# Controllability Analysis of Two-dimensional Systems Using 1D Approaches

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***Abstract*—Working with the 1D form of 2D systems is an alternative strategy to reduce the inherent complexity of 2D systems and their applications. To achieve the 1D form of 2D systems, different from the so-called WAM model, a new row (column) process was proposed recently. The controllability analysis of this new 1D form is explored in this paper. Two new notions of controllability named WAM-controllability and directional controllability for the underlying 2D systems are defined. Corresponding conditions on the WAM-controllability and directional controllability are derived, which are particularly useful for the control problems of 2D systems via 1D framework. According to the presented directional controllability, a directional minimum energy control input is derived for 2D systems. A numerical example demonstrates the applicability of the analysis presented in this note.**

***Index Terms*—2D systems, controllability analysis, local controllability, directional controllability, first FM model, WAM model.**

## I. Introduction

Multidimensional linear systems and in particular 2D systems have attracted much attention since 1970s; see [1] - [2] for two-dimensional linear models. In 1972, *Givone* and *Roesser*, for the first time, introduced a state-space model for a linear iterative circuit which is studied as a spatial system rather than a temporal system [1], [3]. This state-space model is then referred to as GR model. *Fornasini* and *Marchesini* proposed a different state-space realization for the 2D digital filters [4], known as the first FM model. Later in [2], they proposed a new state-space form which is the first-order difference equation and sometimes is called second FM model. Since then, multidimensional systems, especially 2D systems, have been studied in many aspects and in many applications.

Broadly speaking, the time domain analysis, such as controllability, reachability and observability of 2D systems, has been done by various researchers, resulting in a number of new notions such as local, global and causal controllability (reachability) [1] - [5]. Necessary and sufficient conditions for the exact reconstructibility of the state of the second FM model have been presented in [5]. Other necessary and sufficient conditions with respect to 2D matrix polynomial equations for the local controllability and the causal reconstructibility of 2D linear systems are proposed in [6]. Reference [7] extends notions of the local controllability, reachability, and reconstructibility for the general singular model of 2D linear systems.

Another strategy to work with 2D systems is to transfer them to a 1D form. Wave advance model (WAM) is a 1D form of 2D systems established in [8]. From the view point of WAM model, 2D systems are considered as advanced waves and consequently the original stationary 2D system is converted to a time-varying 1D system. Moreover, the system matrices are in rectangular form rather than square form. As a result, the major drawback of this 1D form of 2D systems is the varying dimensions of the defined state vectors. This means that the results developed using this framework are most likely computationally unattractive in terms of possible applications. Motivated by this issue and using stacking vectors, a new approach to converting 2D systems to a 1D form is proposed in [9]. Specifically, in [9], rather than using WAM model, a row (column) processing method is used. Row (column) processing means that the 2D variables which are in the same rows (columns) are used to form 1D stacking vectors. Consequently, the states, inputs and outputs of the obtained 1D system are in the vector form, and more importantly their dimensions are invariant. This framework is basically useful for a class of 2D linear systems in which information propagation in one of the two distinct directions only occurs over a finite horizon. This can be the case of a repetitive process [10] or any inherently 2D system, for instance, Darboux equation [11]. As an illustration, the discrete form of Darboux equation which describes the dynamical processes such as gas absorption, is a first FM model which has a finite propagation over the space direction.

In this paper, firstly, the controllability analysis of WAM model of the first FM model is studied, and a necessary condition for the controllability of this 1D model is given. It should be noted that finding the sufficient condition for the controllability of the WAM model is hard. This fact in addition to the time-varying form of WAM model limits the applicability of WAM model of 2D systems. This prompts us to exploit the row (column) process for converting 2D systems to their 1D models instead. On the other hand, during the procedure of designing the sliding surface in [9], it is assumed that the obtained 1D system is controllable; see e.g. [12] for the similar treatment. But, the controllability of the obtained 1D form and its relation to the original 2D system is an unanswered problem in [9]. Hence, motivated by these issues, in this paper, we focus on the controllability analysis of the proposed 1D form of the underlying 2D systems. Based on the controllability analysis, a new notion, *directional controllability*, for the underlying 2D systems is introduced and studied. More importantly, a necessary and sufficient condition for the directional controllability of 2D systems is presented in this paper.

Also, there is a strong connection between controllability and the theory of minimal realization of linear time-invariant (LTI) control systems. Hence, the controllability result of this paper would also provide useful insight into the observability and realization analysis of the underlying 2D systems using the developed 1D framework. Furthermore, note that the so-called minimum energy control problem is explicitly connected with controllability analysis [13]. Therefore, one application of the presented controllability analysis is the design of a specific 1D minimum energy control input for 2D systems called *directional minimum energy control input*. It should be noted that the results of this paper are particularly useful for those who want to control the 2D systems via the proposed 1D framework.

The rest of this paper is as follows. In the next section, the 1D WAM model of first FM systems and its controllability analysis are presented. Section III gives the procedure of our new proposed 1D model of 2D systems. The controllability analysis of this 1D model is represented in Section IV. Besides, a numerical example is given in Section V. Finally, Section VI concludes this paper.

## II. WAM model of First FM Model

Consider the first FM model with the following formulation,

$$x(i+1,j+1) = A_1x(i+1,j)+A_2x(i,j+1)+A_0x(i,j)+Bu(i,j), \qquad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are respectively local state and control input, $A_1 \in \mathbb{R}^{n\times n}$, $A_2 \in \mathbb{R}^{n\times n}$, $A_0 \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

In this section, a brief review on the WAM model [8] of the first FM model (1) is given. Then, the drawbacks of this method are explained, which motivate us to investigate an alternative 1D form for the 2D systems in Section III which is more effective.

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Define the state vectors $\phi(k)$ and $v(k)$ as

$$\begin{aligned}\phi(k) &= \begin{bmatrix} x^T(k,0) & x^T(k-1,1) & \cdots & x^T(0,k)\end{bmatrix}^T,\\ v(k) &= \begin{bmatrix} u^T(k,0) & u^T(k-1,1) & \cdots & u^T(0,k)\end{bmatrix}^T.\end{aligned} \tag{2}$$

The resulting WAM form of the first FM model (1) is as

$$\phi(k+1) = M(k)\phi(k) + N(k-1)\phi(k-1) + E(k-1)v(k-1). \tag{3}$$

Here, $M(k)$, $N(k-1)$ and $E(k-1)$ are determined by

$$\begin{aligned}M(k) &= \begin{bmatrix} I_{k+1} \\ 0_{1\times(k+1)}\end{bmatrix} \otimes A_2 + \begin{bmatrix} 0_{1\times(k+1)} \\ I_{k+1}\end{bmatrix} \otimes A_1,\\ N(k-1) &= T(k)\otimes A_0, \quad E(k-1) = T(k)\otimes B,\end{aligned} \tag{4}$$

where

$$T(k) = \begin{bmatrix} 0^T_{1\times k} & I_k & 0^T_{1\times k}\end{bmatrix}^T, \tag{5}$$

and $I_k$ is the identity matrix of order $k$. Defining

$$r(k) = N(k-1)\phi(k-1) + E(k-1)v(k-1), \tag{6}$$

a 1D state space model is obtained as

$$\begin{bmatrix}\phi(k+1)\\ r(k+1)\end{bmatrix} = \begin{bmatrix} M(k) & I \\ N(k) & 0\end{bmatrix}\begin{bmatrix}\phi(k)\\ r(k)\end{bmatrix} + \begin{bmatrix} 0 \\ E(k)\end{bmatrix} v(k). \tag{7}$$

*Remark 1:* The state vector in (7) is a linear combination of the local states and inputs. However, in some applications, having state space equations with direct access to the local states is required. In this case, by introducing a new state vector,

$$\begin{aligned}\hat{\phi}(k) =& [x^T(k,0), x^T(k,1), x^T(k-1,1), x^T(k-1,2),\\ & \cdots, x^T(1,k-1), x^T(1,k), x^T(0,k)]^T,\end{aligned} \tag{8}$$

a 1D state space equation with direct access to the state vectors $\phi(k)$ and $\phi(k+1)$ is acquired.

*Remark 2:* In the definition of state vectors (8), instead of using the local states just on the line $i+j=k+1$, the local states located on the line $i+j=k$ are also used to form state vectors. Generally, for WAM description of 2D systems which are of at least second order, using the state vector (8) is useful. However, obtaining WAM method for second order 2D systems (for instance FM model) and especially for large scale 2D systems is complicated and, more importantly, the dimension of the state vector (8) is varying.

*Remark 3:* In the case that the boundary conditions are assumed to be constant, the state vector (8) should get rid of the boundary condition terms $x(k,0)$ and $x(0,k)$ as

$$\begin{aligned}\bar{\phi}(k) =& [x^T(k,1), x^T(k-1,1), x^T(k-1,2),\\ & \cdots, x^T(1,k-1), x^T(1,k)]^T \in \mathbb{R}^{[(2k-1)\cdot n]}, \quad \forall k \geq 1.\end{aligned} \tag{9}$$

Hence, the 1D model is as follows

$$\bar{\phi}(k+1) = \bar{M}(k)\bar{\phi}(k) + \bar{E}(k)v(k) + \bar{V}(k), \tag{10}$$

where

$$\bar{M}(k) = \begin{bmatrix} A_2 & 0 & 0 & 0 & 0 & \cdots & 0\\ I & 0 & 0 & 0 & 0 & \cdots & 0\\ A_1 & A_0 & A_2 & 0 & 0 & \cdots & 0\\ 0 & 0 & I & 0 & 0 & \cdots & 0\\ 0 & 0 & A_1 & A_0 & A_2 & \cdots & 0\\ \vdots & \vdots & \vdots & & & \cdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & I\\ 0 & 0 & 0 & 0 & 0 & \cdots & A_1\end{bmatrix}, \tag{11}$$

$$\bar{E}(k) = \begin{bmatrix} B & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ 0 & B & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & B\end{bmatrix}, \bar{V}(k) = \begin{bmatrix} A_1x(k+1,0) + A_0x(k,0)\\ 0\\ \vdots\\ 0\\ A_2x(0,k+1) + A_0x(0,k)\end{bmatrix}, \ \forall k \geq 1,$$

and, $\bar{M}(k) \in \mathbb{R}^{[(2k+1)\cdot n]\times[(2k-1)\cdot n]}$, $\bar{E}(k) \in \mathbb{R}^{[(2k+1)\cdot n]\times[(k+1)\cdot m]}$, $\bar{V}(k) \in \mathbb{R}^{(2k+1)\cdot n}$, for all $k \geq 1$. Note that, here, $\bar{M}(0) = A_2$, $\bar{E}(0) = B$, $\bar{V}(0) = A_1x(1,0) + A_0x(0,0)$ and $\bar{\phi}(0) = x(0,1)$.

### A. Controllability analysis of WAM model

This subsection aims to analyze the controllability of the 1D WAM model presented in (10). To this end, define the so-called state transition matrix $A^{i,j}$ as

$$\begin{aligned}A^{i,j} &= A_0A^{i-1,j-1} + A_1A^{i,j-1} + A_2A^{i-1,j}\\ &= A^{i-1,j-1}A_0 + A^{i,j-1}A_1 + A^{i-1,j}A_2, \quad \forall i,j > 0.\end{aligned} \tag{12}$$

Furthermore, it is assumed that

$$A^{0,0} = I_n,\ A^{-i,j} = A^{i,-j} = A^{-i,-j} = 0, \quad \forall i,j > 0. \tag{13}$$

Now, from (10) and with some recursive manipulations, we have

$$\bar{\phi}(k+1) - \hat{\mathscr{C}}_w(k)\mathscr{V}_w(k) - \prod_{i=0}^{k}\bar{M}(i)\bar{\phi}(0) = \mathscr{C}_w(k)\mathscr{U}_w(k), \tag{14}$$

where

$$\begin{aligned}\mathscr{C}_w(k) &= [\{\prod_{i=1}^{k}\bar{M}(i)\}\bar{E}(0) \mid \{\prod_{i=2}^{k}\bar{M}(i)\}\bar{E}(1) \mid \cdots \mid \bar{M}(k)\bar{E}(k-1) \mid \bar{E}(k)],\\ \hat{\mathscr{C}}_w(k) &= [\prod_{i=1}^{k}\bar{M}(i) \mid \prod_{i=2}^{k}\bar{M}(i) \mid \cdots \mid \bar{M}(k) \mid I_{\{(2k+1)\cdot n\}}],\\ \mathscr{U}_w(k) &= [v^T(0) \mid v^T(1) \mid \cdots \mid v^T(k-1) \mid v^T(k)],\\ \mathscr{V}_w(k) &= [\bar{V}^T(0) \mid \bar{V}^T(1) \mid \cdots \mid \bar{V}^T(k-1) \mid \bar{V}^T(k)].\end{aligned} \tag{15}$$

As $\mathscr{V}_w(k)$ is determined by boundary conditions only (not a function of control), we neglect the second item of the equation (14) during the controllability analysis.

*Theorem 1:* The 1D WAM model (10) is not controllable unless $B$ is of full row rank.

*Proof:* Matrix $\mathscr{C}_w(k)$ in (15) can be found to be as (16). Left multiplying this matrix by

$$\mathfrak{L}(k) = \begin{bmatrix} I_n & -A_2 & 0 & \cdots & 0\\ 0 & I_n & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & I_n\end{bmatrix}, \tag{17}$$

where $\mathfrak{L}(k) \in \mathbb{R}^{[(2k+1)\cdot n]\times[(2k+1)\cdot n]}$, it is obtained that

$$\mathfrak{L}(k)\mathscr{C}_w(k) = \begin{bmatrix} 0 & 0 & \cdots & 0 & B & 0 & \cdots & 0\\ A^{k-1,0}B & A^{k-2,0}B & \cdots & 0 & 0 & 0 & \cdots & 0\\ * & * & \cdots & * & * & * & \cdots & *\\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ * & * & \cdots & * & * & * & \cdots & *\end{bmatrix}. \tag{18}$$

where $\{*\}$ means irrelevant entries. Note that since $\mathfrak{L}(k)$ is invertible, it does not change the row rank of the obtained matrix $\mathfrak{L}(k)\mathscr{C}_w(k)$ compared to $\mathscr{C}_w(k)$. Clearly, if $B$ is not of full row rank, then, $\mathfrak{L}(k)\mathscr{C}_w(k)$, and consequently, $\mathscr{C}_w(k)$ is not of full row rank. Thus, the WAM model (10) is not controllable. ■

From Theorem 1, it can be seen that the necessary condition for the controllability of WAM model (10) is that $B$ has full row rank. However, this condition is very restrictive. As mentioned in Remark 2, in order to construct the state vector $\bar{\phi}(k)$, the local states on the line $i+j=k+1$, and $i+j=k$ are both used. In other words, the even elements of the state vector $\bar{\phi}(\cdot)$ are carried elements from the previous step and only local states on the line $i+j=k+1$ have new information. Besides, the local states on the line $i+j=k+1$ will cover the whole space when $k$ increases. As a result of this fact, the even block rows of the matrix $\mathscr{C}_w(k)$ are removed and the

$$
\mathscr{C}_w(k) = \left[\begin{array}{c|cc|c|cccc|ccccc}
A^{k,0}B & A^{k-1,0}B & 0 & \cdots & A^{1,0}B & \cdots & 0 & 0 & B & 0 & \cdots & 0 & 0 \\
A^{k-1,0}B & A^{k-2,0}B & 0 & \cdots & B & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
A^{k-1,1}B & A^{k-2,1}B & A^{k-1,0}B & \cdots & A^{0,1}B & \cdots & 0 & 0 & 0 & B & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
A^{1,k-1}B & A^{0,k-1}B & A^{1,k-2}B & \cdots & 0 & \cdots & A^{0,1}B & A^{1,0}B & 0 & 0 & \cdots & B & 0 \\
A^{0,k-1}B & 0 & A^{0,k-2}B & \cdots & 0 & \cdots & 0 & B & 0 & 0 & \cdots & 0 & 0 \\
A^{0,k}B & 0 & A^{0,k-1}B & \cdots & 0 & \cdots & 0 & A^{0,1}B & 0 & 0 & \cdots & 0 & B
\end{array}\right]. \tag{16}
$$

remaining matrix can be written as $\bar{\mathscr{C}_w}(k)$ in (20), which will be used to determine WAM-controllability defined below. This is equivalent to the output controllability with the following WAM output matrix for the system (10),

$$
C_w(k) = \begin{bmatrix} I_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \end{bmatrix} \tag{19}
$$

where $C_w(k) \in \mathbb{R}^{[k \cdot n] \times [(2k-1) \cdot n]}$. This equivalence to the special output controllability is in particular useful to the control/tracking problems for the 1D WAM model of the form (10), however it is beyond the scope of this paper.

*Definition 1:* The 2D system in (1) is said to be *WAM-controllable* if there exists a $k \geq k^+ = \lceil \frac{2n}{m} - 2 \rceil \triangleq \min\{k \in \mathbb{N} \mid k \geq \frac{2n}{m} - 2\}$ such that $rank(\bar{\mathscr{C}_w}(k) \cdot \bar{\mathscr{C}_w^T}(k)) = (k+1) \cdot n$.

*Remark 4:* It can be seen that $\bar{\mathscr{C}_w}(k) \in \mathbb{R}^{[(k+1) \cdot n] \times [\frac{(k+1)(k+2)}{2} \cdot m]}$. Besides, in the above definition, the condition $k \geq \frac{2n}{m} - 2$ is arising from the fact that the number of columns of matrix $\bar{\mathscr{C}_w}(k)$ is greater than or equal to the number of its rows if $k \geq \frac{2n}{m} - 2$.
As $\bar{\mathscr{C}_w}(k)$ and its dimension are time-varying, one may ask about the future step's WAM-controllability even if the system (1) is WAM-controllable at the step $k$. Proposition 1, in the following, confirms the WAM-controllability for all the future steps, thus, validating the definition of WAM-controllability in Definition 1. Before it, consider the following lemma which provides a necessary condition for the WAM-controllability.

*Lemma 1:* If the system (1) is WAM-controllable, then the pairs ($A_1$, $B$) and ($A_2$, $B$) are both controllable.

*Proof:* It is obvious that the nonzero blocks of the first and the last block rows of the matrix $\bar{\mathscr{C}_w}(k)$ are equivalent to the $(k+1)$-th step controllability matrices of ($A_2$, $B$) and ($A_1$, $B$), respectively. If either one is not controllable $\bar{\mathscr{C}_w}(k)$ is not of full row rank. Hence, system (1) is not WAM-controllable. ∎

*Proposition 1:* If $\bar{\mathscr{C}_w}(k)$ is of full row rank for any $k \geq k^+ = \lceil \frac{2n}{m} - 2 \rceil$, $\bar{\mathscr{C}_w}(k_1)$ is of full row rank for any $k_1 > k$.

*Proof:* The matrix $\bar{\mathscr{C}_w}(k+1)$ can be rearranged by some column permutation operations (without changing the row rank) as

$$
\left[\begin{array}{ccccc|c}
A^{k+1,0}B & A^{k,0}B & \cdots & A^{1,0}B & B & 0 \\ \cline{1-5}
A^{k,1}B & A^{k-1,1}B & \cdots & A^{0,1}B & 0 & \\
\vdots & \vdots & & \vdots & \vdots & \bar{\mathscr{C}_w}(k) \\
A^{1,k}B & A^{0,k}B & \cdots & 0 & 0 & \\
A^{0,k+1}B & 0 & \cdots & 0 & 0 &
\end{array}\right]. \tag{21}
$$

From (21), it can be seen that if $\bar{\mathscr{C}_w}(k)$ has full row rank, necessarily, the controllability matrix of the pair ($A_2$, $B$) is of full row rank in $(k+1)$-th step from Lemma 1. Therefore, this pair is controllable in $(k+2)$-th step as well. Since the non-zero elements of the first block row of (21) contains the controllability matrix of the pair ($A_2$, $B$) in $(k+2)$-th step, $\bar{\mathscr{C}_w}(k+1)$ is of full row rank. This can be simply extended to the general case of $\bar{\mathscr{C}_w}(k+r)$, $r \geq 1$. ∎

The next result characterizes the WAM-controllability condition in terms of the original system matrices; if in particular $n = 2$, $m = 1$ and thus $\bar{\mathscr{C}_w}(2) \in \mathbb{R}^{6 \times 6}$, we would conclude the necessary and sufficient condition on the full rank of $\bar{\mathscr{C}_w}(2)$.

*Theorem 2:* If $n = 2$, $m = 1$, the matrix $\bar{\mathscr{C}_w}(2)$ is of full row rank if and only if the three pairs ($A_1$, $B$), ($A_2$, $B$) and ($A_0$, $B$) are controllable.

$$
\bar{\mathscr{C}_w}(2) = \left[\begin{array}{c|cc|ccc}
A_2^2 B & A_2 B & 0 & B & 0 & 0 \\
(A_1 A_2 + A_2 A_1 + A_0)B & A_1 B & A_2 B & 0 & B & 0 \\
A_1^2 B & 0 & A_1 B & 0 & 0 & B
\end{array}\right]. \tag{22}
$$

*Proof:* Let $A_3 = A_1 + A_2$ and $\alpha_i$, $\beta_i$ be the scalar coefficients of the characteristic polynomial of $A_i$, satisfying $A_i^2 + \alpha_i A_i + \beta_i I = 0$, for $i = 1, 2, 3$. By noting $trace(A_i) = -\alpha_i$, it follows $\alpha_3 = \alpha_1 + \alpha_2$ for $n = 2$. Then

$$
\begin{aligned}
A_1 A_2 + A_2 A_1 =& (A_1 + A_2)^2 - A_1^2 - A_2^2 \\
=& -\alpha_3 (A_1 + A_2) - \beta_3 I + \alpha_1 A_1 + \beta_1 I + \alpha_2 A_2 + \beta_2 I \\
=& -\alpha_2 A_1 - \alpha_1 A_2 + (\beta_1 + \beta_2 - \beta_3) I.
\end{aligned}
$$

As a result, the matrix in (22) can be rewritten as

$$
\bar{\mathscr{C}_w}(2) = \tag{23}
$$
$$
\left[\begin{array}{c|cc|ccc}
-(\alpha_2 A_2 + \beta_2 I)B & A_2 B & 0 & B & 0 & 0 \\
\{-\alpha_2 A_1 - \alpha_1 A_2 + (\beta_1 + \beta_2 - \beta_3) I + A_0\}B & A_1 B & A_2 B & 0 & B & 0 \\
-(\alpha_1 A_1 + \beta_1 I)B & 0 & A_1 B & 0 & 0 & B
\end{array}\right].
$$

With some elementary column operations on $\bar{\mathscr{C}_w}(2)$, $\text{col}_1 + \alpha_2 \text{col}_2 + \alpha_1 \text{col}_3 + \beta_2 \text{col}_4 + \beta_1 \text{col}_6 + (\beta_1 + \beta_2 - \beta_3)\text{col}_5$, ($\text{col}_i$ is the $i$-th block column of the matrix in (23)), one can change $\bar{\mathscr{C}_w}(2)$ to

$$
\left[\begin{array}{c|cc|ccc}
0 & A_2 B & 0 & B & 0 & 0 \\
A_0 B & A_1 B & A_2 B & 0 & B & 0 \\
0 & 0 & A_1 B & 0 & 0 & B
\end{array}\right]. \tag{24}
$$

With some row and column permutations (24) is converted to

$$
\left[\begin{array}{cc|cc|cc}
A_0 B & B & A_1 B & 0 & A_2 B & 0 \\ \hline
0 & 0 & A_2 B & B & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & A_1 B & B
\end{array}\right]. \tag{25}
$$

It can be realized that all the rows of the above matrix are linearly independent if and only if the pairs ($A_1$, $B$), ($A_2$, $B$) and ($A_0$, $B$) are controllable. Note that, here, we use the fact that $\bar{\mathscr{C}_w}(2)$ is a square matrix and the row rank is equivalent to the column rank. ∎

Theorem 2 provides a necessary and sufficient condition for the WAM-controllability of the special case $n = 2$, $m = 1$. As for the general case, while Lemma 1 presents the necessary condition for the controllability of the WAM model, finding its sufficient condition is hard and this can be the subject of future works. In Section III, the 1D model of 2D systems will be obtained from column (row) process. It will be shown, in Section IV, that the necessary and sufficient condition for the controllability of this 1D model, referred to as *directional controllability*, is only the controllability of one of the two pairs ($A_1$, $B$) and ($A_2$, $B$). The pair ($A_0$, $B$) would have no influence on the directional controllability.

$$
\bar{\mathscr{C}}_w(k) = \left[\begin{array}{c|cc|c|cccc|ccccc}
A^{k,0}B & A^{k-1,0}B & 0 & \cdots & A^{1,0}B & \cdots & 0 & 0 & B & 0 & \cdots & 0 & 0 \\
A^{k-1,1}B & A^{k-2,1}B & A^{k-1,0}B & \cdots & A^{0,1}B & \cdots & 0 & 0 & 0 & B & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
A^{1,k-1}B & A^{0,k-1}B & A^{1,k-2}B & \cdots & 0 & \cdots & A^{0,1}B & A^{1,0}B & 0 & 0 & \cdots & B & 0 \\
A^{0,k}B & 0 & A^{0,k-1}B & \cdots & 0 & \cdots & 0 & A^{0,1}B & 0 & 0 & \cdots & 0 & B
\end{array}\right]. \tag{20}
$$

## III. New 1D Form of 2D first FM model

In this section, our proposed 1D form of the first FM model in [9] is reviewed.

The FM model (1) can be represented in the following form,

$$
x(i+1,j+1) - A_1 x(i+1,j) = A_2 x(i,j+1) + A_0 x(i,j) + Bu(i,j). \tag{26}
$$

*Assumption 1:* In what follows, it is assumed that the $\{j\}$-direction of the 2D system in (1) has a finite horizon, $j = 0,1,\cdots,v$.
Now, we define the following stacking vectors

$$
V(i) = \begin{bmatrix} A_1 x(i+1,0) + A_0 x(i,0) \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ X(i) = \begin{bmatrix} x(i,1) \\ x(i,2) \\ \vdots \\ x(i,v) \end{bmatrix}, U(i) = \begin{bmatrix} u(i,0) \\ u(i,1) \\ \vdots \\ u(i,v-1) \end{bmatrix}, \tag{27}
$$

where $X(i) \in \mathbb{R}^{v \cdot n}$, $V(i) \in \mathbb{R}^{v \cdot n}$ and $U(i) \in \mathbb{R}^{v \cdot m}$. The 2D system (26) can be presented as

$$
JX(i+1) = KX(i) + LU(i) + V(i), \tag{28}
$$

where

$$
\begin{aligned}
J &= I_v \otimes I_n + \begin{bmatrix} 0_{1\times(v-1)} & 0 \\ I_{v-1} & 0_{(v-1)\times 1} \end{bmatrix} \otimes (-A_1), \\
K &= I_v \otimes A_2 + \begin{bmatrix} 0_{1\times(v-1)} & 0 \\ I_{v-1} & 0_{(v-1)\times 1} \end{bmatrix} \otimes A_0,\ L = I_v \otimes B.
\end{aligned} \tag{29}
$$

Here, $x(i+1,0)$ and $x(i,0)$ are state boundary conditions. As seen in (27), the variable $\{j\}$ is hidden in the new defined 1D form.

To have the standard form of a 1D discrete system, left multiply both sides of (28) by $J^{-1}$ to obtain

$$
\Sigma_v \ : X(i+1) = \hat{K}X(i) + \hat{L}U(i) + \hat{R}V(i). \tag{30}
$$

where $\hat{K} = J^{-1}K,\ \hat{L} = J^{-1}L,$ and $\hat{R} = J^{-1}$. Note that a numerical algorithm is given in [9] to compute $J^{-1}$ explicitly. It can be found that $\hat{K}$ and $\hat{L}$ are block lower triangular matrices as

$$
\begin{aligned}
\hat{K} &= \begin{bmatrix} A_2 & 0 & \cdots & 0 & 0 \\ A_1 A_2 + A_0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A_1^{v-1} A_2 + A_1^{v-2} A_0 & A_1^{v-2} A_2 + A_1^{v-3} A_0 & \cdots & A_1 A_2 + A_0 & A_2 \end{bmatrix}, \\
\hat{L} &= \begin{bmatrix} B & 0 & \cdots & 0 & 0 \\ A_1 B & B & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A_1^{v-1} B & A_1^{v-2} B & \cdots & A_1 B & B \end{bmatrix}.
\end{aligned} \tag{31}
$$

As seen, in this new 1D form, the dimension of the state vectors is invariant.

*Remark 5:* In this paper, it is assumed that one of the distinct variables of 2D system is finite. Moreover, the computing limitation has made it inevitable to assume a finite dimension for the other direction of 2D systems. Consequently, in this paper, the dimension of considered 2D system is assumed to be $\mu \times v$ and, as a result, the sizes of $X(i)$ and $U(i)$ in (30) are $v \cdot n$ and $v \cdot m$, respectively. Besides, there are two set of boundary conditions,

$$
\begin{cases} \alpha(i) = x(i,0) & over\ j = 0, \\ \beta(j) = x(0,j) & over\ i = 0. \end{cases} \tag{32}
$$

## IV. Controllability analysis

In this section, different controllability notions of the 2D system (1) are considered, namely, *local controllability* and *directional controllability*. This is achieved by studying the relation between the controllability of the obtained 1D system (30) and that of the 2D system (1).

### A. Notion of local controllability for 2D systems

Different definitions of controllability according to different types of dynamical systems can be found in the literature. Broadly speaking, considering the controllability of 2D systems is relatively more complex compared to 1D systems. Instead of notion of controllability introduced for 1D discrete-time systems, notion of local controllability (reachability) is developed for 2D systems [14]. Here, the controllability of the first FM model (1) is studied referring to [4] and [14].

With the boundary conditions (32) and the given admissible controls sequence, it can be shown

$$
\begin{aligned}
x(i,j) =& A^{i-1,j-1} A_0 x(0,0) \\
&+ \sum_{p=1}^{i} (A^{i-p,j-1} A_1 + A^{i-p-1,j-1} A_0) x(p,0) \\
&+ \sum_{q=1}^{j} (A^{i-1,j-q} A_2 + A^{i-1,j-q-1} A_0) x(0,q) \\
&+ \sum_{p=0}^{i-1} \sum_{q=0}^{j-1} A^{i-p-1,j-q-1} Bu(p,q),
\end{aligned} \tag{33}
$$

where the state transition matrix $A^{i,j}$ is as in (12) and (13). From (33), we have

$$
\begin{aligned}
M(i,j) \triangleq& x(i,j) - A^{i-1,j-1} A_0 x(0,0) \\
&- \sum_{p=1}^{i} (A^{i-p,j-1} A_1 + A^{i-p-1,j-1} A_0) x(p,0) \\
&- \sum_{q=1}^{j} (A^{i-1,j-q} A_2 + A^{i-1,j-q-1} A_0) x(0,q) \\
=& \sum_{p=0}^{i-1} \sum_{q=0}^{j-1} A^{i-p-1,j-q-1} Bu(p,q) \\
=& \mathfrak{C}_{ij} u_{ij},
\end{aligned} \tag{34}
$$

where

$$
\mathfrak{C}_{ij} = [A^{i-1,j-1}B, A^{i-1,j-2}B, \cdots, A^{i-1,0}B, \cdots, A^{0,j-1}B, A^{0,j-2}B, \cdots, B], \tag{35}
$$

and

$$
\begin{aligned}
u_{ij} = [&u^T(0,0), u^T(0,1), \cdots, u^T(0,j-1), \cdots, \\
&u^T(i-1,0), u^T(i-1,1), \cdots, u^T(i-1,j-1)]^T.
\end{aligned} \tag{36}
$$

*Definition 2:* Consider the system (1) with the boundary conditions (32). This system is locally controllable in a given rectangle $[(0,0),(\mu,v)]$ if for every boundary conditions (32) and for every vector $x_d \in \mathbb{R}^n$, there exists a sequence of controls $u_{\mu v}$ as in (36) such that $x(\mu,v) = x_d$.

The matrix $\mathfrak{C}_{ij}$ in (35) is known as local controllability matrix.

*Lemma 2 ( [14]):* The system (1) is locally controllable in a given rectangle $[(0,0),(\mu,v)]$ with *unconstrained* control inputs $u$ if and only if $rank(\mathfrak{C}_{\mu v}\cdot\mathfrak{C}_{\mu v}^T)=n$.

Furthermore, it is shown in [4] that Lemma 2 can be confined to the following lemma.

*Lemma 3:* The system (1) is locally controllable in a given rectangle $[(0,0),(\mu,v)]$ with *unconstrained* control inputs $u$ if and only if $rank(\mathfrak{C}_{nn}\cdot\mathfrak{C}_{nn}^T)=n$ where $\mu\geq n$ and $v\geq n$.

It should be mentioned that this lemma is proven in [4] for reachability case. Indeed, a method similar to proving Cayley-Hamilton theorem for 1D systems can be developed for the 2D case.

### B. Directional controllability with respect to $\{j\}$-direction

In this subsection, the controllability of the 1D system in (30) is considered. Moreover, a new notion of controllability for this special form of the 2D system in (1) is defined.

Now, define

$$\mathscr{M}(i)=[M^T(i,1)\cdots M^T(i,v)]^T=[(\mathfrak{C}_{i1}u_{i1})^T\cdots(\mathfrak{C}_{iv}u_{iv})^T]^T. \tag{37}$$

Since $u_{i1},\ldots,u_{i(v-1)}$ are included in $u_{iv}$, (37) can be rewritten as

$$\mathscr{M}(i)=\mathscr{C}_i u_{iv}, \tag{38}$$

where $\mathscr{C}_i$ is the matrix in (39).

*Lemma 4:* The matrix $\mathscr{C}_i$ in (39) satisfies

$$\mathscr{C}_i=\left[\hat{K}^{i-1}\hat{L}\mid\cdots\mid\hat{K}\hat{L}\mid\hat{L}\right], \tag{40}$$

where $\hat{K}$ and $\hat{L}$ have the form in (31).

*Proof:* From (30) it can be demonstrated that

$$X(i)-\hat{K}^iX(0)-\mathscr{C}_{v_i}\mathscr{V}(i)=\mathscr{C}_{u_i}\mathscr{U}(i), \tag{41}$$

where

$$\begin{aligned}&\mathscr{V}(i)=[V^T(0)\cdots V^T(i-1)]^T,\mathscr{U}(i)=[U^T(0)\cdots U^T(i-1)]^T,\\&\mathscr{C}_{u_i}=\left[\hat{K}^{i-1}\hat{L}\mid\cdots\mid\hat{K}\hat{L}\mid\hat{L}\right],\mathscr{C}_{v_i}=\left[\hat{K}^{i-1}\hat{R}\mid\cdots\mid\hat{K}\hat{R}\mid\hat{R}\right],\end{aligned} \tag{42}$$

and $U(\cdot),V(\cdot)$ are defined in (27). Noting that $\mathscr{U}(i)=u_{iv}$ and comparing (41) with (38), we can conclude (40) as $\mathscr{C}_i=\mathscr{C}_{u_i}$. ■

As $X(0)$ and $\mathscr{V}(i)$ are determined by the boundary and initial conditions, we only need to check $\mathscr{C}_i$ to analyze the controllability of system (41). As seen, the matrix $\mathscr{C}_i$ has the form of the controllability matrix of the 1D system (30), hence, the controllability of the 1D system (30) can be analyzed by checking the rank of this matrix. Furthermore, in the sequel it is shown that the matrix $\mathscr{C}_i$ in (39) has more to do with the local controllability of the 2D system in (1). Note that, in the sequel of this paper, it is assumed that $\mu\geq n$ and $v\geq n$, without loss of generality.

*Lemma 5:* The system (30) is controllable at the $k$-th ($k=1,\cdots,\mu$) step with *unconstrained* control inputs $U$, if and only if $rank(\mathscr{C}_k\cdot\mathscr{C}_k^T)=v\cdot n$.

*Proof:* From Lemma 4, the $k$-th step controllability matrix of (30) is equivalent to $\mathscr{C}_k$. Hence, this system is controllable if and only if $\mathscr{C}_k$ has full row rank. ■

Moreover, in the following theorem it will be shown that when $\mu\geq n$, $v\geq n$ and $\mathscr{C}_\mu$ is of full row rank, the local controllability matrix $\mathfrak{C}_{nn}$, and hence, $\mathfrak{C}_{\mu v}$ will be of full row rank. However, the converse of this issue is not always true.

*Theorem 3:* The local controllability matrix $\mathfrak{C}_{nn}$ has full row rank if the matrix $\mathscr{C}_\mu$ has full row rank where $\mu\geq n$ and $v\geq n$.

*Proof:* $\mathscr{C}_\mu$ has $v$ block rows with each block having the dimension $\{n\times(\mu\cdot v\cdot m)\}$. It is not hard to show that the nonzero blocks of the $n$-th block row of $\mathscr{C}_\mu$ is equivalent to the controllability matrix $\mathfrak{C}_{\mu n}$. Hence, if $\mathscr{C}_\mu$ has full row rank, $\mathfrak{C}_{\mu n}$ and thus $\mathfrak{C}_{\mu v}$ has full row rank. From Lemma 2, the 2D system (1) is locally controllable in a given rectangle $[(0,0),(\mu,v)]$. According to Lemma 3, it can be concluded that $\mathfrak{C}_{nn}$ is of full row rank. ■

In other words, whenever the matrix $\mathscr{C}_\mu$ has full row rank the 1D form system (30) is controllable and the 2D system (1) is locally controllable in a given rectangle $[(0,0),(\mu,v)]$ with unconstrained control inputs.

Now comes the main result of this section.

*Theorem 4:* The 1D form (30) of the 2D system (1) is controllable if and only if the matrix pair ($A_2$, $B$) is controllable.

*Proof:* By some column permutations (without changing the row rank) the matrix $\mathscr{C}_n$ is rearranged to the matrix in (43). Obviously, the matrix in (43) is a lower-triangular block matrix and its diagonal blocks are the controllability matrix of the pair ($A_2$, $B$). Therefore, the controllability of ($A_2$, $B$) is equivalent to the controllability of $(\hat{K},\hat{L})$. ■

Here, according to Theorem 4, a new notion of controllability for 2D systems is defined.

*Definition 3:* The 2D system in (1) is said to be directionally controllable with respect to the direction $\{j\}$, if its 1D form $\Sigma_v$ in (30) is controllable.

*Proposition 2:* The 2D system in (1) is directionally controllable with respect to the direction $\{j\}$, if and only if the matrix pair ($A_2$, $B$) is controllable.

*Remark 6:* Basically, the notion of local controllability of 2D systems uses the Kalman-controllability notion and extends it to a more general form for 2D systems. Meantime, the notions of WAM controllability and/or directional controllability defined specifically for the 1D form of the 2D system (1) also exploits the standard Kalman-controllability notion. Note that Theorem 4 provides a sufficient and necessary condition for the controllability of the obtained 1D system (30) which is exactly equivalent to the Kalman-controllability of the matrix pair ($A_2$, $B$).

### C. Directional controllability with respect to $\{i\}$-direction

In the procedure of [9] and this paper, it is assumed that the $\{j\}$-direction is finite, and hence, the local states located in the same $\{j\}$-direction form the 1D stacking vectors. In the case that the $\{i\}$-direction is of finite dimension, the local states located in the same $\{i\}$-direction can be stacked to form the 1D stacking vectors. Similarly, a sufficient and necessary condition of the directional controllability with respect to $\{i\}$-direction can be obtained as follows.

*Proposition 3:* The 2D system in (1) is directionally controllable with respect to the direction $\{i\}$, if and only if the matrix pair ($A_1$, $B$) is controllable.

### D. Directional minimum energy control input

For 1D LTI systems the controllability analysis is strongly related to the so-called minimum energy control problem [13]. In this subsection, a specific minimum energy control input is proposed for 2D systems according to the directional controllability notion given in the previous subsections. This control input will be denoted in this note as the *directional minimum energy control input*.

Suppose that ($A_2$, $B$) is controllable. From Theorem 4 we have the matrix pair ($\hat{K}$, $\hat{L}$) of the system in (30) is controllable with $\mathscr{C}_{i_f}$, the controllability matrix, where $i_f>n$. Let

$$\tilde{\mathscr{U}}(i_f)=-\mathscr{C}_{i_f}^T(\mathscr{C}_{i_f}\mathscr{C}_{i_f}^T)^{-1}[X(i_f)-\hat{K}^{i_f}X(0)-\hat{\mathscr{C}}_{i_f}\mathscr{V}(i_f)], \tag{44}$$

($\tilde{\mathscr{U}}_{i_f}$ is as in (42)), then $\tilde{\mathscr{U}}(i_f)$ has the minimum energy $\left\|\tilde{\mathscr{U}}(i_f)\right\|^2$ among all possible control input sequences which can steer the system state from $X(0)$ to $X(i_f)$ [15]. The control input sequence given in (44) is said to have the directional minimum energy with respect to $\{j\}$-direction.

$$\mathscr{C}_i = \left[\begin{array}{ccccc|c|ccccc} A^{i-1,0}B & 0 & \cdots & 0 & 0 & \cdots & B & 0 & \cdots & 0 & 0 \\ A^{i-1,1}B & A^{i-1,0}B & \cdots & 0 & 0 & \cdots & A^{0,1}B & B & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ A^{i-1,\nu-2}B & A^{i-1,\nu-3}B & \cdots & A^{i-1,0}B & 0 & \cdots & A^{0,\nu-2}B & A^{0,\nu-3}B & \cdots & B & 0 \\ A^{i-1,\nu-1}B & A^{i-1,\nu-2}B & \cdots & A^{i-1,1}B & A^{i-1,0}B & \cdots & A^{0,\nu-1}B & A^{0,\nu-2}B & \cdots & A^{0,1}B & B \end{array}\right]. \tag{39}$$

$$\left[\begin{array}{ccc|ccc|c|ccc} A^{n-1,0}B & \cdots & B & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ A^{n-1,1}B & \cdots & A^{0,1}B & A^{n-1,0}B & \cdots & B & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A^{n-1,\nu-2}B & \cdots & A^{0,\nu-2}B & A^{n-1,\nu-3}B & \cdots & A^{0,\nu-3}B & \cdots & 0 & \cdots & 0 \\ A^{n-1,\nu-1}B & \cdots & A^{0,\nu-1}B & A^{n-1,\nu-2}B & \cdots & A^{0,\nu-2}B & \cdots & A^{n-1,0}B & \cdots & B \end{array}\right]. \tag{43}$$

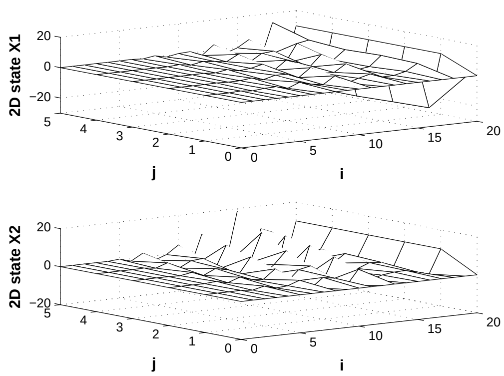


Fig. 1. 2D system state

## V. Numerical Example

Consider the following 2D first FM model

$$A_1 = \begin{bmatrix} -0.56 & 0 \\ 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.33 & -0.54 \\ 0.26 & -0.41 \end{bmatrix},$$
$$A_0 = \begin{bmatrix} 0.51 & -0.09 \\ 0.00 & 0.04 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{45}$$

Here, $x \in \mathbb{R}^2$ and $u \in \mathbb{R}$. We assume this 2D system over the rectangle $\mu \times \nu$ ($\mu = 20$ and $\nu = 5$). It is supposed that $x(0, j) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \; 0 \le j \le 5$, $x(i, 0) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \; 0 \le i \le 20$. It can be seen that despite the uncontrollability of the pair ($A_1$, $B$), and, since the pair ($A_2$, $B$) is controllable, the pair ($\hat{K}$, $\hat{L}$) is controllable. Also, $\mathscr{C}_2$ and $\mathfrak{C}_{22}$ have full row rank ($rank(\mathscr{C}_2) = 6$ and $rank(\mathfrak{C}_{22}) = 2$). As a result, this 2D system can be said to be *directionally controllable with respect to the* $\{j\}$-direction. Note that since $rank(\bar{\mathscr{C}}_w(2)) = 5 < 6$ this system is not WAM controllable in 3*rd* step. Also, since ($A_1$, $B$) is not controllable, from Lemma 1, this system is not WAM controllable in general.

Now, the results of applying the open-loop minimum energy control input sequences in (44), with $i_f = \mu = 20$, $X(0) = \mathbf{0}_{10\times 1}$ and $X(20) = 10 \times \mathbf{1}_{10\times 1}$, to the system (30) are shown in Fig. 1.

## VI. Conclusions

In this paper, a new WAM-controllability notion has been defined for 2D systems and a necessary condition is given accordingly. Then, using a row (column) process, the original 2D system is replaced by a 1D virtual system which can be controlled easily. A necessary and sufficient condition has been derived for the controllability of the newly proposed 1D model. Accordingly a new notion, directional controllability, has been defined for the underlying 2D systems. The directional controllability analysis presented in this work is beneficial in terms of designing the so-called minimum energy control input.

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