Evaluating the Impact of Inequality Constraints and Parameter Uncertainty on Optimal Portfolio Choice

A. D. Hall
University of Technology Sydney
tony.hall@uts.edu.au

S. E. Satchell
Trinity College, Cambridge and University of Sydney
ses999gb@yahoo.co.uk

P. J. Spence*
Sustainable Insight Capital Management, Sydney
Paul.Spence@sicm.com

Abstract
We present new analytical results for the impact of portfolio weight constraints on an investor's optimal portfolio when parameter uncertainty is taken into account. While it is well known that parameter uncertainty and imposing weight constraints results in reduced certainty equivalent returns, in the general case there are no analytical results. In a special case, commonly used in the funds management literature, we derive analytical expression for the certainty equivalent loss that does not depend on the risk aversion parameter. We illustrate our theoretical results using hedge fund data, from the perspective of a fund-of-fund manager. Our contribution is to formalise the framework to investigate this problem, as well as providing tractable analytical solutions that can be implemented using either simulated or asset manager returns.

To appear in *Applied Economics*: DOI: 10.1080/00036846.2015.1034845

This Version March 2015

INTRODUCTION

The problem of how to find an analytical solution for the impact of portfolio weight inequality constraints on the properties of the investor's optimal portfolio with parameter uncertainty has not been resolved in the literature because of its seeming intractability. There appears to be no analytic results that would allow the examination of this problem in a distributional setting, even under the simplifying assumption of multivariate normality. A number of authors have derived properties of the optimal portfolio under exact constraints, i.e. where portfolio weights add to one or zero, see *inter alia*, Jagannathan and Ma (2003), Hillier and Satchell (2003), Knight and Satchell (2010). Other authors, such as Best and Grauer (1991a, 1991b) and Scherer and Xu (2007), have discussed what happens to the efficient frontier when linear inequality constraints are imposed, but with no consideration of parameter uncertainty. Parameter uncertainty has been considered by a number of authors including Best and Grauer (1991a, 1991b), Chopra and Ziemba (1993), Bengtsson (2004), and Kan and Zhou (2007). Bengtsson extended Chopra and Ziemba's study by relaxing the long-only constraint and simultaneously considering errors in parameter estimates of the means, variances and covariances. While he demonstrated the importance of considering the effect of estimation error in variance and covariance estimates for long-short, and highly risk-averse investors, this analysis did not provide analytical solutions. To the best of our knowledge no one has combined all these approaches, possibly because of the complexity of the problem.

There is a special case where analytical results are possible. This is the setup of Treynor and Black (1973) and subsequently employed by Grinold (1989) and Grinold and Kahn (1999, page 28) when dealing with characteristic portfolios. Other authors have subsequently extended this analysis (see for example, Clarke, de Silva and Thorley (2002)). In this special case the "assets" are assumed to be independent and the sum of the weights equalling one or zero is not imposed. The relevant interpretation here is that this is an active fund where all common factors have been hedged out, and where there are other assets available outside the problem that offer the possibility of full investment. Here we can impose simple inequality constraints corresponding to long-only active investment whilst the unconstrained case would correspond to long-short active investment.

In practise, funds apply different portfolio constraints at different times. Furthermore, the availability of "soft constraint" technologies makes it virtually impossible to know what constraints are applied by any fund in a universe of funds. Here we proxy this unknowable problem by assuming that all managers hold only long-only portfolios. Whilst this is very unlikely to be true at all times, indeed at any point in time, it should capture, at least in certainty-equivalent terms, a measure of the impact of constraints on portfolio optimisation and performance measurement. The problem has an interesting application to combining hedge fund strategies where we can regard the individual funds/strategies as "market-neutral". If in addition it is assumed that the funds' alphas are positive, then long-only combining strategies is the only option considered. However, alternative explanations could consider the combination of risk-adjusted strategies with some strategies having negative alphas.

In the next section, we set up the problem and calculate the expectations of estimated sample portfolio means and variances. The approach of Kan and Zhou (2007) using a utility-based loss function is employed in order to calculate the loss of utility resulting from estimation risk in both the constrained and unconstrained cases. We present an empirical example based on a sample of hedge fund returns before concluding.

FRAMEWORK AND NOTATION

In this section we utilise the framework of Clarke, de Silva and Thorley (2002). Given a benchmark portfolio, the total excess return (i.e., return in excess of the risk free rate) on any stock k can be decomposed into a systematic portion that is correlated with the benchmark excess return and a residual that is not by:

$$r_k^{Total} = \beta_k R_B + r_k$$

where

R_B = the benchmark excess return,
β_k = the beta of stock k with respect to the benchmark, and,
r_k = the security residual return with standard deviation σ_k.

The benchmark portfolio is defined by the weights, $\omega_{B,k}$, assigned to each of the N stocks in the investable universe. The benchmark excess return is

$$R_B = \sum_{k=1}^{N} \omega_{B,k} r_k^{Total}.$$

The excess return on an actively managed portfolio is determined by the weights, $\omega_{P,k}$, on each stock:

$$R_P = \sum_{k=1}^{N} \omega_{P,k} r_k^{Total}.$$

Define the active return as the managed portfolio excess return minus the benchmark excess return, adjusted for the managed portfolio's beta with respect to the benchmark; that is

$$R_A = R_P - \beta_P R_B.$$

The managed portfolio's beta, β_P, is simply the weighted average beta of the stocks in the managed portfolio:

$$\beta_P = \sum_{k=1}^{N} \omega_{P,k} \beta_k.$$

It can be shown that the active return is

$$R_A = \sum_{k=1}^{N} \omega_k r_k,$$

where the active weight for each stock is defined as the difference between the managed portfolio weight and the benchmark weight; that is

$$\omega_k = \omega_{P,k} - \omega_{B,k}.$$

The formulation of the active return is the focus of our analysis. The active weights, ω_k, sum to zero because they are the differences in two sets of weights that each sum to unity. Also note that the stock returns used to define the active return are residual, not total, excess returns. The residual are the relevant component of security returns when performance is measured against a benchmark on a beta-adjusted basis.

Unconstrained Optimisation Problem

We make the important simplifying assumptions that the residual return of stock k is normally distributed with mean α_k, variance σ_k^2, and residual stock returns are uncorrelated (i.e., the residual return covariance matrix is diagonal). We assume that portfolio optimisation is based on choosing active weights, $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_N)^T$, that maximise the mean-variance utility function:

$$U(\boldsymbol{\omega}) = E[R_A] - \frac{\gamma}{2}\sigma_A^2, \tag{1}$$

where

$E[R_A]$ = expected active return,
σ_A^2 = active return variance, and,
γ = a risk aversion parameter.

Under the simplifying assumptions the expected active return for the portfolio is

$$E[R_A] = \sum_{k=1}^{N} \omega_k \alpha_k, \tag{2}$$

and the active portfolio return variance is

$$\sigma_A^2 = \sum_{k=1}^{N} \omega_k^2 \sigma_k^2. \tag{3}$$

Substituting the equations (2) and (3) into the optimisation problem (1) leads to the optimal weights given by the formula

$$\omega_k^* = \frac{1}{\gamma}\frac{\alpha_k}{\sigma_k^2}. \tag{4}$$

An intuitive property of this equation is that the optimal active weight for each stock is proportional to the ratio of the expected residual return to the residual return variance. The constant of proportionality, common to all securities, is the inverse of the risk-aversion factor. Lower values of γ lead to more aggressive portfolios with proportionately larger absolute active weights, higher expected active return and higher active risk.

The active return variance associated with the optimal portfolio is:

$$\sigma_A^2 = \sum_{k=1}^{N} (\omega_k^*)^2 \sigma_k^2 = \sum_{k=1}^{N} \left(\frac{1}{\gamma^2}\frac{\alpha_k^2}{\sigma_k^2}\right).$$

We can insert the optimal weights from equation (4) into the definition of the active return variance in equation (3) and solve for the risk aversion parameter, γ. Using this solution, the optimal active weights in terms of the active portfolio risk, σ_A, are:

$$\omega_k^* = \frac{\alpha_k}{\sigma_k^2}\frac{\sigma_A}{\sqrt{\sum_{k=1}^{N}(\alpha_k/\sigma_k)^2}}.$$

The simple closed form solution to the optimal weights is based on two simplifying assumptions. First, we have assumed a diagonal covariance matrix for the residual returns. Second, following Clarke, de Silva and Thorley (2002), the formal optimisation problem has a budget constraint that the active weights must sum to zero, but no such condition has been imposed in the optimisation of equation (1).

Constrained Optimisation Problem

Suppose that we now wish to consider the constrained problem with long-only weights. The quadratic function is to be maximised subject to the constraint that the weights must be non-negative:

$$U = \boldsymbol{\omega}^T \boldsymbol{\alpha} - \frac{\gamma}{2} \boldsymbol{\omega}^T \boldsymbol{\Omega} \boldsymbol{\omega} \quad \text{and} \quad \boldsymbol{\omega} \geq \mathbf{0}.$$

The Kuhn-Tucker conditions (see for example Chapter 13 of Chiang and Wainwright (2005)) are that:

$$\frac{\partial U}{\partial \omega_k} = \alpha_k - \gamma \sigma_k^2 \omega_k \leq 0\,, \quad \omega_k \geq 0 \;\; and \;\; \omega_k \frac{\partial U}{\partial \omega_k} = \; 0 \;\; for\, all\, k = 1, \dots, N.$$

Thus, either $\omega_k = 0$, or $\frac{\partial U}{\partial \omega_k} = \alpha_k - \gamma \sigma_k^2 \omega_k = 0,$ or $\omega_k = \frac{\alpha_k}{\gamma \sigma_k^2}$, so that the optimal constrained weights are:

$$\omega_k^{*,c} = \begin{cases} \frac{\alpha_k}{\gamma \sigma_k^2} & if\; \alpha_k > 0 \\ 0 & otherwise \end{cases}.$$

These conditions are necessary and sufficient as long as $\boldsymbol{\Omega}$ is positive semi-definite, a condition met in this example.

The Impact of Estimation Error

Now consider the above problems with the weights based on unbiased estimates of each stock residual return mean and variance. Specifically with a sample of size T, we employ the sample mean and sample variances:

$$\bar{r}_k = \frac{1}{T} \sum_{t=1}^{T} r_{k,t} \quad \text{and} \quad s_k^2 = \frac{1}{(T-1)} \sum_{t=1}^{T} \left(r_{k,t} - \bar{r}_k \right)^2 .$$

In the unconstrained case we have the following estimates:

Estimated weights: $\hat{\omega}_k^{*,u} = \frac{1}{\gamma} \frac{\bar{r}_k}{s_k^2}$,

Estimated expected return: $\hat{E}[R_A^u] = \sum_{k=1}^{N} \frac{\bar{r}_k^2}{\gamma s_k^2}$,

Estimated portfolio variance: $\hat{V}[R_A^u] = \sum_{k=1}^{N} \frac{\bar{r}_k^2}{\gamma^2 s_k^2}$,

and,

Estimated standard deviation: $\widehat{sd}[R_A^u] = \sqrt{\sum_{k=1}^{N} \frac{\bar{r}_k^2}{\gamma^2 s_k^2}}$.

In contrast for the constrained case, define the binary variable J_k , such that

$$J_k = \begin{cases} 1 & if\;\; \bar{r}_k > 0 \\ 0 & otherwise \end{cases}.$$

In the long-only constrained case we have the following estimates

Estimated weights: $\hat{\omega}_k^{*,c} = \frac{1}{\gamma} \frac{\bar{r}_k}{s_k^2} J_k$,

Estimated expected return: $\hat{E}[R_A^c] = \sum_{k=1}^{N} \frac{\bar{r}_k^2}{\gamma s_k^2} J_k$,

Estimated portfolio variance: $\hat{V}[R_A^c] = \sum_{k=1}^{N} \frac{\bar{r}_k^2}{\gamma^2 s_k^2} J_k$,

and,

Estimated standard deviation: $\widehat{sd}[R_A^c] = \sqrt{\sum_{k=1}^{N} \frac{\bar{r}_k^2}{\gamma^2 s_k^2} J_k}$.

We wish to calculate the expectations of the estimated portfolio returns and estimated portfolio variances of the unconstrained and constrained portfolios, in order to compare the outcomes for long-short and long-only portfolios. The independence of the sample moments $\bar{r}_k$ and s_k^2 follows from the normality assumption, but this also implies the independence of $\bar{r}_k^2$ and s_k^2 as well as the independence of $\bar{r}_k^2 J_k$ and s_k^2.

The sampling distributions of the sample moments are:

$$\bar{r}_k \sim N\left(\alpha_k, \frac{\sigma_k^2}{T}\right) \quad \text{and} \quad s_k^2 \sim \frac{\sigma_k^2}{T-1}\chi^2_{(T-1)}.$$

For $T > 5$, using the inverse moments of the chi-squared distribution, it follows that:

$$E\left[\frac{1}{s_k^2}\right] = \frac{(T-1)}{(T-3)\sigma_k^2} \quad \text{and} \quad E\left[\frac{1}{s_k^4}\right] = \frac{(T-1)^2}{(T-3)(T-5)\sigma_k^4}.$$

We can evaluate the expected value of the estimated expected return of the unconstrained portfolio as:

$$E\left[\hat{E}[R_A^u]\right] = \sum_{k=1}^N E\left[\frac{\bar{r}_k^2}{\gamma s_k^2}\right] = \sum_{k=1}^N \frac{1}{\gamma} E\left[\bar{r}_k^2\right] E\left[\frac{1}{s_k^2}\right],$$

and by substituting the known expectations of these functions of the sample moments and simplifying we obtain:

$$E\left[\hat{E}[R_A^u]\right] = \frac{(T-1)}{(T-3)}\frac{1}{\gamma}\sum_{k=1}^N \left[\frac{\alpha_k^2}{\sigma_k^2} + \frac{1}{T}\right].$$

Similarly, the expected value of the estimated variance of this unconstrained portfolio is:

$$E\left[\hat{V}[R_A^u]\right] = \frac{(T-1)}{(T-3)}\frac{1}{\gamma^2}\sum_{k=1}^N \left[\frac{\alpha_k^2}{\sigma_k^2} + \frac{1}{T}\right].$$

In order to evaluate the equivalent expectations for the long-only portfolio we require the moments of a truncated normal distribution. For the required well known results (see Olsen (1980) and Cameron and Trivedi (2005) for a derivation) consider the normal random variable, $x \sim N(\mu, \sigma^2)$ with $\mu > 0$ and define the binary variable $I = 1 \; if \; x > 0, and \; I = 0 \; otherwise$. Then,

$$E[xI] = \mu\Phi\left(\frac{\mu}{\sigma}\right) + \sigma\phi\left(\frac{\mu}{\sigma}\right) \quad \text{and} \quad E[x^2 I] = (\mu^2 + \sigma^2)\Phi\left(\frac{\mu}{\sigma}\right) + \mu\sigma\phi\left(\frac{\mu}{\sigma}\right),$$

where $\phi(.)$ and $\Phi(.)$ are respectively the normal density and distribution functions. In the context of our problem, by replacing μ by α_k and σ^2 by σ_k^2/T, we see that:

$$E[\bar{r}_k J_k] = \alpha_k \Phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k}\right) + \frac{\sigma_k}{\sqrt{T}}\phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k}\right) \quad \text{and} \quad E\left[\bar{r}_k^2 J_k\right] = \left(\alpha_k^2 + \frac{\sigma_k^2}{\sqrt{T}}\right)\Phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k}\right) + \frac{\alpha_k\sigma_k}{\sqrt{T}}\phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k}\right)$$

Using these expressions we can derive the expected values of the estimated mean and variance of the long-only, constrained portfolios as:

$$E\left[\hat{E}[R_A^c]\right] = \frac{(T-1)}{(T-3)}\frac{1}{\gamma}\sum_{k=1}^N \left[\left(\frac{\alpha_k^2}{\sigma_k^2} + \frac{1}{T}\right)\Phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k}\right) + \frac{\alpha_k}{\sigma_k\sqrt{T}}\phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k}\right)\right],$$

and,

$$E\left[\hat{V}[R_A^c]\right] = \frac{(T-1)}{(T-3)}\frac{1}{\gamma^2}\sum_{k=1}^N\left[\left(\frac{\alpha_k^2}{\sigma_k^2}+\frac{1}{T}\right)\Phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k}\right)+\frac{\alpha_k}{\sigma_k\sqrt{T}}\phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k}\right)\right].$$

It is interesting to note that these formulae depend only on $\left(\frac{\alpha_k}{\sigma_k}\right)$, that is the "information ratios" of each asset. The expected return and the expected variance of the long-only portfolio will be less than those of the long-short portfolios. The interpretation is that the long-only portfolio moves the manager to a different point in the mean-variance space and that a portfolio which is efficient on one frontier may not be for the other.

The cost of Estimation Error

For decision purposes it is important to consider the average losses involving any portfolio decisions taken under various samples of historical returns, so following Kan and Zhou (2007), we define the expected loss function as:

$$\rho(\boldsymbol{\omega}, \widehat{\boldsymbol{\omega}}) = U(\boldsymbol{\omega}) - E[U(\widehat{\boldsymbol{\omega}})],$$

where the expectation is taken with respect to the true distribution of the sample returns. This measure, which is non-negative by construction, represents the expected loss over all realizations that are incurred in using the portfolio rule $\widehat{\boldsymbol{\omega}}$.

For the unconstrained case with known parameters,

$$U(\boldsymbol{\omega}^{\boldsymbol{u}}) = \frac{1}{2\gamma}\sum_{k=1}^N\left(\frac{\alpha_k}{\sigma_k}\right)^2$$

while for the unconstrained case with unknown parameters we have,

$$E[U(\widehat{\boldsymbol{\omega}}^{\boldsymbol{u}})] = E\left[\sum_{k=1}^N\frac{\bar{r}_k\alpha_k}{\gamma s_k^2} - \frac{\gamma}{2}\sum_{k=1}^N\frac{\bar{r}_k^2\sigma_k^2}{\gamma^2 s_k^4}\right].$$

Utilising the expectations of the functions of the sample moments above, this expression can be evaluated as:

$$E[U(\widehat{\boldsymbol{\omega}}^{\boldsymbol{u}})] = \frac{(T-1)}{2\gamma(T-3)(T-5)}\left[(T-9)\sum_{k=1}^N\left(\frac{\alpha_k}{\sigma_k}\right)^2 - N\frac{(T-1)}{T}\right].$$

Substituting this result into the expected loss function we see that:

$$\rho(\boldsymbol{\omega}^{\boldsymbol{u}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{u}}) = \frac{1}{2\gamma(T-5)}\left[\frac{2(T+3)}{(T-3)}\sum_{k=1}^N\left(\frac{\alpha_k}{\sigma_k}\right)^2 + N\frac{(T-1)}{T}\right].$$

The expected loss function in the unconstrained case is positive for $T > 5$, decreasing in risk aversion and sample size, but increasing in the average strategy information ratio.

These derivations can be repeated for the constrained long-only case, where

$$U(\boldsymbol{\omega}^{\boldsymbol{c}}) = \frac{1}{2\gamma}\sum_{k=1}^N\left(\frac{\alpha_k}{\sigma_k}\right)^2 J_k,$$

and,

$$E[U(\widehat{\boldsymbol{\omega}}^{\boldsymbol{c}})] = E\left[\sum_{k=1}^N\frac{\bar{r}_k J_k\alpha_k}{\gamma s_k^2} - \frac{\gamma}{2}\sum_{k=1}^N\frac{\bar{r}_k^2 J_k\sigma_k^2}{\gamma^2 s_k^4}\right].$$

Utilising the expectations of the functions of the sample moments above, this expectation can be evaluated as:

$$E[U(\widehat{\boldsymbol{\omega}}^{\boldsymbol{c}})] = \frac{(T-1)}{2\gamma(T-3)(T-5)} \sum_{k=1}^{N} \left[\left(\frac{\alpha_k^2(T-9)}{\sigma_k^2} - \frac{(T-1)}{T} \right) \Phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k} \right) + \frac{\alpha_k(T-9)}{\sigma_k\sqrt{T}} \phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k} \right) \right].$$

It follows that in the constrained long-only case the expected loss function is:

$$\rho(\boldsymbol{\omega}^{\boldsymbol{c}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{c}}) = \frac{1}{2\gamma} \sum_{k=1}^{N} \left(\frac{\alpha_k}{\sigma_k} \right)^2 J_k - \left\{ \frac{(T-1)}{2\gamma(T-3)(T-5)} \sum_{k=1}^{N} \left[\left(\frac{\alpha_k^2(T-9)}{\sigma_k^2} - \frac{(T-1)}{T} \right) \Phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k} \right) + \frac{\alpha_k(T-9)}{\sigma_k\sqrt{T}} \phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k} \right) \right] \right\}.$$

This function is non-negative. If the positive truncated sample means accurately forecast the positive population mean, then we might expect the expected loss to be small.

Now consider the ratio of the difference between the expected loss in the constrained case and the expected loss in the unconstrained case to the expected loss in the unconstrained case to obtain:

$$\delta(\widehat{\boldsymbol{\omega}}^{\boldsymbol{c}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{u}}) = \frac{\rho(\boldsymbol{\omega}^{\boldsymbol{c}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{c}}) - \rho(\boldsymbol{\omega}^{\boldsymbol{u}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{u}})}{\rho(\boldsymbol{\omega}^{\boldsymbol{u}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{u}})} = \frac{\rho(\boldsymbol{\omega}^{\boldsymbol{c}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{c}})}{\rho(\boldsymbol{\omega}^{\boldsymbol{u}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{u}})} - 1$$

so that,

$$\delta(\widehat{\boldsymbol{\omega}}^{\boldsymbol{c}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{u}}) = \frac{\left[\sum_{k=1}^{N} \left(\frac{\alpha_k}{\sigma_k} \right)^2 J_k - \left\{ \frac{(T-1)}{(T-3)(T-5)} \sum_{k=1}^{N} \left[\left(\frac{\alpha_k^2(T-9)}{\sigma_k^2} - \frac{(T-1)}{T} \right) \Phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k} \right) + \frac{\alpha_k(T-9)}{\sigma_k\sqrt{T}} \phi\left(\frac{\sqrt{T}\alpha_k}{\sigma_k} \right) \right] \right\} \right]}{\left[\frac{2(T+3)}{(T-3)(T-5)} \sum_{k=1}^{N} \left(\frac{\alpha_k}{\sigma_k} \right)^2 + N \frac{(T-1)}{T(T-5)} \right]} - 1. \qquad (5)$$

The Information Loss Statistic, $\delta(\widehat{\boldsymbol{\omega}}^{\boldsymbol{c}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{u}})$, represents the percentage loss due to the imposition of the long-only constraint once we have taken into consideration parameter uncertainty. This statistic is independent of the risk aversion parameter.

EMPIRICAL EVIDENCE FROM GLOBAL HEDGE FUNDS

We investigate the loss in portfolio efficiency, using estimates of the information ratios of the managers in a sample of returns for global hedge fund managers. The data, sourced from Morningstar, consists of the gross returns on 288 hedge funds, being all those global equity funds with a complete record of returns from January 2000 to June 2014. We evaluate the Information Loss Statistic by evaluating equation (5) using rolling windows of 60 monthly returns.

The context we consider is a mythical fund of funds manager who wishes to hold (long) the style hedged returns of each global hedge fund while taking a zero position in them. By style hedged returns we mean factor neutral returns. As a point of comparison, we also include the raw returns of our universe of hedge funds.

In order to obtain some insight into the properties of the hedge fund raw returns used in this analysis, Table 1 reports average moments and statistics for various quantiles of the hedge fund returns. As expected, the returns display on average negative skewness and excess kurtosis with 95% of the funds displaying significant departure from normality (on the basis of the Jarque-Bera test at the 5% significance level). There is on average some positive autocorrelation in the sample returns, particularly at the first lag. Across the 288 funds the average annualised information ratio is 0.73. The average pairwise correlation coefficient between the hedge fund returns is 0.44. It is clear that these raw hedge fund returns deviate somewhat from the assumptions used in deriving the information loss ratio. In an attempt to minimise the impact of these issues we have estimated two different factor models and analysed the residuals observed from these models after accounting for the factors. The two factor models considered were those of Fung and Hsieh (2001) and the Barra Global Equity Risk Model (GEM3) of MSCI (2013).

The first factor model utilises the *Trend Following Risk Factors* of Fung and Hsieh (2001). Specifically the factors considered here were their primitive trend following factors based on bond, currency, commodity, short term interest rate and stock index look-back straddles. Table 2 reports summary statistics on the estimated Fung-Hsieh factor loadings. On average these loadings are small in value and with the model explaining only a small part (an average of 13.6%) of the variation in the hedge fund returns. Perhaps these results are not surprising given the construction of the Fung-Hsieh factors and the nature of hedge funds being analysed. Table 3 reports average moments and statistics for various quantiles of the residuals from this factor model. The average skewness (now positive) and kurtosis have been reduced, with 76% of the funds displaying significant departure from normality (on the basis of the Jarque-Bera test at the 5% significance level). There is on average a reduction in the degree of autocorrelation in the residuals, particularly at the first lag. The average pairwise correlation coefficient between the residual returns is 0.41. Across the 288 funds, after the Fung-Hsieh factors have been accounted for, the average annualised information ratio is 0.53.

The factor returns from the Barra GEM3 global equity risk model represent the set of common factors defined by Barra that drive the common sources of stock returns. The twelve factors used in our analysis are beta, book-to-price, dividend yield, earnings yield, growth, leverage, liquidity, momentum, nonlinear size, volatility, size and the World Equity Index. Table 4 summarises the estimated factor loadings from the regressions, and demonstrate that the factors are able to explain a (perhaps surprisingly) substantial amount of the hedge fund returns with an average R-squared of 53.3% and estimates of factor loadings that are generally large in magnitude. Table 5 reports average moments and statistics for various quantiles of the residuals from this factor model. The skewness (now positive) and kurtosis of the residuals have been substantially reduced with 64% of the funds displaying significant departure from normality (on the basis of the Jarque-Bera test at the 5% significance level). There is on average a substantial reduction in the degree of autocorrelation in the residuals. The average pairwise correlation coefficient between returns after extracting the impact of the factors is 0.16. Across the 288 funds, after accounting for the GEM3 factors, the average annualised information ratio is 0.70.

These results suggest that for this group of equity hedge funds factors based on option payoffs are perhaps less appropriate than the more traditional, fundamentally based structure of models such as the Barra GEM3 model designed to measure the risk measures in an equity universe. This is not to infer that one model is better or worse, simply that one needs to consider the context of the problem and the universe of assets when applying any risk model.

The information loss statistic was estimated using rolling samples of 60 observations ending in December 2004 through to June 2014 and these values (expressed as percentages) are plotted in Figure 1. To assist interpretation of the precision these estimates, a Monte Carlo study was conducted to estimate the sampling distribution. Plots of quantiles of the information loss statistic as a function of the information ratio are reported in the Appendix.

In the period up until the middle of 2008, the information loss measures based on the raw hedge fund returns display relatively low information losses of less than 10%, consistent with information ratios exceeding 0.8. Subsequently, the information loss measures decreased rapidly, with losses varying between 15% and 35% up until the end of 2011. These information loss values correspond to information ratios of between 0.75 and 0.25. Since late 2010, the information loss measure has declined steadily reaching a loss value exceeding 40% by the middle of 2012, which is consistent with an information ratio of less than 0.25. By stark comparison, the residuals based on the GEM3 factor model have shown greater stability, fluctuating between losses of 15% to 25% for most of the period, consistent with information ratios in the range of 0.7 to 0.4. It is also the case that since late 2010 this information loss measure has declined consistently until the end of 2012 but recovering in eth subsequent period. The size of the implied information ratios are consistent with those reported elsewhere in the literature (e.g. Clarke, de Silva and Thorley, 2002).

It is generally accepted that the portfolio information ratio is a proxy for the manger's skill. The period from June 2008 to March 2009 corresponds broadly to the Global Financial Crisis (GFC). The cause for the increasing information loss over this period observed in the raw returns data may be a general decline in manager skill. In contrast, this is no corresponding increase in information loss (and decline in skill) when adjusting the raw returns data for the global risk factors sourced from the Barra GEM3 model. These results suggest that in this period hedge fund managers were not necessarily market neutral. Their portfolios (and skill) were exposed to market influences that closely followed the general and rapid decline in the market during this period, and the subsequent recovery of the equity markets beyond the GFC. Furthermore, the Barra GEM3 global risk factors do seem to successfully control for effects in the broad market. The information losses based on the residuals produced by controlling for these factors were relatively constant throughout the observation period.

We can infer the level manager skill (via the information ratio of the raw returns) by looking at Figure 1 in light of the relationship between the information ratio and the sampling distribution of the information loss statistic illustrated in Figure 2 of the Appendix. Considering the period up to June 2008, we may infer that the average information ratio is around 1.2. As the general market declines, the information loss statistics fall for the next nine months to March 2009, and we infer that the information ratios decline to an average of 0.3. This represents a significant decline in the skill of the managers taken as a cohort. Moving beyond the GFC and the subsequent recovery we see evidence of a slow decline in skill as the information losses of the raw returns increase from 15% to 45% or a decline in the average information ratio from around 0.75 to below 0.25. This leads to a different conclusion: hedge fund managers were unable to apply their skill to factor returns, but there was a negligible decrease in their skill in managing idiosyncratic returns.

CONCLUSIONS

It is well known that estimation error and imposing constraints lead to a loss of certainty equivalent returns. In general, the presence of estimation risk results in optimised portfolios taking larger positions that the optimal portfolio weights. Imposing long only constraints may mitigate the effects of estimation risk. No analytic results are available in the general case and measures of any loss in certainty equivalent returns depend on the value of the risk aversion parameter. In this paper, in a special case, we have derived analytical expressions for the effect of portfolio long only constraints in the face of parameter uncertainty which is independent of the value of the risk aversion parameter. Our results are widely applicable, as a natural setting for empirical work is in the fund-of-funds domain, and we have illustrated the results using hedge fund return data. We demonstrate that, depending on the time period used to estimate portfolio statistics, the combined effects of long-only constraints and uncertainty in the estimated information ratios have resulted in information losses of up to 45%.

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Figure 1: The Estimated Information Loss for 288 Hedge Funds.

Plots of the estimated Information Loss Statistic, based on 288 Hedge Funds, using a rolling window of 60 monthly returns. The plots report the Information Loss estimated using the raw hedge fund returns, as well as the residuals after estimating the MCSI-Barra GEM3 model. The data period used for the calculations is January 2000 to June 2014.

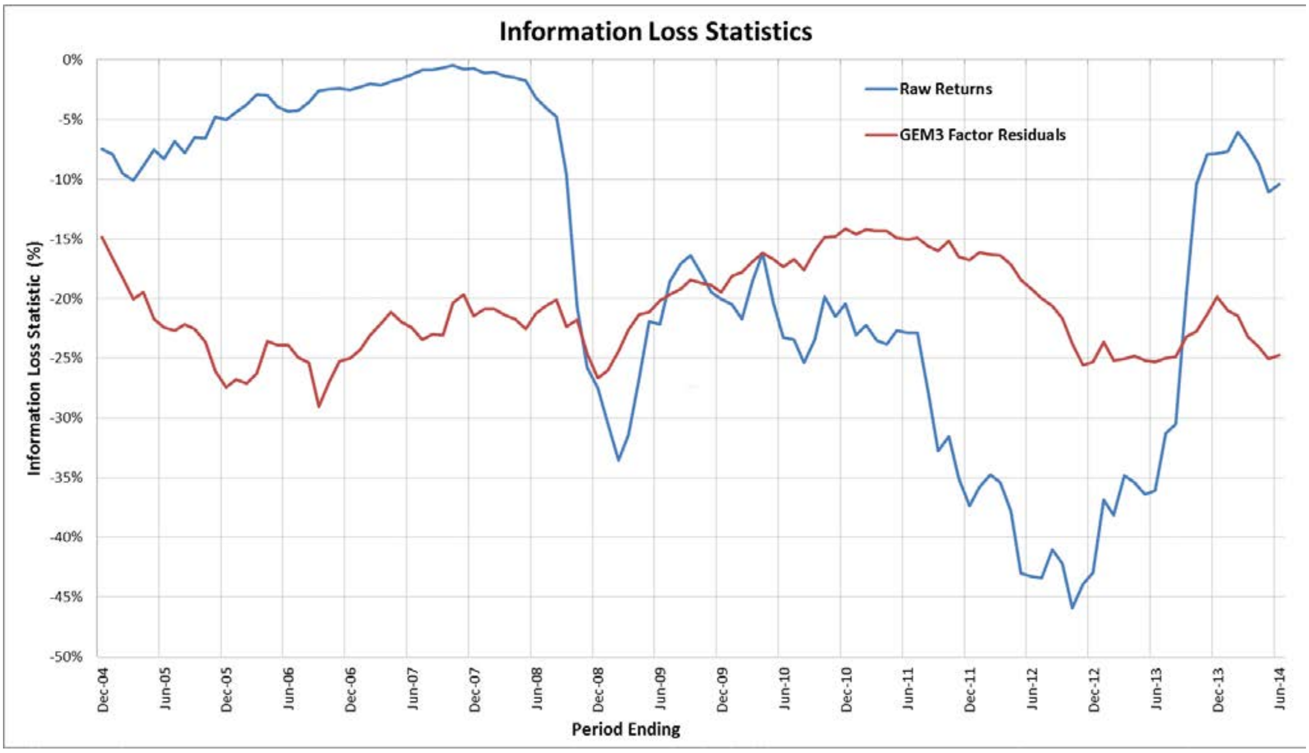

Table 1: Summary Statistics: Hedge Fund Returns

This table reports summary statistics for the returns (% per month) on 288 hedge funds using 174 monthly observations from January 2000 to June 2014. The column labelled Mean reports the sample mean across the cross-section of 288 funds of a variety of statistics. Similarly the other columns report the sample standard deviation, the sample minimum, the sample quantiles Q(q) for q = 0.05, 0.25, 0.50, 0.75 and 0.95 as well as the sample maximum. The rows define the sample statistics, computed for each fund based on 174 monthly observations, which are the sample mean, the sample standard deviation, the annualised information ratio, the minimum, the median, the maximum, the skewness, the kurtosis, the Jarque-Bera test for normality and its p-value, as well as the first three sample autocorrelation coefficients rho-1, rho-2 and rho-3.

	Mean	Std. Dev.	Minimum	Q(0.05)	Q(0.25)	Q(0.50)	Q(0.75)	Q(0.95)	Maximum
Mean	0.71	0.41	-0.12	0.23	0.46	0.62	0.87	1.35	2.87
Std. Dev.	3.89	2.26	0.80	1.28	2.11	3.66	4.91	8.02	14.29
Information ratio	0.73	0.37	-0.24	0.17	0.50	0.67	0.92	1.45	2.35
Minimum	-14.54	9.11	-63.27	-32.35	-17.61	-12.53	-8.12	-4.40	-1.96
Median	0.81	0.44	-0.32	0.28	0.52	0.71	1.00	1.63	2.70
Maximum	15.44	11.62	1.91	3.47	7.75	12.60	19.88	31.93	101.06
Skewness	-0.25	1.10	-6.01	-1.83	-0.75	-0.31	0.19	1.61	4.78
Kurtosis	7.83	6.22	2.76	3.71	4.60	6.06	8.71	18.34	69.64
Jarque-Bera Test (*p*-value)			0.55 (0.760)	6.12 (0.047)	24.21 (<0.001)	74.54 (<0.001)	258.86 (<0.001)	1771.61 (<0.001)	33,240 (<0.001)
rho-1	0.18	0.14	-0.16	-0.02	0.09	0.16	0.24	0.41	0.64
rho-2	0.04	0.11	-0.21	-0.11	-0.04	0.03	0.11	0.26	0.44
rho-3	0.07	0.09	-0.16	-0.08	0.02	0.08	0.13	0.21	0.30
Observations	174	174	174	174	174	174	174	174	174

Table 2: Summary Statistics: Fung-Hsieh Hedge Fund Model Factor Loadings.

This table reports summary statistics for estimated Fung-Hsieh factor models for the returns on 288 hedge funds. The column labelled Mean reports the sample mean across the cross-section of 288 funds of a variety of statistics. Similarly the other columns report the sample minimum, the sample quantiles Q(q) for q = 0.25, 0.50 and 0.75 as well as the sample maximum. The rows report on the estimated alpha and loadings for the primitive trend following factors based on bond (PTFSBD), currency (PTFSFX), commodity (PTFSCOM), short term interest rate (PTFSIR) and stock index (PTFSSTK) look-back straddles as well as the R-squared for factor models estimated using 174 monthly observations from January 2000 to June 2014. Data source: https://faculty.fuqua.duke.edu/~dah7/HFRFData.htm.

	Mean	**Minimum**	**Q(0.25)**	**Q(0.50)**	**Q(0.75)**	**Maximum**
Alpha	0.430	-1.035	0.242	0.401	0.615	2.475
PTFSBD	-0.033	-0.131	-0.048	-0.029	-0.017	0.026
PTFSX	0.003	-0.049	-0.006	0.003	0.011	0.045
PTFSCOM	0.000	-0.114	-0.010	0.002	0.013	0.108
PTFSIR	-0.026	-0.096	-0.037	-0.023	-0.015	0.027
PTFSSTK	-0.029	-0.185	-0.046	-0.020	-0.005	0.095
R^2	0.136	0.012	0.091	0.129	0.179	0.334

Table 3: Summary Statistics: Fung-Hsieh Factor Model Residuals.

This table reports summary statistics for the estimated residuals plus alpha from the estimated Fung-Hsieh factor models for the 288 hedge funds using 174 monthly observations from January 2000 to June 2014. The column labelled Mean reports the sample mean across the cross-section of 288 funds of a variety of statistics. Similarly the other columns report the sample standard deviation, the sample minimum, the sample quantiles Q(q) for q = 0.05, 0.25, 0.50, 0.75 and 0.95 as well as the sample maximum. The rows define the sample statistics, computed for each fund based on 174 monthly observations, which are the sample mean, the sample standard deviation, the annualised information ratio, the minimum, the median, the maximum, the skewness, the kurtosis, the Jarque-Bera test for normality and its p-value, as well as the first three sample autocorrelation coefficients rho-1, rho-2 and rho-3.

	Mean	**Std. Dev.**	**Minimum**	**Q(0.05)**	**Q(0.25)**	**Q(0.50)**	**Q(0.75)**	**Q(0.95)**	**Maximum**
Mean	0.43	0.38	-1.04	-0.10	0.24	0.40	0.61	1.02	2.48
Std. Dev.	3.63	2.15	0.78	1.12	1.97	3.43	4.61	7.41	14.06
Information ratio	0.53	0.43	-0.42	-0.08	0.25	0.49	0.77	1.30	2.04
Minimum	-11.15	6.98	-62.24	-23.20	-13.81	-10.50	-6.11	-3.47	-1.59
Median	0.42	0.38	-1.88	-0.23	0.23	0.44	0.63	1.00	1.53
Maximum	14.92	11.26	1.80	3.35	7.48	12.44	19.36	31.30	98.21
Skewness	0.21	0.91	-5.94	-0.70	-0.23	0.10	0.47	1.83	5.34
Kurtosis	6.65	6.33	2.62	3.06	3.73	4.80	6.89	16.97	68.09
Jarque-Bera Test (*p*-value)			0.03 (0.984)	0.77 (0.681)	6.69 (0.035)	25.28 (<0.001)	126.50 (<0.001)	1550.3 (<0.001)	31,744 (<0.001)
rho-1	0.13	0.12	-0.17	-0.04	0.05	0.11	0.20	0.33	0.62
rho-2	0.01	0.11	-0.25	-0.15	-0.07	-0.01	0.07	0.19	0.44
rho-3	0.07	0.09	-0.14	-0.10	0.02	0.07	0.13	0.22	0.31
Observations	174	174	174	174	174	174	174	174	174

Table 4: Summary Statistics: Barra GEM3 Global Equity Risk Model Factor Loadings

This table reports summary statistics for estimated Barra GEM3 factor models for returns on 288 hedge funds. The column labelled Mean reports the sample mean across the cross-section of 225 funds of a variety of statistics. Similarly the other columns report the sample minimum, the sample quantiles Q(q) for q = 0.25, 0.50 and 0.75 as well as the sample maximum. The rows report on the estimated alpha and factor loadings for the Beta, Book-to-Price, Dividend Yield, Earnings Yield, Growth, Leverage, Liquidity, Momentum, Non-linear Size, Volatility, Size and World Equity Index factors as well as the R-squared for the factor models estimated using using 174 monthly observations from January 2000 to June 2014.

	Mean	Minimum	Q(0.25)	Q(0.50)	Q(0.75)	Maximum
Alpha	0.407	-0.655	0.363	0.588	2.948	0.121
Beta	0.456	-0.362	0.384	0.655	1.739	0.222
Book-To-Price	-0.097	-2.846	-0.132	0.092	1.660	-0.293
Dividend Yield	0.510	-1.667	0.425	0.819	3.993	0.124
Earnings Yield	-0.655	-4.799	-0.751	-0.304	4.663	-1.121
Growth	0.366	-2.719	0.306	0.705	3.823	-0.080
Leverage	0.111	-4.020	0.167	0.625	7.538	-0.367
Liquidity	0.090	-2.515	0.050	0.559	5.998	-0.474
Momentum	0.406	-3.365	0.329	0.703	3.794	0.005
Non-Linear Size	0.249	-2.046	0.252	0.507	2.037	-0.010
Volatility	-0.196	-3.492	-0.103	0.211	2.339	-0.500
Size	0.376	-1.877	0.369	0.704	4.056	0.039
World Equity Index	-0.733	-4.984	-0.535	-0.221	1.455	-1.102
R^2	0.533	0.115	0.552	0.679	0.856	0.397

Table 5: Summary Statistics: Barra GEM3 Global Equity Risk Model Residuals

This table reports summary statistics for the estimated residuals plus alpha from the estimated Barra GEM3 factor models for the 288 hedge funds using 174 monthly observations from January 2000 to June 2014. The column labelled Mean reports the sample mean across the cross-section of 288 funds of a variety of statistics. Similarly the other columns report the sample standard deviation, the sample minimum, the sample quantiles Q(q) for q = 0.05, 0.25, 0.50, 0.75 and 0.95 as well as the sample maximum. The rows define the sample statistics, computed for each fund based on 174 monthly observations, which are the sample mean, the sample standard deviation, the annualised information ratio, the minimum, the median, the maximum, the skewness, the kurtosis, the Jarque-Bera test for normality and its p-value, as well as the first three sample autocorrelation coefficients rho-1, rho-2 and rho-3.

	Mean	Std. Dev.	Minimum	Q(0.05)	Q(0.25)	Q(0.50)	Q(0.75)	Q(0.95)	Maximum
Mean	0.41	0.48	-0.66	-0.26	0.12	0.36	0.59	1.23	2.95
Std. Dev.	2.62	1.74	0.62	0.82	1.36	2.33	3.31	5.53	13.41
Information ratio	0.70	0.68	-0.89	-0.36	0.17	0.63	1.18	1.87	2.39
Minimum	-7.88	6.14	-61.38	-15.49	-9.93	-7.09	-3.79	-2.02	-1.13
Median	0.33	0.47	-0.77	-0.36	0.06	0.30	0.51	1.10	2.72
Maximum	10.19	9.44	2.01	2.87	4.89	8.35	11.86	22.37	96.66
Skewness	0.19	0.74	-6.85	-0.63	-0.04	0.15	0.47	1.29	2.68
Kurtosis	5.70	5.88	2.51	2.83	3.40	4.16	5.76	13.83	75.69
Jarque-Bera Test (*p*-value)			0.00 (0.998)	0.26 (0.878)	2.41 (0.299)	14.19 (0.001)	68.58 (<0.001)	887.8 (<0.001)	39672 (<0.001)
rho-1	0.05	0.12	-0.25	-0.15	-0.03	0.05	0.12	0.25	0.39
rho-2	0.02	0.09	-0.21	-0.11	-0.03	0.01	0.07	0.17	0.31
rho-3	0.04	0.08	-0.20	-0.09	-0.01	0.03	0.10	0.17	0.26
Observations	174	174	174	174	174	174	174	174	174

APPENDIX: SIMULATED QUANTILES OF THE INFORMATION LOSS STATISTIC.

We conducted a simulation study in order to assess the sampling distribution of the information loss estimates reported in Section III. For a cross-section of N = 288 funds we simulated returns for T = 60 observations for a range of values of the annualized information ratio. Figure 2 plots selected quantiles for the distribution of the information loss statistic, assuming normally distributed returns and using 10,000 simulations. The distribution is tightly dispersed about the median, and varies from a loss of 50% when the information ratio is zero, asymptoting to effectively no information loss when the information ratio exceeds 1.5. By conducting simulations using a variety of skewed and kurtotic distributions we have confirmed that these characteristics do not impact on the simulated distribution of the information loss ratio. The loss measure, $\delta(\widehat{\boldsymbol{\omega}}^{\boldsymbol{c}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{u}})$, is a function of the moments of the distribution only through the information ratio so this result is not unexpected.

Figure 2: Simulated Quantiles of the Information Loss

This figure presents the simulated 1%, 50% and 99% quantiles of the Information Loss Statistic, $\delta(\widehat{\boldsymbol{\omega}}^{\boldsymbol{c}}, \widehat{\boldsymbol{\omega}}^{\boldsymbol{u}})$, as a function of the Information Ratio. The simulation employs parameter values that match the sample of hedge funds used in the empirical application, that is we set N = 288 hedge funds and T = 60 monthly returns for each hedge fund. The data is generated from normal distributions to mimic monthly observations with annualised Information Ratios from 0.00 to 2.00 in steps of 0.25. The quantiles plotted in the figure are based on 10,000 replications.