Beyond the Classical Paradigm

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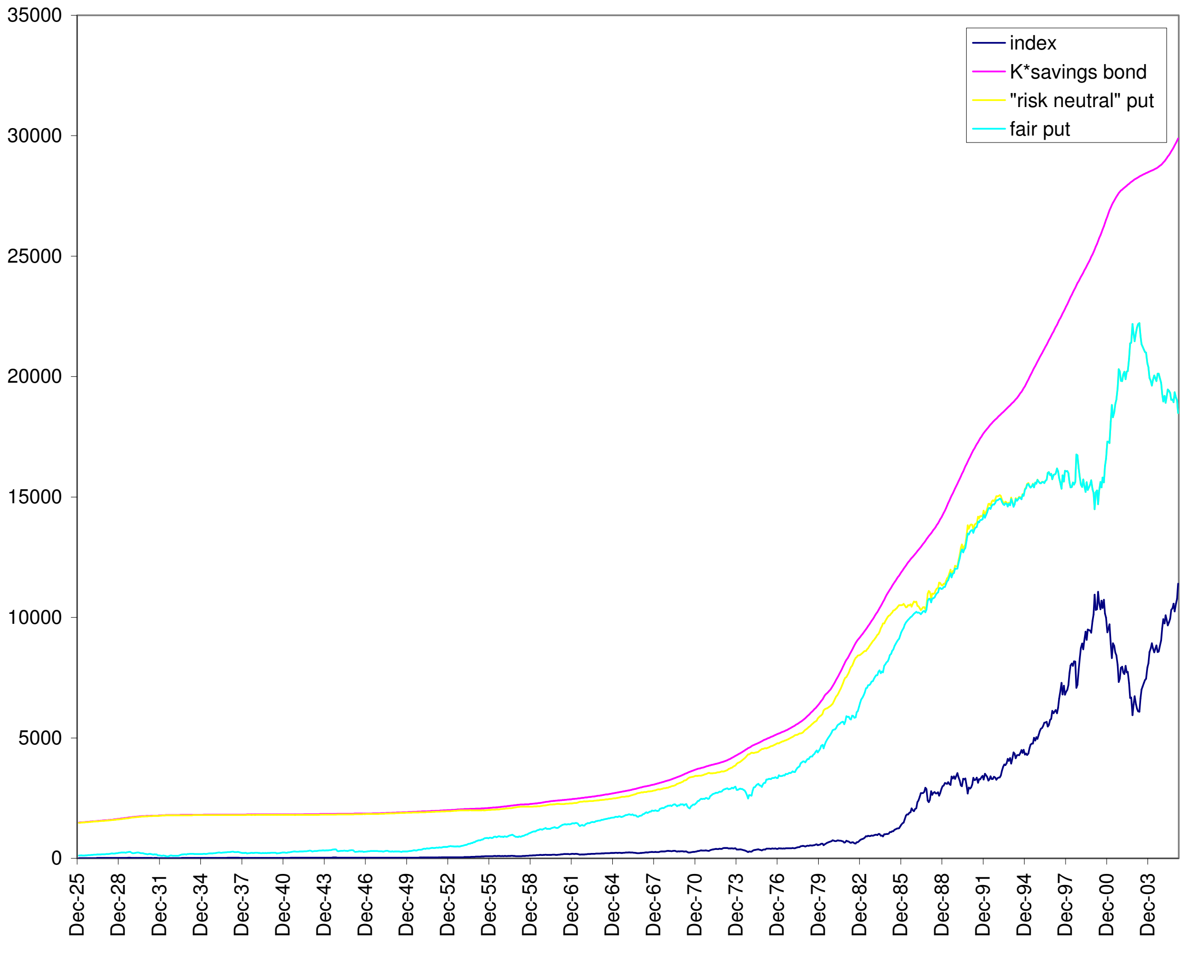
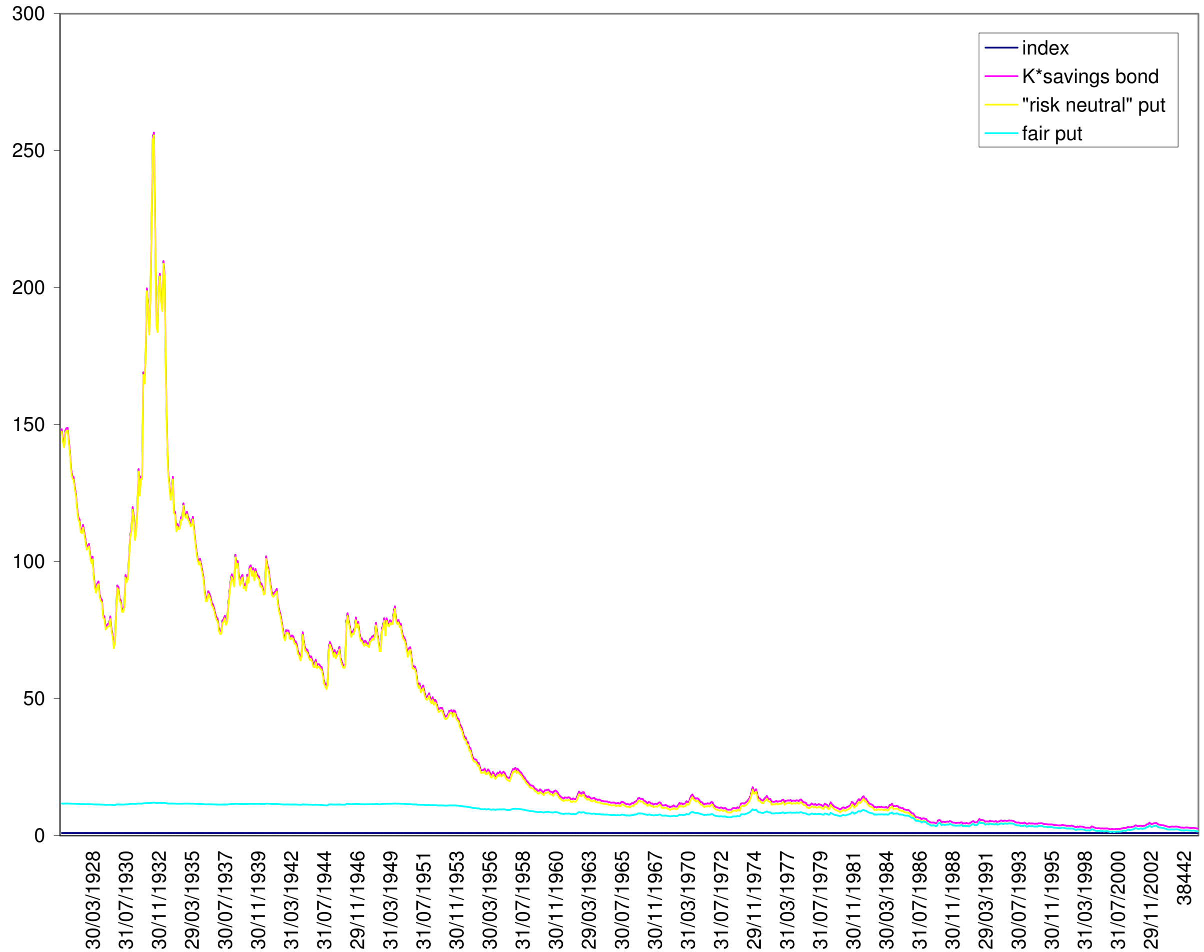
Pl. & Heath (2006, 2010), A Benchmark Approach to Quantitative Finance. Springer

Pl. & Bruti-Liberati (2010), Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer

Baldeaux & Pl. (2013). Functionals of Multidimensional Diffusions with Applications to Finance. Springer



Risk neutral and fair put on index



Benchmarked “risk neutral” and fair put on index

Complications with Classical Approach

- inverse of 3-dimensional Bessel process; Delbaen & Schachermayer (1995)
- $\frac{3}{2}$ volatility model; Pl. (1997), Heston (1997), Lewis (2000), Pl. (2001), ...
- some other stochastic volatility models; Sin (1998), ...
- allowing some classical arbitrage; Loewenstein & Willard (2000), Pl. (2002), Fernholz & Karatzas (2005), ...

Numéraire Portfolio S_t^* as Benchmark

- tradeable for modeling, investing and pricing
- **supermartingale property**:

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^*} \geq E(\hat{S}_s^\delta | \mathcal{F}_t)$$

$0 \leq t \leq s \leq \infty$

Long (1990), Becherer (2001), Pl. (2002), Goll & Kallsen (2003), Pl. & Heath (2006), Christensen & Pl. (2005), Karatzas & Kardaras (2007), ...

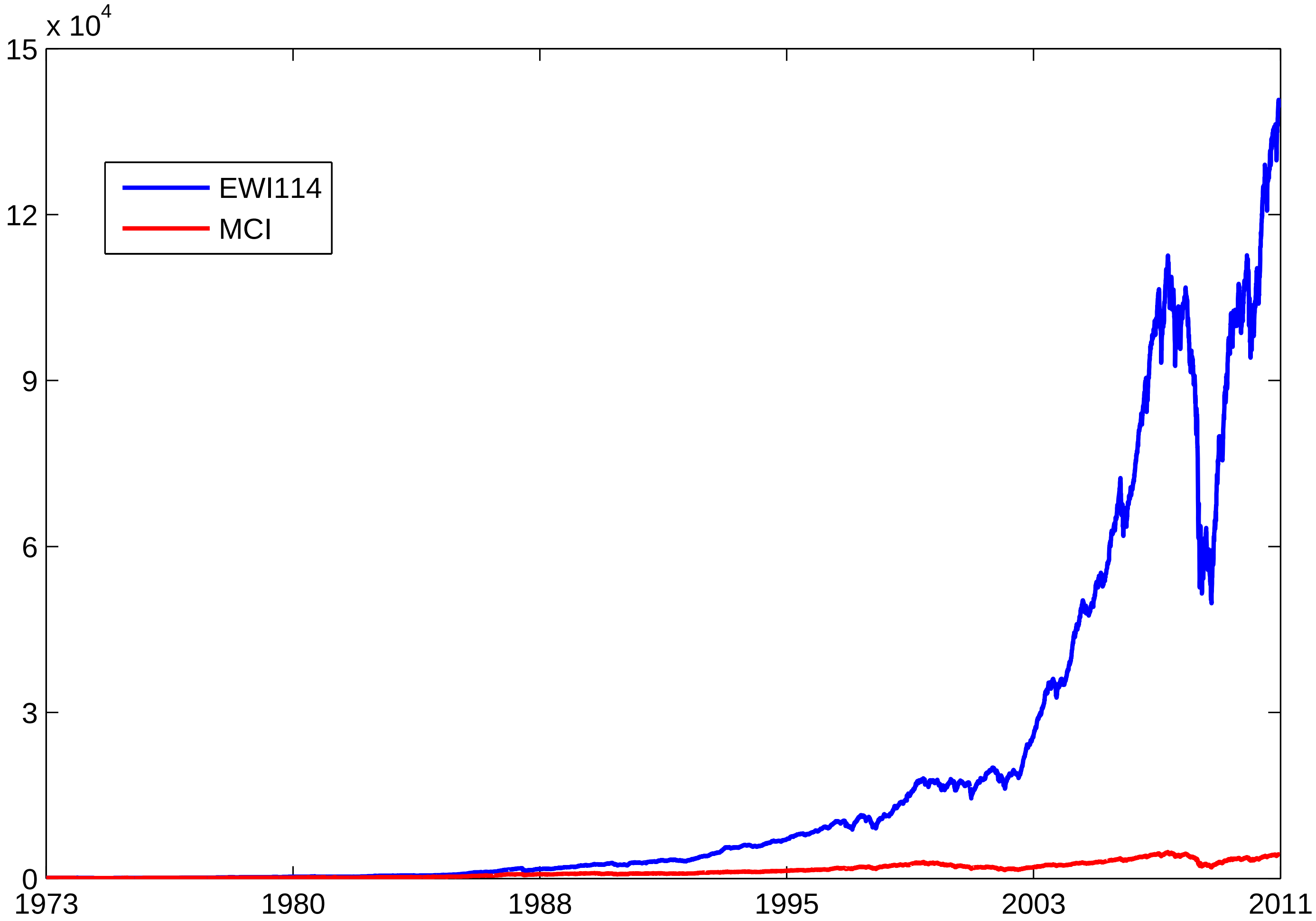
- **growth optimal portfolio** is NP $S^*_{t_i}$
  Kelly (1956), ..., MacLean et al. (2011)

- **long term growth rate**

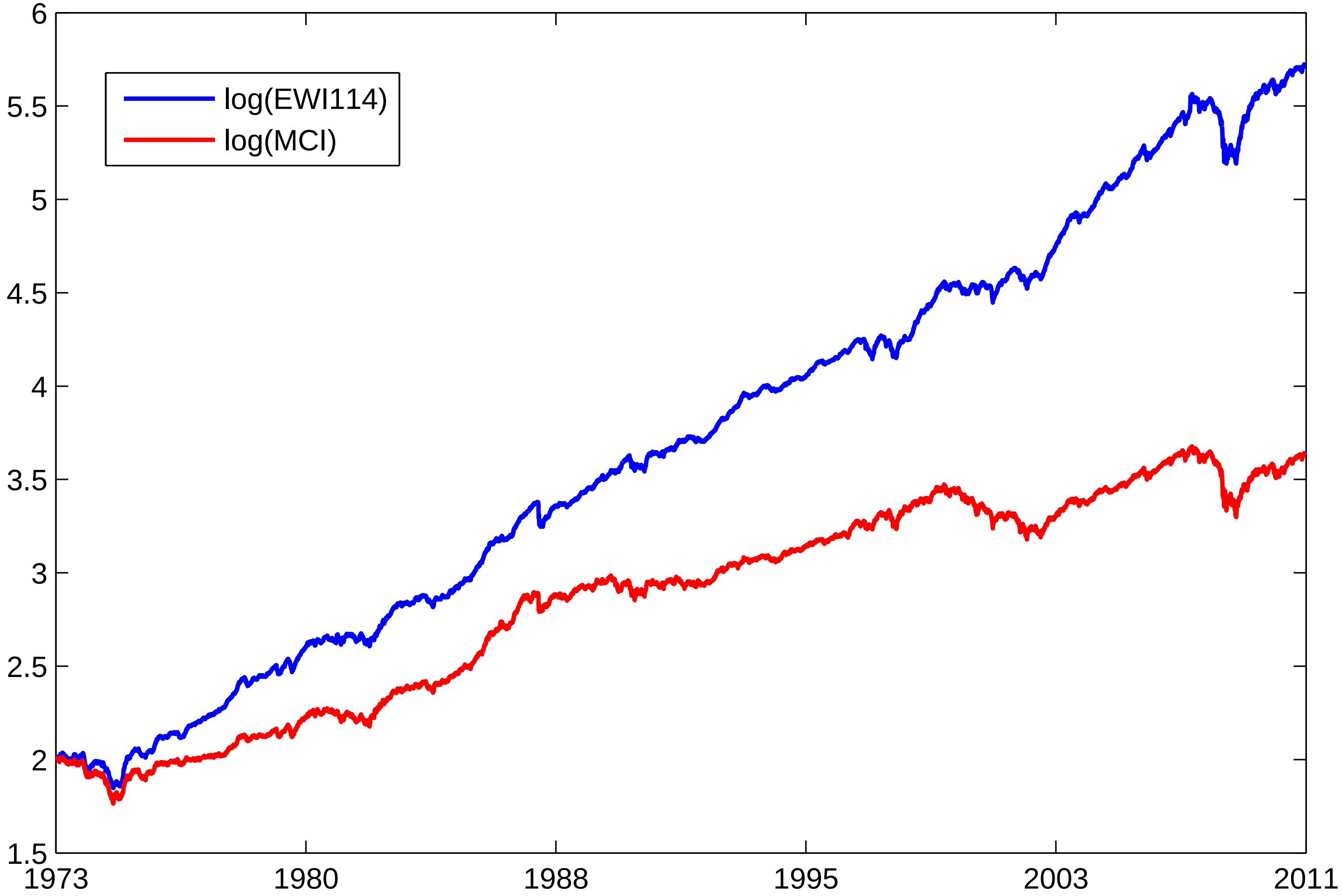
$$g = \lim_{t\to\infty} \sup \frac{1}{t} \ln\left(\frac{S^*_t}{S^*_0}\right) \text{ -maximal P-a.s.}$$

Diversification Approximates NP

- **diversification theorem**
  Pl. (2005), Le & Pl. (2006)
- **naive diversification**
  Pl. & Rendek (2012)
  model independent



EWI114 and MCI



Logarithms of EWI114 and MCI

Maximum Drawdown Constrained Portfolios

Kardaras, Obloj & Pl. (2012)
Cheredito, Nikeghbali & Pl. (2012)

$\mathcal{X}$ - set of nonnegative continuous discounted portfolios

- **running maximum**
  for $X = \{X_t, t \geq 0\} \in \mathcal{X}$

$$X_t^* = \sup_{u \in [0,t]} X_u$$

- **relative drawdown**

$$\frac{X_t}{X_t^*}$$

- **maximum relative drawdown**

express attitude towards risk by restricting to $\boldsymbol{X \in {}^{\alpha}\mathcal{X}}$, where

$$\frac{X_t}{X_t^*} \geq \alpha \;\; , \alpha \in [0, 1)$$

pathwise criterion

- **maximum drawdown constrained portfolio**

  $\alpha \in [0,1)$, $X \in \mathcal{X}$

$$
\begin{aligned}
{}^{\alpha}X_t &= \alpha (X_t^*)^{1-\alpha} + (1-\alpha) X_t (X_t^*)^{-\alpha} \\
&= 1 + \int_0^t (1-\alpha)(X_s^*)^{-\alpha} dX_s \geq \alpha^{\alpha} X_t^*
\end{aligned}
$$

  Grossman & Zhou (1993), Cvitanic & Karatzas (1994)

$\Longrightarrow$ SDE

$$\frac{d^{\alpha}X_t}{^{\alpha}X_t} = {}^{\alpha}\pi_t \frac{dX_t}{X_t}$$

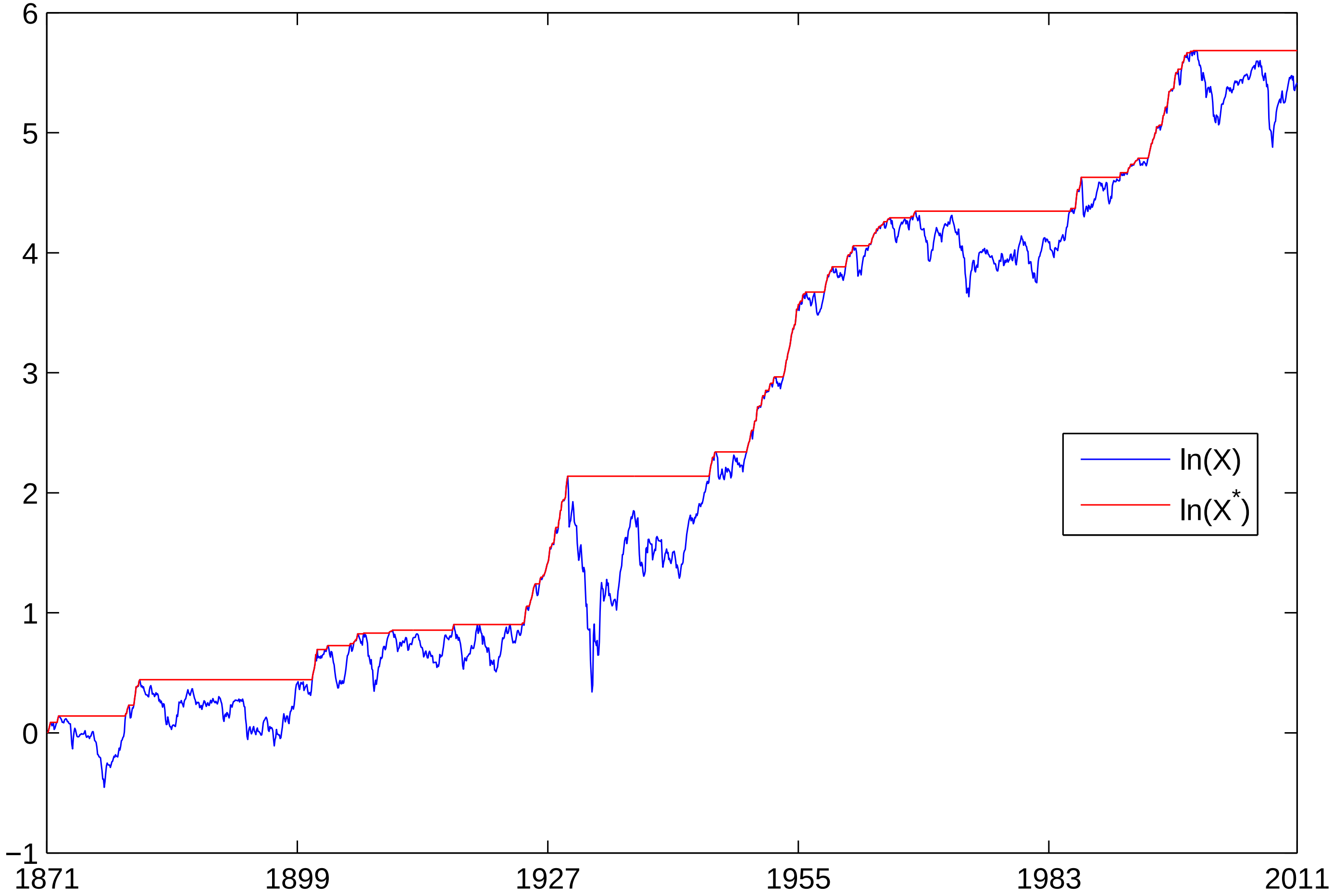
fraction

$${}^{\alpha}\pi_t = 1 - \frac{(1-\alpha)\frac{X_t}{X_t^*}}{\alpha + (1-\alpha)\frac{X_t}{X_t^*}}$$

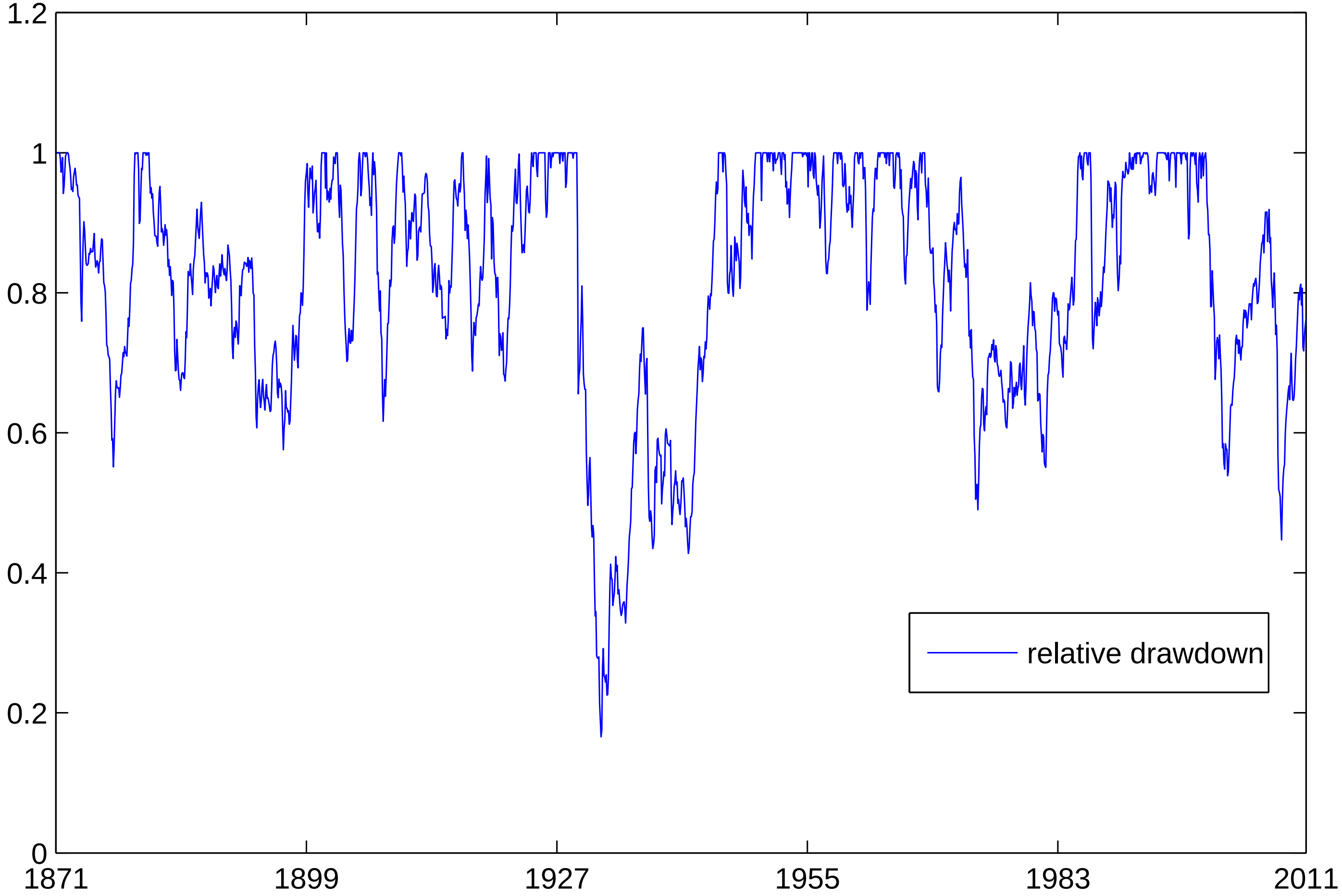
model independent

depends only on $\frac{X_t}{X_t^*}$ and α

${}^{\alpha}\bar{S}_t^*$ - invests in $\bar{S}^*$ and savings account



Logarithm of discounted S&P500 and its running maximum



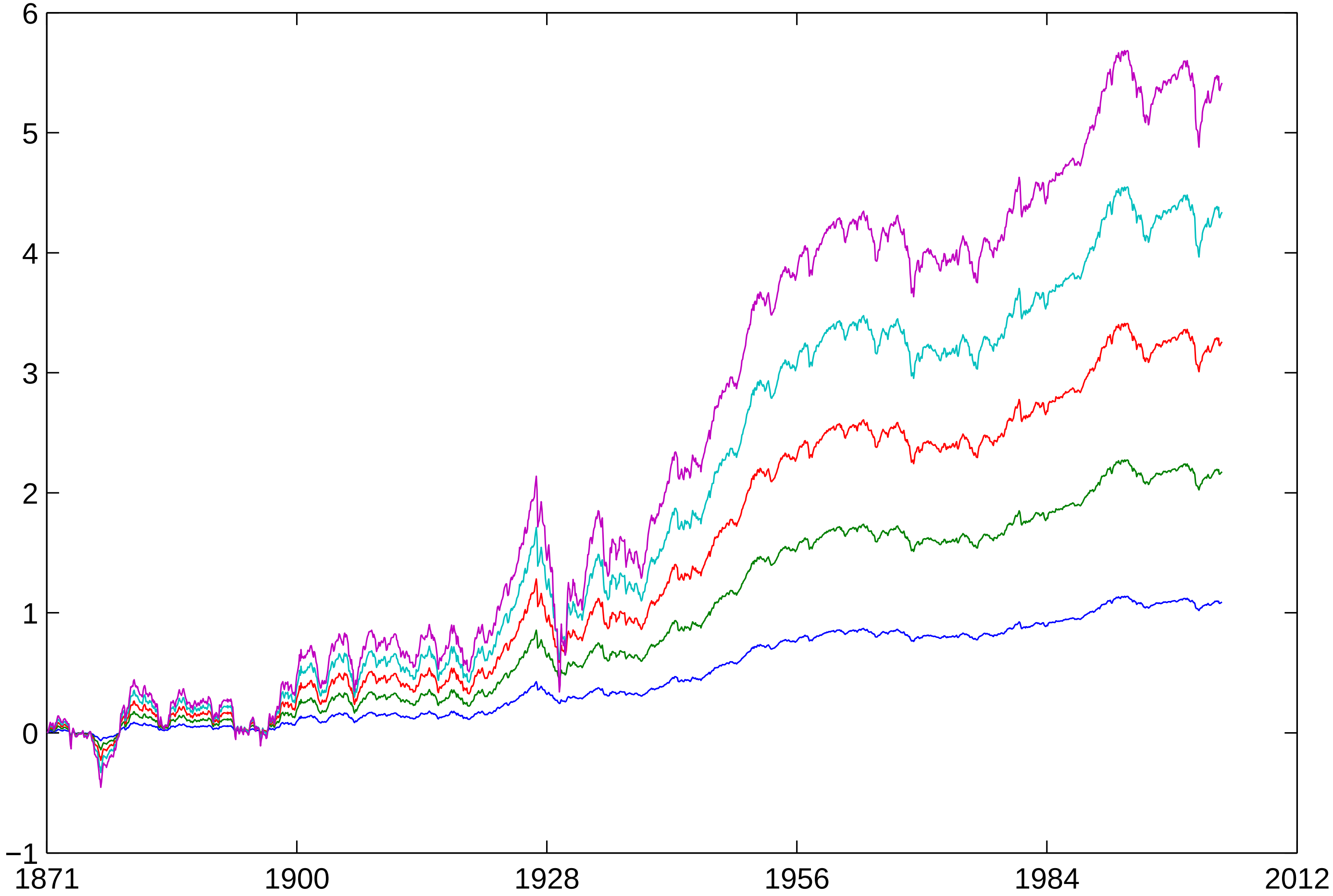
Relative drawdown of discounted S&P500

- **asymptotic maximum long term growth rate**
  Kardaras, Obloj & Pl. (2012)

$$\lim_{t\to\infty} \frac{1}{t}\log({}^{\alpha}S_t^*) = (1-\alpha) \lim_{t\to\infty} \frac{1}{t}\log(S_t^*)$$

$${}^{\alpha}g = (1-\alpha)g$$

restricted drawdown $\Longrightarrow$ reduced maximum growth rate
long term view with short term attitude towards risk
realistic alternative to Markowitz mean-variance approach
and utility maximization



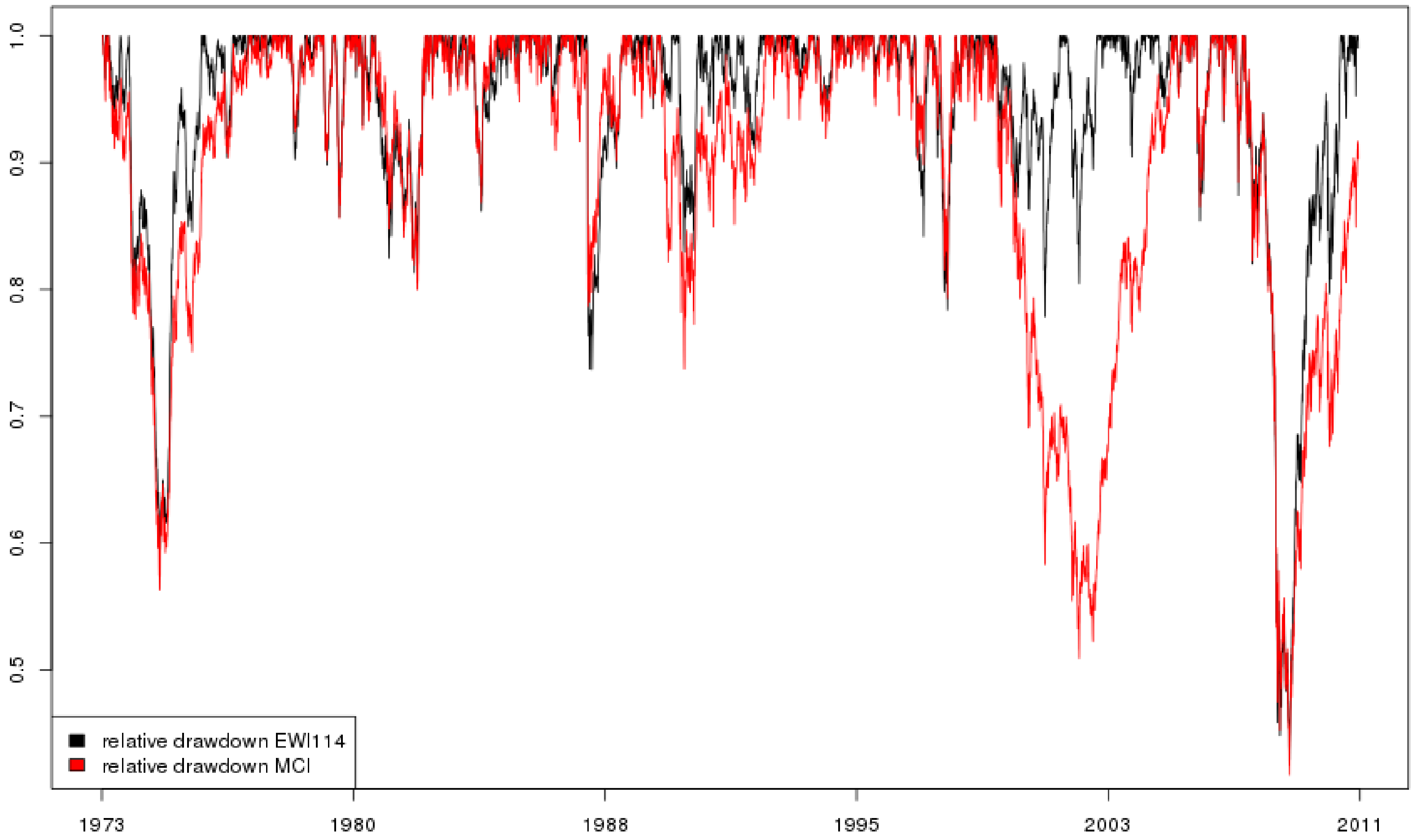
Logarithm of drawdown constrained portfolios

Shortest Expected Market Time to Reach a Level

Kardaras & Pl. (2010)

Works together with maximum drawdown constraint
$\Rightarrow$ new type of fund management

- long term view
- short term attitude
- pathwise properties
- model independent



Relative drawdown of MCI and EWI114

Utility Maximization

Kardaras & Pl. (2013), Pl. & Heath (2006)

$$\hat{S}_t^\delta = E(U'^{-1}(\hat{S}_T^0)\hat{S}_T^0|\mathcal{F}_t)$$

- two fund separation $\Rightarrow$ some efficient frontier
- substituting market portfolio by NP $\Rightarrow$ modified CAPM
- does not maximize Sharpe ratio for jump case
  Christensen & Pl. (2007)

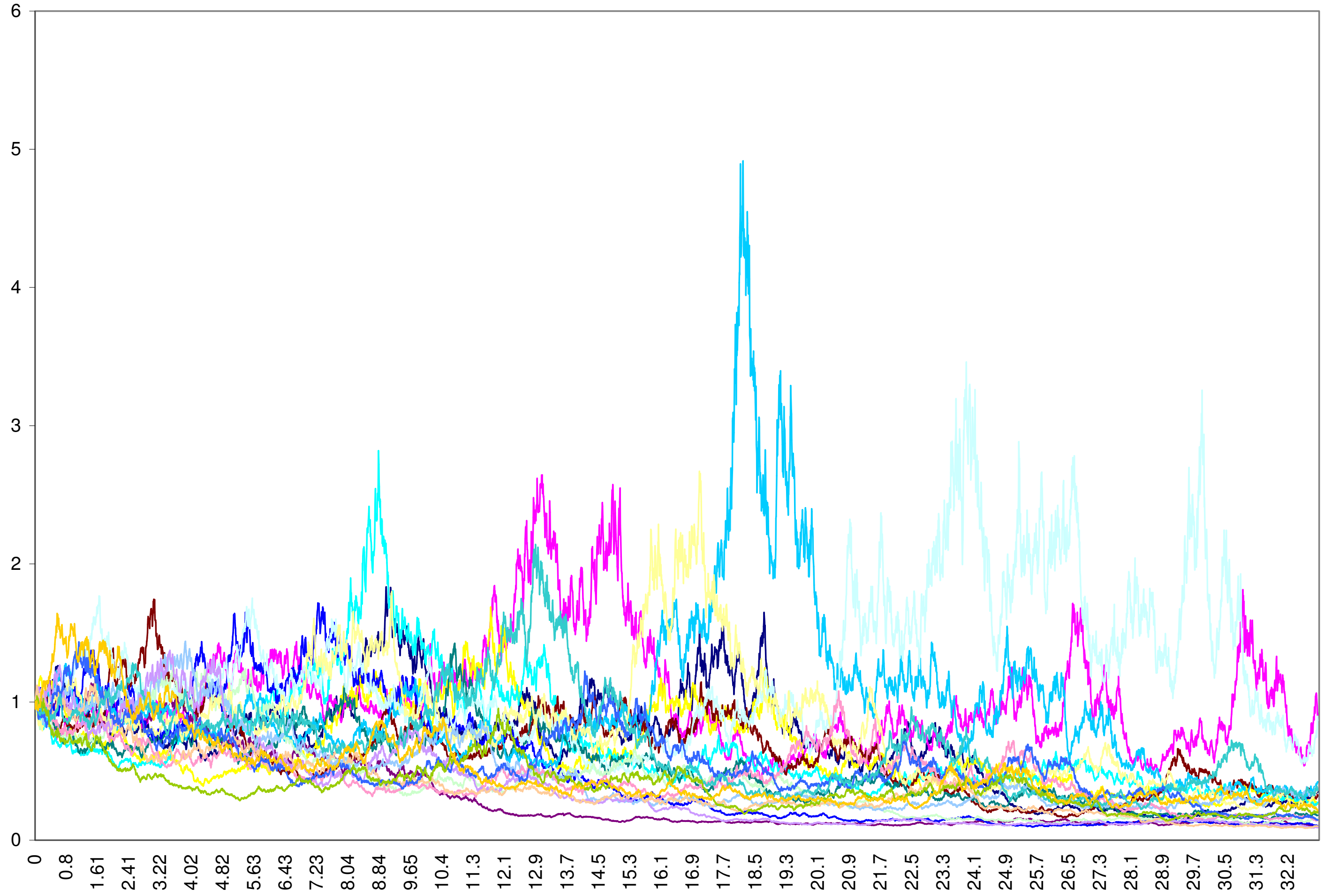
Last Passage Times

Nikeghbali & Pl. (2008, 2013); Profeta, Roynette & Yor (2010)

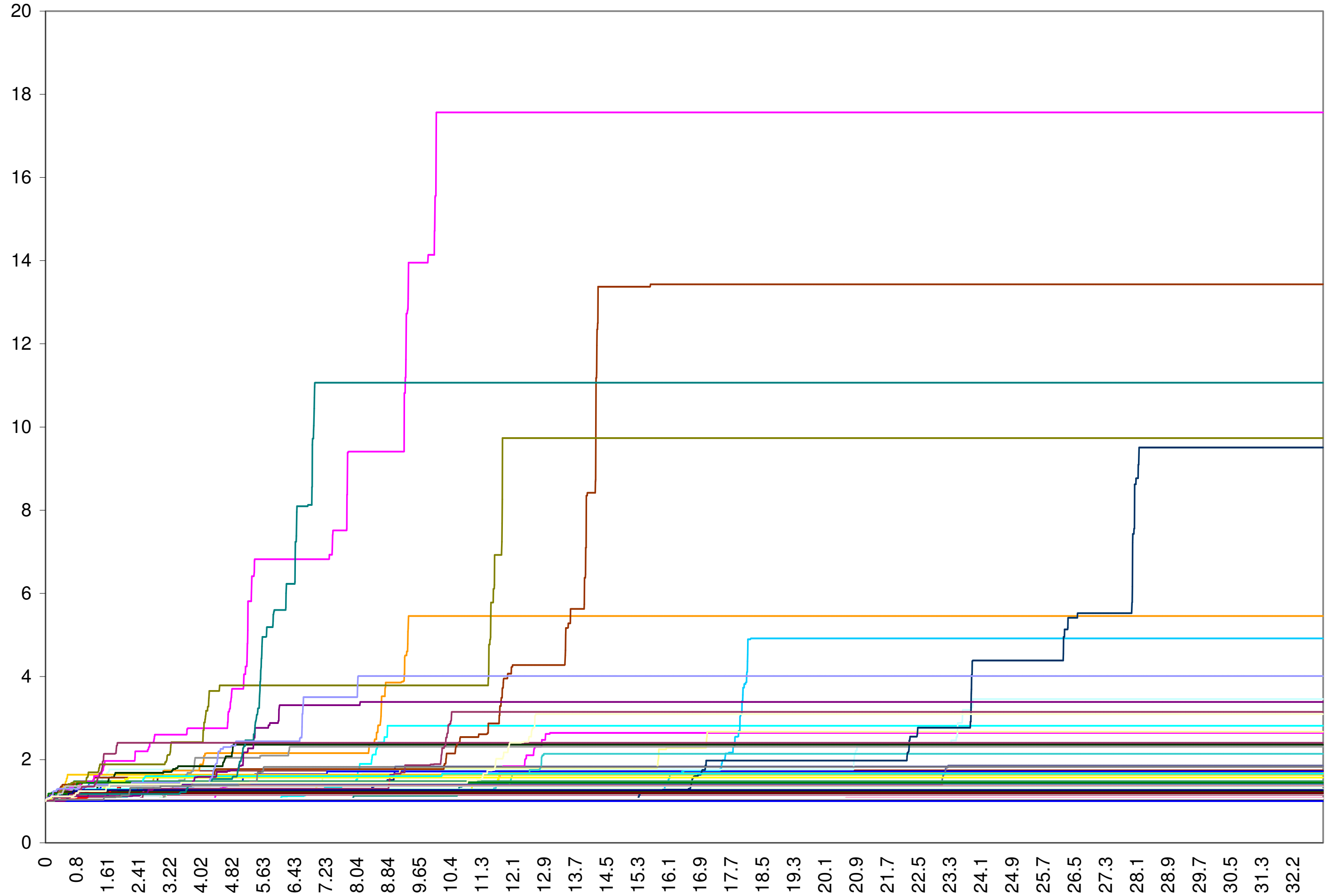
N_t-local martingale, no positive jumps, $\lim_{t\to\infty} N_t = 0$
$\Rightarrow$

$$\frac{N_0}{\lim_{t\to\infty} \sup_{t\geq 0} N_t} \sim U(0,1)$$

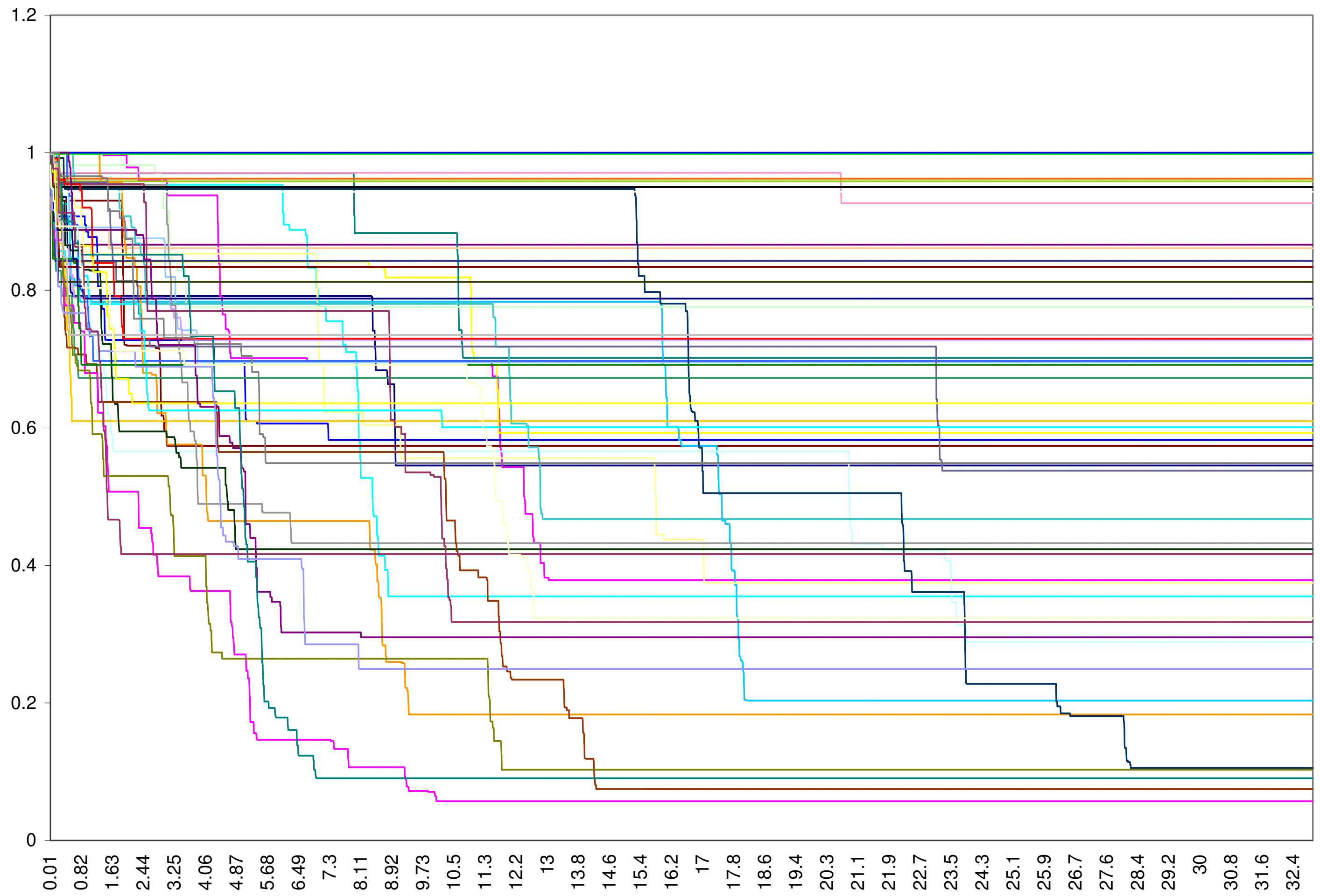
- model independent



Trajectories of N_t under MMM



Running maxima



Inverse of maxima

Valuation and Pricing

- supermartingale property for all price processes $\Rightarrow$ NUPBR

$$\hat{S}_t^{\delta_{H_T}} \geq E\left(\frac{H_T}{S_T^*} \Big| \mathcal{F}_t\right)$$

- martingale is the minimal supermartingale, Du & Pl. (2013)
  $\Rightarrow$ minimal price:
  real world pricing formula

$$\hat{S}_t^{\delta_{H_T}} = E\left(\frac{H_T}{S_T^*} \Big| \mathcal{F}_t\right)$$

$$\Rightarrow S_t^{\delta_{H_T}} = S_t^* E\left(\frac{H_T}{S_T^*} \Big| \mathcal{F}_t\right)$$

If S^*_T and H_T independent $\Rightarrow$ **Actuarial pricing formula**

$$S_t^{\delta_{H_T}} = P(t,T)E(H_T|\mathcal{F}_t)$$

If Radon-Nikodym derivative $\Lambda_t = \frac{\hat{S}_t^0}{\hat{S}_0^0}$ martingale $\Rightarrow$

Risk neutral pricing formula

$$S_t^{\delta_{H_T}} = E(\frac{\Lambda_T}{\Lambda_t}\frac{S_t^0 H_T}{S_T^0}|\mathcal{F}_t) = E_Q(\frac{S_t^0 H_T}{S_T^0}|\mathcal{F}_t)$$

Benchmarked Risk Minimization

Föllmer & Sondermann (1986), Föllmer & Schweizer (1991)
Du & Pl. (2013), Biagini, Cretarola & Pl. (2014)

- jth benchmarked primary security account

$$\hat{S}_t^j$$

local martingale

Dynamic Trading Strategy

$v = \{v_t = (\eta_t, \vartheta_t^1, \dots, \vartheta_t^d)^\top, t \in [0, \infty)\}$ forms benchmarked price process

$$\hat{V}_t^v = \vartheta_t^\top \hat{\mathbf{S}}_t + \eta_t$$

ϑ predictable,

η adapted, $\eta_0 = 0$, monitors non-self-financing part of supermartingale

$$\hat{V}_t^v = \hat{V}_0^v + \int_0^t \vartheta_{\mathrm{s}}^\top \mathrm{d}\hat{\mathbf{S}}_{\mathrm{s}} + \eta_{\mathrm{t}}$$

- benchmarked contingent claim $\hat{H}_T$
  v *delivers* $\hat{H}_T$ if

$$\hat{V}_T^v = \hat{H}_T$$

$\Longrightarrow$ *replicable* if self-financing

Benchmarked P&L

$$\hat{C}_t = \hat{V}_t^v - \sum_{j=1}^{d} \int_0^t \vartheta_u^j d\hat{S}_u^j - \hat{V}_0^v$$

$\Rightarrow \hat{C}_t = \eta_t$ for $t \in [0, \infty)$

- benchmarked P&L usually fluctuating
- intrinsic risk

What criterion would be most natural?

- **symmetric** view with respect to all primary security accounts, in particular the domestic savings account
- **pooling** in large trading book
  $\Longrightarrow$ vanishing total hedge error, P&L

Pooling

$\hat{H}_{T,l}$; $\hat{V}^{v_l}$ with $\hat{C}^{v_l}$ independent square integrable martingale with $E\left(\left(\frac{\hat{C}_t^{v_l}}{\hat{V}_0^{v_l}}\right)^2\right) \leq K_t < \infty$ for $l \in \{1, 2, \dots\}$,
well diversified trading book holds equal fractions at initial time:

total benchmarked wealth $\hat{U}_t = \frac{\hat{U}_0}{m} \sum_{l=1}^{m} \frac{\hat{V}_t^{v_l}}{\hat{V}_0^{v_l}}$

total benchmarked P&L

$$\hat{C}_m(t) = \frac{\hat{U}_0}{m} \sum_{l=1}^{m} \frac{\hat{C}_t^{v_l}}{\hat{V}_0^{v_l}} \; ,$$

$\Rightarrow$

$$\lim_{m \to \infty} \hat{C}_m(t) = 0$$

P-a.s.

- $\mathcal{V}_{\hat{H}_T}$ set of strategies v delivering $\hat{H}_T$ with orthogonal benchmarked P&L

  that is, η and $\eta\hat{S}$ are local martingales

- market participants prefer **more for less**

$\Rightarrow$ **Benchmarked Risk Minimization**

For $\hat{H}_T$ strategy $\widetilde{v} \in \mathcal{V}_{\hat{H}_T}$ *benchmarked risk minimizing* (BRM) if for all $v \in \mathcal{V}_{\hat{H}_T}$ price $\hat{V}_t^{\tilde{v}}$ is minimal

$$\hat{V}_t^{\tilde{v}} \leq \hat{V}_t^{v}$$

P-a.s. for all $t \in [0, T]$.

Regular Benchmarked Contingent Claims

$\hat{H}_T$ is called *regular* if

$$\hat{H}_T = E_t(\hat{H}_T) + \sum_{j=1}^{d} \int_t^T \vartheta^j_{\hat{H}_T}(s) d\hat{S}^j_s + \eta_{\hat{H}_T}(T) - \eta_{\hat{H}_T}(t)$$

$\vartheta_{\hat{\mathbf{H}}_\mathbf{T}}$ - predictable

$\eta_{\hat{H}_T}$ - local martingale, adapted, orthogonal to $\hat{S}$

$\Rightarrow$ regular $\hat{H}_T$ has BRM strategy v with

$$\hat{V}_t^v = E(\hat{H}_T | \mathcal{F}_t) \,,$$

and orthogonal benchmarked P&L: $\hat{C}_t = \eta_{\hat{H}_T}(t)$

- general semimartingale market
- no second moments required
- no risk neutral measure required
- takes evolving information about nonhedgeable part of claim into account
- $\boldsymbol{H_T}$ nonhedgeable $\Rightarrow \boldsymbol{\delta_t^0 = E(H_T|\mathcal{F}_t)}$

- hedge designed for pooling
- under minimal martingale measure, see Schweizer (1995), prices as under local risk-minimization but hedge different

Conjectured Model for Well-diversified Equity Indices

Filipovic & Pl. (2009), Pl. & Rendek (2012)

- well diversified portfolio $\approx$ NP
- discounted NP time transformed squared Bessel process
- **normalized NP**

$$d\boldsymbol{Y}_{\boldsymbol{\tau}} = (1 - \boldsymbol{Y}_{\boldsymbol{\tau}})d\boldsymbol{\tau} + \sqrt{\boldsymbol{Y}_{\boldsymbol{\tau}}}\,d\boldsymbol{W}(\boldsymbol{\tau})$$

- nature of feedback in diversified wealth

- **market activity time**

$$\frac{d\tau_t}{dt} = \frac{1}{Z_t}$$

- **inverse market activity Z_t**

trading behaviour exaggerates volatility, Kahneman & Tversky (1979)
fast moving

$$dZ_t = (\gamma - \varepsilon Z_t)dt + \sqrt{\gamma Z_t}dW_t$$

$$dW_t = \sqrt{Z_t}dW(\tau_t)$$

$\Rightarrow$ **discounted index model**

$$S_t = A_{\tau_t} Y_{\tau_t}$$

$$A_{\tau_t} = A_0 \exp\{a\tau_t\}$$

$$a \geq 1 \Rightarrow \hat{S}^0 \text{supermartingale}$$

$\Rightarrow$

- volatility

$$\sigma_t = \sqrt{\frac{1}{Y_{\tau_t} Z_t}}$$

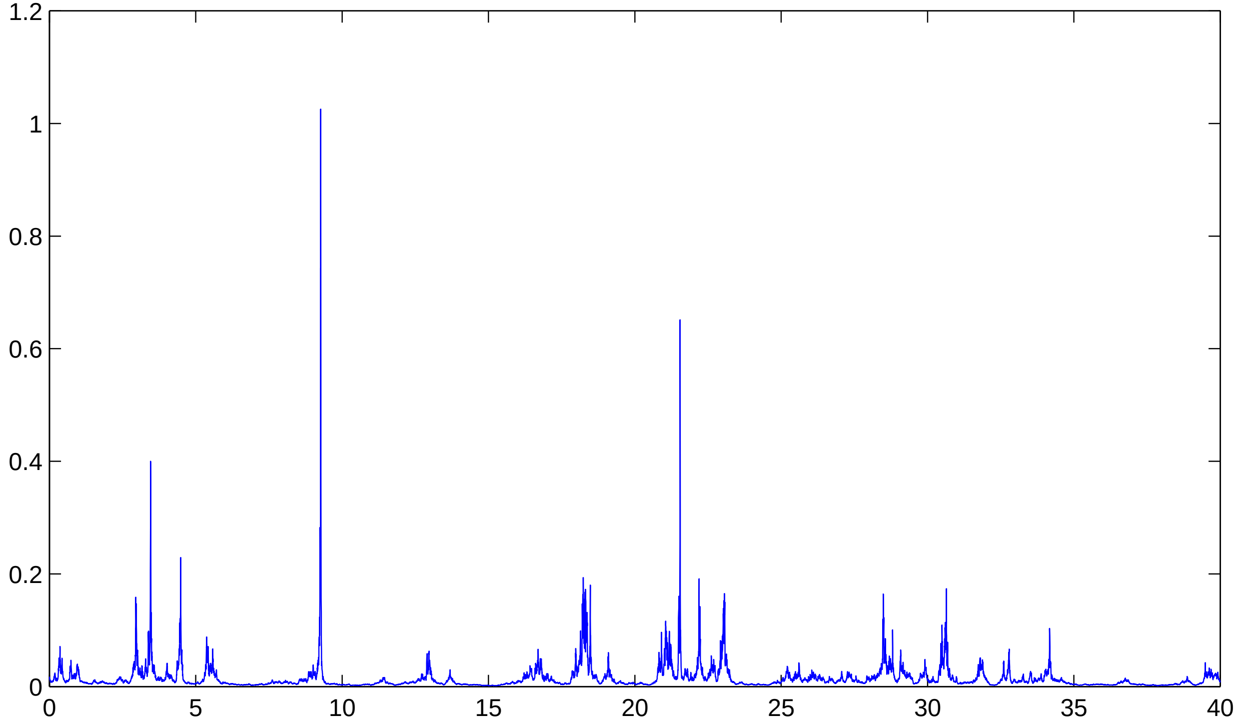
- expected rate of return

$$\mu_t = \frac{a-1}{Z_t} + \sigma_t^2$$

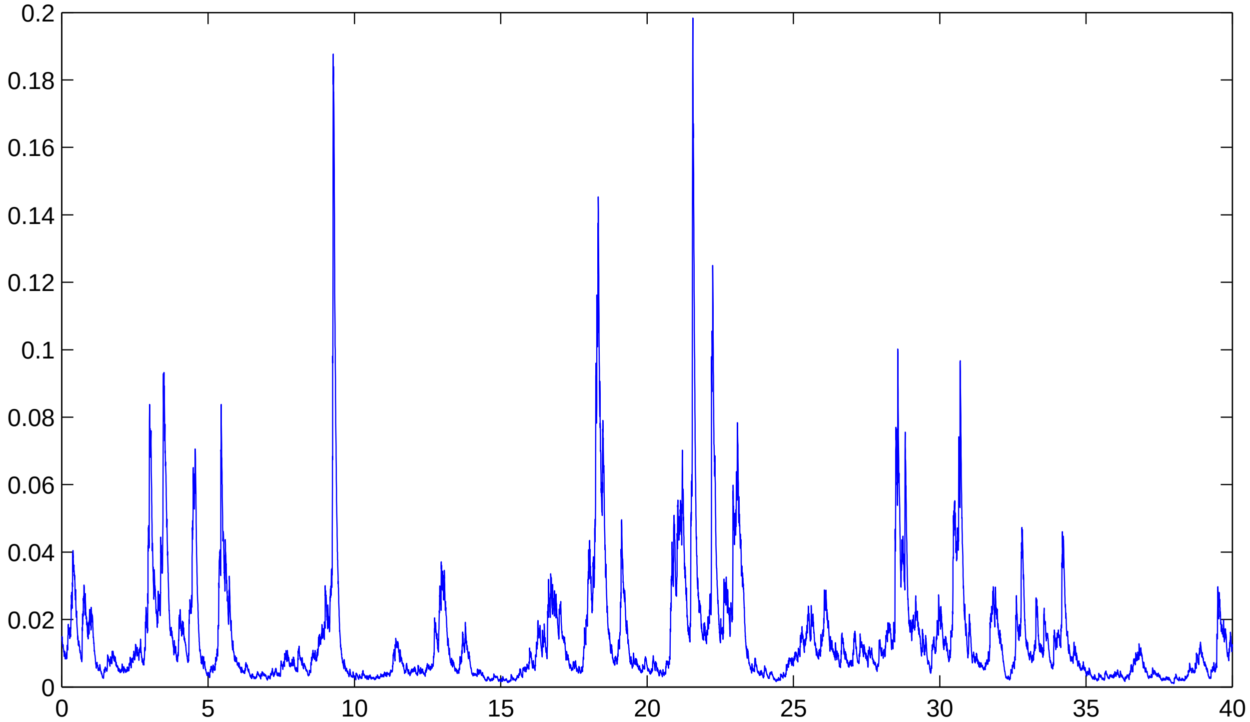
- long term average market time

$$\tau_t \approx \frac{2\varepsilon}{\gamma} t$$

- almost exact simulation



Simulated market activity



Estimated market activity of simulated index

- **conjectured parsimonious model:**

  3 initial parameters: $\boldsymbol{A_0}$, $\boldsymbol{Y_0}$, $\boldsymbol{Z_0}$

  3 structural parameters: $\boldsymbol{a}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\gamma}$

  1 driving Brownian motion (nondiversifiable equity risk)

  highly tractable, Heath & Pl. (2014), Baldeaux & Pl. (2013)

- for long term tasks average market activity $\bar{M} = const.$
  if $\hat{S}^0$ local martingale $\Rightarrow$

  3 initial parameters: $A_0, Y_0, Z_0 = \frac{2}{\bar{M}}$
  1 driving Brownian motion

  analytically tractable
  explicit call, put, binary, bond, ...

- Popper (1935, 2002): one can only falsify models
- conjectured model difficult to falsify: MCI, EWI114, S&P500, ...
- most known index models can be falsified
  by some of **10** listed stylized empirical facts, Pl. & Rendek (2014)
- ideal for pricing **long-dated pension and insurance contracts**
  Baldeaux & Pl. (2013), Heath & Pl. (2014), Baldeaux, Grasselli & Pl. (2014) (FX)
- model leads **beyond classical paradigm**

- BA generalizes classical theory
- several ”puzzles” can be naturally explained
- no representative agent employed
- long term model as simple as BS model

- significant performance improvement in long term fund management
- major savings in pension and insurance products
- systematic, efficient approach to risk measurement and regulation, Pl. & Stahl (2005)