f^2, \overline{\sigma}_f^2)\}. \tag{3.6}$$

Essentially, they avoid the worst case outcome by choosing a portfolio that has the lowest risk. We write the maxmin problem as follow

$$\max_{x_t} \min_{\theta_t} \{R_f W_{f,t} + x_{f,t}(\mathbb{E}_{f,t}(p_{t+1}+d_{t+1}) - R_f p_t) - \frac{1}{2} a_f x_{f,t}^2 \mathbb{V}_{f,t}(p_{t+1}+d_{t+1})\}. \tag{3.7}$$

Since the minimisation problem says that any portfolio's minimum occurs at the maximum possible variance, fundamentalists simply take the highest possible risk $\overline{\sigma}_f^2$ for every possible payoff. Therefore, whether the problem is minimised at the maximum or the minimum mean payoff depends on the whether the fundamentalist is buying or selling the risky asset. Similar to Easley and O'Hara [2009], we show in Appendix B.1 that the optimal demand function is

$$x_t^f = \begin{cases} \frac{\underline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}-p_t R_f}{a_f(\overline{\sigma}_f^2+\sigma_d^2)} > 0 & \text{if} \quad \frac{\underline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}}{R_f} > p_t \\ 0 & \text{if} \quad \frac{\underline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}}{R_f} \leq p_t \leq \frac{\overline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}}{R_f} \\ \frac{\overline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}-p_t R_f}{a_f(\overline{\sigma}_f^2+\sigma_d^2)} < 0 & \text{if} \quad \frac{\overline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}}{R_f} < p_t. \end{cases} \tag{3.8}$$

The optimal demand function only depends on the minimum and maximum of the mean payoff and the maximum variance. From equation (3.4), we know the set of distributions θ_t depends on the range of ambiguity. Intuitively, the larger the range the more likely that fundamentalists will stay out of the market and demand zero position of the risky asset, we refer to this as *market non-participation*[4]. On the other hand, depending on whether the fundamentalists are buying or selling, the demand function is either determined by the first or the last expression in (3.8); and we refer this as the *buy-only* and *sell-only* market participation respectively. A buy (sell) would suggest the fundamentalists expect the worst case scenario (highest risk) with the lowest (highest) possible expected payoff. Finally, the demand function is continuous in price but is kinked at $\underline{\mathbb{E}}_{f,t}$ and $\overline{\mathbb{E}}_{f,t}$ as shown in equation (3.8).

Trend followers believe information lies in past prices and their expected future price of the risky asset can be predicted by extrapolating patterns and trends from those prices. Following He and Li [2007], we let trend followers to update their conditional expectation as well as variance ever period, such that

$$\mathbb{E}_{c,t}(p_{t+1}) = p_{t-1} + \gamma_c[p_{t-1} - u_{t-1}], \quad \textit{and} \;\; \mathbb{V}_{c,t}(p_{t+1}) = \sigma_f^2(1 + v_t). \tag{3.9}$$

[4]Nevertheless, this market non-participation may still generate trades depending on the holding from the previous period. The trade is the difference between the position at time $t+1$ and t, denotes $x_{f,t+1} - x_{f,t}$. If the previous holding is non-zero, the difference is the excess position that is required to be balanced off to zero.

The constant $\gamma_c \geq 0$ measures the strength of the extrapolation. The greater the value the more aggressive the trend followers are in extracting past information. We assume the sample has a mean u_t and variance v_t that follows a geometric decay process (GDP) (see He and Li [2007]). In other words, the extrapolation process is not equally weighted across all periods; rather it weighs more on the most recently observed prices with decreasing weight on the distant prices. The sample mean and variance are thus given by

$$u_t = \delta_c u_{t-1} + (1-\delta_c)p_t, \;\; and \;\; v_t = \delta_c v_{t-1} + \delta_c(1-\delta_c)(p_t - u_{t-1})^2, \tag{3.10}$$

where $\delta_c \in (0,1)$ measures the geometric decay. The greater the value the less past information is taken into account in regards to the expectation of future payoff. In extreme cases, $\delta_c = 1$ means the belief is independent of the most recent price p_t; while in case of $\delta_c = 0$, the trend follower only considers the latest observed price. Intuitively, trend followers extrapolate the most information from the latest prices as they carry the most relevant information about the future. The trend follower is a CARA utility maximiser such that his $\mathbb{CE}$ of the wealth is expressed as

$$\mathbb{CE} = R_f W_{c,t} + x_{c,t}(\mathbb{E}_{c,t}(p_{t+1} + d_{t+1}) - R_f p_t) - \frac{1}{2} a_c x_{c,t}^2 \mathbb{V}_{c,t}(p_{t+1} + d_{t+1}). \tag{3.11}$$

Given information at time t, $\mathbb{E}_{c,t}$ and $\mathbb{V}_{c,t}$ are trend follower's conditional expectation and variance at time t. Therefore, the optimal demand function is

$$x_{c,t} = \frac{\mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - R_f p_t}{a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}. \tag{3.12}$$

Different from the fundamentalists, the trend followers do not experience ambiguity as his belief is based only on past prices. Therefore unlike fundamentalist, trend follower will always demand non-zero positions unless $\mathbb{E}_{c,t}(p_{t+1}) + \bar{d} = R_f p_t$ and the demand is continuous in price.

Noise Investors are assumed to act like non-utility maximisers, they supply and take liquidity randomly from the market for reasons that are non-investment related, e.g. liquidation required from client withdrawing fund. In that sense, we can simply let their demand to be random, $x_{n,t} \sim \mathcal{N}(0, \sigma_n^2)$ where $\sigma_n^2 \in (0, 1)$. In a Walrasian auction market, noise investors both helpful in the later formation of market price as well as economically intuitive. We shall review the advantages in later analysis.

3.2.1 Market Equilibrium

In equilibrium the per capita demand for the risky asset must equal to the per capita supply,

$$\eta_f x_{f,t} + \eta_c x_{c,t} + \eta_n x_{n,t} = \overline{x}, \tag{3.13}$$

where η_f,η_c and η_n are fractions of fundamentalists, trend followers and noise investors in the the market respectively such that $\eta_f + \eta_c + \eta_c = 1$. There are three types of market conditions.

Type I According to equation (3.8), if the price lies between the lowest and highest of the fundamentalists' conditional expectation discounted by risk-free rate, then only the trend followers and noise investors are participating in the market $\eta_c x_{c,t} + \eta_n x_{n,t} = \overline{x}$ and fundamentalists demand zero $x_{f,t} = 0$. Therefore, substituting equation (3.12) yields such market clearing price, we refer this as the non-participation equilibrium price (NP)

$$p_t^{NP} = \frac{1}{R_f}[\mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - \delta_t], \tag{3.14}$$

where $\delta_t = \frac{a_c(\overline{x} - x_{n,t}\eta_n)(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{\eta_c}$. From the fundamentalists' demand equation, we know the price must lie between a range for non-participation price p_t^{NP} denoted $\frac{\underline{\mathbb{E}}_{f,t}(p_{t+1}) + \bar{d}}{R_f} \leq p_t^{NP} \leq \frac{\overline{\mathbb{E}}_{f,t}(p_{t+1}) + \bar{d}}{R_f}$. By substituting equation (3.14) into the condition, it can be re-expressed in terms of $E_{c,t}(p_{t+1})$ such that

$$\underline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t \leq \mathbb{E}_{c,t}(p_{t+1}) \leq \overline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t. \tag{3.15}$$

We express above in term of $\mathbb{E}_{c,t}(p_{t+1})$ as the expectation of trend chasers is not ambiguous, and serves as a basis in relation to the fundamentalists, this is discussed in later section. This particular condition is crucial in equilibrium formation and is affected by several variables. One of the most important factor is the degree of ambiguity $[\underline{\mu}, \bar{\mu}]$ as it directly impacts the range of $[\underline{\mathbb{E}}_{f,t}(p_{t+1}), \overline{\mathbb{E}}_{f,t}(p_{t+1})]$. Secondly, δ_t shifts the range to the right (left) when it is positive (negative); and the sign depends on $\overline{x} - x_{n,t}\eta_n > (<)0$. How far it moves along the horizon of $\mathbb{E}_{c,t}(p_{t+1})$ is affected by η_c, a_c and $\mathbb{V}_{c,t}(p_{t+1})$.

Type II If $\mathbb{E}_{c,t}(p_{t+1})$ lies below the lowest conditional expectation of the fundamentalists $\mathbb{E}_{c,t}(p_{t+1}) < \underline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t$, the fundamentalists participate in the market (to buy). Therefore, by substituting equation (3.12) and the first condition of equation (3.8) into equation (3.13), the buy-only equilibrium price (BO) is determined by

$$p_t^{BO} = \frac{1}{R_f}(\beta_t(\underline{\mathbb{E}}_{f,t}(p_{t+1}) + \bar{d}) + (1-\beta_t)(\mathbb{E}_{c,t}(p_{t+1}) + \bar{d}) - \Delta_t), \tag{3.16}$$

where

$$\beta_t = \frac{\eta_f a_c(\overline{\mathbb{V}}_{c,t}(p_{t+1}) + \sigma_d^2)}{\eta_f a_c(\overline{\mathbb{V}}_{c,t}(p_{t+1}) + \sigma_d^2) + \eta_c a_f(\overline{\sigma}_f^2 + \sigma_d^2)}, \tag{3.17}$$

$$\Delta_t = \frac{(\overline{x} - x_{n,t}\eta_n)a_f a_c(\overline{\sigma}_f^2 + \sigma_d^2)(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{\eta_f a_c(\overline{\mathbb{V}}_{c,t}(p_{t+1}) + \sigma_d^2) + \eta_c a_f(\overline{\sigma}_f^2 + \sigma_d^2)}. \tag{3.18}$$

For equilibrium price to be p_t^{BO}, it must satisfies $p_t^{BO} < \frac{\underline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}}{R_f}$, but this will occurs if and only if the constrain $\mathbb{E}_{c,t}(p_{t+1}) < \underline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t$ is met. The fundamentalists buy the risky asset as the conditional expectation will always be greater than the market price at time t.

Type III Similar to p_t^{BO} from above, the fundamentalists only sell the risky asset when the price lies above their conditional expectation and the sell-only equilibrium price (SO) becomes

$$p_t^{SO} = \frac{1}{R_f}(\beta_t(\overline{\mathbb{E}}_{f,t}(p_{t+1}) + \bar{d}) + (1-\beta_t)(\mathbb{E}_{c,t}(p_{t+1}) + \bar{d}) - \Delta_t). \tag{3.19}$$

The equilibrium price is p_t^{SO} if and only if the condition $\mathbb{E}_{c,t}(p_{t+1}) > \overline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t$ is met. In summary, we have the following results.

Proposition 3.1. *The unique equilibrium price is determined by one of the following types:*

- *Non-participation from the fundamentalists: if* $\underline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t \leq \mathbb{E}_{c,t}(p_{t+1}) \leq \overline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t$*, where* $\delta_t = \frac{a_c(\overline{x} - x_{n,t}\eta_n)(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{\eta_c}$*, then in the equilibrium* $x_{f,t}^* = 0$*,* $x_{c,t}^* = \frac{(\overline{x} - \eta_n x_{n,t})}{\eta_c}$ $x_{n,t}^* \neq 0$ *and* p_t^{NP} *defined in equation (3.14) is the market clearing price.*

- *Buy-only from the fundamentalists: if* $\mathbb{E}_{c,t}(p_{t+1}) < \underline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t$*, then in the equilibrium* $x_{f,t}^* > 0$*,* $x_{c,t}^*, x_{n,t}^* \neq 0$ *and* p_t^{BO} *defined in equation (3.16) is the market clearing price.*

- *Sell-only from the fundamentalists: if* $\mathbb{E}_{c,t}(p_{t+1}) > \overline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t$*, then in the equilibrium* $x_{f,t}^* < 0$*,* $x_{c,t}^*, x_{n,t}^* \neq 0$ *and* p_t^{SO} *defined in equation (3.19) is the market clearing price.*

Therefore, the unique equilibrium price function is given by

$$p_t \begin{cases} p_t^{BO}; & \mathbb{E}_{c,t}(p_{t+1}) < \underline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t \\ p_t^{NP} & \underline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t \leq \mathbb{E}_{c,t}(p_{t+1}) \leq \overline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t \\ p_t^{SO}; & \mathbb{E}_{c,t}(p_{t+1}) > \overline{\mathbb{E}}_{f,t}(p_{t+1}) + \delta_t). \end{cases} \tag{3.20}$$

Ambiguity about the fundamental value is explicitly applied through fundamentalists' belief about the future value of the risky asset. Instead of a single equilibrium price in a typical mean-variance framework, the equilibrium price is determined by this piecewise function shown in equation (3.20), that is, there are two kinks in the price function and smooth everywhere else.

3.3 Characterisation of Equilibrium

To better understand the mechanism of the model and the coupling implications of ambiguity and heterogeneity on the market, we firstly discuss the model in a static setup to provide intuitions behind the derived equilibrium; and proceed to show that our dynamic model is able to generate market stylised facts due to ambiguity rather than heterogeneity (results are based on model simulations). More specially, we study the effect between ambiguity and 1) the non-participation of the fundamentalists, 2) the mean-reverting rate of the fundamentalists, and 3) extrapolation rate of the trend chasers. The implications of those interactions lead to excess volatility in market prices and autocorrelation patterns in returns, absolute returns and squared returns. We show that the model is able to characterise various market behaviours and has a potential power to generate stylised facts observed in financial markets.

We refer to the core model as Model under Ambiguity (MuA). Further, the Benchmark (Bmk) is the instance when both fundamentalists and trend followers are utility maximisers and *do not* experience ambiguity. In this market, there exists a single equilibrium price that clears the market. Non-participation only arises with ambiguity, therefore in absence of it, the demand will be continuous in price without kinks.

Table 3.1: Parameter settings and initial values

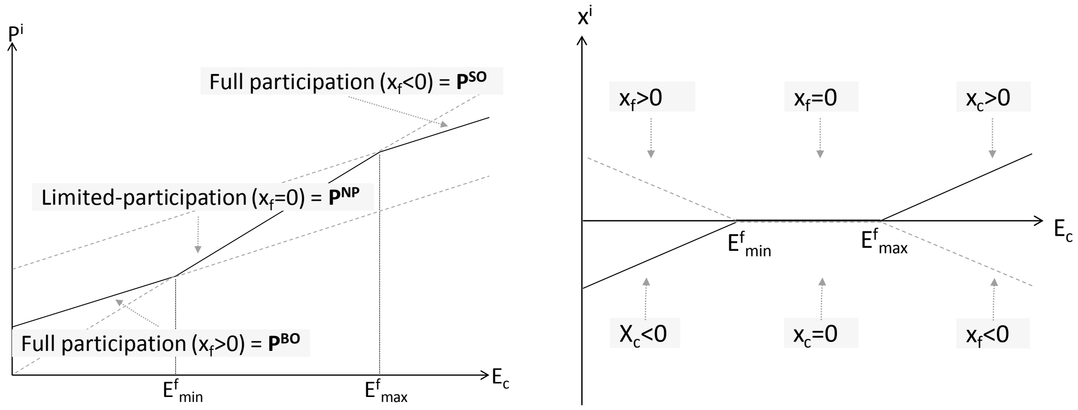
μ	$\#of sim.$	K	T	N	$\overline{p}$	σ_{p^*}	$\overline{\sigma}_f$	σ_n	γ_f
$[\pm 1.25\%, ... \pm 50\%]$	40	250	40	$K * T$	\$100	$\frac{0.2}{\sqrt{K}}$	$\frac{0.25 * \overline{p}}{\sqrt{K}}$	0.5	0.1
γ_c	δ_c	η_f	η_c	η_n	a_f	a_c	r_f	$\overline{x}$	σ_d
0.15	0.85	0.45	0.45	0.1	1	1	0	0	0

A summary of the initial values of all the necessary parameters are provided in Table 3.1. All simulation results are based on this unless otherwise stated. Note that parameters are chosen to match typical market returns. Results of averages are based on the mean of 40 simulations. Since larger number of simulations produces the same results, we have limited it to 40 to reduce calculation time. Further, the same set of random draws is used for noise processes (noise investor demand and fundamental noise) for comparison. Each simulation takes a single period as one trading day, therefore there are 250 trading days in a year stretching over 40 years. For simplicity, we assume net-zero supply and risk-free rate are zero. Further, since the dividend has minimal impact on a daily basis, it is also set to zero. To study the effects of ambiguity at various ranges; we start with a low level of ambiguity i.e. $\mu \in [-1.25\%, 1.25\%]$ at an increment of 1.25% until $\mu \in [-50\%, 50\%]$.

3.3.1 Static Model

In this static setting, we first assume there exists no noise investors $\eta_n = 0$. There is an equal amount of fundamentalists and trend followers. From proposition 3.1, the optimal equilibrium can be visually realised in Figure 3.1(a). As previously mentioned, there are two kinked spots that occur at the minimum $\underline{\mathbb{E}}_f$ and maximum $\overline{\mathbb{E}}_f$ of fundamentalists' conditional expectation and smooth everywhere else. The dotted lines represent each of the condition if there exists no ambiguity. Together, the solid line shows the equilibrium price under different market conditions, i.e. p^{BO}, p^{NP} and finally p^{SO} (from left to right). Visually, we observed that

$p^{BO} < p^{NP} < p^{SO}$. It shows price decreases (increases) when fundamentalist buys (sells). Both investors' demand equal but in opposite direction of trades in Figure 3.1(b). Since there are no noise investors to facilitate trades when fundamentalists do not participate in the market, the trend followers will not trade neither between the given range, i.e. $(\underline{\mathbb{E}}^f, \overline{\mathbb{E}}^f)$. The market becomes illiquid. On the other hand, in absence of ambiguity, demands from both investors are continuous in price and price function is monotonically increasing. Investors will trade each other as long as their conditional expectations are heterogeneous. The only time investors do not trade is when the conditional expectations are the same as in the equilibrium price.



(a) The equilibrium price is in sold line, and the dotted lines are the price function of the each condition. The p^{BO}, p^{NP} and finally p^{SO} is shown from left to right.

(b) Optimal demand is shown without noise investors. Fundamentalists are in dotted lines and trend followers are in solid lines. Investors demand different under different market conditions.

FIGURE 3.1: Static equilibrium price (a) and demand (b) without noise investors.

Now, rather assume $\mathbb{E}_c$ is static, let $\mathbb{E}_{c,t}$ varies with time. The market is also populated with noise investors. Therefore, equilibrium will always prevail because the trend followers are able to trade with noise investors when fundamentalists exit the market. Consequently, trend followers update their expectations at each time period. From Figure 3.1(a), it shows the more different trend followers' expectation about the future value of the asset is from one period to the next, the large the shift in price. Intuitively, these changes in price induce excess volatility.

3.3.2 Dynamic Model

The coupling implications of ambiguity and heterogeneity are discussed in the numerical results below. The analysis is based on a typical simulation unless otherwise stated, and ambiguity range is set as $\mu \in [-18\%, 18\%]$. This is a representative range picked arbitrarily, and reflects more or less similar patterns for other range. In particular, one would expect to observe higher volatility with increasing ambiguity.

The static setup above highlights the lack of trading activities when the ambiguity is significant. That is because with fundamentalists out of the market in presence of ambiguity, trend followers are left with no one to trade. In the dynamic setup, noise investors are introduced to act as a buffer in trading activities. Consequently, Figures 3.2 and 3.3 show price and demand dynamics *without* noise investors, and the results are compared to the market *with* noise investors, in Figures 3.4 and 3.5. They are snapshots of a single simulation over a short time frame for demonstration purpose only.

Figures 3.2 and 3.4 show price, returns and the type of equilibrium price (from top to bottom). In particular, the upper panels show MuA price series (in hollow dot) and the fundamental price (in solid dot). Further, the three types of prices are from equation (3.20), denoted p^{NP}=1 , p^{BO}=2, and p^{SO}=3 in the bottom panels of the figures[5].

In absence of noise inventors, we show in Figure 3.2 that little to known movements can be observed in MuA price and return series due to the lack of trading activities. It is obvious that the market is dominated by a single type of price, p^{NP} [6]. Under this non-participation market, fundamentalist desires zero position in the market

[5]The time period is purposely kept short so that it gives a close-up picture of what the price dynamics would look like and can be readily compared to that of the static setup.

[6]The market price is non-constant, this is more apparently reflected in the return series due to the trend chaser's extrapolation rate. If $\gamma_c = 0$, price under p^{NP} is constant.

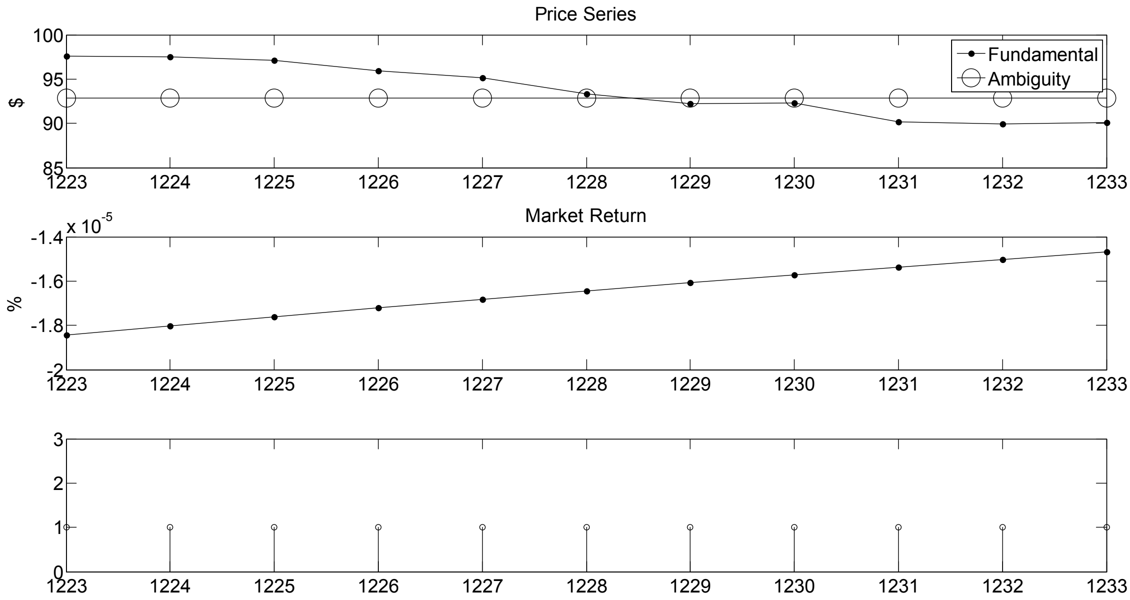


FIGURE 3.2: **Without** noise investors. Price - MuA (hollow), fundamental value(solid); returns, and the type of equilibrium price.

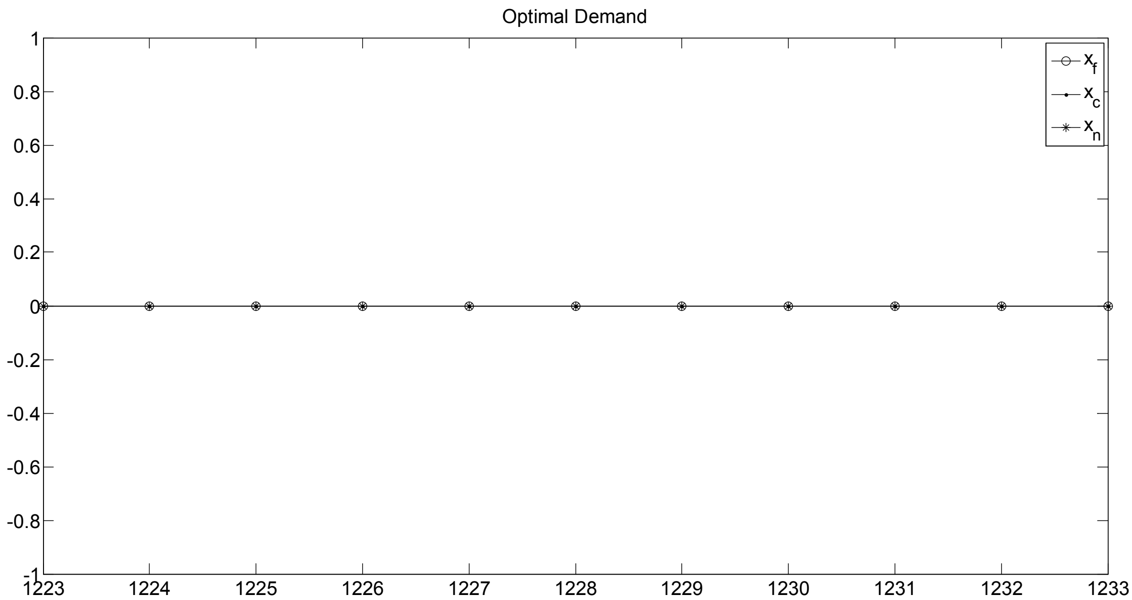


FIGURE 3.3: **Without** noise investors. Demand - fundamentalists (hollow); and trend followers (solid).

and trend follower cannot trade in absence of noise investors. Consequently, the optimal demand shown in Figure 3.3 is zero.

On the other hand, Figures 3.4 and 3.5 show the implication of noise investors

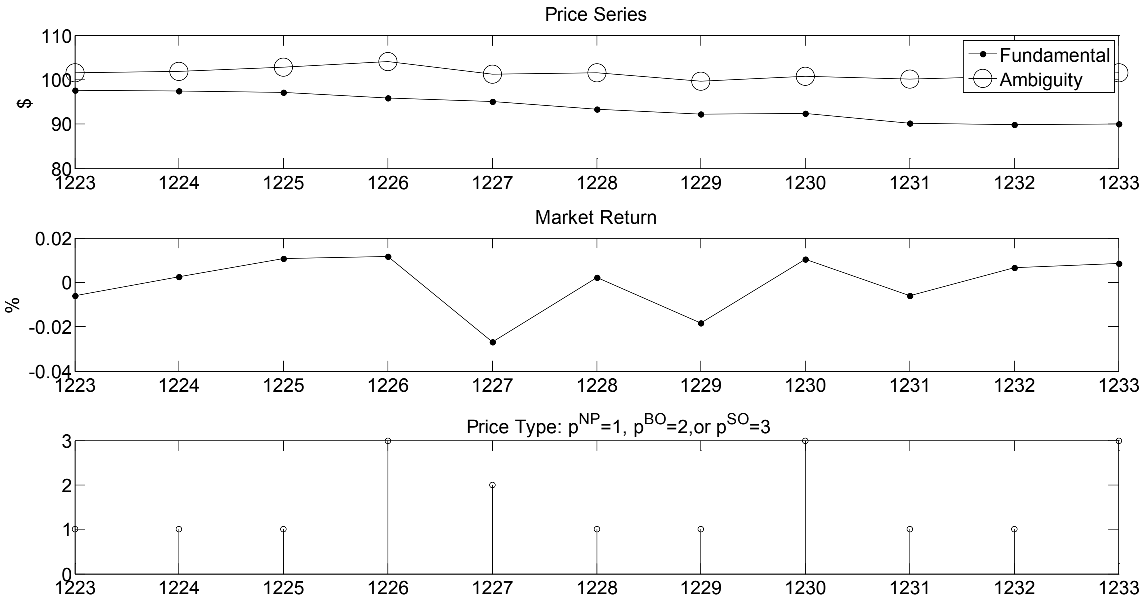


FIGURE 3.4: **With** noise investors. Price - MuA (hollow), fundamental value(solid); returns, and the type of equilibrium price.

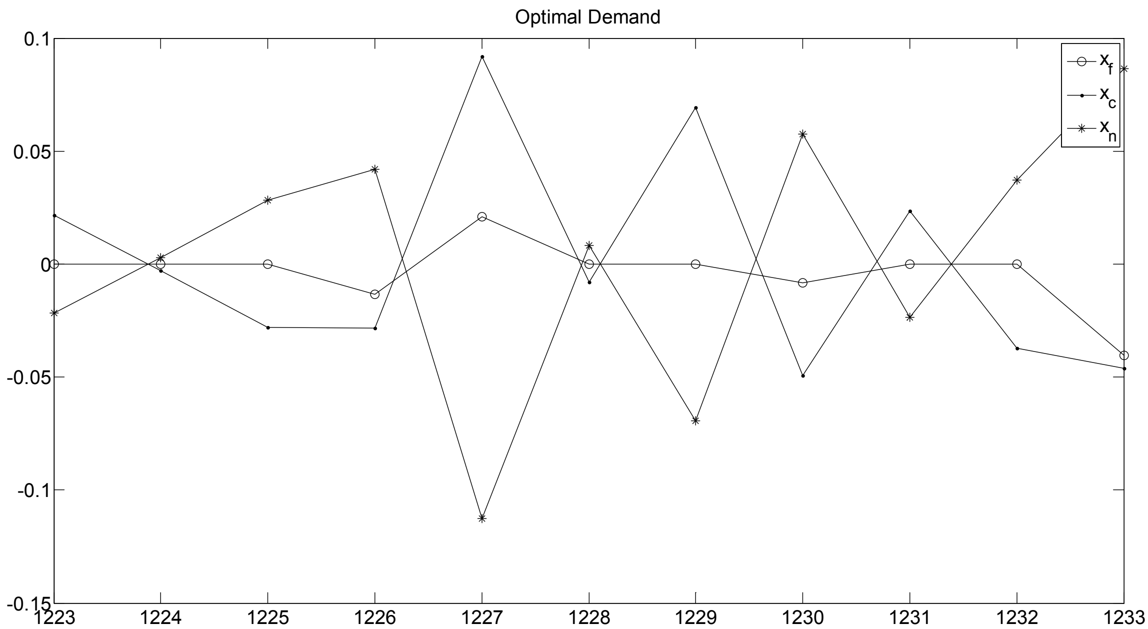


FIGURE 3.5: **With** noise investors. Demand - fundamentalists (hollow); trend followers (solid), and noise investors (star).

on price, return series, and demands for risky asset. Here, noise investors act as counterparts for providing/supplying liquidity to the trend chaser when fundamentalist chooses to stay out of the market. Within the same set of period as above, the price level reflects the increasing trading activities, while returns vary between $\pm 2\%$ over the same period. A notable change is also observed in the type of price determined. Further, price is no longer dominated by p^{NP}, and the switching between different price types, e.g. 1 to 3 to 2 (periods 1225,1226,1227) induces price fluctuation due to increasing trading activities for risky asset shown in Figure 3.5. Recall the demand function for fundamentalists; in order to encourage them to get back into the market, the changes in prices need to be large enough to move them out of the non-participation price range. As a result, we observe increased fluctuations in both price and return series. We now proceed to analyse the interaction effects of ambiguity and some of the key parameters.

3.3.2.1 Implications of Ambiguity and the Key Parameters

A quick illustration of the implication that ambiguity has on price dynamics can be observed in Figure 3.6. Three price series are shown here, fundamental value (dot), the benchmark (light-coloured solid) and MuA (dark solid). Given ambiguity, MuA fluctuates around the fundamental value over time, while the benchmark hardly deviated away from it. In comparison to the benchmark, MuA shows greater fluctuations in price and exhibits a realistic picture of a market price series. That is, the movements observed in MuA price is due to ambiguity rather than heterogeneity, and more specially, it is the switching mechanism of fundamentalists' market participation from one period to the next that induces volatility.

Now, let us observe the return series of the above three scenarios. Figure 3.7 shows return series and the corresponding density distribution (from left to right). While, MuA, fundamental value and the benchmark are demonstrated from top to bottom. To observe if MuA leads to excess volatility, we define excess volatility as the difference between the returns of MuA and the fundamental value. In this

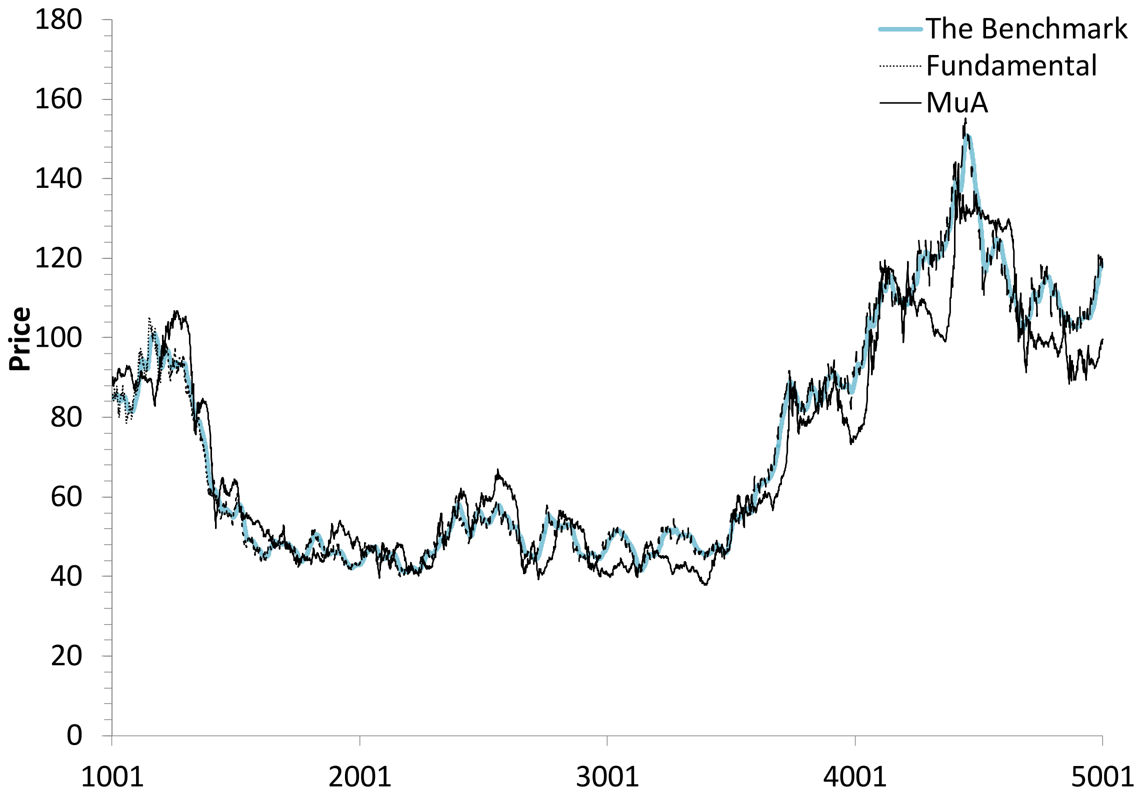


FIGURE 3.6: Price series. Fundamental value of the asset (dot) , MuA (thin solid) and the benchmark (light-coloured solid).

particular instance, the fundamental value is 20%p.a, MuA is 24%p.a., and the difference is 4%p.a..

Further, the difference between the benchmark and MuA is that there exists only one equilibrium price in the former, while the latter is conditional on the market participation from the fundamentalists. The benchmark price oscillates around the fundamental value of the risky asset but does not deviate significantly away from it. As a result, the returns are small and are not clustered. In comparison, MuA exhibits strong deviation from the asset's fundamental value. We observe clustering in returns where high (low) fluctuations are followed by high (low) fluctuations over time. Further, the density distribution exhibits excess kurtosis. In order to provide an in-depth understanding of the implication of ambiguity and heterogeneity on the market, we now direct our attention to some of the key parameters in MuA. Morever, Figure 3.8 shows the corresponding autocorrelations of returns, absolute and squared returns of a typical simulation for the Benchmark (left) and MuA (right). It is clear that ambiguity leads to a variety of stylised facts.

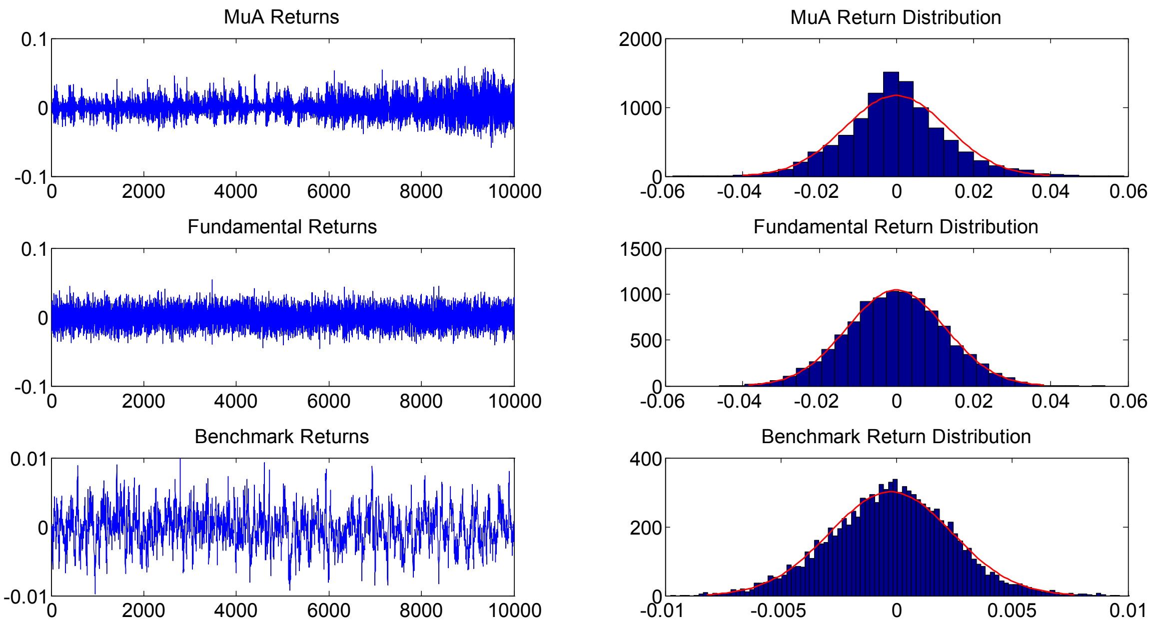


FIGURE 3.7: Returns and distribution (left to right). MuA, fundamental value and the benchmark (from top to bottom).

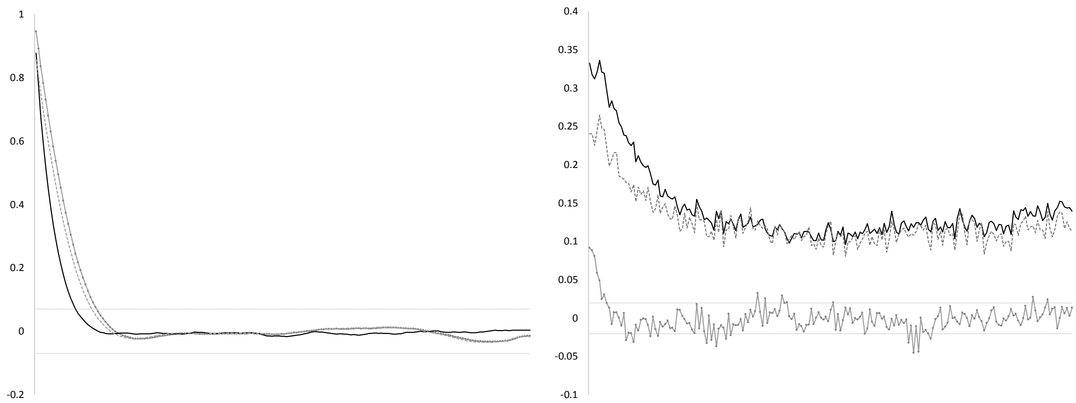


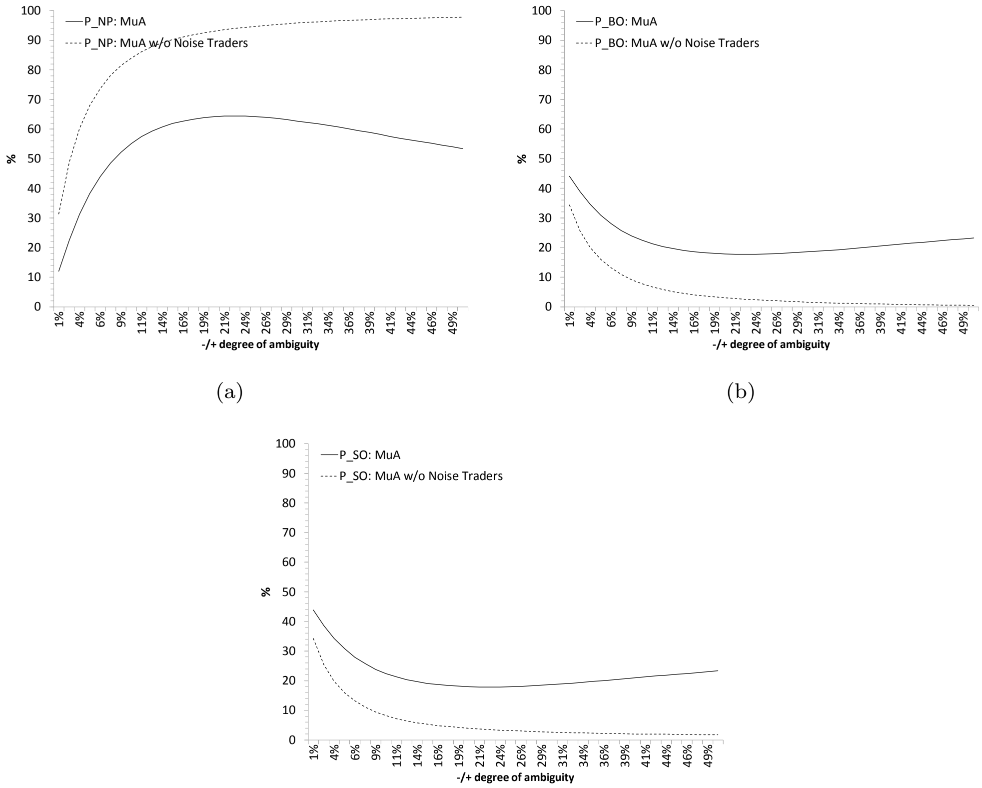
FIGURE 3.8: The Benchmark and MuA are shown from left to right. This is the autocorrelation for returns (dot-solid), absolute returns (solid) and squared returns (dash) over 200 time lags of a typical simulation.

Fraction of Noise Investors - The importance of noise investors in the market is analysed in the previous section. However, in order to understand the joint impact of ambiguity and noise investors, we study two cases. Case 1 studies the

the market when the fraction of noise investor is zero, $\eta_n = 0$, while case 2 shows results for $\eta_n \neq 0$. One way to determine the implication of noise investors on the market is to study the type of price that is determined in equilibrium. That is, without noise investors in the market, trend followers have no one to trade; fundamentalists therefore are less likely to step back into the market due to low fluctuation in prices. Figure 3.9 shows the proportion (%) of the type of price determined, i.e. p^{NP} (a), p^{BO}(b), and p^{SO}(c) over different degrees of ambiguity from left to right respectively. Cases 1 (dot) and 2 (solid) are shown in each of the sub-Figures. Intuitively, we expect to see price is dominated by p^{NP} due to non-participating of the fundamentalists.

There are two observations. First, in all plots, the fraction of all three price types is non-monotonically increasing (in case of p^{NP}) or decreasing (in cases of p^{BO}, and p^{SO}) with increasing ambiguity. This is expected as ambiguity should have greater impact initially when it starts from barely nothing to something substantial, as oppose to a further increment thereafter. Second, by comparing the cases 1 and 2, it is obvious that the market is no longer dominated by p^{NP} from non-participation of the fundamentalists with the introduction of noise investors. In case 1, fundamentalists stay mostly inactive at approximately 90% of the time for ambiguity level larger than 10%. While in the second case, a systematic shift in all three types of prices is observed. A downward shift in the proportion of p^{NP} means it would contribute to an upward shift in the proportion of both p^{BO}, and p^{SO}. The comparison shows noise investors are important in increasing market activities, in particular participation from the fundamentalists, and possibility reduces the effect of ambiguity.

Further, we study the joint impact by observing the volatility of price deviation, denoted as the difference between the market price and the fundamental price



(a) (b)

(c)

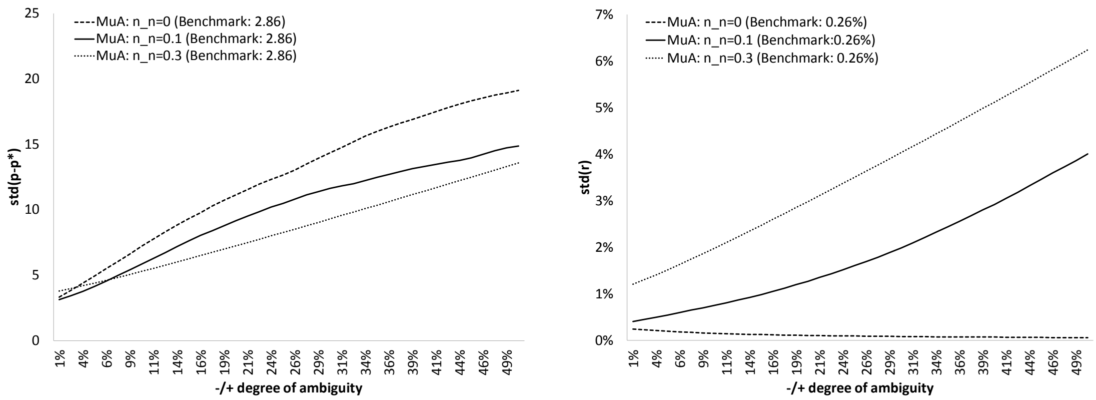
FIGURE 3.9: Equilibrium Price types: p^{NP} (a), p^{BO}(b), and p^{SO} (c).

$\sigma(p_t - p_t^*)$; and the corresponding returns $\sigma(r_t)$ in Figures 3.10 (a) and (b) respectively[7]. The corresponding benchmark is denoted in the legend next to each of the case. Both volatility graphs are plotted over different degree of ambiguity; with each figure showing three different cases, i)$\eta_n = 0$ (dash), ii) $\eta_n = 0.1$ (solid), and iii) $\eta_n = 0.3$ (dot).

With increasing fraction of noise investors in the market, we observe a decreasing volatility in price deviation between the market price and the fundamental price, while an increasing volatility in returns. In comparison to the respective benchmark, the fraction of noise investors has a much greater role in determining volatility in MuA. The benchmark in this case stays relatively stagnant to reflect the lack of movements in market price. This suggests noise investors are important in time of uncertainty; they help to facilitate trades when fundamentalists choose

[7]The market returns is simply calculated as $r_t = \frac{p_t}{p_{t-1}} - 1$.

to stay out of the market. On the other hand, the induced trading activities also increase volatility in market returns.



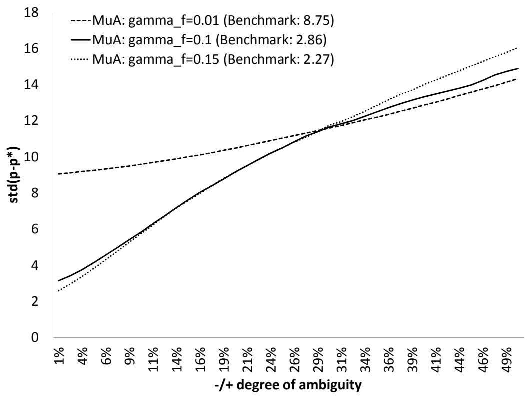
(a) (b)

FIGURE 3.10: (a) volatility of price deviation and (b) volatility of returns.

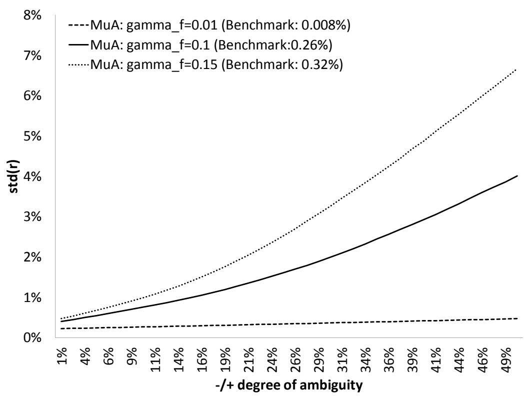
Effect of γ_f - The mean-revering rate γ_f is an important parameter in facilitating MuA to generate stylised facts. The converging rate determines how quickly fundamentalist expects the market price to converge to the true value of the risky asset. The higher the rate the quicker the convergence will take place. Since fundamentalist is generally considered long-term investor, γ_f is usually kept low to reflect this long-term investment horizon.

Figures 3.11 (a) and (b) show the volatility of price deviation and returns with varying $\gamma_f \in (0, 1)$, all else being equal. Similarly, both plots are plotted over ranges of ambiguity. Each plot shows three different cases, i)$\gamma_f = 0.01$ (dash); ii) $\gamma_f = 0.1$ (solid), and iii) $\gamma_f = 0.3$ (dot). The corresponding benchmark is denoted in the legend next to each case.

Firstly, we observe a common increasing trend in volatility with increasing ambiguity in Figure 3.11. Secondly and more importantly, Figure 3.11 (a) shows varying slopes in the three cases. Referring back to equation (3.4), it is obvious to see that γ_f is indeed the slope of the second term from the conditional expectation function. The smaller the rate, the less steeper the slope is going to be. Further, a smaller rate also means the fundamentalist is less sensitive to ambiguity. Since price deviation is the the difference between the market price and the fundamental price, and γ_f directly influences fundamentalists' expectations; we observe volatility in price deviation is increasingly sensitive to ambiguity with increasing value of γ_f. On the other hand, for the same three cases, market volatility in Figure 3.11(b) shows an increasing sensitive with increasing γ_f, reflects that fundamentalists expect the market price to converge faster to the true value of the risky asset. Overall, the level of γ_f is an important factor in determining the sensitiveness of volatility in price deviation and returns.



(a)



(b)

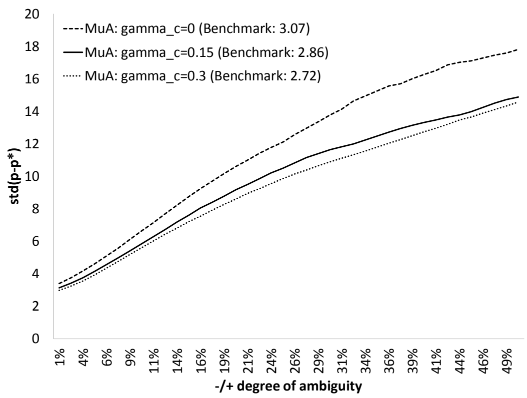
FIGURE 3.11: (a)volatility of price deviation and (b) volatility of market returns with **varying** γ_f.

Effect of γ_c - The extrapolation rate γ_c is another parameter that is equally important as the γ_f as the trend followers extrapolate information from past prices. Typically, the greater the γ_c, the more emphasis they put on past price trend (information). Since they essential are market riders, γ_c is typically set at a higher rate than γ_f to reflect the timely nature of the strategy.

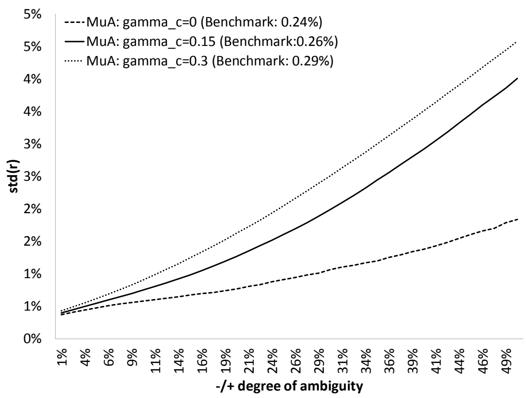
Figures 3.12 (a) and (b) show the volatility of price deviation and returns with varying $\gamma_c > 0$, all else being equal. Similarly, both sub-Figures are plotted over a range of ambiguity. Each plot shows three different cases, i) $\gamma_c = 0$ (dash); ii)

$\gamma_c = 0.15$ (solid), and iii) $\gamma_c = 0.3$ (dot). Again, the corresponding benchmark is denoted in the legend next to each case.

First, again we observe a common increasing trend in volatility with increasing ambiguity. Second, we observe decrease volatility in price deviation and increase in return volatility with increasing parameter γ_c. Given Proposition 3.1, MuA's equilibrium price is conditional on the expectation of the trend followers and therefore the extrapolation rate implicitly. The trend followers' expectation and variance depend on the mean and variance of the geometric decay process. In case of $\gamma_c \neq 0$, we observe an increase in volatility of returns to reflect a large price change over time; at the same time, the consequent high movements in the expectation of trend followers mean the fluctuation in equilibrium price is also higher, therefore enticing the fundamentalists to come back to the market to trade. Therefore, with fundamentalists to correct the market, the deviation between the price and fundamental can be reduced. On the other, if $\gamma_c = 0$, the volatility in returns is reduced due to smaller price difference from one period to the next, therefore, the fundamentalists may be reluctant to participate and the market price is not been corrected and hence reflecting a larger volatility in price deviation.



(a)



(b)

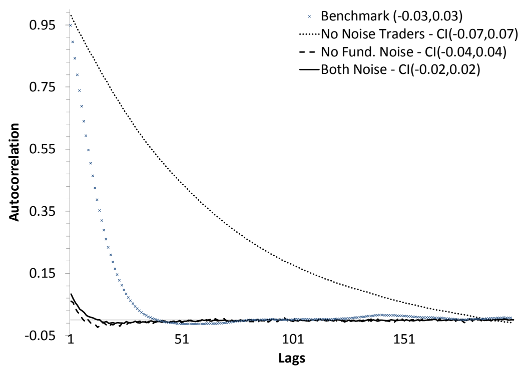
FIGURE 3.12: (a)volatility of price deviation and (b) volatility of market returns with **varying** γ_c.

3.3.3 Analysis of Stylised Facts

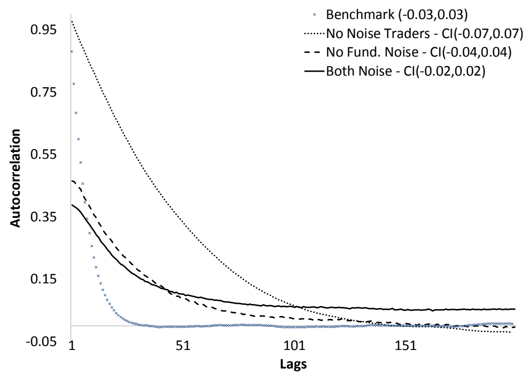
To show that our MuA is able to generate stylised facts due to ambiguity rather than heterogeneity, e.g. autocorrelation patterns of returns, absolute returns and squared returns; we study the effect of ambiguity and the two noises i.e., fundamental noise σ_{p^*} and noise investors σ_n. There are four different cases in scope. Case 1 is MuA (solid); case 2 is the benchmark (*-indicator); case 3 is MuA without noise investors (dot), and finally case 4 is MuA without fundamental noise (dash). Figures 3.13(a), (b) and (c) show the autocorrelation of returns, absolute returns, and squared returns respectively. Further, the confidence interval (CI) corresponding to each case is shown in the legend, denoted CI. While, Figure 3.14 shows the autocorrelations for each case. Finally, the results shown here are the average of 40 simulations and is reported up to 200 lags.

We observe, else being equal and with the same set of random seeds, that MuA is able to generate insignificant autocorrelations in returns and slow decay in both absolute returns and squared returns (where squared returns typically exhibits a lower autocorrelation). While the benchmark on the other hand, is unable to generate the same patterns, even in presence of both noises. In the benchmark, the autocorrelation in returns is extremely large and is significant over many lags, while the absolute and squared returns exhibit little patterns and decays very quickly. As discussed above, the volatility of the benchmark is low and flat while MuA exhibits volatility cluttering.

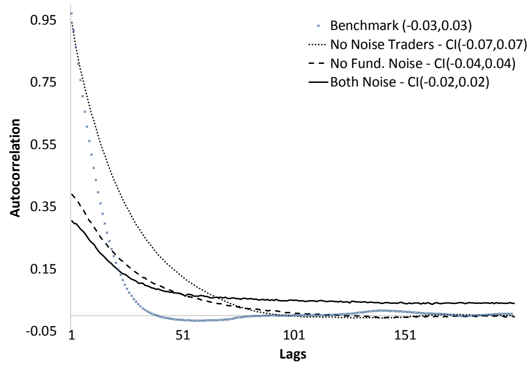
Further, in order for us to discuss the contribution of each noise, we further analyse autocorrelation patterns by studying noises separately, shown in cases 3 and 4. Under ambiguity, although each noise serves a different purpose, it is clear to say that the impact of the noise investors is greater on the overall formation of case 1 than that of the fundamental noise. Case 4 shows MuA without fundamental noise tracks closely to case 1. However, the difference here is the decay speed in



(a) Autocorrelation of returns.



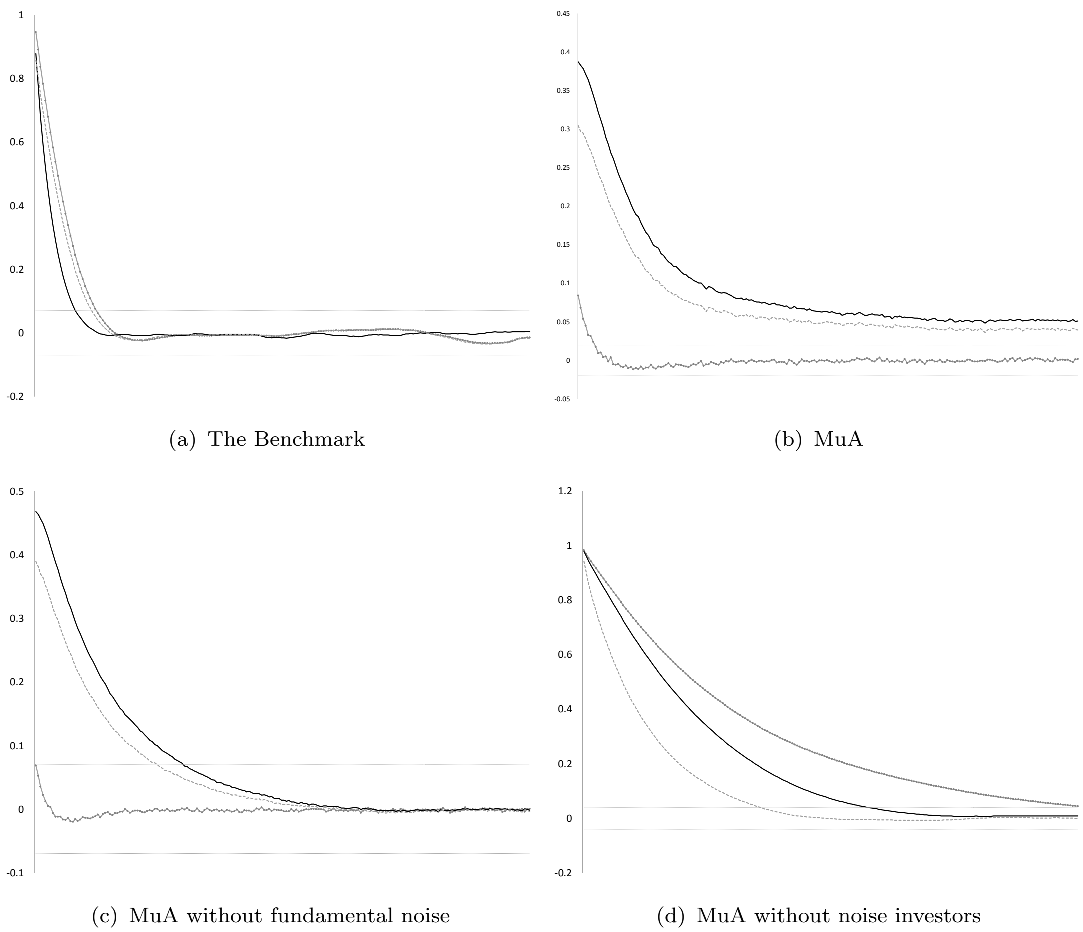
(b) Autocorrelation of absolute returns



(c) Autocorrelation of squared returns.

FIGURE 3.13: (a)autocorrelation of returns. (b) autocorrelation of absolute returns. (c)autocorrelation of squared returns. The benchmark (*), MuA (solid), MuA without Noise Investors (dot), and MuA without Fundamental noise (dash).

absolute and squared returns. In case 4, insignificance in absolute and squared returns are observed approximately after 90 lags while case 1 stays significant over 200 lags. Intuitively, when the market is driven only by the random process of the noise investors, returns volatility should not be predictive and no significant autocorrelation patterns should be observed in absolute and squared returns as in He and Li [2007]. On the other hand, case 3 shows large significant autocorrelation in all, therefore, a combination effect of both noises help to produce results in case 1. As a result, we have shown the important role of ambiguity in generating some of the realistic stylised facts in the market.



(a) The Benchmark

(b) MuA

(c) MuA without fundamental noise

(d) MuA without noise investors

FIGURE 3.14: Autocorrelations are plotted for each case, that is, autocorrelation for returns (dot-solid), absolute returns (solid) and squared returns (dash) are shown over 200 time lags.

3.4 Conclusion

The chapter introduces a dynamic model that studies the coupling effect of ambiguity and heterogeneity on market dynamics and shows that the model is capable to produce some of the most common market stylised fact and anomalies due to ambiguity rather than heterogeneity. We have shown that 1) fundamentalists stay out of the market from time to time due to their aversion to ambiguity, resulting nonparticipation, and that the greater the ambiguity, the more likely the nonparticipation occurs. 2) Increasing ambiguity about the fundamental value of the risky asset leads to excess volatility. Further, the fraction of noise traders in the market, extrapolation rate of the trend followers and mean-reverting rate of the fundamentalists play important roles in market activities and together attributes to excess volatility. Finally, 3) we show *Model under Ambiguity* is able to capture volatility clustering and the long-range dependence of asset returns due to ambiguity rather than heterogeneity. Indeed, the incorporation of ambiguity is to some extend overly simplified. In the next chapter, we improve the structure of the model by embedding ambiguity endogenously, that is fundamentalists receive ambiguous signals about the value of the asset updated at each period, dynamically creating a feedback mechanism.

Chapter 4

Price Dynamics and Excess Volatility under Heterogeneous Beliefs and Ambiguous Information

4.1 Introduction

The previous chapter considers a simple and exogenous approach in structuring ambiguity. Specifically, the fundamentalists experience ambiguity about the fundamental value of the risky asset when the fundamental is not readily observable in the market. Different from Chapter 3, we introduce an ambiguous signal that the fundamentalists receive on the fundamental value in this chapter. Further, the estimation on the signal is updated at every time period using Bayesian rule and is fed into the fundamentalists' conditional beliefs, and consequently the equilibrium price.

This chapter shows the model is capable of producing stylised facts by allowing fundamentalists to receive ambiguous signal about the future value of the risky asset and belief is updated via Bayesian updating mechanism. Similar to the previous chapter, fundamentalists are ambiguous in the sense of Gilboa and Schmeidler [1989], and their preference is governed by the worse-case scenario over a range of asset distributions, and they are ambiguity-averse. We show that excess volatility is comparatively larger under ambiguity than in the case where all investors are utility maximiser (in absence of ambiguity). Further, we show that our model generates stylised facts due to ambiguity rather than heterogeneity, they are characterised by insignificant autocorrelations (ACs) in the returns and significant and long memories ACs in the absolute and squared returns.

This chapter is closest related to Illeditsch [2011] which studies the behaviour of investor who are averse to both risk and ambiguity and receive information with unknown precision. The investor in his model acts according to a worst-case scenario. This leads to a situation where even small shocks can cause drastic changes in the worst-case scenario beliefs about the precision of the signal, which in turn leads to large price changes and therefore excess volatility. The model developed in this chapter has two differences from Illeditsch [2011]. First, we consider a dynamic model instead of a static setup. Second, we consider co-existing of fundamentalists; trend followers, and noise investors. However, only fundamentalists are considered to exhibit ambiguity-aversion. Trend followers are utility maximisers. Noise investors trade for liquidity purpose. The introduction of noise investors plays a crucial role in our dynamic model. Due to the market clearing in a dynamic setting, the noise investors serve the role as liquidity providers/consumers such that there will always be trading activities and the market equilibrium price can be determined.

The remainder of this chapter is organised as follows. Section 4.2 establishes the general framework of the model. Section 4.3 shows the main results, explains the market dynamics as well the model's ability to generate a variety of market stylised facts including volatility clustering, excess volatility, insignificant AC in returns, significant AC in absolute returns, and squared returns. Section 4.4 concludes. Appendix C provides proofs.

4.2 The Model

The key difference between the previous and the current models is the structure of ambiguity been introduced, else remains the same. Consider a market with one risky asset and one risk-free asset. The risk-free asset is perfectly elastically supplied at a gross return of $R_f = 1 + r_f/K$. The r_f is an annual constant risk-free rate, and for simplicity, we assume $r_f = 0$ and hence $R_f = 1$. The market consists three types of investors; ambiguity-aversed *fundamentalists (i = f)*, utility maximising *trend followers (i = c)* and *noise investors (i = n)* who trades for non-profit gaining reasons. Assume the former two types have constant absolute risk aversion (CARA) utility for wealth, i.e. $U_{i,t}(W) = -e^{-a_i W_{i,t}}$. The future wealth of type i investors $W_{i,t+1}$ is given by

$$W_{i,t+1} = x_{i,t}(p_{t+1} + d_{t+1}) + m_{i,t} R_f \tag{4.1}$$

subjected to a budget constraint $m_{i,t} = W_{i,t} - x_{i,t} p_t$, where the risk-free demand is denoted by $m_{i,t}$, and $x_{i,t}$ is the demand of the risky asset. Further, p_{t+1} and d_{t+1} are the future value and dividend of the risky asset, and we assume the dividend process d_t follows a normal distribution $d_t \sim \mathcal{N}(\bar{d}, \sigma_d^2)$.

4.2.1 Optimal Demand

In this subsection, we determine the optimal portfolio for both fundamentalists under ambiguity and trend followers as utility maximisers. In particular, fundamentalists receive an ambiguous signal about the future value of the risky asset; therefore there is some instance where their demand is zero for the risky asset over a range of prices. Consequently, trend followers and noise investors are left in the market to trade each other, showing temporary market non-participation from the fundamentalist due to the ambiguous signals.

Fundamentalists receive a signal s_{t+1} about the future value of the fundamental price f_{t+1} of the asset at time t, therefore $s_{t+1} = f_{t+1} + \epsilon_{t+1}$, where $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ is independent of f_{t+1}, and $\sigma_\epsilon^2 \in [\sigma_a^2, \sigma_b^2]$ such that they are uncertain about the precision of the signal. The bounded variance is to characterise fundamentalists' uncertainty about the future risk of the asset. They are ambiguity-averse investors in the sense of Gilboa and Schmeidler [1989] where they consider the worst case scenario (the set of belief which leads to the lowest expected utility) depending on the position of the asset $x_{f,t}$ and the signal s_{t+1}. The unconditional distribution of f_{t+1} is unambiguous and is given $f_{t+1} \sim \mathcal{N}(\bar{f}, \sigma^2)$. Following from Baysian rule, the conditional belief for f_{t+1} given s_{t+1} for $\epsilon_{t+1} \sim \mathcal{N}(0, \sigma_\epsilon^2)$ leads to

$$\begin{aligned} \mathbb{E}_t(f_{t+1}|s_{t+1}) &= \bar{f} + \phi(s_{t+1} - \bar{f}), \\ \mathbb{V}_t(f_{t+1}|s_{t+1}) &= \sigma^2(1 - \phi), \end{aligned} \tag{4.2}$$

where $\phi = \frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2}$ (refer to Appendix C.1 for some details). The informativeness of the signal can be defined by ϕ. Since ϕ satisfies $\phi \in [\phi_l, \phi_u] \subset [0, 1]$, we have

$$\phi_l = \frac{\sigma^2}{\sigma^2 + \sigma_b^2}, \quad \phi_u = \frac{\sigma^2}{\sigma^2 + \sigma_a^2}. \tag{4.3}$$

The precision of the information is determined by the size of ϕ, in which we say the information is reliable when $\phi = \phi_u$ (i.e. a lower variance σ_a), and unreliable when $\phi = \phi_l$ (i.e. a higher variance σ_b). Further, the difference between the upper and lower bounds $\phi_u - \phi_l$ determines the degree of ambiguity aversion.

Let $\mathbb{CE}_{f,t}$ be the certainty equivalent of ambiguity-averse investors. Then the investors' utility

$$\min_{\phi \in [\phi_l, \phi_u]} \mathbb{E}_{f,\phi,t}[U(W_{f,t} + (p_{t+1} + \bar{d} - p_t)x_{f,t})|s_{t+1}], \tag{4.4}$$

is equivalent to $U_{f,t}((\mathbb{CE})(x))$, such that

$$\mathbb{CE}_{f,t}(x) = \min_{\phi \in [\phi_l, \phi_u]} [\mathbb{E}_{f,t}(W_{f,t+1}|s_{t+1}) - \frac{1}{2}\mathbb{V}_{f,t}(W_{f,t+1}|s_{t+1})]. \tag{4.5}$$

Fundamentalists update their beliefs according to

$$\begin{aligned} \mathbb{E}_{f,t}(p_{t+1}|s_{t+1}) &= p_{t-1} + \gamma_f[\mathbb{E}(f_{t+1}|s_{t+1}) - p_{t-1}], \\ \mathbb{V}_{f,t}(p_{t+1}|s_{t+1}) &= \sigma_f^2[1 + b_f \mathbb{V}(f_{t+1}|s_{t+1})], \end{aligned} \tag{4.6}$$

with parameters $\gamma_f > 0$ and $b_f > 0$. Further, we say the signal is of good (bad) news when the difference is $s_{t+1} - \bar{f} > (<)0$. The estimation is updated according to equation (4.2). Since the investors are ambiguity-averse, a CARA utility and normal belief lead to mean-variance preference over future wealth, and the worse-case scenario of such utility depends on the realisation of the fundamental signal s and the portfolio position x. The certainty equivalent is a continuous and concave function of the risky asset demand x given below, except for two threshold positions, 1) $x_{f,t} = 0$ and 2) $x_{f,t} = \hat{x}_{f,t}$ (refer to Appendix C.2 for proof).

$$\mathbb{CE}_{f,t}(x) = \begin{cases} \underset{\phi_l,f,t}{\mathbb{E}}(W_{f,t+1}|s_{t+1}) - \frac{1}{2}\underset{\phi_l,f,t}{\mathbb{V}}(W_{f,t+1}|s_{t+1}) & \text{if} \quad x_{f,t} \leq min(\hat{x}_{f,t}, 0) \\ \underset{\phi_u,f,t}{\mathbb{E}}(W_{f,t+1}|s_{t+1}) - \frac{1}{2}\underset{\phi_u,f,t}{\mathbb{V}}(W_{f,t+1}|s_{t+1}) & \text{if} \quad min(\hat{x}_{f,t}, 0) < x \leq max(\hat{x}_{f,t}, 0) \\ \underset{\phi_l,f,t}{\mathbb{E}}(W_{f,t+1}|s_{t+1}) - \frac{1}{2}\underset{\phi_l,f,t}{\mathbb{V}}(W_{f,t+1}|s_{t+1}) & \text{if} \quad x_{f,t} > max(\hat{x}_{f,t}, 0)), \end{cases} \tag{4.7}$$

where

$$\hat{x}_{f,t} = \frac{2(\bar{f} - s_{t+1})\gamma_f}{a_f b_f \sigma_f^2 \sigma^2}. \tag{4.8}$$

Denote $x_{f,t}$ the optimal demand function for fundamentalists under ambiguity; and that $x_{f,\phi_l,t}$ and $x_{f,\phi_u,t}$ for utility maximisers who treat the signal as unreliable and reliable respectively. Applying first order condition (FOC) to equation (4.7), the ambiguity-averse fundamentalists who also has CARA utility choose their optimal portfolio according to (see Appendix C.3 for proof)

$$x_{f,t} = \begin{cases} x_{f,\phi_l,t}; & p_t \leq p_{1,t} \\ max(\hat{x}_{f,t}, 0); & p_{1,t} < p_t \leq p_{2,t} \\ x_{f,\phi_u,t}; & p_{2,t} < p_t \leq p_{3,t} \\ min(\hat{x}_{f,t}, 0); & p_{3,t} < p_t \leq p_{4,t} \\ x_{f,\phi_l,t}; & p_t > p_{4,t}, \end{cases} \tag{4.9}$$

such that the optimal position for the risky asset depends on the four threshold prices determined by the informativeness of the signal and the type of news,

$$p_{1,t} = \mathbb{E}_{f,\phi_l,t}(p_{t+1}|s_{t+1}) + \bar{d} - a_f(\mathbb{V}_{f,\phi_l,t}(p_{t+1}|s_{t+1}) + \sigma_d^2)max(\hat{x}_{f,t}, 0), \tag{4.10}$$

$$p_{2,t} = \mathbb{E}_{f,\phi_u,t}(p_{t+1}|s_{t+1}) + \bar{d} - a_f(\mathbb{V}_{f,\phi_u,t}(p_{t+1}|s_{t+1}) + \sigma_d^2)max(\hat{x}_{f,t}, 0), \tag{4.11}$$

$$p_{3,t} = \mathbb{E}_{f,\phi_u,t}(p_{t+1}|s_{t+1}) + \bar{d} - a_f(\mathbb{V}_{f,\phi_u,t}(p_{t+1}|s_{t+1}) + \sigma_d^2)min(\hat{x}_{f,t}, 0), \tag{4.12}$$

$$p_{4,t} = \mathbb{E}_{f,\phi_l,t}(p_{t+1}|s_{t+1}) + \bar{d} - a_f(\mathbb{V}_{f,\phi_l,t}(p_{t+1}|s_{t+1}) + \sigma_d^2)min(\hat{x}_{f,t}, 0). \tag{4.13}$$

In the cases when fundamentalists' demand are $x_{f,\phi_l,t}$ and $x_{f,\phi_u,t}$, they would trade as if they are utility maximisers, $x_{f,\phi,t}$ is continuously non-increasing in p and is given

$$x_{f,\phi,t} = \frac{\mathbb{E}_{f,\phi,t}(p_{t+1}|s_{t+1}) + \bar{d} - p_t}{a_f(\mathbb{V}_{f,\phi,t}(p_{t+1}|s_{t+1}) + \sigma_d^2)}, \tag{4.14}$$

except the decision is influenced by the informativeness of the signal ϕ. From above, we observe that fundamentalists care more about risk when the position is large therefore ϕ_l is considered, that is the two end of the conditions; while they care more about the mean for a lower risk i.e. ϕ_u when the position is moderate.

However, there are instances where investors hold on the threshold portfolios that induce fundamentalists either not to participate in the market completely $x_{f,t} = 0$ or to hold a constant position $x_{f,t} = \hat{x}_{f,t}$ for a range of prices. Figure 4.1 provides a snapshot of fundamentalists demand piecewise function at time t when they receive good news $s_{t+1} - \bar{f} > 0$. The two kinks are attributed to the threshold positions, and the demand decreases in price everywhere else. Note those two positions are persistent over a range of prices where the investors will not move away from the fixed position for a range of prices. The larger the price ranges the greater the resistance the market is going to experience with fundamentalists refusing to trade, we refer to this as inertia positions. The price region of inertia depend mainly on the size of the ambiguity $(\phi_u - \phi_l)$ and the type of signal $(\bar{f} - s_{t+1})$. Therefore, only a large change in price will bring the fundamentalists out of the two regions and as we show later this fluctuation in price attributes to market volatility.

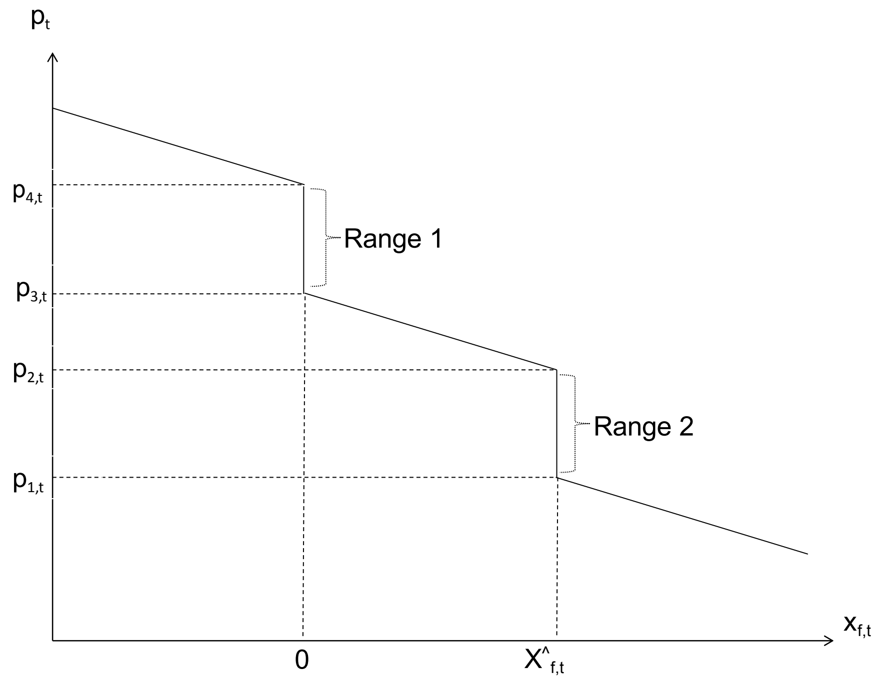


FIGURE 4.1: When news is positive i.e. $\hat{x}_{f,t} > 0$, we show the optimal demand of the fundamentalists in equation (4.9) will choose to hold depending on the conditions above.

Essentially, how the fundamentalists choose the worse-case scenario depends on the position of the risky asset and the signal observed. From equation (4.6), we know that the fundamentalists take a long position in the risky asset, i.e. $x_{f,t} > 0$ when news is bad (good) i.e. $s_{t+1} - \bar{f} < 0 (> 0)$, and the information is reliable (unreliable) $\phi_u(\phi_l)$. The opposite applies to a short position $x_{f,t} < 0$. As for risky asset variance, the worse-case scenario is always an unreliable signal (ϕ_l) because more risk is accounted for.

Trend followers do not experience ambiguity and have CARA utility over future wealth $W_{c,t+1}$, i.e. $U_{c,t}(W_{c,t+1}) = -e^{-a_c W_{c,t+1}}$ with $a_c > 0$. Also denote $\mathbb{CE}_c(x)$ to be the certainty equivalent of an utility maximiser. Then the investors' utility

$\mathbb{E}_{c,t}[U(W_{c,t} + (p_{t+1} + \bar{d} - p_t)x_{c,t})]$ is equal to $U_{c,t}((\mathbb{CE})(x))$, with

$$\mathbb{CE}_{c,t}(x) = \mathbb{E}_{c,t}(W_{c,t+1}) - \frac{1}{2}\mathbb{V}_{c,t}(W_{c,t+1}). \tag{4.15}$$

Here, $\mathbb{E}_{c,t}$ and $\mathbb{V}_{c,t}$ are the conditional expectation and variance of the trend followers. As in chapter 3, we assume

$$\mathbb{E}_{c,t}(p_{t+1}) = p_{t-1} + \gamma_c(p_{t-1} - u_{t-1}), \quad \textit{and} \quad \mathbb{V}_{c,t}(p_{t+1}) = \sigma_f^2(1 + b_c v_{t-1}), \tag{4.16}$$

where $\gamma_c > 0$ and $b_c > 0$. The mean and variance of the geometric decaying process are

$$u_t = \delta_c u_{t-1} + (1 - \delta_c)p_{t-1} \quad \textit{and} \quad v_t = \delta_c v_{t-1} + \delta_c(1 - \delta_c)(p_{t-1} - u_{t-1})^2$$

respectively with $\delta_c > 0$. Since ambiguity does not affect the decision-making of the trend followers, the demand of the trend followers is given

$$x_{c,t} = \frac{\mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - p_t}{a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}. \tag{4.17}$$

4.2.2 Market Equilibrium

In equilibrium we assume that the per capita demand for the risky asset must equal to the per capita supply. Since there are a range of prices fundamentalists do not participate in the market, the equilibrium determined varies accordingly. Let $p_{\phi_l,t}$ and $p_{\phi_u,t}$ denote the equilibrium stock price when fundamentalists is a standard expected utility maximiser with subject belief ϕ_l and ϕ_u respectively. While $p_{\hat{x},t}$ and $p_{o,t}$ denote the prices when the risky asset position of the fundamentalists is associated with the threshold positions i.e $x_{f,t} = \hat{x}_{f,t}$ and $x_{f,t} = 0$. Let η_f, η_c, and η_n be the fractions (%) of fundamentalists, trend followers and noise investors in the market respectively, summing up to one, and let demand equal to a supply $\bar{x}$

$$\eta_f x_{f,t} + \eta_c x_{c,t} + \eta_n x_{n,t} = \overline{x}, \tag{4.18}$$

where x_n is normally distributed with $x_n \sim \mathcal{N}(0, \sigma^2_{x_n})$. Depending on fundamentalists' participation in the market, and conditional on the expectation of the trend followers (refer to Appendix C.3 for proofs), there are four types of market condition.

Type I - According to equation (4.9), if the fundamentalists demand is $\hat{x}_{f,t} > 0(< 0)$, we know they will only demand a fixed position of the risky asset for a range of prices $p_{1,t} < p_t < p_{2,t}$ $(p_{3,t} < p_t < p_{4,t})$. The market clears according to $\eta_c \hat{x}_{f,t} + \eta_c x_{c,t} = \bar{X}$. Therefore, substituting equation (4.17) yields a market clearing price, we refer this as the $p_{\hat{x},t}$

$$p_{\hat{x},t} = \mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - \frac{(X - n_f \hat{x}_{f,t}) a_c (\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{n_c}, \tag{4.19}$$

where $X = \bar{x} - \eta_n x_{n,t}$. Similar to chapter 3, we substitute (4.19) into the condition $p_{1,t} < p_t < p_{2,t}$ $(p_{3,t} < p_t < p_{4,t})$ to re-express in terms of $\mathbb{E}_{c,t}$ such that we have $\mathbb{E}_{1,a,t}(p_{t+1}) < \mathbb{E}_{c,t}(p_{t+1}) \leq \mathbb{E}_{1,b,t}(p_{t+1})$ or $\mathbb{E}_{2,b,t}(p_{t+1}) < \mathbb{E}_{c,t}(p_{t+1}) \leq \mathbb{E}_{2,a,t}(p_{t+1})$ depending on whether the signal is good news or bad news. Here we denote $\mathbb{E}_{1,a,t}$, $\mathbb{E}_{2,a,t}$, $\mathbb{E}_{1,b,t}$, and $\mathbb{E}_{2,b,t}$ by

$$\begin{aligned}
\mathbb{E}_{1,a,t} &= \mathbb{E}_{f,\phi_l,t} + \bar{d} + \frac{X a_c(\mathbb{V}_{c,t} + \sigma_d^2) - max(\hat{x}_{f,t}, 0)(n_f a_c(\mathbb{V}_{c,t} + \sigma_d^2) + n_c a_f(\mathbb{V}_{f,\phi_l,t} + \sigma_d^2))}{n_c} \\
\mathbb{E}_{2,a,t} &= \mathbb{E}_{f,\phi_l,t} + \bar{d} + \frac{X a_c(\mathbb{V}_{c,t} + \sigma_d^2) - min(\hat{x}_{f,t}, 0)(n_f a_c(\mathbb{V}_{c,t} + \sigma_d^2) + n_c a_f(\mathbb{V}_{f,\phi_l,t} + \sigma_d^2))}{n_c} \\
\mathbb{E}_{1,b,t} &= \mathbb{E}_{f,\phi_u,t} + \bar{d} + \frac{X a_c(\mathbb{V}_{c,t} + \sigma_d^2) - max(\hat{x}_{f,t}, 0)(n_f a_c(\mathbb{V}_{c,t} + \sigma_d^2) + n_c a_f(\mathbb{V}_{f,\phi_u,t} + \sigma_d^2))}{n_c} \\
\mathbb{E}_{2,b,t} &= \mathbb{E}_{f,\phi_u,t} + \bar{d} + \frac{X a_c(\mathbb{V}_{c,t} + \sigma_d^2) - min(\hat{x}_{f,t}, 0)(n_f a_c(\mathbb{V}_{c,t} + \sigma_d^2) + n_c a_f(\mathbb{V}_{f,\phi_u,t} + \sigma_d^2))}{n_c},
\end{aligned} \tag{4.20}$$

where for easy expression let us simply express the expectation and variance in terms of $\mathbb{E}$ and $\mathbb{V}$ in price.

Type II - If the fundamentalists demand is $x_{f,t} = 0$, then the market is only populated with the trend followers and noise investors. The market clearing condition becomes $\eta_c x_{c,t} = \bar{X}$, and the equilibrium price becomes

$$p_{o,t} = \mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - \frac{X a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{n_c}. \tag{4.21}$$

Again, the condition can be re-expressed as $\mathbb{E}_{1,a,t}(p_{t+1}) < \mathbb{E}_{c,t}(p_{t+1}) \leq \mathbb{E}_{1,b,t}(p_{t+1})$ or $\mathbb{E}_{2,b,t}(p_{t+1}) < \mathbb{E}_{c,t}(p_{t+1}) \leq \mathbb{E}_{2,a,t}(p_{t+1})$ depending on the signal the fundamentalists receive.

Type III - Now, other than the above two threshold demand functions, fundamentalists participate the market like utility maximisers conditional on the type of news and the quality of the signal. Apply market clearing condition $\eta_c x_{f,\phi_u,t} + \eta_c x_{c,t} = \bar{X}$, the equilibrium price when fundamentalists are subjected to ϕ_u is given by

$$p_{\phi_u,t} = \beta_{\phi_u,t}(\mathbb{E}_{f,\phi_u,t}(p_{t+1}) + \bar{d}) + (1 - \beta_{\phi_u,t})(\mathbb{E}_{c,t}(p_{t+1}) + \bar{d}) - \Delta_{\phi_u,t}, \tag{4.22}$$

where

$$\beta_{\phi_u,t} = \frac{n_f}{n_f + n_c \frac{a_f(\mathbb{V}_{f,\phi_u,t}(p_{t+1}) + \sigma_d^2)}{a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}}, \tag{4.23}$$

$$\Delta_{\phi_u,t} = \frac{X a_f a_c(\mathbb{V}_{f,\phi_u,t}(p_{t+1}) + \sigma_d^2)(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{n_f a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2) + n_c a_f(\mathbb{V}_{f,\phi_u,t}(p_{t+1}) + \sigma_d^2)}.$$

The condition is therefore re-expressed as $\mathbb{E}_{1,b,t}(p_{t+1}) < \mathbb{E}_{c,t}(p_{t+1}) \leq \mathbb{E}_{2,b,t}(p_{t+1})$.

Type IV - Finally, the equilibrium price when fundamentalists are subjected to ϕ_l after applying market condition $\eta_c x_{f,\phi_l,t} + \eta_c x_{c,t} = \bar{X}$ becomes

$$p_{\phi_l,t} = \beta_{\phi_l,t}(\mathbb{E}_{f,\phi_l,t}(p_{t+1}) + \bar{d}) + (1 - \beta_{\phi_l,t})(\mathbb{E}_{c,t}(p_{t+1}) + \bar{d}) - \Delta_{\phi_l,t}, \tag{4.24}$$

where

$$\beta_{\phi_l,t} = \frac{n_f}{n_f + n_c \frac{a_f(\mathbb{V}_{f,\phi_l,t}(p_{t+1})+\sigma_d^2)}{a_c(\mathbb{V}_{c,t}(p_{t+1})+\sigma_d^2)}}, \tag{4.25}$$

$$\Delta_{\phi_l,t} = \frac{X a_f a_c(\mathbb{V}_{f,\phi_l,t}(p_{t+1}) + \sigma_d^2)(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{n_f a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2) + n_c a_f(\mathbb{V}_{f,\phi_l,t}(p_{t+1}) + \sigma_d^2)},$$

such that the price either satisfies $\mathbb{E}_{c,t}(p_{t+1}) \leq \mathbb{E}_{1,a,t}(p_{t+1})$ or $\mathbb{E}_{c,t}(p_{t+1}) > \mathbb{E}_{2,a,t}(p_{t+1})$.

Due to the structure of the model in this chapter, we observe two portfolio inertia subjected to the fundamentalists being ambiguity averse to the signal received. This is different to the previous chapter where a single inertia is observed, in particular, the fundamentalist will demand zero position. The co-existing of different investors is the key to our equilibrium. Consequently, the market will always arrive at equilibrium regardless of the participation from the fundamentalists. Notice $p_{o,t}$ is completely independent of the fundamentalists' belief as they demand zero position and hence the market price is determined through the interaction between the trend followers and the noise investors. While $p_{\hat{x},t}$ is affected by threshold position that fundamentalists would hold for a range of prices under uncertainty, $\hat{x}_{f,t} = \frac{2(\bar{d}-f_{t+1})\gamma_f}{a_f b_f \sigma_f^2 \sigma_d^2}$. Consequently, the size and the type of signal (i.e. good or bad news) about the future fundamental value of the risky asset received by the fundamentalists impacts the value of the equilibrium price over different price range.

A summary of the market equilibrium price is characterised as

$$p_t = \begin{cases} p_{\phi_l,t} & \mathbb{E}_{c,t}(p_{t+1}) \leq \mathbb{E}_{1,a,t}(p_{t+1}) \\ max(p_{o,t}, p_{\hat{x},t}); & \mathbb{E}_{1,a,t}(p_{t+1}) < \mathbb{E}_{c,t}(p_{t+1}) \leq \mathbb{E}_{1,b,t}(p_{t+1}) \\ p_{\phi_u,t}; & \mathbb{E}_{1,b,t}(p_{t+1}) < \mathbb{E}_{c,t}(p_{t+1}) \leq \mathbb{E}_{2,b,t}(p_{t+1}) \\ min(p_{o,t}, p_{\hat{x},t}); & \mathbb{E}_{2,b,t}(p_{t+1}) < \mathbb{E}_{c,t}(p_{t+1}) \leq \mathbb{E}_{2,a,t}(p_{t+1}) \\ p_{\phi_l,t}; & \mathbb{E}_{c,t}(p_{t+1}) > \mathbb{E}_{2,a,t}(p_{t+1}). \end{cases} \tag{4.26}$$

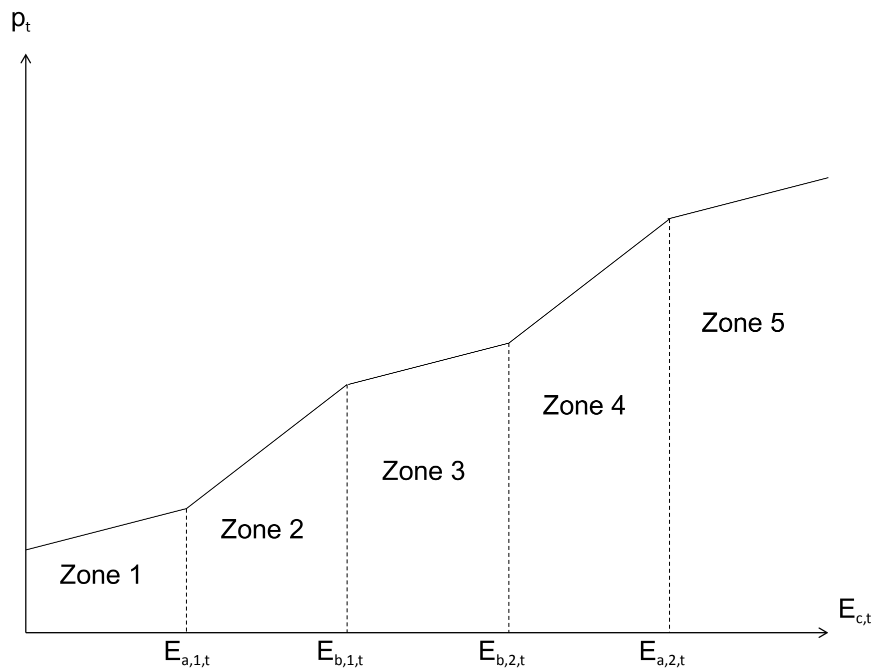


FIGURE 4.2: Five zones represents the five intervals as in equation (4.26).

Figure 4.2 is a snapshot of the equilibrium price at time t, such that the x-axis is in terms of the expectation of the trend followers, while the y-axis is the level of price. Depending on which condition is satisfied at time t, the equilibrium price is determined by one of the five prices in equation (4.26). The figure shows that the size of the zone determines how likely the type of price is determined and that price increases with increasing expectation of the trend followers about the future value of the risky asset with kinks in between. Intuitively, as the model moves into a multi-period framework (as it is with our model), the change in the expectation of the trend followers will shift the equilibrium to different zones. The further is the shifts from one zone to another, the greater the change in price leading to an increase in volatility. In fact, as we shall show in the simulations later, it is indeed the case where price shifts drastically from one zone to another creating excess volatility.

4.3 Market Dynamics and Stylised Facts

In this section, we show that our model is able to characterise various market behaviours and provide evidence to excess volatility puzzle due to the rises of ambiguity in the market, as well as insignificant ACs in returns, significant ACs in absolute and squared returns, as well as volatility clustering. In particular, we discuss some important parameters in greater details that are vital to our results.

A summary of the initial parameters is provided in the Table 4.1. All results discussed below are based on this set of parameters unless otherwise stated. For results that are based on averages, it is the mean of 40 simulations. Random draw is fixed to allow consistent comparison between results. Each simulation takes a single period as one trading day, therefore there are 250 trading days in a year stretching over 30 years. For simplicity, we assume net-zero supply; dividend and risk-free rate are zero (as the impact of the dividend is minimal). All initial price levels start at 100. There are two sources of randomness in this model, 1) noise investors randomly trade according to a normal distribution, and 2) the noisy signal received by the fundamentalists. We will see later that these two sources of randomness are vital in our model as they both help to contribute to the arrival of the stylised facts discussed below.

TABLE 4.1: Parameter settings and initial values

$\#of\,sim.$	K	T	N	$\bar{f}$	σ_a	σ_b	σ	σ_s	(ϕ_l, ϕ_u)
40	250	30	$K*T$	100	$\frac{\bar{f}*0.001}{\sqrt{K}}$	$\frac{\bar{f}*0.15}{\sqrt{K}}$	$\frac{\bar{f}*0.015}{\sqrt{K}}$	$\frac{\bar{f}*0.2}{\sqrt{K}}$	(0.01,0.9)
γ_f	γ_c	δ_c	η_f	η_c	σ_{x_n}	a_f	a_c	r_f	$\overline{x}$
0.001	0.17	0.85	0.45	0.45	1	1	1	0	0

4.3.1 Excess Volatility and Volatility Clustering

The pre and post crisis markets indicate strong increase in excess volatility and volatility clustering, which can be intuitively related to the increase in market uncertainty. Here, we study the mechanism behind our model in generating excess volatility and volatility clustering in returns. Figure 4.3 shows the fraction of five types of equilibrium prices that make up a typical simulation, where each zone represents the price that is conditional on the demand of the fundamentalists. Recall Figure 4.2 shows a snapshot of the market at time t, the greatest change in price is observed from zone 1 to zone 5 (or vice versa), and therefore leads to volatility. Here, we observe that excess volatility is indeed induced by this switching mechanism. The majority of the prices here are determined by the either end of the conditions as fundamentalists factor in the worst case scenario with the largest possible risk involved (and assumes the information is most unreliable). Further, the lower panel shows a sample of period, where the price switches between zone 1 and 5, and induces high volatility. This large movement in returns is further demonstrated in Figure 4.4 where the moving-average over a rolling 200-window of the return series is shown for Cases(0,1,1) and (1,1,1). Here we denote Case(0,1,1) to be a "normal market" and Case(1,1,1) to be a "market of turmoil". They are plotted against the noisy signal received by the fundamentalists for comparison purpose. Here, the fundamental noisy cannot explain the excess volatility that exhibits in both cases. The returns series in the normal-state of the market shows an excess volatility, however, the ambiguity amplifies the excess volatility.

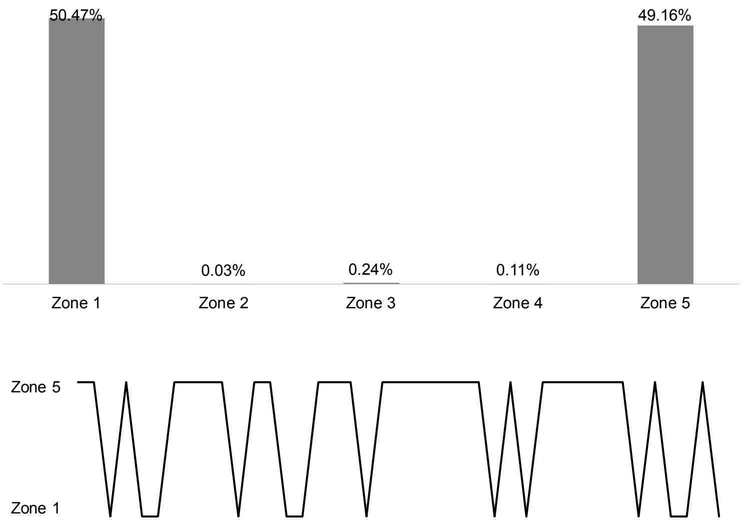


FIGURE 4.3: The five different zones of prices are shown here with percentage as a portion of the whole period for each zone. Whereas the second sub-figure is a snapshot of the change in price from zone 1 to zone 5 over a short period of time.

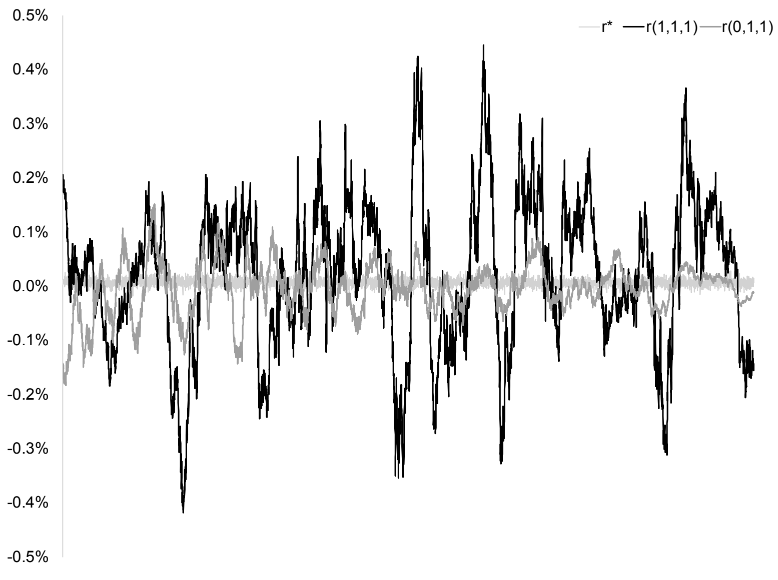


FIGURE 4.4: Moving average return series over a rolling-window of 200 days for Case(0,1,1) and Case(1,1,1). The fundamental signal is also plotted for comparison purpose.

4.3.2 Volatility Clustering and Autocorrelation Patterns

The aim of this section is to explore the possible explanations to volatility fluctuations and more specifically the difference between in presence of ambiguity and non-ambiguity, as well as the important role noise investors have in this model. To do so, we look at four different scenarios. Let each scenario be denoted by Case(amb, f, n), such that the indicator of each parameter represents 1 or 0. Case(0,0,0) corresponds to a market in absence of ambiguity, and no noises, i.e a deterministic case. Case(1,1,0) corresponds to a market in presence of ambiguity with noisy fundamental signals. Case(0,1,1) corresponds to a market in absence of ambiguity but receives both noises. Finally, Case(1,1,1) corresponds to a market in presence of ambiguity with both noises. Remembering σ_ϵ is bounded by some upper and lower bounds and is randomly drawn at each period. Figures 4.5 and 4.6 show a typical simulation of the price and return series over the whole periods for each of the four cases.

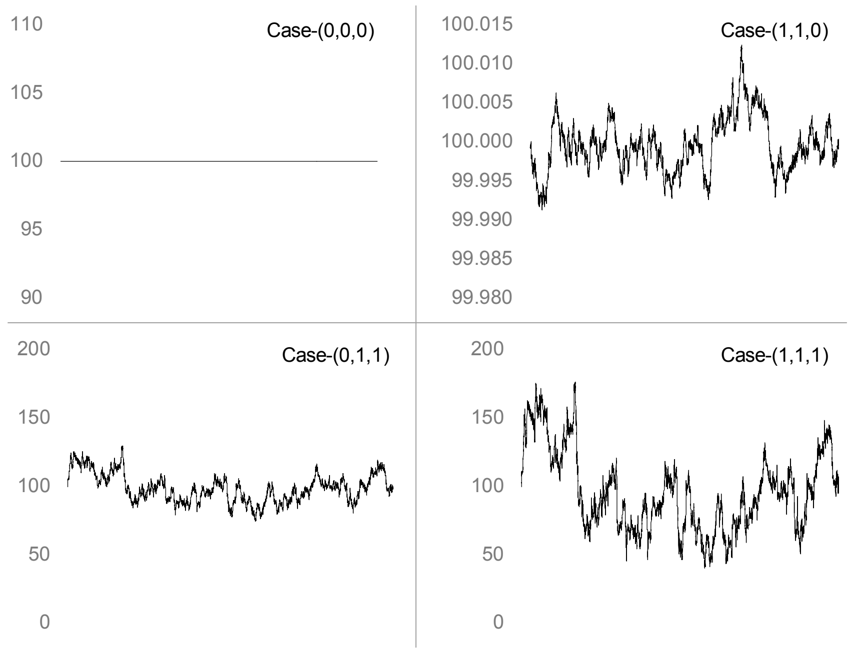


Figure 4.5: Price series for four different cases: case(0,0,0); case(1,1,0); case(0,1,1); and case(1,1,1).

When there is no randomness and ambiguity, i.e. Case(0,0,0), the top left of Figure 4.5 shows that no market movement is expected as all price values are initialised at 100. In Case(1,1,0), we observe small but nevertheless movements in price. The small fluctuations are mainly due to the fact that market is only populated with ambiguity-averse fundamentalists and trend followers. Since the fundamentalists only depend on the random signal which is a white noise. Without noise investors to distract and deviate the market by trading with the trend followers, the market moves in little motions. Analytically, we may observe from equation (4.26) that the prices bounded by the conditions do not vary much between the four equilibrium types due to $X = 0$ and small converging rates of both fundamentalists and chartists, i.e. γ_f and γ_c mean there are little movement in the expectation about the future value of the risk asset with any of investors. Consequently, changing from one zone to another (i.e. Figure 4.2) from this period to the next results in minimal change in price. However, the important thing to notice in this case is the mechanism for potential large jumps exist in the model itself and with a realistic approach as in Case(1,1,1) as we factor in ambiguity and consequently induce large price movements in the market. Case(0,1,1) shows a "normal" market in the sense that it is free of ambiguity signal. The market is populated with two types of utility maximisers and noise investors, thus there will always be a single equilibrium price. Let us also denote this case to be the pre-crisis scenario and compares the results to Case(1,1,1), which we would refer to as a "period of turmoil" during the financial crisis. It is quite evident in Case(1,1,1) that when the fundamentalists are subjected to uncertainty, price movements are amplified. The amplification observed in this type of market is due to the interactions between the ambiguity-averse fundamentalists and the rest of the investors in the market.

We further this analysis by looking at the return series of the last three cases shown in Figures 4.6 and 4.7. Again the return series are shown for all periods while autocorrelation of returns, absolute returns and squared returns are shown with 200 lags. In particular, Case(0,1,1) and (1,1,1) replicate the pre and during crisis patterns observed in the real market indices discussed in chapter 1. The return series in Case(0,1,1) is bounded by a similar range of returns and exhibit

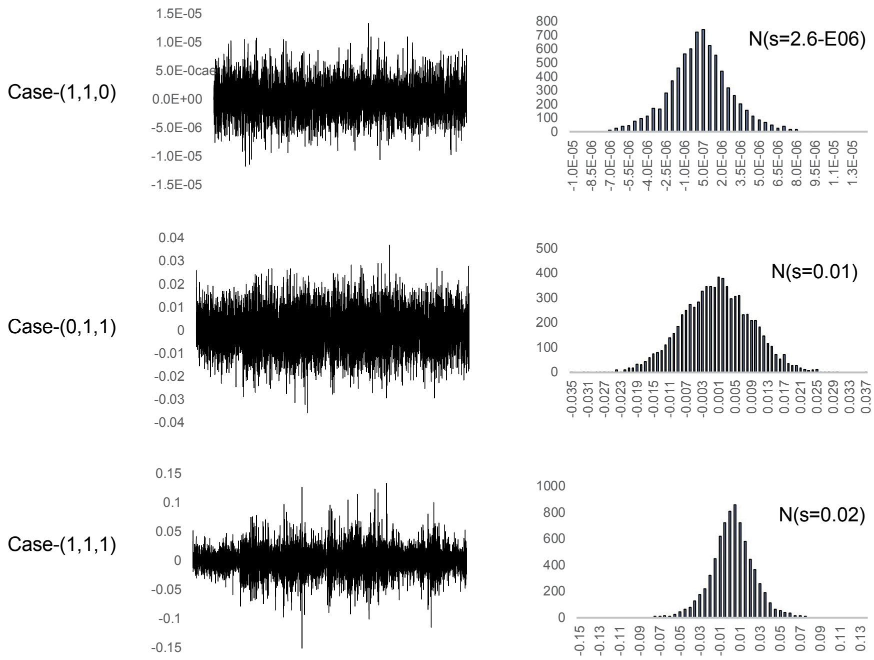


FIGURE 4.6: Return series for case 2-(1,1,0); case 3-(0,1,1); and case 4-(1,1,1).

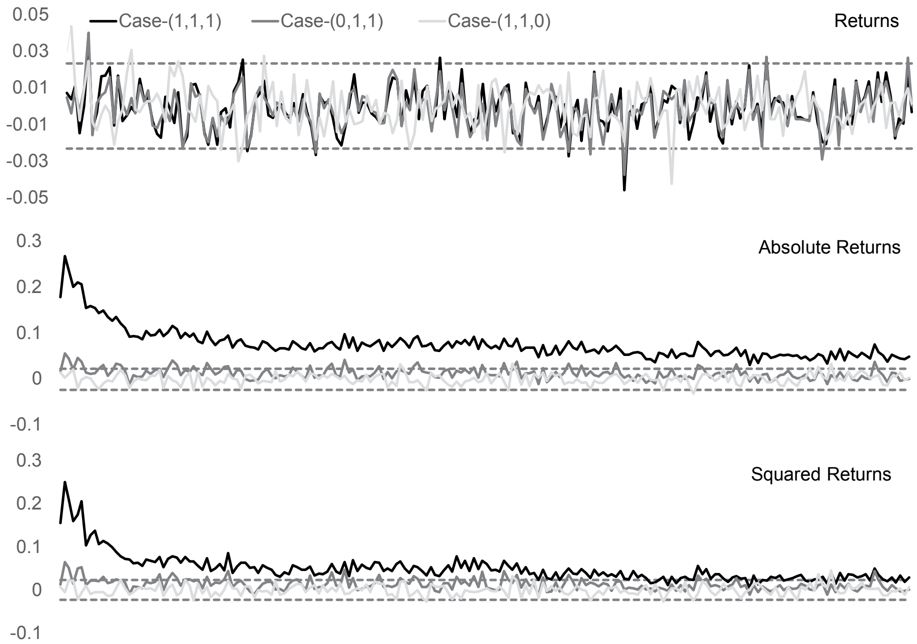


FIGURE 4.7: Autocorrelations for case 2-(1,1,0); case 3-(0,1,1); and case 4-(1,1,1).

little volatility clustering as well as insignificant autocorrelations in all return, absolute return, and squared returns series. What is really interesting is when ambiguity is added to the picture as in Case(1,1,1), the return series start to exhibit volatility clustering with a volatility range of about ±10%, while showing a higher kurtosis than the rest. Insignificant autocorrelation in returns is observed, while absolute and squared return series show strong indication of long memories starting off around 0.25 and slowly decays to insignificance over a lag of 200.

4.4 Conclusion

We propose a new model under ambiguity with updating mechanism, and study the implication of ambiguity on the market coupling with investors heuristic behaviours. In particular, rather than to exogenously apply ambiguity to the fundamental value of the risky asset as in the previous chapter, here we allow ambiguity to be endogenously embedded through an ambiguous signal received by fundamentalists and a Bayesian updating mechanism. This model assumes fundamentalists only observe an ambiguous signal about the fundamental value as oppose to the true process and the estimation is updated at every time period. The switching between the perceived reliability of the signal over time gives rise to excessive volatility and other stylised facts such as insignificant ACs in returns, significant ACs in absolute and squared returns, as well as volatility clustering.

Chapter 5

Conclusion

Traditional asset pricing theory based on rational expectation and representative agent paradigm has been unsuccessful in providing plausible explanations to many of the stylised facts and anomalies observed in todays ever changing and complex markets. A growing body of empirical evidence suggests investors behaviours are often ambiguity-averse and they may choose not to participate in some market conditions which results in market freeze or market non-participation. Further, market evidences have suggested that market uncertainty rose during the financial crisis, and we have shown that market stylised facts are amplified when compared to the pre-crisis period. On the other hand, evidence has suggested that markets are actually populated with investors where their trading strategies are often described by simple heuristic rules. This is an important feature in the popular HAMs where market observations can be explain in terms of simple, behavioural rules of market participants, which is rather intuitive. Since both ambiguity-aversion and heterogeneity are natural parts of investors' behaviours in financial markets, this thesis explores the joint impact of heterogeneity and ambiguity in particular on price dynamics in financial markets and provides some insight into various market behaviours and anomalies. This thesis contributes to the development of financial market modelling and asset price dynamics under ambiguity and heterogeneous beliefs to tackle two main issues. First, we provide a multi-asset setup to understand the implication of correlation on ambiguity, and therefore market liquidity in time of uncertainty. Second, we propose two dynamic ambiguity models to characterise market anomalies and stylised facts that have yet to be studied in the ambiguity literature. The main contributions of the three chapters and related future research are summarised below.

Market liquidity under ambiguity is studied in a static setup with a single risky asset and risk-free asset. However there is usually more than one risky asset in the market, the implication of correlation therefore plays an important role in capturing this relationship. Chapter 2 adds to the ambiguity literature by exploring a multi-asset setup under ambiguity and heterogeneity, and studies the consequent implication on market illiquidity during a market downturn. Further, in a typical HAM setup, trading activities can be observed between heterogeneous investors as they differ in their expectation about the future value of the risky assets. However under ambiguity, heterogeneous investors become uncertain about the future values, market fails to arrive at equilibrium and become illiquid. The results show when risky assets are negatively correlated, the market illiquidity is reduced due to diversification effect. Further, despite the correlation coefficients between risky assets, ambiguity affects all assets such that the riskier asset is more illiquid than a relative safer asset. While heterogeneity does facilitate trades (with smaller degree of ambiguity), it is not always true when the market experiences heightened uncertainty, in fact, illiquidity worsen in some cases. Heterogeneity is more effective in reducing illiquidity when it is about the second moment of the risky asset, i.e. variance and covariance, than when it is applied to the first moment. In general, market freeze is resilient in case of heightened uncertainty, however, the higher the diversity between beliefs and correlations between risky assets the greater the reduction in market illiquidity due to the effect of variance-covariance matrix. In this chapter, we only consider a systematic ambiguous shock to the market; it would be interesting to look at an idiosyncratic ambiguous shock in a multi-risky asset setup to study the spillover effect from one risky asset to another.

Market anomalies and stylised facts have long been found and studied in the literature. Due to the restrictive assumptions of the traditional asset pricing models, they cannot provide satisfactory explanations to what we observe empirically in the markets. Such problems have been well addressed in the HAMs space, in which they justify the empirical evidence by relating to the heuristic behaviours that investors typically exhibit. Although some researchers have discussed ambiguity and volatility, most of these models are in a static setting and do not explicitly demonstrate models' abilities to generate anomalies and stylised facts in price and return series. Chapters 3 and 4 contribute to the ambiguity literature by filling this gap. We provide results to show that ambiguity leads to a variety of market anomalies and stylised facts. The two chapters introduce ambiguity models by imposing typical heuristic behaviours from HAMs and analyse market dynamics in a multi-period setup. The models allow for three different types of strategists and each type follows a specific heuristic updating rule, namely fundamentalists, trend chasers and noise traders. Since the market is perceived to be more uncertain during the financial crisis, we also assume the fundamentalists became uncertainty about the fundamental value of the risky asset. Based on this, fundamentalists are ambiguity-averse, while allowing trend chasers to update based on past prices (in absence of ambiguity). The key difference between chapters 3 and 4 is the structure of ambiguity imposed in each of the model. In chapter 3, we consider a simple and exogenous approach in structuring ambiguity, whereas chapter 4 allows ambiguity to be endogenously embedded through an ambiguous signal received by fundamentalists while allowing an updating mechanism. The results show that ambiguity leads to market participation of fundamentalists in some periods and not in others. Specifically, if fundamentalists exhibit portfolio inertia, the change in price needs to be significant enough to entice them to return to the market. Consequently, the switching in participation, as well as the interaction between investors under the influence of uncertainty over time induces and amplifies excessive volatility and other stylised facts. For future study, it would prove fruitful to introduce multi-risky assets in a dynamic setting to understand the implication of ambiguity on cross-sectional assets or markets, with investors follow some particular heuristic rules.

Appendix A

for Chapter 2

A.1 Bid and ask prices of two assets

To find bid and ask prices in terms of the equilibrium price vector for equation (2.20) and (2.21), we use equations (2.5) and (2.7). The bid price becomes

$$
\begin{aligned}
\mathbf{p_{bid}} &= max_i\{\frac{1}{R_f}(\underline{\alpha}\mathbf{v_i} - \frac{1}{\tau_i}\mathbf{\Omega_i}\mathbf{x^*_{io}})\} \\
&= max_i\{\frac{1}{R_f}(\underline{\alpha}\mathbf{v_i} - \frac{1}{\tau_i}\mathbf{\Omega_i}\tau_\mathbf{i}\mathbf{\Omega_i^{-1}}[(\mathbf{v_i} - \mathbf{v_a}) + \frac{\mathbf{\Omega_a x_m}}{\tau_\mathbf{a}}])\} \\
&= max_i\{\frac{1}{R_f}(\underline{\alpha}\mathbf{v_i} - [(\mathbf{v_i} - \mathbf{v_a}) + \frac{\mathbf{\Omega_a x_m}}{\tau_a}])\} \\
&= max_i\{\frac{1}{R_f}(\underline{\alpha}\mathbf{v_i} - \mathbf{v_i} + (\mathbf{v_a} - \frac{\mathbf{\Omega_a x_m}}{\tau_a}))\} \\
&= max_i\{\frac{1}{R_f}((\underline{\alpha} - 1)\mathbf{v_i} + \mathbf{p^*_o})\} \\
&= \mathbf{p^*_o} + \frac{1}{R_f}max_i\{-(1 - \underline{\alpha})\mathbf{v_i}\} \\
&= \mathbf{p^*_o} - \frac{1}{R_f}min_i\{(1 - \underline{\alpha})\mathbf{v_i}\}
\end{aligned}
$$

and the ask price becomes

$$
\begin{aligned}
\mathbf{p_{ask}} &= min_i\{\frac{1}{R_f}(\bar{\alpha}\mathbf{v_i} - \frac{1}{\tau_i}\mathbf{\Omega_i x^*_{io}}\} \\
&= min_i\{\frac{1}{R_f}(\bar{\alpha}\mathbf{v_i} - \frac{1}{\tau_i}\mathbf{\Omega_i}\tau_\mathbf{i}\mathbf{\Omega_i^{-1}}[(\mathbf{v_i} - \mathbf{v_a}) + \frac{\mathbf{\Omega_a x_m}}{\tau_\mathbf{a}}])\} \\
&= min_i\{\frac{1}{R_f}(\bar{\alpha}\mathbf{v_i} - [(\mathbf{v_i} - \mathbf{v_a}) + \frac{\mathbf{\Omega_a x_m}}{\tau_a}])\} \\
&= min_i\{\frac{1}{R_f}(\bar{\alpha}\mathbf{v_i} - \mathbf{v_i} + (\mathbf{v_a} - \frac{\mathbf{\Omega_a x_m}}{\tau_a}))\} \\
&= min_i\{\frac{1}{R_f}((\bar{\alpha} - 1)\mathbf{v_i} + \mathbf{p^*_o})\} \\
&= \mathbf{p^*_o} + \frac{1}{R_f}min_i\{-(1-\bar{\alpha})\mathbf{v_i}\} \\
&= \mathbf{p^*_o} - \frac{1}{R_f}max_i\{(1-\bar{\alpha})\mathbf{v_i}\}.
\end{aligned}
$$

A.2 The Threshold Δ_o

To find the threshold value in equation (2.24), we let the bid price to equate to ask price, and substitute equation (2.23). We have

$$
\begin{aligned}
p_{ask} &= p_{bid} \\
\mathbf{p^*_o} - \frac{1}{R_f}(1-\underline{\alpha})max_i\{\mathbf{v_i}\} &= \mathbf{p^*_o} - \frac{1}{R_f}(1-\bar{\alpha})min_i\{\mathbf{v_i}\} \\
\mathbf{p^*_o} - \frac{1}{R_f}(1-\alpha_o(1-\Delta_o))max_i\{\mathbf{v_i}\} - (\mathbf{p^*_o} - \frac{1}{R_f}(1-\alpha_o(1+\Delta_o))min_i\{\mathbf{v_i}\}) &= 0 \\
\frac{[max_i\{\mathbf{v_i}\} - min_i\{\mathbf{v_i}\}](1-\alpha_o)}{[max_i\{\mathbf{v_i}\} + min_i\{\mathbf{v_i}\}]\alpha_o} &= \Delta_o
\end{aligned}
$$

A.3 Bid and ask prices

A.3.1 Case 1 - bid and ask price for equation (2.32) and (2.33)

Since we are only allowing risky asset 2 to differ about the expected payoff, $min\{v_{i1}\} = max\{v_{i1}\} = v_1$, while $min\{v_{i2}\} = v_2(1-\gamma_v)$ and $max\{v_{i2}\} = v_2(1+\gamma_v)$.

$$\begin{aligned}
\mathbf{p_{bid_{case1}}} &= \left(p^*_{o1_{case1}} - \frac{(1-\underline{\alpha})min\{v_{i1}\}}{R_{f_{case1}}}, p^*_{o2_{case1}} - \frac{(1-\underline{\alpha})min\{v_{i2}\}}{R_{f_{case1}}}\right)^\top \\
&= \frac{1}{R_{f_{case1}}}\left(\frac{\alpha_o\tau_o(1-\Delta)v_1 - \sigma_1(\sigma_1+\rho\sigma_2)}{\tau_o},\right. \\
&\left.\frac{\tau_o((1-\Delta)(1-\gamma_v)\alpha_o + (1+\gamma)\gamma_v)v_2 - \sigma_2(\sigma_1+\rho\sigma_2)}{\tau_o}\right)^\top,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{p_{ask_{case1}}} &= \left(p^*_{o1_{case1}} - \frac{(1-\bar{\alpha})max\{v_{i1}\}}{R_{f_{case1}}}, p^*_{o2_{case1}} - \frac{(1-\bar{\alpha})max\{v_{i2}\}}{R_{f_{case1}}}\right)^\top \\
&= \frac{1}{R_{f_{case1}}}\left(\frac{\alpha_o\tau_o(1+\Delta)v_1 - \sigma_1(\sigma_1+\rho\sigma_2)}{\tau_o},\right. \\
&\left.\frac{\tau_o((1+\Delta)(1+\gamma_v)\alpha_o - (1-\gamma)\gamma_v)v_2 - \sigma_2(\sigma_1+\rho\sigma_2)}{\tau_o}\right)^\top,
\end{aligned}$$

A.3.2 Case 2 - bid and ask price for equation (2.36) and (2.37)

Since we are only allowing risky asset 2 to differ about the variance, we let $min\{v_{i1}\} = max\{v_{i1}\} = v_1$, and $min\{v_{i2}\} = max\{v_{i2}\} = v_2$.

$$
\begin{aligned}
\mathbf{p_{bid_{case2}}} &= \left(p^*_{o1_{case2}} - \frac{(1-\underline{\alpha})v_{min_1}}{R_{f_{case2}}}, p^*_{o2_{case2}} - \frac{(1-\underline{\alpha})v_{min_2}}{R_{f_{case2}}}\right)^\top \\
&= \frac{1}{R_{f_{case2}}}\left(\alpha_o(1-\Delta)v_1 - \frac{(1-\rho^2)((1+\gamma_\sigma^2)\sigma_1^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}\right., \\
&\left.\alpha_o(1-\Delta)v_2 - \frac{(1-\rho^2)((1-\gamma_\sigma^2)^2\sigma_2^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}\right)^\top
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{p_{ask_{case2}}} &= (p^*_{o1_{case2}} - \frac{(1-\bar{\alpha})v_{max_1}}{R_{f_{case2}}}, p^*_{o2_{case2}} - \frac{(1-\bar{\alpha})v_{max_2}}{R_{f_{case2}}})^\top \\
&= \frac{1}{R_{f_{case2}}}\left(\alpha_o(1+\Delta)v_1 - \frac{(1-\rho^2)((1+\gamma_\sigma^2)\sigma_1^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}\right., \\
&\left.\alpha_o(1+\Delta)v_2 - \frac{(1-\rho^2)((1-\gamma_\sigma^2)^2\sigma_2^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}\right)^\top.
\end{aligned}
$$

A.3.3 Case 3 - bid and ask price

Since we are allowing risky asset 2 to differ about the expected payoff and the variance, $min\{v_{i1}\} = max\{v_{i1}\} = v_1$, while $min\{v_{i2}\} = v_2(1-\gamma_v)$ and $max\{v_{i2}\} = v_2(1+\gamma_v)$, hence

$$
\begin{aligned}
\mathbf{p_{bid_{case3}}} &= \left(p^*_{o1_{case3}} - \frac{(1-\underline{\alpha})v_{min_1}}{R_{f_{case3}}}, p^*_{o2_{case3}} - \frac{(1-\underline{\alpha})v_{min_2}}{R_{f_{case3}}}\right)^\top \\
&= \frac{1}{R_{f_{case3}}}\left(\alpha_o(1-\Delta)v_1 - \frac{\gamma_\sigma\gamma_v\rho\sigma_1}{(1+\gamma_\sigma^2-\rho^2)\sigma_2}v_2 - (1-\rho^2)\frac{((1+\gamma_\sigma^2)\sigma_1^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}\right), \\
&(1-(1-\gamma_v)(1-(1-\Delta)\alpha_o) \\
&\left.-\frac{\gamma_\sigma\gamma_v(2-\rho^2)}{1+\gamma_\sigma^2-\rho^2})v_2 - (1-\rho^2)\frac{((1-\gamma_\sigma^2)^2\sigma_2^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)}\right)^\top,
\end{aligned}
$$

and the ask prices are

$$
\begin{aligned}
\mathbf{p_{ask_{case3}}} =& \left(p^*_{o1_{case3}} - \frac{(1-\bar{\alpha})v_{max_1}}{R_{f_{case3}}}, p^*_{o2_{case3}} - \frac{(1-\bar{\alpha})v_{max_2}}{R_{f_{case3}}} \right)^\top \\
=& \frac{1}{R_{f_{case3}}} \left(\alpha_o(1+\Delta)v_1 - \frac{\gamma_\sigma \gamma_v \rho \sigma_1 v_2}{(1+\gamma_\sigma^2-\rho^2)\sigma_2} - (1-\rho^2)\frac{((1+\gamma_\sigma^2)\sigma_1^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)} \right), \\
& (1-(1+\gamma_v)(1-(1+\Delta)\alpha_o) \\
& - \frac{\gamma_\sigma\gamma_v(2-\rho^2)}{1+\gamma_\sigma^2-\rho^2})v_2 - (1-\rho^2)\frac{((1-\gamma_\sigma^2)^2\sigma_2^2 + (1-\gamma_\sigma^2)\rho\sigma_1\sigma_2)}{\tau_o(1+\gamma_\sigma^2-\rho^2)} \Bigg)^\top .
\end{aligned}
$$

Appendix B

for Chapter 3

B.1 Demand function for fundamentalists

To derive demand function for fundamentalists, we take equation (3.8) and find the first derivative of the function with respect to $x_{f,t}$ and set to zero, we show

$$(\mathbb{E}_{f,t}(p_{t+1}+d_{t+1})-R_f p_t)-a_f\mathbb{V}_{f,t}(p_{t+1}+d_{t+1})x_{f,t}=0 \tag{B.1}$$

$$\begin{aligned} x_{f,t} &= \frac{(\mathbb{E}_{f,t}(p_{t+1}+d_{t+1})-R_f p_t)}{a_f\mathbb{V}_{f,t}(p_{t+1}+d_{t+1})} \\ &= \frac{\mathbb{E}_{f,t}(p_{t+1})+\bar{d}-p_t R_f}{a_f(\sigma_f^2+\sigma_d^2)}, \end{aligned}$$

where $\mathbb{E}(d_{t+1})$ is $\bar{d}$ and $\mathbb{V}(d_{t+1})$ is σ_d^2.

Since fundamentalists experience ambiguity about the true value of the risky asset, their expectation is expressed as a set of beliefs

$$\mathbb{E}_{f,t}(p_{t+1})\in(\underline{\mathbb{E}}_{f,t}(p_{t+1}),\overline{\mathbb{E}}_{f,t}(p_{t+1})), \quad \mathbb{V}_{f,t}(p_{t+1})\in(\underline{\sigma}_f^2,\overline{\sigma}_f^2), \tag{B.2}$$

Consequently, under uncertainty, what matters to the fundamentalists is the largest possible risk, i.e. $\overline{\sigma}_f^2$. Whether the minimum occurs at the maximum or minimum mean payoff depends on their position in the asset. The minimum occurs at minimum mean payoff for the risky asset if fundamentalists expect the value to be greater, and maximum mean payoff occurs if they expect the price to fall. Therefore, the demand function becomes

$$x_t^f = \begin{cases} \frac{\underline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}-p_t R_f}{a_f(\overline{\sigma}_f^2+\sigma_d^2)} > 0 & \text{if} \quad \frac{\underline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}}{R_f} > p_t \\ 0 & \text{if} \quad \frac{\underline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}}{R_f} \leq p_t \leq \frac{\overline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}}{R_f} \\ \frac{\overline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}-p_t R_f}{a_f(\overline{\sigma}_f^2+\sigma_d^2)} < 0 & \text{if} \quad \frac{\overline{\mathbb{E}}_{f,t}(p_{t+1})+\bar{d}}{R_f} < p_t. \end{cases} \tag{B.3}$$

Appendix C

for Chapter 4

C.1 Conditional beliefs on f

The conditional expectation for f given s for each $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ is

$$\begin{aligned}
\mathbb{E}_t(f_{t+1}|s_{t+1}) &= \mathbb{E}_t(f_{t+1}) + \frac{\mathbb{C}ov(f_{t+1}, s_{t+1})}{\mathbb{V}(s_{t+1})}(s_{t+1} - \mathbb{E}(s_{t+1})) \\
&= \bar{f} + \frac{\mathbb{C}ov(f_{t+1}, (f_{t+1} + \epsilon_{t+1}))}{\sigma^2 + \sigma_\epsilon^2}(s_{t+1} - \bar{f}) \\
&= \bar{f} + \frac{\sigma^2}{\sigma^2 + \sigma_\epsilon^2}(s_{t+1} - \bar{f}) \\
&= \bar{f} + \phi(s_{t+1} - \bar{f}),
\end{aligned}$$

where $\phi = \frac{\sigma^2}{\sigma^2 + \sigma_\epsilon^2}$. While, the variance for f given s is

$$\begin{aligned}
\mathbb{V}_t(f_{t+1}|s_{t+1}) &= \mathbb{V}(f_{t+1}) - \frac{\mathbb{C}ov(f_{t+1}, s_{t+1})^2}{\mathbb{V}(s_{t+1})} \\
&= \sigma^2 - \frac{(\sigma^2)^2}{\sigma^2 + \sigma_\epsilon^2} \\
&= \sigma^2(1 - \phi).
\end{aligned}$$

C.2 Certainty Equivalent of Fundamentalists

We minimise $\mathbb{CE}_{f,t}(x)$ of the fundamentalists in ϕ

$$\min_{\phi\in[\phi_l,\phi_u]} \mathbb{E}_{f,\phi,t}[U(W_{f,t} + (p_{t+1} + \bar{d} - p_t)x_{f,t})|s_{t+1}].$$

But first, let us derive the expected future wealth

$$\begin{aligned}\mathbb{E}(W_{f,t+1}|s_{t+1}) &= \mathbb{E}(W_{f,t} + (p_{t+1} + d_{t+1} - p_t)x_{f,t}|s_{t+1})\\ &= W_{f,t} + (\mathbb{E}_{f,t}(p_{t+1}) + \bar{d} - p_t)x_{f,t},\end{aligned}$$

and variance

$$\begin{aligned}\mathbb{V}(W_{f,t+1}|s_{t+1}) &= \mathbb{V}_{f,t}(W_{f,t} + (p_{t+1} + d_{t+1} - p_t)x_{f,t}|s_{t+1})\\ &= (\mathbb{V}_{f,t}(p_{t+1}) + \sigma_d^2)x_{f,t}^2.\end{aligned}$$

Therefore, we can substitute the above expectation and variance into the $\mathbb{CE}_{f,t}(x)$ and show that

$$\begin{aligned}&\mathbb{CE}_{f,t}(x_{f,t},\phi)\\ &= \mathbb{E}(W_{f,t+1}|s_{t+1}) - \frac{1}{2}a_f\mathbb{V}(W_{f,t+1}|s_{t+1})\\ &= W_{f,t} + (\mathbb{E}_{f,t}(p_{t+1}) + \bar{d} - p_t)x_{f,t} - \frac{1}{2}a_f x_{f,t}^2(\mathbb{V}_{f,t}(p_{t+1}) + \sigma_d^2)\\ &= W_{f,t} + (p_{t-1} + \gamma_f[\mathbb{E}(f_{t+1}|s_{t+1}) - p_{t-1}] + \bar{d} - p_t)x_{f,t} - \frac{1}{2}a_f x_{f,t}^2(\sigma_f^2[1 + b_f\mathbb{V}(f_{t+1}|s_{t+1})] + \sigma_d^2)\\ &= W_{f,t} + (p_{t-1} + \gamma_f(\bar{f} + \phi(s_{t+1} - \bar{f}) - p_{t-1}) + \bar{d} - p_t)x_{f,t} - \frac{1}{2}a_f x_{f,t}^2(\sigma_f^2(1 + b_f\sigma^2(1-\phi)) + \sigma_d^2).\end{aligned}$$

We applying the F.O.C with respect to ϕ,

$$\frac{\partial\mathbb{CE}_{f,t}(x_{f,t},\phi)}{\partial\phi} = (s_{t+1} - \bar{f})x_{f,t}\gamma_f + \frac{1}{2}a_f x_{f,t}^2\sigma_f^2 b_f\sigma^2 = 0.$$

Hence

$$x_{f,t}((s_{t+1} - \bar{f})\gamma_f + \frac{1}{2}a_f x_{f,t}\sigma_f^2 b_f \sigma^2) = 0$$

Therefore we have two solutions

$$x_{f,t} = 0$$

and

$$x_{f,t} = \frac{2(s_{t+1} - \bar{f})\gamma_f}{a_f b_f \sigma_f^2 b_f \sigma^2} = \hat{x}_{f,t},$$

they two thresholds where the demand of the fundamentalists are independent of the price. The $\mathbb{CE}_{f,t}(p_{t+1})$ is therefore expressed such that it depends on the above two inertia demand.

$$\mathbb{CE}_{f,t}(x) = \begin{cases} \underset{\phi_l,f,t}{\mathbb{E}}(W_{f,t+1}|s_{t+1}) - \frac{1}{2}\underset{\phi_l,f,t}{\mathbb{V}}(W_{f,t+1}|s_{t+1}) & \text{if} \quad x_{f,t} \leq min(\hat{x}_{f,t}, 0) \\ \underset{\phi_u,f,t}{\mathbb{E}}(W_{f,t+1}|s_{t+1}) - \frac{1}{2}\underset{\phi_u,f,t}{\mathbb{V}}(W_{f,t+1}|s_{t+1}) & \text{if} \quad min(\hat{x}_{f,t}, 0) < x \leq max(\hat{x}_{f,t}, 0) \\ \underset{\phi_l,f,t}{\mathbb{E}}(W_{f,t+1}|s_{t+1}) - \frac{1}{2}\underset{\phi_l,f,t}{\mathbb{V}}(W_{f,t+1}|s_{t+1}) & \text{if} \quad x_{f,t} > max(\hat{x}_{f,t}, 0)). \end{cases}$$

C.3 Equilibrium Price

To find the equilibrium price function, we apply market clearing condition

$$\eta_f x_{f,t} + \eta_c x_{c,t} + \eta_n x_{n,t} = \overline{x}.$$

Given the demand of the fundamentalists

$$x_{f,t} = \begin{cases} x_{f,\phi_l,t}; & p_t \leq p_{1,t} \\ max(\hat{x}_{f,t}, 0); & p_{1,t} < p_t \leq p_{2,t} \\ x_{f,\phi_u,t}; & p_{2,t} < p_t \leq p_{3,t} \\ min(\hat{x}_{f,t}, 0); & p_{3,t} < p_t \leq p_{4,t} \\ x_{f,\phi_l,t}; & p_t > p_{4,t}, \end{cases}$$

where

$$x_{f,\phi,t} = \frac{\mathbb{E}_{f,\phi,t}(p_{t+1}|s_{t+1}) + \bar{d} - p_t}{a_f(\mathbb{V}_{f,\phi,t}(p_{t+1}|s_{t+1} + \sigma_d^2)}.$$

While the trend followers' demand is

$$x_{c,t} = \frac{\mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - p_t}{a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}.$$

We substitute the conditional demand of the fundamentalists along with the trend followers into the above market clearing condition, and re-arrange for p_t to find the equilibrium price

$$\eta_f \frac{\mathbb{E}_{f,\phi,t}(p_{t+1}|s_{t+1}) + \bar{d} - p_t}{a_f(\mathbb{V}_{f,\phi,t}(p_{t+1}|s_{t+1} + \sigma_d^2)} + \eta_c \frac{\mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - p_t}{a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)} = \bar{X},$$

where $\overline{x} - \eta_n x_{n,t}$. Hence

$$p_t = \beta_{\phi,t}(\mathbb{E}_{f,\phi,t}(p_{t+1}) + \bar{d}) + (1 - \beta_{\phi_l,t})(\mathbb{E}_{c,t}(p_{t+1}) + \bar{d}) - \Delta_{\phi,t},$$

where

$$\beta_{\phi,t} = \frac{n_f}{n_f + n_c \frac{a_f(\mathbb{V}_{f,\phi,t}(p_{t+1}) + \sigma_d^2)}{a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}} > 0$$

and

$$\Delta_{\phi,t} = \frac{Xa_f a_c(\mathbb{V}_{f,\phi,t}(p_{t+1}) + \sigma_d^2)(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{n_f a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2) + n_c a_f(\mathbb{V}_{f,\phi,t}(p_{t+1}) + \sigma_d^2)} > 0.$$

Since the reliability of the information is estimated, $\phi \in (\phi_l, \phi_u)$, each of the price corresponding to the reliability of signal for ϕ_l and ϕ_u is given respectively by

$$p_{\phi_l,t} = \beta_{\phi_l,t}(\mathbb{E}_{f,\phi_l,t}(p_{t+1}) + \bar{d}) + (1 - \beta_{\phi_l,t})(\mathbb{E}_{c,t}(p_{t+1}) + \bar{d}) - \Delta_{\phi_l,t},$$

$$p_{\phi_u,t} = \beta_{\phi_u,t}(\mathbb{E}_{f,\phi_u,t}(p_{t+1}) + \bar{d}) + (1 - \beta_{\phi_u,t})(\mathbb{E}_{c,t}(p_{t+1}) + \bar{d}) - \Delta_{\phi_u,t}.$$

We know there are two inertia demand that fundamentalists will hold for a range of prices. Similar to above, we substitute the demand function into the market clearing conditions

$$p_{\hat{x},t} = \mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - \frac{(X - n_f \hat{x}_{f,t})a_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{n_c}$$

such that this price corresponds to the case when the fundamentalists demand a fixed position $\hat{x}_{f,t} = \frac{2(s_{t+1} - \bar{f})\gamma_f}{a_f b_f \sigma_f^2 b_f \sigma^2}$ of the risky asset that depends on the size of the signal, as well as on whether the new is good or bad. Finally, when fundamentalists demand zero position of the risky asset, the corresponding equilibrium price becomes

$$p_{o,t} = \mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - \frac{Xa_c(\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{n_c},$$

in which the market is independent of the fundamentalists expectation as they choose to stay out of the market. The market thus only consists trend followers and noise investors to trade each other.

According to the above conditional price, the price function can be summarised,

$$p_t = \begin{cases} p_{\phi_l,t} = \beta_{\phi_l,t}(\mathbb{E}_{f,\phi_l,t}(p_{t+1}) + \bar{d}) + (1-\beta_{\phi_l,t})(\mathbb{E}_{c,t}(p_{t+1}) + \bar{d}) - \Delta_{\phi_l,t} \\ p_{\phi_u,t} = \beta_{\phi_u,t}(\mathbb{E}_{f,\phi_u,t}(p_{t+1}) + \bar{d}) + (1-\beta_{\phi_u,t})(\mathbb{E}_{c,t}(p_{t+1}) + \bar{d}) - \Delta_{\phi_u,t}; \\ p_{\hat{x},t} = \mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - \frac{(X - n_f \hat{x}_{f,t}) a_c (\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{n_c}; \\ p_{o,t} = \mathbb{E}_{c,t}(p_{t+1}) + \bar{d} - \frac{X a_c (\mathbb{V}_{c,t}(p_{t+1}) + \sigma_d^2)}{n_c}. \end{cases}$$

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