# Chapter 6

# Tri-level Multi-follower Decision Making

In a tri-level hierarchical decision problem, each decision entity at one level has its objective, constraints and decision variables affected in part by the decision entities at the other two levels. The choice of values for its variables may allow it to influence the decisions made at other levels, and thereby improve its own objective. We called this a tri-level decision problem. When multiple decision entities are involved at the middle and bottom levels, the top-level entity's decision will be affected not only by these followers' individual reactions but also by the relationships among the followers. We call this problem a *tri-level multi-follower* (TLMF) decision.

In this chapter, we first identify tri-level decision problems from real world cases in Section 6.1.We then introduce basic tri-level decision-making models in Section 6.2. Section 6.3 presents a framework for the TLMF decision through analyzing various kinds of relationships between decision entities in a tri-level decision problem. The TLMF decision framework contains 64 standard TLMF decision-making situations. To model these TLMF decision situations, we extend the bi-level *decision entity-relationship diagram* (DERD) approach introduced in Chapter 4 to describe tri-level decision problems. Furthermore, we establish a set of standard and hybrid TLMF decision models using a mathematical programming approach in Section 6.4. A set of case studies illustrates the development of TLMF decision models by DERD, as well as programming approaches, in Section 6.5. Section 6.6 gives solution concepts for a linear tri-level decision problem. It also presents a set of tri-level programming algorithms including a tri-level *K*th-Best algorithm. Section 6.7 focuses on solution methods for the proposed 64 kinds of TLMF decision model. To discuss this in detail, we take the TLMF decision model *S*12 in its linear version as a representative to illustrate solution concepts and theoretical properties, and to describe a TLMF *K*th-Best algorithm for TLMF decision-making. Finally, Section 6.8 summarizes this chapter.

## 6.1 Problem Identification

Some decision problems require making a compromise between the objectives of several interacting *decision entities*(DE) allocated in a three-level hierarchy. The execution of decisions is sequential, from top to middle and then to bottom levels.

Each decision entity independently optimizes (maximizes or minimizes) its own objective but is affected by the actions of other decision entities at the other two levels. Such a hierarchical decision process appears naturally in many organizations and business systems.

We use a university example here to explain the nature of the problem.

**Example 6.1** A university is organized with three faculties (Information, Business, Science) and each faculty has 2-4 departments. The university aims to improve its research quality through creating new research development strategies in 2013. The strategies made at university level directly affect the research strategy-making in its faculties. This process continues within a hierarchy of decision entities, including its departments and research centers. In the meantime, the actions at the faculty level may affect the research development strategies sought by the university and the actions at department level may affect those of its faculty. Each related decision entity in this university wishes to optimize its individual research development objective in view of the partial control exercised at other levels. The university's decision makers can control this effect by exercising preemptive-partial control over the university through budget modifications or regulations, but subject to possible reactions from its faculties and also departments. This kind of decision problem is called a multi-level decision problem or multi-level optimization problem.

The complexity of decision problems increases significantly when the number of levels ($n$) is greater than two (Blair 1992). The tri-level decision is the most typical form of multi-level decision ($n > 2$). In a tri-level decision, the decision entity at the top level is called the *leader*, while entities at the middle and bottom levels are the *followers*. However, a decision entity at the middle level is also the leader for associate entities at the bottom level. As a tri-level decision reflects the main features of multi-level decision problems, the models and methods developed for tri-level decisions can be easily extended to other multi-level decision problems.

The tri-level decision problem has been studied by researchers such as White (1997), Bard and Falk (1982a), Lai (1996) and Shih et al. (1996). The existing research results are mostly limited to the *one-level one-entity* situation. In real world tri-level decision applications, decisions are often made in situations where several decision entities are at the middle and bottom levels and interact with one another in some way. Consider Example 6.1. As these three faculties may have different objectives and different reactions to each possible decision made by the university, they should be treated as multiple entities at the middle level. These faculties may also have various relationships between each other, such as sharing their decision variables or not, and sharing their constraints or not, which may create different decision situations. As a result, the university's decision will be affected not only by its faculties' individual optimal reactions but also by the relationships between faculties and related departments. Some research, such as Shih et al. (1996), considered tri-level decision problems with multiple followers. However, very few studies classify the possible relationships among these followers and discuss different models to handle different situations.

Another issue related to tri-level decision-making is the relationship between the top-level decision entity and the bottom-level entities. In general, in a tri-level decision problem, the top-level decision entity's solution will be directly affected by the middle-level decision entities but indirectly affected by the bottom-level decision entities. However, in some cases, the solution of the top-level decision entity can be directly affected by the bottom-level entities' reactions as well. Considering Example 6.1, this university leader may also take a department's feedback in strategy making and in such a situation its decisions will be directly affected by its departments' reactions.

A more complex situation occurs when different entities at the same level have different decision situations. Considering Example 6.1, some faculties' departments make decisions (reactions) cooperatively while others do not. For example, all the departments in the Business Faculty react cooperatively to the decisions of the faculty, whereas the departments in the Information Faculty react uncooperatively to decisions made by the Faculty.

In summary, tri-level decisions involve a variety of situations caused by various possible relationships among multiple decision entities at two lower levels. The following sections will first provide basic tri-level decision models and will then model TLMF decision problems in various situations.

## 6.2 Basic Tri-level Decision Models

Basic tri-level decision focuses on a one-level one-entity situation and therefore has only three decision entities: DE1, DE2, and DE3. It can be described as follows (Bard and Falk 1982a):

$$\min_{x \in X} f_1(x, y, z) \text{ (DE1)}$$

$$\text{s.t. } g_1(x, y, z) \leq 0,$$

where $y, z$ solve:

$$\min_{y \in Y} f_2(x, y, z) \text{ (DE2)}$$

$$\text{s.t. } g_2(x, y, z) \leq 0,$$

where$z$ solves:

$$\min_{z \in Z} f_3(x, y, z) \text{ (DE3)}$$

$$\text{s.t. } g_3(x, y, z) \leq 0, \tag{6.1}$$

where $x \in X \subset R^n$, $y \in Y \subset R^m$, $z \in Z \subset R^p$, $f_i: X \times Y \times Z \to R$, $i = 1,2,3$, variables $x$, $y$, $z$ are called the top-level, middle-level and bottom-level variables, and $f_1(x,y,z), f_2(x,y,z)$, $f_3(x,y,z)$ are the top-level, middle-level and bottom-level objective functions respectively.

From the tri-level decision model (6.1), we can see that this decision problem has three optimization sub-problems (objective functions). Each level has individual control variables within its optimization sub-problem, but also considers other levels' variables in its optimization sub-problem. This decision process is sequential: decision entity DE 1, at the top-level, selects an action within its specified constraint set, then DE 2, at the middle-level, responds within its constraint set, and lastly DE 3 responds.

To solve the tri-level decision problem, Bard and Falk (1982a) first developed a cutting plane algorithm and White (1997) developed a penalty function approach. In the meantime, Lai (1996) and Shih et al. (1996) extended the tri-level decision research in two aspects. One is that they developed a fuzzy approach to solve multi-level programming problems. The other is that a TLMF decision model is proposed in which multiple followers are at both middle and bottom levels. Below is a TLMF model. It assumes three sub-problems as centre $f_1 \to$ division $f_{2i} \to$subdivision, $f_{3t}$, $t = 1, 2, \ldots, t_i$, $i = 1,2, \ldots, s$ (Shih et al. 1996):

$$\min_{x_1} f_1(x) = \sum\nolimits_j c_{1j} x_j \qquad \text{(top level)}$$

where$x_{2i}, x_{3i1}, \ldots, x_{3it_i}$solve

$$\min_{x_{2i}} f_{2i}(x) = \sum\nolimits_j c_{2ij} x_j \qquad \text{(middle level)}$$

where$x_{3i1}, \ldots, x_{3it_i}$solve

$$\min_{x_{3i1}} f_{3i1}(x) = \sum\nolimits_j c_{3i1j} x_j \qquad \text{(bottom level)}$$

$$\vdots$$

$$\min_{x_{3it_i}} f_{3it_i}(x) = \sum\nolimits_j c_{3it_ij} x_j$$

$$\text{s.t. } A_1 x_1 + A_{2i} x_{2i} + A_{3i1} x_{3i1} + \cdots + A_{3it_i} x_{3it_i} \leq b,$$

$$x_j \geq 0, j = 1,2, \ldots, n, \qquad (6.2)$$

In this model, there is one decision entity at the top level, $s$ decision entities at the middle level and $t = \sum_i t_i$ decision entities at the bottom level. This is a general TLMF decision model with uncooperative relationships which adopts the decisions of other decision entities as references.

In the following section, we will provide more discussion on the TLMF decision models and solution methods.

## 6.3 Tri-level Multi-follower Decision Framework

This section first identifies seven issues which are related to the TLMF decision classification, and then presents a TLMF decision framework and a DERD modeling approach for TLMF decision situations.

### *6.3.1 TLMF Decision Concepts*

When a tri-level decision problem has multiple followers at the middle level and/or the bottom level, we call it a TLMF decision problem. The model given in (6.1) describes a basic situation of tri-level decision, that is, each level has one decision entity only. Problem (6.2) presents the model for a general TLMF decision problem. In order to identify and classify TLMF decision situations, we first introduce the following concepts:

(1) *Neighborhood entity*: two decision entities are at the same level, led by the same decision entity. All neighborhood entities under the same leader are called a neighborhood entity set (NES).

(2) *Cooperative entity*: two neighborhood entities share their decision variables and have the same objective and constraint functions. In such a case, we consider the two entities as one.

(3) *Semi-cooperative entity*: two neighborhood entities share their decision variables but have distinct objectives and constraint functions.

(4) *Uncooperative entity*: two neighborhood entities have distinct decision variables, objectives, and constraints.

(5) *Reference-uncooperative entity*: two neighborhood entities have distinct decision variables, objectives and constraints but take account of others' variables as references; that is, they include others' variables in their objective/constraint functions, but not as control variables.

(6) *Direct* and *secondary follower*: all decision entities at the middle level are direct followers of the top-level decision entity (similarly, each bottom-level entity is a direct follower of an entity at the middle level); and all entities at the bottom level are secondary followers of the top-level decision entity.

(7) *Direct leader and secondary leader*: a decision entity at the top level is the direct leader of all decision entities at the middle level (similarly, each bottom-level entity has a direct leader at the middle level) and is the secondary leader of all decision entities at the bottom level.

### 6.3.2 TLMF Decision Problem Classification

In a TLMF decision problem, a middle-level decision entity has two roles in decision-making process, that is, it reacts to each possible strategy made by the top-level entity and is influenced by the decisions of the followers at the bottom level. Different relationships between the decision entities at the middle level and bottom level could result in different processes for deriving an optimal solution for the decision entity at the top level. The top level's decision will also sometimes be affected by the reactions of its secondary followers as well as those of its direct followers. We therefore list the following relationships between decision entities for TLMF decision problems:

(1) *Leader-follower relationship*: if an entity is a direct follower of another entity (leader), we say there is a leader-follower relationship or leadership relationship between the two entities.

(2) *Secondary leadership relationship*: if the top-level decision entity directly considers the reactions of an entity at the bottom level, that is, includes a control variable of this bottom-level entity in its objective and/or constraints, we say that this top-level entity and the bottom-level entity's NES have a secondary leadership relationship.

(3) *Uncooperative relationship*: if there are uncooperative entities but no reference-uncooperative entities in a NES, we say there is an uncooperative relationship in this NES.

(4) *Reference-uncooperative relationship*: if there are reference-uncooperative entities in a NES and the rest are uncooperative, we say there is a reference-uncooperative relationship in this NES.

(5) *Cooperative relationship at the middle level*: if all entities in a NES are cooperative, we say there is a cooperative relationship in this NES.

(6) *Semi-cooperative relationship at the middle level*: if there are semi-cooperative entities in a NES and the rest, if any, are cooperative entities, we say there is a *semi-cooperative relationship* in this NES.

(7) *Secondary followership relationship*: if a bottom-level decision entity includes the control variables of the top-level decision entity in its objective and/or constraints, we call the relationship between this bottom-level entity's NES and the top-level entity a secondary followership relationship.

### 6.3.3 TLMF Decision Framework

Based on the above seven relationships defined, a TLMF decision framework is established as shown in Table6.1.The framework also presents a classification for TLMF decision problems. Under the eight features (SL, ML-V, ML-O, ML-R, SF, BL-V, BL-O, and BL-R) given in Table 6.1, "*Y*" means "yes", "*N*" means "no", and blank means ‘not applicable’. A total of 64 standard situations of TLMF decision problems are identified, named *S*1, *S*2,…, and *S*64 (note that some combinations of these features are not applicable). Each situation is described by using these seven relationships. We can describe any complex TLMF decision problem by combining two or more of these standard situations. For example, in a TLMF decision problem, a set of bottom-level entities are in the *S*1 situation and another set of bottom-level entities match the features of *S*2. We describe this problem of the combination of *S*1 and *S*2 as a hybrid situation.

The abbreviations used in Table 6.1 for the features are explained as follows:

(1) SL: secondary leadership relationship;
(2) ML-V: middle-level entities have the same variables;
(3) ML-O: middle-level entities have the same objectives and constraints;
(4) ML-R: middle-level entities include others’ variables as references;
(5) SF: secondary followership relationship;
(6) BL-V: bottom-level entities have the same variables;
(7) BL-O: bottom-level entities have the same objectives and constraints;
(8) BL-R: bottom-level entities include others’ variables as references.

**Table 6.1** TLMF decision framework with 64 standard situations

| Feature Situation | SL | ML-V | ML-O | ML-R | SF | BL-V | BL-O | BL-R | Decision situation description |
|---|---|---|---|---|---|---|---|---|---|
| *S*1 | Y | Y | Y | | Y | Y | Y | | Both middle and bottom levels cooperative; both secondary leadership and followership |
| *S*2 | Y | Y | Y | | Y | Y | N | | Middle-level cooperative, bottom-level semi-cooperative; both secondary leadership and followership |
| *S*3 | Y | Y | Y | | Y | N | | Y | Middle-level cooperative, bottom-level reference-uncooperative; both secondary leadership and followership |
| *S*4 | Y | Y | Y | | Y | N | | N | Middle-level cooperative, bottom-level uncooperative; both secondary leadership and followership |
| *S*5 | Y | Y | Y | | N | Y | Y | | Both middle and bottom levels cooperative; secondary leadership only |
| *S*6 | Y | Y | Y | | N | Y | N | | Middle-level cooperative, bottom-level semi-cooperative; secondary leadership only |
| *S*7 | Y | Y | Y | | N | N | | Y | Middle-level cooperative, bottom-level reference-uncooperative; secondary leadership only |
| *S*8 | Y | Y | Y | | N | N | | N | Middle-level cooperative, bottom-level uncooperative; secondary leadership only |
| *S*9 | Y | Y | N | | Y | Y | Y | | Middle-level semi-cooperative; bottom-level cooperative; both secondary leadership and followership |
| *S*10 | Y | Y | N | | Y | Y | N | | Both middle and bottom levels semi-cooperative; both secondary leadership and followership |
| *S*11 | Y | Y | N | | Y | N | | Y | Middle-level semi-cooperative, bottom-level reference uncooperative; both secondary leadership and followership |
| *S*12 | Y | Y | N | | Y | N | | N | Middle-level semi-cooperative, bottom-level uncooperative; both secondary leadership and followership |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| *S*13 | Y | Y | N | | N | Y | Y | | Middle-level semi-cooperative; bottom-level cooperative; secondary leadership only |
| *S*14 | Y | Y | N | | N | Y | N | | Both middle and bottom levels semi-cooperative; secondary leadership only |
| *S*15 | Y | Y | N | | N | N | | Y | Middle-level semi-cooperative, bottom-level reference-uncooperative; secondary leadership only |
| *S*16 | Y | Y | N | | N | N | | N | Middle-level semi-cooperative, bottom-level uncooperative; secondary leadership only |
| *S*17 | Y | N | | Y | Y | Y | Y | | Middle-level reference-uncooperative, bottom-level cooperative; both secondary leadership and followership |
| *S*18 | Y | N | | Y | Y | Y | N | | Middle-level reference-uncooperative, bottom-level semi-cooperative; both secondary leadership and followership |
| *S*19 | Y | N | | Y | Y | N | | Y | Both middle and bottom levels reference-uncooperative; both secondary leadership and followership |
| *S*20 | Y | N | | Y | Y | N | | N | Middle-level reference-uncooperative, bottom-level uncooperative; both secondary leadership and followership |
| *S*21 | Y | N | | Y | N | Y | Y | | Middle-level reference-uncooperative, bottom-level cooperative; secondary leadership only |
| *S*22 | Y | N | | Y | N | Y | N | | Middle-level reference-uncooperative, bottom-level semi-cooperative; secondary leadership only |
| *S*23 | Y | N | | Y | N | N | | Y | Both middle and bottom levels reference-uncooperative; secondary leadership only |
| *S*24 | Y | N | | Y | N | N | | N | Middle-level reference-uncooperative, bottom-level uncooperative; secondary leadership only |
| *S*25 | Y | N | | N | Y | Y | Y | | Middle-level uncooperative; bottom-level cooperative; both secondary leadership and followership |
| *S*26 | Y | N | | N | Y | Y | N | | Middle-level uncooperative; bottom-level semi-cooperative; both |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | secondary leadership and followership |
| *S*27 | Y | N | | N | Y | N | | Y | Middle-level uncooperative, bottom-level reference-uncooperative; both secondary leadership and followership |
| *S*28 | Y | N | | N | Y | N | | N | Both middle and bottom levels uncooperative; both secondary leadership and followership |
| *S*29 | Y | N | | N | N | Y | Y | | Middle-level uncooperative; bottom-level cooperative; secondary leadership only |
| *S*30 | Y | N | | N | N | Y | N | | Middle-level uncooperative, bottom-level semi-cooperative, secondary leadership only |
| *S*31 | Y | N | | N | N | N | | Y | Middle-level uncooperative, bottom-level reference-uncooperative; secondary leadership only |
| *S*32 | Y | N | | N | N | N | | N | Both middle and bottom levels uncooperative; secondary leadership only |
| *S*33 | N | Y | Y | | Y | Y | Y | | Both middle and bottom levels cooperative; secondary followership only |
| *S*34 | N | Y | Y | | Y | Y | N | | Middle-level cooperative, bottom-level semi-cooperative; secondary followership only |
| *S*35 | N | Y | Y | | Y | N | | Y | Middle-level cooperative, bottom-level reference-uncooperative; secondary followership only |
| *S*36 | N | Y | Y | | Y | N | | N | Middle-level cooperative, bottom-level uncooperative; secondary followership only |
| *S*37 | N | Y | Y | | N | Y | Y | | Both middle and bottom levels cooperative; no secondary relationships |
| *S*38 | N | Y | Y | | N | Y | N | | Middle-level cooperative, bottom-level semi-cooperative; no secondary relationships |
| *S*39 | N | Y | Y | | N | N | | Y | Middle-level cooperative, bottom-level reference-uncooperative; no secondary relationships |
| *S*40 | N | Y | Y | | N | N | | N | Middle-level cooperative, bottom-level uncooperative; no secondary |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | relationships |
| *S*41 | N | Y | N | | Y | Y | Y | | Middle-level semi-cooperative; bottom-level cooperative; secondary followership only |
| *S*42 | N | Y | N | | Y | Y | N | | Both middle and bottom levels semi-cooperative; secondary followership only |
| *S*43 | N | Y | N | | Y | N | | Y | Middle-level semi-cooperative, bottom-level reference-uncooperative; secondary followership only |
| *S*44 | N | Y | N | | Y | N | | N | Middle-level semi-cooperative, bottom-level uncooperative; secondary followership only |
| *S*45 | N | Y | N | | N | Y | Y | | Middle-level semi-cooperative; bottom-level cooperative; no secondary relationships |
| *S*46 | N | Y | N | | N | Y | N | | Both middle and bottom levels semi-cooperative, no secondary relationships |
| *S*47 | N | Y | N | | N | N | | Y | Middle-level semi-cooperative, bottom-level reference-uncooperative; no secondary relationships |
| *S*48 | N | Y | N | | N | N | | N | Middle-level semi-cooperative, bottom-level uncooperative; no secondary relationships |
| *S*49 | N | N | | Y | Y | Y | Y | | Middle-level reference-uncooperative, bottom-level cooperative; secondary followership only |
| *S*50 | N | N | | Y | Y | Y | N | | Middle-level reference-uncooperative, bottom-level semi-cooperative; secondary followership only |
| *S*51 | N | N | | Y | Y | N | | Y | Both middle and bottom levels reference-uncooperative; secondary followership only |
| *S*52 | N | N | | Y | Y | N | | N | Middle-level reference-uncooperative, bottom-level uncooperative; secondary followership only |
| *S*53 | N | N | | Y | N | Y | Y | | Middle-level reference-uncooperative, bottom-level cooperative; no secondary relationships |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| *S*54 | N | N | | Y | N | Y | N | | Middle-level reference-uncooperative, bottom-level semi-cooperative; no secondary relationships |
| *S*55 | N | N | | Y | N | N | | Y | Both middle and bottom levels reference-uncooperative; no secondary relationships |
| *S*56 | N | N | | Y | N | N | | N | Middle-level reference-uncooperative, bottom-level uncooperative; no secondary relationships |
| *S*57 | N | N | | N | Y | Y | Y | | Middle-level uncooperative; bottom-level cooperative; secondary followership only |
| *S*58 | N | N | | N | Y | Y | N | | Middle-level uncooperative; bottom-level semi-cooperative; secondary followership only |
| *S*59 | N | N | | N | Y | N | | Y | Middle-level uncooperative, bottom-level reference-uncooperative; secondary followership only |
| *S*60 | N | N | | N | Y | N | | N | Both middle and bottom levels uncooperative; secondary followership only |
| *S*61 | N | N | | N | N | Y | Y | | Middle-level uncooperative; bottom-level cooperative; no secondary relationships |
| *S*62 | N | N | | N | N | Y | N | | Middle-level uncooperative, bottom-level semi-cooperative, no secondary relationships |
| *S*63 | N | N | | N | N | N | | Y | Middle-level uncooperative, bottom-level reference-uncooperative; no secondary relationships |
| *S*64 | N | N | | N | N | N | | N | Both middle and bottom levels uncooperative; no secondary relationships |

### *6.3.4 TLMF Decision Entity-Relationship Diagrams*

We have identified seven decision-entity relationships: a normal leader-follower relationship and six implicit relationships. These seven relationships are capable of fully reflecting the features of the TLMF decision problems identified in Table 6.1 and any of their combinations. Based on this, we introduce a TLMF *Decision Entity-Relationship Diagrams* (TLMF-DERD) approach and use it in TLMF modeling. Figure 6.1 presents diagrammatic notations of the TLMF-DERD approach.

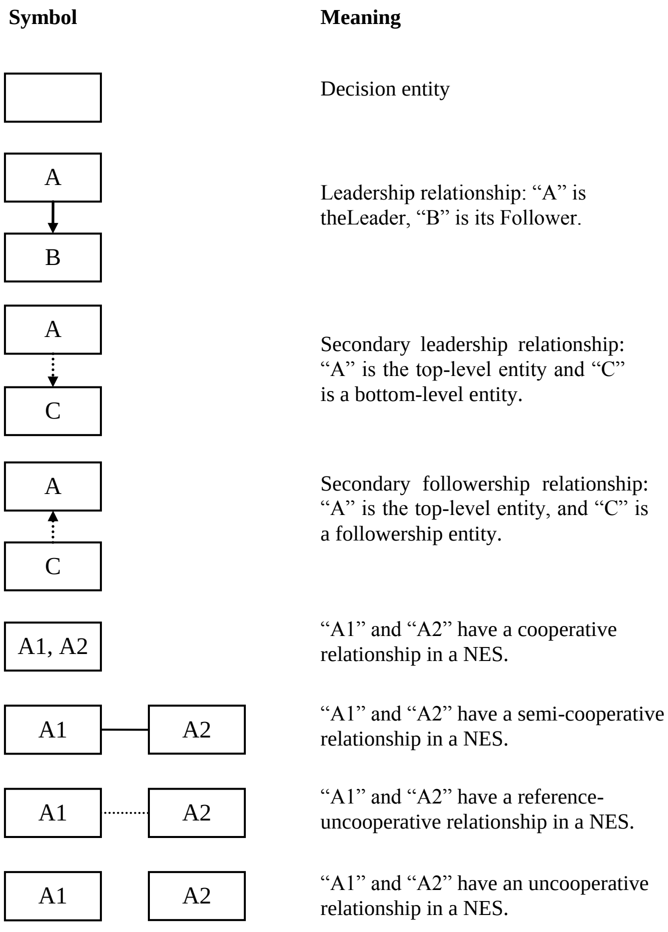


**Figure6.1** Notations for TLMF decision entity-relationship diagrams

This TLMF-DERD approach is a concept modeling of TLMF decision problems. In the following sections, we will show how a TLMF decision problem is first described by the DERD approach and then presented in a tri-level programming model.

## 6.4 Tri-level Multi-follower Decision Models

This section first describes a general TLMF decision model using multi-level programming. It then presents a set of specific models for some standard TLMF decision problems including *S*9, *S*12, *S*15, *S*18, *S*20, *S*25 and *S*32 selected from Table 6.1. We also give a hybrid TLMF decision model for a decision situation which is the combination of *S*63 and *S*64.

### *6.4.1 General Model for TLMF Decision*

A general TLMF decision model, which covers all the 64 TLMF decision situations, is given as follows:

$$\min_{x\in X} f^{(1)}(x, y_1, \dots, y_n, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n})$$

$$\text{s.t. } g^{(1)}(x, y_1, \dots, y_n, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) \le 0,$$

where $y_i, z_{i1}, \dots, z_{im_i} (i = 1, \dots, n)$, solve the *i*th middle-level follower's and its bottom-level followers' problems :

$$\min_{y_i\in Y_i} f_i^{(2)}\left(x, y_1, \dots, y_i, \dots, y_n, z_{i1}, \dots, z_{im_i}\right)$$

$$\text{s.t. } g_i^{(2)}(x, y_1, \dots, y_i, \dots y_n, z_{i1}, \dots, z_{im_i}) \le 0,$$

where $z_{ij}$ $(j = 1, \dots, m_i)$ solves the *i*th middle-level follower's *j*th bottom-level follower's problem:

$$\min_{z_{ij}\in Z_{ij}} f_{ij}^{(3)}(x, y_i, z_{i1}, \dots, z_{ij}, \dots, z_{im_i})$$

$$\text{s.t. } g_{ij}^{(3)}(x, y_i, z_{i1}, \dots, z_{im_i}) \le 0,$$

$$i = 1, \dots, n, j = 1, \dots, m_i, \tag{6.3}$$

where $x \in X \subset R^{l_1}, y_i \in Y_i \subset R^{l_{2i}}, z_{ij} \in Z_{ij} \subset R^{l_{3ij}}, f^{(1)}: X \times \prod_{i=1}^{n} Y_i \times \prod_{i=1}^{n} \prod_{j=1}^{m_i} Z_{ij} \to R, f_i^{(2)}: X \times \prod_{i=1}^{n} Y_i \times \prod_{j=1}^{m_i} Z_{ij} \to R, f_{ij}^{(3)}: X \times Y_i \times \prod_{j=1}^{m_i} Z_{ij}$ , $i = 1, \dots, n, j = 1, \dots, m_i$.

In this model, there is one top decision entity $f^{(1)}$ and $n$ middle decision entities with objectives $f_1^{(2)}, \dots, f_n^{(2)}$. For the *i*th middle decision problem, there are $m_i$ sub-problems $f_{i1}^{(3)}, \dots, f_{im_i}^{(3)}$ to optimize. Based on this model, we can establish models, also supported by DERD for all the 64 standard TLMF decision situations presented in Table6.1.

### *6.4.2 Typical Standard Models for TLMF Decision*

This section will present seven typical TLMF decision models from the 64 models proposed in Section 6.3.3 by using both DERD and tri-level programming approaches.

**(1) *S*9 Model**

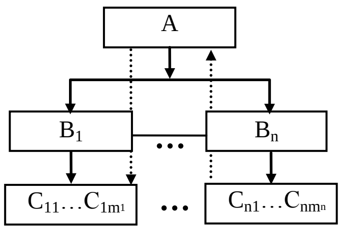


**Figure 6.2** The DERD of TLMF decision situation *S9*

This model presents a TLMF decision problem which has the following features and is described by DERD in Figure 6.2:

1) The top level entity takes the control variables of the decision entities at both middle and bottom levels into consideration in its objectives, that is, there is a secondary leadership relationship;
2) The middle-level decision entities have the same variables;
3) The middle-level decision entities have individual objective functions and constraints, that is, they have a semi-cooperative relationship;
4) The bottom-level decision entities include the control variables of the top-level entity, that is, there is a secondary followership relationship;
5) The bottom-level decision entities have the same variables;
6) The bottom-level decision entities have the same objective functions and constraints, that is, they have a cooperative relationship.

We describe the *S*9 model by the tri-level programming approach as follows:

$$\min_{x \in X} f^{(1)}(x, y, z_1, \ldots, z_n)$$
$$\text{s.t. } g^{(1)}(x, y, z_1, \ldots, z_n) \leq 0,$$

where $y, z_i (i = 1, \ldots, n)$ solve the *i*th middle-level follower's problem and its bottom-level followers' problems :

$$\min_{y \in Y_i} f_i^{(2)}(x, y, z_i)$$
$$\text{s.t. } g_i^{(2)}(x, y, z_i) \le 0,$$

where $z_i$ $(i = 1, \dots, n)$ solves the $i$th middle-level follower's bottom-level follower's problem:

$$\min_{z_i \in Z_i} f_i^{(3)}(x, y, z_i)$$
$$\text{s.t. } g_i^{(3)}(x, y, z_i) \le 0, \tag{6.4}$$

where $x \in X \subset R^{l_1},\ y \in Y_i \subset R^{l_2},\ z_i \in Z_i \subset R^{l_{3i}}, Y = Y_1 \cap \cdots \cap Y_n, f^{(1)}: X \times Y \times \prod_{i=1}^{n} Z_i \to R, f_i^{(2)}: X \times Y_i \times Z_i \to R, f_i^{(3)}: X \times Y_i \times Z_i \to R, i = 1, \dots, n.$

In this model, there is one top-level decision entity $f^{(1)}$ and $n$ middle-level decision entities with objectives $f_1^{(2)}, \dots, f_n^{(2)}$ respectively. Since these middle-level entities have a semi-cooperative relationship, we describe all middle-level followers as sharing a decision variable $y \in Y_i$ and having individual objective functions $f_i^{(2)}$ and the individual constraints $g_i^{(2)} \le 0$. For any middle-level decision problem $f_i^{(2)}$, there are $m_i$ sub-problems $f_{i1}^{(3)}, \dots, f_{im_i}^{(3)}$ at the bottom level. As all bottom-level neighborhood decision entities attached to the $i$th middle-level follower share variables, objective functions and constraints, that is, they are in a cooperative relationship. We describe this feature as the shared variable $z_i \in Z_i$ and $f_{i1}^{(3)} = \cdots = f_{im_i}^{(3)} = f_i^{(3)}$, $g_{i1}^{(3)} = \cdots = g_{im_i}^{(3)} = g_i^{(3)}$. To describe the secondary leadership relationship, we have $z_1, \dots, z_n$ in the objective functions and constraints of the top-level decision entity.

**(2) *S*12 Model**

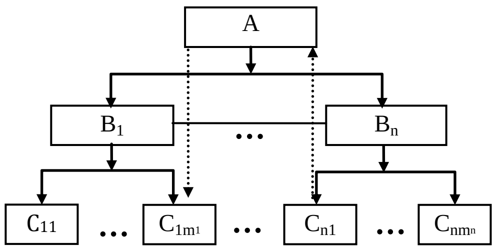


**Figure 6.3** The DERD of TLMF decision situation S12

This model presents a TLMF decision problem which has the following features and is described by DERD in Figure 6.3:

1) There is a secondary leadership relationship;
2) The decision entities at the middle level have the same variables;

3) The middle-level decision entities have a semi-cooperative relationship;
4) There is a secondary followership relationship;
5) The bottom-level decision entities have individual variables;
6) The bottom-level decision entities have an uncooperative relationship.

We describe the *S12* model by the tri-level programming approach as follows:

$$\min_{x \in X} f^{(1)}(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n})$$
$$\text{s.t. } g^{(1)}(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) \leq 0,$$

where $y, z_{i1}, \dots, z_{im_i} (i = 1, \dots, n)$ solve the *i*th middle-level follower's problem and its bottom-level followers' problems:

$$\min_{y \in Y_i} f_i^{(2)}(x, y, z_{i1}, \cdots, z_{im_i})$$
$$\text{s.t. } g_i^{(2)}(x, y, z_{i1}, \cdots, z_{im_i}) \leq 0,$$

where $z_{ij}$ $(j = 1, \cdots, m_i)$ solves the *i*th middle-level follower's *j*th bottom-level follower's problem:

$$\min_{z_{ij} \in Z_{ij}} f_{ij}^{(3)}(x, y, z_{ij})$$
$$\text{s.t. } g_{ij}^{(3)}(x, y, z_{ij}) \leq 0, \tag{6.5}$$

where $x \in X \subset R^{l_1},\ y \in Y_i \subset R^{l_2},\ z_{ij} \in Z_{ij} \subset R^{l_{3ij}}, Y = Y_1 \cap \cdots \cap Y_n, f^{(1)}: X \times Y \times \prod_{i=1}^{n} \prod_{j=1}^{m_i} Z_{ij} \to R, f_i^{(2)}: X \times Y_i \times \prod_{j=1}^{m_i} Z_{ij} \to R,\ f_{ij}^{(3)}: X \times Y_i \times Z_{ij} \to R, i = 1, \dots, n, j = 1, \dots, m_i$.

In this model, for the *i*th middle-level decision problem, there are $m_i$ sub-problems $f_{i1}^{(3)}, \dots, f_{im_i}^{(3)}$ at the bottom level. As the bottom-level decision entities are uncooperative, that is, they have the individual decision variables $z_{ij} \in Z_{ij}$, objective $f_{ij}^{(3)}$ and constraint $g_{ij}^{(3)}$ for $i = 1, \dots, n, j = 1, \dots, m_i$.

**(3) *S*15 Model**

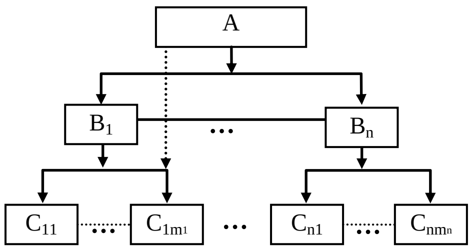


**Figure 6.4** The DERD of TLMF decision situation S15

This model presents a TLMF decision problem which has the following features and is described by DERD in Figure 6.4:

1) There is a secondary leadership relationship;
2) The middle-level decision entities have the same variables;
3) The middle-level decision entities have a semi-cooperative relationship;
4) There is no secondary followership relationship;
5) The bottom-level decision entities have individual variables;
6) The bottom-level decision entities are reference-uncooperative.

We describe the $S15$ model by the tri-level programming approach as follows:

$$\min_{x \in X} f^{(1)}(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n})$$
$$\text{s.t. } g^{(1)}(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) \leq 0,$$

where$y, z_{i1}, \dots, z_{im_i}$ $(i = 1, \dots, n)$ solve the $i$th middle-level follower's problem and its bottom-level followers' problems:

$$\min_{y \in Y_i} f_i^{(2)}(x, y, z_{i1}, \dots, z_{im_i})$$
$$\text{s.t. } g_i^{(2)}(x, y, z_{i1}, \dots, z_{im_i}) \leq 0,$$

where $z_{ij}$ $(j = 1, \dots, m_i)$ solves the $i$th middle-level follower's $j$th bottom-level follower's problem:

$$\min_{z_{ij} \in Z_{ij}} f_{ij}^{(3)}(y, z_{i1}, \dots, z_{im_i})$$
$$\text{s.t. } g_{ij}^{(3)}(y, z_{i1}, \dots, z_{im_i}) \leq 0, \tag{6.6}$$

where $x \in X \subset R^{l_1}$, $y \in Y_i \subset R^{l_2}$, $z_{ij} \in Z_{ij} \subset R^{l_{3ij}}$, $Y = Y_1 \cap \cdots \cap Y_n$, $f^{(1)}: X \times Y \times \prod_{i=1}^{n} \prod_{j=1}^{m_i} Z_{ij} \to R, f_i^{(2)}: X \times Y_i \times \prod_{j=1}^{m_i} Z_{ij} \to R$, $f_{ij}^{(3)}: Y_i \times Z_{ij} \to R, i = 1, \dots, n, j = 1, \dots, m_i$.

In this model, the bottom level has no secondary followership relationship to the top level entity; there are only $y, z_{ij}$ as variables in the objectives $f_{ij}^{(3)}$ and constraints $g_{ij}^{(3)}$ of the bottom level. As the bottom-level decision entities attached to the $i$th middle-level follower are reference-uncooperative, we have $z_{i1}, \dots, z_{im_i}$ in all objective functions $f_{i1}^{(3)}, \dots, f_{im_i}^{(3)}$ and constraints $g_{i1}^{(3)}, \dots, g_{im_i}^{(3)}$ of the bottom level for $i = 1, \dots, n$.

**(4) *S*18 Model**

This model will present a TLMF decision problem which has the following features and is described by DERD in Figure 6.5:

1) There is a secondary leadership relationship;

2) The middle-level decision entities have individual variables;
3) The middle-level entities have a reference-uncooperative relationship;
4) There is a secondary followership relationship;
5) The bottom-level decision entities have the same variables;
6) The bottom-level entities have a semi-cooperative relationship.

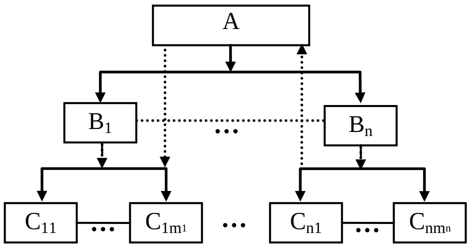


**Figure 6.5** The DERD of TLMF decision situation S18

We describe the $S18$ model by the tri-level programming approach as follows:

$$\min_{x \in X} f^{(1)}(x, y_1, \dots, y_n, z_1, \dots, z_n)$$

$$\text{s.t. } g^{(1)}(x, y_1, \dots, y_n, z_1, \dots, z_n) \leq 0,$$

where $y_i, z_i (i = 1, \dots, n)$ solve the $i$th middle-level follower's problem and its bottom-level followers' problems:

$$\min_{y_i \in Y_i} f_i^{(2)}(x, y_1, \dots, y_n, z_i)$$

$$\text{s.t. } g_i^{(2)}(x, y_1, \dots, y_n, z_i) \leq 0,$$

where $z_i$ solves the $i$th middle-level follower's $j$th$(j = 1,2, \dots, m_i)$ bottom-level follower's problem:

$$\min_{z_i \in Z_{ij}} f_{ij}^{(3)}(x, y_i, z_i)$$

$$\text{s.t. } g_{ij}^{(3)}(x, y_i, z_i) \leq 0, \tag{6.7}$$

where $x \in X \subset R^{l_1}$, $y_i \in Y_i \subset R^{l_{2i}}$, $z_i \in Z_{ij} \subset R^{l_{3i}}$, $Z_i = Z_{i1} \cap \cdots \cap Z_{im_i}$, $f^{(1)}: X \times \prod_{i=1}^{n} Y_i \times \prod_{i=1}^{n} Z_i \to R$, $f_i^{(2)}: X \times Y_i \times Z_i \to R$, $f_{ij}^{(3)}: X \times Y_i \times Z_{ij} \to R$, $i = 1, \dots, n, j = 1, \dots, m_i$.

In this model, the middle-level decision entities have a reference-uncooperative relationship so we have $y_1, \dots, y_n$ in all objective functions $f_i^{(2)}$ and constraints $g_i^{(2)}$ of the middle level. As all decision entities at the bottom level have a semi-cooperative relationship, we have the shared variable $z_i \in Z_i$ for the $i$th middle-level follower's NES, $j = 1, \dots, m_i, i = 1, \dots, n$.

**(5) *S*20 Model**

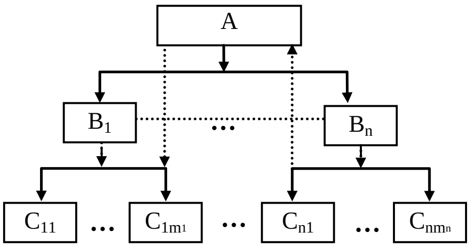


**Figure 6.6** The DERD of TLMF decision situation S20

This model will present a TLMF decision problem which has the following features and is described by DERD in Figure 6.6:

1) There is a secondary leadership relationship;
2) The decision entities at the middle level have individual variables;
3) The middle-level entities have a reference-uncooperative relationship;
4) There is a secondary followership relationship;
5) The bottom-level entities have individual variables;
6) The bottom-level entities have an uncooperative relationship.

We describe the *S*20 model by the tri-level programming approach as follows:

$$\min_{x\in X} f^{(1)}(x, y_1, \dots, y_n, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n})$$
$$\text{s.t. } g^{(1)}(x, y_1, \dots, y_n, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) \leq 0,$$

where $y_i, z_{i1}, \dots, z_{im_i}$ $(i = 1, \dots, n)$ solve the *i*th middle-level follower's problem and its bottom-level followers' problems:

$$\min_{y_i\in Y_i} f_i^{(2)}(x, y_1, \dots, y_n, z_{i1}, \dots, z_{im_i})$$
$$\text{s.t. } g_i^{(2)}(x, y_1, \dots, y_n, z_{i1}, \dots, z_{im_i}) \leq 0,$$

where $z_{ij}$ $(j = 1, \dots, m_i)$ solves the *i*th middle-level follower's *j*th bottom-level follower's problem:

$$\min_{z_{ij}\in Z_{ij}} f_{ij}^{(3)}(x, y_i, z_{ij})$$
$$\text{s.t. } g_{ij}^{(3)}(x, y_i, z_{ij}) \leq 0, \tag{6.8}$$

where $x \in X \subset R^{l_1}$, $y_i \in Y_i \subset R^{l_{2i}}$, $z_{ij} \in Z_{ij} \subset R^{l_{3ij}}$, $f^{(1)}: X \times \prod_{i=1}^{n} Y_i \times \prod_{i=1}^{n}\prod_{j=1}^{m_i} Z_{ij} \to R, f_i^{(2)}: X \times \prod_{i=1}^{n} Y_i \times \prod_{j=1}^{m_i} Z_{ij} \to R$, $f_{ij}^{(3)}: X \times Y_i \times Z_{ij} \to R, i = 1, \dots, n, j = 1, \dots, m_i$.

In this model, as all decision entities at the middle level have a reference-uncooperative relationship, we have $y_1, \dots, y_n$ in the objective $f_i^{(2)}$ and constraint

$g_i^{(2)}$ for $i = 1, \dots, n$. While the bottom-level followers attached to the same middle-level follower have an uncooperative relationship, each bottom-level entity's objective function $f_{ij}^{(3)}$ and constraint $g_{ij}^{(3)}$ have no other counterparts' variables for $j = 1, \dots, m_i, i = 1, \dots, n$.

**(6) *S*25 Model**

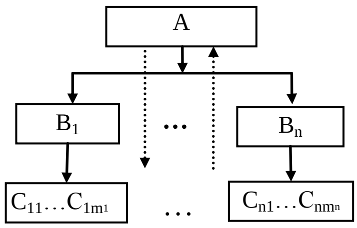


**Figure 6.7** The DERD of TLMF decision situation S25

This model will present a TLMF decision problem which has the following features and is described by DERD in Figure 6.7:

1) There is a secondary leadership relationship;
2) The middle level entities have individual variables;
3) The middle-level entities have an uncooperative relationship;
4) There is a secondary followership relationship;
5) The bottom-level decision entities have the same variables;
6) The bottom-level neighborhood decision entities have a cooperative relationship.

We describe the *S25* model by the tri-level programming approach as follows:

$$\min_{x \in X} f^{(1)}(x, y_1, \dots, y_n, z_1, \dots, z_n)$$

$$\text{s.t. } g^{(1)}(x, y_1, \dots, y_n, z_1, \dots, z_n) \leq 0,$$

where$y_i, z_i (i = 1, \dots, n)$ solve the *i*th middle-level follower's problem and its bottom-level followers' problems:

$$\min_{y_i \in Y_i} f_i^{(2)}(x, y_i, z_i)$$

$$\text{s.t. } g_i^{(2)}(x, y_i, z_i) \leq 0,$$

where$z_i$ solves the *i*th middle-level follower's bottom-level follower's problem:

$$\min_{z_i \in Z_i} f_i^{(3)}(x, y_i, z_i)$$

$$\text{s.t. } g_i^{(3)}(x, y_i, z_i) \leq 0, \tag{6.9}$$

where $x \in X \subset R^{l_1}$, $y_i \in Y_i \subset R^{l_{2i}}$, $z_i \in Z_i \subset R^{l_{3ij}}$, $f^{(1)}: X \times \prod_{i=1}^{n} Y_i \times \prod_{i=1}^{n} Z_i \to R, f_i^{(2)}: X \times Y_i \times Z_i \to R, f_{ij}^{(3)}: X \times Y_i \times Z_i \to R, i = 1, \dots, n, j = 1, \dots, m_i$.

In this model, as the middle-level decision entities have an uncooperative relationship, each middle-level entity objective function $f_i^{(2)}$ and constraint $g_i^{(2)}$ have no other counterparts' variables for $i = 1, \dots, n$.

**(7) *S*32 Model**

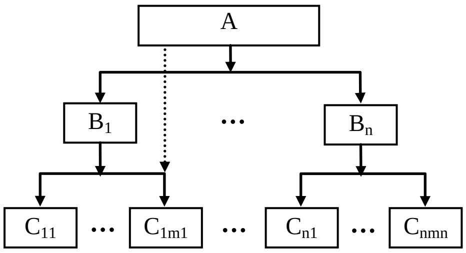


**Figure 6.8** The DERD of TLMF decision situation S32

This model will present a TLMF decision problem which has the following features and is described by DERD in Figure 6.8:

1) There is a secondary leadership relationship;
2) The middle-level decision entities have individual variables;
3) The middle-level decision entities have an uncooperative relationship;
4) There is no secondary followership relationship;
5) The bottom-level decision entities have individual variables;
6) The bottom-level decision entities have an uncooperative relationship.

We describe the *S*32 model by the tri-level programming approach as follows:

$$\min_{xx \in X} f^{(1)}(x, y_1, \dots, y_n, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n})$$

$$\text{s.t. } g^{(1)}(x, y_1, \dots, y_n, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) \le 0,$$

where $y_i, z_{i1}, \dots, z_{im_i} (i = 1, \dots, n)$ solve the *i*th middle-level follower's problem and its bottom-level followers' problems :

$$\min_{y_i \in Y_i} f_i^{(2)}(x, y_i, z_{i1}, \dots, z_{im_i})$$

$$\text{s.t. } g_i^{(2)}(x, y_i, z_{i1}, \dots, z_{im_i}) \le 0,$$

where $z_{ij}$ $(j = 1, \dots, m_i)$ solves the *i*th middle-level follower's *j*th bottom-level follower's problem:

$$\min_{z_{ij} \in Z_{ij}} f_{ij}^{(3)}(y_i, z_{ij})$$

$$\text{s.t. } g_{ij}^{(3)}(y_i, z_{ij}) \le 0, \tag{6.10}$$

where $x \in X \subset R^{l_1}, y_i \in Y_i \subset R^{l_{2i}}, z_{ij} \in Z_{ij} \subset R^{l_{3ij}}, f^{(1)}: X \times \prod_{i=1}^{n} Y_i \times$

$$\prod_{i=1}^{n} \prod_{j=1}^{m_i} Z_{ij} \to R, f_i^{(2)}: X \times Y_i \times \prod_{j=1}^{m_i} Z_{ij} \to R, f_{ij}^{(3)}: Y_i \times Z_{ij} \to R, i = 1, \ldots, n,$$

$j = 1, \ldots, m_i$.

In this model, $z_{ij}$ are included in the objective functions and constraints of the top-level decision entity to describe the secondary leadership relationship. As there is no secondary followership, however, the top-level variable $x$ is not included in the objectives $f_{ij}^{(3)}(y_i, z_{ij})$ and constraints $g_{ij}^{(3)}(y_i, z_{ij})$ of the bottom level decision problem. The decision entities at both middle and bottom level are uncooperative, so each entity's objective and constraints have only its variables, that is, $f_i^{(2)}$ and $g_i^{(2)}$ have only $y_i$, $f_{ij}^{(3)}$ and $g_{ij}^{(3)}$ have only $z_{ij}$, not other variables of the same level entities.

### *6.4.3 Hybrid TLMF Decision Models*

Note that each of the 64 standard situations listed in Table 6.1 supposes that all entities at the same level have the same situations. For example, all the departments in all faculties of the university are uncooperative. However, in some real-world applications, the departments in the Faculty of Science are cooperative, and the departments in the Faculty of Business are uncooperative. We call this a hybrid TLMF decision problem and will describe it by a hybrid TLMF decision model. As an example of such hybrid problems, we present a TLMF decision problem in this section, which is described by DERD in Figure 6.9:

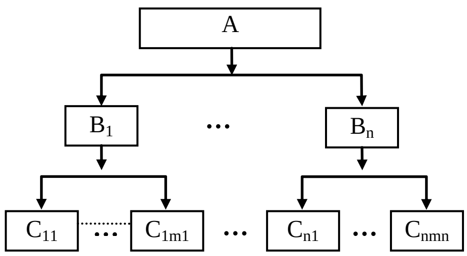


**Figure 6.9** The DERD of a hybrid TLMF decision situation

1) The top-level decision entity is not in a secondary leadership relationship;
2) The middle-level decision entities have individual variables;
3) The middle-level decision entities are uncooperative;
4) There is no secondary followership;
5) The first NES at the bottom level are reference-uncooperative;

6) The rest of the NES at the bottom level have an uncooperative relationship.

This problem is described by a hybrid model combining *S*63 and *S*64 as follows:

$$\min_{x\in X} f^{(1)}(x, y_1, \dots, y_n)$$
$$\text{s.t. } g^{(1)}(x, y_1, \dots, y_n) \leq 0,$$

where $y_i, z_{i1}, \dots, z_{im_i} (i = 1, \dots, n)$ solve the *i*th middle-level follower's problem and its bottom-level followers' problems:

$$\min_{y_i\in Y_i} f_i^{(2)}(x, y_i, z_{i1}, \dots, z_{im_i})$$
$$\text{s.t. } g_i^{(2)}(x, y_i, z_{i1}, \dots, z_{im_i}) \leq 0,$$

where $z_{1j}\,(j = 1, \dots, m_1)$ solves the first middle-level follower's *j*th bottom-level follower's problem in a reference-uncooperative situation:

$$\min_{z_{1j}\in Z_{1j}} f_{1j}^{(3)}(y_1, z_{11}, \dots, z_{1m_1})$$
$$\text{s.t. } g_{1j}^{(3)}(y_1, z_{11}, \dots, z_{1m_1}) \leq 0,$$

where $z_{ij}\,(i \neq 1, j = 1, \dots, m_i)$ solves the *i*th middle-level follower's *j*th bottom-level follower's problem in an uncooperative situation:

$$\min_{z_{ij}\in Z_{ij}} f_{ij}^{(3)}(y_i, z_{ij})$$
$$\text{s.t. } g_{ij}^{(3)}(y_i, z_{ij}) \leq 0, \tag{6.11}$$

where $x \in X \subset R^{l_1}$, $y_i \in Y_i \subset R^{l_{2i}}$, $z_{ij} \in Z_{ij} \subset R^{l_{3ij}}$, $f^{(1)}: X \times \prod_{i=1}^{n} Y_i \to R$, $f_i^{(2)}: X \times Y_i \times \prod_{j=1}^{m_i} Z_{ij} \to R$, $f_{1j}^{(3)}: Y_1 \times \prod_{j=1}^{m_1} Z_{1j} \to R$, $f_{ij}^{(3)}: Y_i \times Z_{ij} \to R$, $i = 1, \dots, n, j = 1, \dots, m_i$.

In this model, as there is no secondary leadership, $z_{ij}$ are not variables in the objective function and constraints of the top-level decision entity. Similarly, as there is no secondary followership, $x$ is not in the objectives and constraints of the bottom level decision entity. The decision entities at the middle level have an uncooperative relationship, so the$i$th entity has only its variables $y_i$ in $f_i^{(2)}$ and constraints $g_i^{(2)}$. The bottom-level entities have two kinds of relationship: the first NES is reference-cooperative (refer to *S*63) and the others are uncooperative (refer to *S*64). Therefore, we have $f_{1j}^{(3)}(y_1, z_{11}, z_{12}, \dots, z_{1m_1})$ for the first NES and $f_{ij}^{(3)}(y_i, z_{ij})(i = 2, 3, \dots, n)$ for other NESs at the bottom level.

From the above analysis and discussions, using the standard and hybrid TLMF decision models, we can easily give the rest of the TLMF decision models according to the situations described in Table 6.1, as well as their hybrid models, based on the features of a decision problem.

## 6.5 Case Studies for TLMF Decision Modeling

In this section, we consider four tri-level multi-follower decision cases concerning research development strategy-making within a university to illustrate both DERD and programming approaches for TLMF decision modeling.

### *6.5.1 Case 1: S28 Model*

Assume that the university's research strategy involves the university, its three faculties and departments. All three faculties have individual objectives, constraints, variables and do not take each other into consideration. The departments within each faculty are also uncooperative. The university takes the responses of both faculties and departments into account. At the same time, the faculties and departments fully consider the research strategies of the university. This TLMF decision problem is described in Figure 6.10.

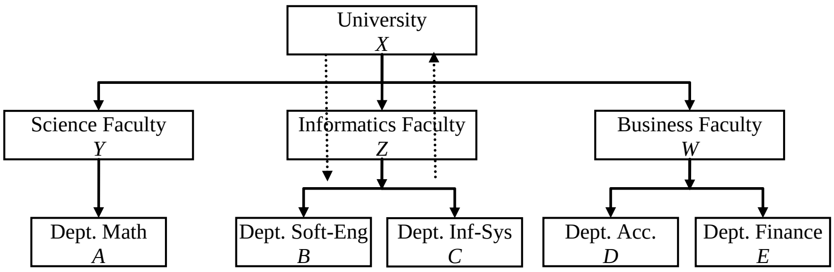


**Figure 6.10** Case 1 of the university research development strategy-making

We give the variables, objectives and constraints of these decision entities as follows:

***1) The university*** **(*leader*):** Objective $f^{(1)}$ is to maximize research quantum which includes the number of publications (can be transformed to points) and research grant income (can be transformed to points). To achieve this aim, the main strategy of the university is to achieve a good balance between rewarding research performance and building a long-term research development environment. It has

Variable $x = (x_1, x_2)$:

$x_1$: How much is used to reward the faculties' research performance, with the aim of encouraging faculties to attract more research grants and generate more publications;

$x_2$: How much is used for the university's long-term research investment, such as earlier career researcher development, campus Intranet construction and lab establishment;

Constraints:

$g_1^{(1)} \leq 0$: annual research budget;

$g_2^{(1)} \leq 0$: a fixed number of students;

$g_3^{(1)} \leq 0$: a fix salary budget which is linked to total working hours.

***2) The three faculties (followers):***

*Science Faculty*: Objective $f_1^{(2)}$ is to maximize the faculty's research budget from the university.

Variables: $y = (y_1, y_2)$:

$y_1$: the points granted to reward publication;

$y_2$: the points granted to reward the securing of research grant income;

*Informatics Faculty:* Objective$f_2^{(2)}$is to maximize the research budget from the university.

Variable:

$z$: how much is used to encourage publication;

*Business Faculty*: Objective$f_3^{(2)}$ is to maximize its research quantum by using the research budget from the university. It is developing a working load policy to reduce the teaching load for researchers who have a high research quantum;

Variable:

$w$: how many points of research quantum per $ of research budget?

***3) The five departments in the three faculties (bottom followers):***

Objectives $f_{ij}^{(3)}, i = 1,2,3, j = 1,2$: all departments have the same objective, that is, to maximize the department's research performance;

Constraints $g_{ij}^{(3)}, i = 1,2,3, j = 1,2$: departments' constraints respectively;

Variables: $a$, $b$, $c$, $d$, and $e$ are variables of the five departments respectively.

Clearly, this TLMF decision case meets the features of S28 in Table 6.1.We give this case's TLMF model as follows:

$$\max_x f^{(1)}(x_1, x_2, y_1, y_2, z, w, a, b, c, d, e) \text{ (University level)}$$

$$\text{s.t. } g^{(1)}(x_1, x_2, y_1, y_2, z, w, a, b, c, d, e) \leq 0,$$

$$\max_y f_1^{(2)}(x_1, x_2, y_1, y_2, a) \text{ (Science faculty)}$$

$$\text{s.t. } g_1^{(2)}(x_1, x_2, y_1, y_2, a) \leq 0,$$

$$\max_a f_1^{(3)}(x_1, x_2, y_1, y_2, a) \text{ (Mathematics department)}$$

$$\text{s.t. } g_1^{(3)}(x_1, x_2, y_1, y_2, a) \leq 0,$$

$$\max_z f_2^{(2)}(x_1, x_2, z, b, c) \text{ (Informatics faculty)}$$

$$\text{s.t. } g_2^{(2)}(x_1, x_2, z, b, c) \leq 0,$$

$$\max_b f_{21}^{(3)}(x_1, x_2, z, b) \text{ (Soft} - \text{Eng department)}$$

$$\text{s.t. } g_{21}^{(3)}(x_1, x_2, z, b) \leq 0,$$

$$\max_c f_{22}^{(3)}(x_1, x_2, z, c) \text{ (Inf} - \text{Sys department)}$$

$$\text{s.t. } g_{22}^{(3)}(x_1, x_2, z, c) \leq 0,$$

$$\max_w f_3^{(2)}(x_1, x_2, w, d, e) \text{ (Business faculty)}$$

$$\text{s.t. } g_3^{(2)}(x_1, x_2, w, d, e) \leq 0,$$

$$\max_d f_{31}^{(3)}(x_1, x_2, w, d) \text{ (Acc. department)}$$

$$\text{s.t. } g_{31}^{(3)}(x_1, x_2, w, d) \leq 0,$$

$$\max_e f_{32}^{(3)}(x_1, x_2, w, e) \text{ (Finance department)}$$

$$\text{s.t. } g_{32}^{(3)}(x_1, x_2, w, e) \leq 0,$$

where $x_1, x_2 \in R$ are the decision variables of the university; $y_1, y_2 \in R, z \in R, w \in R$ are of the three faculties respectively, $a, b, c, d, e \in R$ are of the five departments respectively, and $X = \{(x_1, x_2) | x_1 > 0, x_2 > 0\}, Y = \{(y_1, y_2) | y_1 >$ *0,y2>0,Z=z|z>0, W=w|w>0, A=a|a>0, B=b|b>0, C=c|c>0, D=d|d>0,* $E = \{e | e > 0\}$. As there is a secondary leadership relationship, both objective functions $\max_x f^{(1)}(x_1, x_2, y_1, y_2, z, w, a, b, c, d, e)$ and constraint $g^{(1)}(x_1, x_2, y_1, y_2, z, w, a, b, c, d, e) \leq 0$ of the university include the decision variables of departments $a$, $b$, $c$, $d$, $e$. Similarly, by the secondary followership, $x = (x_1, x_2)$ is included in all departments' objectives and constraints.

### *6.5.2 Case 2: S27 Model*

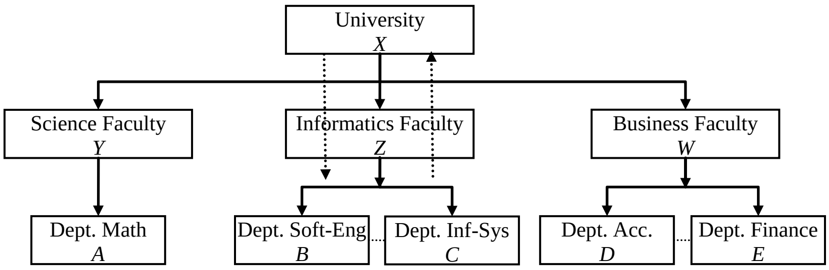


**Figure 6.11** Case 2 of the university research development strategy-making

In this case, we suppose that all three faculties have uncooperative relationships and all the departments of each faculty have reference-uncooperative relationships. As in Case 1, the university takes into account the reactions of the faculties and of all departments. These departments fully consider both their faculty's and the university's strategies. From the TLMF decision framework in Table 1, this case refers to situation *S*27 and is described by DERD in Figure 6.11.

We suppose that the variables, objectives and constraints of decision entities in this case are the same as those of Case 1. This case's TLMF decision model is written as follows:

$$
\begin{aligned}
&\max_{x} f^{(1)}(x,y,z,w,a,b,c,d,e)\\
&\text{s.t. } g^{(1)}(x,y,z,w,a,b,c,d,e) \le 0,\\
&\quad \max_{y} f_1^{(2)}(x,y,a)\\
&\quad \text{s.t. } g_1^{(2)}(x,y,a) \le 0,\\
&\qquad \max_{a} f_1^{(3)}(x,y,a)\\
&\qquad \text{s.t. } g_1^{(3)}(x,y,a) \le 0,\\
&\quad \max_{z} f_2^{(2)}(x,z,b,c)\\
&\quad \text{s.t. } g_2^{(2)}(x,z,b,c) \le 0,\\
&\qquad \max_{b} f_{21}^{(3)}(x,z,b,c)\\
&\qquad \text{s.t. } g_{21}^{(3)}(x,z,b,c) \le 0,\\
&\qquad \max_{c} f_{22}^{(3)}(x,z,b,c)\\
&\qquad \text{s.t. } g_{22}^{(3)}(x,z,b,c) \le 0,
\end{aligned}
$$

$$
\begin{aligned}
&\max_{w} f_3^{(2)}(x,w,d,e)\\
&\text{s.t. } g_3^{(2)}(x,w,d,e) \le 0,\\
&\quad \max_{d} f_{31}^{(3)}(x,w,d,e)\\
&\quad \text{s.t. } g_{31}^{(3)}(x,w,d,e) \le 0,\\
&\quad \max_{e} f_{32}^{(3)}(x,w,d,e)\\
&\quad \text{s.t. } g_{32}^{(3)}(x,w,d,e) \le 0.
\end{aligned}
$$

As there is a secondary leadership relationship, the university's objective function $f^{(1)}(x,y,z,w,a,b,c,d,e)$ and constraint $g^{(1)}(x,y,z,w,a,b,c,d,e)$ include the variables of departments $a,b,c,d,e$. Similarly, by the secondary followership, $x$ is included in all departments' objectives and constraints. As departments take into account their neighborhood decisions (reference-uncooperative), we have variable $d$in the objective $f_{32}^{(3)}(x,w,d,e)$ and constraint $g_{32}^{(3)}(x,w,d,e)$ of departments $E$, and $e$ in the objective $f_{31}^{(3)}(x,w,d,e)$ and constraint $g_{31}^{(3)}(x,w,d,e)$ of departments $D$.

### *6.5.3 Case 3: S54 Model*

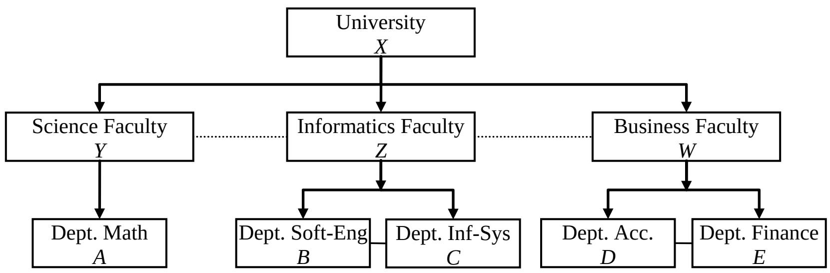


**Figure 6.12** Case 3 of the university research development strategy-making

In this case, all three faculties have a reference-uncooperative relationship and all the departments of each faculty have a semi-cooperative relationship. Unlike Case 2, the university does not take the departments' decisions directly into account, nor do all departments directly consider the university's research strategies during their decision process. It can be seen from Table 6.1 that this relates to situation *S*54. This problem's DERD is shown in Figure 6.12. We use the same notations of variables, objectives and constraints used in Case 1, but the Departments of Soft Eng. and Inf. Sys share variables $b$, and the Departments of Acc. and Finance share variables $e$. We have this case's TLMF decision model as follows:

$$
\begin{aligned}
&\max_{x} f^{(1)}(x, y, z, w)\\
&\text{s.t. } g^{(1)}(x, y, z, w) \leq 0,\\
&\quad \max_{y} f_1^{(2)}(x, y, z, w, a)\\
&\quad \text{s.t. } g_1^{(2)}(x, y, z, w, a) \leq 0,\\
&\qquad \max_{a} f_1^{(3)}(y, a)\\
&\qquad \text{s.t. } g_1^{(3)}(y, a) \leq 0,\\
&\quad \max_{z} f_2^{(2)}(x, y, z, w, b)\\
&\quad \text{s.t. } g_2^{(2)}(x, y, z, w, b) \leq 0,\\
&\qquad \max_{b} f_{21}^{(3)}(z, b)\\
&\qquad \text{s.t. } g_{21}^{(3)}(z, b) \leq 0,\\
&\qquad \max_{c} f_{22}^{(3)}(z, b)\\
&\qquad \text{s.t. } g_{22}^{(3)}(z, b) \leq 0,\\
&\quad \max_{w} f_3^{(2)}(x, y, z, w, d)\\
&\quad \text{s.t. } g_3^{(2)}(x, y, z, w, d) \leq 0,\\
&\qquad \max_{d} f_{31}^{(3)}(w, d)\\
&\qquad \text{s.t. } g_{31}^{(3)}(w, d) \leq 0,\\
&\qquad \max_{e} f_{32}^{(3)}(w, e)\\
&\qquad \text{s.t. } g_{32}^{(3)}(w, e) \leq 0.
\end{aligned}
$$

As these faculties have a reference-cooperative relationship, their variables $y$, $z, w$ are included in all faculties' objective and constraints such as $f_1^{(2)}(x, y, z, w, a)$ and $g_1^{(2)}(x, y, z, w, a)$ . To describe the semi-cooperative relationship between departments, we have $f_{2i}^{(3)}(z, b)$ and $f_{3i}^{(3)}(w, d)$ where variables $b, d$ are shared by two departments respectively. This case has no secondary relationships, so $x$ is not included in department functions and $a, b, d$ are not included in the university's objective function.

### 6.5.4 Case 4: Hybrid of S41, S45 and S48 Models

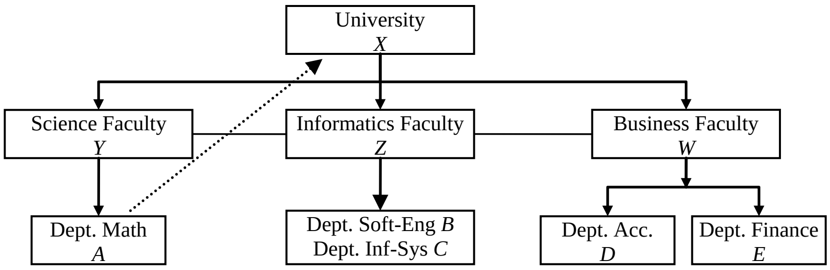


**Figure 6.13** Case 4 of the university research development strategies making

In this case, the three faculties have a semi-cooperative relationship by sharing the same variable $y$.The departments have different relationships in different faculties. In the Science Faculty, the Math department has a second followership relationship with the university. In the Informatics Faculty, the two departments have a cooperative relationship and no secondary relationship. Two departments in the Business Faculty have an uncooperative relationship and no secondary relationship. The three different situations refer to $S41$, $S45$, and $S48$ respectively. This is a hybrid TLMF decision problem. Figure 6.13 describes its DERD. By using the same variables, objectives and constraints of decision entities used in previous cases, we have the following TLMF decision model:

$$\max_{x} f^{(1)}(x,y)$$

$$\text{s.t. } g^{(1)}(x,y) \leq 0,$$

$$\max_{y} f_1^{(2)}(x,y,a)$$

$$\text{s.t. } g_1^{(2)}(x,y,a) \leq 0,$$

$$\max_{a} f_1^{(3)}(x,y,a)$$

$$\text{s.t. } g_1^{(3)}(x,y,a) \leq 0,$$

$$\max_{y} f_2^{(2)}(x,y,b)$$

$$\text{s.t. } g_2^{(2)}(x,y,b) \leq 0,$$

$$\max_{b} f_{21}^{(3)}(y,b)$$

$$\text{s.t. } g_{21}^{(3)}(y,b) \leq 0,$$

$$\max_{y} f_3^{(2)}(x, y, d, e)$$
$$\text{s.t. } g_3^{(2)}(x, y, d, e) \leq 0,$$
$$\max_{d} f_{31}^{(3)}(y, d)$$
$$\text{s.t. } g_{31}^{(3)}(y, d) \leq 0,$$
$$\max_{e} f_{32}^{(3)}(y, e)$$
$$\text{s.t. } g_{32}^{(3)}(y, e) \leq 0.$$

We can see that these three faculties share the same variable $y$ but have individual objectives. To describe the cooperative relationship between the departments in the Informatics Faculty, the two departments share variable $b$, objective function $f_{21}^{(3)}(z, b)$ and constraint $g_{21}^{(3)}(z, b)$ . To describe the uncooperative relationship in the Faculty of Business, its two departments' objective functions $f_{31}^{(3)}(y, d)$ and $f_{32}^{(3)}(y, e)$, have individual variables. As only the Math Department has a secondary relationship with the university level, $x$ is only included in the Math Department's functions.

Through these four cases, we present a way to model real-world TLMF decision problems by both DERD and programming approaches.

## 6.6 Tri-level Decision Solution Methods

This section focuses on a linear version of tri-level decision problems with a single decision entity at each level.

### *6.6.1 Solution Concepts*

According to the basic tri-level decision model (6.1) in a one-level one-entity situation, we present a linear tri-level programming (decision model) as follows.

For $x \in X \subset R^n, y \in Y \subset R^m, z \in Z \subset R^p, f^{(1)}, f^{(2)}, f^{(3)}: X \times Y \times Z \longrightarrow R$,

$$\min_{x \in X} f^{(1)}(x, y, z) = \alpha_1 x + \beta_1 y + \mu_1 z$$
$$\text{s.t. } A_1 x + B_1 y + C_1 z \leq b_1,$$
$$\min_{y \in Y} f^{(2)}(x, y, z) = \alpha_2 x + \beta_2 y + \mu_2 z$$
$$\text{s.t. } A_2 x + B_2 y + C_2 z \leq b_2,$$
$$\min_{z \in Z} f^{(3)}(x, y, z) = \alpha_3 x + \beta_3 y + \mu_3 z$$
$$\text{s.t. } A_3 x + B_3 y + C_3 z \leq b_3, \tag{6.12}$$

where $\alpha_i \in R^n, \beta_i \in R^m, \mu_i \in R^p, b_i \in R^{q_i}, A_i \in R^{q_i \times n}, B_i \in R^{q_i \times m}, C_i \in R^{q_i \times p}$,

$i = 1,2,3.$

The variables $x, y, z$ are called the top-level, middle-level, and bottom-level variables respectively, and $f^{(1)}(x,y,z), f^{(2)}(x,y,z), f^{(3)}(x,y,z)$ the top-level, middle-level, and bottom-level objective functions, respectively. In this model, the decision problem consists of three optimization sub-problems (represented by three objective functions) in a three-level hierarchy. Each level has individual control variables, but also takes account of other levels in its optimization function.

To obtain an optimal solution to the *Linear Tri-level Programming* (LTLP) problem (6.12) based on the solution concept of bi-level programming (Bard 1998), a solution definition is first proposed as follows:

**Definition6.1**

(a) Constraint region of the LTLP:

$$S = \{(x,y,z) | x \in X, y \in Y, z \in Z, A_i x + B_i y + C_i z \leq b_i, i = 1,2,3\}.$$

(b) Constraint region of the middle level for each fixed $x \in X$:

$$S(x) = \{(y,z) \in Y \times Z | B_i y + C_i z \leq b_i - A_i x, i = 2,3\}.$$

(c) Feasible set for the bottom level for each fixed $(x,y) \in X \times Y$:

$$S(x,y) = \{z \in Z | C_3 z \leq b_3 - A_3 x - B_3 y\}.$$

(d) Projection of *S* onto the top level's decision space:

$$S(X) = \{x \in X | \exists (y,z) \in Y \times Z, (x,y,z) \in S\}.$$

(e) Projection of *S* onto the top and middle levels' decision space:

$$S(X,Y) = \{(x,y) \in X \times Y | \exists z \in Z, (x,y,z) \in S\}.$$

(f) Rational reaction set of the bottom level for $(x,y) \in S(X,Y)$:

$$P(x,y) = \{z | z \in \arg\min [f_3(x,y,\hat{z}) | \hat{z} \in S(x,y)\}.$$

(g) Rational reaction set for the middle level for $x \in S(X)$:

$$P(x) = \{(y,z) | (y,z) \in \arg\min [f_2(x,\hat{y},\hat{z}) | (\hat{y},\hat{z}) \in S(x),$$
$$\hat{z} \in P(x,\hat{y})]\}.$$

(h) Inducible region ($IR$):

$$IR = \{(x,y,z) | (x,y,z) \in S, (y,z) \in P(x)\}.$$

Therefore, problem (6.12) is equivalent to the following problem:

$$\min\{f_1(x,y,z) | (x,y,z) \in IR\}. \quad (6.13)$$

### *6.6.2 Theoretical Properties*

The three assumptions stated below serve as an introduction to the solution existence theorem.

**Assumption 6.1**
(1) $S$ is non-empty and compact.
(2) $IR$ is non-empty.
(3) $P(x)$ and $P(x, y)$ are point-to-point maps with respect to $x$ and $(x, y)$ respectively.

Three important LTLP theorems are proposed here. Theorem 6.1 proves the existence of an optimal solution of the LTLP model. Theorem 6.2 presents a way to obtain a solution to the LTLP problem. Theorem 6.3 provides the necessary foundations for developing a tri-level $K$th-Best algorithm.

**Theorem 6.1** If the above assumptions are satisfied, there exists an optimal solution to the linear tri-level decision model (6.13).

**Proof:** Since neither $S$ or $IR$ is empty, there is at least one parameter value $x^* \in S(X)$ and $P(x^*) \neq \emptyset$. Consider a sequence $\{(x^t, y^t, z^t)\}_{t=1}^{\infty} \subseteq IR$ converging to $(x^*, y^*, z^*)$. Then, by the well-known results of linear parametric optimization, $(y^*, z^*) \in P(x^*)$ . Hence, $(x^*, y_1^*, \dots, y_k^*) \in IR$ that shows $IR$ is closed. By Assumption 6.1(1) and $IR \subseteq S$, $IR$ is also bounded. $IR$ is non-empty, so the problem (6.13) consists of minimizing a continuous function over a compact non-empty set, which implies that the problem has an optimal solution.

**Theorem 6.2** The inducible region can be written equivalently as a piecewise linear equality constraint comprised of support hyper-planes of $S$.

**Proof:** Using the notations in the proof of Theorem 6.1, the inducible region $IR$ can be rewritten as follows:

$$IR = \{(x, y, z) \in S | \beta_2 y + \mu_2 z = \min\langle \beta_2 \hat{y} + \mu_2 \hat{z} | B_i \hat{y} + C_i \hat{z} \leq b_i - A_i x, \hat{y} \geq 0, \\ z{\geq}0, i{=}2{,}3, \mu 3z{=}\min \mu 3 z / C3 z {\leq} b3{-}A3x{-}B3y, z{\geq}0\}. \quad (6.14)$$

Let

$$Q(x) = \min[\, \beta_2 \hat{y} + \mu_2 \hat{z} | (\hat{y}, \hat{z}) \in S(x), \hat{z} \in arg\, min[\mu_3 \tilde{z} | \tilde{z} \in S(x, \hat{y})]], \quad (6.15)$$

$$Q(x, y) = min[\mu_3 \tilde{z} | \tilde{z} \in S(x, y)]. \quad (6.16)$$

It is then necessary to prove that $Q(x)$ is a piecewise linear equality constraint.

According to the expressions for $Q(x)$ and $Q(x,y)$, the first step is to prove that $Q(x,y)$ is a piecewise linear equality constraint for any given $x$ and $y$. Because $Q(x,y)$ can be seen as a linear programming problem with parameters $x$ and $y$, the dual problem of $Q(x,y)$ is

$$\max\{u(A_3x + B_3y - b_3)|uC_3 \geq -\mu_3, u \geq 0\}. \tag{6.17}$$

This problem has the same optimal values as $Q(x,y)$ at the solution $u^*$. Let $u^1, \ldots, u^t$ be a listing of all the vertices of the constraint region of the dual problem given by $U = \{u|uC_3 \geq -\mu_3\}$. Because a solution of the dual problem occurs at a vertex of $U$, the equivalent problem is

$$\max\{u(A_3x + B_3y - b_3)|u \in \{u^1, \ldots, u^t\}\}. \tag{6.18}$$

This means that $Q(x,y)$ is a piecewise linear function.

Next, it will be proved that $Q(x)$ is a piecewise linear function. Suppose that $z^1, z^2, \ldots, z^s$ are solutions of problem $Q(x,y)$. For each $z^i$, $Q(x)$ becomes a programming problem with parameters $x$ and $z^i$. Therefore, there are $s$ parameterized programming problems, $Q(x)|_{z^1}, \ldots, Q(x)|_{z^s}$. Similarly, each $Q(x)|_{z^i}$ is a piecewise linear function. Hence, the set $IR$ can be rewritten as

$$IR = \cup_{i=1}^{s}\{(x,y,z^i)|\beta_2 y = Q(x)|_{z^i} - \mu_2 z^i\} \tag{6.19}$$

which is a piecewise linear equality constraint.

**Corollary 6.1** A solution to the LTLP problem (6.12) occurs at a vertex of the $IR$.

**Theorem 6.3** The solution $(x^*, y^*, z^*)$ of the linear tri-level programming problem occurs at a vertex of $S$.

**Proof:** Let $(x^1, y^1, z^1), \ldots, (x^t, y^t, z^t)$ be the distinct vertices of $S$. Because any point in $S$ can be written as a convex combination of these vertices, let $(x^*, y^*, z^*) = \sum_{i=1}^{\bar{t}} \delta_i (x^i, y^i, z^i)$, where $\sum_{i=1}^{\bar{t}} \delta_i = 1, \delta_i > 0, i = 1, \ldots, \bar{t}$ and $\bar{t} \leq t$. It must be shown that $\bar{t} = 1$. Let us write the constraints of (6.12) at $(x^*, y^*, z^*)$ in their piecewise linear form (6.19):

$$\begin{aligned}
0 &= Q(x^*)|_{z^*} - \beta_2 y^* - \mu_2 z^* \\
&= Q\left(\sum_{i=1}^{\bar{t}} \delta_i x^i\right)|_{z^*} - \beta_2\left(\sum_{i=1}^{\bar{t}} \delta_i y^i\right) - \mu_2\left(\sum_{i=1}^{\bar{t}} \delta_i z^i\right) \\
&\leq \sum_{i=1}^{\bar{t}} \delta_i Q(x^i)|_{z^*} - \sum_{i=1}^{\bar{t}} \delta_i \beta_2 y^i - \sum_{i=1}^{\bar{t}} \delta_i \mu_2 z^i
\end{aligned}$$

$$= \sum_{i=1}^{\bar{t}} \delta_i (Q(x^i)|_{z^*} - \beta_2 y^i - \mu_2 z^i)$$

by the convexity of $Q(x)$. However, by definition$Q(x^i)|_{z^*}$,

$$Q(x^i)|_{z^*} = \min_{\substack{(y,z)\in S(x^i) \\ z\in p(x^i,y)}} (\beta_2 y + \mu_2 z) \le \beta_2 y^i + \mu_2 z^i.$$

Therefore, $Q(x^i)|_{z^*} - \beta_2 y^i - \mu_2 z^i \le 0, i = 1, \ldots, \bar{t}$ . Noting that $\delta_i > 0, i = 1, \ldots, \bar{t}$, the equality in the preceding expression must hold, or else a contradiction would result in the sequence above. Consequently, $Q(x^i)|_{z^*} - \beta_2 y^i - \mu_2 z^i = 0$ for all $i$. These statements imply that $(x^i, y^i, z^i) \in IR, i = 1, \ldots, \bar{t}$, and that $(x^*, y^*, z^*)$ can be written as a convex combination of points in the *IR*. Because $(x^*, y^*, z^*)$ a vertex of the $IR$ by Corollary 6.1 and $P(x)$ and $P(x, \mathrm{y})$ are single-valued, a contradiction results unless $\bar{t} = 1$.

**Corollary 6.2** If $(x, y, z)$ is a vertex of $IR$, then it is also a vertex of $S$.

### *6.6.3 Tri-level Kth-Best Algorithm*

This section will introduce the tri-level *K*th-Best algorithm for solving the linear tri-level programming problem (6.12).

Theorem 6.3 in Section 6.6.2 provides a theoretical foundation and a suitable way to solve problem (6.12). Therefore, it is necessary only to search the extreme points of the constraint region $S$ to find an optimal solution for the LTLP problem (6.12). The main principle of the tri-level *K*th-Best algorithm is shown as follows.

Consider the linear programming problem below:

$$\min\{\alpha_1 x + \beta_1 y + \mu_1 z | (x, y, z) \in S\}. \tag{6.20}$$

The *N*-ranked basic feasible solutions to (6.20) are:

$$(x_{[1]}, y_{[1]}, z_{[1]}), (x_{[2]}, y_{[2]}, z_{[2]}), \ldots, (x_{[N]}, y_{[N]}, z_{[N]}),$$

such that $\alpha_1 x_{[i]} + \beta_1 y_{[i]} + \mu_1 z_{[i]} \le \alpha_1 x_{[i+1]} + \beta_1 y_{[i+1]} + \mu_1 z_{[i+1]}, i = 1, \ldots, N-1$. Then solving the problem (6.12) is equivalent to finding the index

$$K^* = \min\{i \in \{1, \cdots N\} | (x_{[i]}, y_{[i]}, z_{[i]}) \in IR\}.$$

Therefore, a global solution is$(x_{[K^*]}, y_{[K^*]}, z_{[K^*]})$. Similarly, for fixing $x = x_{[i]}$, we have the middle-level and bottom-level problem (6.21) as follows:

$$\begin{aligned} &\min_{y\in Y} \beta_2 y + \mu_2 z \\ &\text{s.t. } A_2 x + B_2 y + C_2 z \le b_2, \end{aligned} \tag{6.21}$$

$$\min_{z \in Z} \mu_3 z$$
$$\text{s.t. } A_3 x + B_3 y + C_3 z \le b_3.$$

Clearly, problem (6.21) is a general bi-level programming scenario which has been discussed in Chapter 3. We can use the *K*th-Best algorithm, the Kuhn-Tucker approach or the Branch-and-bound algorithm to solve this problem.

The procedure of the tri-level *K*th-Best algorithm is described as follows:

**Algorithm 6.1: Tri-level *K*th-Best Algorithm**

[Begin]

**Step 1:** Set $i \leftarrow 1$. Solve problem (6.20) using the simplex method to obtain the optimal solution, $\left(x_{[1]}, y_{[1]}, z_{[1]}\right)$. Let $W = \{\left(x_{[1]}, y_{[1]}, z_{[1]}\right)\}$ and $T = \emptyset$. Go to Step 2.

**Step 2:** Treat the problem as a top- (middle-, bottom-) level problem. This step is equivalent to solving the follower's (middle-, bottom-) decision problem (6.21) for $x = x_{[i]}$. Let $(\tilde{y}, \tilde{z})$ denote the optimal solution to (6.21). If $\tilde{y} = y_{[i]}$ and $\tilde{z} = z_{[i]}$, stop, and $\left(x_{[i]}, y_{[i]}, z_{[i]}\right)$ is the globally optimal solution of (6.12) with $K^* = i$; otherwise, go to Step 3.

**Step 3:** Let $W_{[i]}$ denote the set of adjacent vertices of $\left(x_{[i]}, y_{[i]}, z_{[i]}\right)$ such that $(x, y, z) \in W_{[i]}$ implies $\alpha_1 x + \beta_1 y + \mu_1 z \ge \alpha_1 x_{[i]} + \beta_1 y_{[i]} + \mu_1 z_{[i]}$ . Let $T = T \cup \left\{\left(x_{[i]}, y_{[i]}, z_{[i]}\right)\right\}$ and $W = (W \cup W_{[i]}) \setminus T$. Go to Step 4.

**Step 4:** Set $i \leftarrow i + 1$ and choose $\left(x_{[i]}, y_{[i]}, z_{[i]}\right)$ so that

$$\alpha_1 x_{[i]} + \beta_1 y_{[i]} + \mu_1 z_{[i]} = \min\{\alpha_1 x + \beta_1 y + \mu_1 z | (x, y, z) \in W\}.$$

Go to Step 2.

[End]

The tri-level *K*th-Best algorithm uses two sub-algorithms: (1) the simplex algorithm, which can obtain an optimal solution for a linear programming problem, and (2) the algorithm for finding the adjacent vertices of a selected vertex. According to the results given by Bard (1984), a vertex is a geometrical interpretation of a feasible solution. Hence, enumerating the adjacent vertices is equivalent to enumerating all the basic feasible solutions for the decision problem.

### *6.6.4 A Numerical Example*

We give an example to illustrate how the tri-level *K*th-Best algorithm can be used to solve a tri-level decision problem.

**Example 6.2** For $x \in X = \{x|x \geq 0\}, y \in Y = \{y|y \geq 0\}, z \in Z = \{z|z \geq 0\}$, $f^{(1)}, f^{(2)}, f^{(3)}: X \times Y \times Z \to R$,

$$\min_{x \in X} f^{(1)} = x + y + 2z$$
$$\text{s.t. } 2x + y + z \geq 14,$$
$$\min_{y \in Y} f^{(2)} = x + y + 3z$$
$$\text{s.t. } x + y \geq 4,$$
$$y \leq 6,$$
$$\min_{z \in Z} f^{(3)} = x + y - z$$
$$\text{s.t. } y + z \leq 8,$$
$$y + 4z \geq 8,$$
$$y + 2z \leq 13.$$

Now it is possible to use the tri-level *K*th-Best algorithm to obtain a solution for this problem. According to the tri-level *K*th-Best algorithm, solving this problem first requires consideration of the middle level and the bottom level as a whole (middle, bottom) and then solving the problem using the bi-level *K*th-Best algorithm.

From (6.20), let us consider a linear programming problem as follows:

$$\min_{x \in X} f^{(1)} = x + y + 2z$$
$$\text{s.t. } 2x + y + z \geq 14,$$
$$x + y \geq 4,$$
$$y \leq 6,$$
$$y + z \leq 8,$$
$$y + 4z \geq 8,$$
$$y + 2z \leq 13.$$

Now we go through the tri-level *K*th-Best algorithm from Step 1 to Step 4.

**Step 1:** Set $i \leftarrow 1$. Solve the above problem using the simplex method to obtain the optimal solution, $(x_{[1]}, y_{[1]}, z_{[1]}) = (6,0,2)$. Let $W = \{(x_{[1]}, y_{[1]}, z_{[1]})\}$ and $T = \emptyset$. Go to Step 2.

**Step 2:** By the problem (6.21), we have the problem:

$$\min_{y \in Y} f^{(2)} = x + y - z$$
$$\text{s.t. } x + y \geq 4,$$
$$y \leq 6,$$
$$x = 6,$$
$$\min_{z \in Z} f^{(3)} = x + y - z$$
$$\text{s.t. } y + z \leq 8,$$
$$y + 4z \geq 8,$$
$$y + 2z \leq 13,$$
$$x = 6.$$

Using the bi-level *K*th-Best algorithm, we have $(\tilde{y}_{[1]}, \tilde{z}_{[1]}) = (6,2) \neq (y_{[1]}, z_{[1]})$ and go to Step 3.

**Step 3:** Find the adjacent vertices of $(x_{[1]}, y_{[1]}, z_{[1]})$ and we have $W_{[1]} = \{(3.75,6,0.5), (4,0,6)\}$, *T*={(6,0,2)} and *W*=$\{(3.75,6,0.5), (4,0,6)\}$. Go to Step 4.

**Step 4:** Update *i*=*i*+1, choose $(x_{[2]}, y_{[2]}, z_{[2]}) = (3.75,6,0.5)$ and go back to Step 2.

**Step 2:** By the problem (6.21), we have the problem:

$$\min_{y \in Y} f^{(2)} = x + y - z$$
$$\text{s.t. } x + y \geq 4,$$
$$y \leq 6,$$
$$x = 3.75,$$
$$\min_{z \in Z} f^{(3)} = x + y - z$$
$$\text{s.t. } y + z \leq 8,$$
$$y + 4z \geq 8,$$
$$y + 2z \leq 13,$$
$$x = 3.75.$$

Using the bi-level *K*th-Best algorithm, we have $(\tilde{y}_{[2]}, \tilde{z}_{[2]}) = (6,2) \neq (y_{[2]}, z_{[2]})$ and go to Step 3.

**Step 3:** Find the adjacent vertices of $(x_{[2]}, y_{[2]}, z_{[2]})$ and we have $W_{[2]} = \{(3,6,2)\}$, *T*={(6,0,2),(3.75,6,0.5)} and $W = \{(4,0,6), (3,6,2)\}$. Go to Step 4.

**Step 4:** Update *i*=*i*+1, choose $(x_{[3]}, y_{[3]}, z_{[3]}) = (3,6,2)$ and go back to Step 2.

**Step 2:** By the problem (6.21), we have the problem:

$$\min_{y \in Y} f^{(2)} = x + y - z$$
$$\text{s.t. } x + y \geq 4,$$
$$y \leq 6,$$
$$x = 3,$$
$$\min_{z \in Z} f^{(3)} = x + y - z$$
$$\text{s.t. } y + z \leq 8,$$
$$y + 4z \geq 8,$$
$$y + 2z \leq 13,$$
$$x = 3.$$

Using the bi-level *K*th-Best algorithm, we have $(\tilde{y}_{[3]}, \tilde{z}_{[3]}) = (6,2) = (y_{[3]}, z_{[3]})$. Therefore, $(x_{[3]}, y_{[3]}, z_{[3]})$ is an optimal solution of Example 6.2 with $K^* = i = 3$. For the global solution, the objective value of $f_1$ is 13, and the objective function values of $f_2$ and $f_3$ are 15 and 7 respectively. Therefore, the *K*th-Best algorithm provides an useful way to solve the linear tri-level decision problem.

## 6.7 Tri-level Multi-follower Decision Solution Methods

We have proposed 64 kinds of TLMF decision model and this section aims to present solution methods for these models. We take the TLMF decision model S12 in its linear version as representative, to illustrate solution concepts and theoretical properties, and describe a TLMF *K*th-Best algorithm for TLMF decision.

### *6.7.1 Solution Concepts*

According to the general model*S12*shownin Section 6.4, the model in linear version can be expressed as follows.

For $x \in X \subset R^k$, $y \in Y_i \subset R^{k_0}$, $Y = Y_1 \cap \cdots \cap Y_n$, $y \in Y$, $z_{ij} \in Z_{ij} \subset R^{k_{ij}}$, $f^{(1)}: X \times Y \times Z_{11} \times \cdots Z_{1m_1} \times \cdots \times Z_{n1} \times \cdots \times Z_{nm_n} \to R, f_i^{(2)}: X \times Y_i \times Z_{i1} \times \cdots \times Z_{im_i} \to R, f_{ij}^{(3)}: X \times Y_i \times Z_{ij} \to R$, and $j = 1, \ldots, m_i, i = 1, \ldots, n,$

$$\min_{x\in X} f^{(1)}\big(x,y,z_{11},\dots,z_{1m_1},\dots,z_{n1},\dots,z_{nm_n}\big) = cx + dy + \sum_{i=1}^{n}\sum_{j=1}^{m_i} e_{ij}\,z_{ij} \quad (6.22a)$$

$$\text{s.t. } Ax + By + \textstyle\sum_{i=1}^{n}\sum_{j=1}^{m_i} C_{ij}\,z_{ij} \ \le b, \quad (6.22b)$$

where $(y,z_{i1},\dots,z_{im_i})$ $(i = 1,\dots,n)$ is the solution to the $i$th middle-level follower's problem and its bottom-level followers' problems (6.22c-6.22f):

$$\min_{y\in Y_i} f_i^{(2)}\big(x,y,z_{i1},\dots,z_{im_i}\big) = c_i x + d_i y + \sum_{j=1}^{m_i} g_{ij}\,z_{ij} \quad (6.22c)$$

$$\text{s.t. } A_i x + B_i y + \textstyle\sum_{j=1}^{m_i} D_{ij}\,z_{ij} \ \le b_i, \quad (6.22d)$$

where $z_{ij}\,(j = 1,\dots,m_i)$ is the solution to the $i$th middle-level follower's $j$th bottom-level follower's problem (6.22e-6.22f):

$$\min_{z_{ij}\in Z_{ij}} f_{ij}^{(3)}\big(x,y,z_{ij}\big) = c_{ij}\,x + d_{ij}\,y + h_{ij}\,z_{ij} \quad (6.22e)$$

$$\text{s.t. } A_{ij}\,x + B_{ij}\,y + E_{ij}\,z_{ij} \ \le b_{ij}, \quad (6.22f)$$

where $c, c_i, c_{ij} \in R^k, d, d_i, d_{ij} \in R^{k_0}, e_{ij}, g_{ij}, h_{ij} \in R^{k_{ij}}, A \in R^{s\times k}, A_i \in R^{s_i\times k}$, $A_{ij} \in R^{s_{ij}\times k}, B \in R^{s\times k_0}, B_i \in R^{s_i\times k_0}, B_{ij} \in R^{s_{ij}\times k_0}, C_{ij} \in R^{s\times k_{ij}}, D_{ij} \in R^{s_i\times k_{ij}}$, $E_{ij} \in R^{s_{ij}\times k_{ij}}, b \in R^s, b_i \in R^{s_i}, b_{ij} \in R^{s_{ij}}, j = 1,\dots,m_i, i = 1,\dots,n.$

To find an optimal solution for the decision model, relevant solution concepts are proposed as follows, based on definitions of bi-level programming and tri-level programming.

**Definition 6.1**

(a) Constraint region of the TLMF decision model:

$$\begin{aligned} S = \{&(x,y,z_{11},\dots,z_{1m_1},\dots,z_{n1},\dots,z_{nm_n}) \in X\times Y\times Z_{11}\times\cdots Z_{1m_1}\times\cdots\times \\ &Z_{n1}\times\cdots\times Z_{nm_n} \,|\, Ax + By + \textstyle\sum_{i=1}^{n}\sum_{j=1}^{m_i} C_{ij}\,z_{ij} \ \le b, \\ &A_i x + B_i y + \textstyle\sum_{j=1}^{m_i} D_{ij}\,z_{ij} \ \le b_i, A_{ij}\,x + B_{ij}\,y + E_{ij}\,z_{ij} \ \le b_{ij}, \\ &j = 1,\dots,m_i, i = 1,\dots,n\}. \end{aligned}$$

(b) Constraint region of the $i$th middle-level follower for each fixed $x \in X$:

$$\begin{aligned} S_i(x) = \{&(y,z_{i1},\dots,z_{im_i}) \in Y_i\times Z_{i1}\times\cdots\times Z_{im_i} \,|\, A_i x + B_i y + \\ &\textstyle\sum_{j=1}^{m_i} D_{ij}\,z_{ij} \ \le b_i, A_{ij}\,x + B_{ij}\,y + E_{ij}\,z_{ij} \ \le b_{ij}, j = 1,\dots,m_i\}. \end{aligned}$$

(c) Feasible set of the $i$th middle-level follower's $j$th bottom-level follower for each fixed $(x,y) \in X\times Y_i$:

$$S_{ij}(x,y) = \{z_{ij} \in Z_{ij} \,|\, A_{ij}\,x + B_{ij}\,y + E_{ij}\,z_{ij} \ \le b_{ij}\}.$$

(d) Projection of $S$ onto the leader's decision space:
$$S(X) = \{x \in X | \exists (y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}),$$
$$(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) \in S\}.$$

(e) Projection of $S$ onto the top-level leader's and the $i$th middle-level follower's decision space:
$$S_i(X, Y) = \{(x, y) | \exists (z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}),$$
$$(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) \in S\}.$$

(f) Rational reaction set of the $i$th middle-level follower's $j$th bottom-level follower for $(x, y) \in S_i(X, Y)$:
$$P_{ij}(x, y) = \{z_{ij} \in Z_{ij} | z_{ij} \in \operatorname{argmin}[f_{ij}^{(3)}(x, y, \hat{z}_{ij}) : \hat{z}_{ij} \in S_{ij}(x, y)]\}.$$

(g) Rational reaction set of the $i$th middle-level follower for $x \in S(X)$:
$$P_i(x) = \{(y, z_{i1}, \dots, z_{im_i}) | (y, z_{i1}, \dots, z_{im_i}) \in \operatorname{argmin}[f_i^{(2)}(x, \hat{y}, \hat{z}_{i1}, \dots, \hat{z}_{im_i}) |$$
$$(\hat{y}, \hat{z}_{i1}, \dots, \hat{z}_{im_i}) \in S_i(x), \hat{z}_{ij} \in P_{ij}(x, \hat{y}), j = 1, \dots, m_i]\}.$$

(h) Inducible region:
$$IR = \{(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) |$$
$$(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) \in S, (y, z_{i1}, \dots, z_{im_i}) \in P_i(x),$$
$$i = 1, \dots, n\}.$$

Therefore, based on the notations, the TLMF decision model (6.22) can be written as:
$$\min_{x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}} f^{(1)}(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n})$$
$$\text{s.t.} \quad (x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) \in IR. \tag{6.23}$$

## 6.7.2 Theoretical Properties

For the sake of assuring that an optimal solution to the model (6.22) exists, we give the following assumption.

**Assumption 6.2**
(1) $S$ is non-empty and compact.
(2) $IR$ is non-empty.
(3) $P_i(x)$ and $P_{ij}(x, y)$ are point-to-point maps with respect to $x$ and $(x, y)$ respectively, where $j = 1, \dots, m_i, i = 1, \dots, n$.

**Theorem 6.4** If the TLMF decision model (6.22) meets Assumption 6.2, then there exists an optimal solution.

**Proof:** Let
$P(x) = \{(y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) : (y, z_{i1}, \dots, z_{im_i}) \in P_i(x), i = 1, \dots, n\}$.
Since neither $S$ nor $IR$ is empty, there is at least one parameter value $x^* \in S(X)$ and $P(x^*) \neq \emptyset$.
Consider a sequence $\{(x^t, y^t, z_{11}^t, \dots, z_{1m_1}^t, \dots z_{n1}^t, \dots, z_{nm_n}^t)\}_{t=1}^{\infty} \subseteq IR$ converging to $(x^*, y^*, z_{11}^*, \dots, z_{1m_1}^*, \dots z_{n1}^*, \dots, z_{nm_n}^*)$. Then, by the well-known results of linear parametric optimization, we have $(x^*, y^*, z_{11}^*, \dots, z_{1m_1}^*, \dots z_{n1}^*, \dots, z_{nm_n}^*) \in P(x^*)$. Hence, $(x^*, y^*, z_{11}^*, \dots, z_{1m_1}^*, \dots z_{n1}^*, \dots, z_{nm_n}^*) \in IR$ which shows that $IR$ is closed. By Assumption 6.2(1) and $IR \subseteq S$, $IR$ is therefore also bounded, and $IR$ is nonempty, so the problem (6.22) consists of minimizing a continuous function over a compact nonempty set, which implies that the problem has an optimal solution.

**Theorem 6.5** The inducible region $IR$ can be expressed equivalently as a piecewise linear equality constraint comprised of supporting hyperplanes of $S$.

**Proof:** First, denote the optimal value of the *i*th middle-level follower's *j*th bottom-level follower by

$$F_{ij}(x, y) = \min\{h_{ij}\hat{z}_{ij} \mid \hat{z}_{ij} \in S_{ij}(x, y)\}, j = 1, \dots, m_i, i = 1, \dots, n,$$

and define

$$F_i(x) = \min\{d_i y + \sum_{j=1}^{m_i} g_{ij} z_{ij} \mid (y, z_{i1}, \dots, z_{im_i}) \in S_i(x),$$
$$h_{ij} z_{ij} = F_{ij}(x, y), j = 1, \dots, m_i\}, i = 1, \dots, n.$$

Since $F_{ij}(x, y)$ can be seen as a linear programming problem with parameters $x$ and$y$, the dual problem of $F_{ij}(x, y)$ can be written as

$$\max\{(A_{ij}x + B_{ij}y - b_i)u_{ij} \mid E_{ij}u_{ij} \geq -h_{ij}, u_{ij} \geq 0\}. \quad (6.24)$$

If both $F_{ij}(x, y)$ and problem (6.24) have feasible solutions, by the dual theorem of linear programming, both have optimal solutions and the same optimal objective function value. Since a solution to problem (6.24) occurs at a vertex of its constraint region $U_{ij} = \{u_{ij} \mid E_{ij}u_{ij} \geq -h_{ij}, u_{ij} \geq 0\}$, adopting $u_{ij}^1, \dots, u_{ij}^{k_{ij}}$ to express all the vertices of $U_{ij}$, then problem (6.24) can be written as:

$$\max\left\{(A_{ij}x + B_{ij}y - b_i)u_{ij} \mid u_{ij} \in \{u_{ij}^1, \dots, u_{ij}^{k_{ij}}\}\right\}. \quad (6.25)$$

Clearly, $F_{ij}(x,y)$ is a piecewise linear function according to problem (6.25).

Next, we prove that $F_i(x)$ is also a piecewise linear function. Assume that $(z_{i1}^1, \dots, z_{im_i}^1), \dots, (z_{i1}^{p_i}, \dots, z_{im_i}^{p_i})$ are solutions to the problem $F_{ij}(x,y)$ for $i = 1, \dots, n$. For each fixed $i$ and a solution $(z_{i1}^{t_i}, \dots, z_{im_i}^{t_i})$ where $t_i = 1, \dots, p_i$, $F_i(x)$ becomes a programming problem with parameters $x$ and $(z_{i1}^{t_i}, \dots, z_{im_i}^{t_i})$, and there are $p_i$ parameterized programming problems such as $F_i(x)|_{(z_{i1}^1,\dots,z_{im_i}^1)}, \dots, F_i(x)|_{(z_{i1}^{p_i},\dots,z_{im_i}^{p_i})}$. Considering different combinations of $(z_{i1}^{t_i}, \dots, z_{im_i}^{t_i})$ for $i = 1, \dots, n$, there are $\prod_{i=1}^n p_i$ parameterized programming problems $F_i(x)|_{(z_{i1}^{t_i},\dots,z_{im_i}^{t_i})}$. Therefore, $F_i(x)$ is also a piecewise linear function as $F_{ij}(x,y)$.

Lastly, according to the above definition of $F_i(x)$, the inducible region*IR* can be rewritten as

$$IR = \{(x, y, z_{11}^{t_1}, \dots, z_{1m_1}^{t_1}, \dots, z_{n1}^{t_n}, \dots, z_{nm_n}^{t_n}) \in S | d_i y + \sum_{j=1}^{m_i} g_{ij} z_{ij} = F_i(x)|_{(z_{i1}^{t_i},\dots,z_{im_i}^{t_i})}, t_i = 1, \dots, p_i, i = 1, \dots, n\}. \tag{6.26}$$

and it can be seen as a piecewise linear equality constraint.

**Corollary 6.3** The TLMF decision model (6.22) is equivalent to optimizing$f^{(1)}$ over a feasible region comprised of a piecewise linear equality constraint.

**Corollary 6.4** An optimal solution to the TLMF decision model (6.22) occurs at a vertex of *IR*.

**Proof:** According to the equivalent form (6.23) of the TLMF decision model, and since $f^{(1)}$ is linear, an optimal solution to the problem must occur at a vertex of IR if it exists.

**Theorem 6.6** The optimal solution $(x^*, y^*, z_{11}^*, \dots, z_{1m_1}^*, \dots z_{n1}^*, \dots, z_{nm_n}^*)$ to the TLMF decision model (6.22) occurs at a vertex of *S*.

**Proof:** Let $(x^1, y^1, z_{11}^1, \dots, z_{1m_1}^1, \dots z_{n1}^1, \dots, z_{nm_n}^1), \dots, (x^t, y^t, z_{11}^t, \dots, z_{1m_1}^t, \dots z_{n1}^t, \dots, z_{nm_n}^t)$ indicate the distinct vertices of *S*. Since any point in *S* can be written as a convex combination of these vertices, we have

$$(x^*, y^*, z_{11}^*, \dots, z_{1m_1}^*, \dots z_{n1}^*, \dots, z_{nm_n}^*) = \sum_{r=1}^{\bar{t}} \delta_r \left(x^r, y^r, z_{11}^r, \dots, z_{1m_1}^r, \dots z_{n1}^r, \dots, z_{nm_n}^r\right)$$

where $\sum_{r=1}^{\bar{t}} \delta_r = 1, \delta_r > 0, r = 1, \dots, \bar{t}$ and $\bar{t} \le t$.

We can write the constraints of (6.22) in the piecewise linear form (6.26) discussed in Theorem 6.6:

$$0 = F_i(x^*)|_{(z_{i1}^*,\dots,z_{im_i}^*)} - d_i y^* - \sum_{j=1}^{m_i} g_{ij} z_{ij}^*$$
$$= F_i\left(\sum_{r=1}^{\bar{t}} \delta_r x^r\right)|_{(z_{i1}^*,\dots,z_{im_i}^*)} - d_i \sum_{r=1}^{\bar{t}} \delta_r y_i^r - \sum_{j=1}^{m_i} g_{ij} \sum_{r=1}^{\bar{t}} \delta_r z_{ij}^r\ , i = 1,..,n.$$

Because of the convexity of $F_i(x^*)$, we have

$$0 \le \sum_{r=1}^{\bar{t}} \delta_r\, F_i(x^r)|_{(z_{i1}^*,\dots,z_{im_i}^*)} - \sum_{r=1}^{\bar{t}} \delta_r\, d_i y_i^r - \sum_{r=1}^{\bar{t}} \delta_r \sum_{j=1}^{m_i} g_{ij} z_{ij}^r$$
$$= \sum_{r=1}^{\bar{t}} \delta_r\, [F_i(x^r)|_{(z_{i1}^*,\dots,z_{im_i}^*)} - d_i y_i^r - \sum_{j=1}^{m_i} g_{ij} z_{ij}^r]\,, i = 1,..,n. \quad (6.27)$$

By the definition of $F_i(x)|_{(z_{i1}^{t_i},\dots,z_{im_i}^{t_i})}$, we have

$$F_i(x^r)|_{(z_{i1}^*,\dots,z_{im_i}^*)} = \min(d_i y + \sum_{j=1}^{m_i} g_{ij} z_{ij}) \le d_i y_i^r - \sum_{j=1}^{m_i} g_{ij} z_{ij}^r, i = 1,..,n.$$

Thus, $F_i(x^r)|_{(z_{i1}^*,\dots,z_{im_i}^*)} - d_i y_i^r - \sum_{j=1}^{m_i} g_{ij} z_{ij}^r \le 0, r = 1,\dots,\bar{t}, i = 1,..,n.$

Since the above expression (6.27) must be held with $\delta_r > 0, r = 1,\dots,\bar{t}$,, there exist $F_i(x^r)|_{(z_{i1}^*,\dots,z_{im_i}^*)} - d_i y_i^r - \sum_{j=1}^{m_i} g_{ij} z_{ij}^r \le 0, r = 1,\dots,\bar{t}, i = 1,..,n.$

These statements imply that $(x^r, y^r, z_{11}^r, \dots, z_{1m_1}^r, \dots z_{n1}^r, \dots, z_{nm_n}^r) \in IR$ , $r = 1,\dots,\bar{t}$ and that $(x^*, y^*, z_{11}^*, \dots, z_{1m_1}^*, \dots z_{n1}^*, \dots, z_{nm_n}^*)$ can be denoted as a convex combination of the points in the *IR*. Since $(x^*, y^*, z_{11}^*, \dots, z_{1m_1}^*, \dots z_{n1}^*, \dots, z_{nm_n}^*)$ is a vertex of the *IR* according to Corollary 6.4 and Assumption 6.2(3), there must exist $\bar{t} = 1$, which means $(x^*, y^*, z_{11}^*, \dots, z_{1m_1}^*, \dots z_{n1}^*, \dots, z_{nm_n}^*)$ is a vertex of *S*.

**Corollary 6.5** If $(x^*, y^*, z_{11}^*, \dots, z_{1m_1}^*, \dots z_{n1}^*, \dots, z_{nm_n}^*)$ is a vertex of the *IR*, it is also a vertex of *S*.

### *6.7.3 TLMF Kth-Best Algorithm*

The above theorems and corollaries provide a theoretical foundation to extend the tri-level *K*th-Best algorithm proposed in Section 6.6.3 for solving the TLMF decision problem (6.22). The main principle of the TLMF *K*th-Best algorithm is showed as follows.

First, consider the following linear programming problem:

$$\min_{(x,y,z_{11},\ldots,z_{1m_1},\ldots,z_{n1},\ldots,z_{nm_n})\in S} f^{(1)}(x, y, z_{11}, \ldots, z_{1m_1}, \ldots, z_{n1}, \ldots, z_{nm_n}) \tag{6.28}$$

and let

$$\left(x^1, y^1, z_{11}^1, \ldots, z_{1m_1}^1, \ldots z_{n1}^1, \ldots, z_{nm_n}^1\right), \ldots, \left(x^N, y^N, z_{11}^N, \ldots, z_{1m_1}^N, \ldots z_{n1}^N, \ldots, z_{nm_n}^N\right)$$

denote the $N$-ranked basic feasible solutions to (6.28), such that

$$f^{(1)}\left(x^K, y^K, z_{11}^K, \ldots, z_{1m_1}^K, \ldots z_{n1}^K, \ldots, z_{nm_n}^K\right)$$
$$\leq f^{(1)}\left(x^{K+1}, y^{K+1}, z_{11}^{K+1}, \ldots, z_{1m_1}^{K+1}, \ldots z_{n1}^{K+1}, \ldots, z_{nm_n}^{K+1}\right), K = 1, \ldots, N-1.$$

Then solving the problem (6.28) is equivalent to searching the index $K^* = \min\left\{K \middle| K \in \{1, \ldots, N\}, \left(x^K, y^K, z_{11}^K, \ldots, z_{1m_1}^K, \ldots z_{n1}^K, \ldots, z_{nm_n}^K\right) \in \mathrm{IR}\right\}$, which ensures that $\left(x^{K^*}, y^{K^*}, z_{11}^{K^*}, \ldots, z_{1m_1}^{K^*}, \ldots z_{n1}^{K^*}, \ldots, z_{nm_n}^{K^*}\right)$ is the global solution to the TLMF problem.

To get $(x^{K^*}, y^{K^*}, z_{11}^{K^*}, \ldots, z_{1m_1}^{K^*}, \ldots z_{n1}^{K^*}, \ldots, z_{nm_n}^{K^*})$, we must obtain $\left(y^{K^*}, z_{11}^{K^*}, \ldots, z_{1m_1}^{K^*}, \ldots z_{n1}^{K^*}, \ldots, z_{nm_n}^{K^*}\right)$ by solving a set of uncooperative linear multi-follower bi-level (MFBL) decision problems at the middle and bottom level, so next, for $i = 1, \ldots, n$ and the fixing $x = x^{K^*}$, the middle-level and bottom-level problem becomes:

$$\min_{y\in Y_i} f_i^{(2)}\left(x, y, z_{i1}, \ldots, z_{im_i}\right) = c_i x + d_i y + \sum_{j=1}^{m_i} g_{ij} z_{ij}$$
$$\text{s.t. } A_i x + B_i y + \textstyle\sum_{j=1}^{m_i} D_{ij} z_{ij} \leq b_i,$$

where $z_{ij}$ $(j = 1, \ldots, m_i)$ is the solution to the $i$th middle-level follower's $j$th bottom-level follower's problem:

$$\min_{z_{ij}\in Z_{ij}} f_{ij}^{(3)}\left(x, y, z_{ij}\right) = c_{ij} x + d_{ij} y + h_{ij} z_{ij}$$
$$\text{s.t. } A_{ij} x + B_{ij} y + E_{ij} z_{ij} \leq b_{ij}. \tag{6.29}$$

Clearly, problem (6.29) is an uncooperative MFBL decision problem. It can be solved by the multi-follower (uncooperative) *K*th-Best algorithm given in Chapter 4.4.3, or the multi-follower (uncooperative) Kuhn-Tucker approach given in Chapter 4.4.4.

Detailed procedures of the TLMF *K*th-Best algorithm are presented as follows:

**Algorithm 6.2: TLMF *K*th-Best Algorithm**

[Begin]

**Step 1:** Set $k = 1$, adopt the simplex method to obtain the optimal solution $\left(x^1, y^1, z_{11}^1, \dots, z_{1m_1}^1, \dots z_{n1}^1, \dots, z_{nm_n}^1\right)$ to the linear programming problem (6.28). Let $W = \left\{\left(x^1, y^1, z_{11}^1, \dots, z_{1m_1}^1, \dots z_{n1}^1, \dots, z_{nm_n}^1\right)\right\}$ and $T = \emptyset$. Go to Step 2.

**Step 2:** Put $x = x^k$ and $i = 1$, solve the uncooperative BLMF decision problems (6.29) and obtain the optimal solution $\left(\hat{y}, \hat{z}_{i1}, \dots, \hat{z}_{im_i}\right)$ using the BLMF *K*th-Best algorithm shown as the following subroutine Step 2.1-Step 2.5. Then go to Step 3.

**Step 2.1:** Set $x = x^k$ and $k_i = 1$, adopt the simplex method to obtain the optimal solution $(y^{i1}, z_{i1}^{i1}, \dots, z_{im_i}^{i1})$ to the linear programming problem (6.30):

$$\min\{f_i^{(2)}\left(x, y, z_{i1}, \dots, z_{im_i}\right) | \left(y, z_{i1}, \dots, z_{im_i}\right) \in S_i(x)\}. \tag{6.30}$$

Let $W_i^{'} = \left\{\left(y^{i1}, z_{i1}^{i1}, \dots, z_{im_i}^{i1}\right)\right\}$ and $T_i = \emptyset$. Go to Step 2.2.

**Step 2.2:** Put $x = x^k, y = y^{ik}$, and $j = 1$. Adopt the simplex method to solve the problem (6.31):

$$\min\{f_{ij}^{(3)}\left(x, y, z_{ij}\right) | z_{ij} \in S_{ij}(x, y)\}. \tag{6.31}$$

and obtain the optimal solution $\tilde{z}_{ij}$.

**Step 2.3:** If $\tilde{z}_{ij} \neq z_{ij}^{ik_i}$, go to Step 2.4. If $\tilde{z}_{ij} = z_{ij}^{ik_i}$ and $j \neq m_i$, set $j = j + 1$ and go to Step 2.2. If $\tilde{z}_{ij} = z_{ij}^{ik_i}$ and $j = m_i$, stop the subroutine, $K_i^* = k_i$ and go to Step 2 with $\left(\hat{y}, \hat{z}_{i1}, \dots, \hat{z}_{im_i}\right) = (y^{ik_i}, z_{i1}^{ik_i}, \dots, z_{im_i}^{ik_i})$.

**Step 2.4:** Let $W_{k_i}$ denote the set of adjacent vertices of $\left(y^{ik_i}, z_{i1}^{ik_i}, \dots, z_{im_i}^{ik_i}\right)$ such that $\left(y, z_{i1}, \dots, z_{im_i}\right) \in W_{k_i}$ implies

$$\{f_i^{(2)}(x^k, y, z_{i1}, \dots, z_{im_i}) \geq f_i^{(2)}(x^k, y^{ik_i}, z_{i1}^{ik_i}, \dots, z_{im_i}^{ik_i}).$$

Let $T_i = T_i \cup \{\left(y^{ik_i}, z_{i1}^{ik_i}, \dots, z_{im_i}^{ik_i}\right)\}$ and $W_i^{'} = W_i^{'} \cup W_{k_i}/T_i$. Go to Step 2.5.

**Step 2.5:** Set $k_i = k_i + 1$ and choose $(y^{ik_i}, z_{i1}^{ik_i}, \dots, z_{im_i}^{ik_i})$ such that

$$f_i^{(2)}(x^k, y^{ik_i}, z_{i1}^{ik_i}, \dots, z_{im_i}^{ik_i}) = \min\{f_i^{(2)}(x^k, y, z_{i1}, \dots, z_{im_i}) | \left(y, z_{i1}, \dots, z_{im_i}\right) \in W_i^{'}\}.$$

Go to Step 2.2.

**Step 3:** If $\left(\hat{y}, \hat{z}_{i1}, \dots, \hat{z}_{im_i}\right) \neq \left(y^k, z_{i1}^k, \dots, z_{im_i}^k\right)$, go to Step 4. If $\left(\hat{y}, \hat{z}_{i1}, \dots, \hat{z}_{im_i}\right) = \left(y^k, z_{i1}^k, \dots, z_{im_i}^k\right)$ and $i \neq n$, set $i = i + 1$ and go to Step 2. If $\left(\hat{y}, \hat{z}_{i1}, \dots, \hat{z}_{im_i}\right) = \left(y^k, z_{i1}^k, \dots, z_{im_i}^k\right)$ and $i = n$, stop and $\left(x^k, y^k, z_{11}^k, \dots, z_{1m_1}^k, \dots z_{n1}^k, \dots, z_{nm_n}^k\right)$ is the optimal solution to the TLMF decision problem (6.22) and $K^* = k$.

**Step 4:** Let $W_k$ denote the set of adjacent vertices of $(x^k, y^k, z_{11}^k, \dots, z_{1m_1}^k, \dots z_{n1}^k, \dots, z_{nm_n}^k)$ such that $(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}) \in W_k$ implies

$$f^{(1)}\left(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}\right) \geq f^{(1)}\left(x^k, y^k, z_{11}^k, \dots, z_{1m_1}^k, \dots z_{n1}^k, \dots, z_{nm_n}^k\right).$$

Let $T = T \cup \left\{\left(x^k, y^k, z_{11}^k, \dots, z_{1m_1}^k, \dots z_{n1}^k, \dots, z_{nm_n}^k\right)\right\}$ and $W = W \cup W_k/T$. Go to Step 5.

**Step 5:** Set $k = j + k$ and choose $\left(x^k, y^k, z_{11}^k, \dots, z_{1m_1}^k, \dots z_{n1}^k, \dots, z_{nm_n}^k\right)$ such that

$$f^{(1)}(x^k, y^k, z_{11}^k, \dots, z_{1m_1}^k, \dots z_{n1}^k, \dots, z_{nm_n}^k) =$$
$$= \min_{(x,y,z_{11},\dots,z_{1m_1},\dots,z_{n1},\dots,z_{nm_n}) \in W} f^{(1)}(x, y, z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}).$$

Go to Step 2.
[End]

### *6.7.4 A Numerical Example*

A numerical example is adopted to illustrate how the TLMF *K*th-Best algorithm works.

**Example 6.3** Consider a TLMF decision problem in a linear version shown as follows with $x \in R$, $y \in R$, $z_{ij} \in R$ and $X = \{x | x \geq 0\}$, $Y_i = \{y | y \geq 0\}$, $Z_{ij} = \left\{z_{ij} \middle| z_{ij} \geq 0\right\}$, $i = 1,2, m_i = 2, j = 1, \dots, m_i$.

$$\min_{x \in X} f^{(1)}(x, y, z_{11}, z_{12}, z_{21}, z_{22}) = -1.5x - y + 2z_{11} + z_{12} - z_{21} - 1.5z_{22}$$
$$\text{s.t. } x + y + z_{11} + z_{12} + z_{21} + z_{22} \geq 10,$$
$$x \leq 1.5,$$
$$\min_{y \in Y_1} f_1^{(2)}(x, y, z_{11}, z_{12}) = x + y + z_{11} + z_{12}$$
$$\text{s.t. } x + y + z_{11} + z_{12} \geq 6.5,$$
$$\min_{z_{11} \in Z_{11}} f_{11}^{(3)}(x, y, z_{11}) = x + y + 3z_{11}$$
$$\text{s.t. } x + y + z_{11} \geq 3.5,$$
$$z_{11} \leq 2,$$
$$\min_{z_{12} \in Z_{12}} f_{12}^{(3)}(x, y, z_{12}) = x + y + 2z_{12}$$
$$\text{s.t. } x + y + z_{12} \geq 5,$$
$$z_{12} \leq 4,$$

$$\min_{y \in Y_2} f_2^{(2)}(x, y, z_{21}, z_{22}) = x - y + 2z_{21} + 3z_{22}$$
$$\text{s.t. } x + y + z_{21} + z_{22} \geq 5.5,$$
$$x + y \leq 2,$$
$$\min_{z_{21} \in Z_{21}} f_{21}^{(3)}(x, y, z_{21}) = x + y + 2z_{21}$$
$$\text{s.t. } x + y + z_{21} \geq 3,$$
$$z_{21} \leq 2,$$
$$\min_{z_{22} \in Z_{22}} f_{22}^{(3)}(x, y, z_{22}) = x + y + z_{22}$$
$$\text{s.t. } x + y + z_{22} \geq 4.5,$$
$$z_{22} \leq 3.$$

We can adopt the TLMF *K*th-Best algorithm to solve the linear semi-cooperative decision problem. First, we have to solve a linear programming problem in the format (6.28) of the leader.

**Step 1:** Set $k = 1$ and adopt the simplex method to obtain the optimal solution to the problem (6.28). The optimal solution to (6.28) is $(x^1, y^1, z_{11}^1, z_{12}^1, z_{21}^1, z_{22}^1) =$ $(1.5,0.5,1.5,3,2,3)$ and now $W = \{(1.5,0.5,1.5,3,2,3)\}$ and $T = \emptyset$. Go to Step 2 and iteration 1 will start.

**Step 2:** Put $x = 1.5$ and $i = 1$, and solve the BLMF decision problem in the form of (6.29). We can obtain the optimal solution $(\hat{y}, \hat{z}_{11}, \hat{z}_{12}) = (0.5,1.5,3)$ to (6.29) and go to Step 3.

**Step 3:** Evidently, $\left(\hat{y}, \hat{z}_{i1}, \dots, \hat{z}_{im_i}\right) = \left(y^k, z_{i1}^k, \dots, z_{im_i}^k\right), i = 1$ and $n = 2$, so $i \neq n$, set $i = 2$ and go to Step 2.

**Step 2:** Put $x = 1.5$ and $i = 2$, and solve the BLMF decision problem (6.29). We can obtain the optimal solution $(\hat{y}, \hat{z}_{21}, \hat{z}_{22}) = (0.5,1,2.5)$ to (6.29) and go to Step 3.

**Step 3:** Now, $(\hat{y}, \hat{z}_{i1}, \dots, \hat{z}_{im_i}) \neq (y^k, z_{i1}^k, \dots, z_{im_i}^k)$ and go to Step 4.

**Step 4:** Find the adjacent vertices of $(x^1, y^1, z_{11}^1, z_{12}^1, z_{21}^1, z_{22}^1)$and the set of adjacent vertices $W_1 = \{(0,2,1.5,3,2,3), (1.5,0.5,1.5,3,1,3), (1.5,0.5,1.5,3,2,2.5)\}$, $T = \{(1.5,0.5,1.5,3,2,3)\}, W = \{(0,2,1.5,3,2,3), (1.5,0.5,1.5,3,1,3), (1.5,0.5,1.5,$ $3,2,2.5)\}$. Go to Step 5.

**Step 5:** Set $k = 2$ and choose $(x^2, y^2, z_{11}^2, z_{12}^2, z_{21}^2, z_{22}^2) =$ $(0,2,1.5,3,2,3)$ and go to Step 2. This step means that iteration 1 has stopped and we cannot obtain an optimal solution through the iteration. The next iteration will be then executed.

In this way, we ultimately achieve the optimal solution through seven iterations. The searched vertices and the detailed computing process of iterations 2-7 are shown as Table 6.2.

**Table 6.2** The detailed computing process of the TLMF *K*th-Best algorithm

| Iteration $k$ | $(x^k, y^k, z_{11}^k, z_{12}^k, z_{21}^k, z_{22}^k)$ | $W_k$ | $T$ | $W$ |
|---|---|---|---|---|
| 2 | (0,2,1.5,3,2,3) | {(0,2,1.5,3,1,3), (0,2,1.5,3,2,2.5)} | {(1.5,0.5,1.5,3,1,3), (0,2,1.5,3,2,3)} | {(1.5,0.5,1.5,3,1,3), (1.5,0.5,1.5,3,2,2.5), (0,2,1.5,3,1,3), (0,2,1.5,3,2,2.5)} |
| 3 | (1.5,0.5,1.5,3,2,2.5) | {(1.5,0.5,1.5,3,1,2.5)} | {(1.5,0.5,1.5,3,1,3), (0,2,1.5,3,2,3), (1.5,0.5,1.5,3,2,2.5)} | {(1.5,0.5,1.5,3,1,3), (0,2,1.5,3,1,3), (0,2,1.5,3,2,2.5), (1.5,0.5,1.5,3,1,2.5)} |
| 4 | (1.5,0.5,1.5,3,1,3) | $\emptyset$ | {(1.5,0.5,1.5,3,1,3), (0,2,1.5,3,2,3), (1.5,0.5,1.5,3,2,2.5), (1.5,0.5,1.5,3,1,3)} | {(0,2,1.5,3,1,3), (0,2,1.5,3,2,2.5), (1.5,0.5,1.5,3,1,2.5)} |
| 5 | (0,2,1.5,3,2,2.5) | {(0,2,1.5,3,1,2.5)} | {(1.5,0.5,1.5,3,1,3), (0,2,1.5,3,2,3), (1.5,0.5,1.5,3,2,2.5), (1.5,0.5,1.5,3,1,3), (0,2,1.5,3,2,2.5)} | {(0,2,1.5,3,1,3), (1.5,0.5,1.5,3,1,2.5), (0,2,1.5,3,1,2.5)} |
| 6 | (0,2,1.5,3,1,3) | $\emptyset$ | {(1.5,0.5,1.5,3,1,3), (0,2,1.5,3,2,3), (1.5,0.5,1.5,3,2,2.5), (1.5,0.5,1.5,3,1,3), (0,2,1.5,3,2,2.5), (0,2,1.5,3,1,3)} | {(1.5,0.5,1.5,3,1,2.5), (0,2,1.5,3,1,2.5)} |
| 7 | (1.5,0.5,1.5,3,1,2.5) | | | |

In iteration 7, $(x^7, y^7, z_{11}^7, z_{12}^7, z_{21}^7, z_{22}^7) = (1.5,0.5,1.5,3,1,2.5)$ is the optimal solution to the TLMF decision problem and the objective function values of all decision entities are $f^{(1)} = -1.5, f_1^{(2)} = 6.5, f_2^{(2)} = 10.5, f_{11}^{(3)} = 6.5, f_{12}^{(3)} = 8,$ $f_{21}^{(3)} = 4, f_{22}^{(3)} = 4.5.$

It is worthwhile to note that $W_4 = \emptyset$ and $W_6 = \emptyset$ in Table 1 do not mean that adjacent vertices of $(x^4, y^4, z_{11}^4, z_{12}^4, z_{21}^4, z_{22}^4)$ and $(x^6, y^6, z_{11}^6, z_{12}^6, z_{21}^6, z_{22}^6)$ do not exist but may imply that their adjacent vertices have been found in previous iterations and have been involved in $W$.

The results show that the TLMF *K*th-Best algorithm provides a practical way of solving the proposed TLMF decision problem. However, the computational load of the algorithm may grow steeply with the number of variables and constraints. Therefore, the execution efficiency of the TLMF *K*th-Best algorithm is needed to explore sufficient numeric experiments.

## 6.8 Summary

In a hierarchical organization, interactive decision entities exist within a predominantly hierarchical structure and the execution of decisions is sequential, from the top to the middle and then to the bottom levels. Each entity independently maximizes its own objective, but is affected by the actions of other entities at the same or different levels through externalities. Multiple followers commonly appear in both middle and bottom levels and have various relationships with each other, which results in the complication of this problem.

This chapter presents four main issues in the area: (1) it establishes a TLMF decision framework which identifies64 standard situations and their possible combinations of TLMF decision problems; (2) it develops a DERD approach to effectively model various TLMF decision problems; (3) it gives a general and standard set of models using both DERD and programming modeling approaches, as well as hybrid TLMF decision models; (4) it presents solution concepts, theoretical properties and related algorithms for a TLMF decision problem.