Strict Local Martingales in Continuous Financial Market Models

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Abstract

It is becoming increasingly clear that strict local martingales play a distinctive and important role in stochastic finance. This thesis presents a detailed study of the effects of strict local martingales on financial modelling and contingent claim valuation, with the explicit aim of demonstrating that some of the apparently strange features associated with these processes are in fact quite intuitive, if they are given proper consideration.

The original contributions of the thesis may be divided into two parts, the first of which is concerned with the classical probability-theoretic problem of deciding whether a given local martingale is a uniformly integrable martingale, a martingale, or a strict local martingale. With respect to this problem, we obtain interesting results for general local martingales and for local martingales that take the form of time-homogeneous diffusions in natural scale.

The second area of contribution of the thesis is concerned with the impact of strict local martingales on stochastic finance. We identify two ways in which strict local martingales may appear in asset price models: Firstly, the density process for a putative equivalent risk-neutral probability measure may be a strict local martingale. Secondly, even if the density process is a martingale, the discounted price of some risky asset may be a strict local martingale under the resulting equivalent risk-neutral probability measure. The minimal market model is studied as an example of the first situation, while the constant elasticity of variance model gives rise to the second situation (for a particular choice of parameter values).

Certificate of Authorship and Originality

I certify that the work presented in this thesis has not previously been submitted for another degree, and nor has it been submitted in partial fulfilment of the requirements of another degree. I also certify that the thesis has been written by me. All help received with my research, or in the preparation of the thesis itself, has been acknowledged. In addition, I certify that all information and results obtained from the literature and other sources have been properly credited.

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Preface

Local martingales are ubiquitous in stochastic finance. For example, after formulating a stochastic model for asset prices, the traditional first step is to change probability measures from the so-called "real-world" probability measure to an equivalent risk-neutral probability measure. A necessary precondition for this operation is that a certain exponential local martingale—referred to as the density process for the measure transformation—must be a (positive) martingale.[1] If not, the risk-neutral approach to contingent claim pricing is invalid. However, even when the risk-neutral approach is applicable, local martingales continue to feature, since the discounted asset prices are then local martingales under the equivalent risk-neutral probability measure. An asset price model for which these local martingales are strict exhibits all manner of strange behaviour.[2]

For the reasons cited above, the question of whether a given local martingale is in fact a martingale is clearly fundamental to stochastic finance. Unfortunately, it is also an extremely delicate question, whose answer usually hinges on very technical conditions. This may explain why financial engineers appear to adopt the practice of simply assuming that the local martingales under consideration are actually martingales. It is becoming increasingly clear, however, that strict local martingales have an important role to play in financial modelling, and should be better understood by financial modellers. In particular, the above-mentioned practice of blurring the distinction between local martingales and martingales is potentially limiting and rather dangerous.

As already mentioned, the density process associated with a hypothetical equivalent risk-neutral probability measure is a local martingale. The assumption that it is a martingale is typically justified by appealing to the Fundamental Theorem of Asset

[1]Note that we are abusing terminology a little, since the implicit convention in the literature is that density processes are martingales, by definition. It would therefore probably be more correct to described the exponential local martingale in question as a "candidate density process," to cater for the possibility that it may not be a martingale. We choose, however, to avoid this extra terminological overhead, and trust instead that the reader is sufficiently flexible on this point.

[2]A local martingale is said to be strict if it is not a martingale.

Pricing, the most comprehensive formulation of which asserts an equivalence between the following conditions (see e.g. Dybvig and Ross [1998]):[3]

(A) the absence of arbitrage opportunities;

(B) the existence of a positive linear pricing rule;

(C) the existence of optimal demand for some agent who prefers more to less;

(D) the existence of an equivalent risk-neutral probability measure; and

(E) the existence of a strictly positive state-price density.

Since viable asset price models must admit equilibria, the equivalence (A) $\Leftrightarrow$ (C) indicates that such models should be arbitrage-free in some sense.[4] Most authors therefore feel justified in assuming the non-existence of arbitrage opportunities as their starting point, after which (A) $\Leftrightarrow$ (D) is invoked to assert the existence of an equivalent risk-neutral probability measure. For sufficiently simple asset price models this reasoning is sound, but for more sophisticated models a stronger no-arbitrage condition is required for the equivalence (A) $\Leftrightarrow$ (D) than for (A) $\Leftrightarrow$ (C). In fact, for the most general class of models, Delbaen and Schachermayer [1994, 1998] establish that (A) $\Leftrightarrow$ (D) only holds if one subscribes to the rather austere "no free lunch with vanishing risk" (NFLVR) definition of no-arbitrage.

The upshot of the discussion above is that it is possible for a model to satisfy a no-arbitrage condition that is both strong enough to ensure equilibrium and yet too weak to support an equivalent risk-neutral probability measure. This insight apparently originated with the publication of Loewenstein and Willard [2000a,b]. Since then, a steady trickle of research has suggested that the NFLVR condition is probably too strong to be realistic. For example, by taking into account the effect of margin calls, Liu and Longstaff [2004] recently showed that log-utility investors can construct optimal portfolios even when NFLVR fails. However, probably the most definitive result in this direction is due to Karatzas and Kardaras [2007], who demonstrate that the "no unbounded profit with bounded risk" (NUPBR) condition—which is weaker than NFLVR—is sufficient for utility maximization in a general semimartingale model. The same article also establishes that NUPBR is equivalent to the existence of a numéraire portfolio. Finally, Heath and Platen [2002a,b, 2005a,b], Platen [2002, 2006] and Platen and Heath [2006] consider asset price models that contain a numéraire portfolio, without satisfying the NFLVR condition. These publications collectively demonstrate that contingent claims can be priced and hedged in such models, without the need for an equivalent change of probability measure. In conclusion, the research cited above appears to suggest that requiring the density process associated with an asset price model to be a martingale imposes an unjustified restriction on modelling freedom.

[3]Actually, the equivalence below is a combination of the Fundamental Theorem of Asset Pricing and the Pricing Rule Representation Theorem (see e.g. Dybvig and Ross [1998]).

[4]The existence of an equilibrium is essentially condition (C).

We now consider asset price models that admit equivalent risk-neutral probability measures, under which the discounted price of at least one risky security is a strict local martingale. The first publication devoted explicitly to this phenomenon appears to have been Sin [1998], where it was studied in the context of stochastic volatility models. Related investigations of stochastic volatility models followed, including Andersen and Piterbarg [2007], Lewis [2000] and Wong and Heyde [2006].

A recent idea, which has become quite fashionable, is to interpret a risky security whose discounted risk-neutral dynamics are those of a strict local martingale as an asset price bubble (see e.g. Cox and Hobson [2005], Ekström and Tysk [2009], Heston et al. [2007], Jarrow et al. [2007a,b] and Pal and Protter [2008]). This interpretation may be justified by first noting that the price of the discounted risky security in question is in fact a strict supermartingale under the equivalent risk-neutral probability measure. Its risk-neutral value must therefore be less than its current price—the disparity between price and value being the bubble.

Financial market models containing asset price bubbles exhibit a number of anomalies. These include the failure of put-call parity for European options; non-convexity of the prices of options with convex payoffs, with respect to the underlying asset price; non-monotonicity of option prices, with respect to volatility; and (in a diffusion setting) multiple polynomial-bounded solutions of the Black-Scholes partial differential equation, for certain payoffs. The failure to recognize these anomalies has led to the publication of incorrect option pricing formulae in at least one documented instance (see Emanuel and Macbeth [1982]).

This thesis aspires to present a detailed and systematic account of strict local martingales in continuous financial market models.[5] Since the mathematical theory underlying this subject tends to be rather involved, a conscious decision has been made to supply too many details, rather than too few.[6] Furthermore, it is our ambition to invest the presentation with intuitive content—something that is perhaps lacking in the published literature on the subject. To this end, we have included numerous examples and illustrations.

The structure of the thesis is as follows: there are five chapters, followed by three technical appendices. Each chapter is devoted to a single coherent topic, and aims to integrate the author's own original research into an up-to-date exposition. The exception is Chapter 1, which is almost entirely expository. The appendices provide technical reviews of some topics that feature repeatedly throughout the body of the thesis. As such, they are mainly expository by nature, although they do contain a number of original results that would fit uncomfortably into the chapters themselves. We conclude this preface with a brief overview of the remainder of the thesis:

[5] By this we mean continuous-time models in which asset prices have continuous sample paths.

[6] To the reader who is already familiar with this material, and who therefore finds the style of presentation somewhat laborious, we apologize in advance.

Chapter 1: A Survey of Continuous Financial Market Models. This chapter consists of a comparative study of no-arbitrage conditions in continuous asset price models; an investigation of the economic purpose of no-arbitrage conditions in such models; and an examination of the impact of strict local martingales in stochastic finance. Although it is essentially expository by nature, it provides important background for the remainder of the thesis—especially Chapters 4 and 5.

Chapter 2: Classification of Local Martingales: I. General Processes. The topic here is the classical probability-theoretic problem of how to determine whether a given scalar-valued local martingale is a uniformly integrable martingale, a non-uniformly integrable martingale, or a strict local martingale. Our first significant result is Thoerem 2.14, which obtains necessary and sufficient conditions for a continuous local martingale to be a martingale or a uniformly integrable martingale. This theorem also serves as a template for later results in Chapter 3. However, the highlights of Chapter 2 are Theorems 2.17 and 2.20, which obtain necessary and sufficient conditions for an arbitrary local martingale to be a martingale or a uniformly integrable martingale, respectively. These conditions—which are expressed in terms of the weak tail of the supremum of a given local martingale—have been studied extensively by Elworthy et al. [1997, 1999], Galtchouk and Novikov [1997], Kaji [2007], Liptser and Novikov [2006], Novikov [1997], and Takaoka [1999]. However, those articles all assume some form of sample-path regularity (usually continuity), whereas we commit to no such assumption. It appears therefore that Theorems 2.17 and 2.20 make a significant contribution to the literature.

Chapter 3: Classification of Local Martingales: II. Diffusions. This chapter revisits the problem of Chapter 2, and specializes the results presented there to the case of time-homogeneous scalar diffusions. To begin with, Proposition 3.2 demonstrates that all local martingales, within the class of processes under consideration, may be characterized as scalar diffusions in natural scale, whose finite boundaries are natural or absorbing and whose infinite boundaries are natural or entrance. Theorem 3.6 then shows that all such processes are essentially integrable, after which Theorem 3.9 leads to Theorem 3.10, which is the first main result of the chapter. This theorem obtains concrete and testable necessary and sufficient conditions—expressed in terms of the fundamental solutions of an ordinary differential equation—for determining whether a local martingale of the type identified by Proposition 3.2 is in fact a martingale. Thereafter, three technical results (see Propositions 3.11–3.13) yield the following remarkable conclusions: First, Theorem 3.15 demonstrates that the question of whether or not a local martingale of the type identified by Proposition 3.2 is a martingale is purely a matter of its boundary behaviour—in particular, such a process is a martingale if and only if its infinite boundaries are natural. Next, Theorem 3.16 establishes that the criteria identified by Theorem 3.10 are equivalent to conditions obtained by Kotani

[2006] for classifying a time-homogeneous diffusion in natural scale as a martingale or a strict local martingale. Finally, Theorem 3.17 demonstrates that the class of local martingales under consideration contains no uniformly integrable martingales, if one of its boundaries is finite. To complete the chapter, we consider a number of examples where Theorem 3.10 is used to classify a given local martingale as a martingale or a strict local martingale.

Chapter 4: Partial Differential Equations and Strict Local Martingales. Here we are concerned mainly with the relationship between certain terminal-boundary-value problems and strict local martingales. After some preliminaries, Theorem 4.7 formulates and proves a version of the Feynman-Kac theorem for time-homogeneous diffusions. Although this result is extremely well-known, we feel that its true meaning is often obscured by textbook presentations, which tend to clutter its formulation with extraneous details. By contrast, we argue that the Feynman-Kac theorem should really be interpreted simply as a statement about the structure of a certain family of local martingales associated with a terminal-boundary-value problem. We then set about obtaining a converse of the Feynman-Kac theorem. This is actually a more difficult result than the Feynman-Kac theorem itself—as well as being more important for applications.[7] The main theorem of the chapter is Theorem 4.10, which is inspired by a comparatively obscure result in Kac [1951], for the case of Brownian motion. Theorem 4.10 then leads to Theorem 4.11, which establishes the desired converse of the Feynman-Kac theorem, in the setting of Chapter 4. Next, we elaborate on the relationship between strict local martingales and badly-behaved terminal-boundary-value problems, before moving on to study the constant elasticity of variance model. Our main result in this regard is Proposition 4.15, which corrects an error by Emanuel and Macbeth [1982].[8] In particular, that article derived an incorrect expression for the price of a European call on an underlying asset with constant elasticity of variance dynamics, for the case when the discounted price of the asset is a strict local martingale under the equivalent risk-neutral probability measure.

Chapter 5: The Minimal Market Model. This chapter begins by considering a continuous financial market model, in which uncertainty is provided by a vector-valued Brownian motion. The initial objective is to construct the "growth-optimal portfolio," and to understand its properties. In this regard, Lemma 5.7 identifies the trading strategy that generates the growth-optimal portfolio, while Propositions 5.8 and 5.9 demonstrate that the growth-optimal portfolio admits a representation as a time-changed squared Bessel process of dimension four. This representation leads to the formulation

[7]It should also be said that very few textbooks recognize the converse of the Feynman-Kac theorem, and that some even confuse the Feynman-Kac theorem with its converse.

[8]Unfortunately, the same result was obtained independently by Heston et al. [2007]. Nevertheless, we did present the contents of Proposition 4.15 at the Fifth National Symposium on Financial Mathematics, which was held in Melbourne in 2006, sometime before the appearance of that article.

of the "minimal market model" for a diversified equity accumulation index. Proposition 5.12 then establishes a number of properties of the index volatility, under the assumptions of this model. Since the minimal market model does not admit an equivalent risk-neutral probability measure, Propositions 5.13–5.17 use the so-called "real-world" pricing approach to derive pricing formulae for a number of standard European claims on the index. Next, Proposition 5.18 derives an expression for the price of a European option to exchange one risky security for another, under the assumptions of the minimal market model. This simplifies the pricing formula for a similar instrument that appeared in Hulley et al. [2005]. Finally, the chapter ends by considering a number of path-dependent claims on the index, under the assumptions of the minimal market model. In this regard, Propositions 5.19 and 5.20 obtain expressions for the real-world prices of certain perpetual and finite-maturity rebates, while Proposition 5.21 does the same for knock-out calls on the index. These propositions are based on similar results in Hulley and Platen [2008b].

Appendix A: Time-Homogeneous Scalar Diffusions. This appendix is devoted to a comprehensive survey of time-homogeneous scalar diffusions. We have chosen to cover this subject carefully, since we make fairly sophisticated use of a number of specialized results about time-homogeneous scalar diffusions throughout Chapters 3–5. Of special interest are a number of Laplace transform identities associated with such processes, as well as their boundary behaviour. Although most of the material in this appendix is expository by nature, Propositions A.4 and A.6 appear to be original.

Appendix B: Squared Bessel Processes. Squared Bessel processes are used in Sections 4.5 and 5.3 to obtain convenient representations of the constant elasticity of variance model and the minimal market model, respectively. It is thus appropriate that we should provide some background on these processes. Although this appendix is essentially a survey, Proposition B.2 does provide an original derivation of the transition density of a squared Bessel process with absorption at the origin.

Appendix C: Some Important Distributions and Their Computation. The pricing formulae for European claims on an asset with constant elasticity of variance dynamics are expressed in terms of the non-central chi-square cumulative distribution function in Section 4.5. The same is true for the pricing formulae for European claims on an equity index, under the assumptions of the minimal market model, which we derive in Section 5.4. The first part of this appendix therefore provides a detailed account of the non-central chi-square distribution, with particular emphasis on the computation of its cumulative distribution function. The second part is concerned with the doubly non-central beta distribution. Here we make two apparently new contributions to the literature: Firstly, Proposition C.2 demonstrates that the doubly non-central beta distribution may be extended to include the situation where one of its shape parameters is zero, while Algorithm C.2 presents a routine for computing the (extended) doubly non-central beta cumulative distribution function. These results are used in Section 5.5, where we show that the price of a European option to exchange one risky security for another, under the assumptions of the minimal market model, may be expressed in terms of the cumulative distribution function of an (extended) doubly non-central beta random variable.

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