Beyond the Classical Paradigm

Eckhard Platen
University of Technology Sydney


Pl. & Bruti-Liberati (2010), Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer

Baldeaux & Pl. (2013). Functionals of Multidimensional Diffusions with Applications to Finance. Springer
Risk neutral and fair put on index
Benchmarked “risk neutral” and fair put on index
Complications with Classical Approach

- inverse of 3-dimensional Bessel process; Delbaen & Schachermayer (1995)
- $\frac{3}{2}$ volatility model; Pl. (1997), Heston (1997), Lewis (2000), Pl. (2001), ...
- some other stochastic volatility models; Sin (1998), ...
- allowing some classical arbitrage; Loewenstein & Willard (2000), Pl. (2002), Fernholz & Karatzas (2005), ...
Numéraire Portfolio $S_t^*$ as Benchmark

- tradeable for modeling, investing and pricing
- supermartingale property:

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^*} \geq E(\hat{S}_s^\delta | \mathcal{F}_t)$$

$0 \leq t \leq s \leq \infty$


...
• growth optimal portfolio is NP $S_{t_i}^*$
  Kelly (1956), ..., MacLean et al. (2011)

• long term growth rate

$$g = \lim_{t \to \infty} \sup \frac{1}{t} \ln\left(\frac{S_t^*}{S_0^*}\right)$$
  -maximal P-a.s.
Diversification Approximates NP

- *diversification theorem*
  Pl. (2005), Le & Pl. (2006)

- *naive diversification*
  Pl. & Rendek (2012)
  model independent
EWI114 and MCI
Logarithms of EWI114 and MCI
Maximum Drawdown Constrained Portfolios

Kardaras, Obloj & Pl. (2012)
Cheredito, Nikeghbali & Pl. (2012)

\( \mathcal{X} \) - set of nonnegative continuous discounted portfolios

- running maximum
  
  for \( X = \{ X_t, t \geq 0 \} \in \mathcal{X} \)

  \[ X^*_t = \sup_{u \in [0, t]} X_u \]

- relative drawdown

  \[ \frac{X_t}{X^*_t} \]
• maximum relative drawdown

express attitude towards risk by restricting to $X \in \mathcal{X}^\alpha$, where

$$\frac{X_t}{X_t^*} \geq \alpha, \quad \alpha \in [0, 1)$$

pathwise criterion
• maximum drawdown constrained portfolio
  \( \alpha \in [0, 1), \ X \in \mathcal{X} \)

  \[
  \alpha X_t = \alpha (X_t^*)^{1-\alpha} + (1 - \alpha) X_t (X_t^*)^{-\alpha}
  \]

  \[
  = 1 + \int_0^t (1 - \alpha) (X_s^*)^{-\alpha} dX_s \geq \alpha^\alpha X_t^*
  \]

SDE

\[ \frac{d^\alpha X_t}{\alpha X_t} = \alpha \pi_t \frac{dX_t}{X_t} \]

fraction

\[ \alpha \pi_t = 1 - \frac{(1 - \alpha) \frac{X_t}{X_t^*}}{\alpha + (1 - \alpha) \frac{X_t}{X_t^*}} \]

model independent
depends only on \( \frac{X_t}{X_t^*} \) and \( \alpha \)

\( \alpha \bar{S}_t^* \) - invests in \( \bar{S}^* \) and savings account
Logarithm of discounted S&P500 and its running maximum
Relative drawdown of discounted S&P500
- asymptotic maximum long term growth rate
  Kardaras, Obloj & Pl. (2012)

\[
\lim_{t \to \infty} \frac{1}{t} \log(\alpha S_t^*) = (1 - \alpha) \lim_{t \to \infty} \frac{1}{t} \log(S_t^*)
\]

\[\alpha g = (1 - \alpha)g\]

restricted drawdown $\implies$ reduced maximum growth rate
long term view with short term attitude towards risk
realistic alternative to Markowitz mean-variance approach
and utility maximization
Logarithm of drawdown constrained portfolios
Shortest Expected Market Time to Reach a Level

Kardaras & Pl. (2010)

Works together with maximum drawdown constraint
⇒ new type of fund management

- long term view
- short term attitude
- pathwise properties
- model independent
Relative drawdown of MCI and EWI114
Utility Maximization


\[ \hat{S}_t^\delta = E(U'^{-1}(\hat{S}_T^0)\hat{S}_T^0|\mathcal{F}_t) \]

- two fund separation \( \Rightarrow \) some efficient frontier
- substituting market portfolio by NP \( \Rightarrow \) modified CAPM
- does not maximize Sharpe ratio for jump case

Christensen & Pl. (2007)
Last Passage Times


$N_t$-local martingale, no positive jumps, $\lim_{t \to \infty} N_t = 0$

$\Rightarrow$

$$\frac{N_0}{\lim_{t \to \infty} \sup_{t \geq 0} N_t} \sim U(0, 1)$$

- model independent
Trajectories of $N_t$ under MMM
Running maxima
Inverse of maxima
Valuation and Pricing

- supermartingale property for all price processes ⇒ NUPBR

\[ \hat{S}_t^{\delta_{HT}} \geq E\left( \frac{H_T}{S_T^*} \mid \mathcal{F}_t \right) \]

- martingale is the minimal supermartingale, Du & Pl. (2013) ⇒ minimal price:

  real world pricing formula

\[ \hat{S}_t^{\delta_{HT}} = E\left( \frac{H_T}{S_T^*} \mid \mathcal{F}_t \right) \]

\[ \Rightarrow S_t^{\delta_{HT}} = S_t E\left( \frac{H_T}{S_T^*} \mid \mathcal{F}_t \right) \]
If $S^*_T$ and $H_T$ independent $\Rightarrow$ Actuarial pricing formula

$$S^\delta_{H_T} = P(t, T)E(H_T | \mathcal{F}_t)$$
If Radon-Nikodym derivative $\Lambda_t = \frac{S_t^0}{S_0^0}$ martingale $\Rightarrow$

Risk neutral pricing formula

$$S_t^{\delta_{HT}} = E\left( \frac{\Lambda_T}{\Lambda_t} \frac{S_t^0 H_T}{S_T^0} | \mathcal{F}_t \right) = E_Q\left( \frac{S_t^0 H_T}{S_T^0} | \mathcal{F}_t \right)$$
Benchmarked Risk Minimization

Du & Pl. (2013), Biagini, Cretarola & Pl. (2014)

- $j$th benchmarked primary security account

\[ \hat{S}_t^j \]

local martingale
Dynamic Trading Strategy

\[ \nu = \{ \nu_t = (\eta_t, \vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0, \infty) \} \] forms benchmarked price process

\[ \hat{V}_t^\nu = \vartheta_t^\top \hat{S}_t + \eta_t \]

\( \vartheta \) predictable,

\( \eta \) adapted, \( \eta_0 = 0 \), monitors non-self-financing part of supermartingale

\[ \hat{V}_t^\nu = \hat{V}_0^\nu + \int_0^t \vartheta_s^\top d\hat{S}_s + \eta_t \]
• benchmarked contingent claim $\hat{H}_T$
  \( \nu \) delivers $\hat{H}_T$ if

\[
\hat{V}^\nu_T = \hat{H}_T
\]

\[\implies\text{replicable if self-financing}\]
Benchmarked P&L

\[ \hat{C}_t = \hat{V}_t^\nu - \sum_{j=1}^{d} \int_0^t \Theta_u^j d\hat{S}_u^j - \hat{V}_0^\nu \]

\[ \Rightarrow \hat{C}_t = \eta_t \text{ for } t \in [0, \infty) \]
- benchmarked P&L usually fluctuating

- intrinsic risk
What criterion would be most natural?

- **symmetric** view with respect to all primary security accounts, in particular the domestic savings account

- **pooling** in large trading book
  \[\implies\text{vanishing total hedge error, P&L}\]
Pooling

\( \hat{H}_T, l; \hat{V}^{vl} \) with \( \hat{C}^{vl} \) independent square integrable martingale with
\[
E \left( \left( \frac{\hat{C}_{t}^{vl}}{\hat{V}_{0}^{vl}} \right)^2 \right) \leq K_t < \infty \text{ for } l \in \{1, 2, \ldots \},
\]
well diversified trading book holds equal fractions at initial time:

\[
\text{total benchmarked wealth } \hat{U}_t = \frac{\hat{U}_0}{m} \sum_{l=1}^{m} \frac{\hat{V}_{t}^{vl}}{\hat{V}_{0}^{vl}}
\]

\[
\text{total benchmarked P&L}
\]
\[
\hat{C}_m(t) = \frac{\hat{U}_0}{m} \sum_{l=1}^{m} \frac{\hat{C}_{t}^{vl}}{\hat{V}_{0}^{vl}},
\]
\[
\Rightarrow
\]
\[
\lim_{m \to \infty} \hat{C}_m(t) = 0
\]
\( P \)-a.s.
• $\mathcal{V}_{\hat{H}_T}$ set of strategies $\nu$ delivering $\hat{H}_T$ with orthogonal benchmarked P&L

that is, $\eta$ and $\eta\hat{S}$ are local martingales
market participants prefer **more for less**

$$\Rightarrow \textbf{Benchmarked Risk Minimization}$$

For $\hat{H}_T$ strategy $\tilde{v} \in \mathcal{V}_{\hat{H}_T}$ *benchmark risk minimizing* (BRM) if for all $v \in \mathcal{V}_{\hat{H}_T}$ price $\hat{V}_t^{\tilde{v}}$ is minimal

$$\hat{V}_t^{\tilde{v}} \leq \hat{V}_t^{v}$$

$P$-a.s. for all $t \in [0, T]$. 
Regular Benchmarked Contingent Claims

$\hat{H}_T$ is called regular if

$$\hat{H}_T = E_t(\hat{H}_T) + \sum_{j=1}^{d} \int_{t}^{T} \vartheta_{\hat{H}_T}^j(s)d\hat{S}_s^j + \eta_{\hat{H}_T}(T) - \eta_{\hat{H}_T}(t)$$

$\vartheta_{\hat{H}_T}$ - predictable

$\eta_{\hat{H}_T}$ - local martingale, adapted, orthogonal to $\hat{S}$
$\Rightarrow$ regular $\hat{H}_T$ has BRM strategy $\nu$ with

$$\hat{V}_t^{\nu} = E(\hat{H}_T|\mathcal{F}_t),$$

and orthogonal benchmarked P&L: $\hat{C}_t = \eta_{\hat{H}_T}(t)$
• general semimartingale market
• no second moments required
• no risk neutral measure required
• takes evolving information about nonhedgeable part of claim into account
• $H_T$ nonhedgeable $\Rightarrow \delta_t^0 = E(H_T | \mathcal{F}_t)$
• hedge designed for pooling

• under minimal martingale measure, see Schweizer (1995), prices as under local risk-minimization but hedge different
Conjectured Model for Well-diversified Equity Indices

Filipovic & Pl. (2009), Pl. & Rendek (2012)

- well diversified portfolio $\approx$ NP
- discounted NP time transformed squared Bessel process
- normalized NP

$$dY_\tau = (1 - Y_\tau) d\tau + \sqrt{Y_\tau} dW(\tau)$$

- nature of feedback in diversified wealth
market activity time

\[
\frac{d\tau_t}{dt} = \frac{1}{Z_t}
\]

inverse market activity \( Z_t \)

trading behaviour exaggerates volatility, Kahneman & Tversky (1979)

fast moving

\[
dZ_t = (\gamma - \varepsilon Z_t)dt + \sqrt{\gamma Z_t}dW_t
\]

\[
dW_t = \sqrt{Z_t}dW(\tau_t)
\]
⇒ discounted index model

\[ S_t = A_{\tau_t} Y_{\tau_t} \]

\[ A_{\tau_t} = A_0 \exp\{a \tau_t\} \]

\[ a \geq 1 \Rightarrow \hat{S}^0 \text{ supermartingale} \]
\[ \sigma_t = \sqrt{\frac{1}{Y_{\tau_t} Z_t}} \]

- volatility

\[ \mu_t = \frac{a - 1}{Z_t} + \sigma_t^2 \]

- expected rate of return

\[ \tau_t \approx \frac{2\varepsilon}{\gamma} t \]

- long term average market time

- almost exact simulation
Simulated market activity
Estimated market activity of simulated index
• conjectured parsimonious model:

3 initial parameters: $A_0$, $Y_0$, $Z_0$
3 structural parameters: $a$, $\varepsilon$, $\gamma$
1 driving Brownian motion (nondiversifiable equity risk)

• for long term tasks average market activity $\bar{M} = \text{const.}$

if $\hat{S}_0$ local martingale $\Rightarrow$

3 initial parameters: $A_0, Y_0, Z_0 = \frac{2}{\bar{M}}$

1 driving Brownian motion

analytically tractable

explicit call, put, binary, bond, ...
• Popper (1935, 2002): one can only falsify models

• conjectured model difficult to falsify: MCI, EWI114, S&P500, ...

• most known index models can be falsified by some of 10 listed stylized empirical facts, Pl. & Rendek (2014)

• ideal for pricing long-dated pension and insurance contracts

• model leads beyond classical paradigm
• BA generalizes classical theory
• several ”puzzles” can be naturally explained
• no representative agent employed
• long term model as simple as BS model
• significant performance improvement in long term fund management

• major savings in pension and insurance products

• systematic, efficient approach to risk measurement and regulation, Pl. & Stahl (2005)