NUMERICAL METHODS OF QUANTITATIVE FINANCE

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Contents

1 Stochastic Expansions

Stochastic Taylor Expansions

Deterministic Taylor Formula

- ordinary differential equation

$$\frac{d}{dt} X_t = a(X_t)$$

- integral form

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)\, ds$$

deterministic chain rule

$\Longrightarrow$

$$\frac{d}{dt} f(X_t) = a(X_t) \frac{\partial}{\partial x} f(X_t)$$

- operator

$$L = a \, \frac{\partial}{\partial x}$$

$\Longrightarrow$

integral equation

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} L\, f(X_s)\, ds$$

- special case

  $f(x) \equiv x$

  $Lf = a, LLf = (L)^2 f = La, \ldots$

- apply to $f = a$

$\Longrightarrow$

$$
\begin{aligned}
\boldsymbol{X_t} &= \boldsymbol{X_{t_0} + \int_{t_0}^{t} \left(a(X_{t_0}) + \int_{t_0}^{s} L\, a(X_z)\, dz \right) ds} \\
&= \boldsymbol{X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + \int_{t_0}^{t} \int_{t_0}^{s} L\, a(X_z)\, dz\, ds}
\end{aligned}
$$

$\Longrightarrow$

nontrivial Taylor expansion

- apply $f = La$

$\Longrightarrow$

$$X_t = X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + L\, a(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dz\, ds + R_1$$

remainder term

$$R_1 = \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} (L)^2 a(X_u)\, du\, dz\, ds$$

- continuing

$\Longrightarrow$

classical deterministic Taylor formula

$$f(X_t) = f(X_{t_0}) + \sum_{l=1}^{r} \frac{(t-t_0)^\ell}{l!} (L)^\ell f(X_{t_0})$$

$$+ \int_{t_0}^{t} \cdots \int_{t_0}^{s_2} (L)^{r+1} f(X_{s_1})\, ds_1 \ldots ds_{r+1}$$

for $t \in [t_0, T]$ and $r \in \mathcal{N}$

Wagner-Platen Expansion

Wagner & Pl. (1978), Pl. (1982),

Pl. & Wagner (1982) and Kloeden & Pl. (1992a)

- SDE

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)\, ds + \int_{t_0}^{t} b(X_s)\, dW_s$$

- Itô formula

$$
\begin{aligned}
f(X_t) &= f(X_{t_0}) \\
&\quad + \int_{t_0}^{t} \left(a(X_s) \frac{\partial}{\partial x} f(X_s) + \frac{1}{2}\, b^2(X_s) \frac{\partial^2}{\partial x^2} f(X_s) \right) ds \\
&\quad + \int_{t_0}^{t} b(X_s) \frac{\partial}{\partial x} f(X_s)\, dW_s \\
&= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s)\, ds + \int_{t_0}^{t} L^1 f(X_s)\, dW_s
\end{aligned}
$$

operators

$$L^0 = a \frac{\partial}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2}{\partial x^2}$$

and

$$L^1 = b \frac{\partial}{\partial x}$$

$$f(x) \equiv x \quad \Longrightarrow \quad L^0 f = a \text{ and } L^1 f = b$$

- apply Itô formula to $f = a$ and $f = b$

$$
\begin{aligned}
X_t &= X_{t_0} \\
&\quad + \int_{t_0}^{t} \left(a(X_{t_0}) + \int_{t_0}^{s} L^0 a(X_z)\, dz + \int_{t_0}^{s} L^1 a(X_z)\, dW_z \right) ds \\
&\quad + \int_{t_0}^{t} \left(b(X_{t_0}) + \int_{t_0}^{s} L^0 b(X_z)\, dz + \int_{t_0}^{s} L^1 b(X_z)\, dW_z \right) dW_s \\
&= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + R_2
\end{aligned}
$$

remainder term

$$R_2 = \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(X_z)\,dz\,ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(X_z)\,dW_z\,ds$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z)\,dz\,dW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(X_z)\,dW_z\,dW_s$$

- apply Itô formula to $f = L^1 b$

$\Longrightarrow$

$$X_t = X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + L^1 b(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dW_z \, dW_s + R_3$$

remainder term

$$
\begin{aligned}
R_3 \;=\; & \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(X_z)\, dz\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(X_z)\, dW_z\, ds \\
& + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z)\, dz\, dW_s \\
& + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^0 L^1 b(X_u)\, du\, dW_z\, dW_s \\
& + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^1 L^1 b(X_u)\, dW_u\, dW_z\, dW_s
\end{aligned}
$$

example for Wagner-Platen expansion

- multiple Itô integrals

$$\begin{aligned}\int_{t_0}^{t} ds &= t - t_0,\\ \int_{t_0}^{t} dW_s &= W_t - W_{t_0},\\ \int_{t_0}^{t}\int_{t_0}^{s} dW_z\, dW_s &= \frac{1}{2}\Big((W_t - W_{t_0})^2 - (t - t_0)\Big)\end{aligned}$$

- remainder term R_3

  consisting of next following multiple Itô integrals

  with nonconstant integrands

Example

$d = m = 1$

for function $f(t, x) \equiv x$,

times $\varrho = 0$, $\tau = t$

hierarchical set $\quad A = \{\alpha \in \mathcal{M}_m : \ell(\alpha) \leq 3\}$

drift $\quad a(t, x) = a(x)$

diffusion coefficient $\quad b(t, x) = b(x)$

$$dX_t = a\,(X_t)\,dt + b(X_t)\,dW_t$$

$\Longrightarrow$

Wagner-Platen expansion in the form:

$$
\begin{aligned}
X_t \;=\; & X_0 + a\, I_{(0)} + b\, I_{(1)} + \left(a\, a' + \frac{1}{2}\, b^2 a'' \right) I_{(0,0)} \\
& + \left(a\, b' + \frac{1}{2}\, b^2 b'' \right) I_{(0,1)} + b\, a' I_{(1,0)} + b\, b' I_{(1,1)} \\
& + \Bigg[a \left(a\, a'' + (a')^2 + b\, b' a'' + \frac{1}{2}\, b^2 a''' \right) + \frac{1}{2}\, b^2 \left(a\, a''' + 3\, a' a'' \right. \\
& \quad \left. + \left((b')^2 + b\, b'' \right) a'' + 2\, b\, b' a''' \right) + \frac{1}{4}\, b^4 a^{(4)} \Bigg] I_{(0,0,0)} \\
& + \Bigg[a \left(a' b' + a\, b'' + b\, b' b'' + \frac{1}{2}\, b^2 b''' \right) + \frac{1}{2}\, b^2 \Big(a'' b' + 2\, a' b'' \\
& \quad + a\, b''' + \left((b')^2 + b\, b'' \right) b'' + 2\, b\, b' b''' + \frac{1}{2}\, b^2 b^{(4)} \Big) \Bigg] I_{(0,0,1)}
\end{aligned}
$$

$$
+ \left[a\left(b'a' + b\,a''\right) + \frac{1}{2}\,b^2\left(b''a' + 2\,b'a'' + b\,a'''\right)\right] I_{(0,1,0)}
$$

$$
+ \left[a\left((b')^2 + b\,b''\right) + \frac{1}{2}\,b^2\left(b''b' + 2\,b\,b'' + b\,b'''\right)\right] I_{(0,1,1)}
$$

$$
+ \,b\left(a\,a'' + (a')^2 + b\,b'a'' + \frac{1}{2}\,b^2a'''\right) I_{(1,0,0)}
$$

$$
+ \,b\left(a\,b'' + a'b' + b\,b'b'' + \frac{1}{2}\,b^2b'''\right) I_{(1,0,1)}
$$

$$
+ \,b\left(a'b' + a''b\right) I_{(1,1,0)} + b\left((b')^2 + b\,b''\right) I_{(1,1,1)} + R_6
$$

Examples for Wagner Platen Expansions

- **Vasicek interest rate model**

$$dr_t = \gamma\,(\bar{r} - r_t)\,dt + \beta\,dW_t$$

for $t \in [0, T]$, $r_0 \geq 0$

- hierarchical set

$$A = \{\alpha \in \mathcal{M}_1 : \ell(\alpha) \leq 3\}$$

$$\begin{aligned} r_t \quad = \quad & r_0 + \gamma\,(\bar{r} - r_0)\,t + \beta\,W_t - \gamma^2\,(\bar{r} - r_0)\,\frac{t^2}{2} \\ & - \beta\,\gamma \int_0^t W_s\,ds + \gamma^3\,(\bar{r} - r_0)\,\frac{t^3}{6} \\ & + \beta\,\gamma^2 \int_0^t \int_0^{s_2} W_{s_1}\,ds_1\,ds_2 + R_6 \end{aligned}$$

- **Black-Scholes dynamics**

$$dS_t = S_t\,(a\,dt + \sigma\,dW_t)$$

for $t \in [0, T]$, $S_0 \geq 0$

- Wagner-Platen expansion

hierarchical set

$$A = \{\alpha \in \mathcal{M}_1 : \ell(\alpha) \leq 3\}$$

is given by

$$
\begin{aligned}
S_t \;=\; & S_0 \biggl(1 + a\,t + \sigma\,W_t + a^2\,\frac{t^2}{2} + a\,\sigma\,(I_{(0,1)} + I_{(1,0)}) \\
& + \sigma^2\,\frac{1}{2}\,\bigl((W_t)^2 - t\bigr) + a^3\,\frac{t^3}{6} \\
& + a^2\,\sigma\,(I_{(0,0,1)} + I_{(0,1,0)} + I_{(1,0,0)}) \\
& + a\,\sigma^2\,(I_{(0,1,1)} + I_{(1,0,1)} + I_{(1,1,0)}) \\
& + \sigma^3\,\frac{1}{6}\,\bigl((W_t)^3 - 3\,t\,W_t\bigr) \biggr) \\
& + R_6
\end{aligned}
$$

- relationship

$$\begin{aligned} t\,W_t &= I_{(0)}\,I_{(1)} = I_{(0,1)} + I_{(1,0)} \\ W_t\,\frac{t^2}{2} &= I_{(1)}\,I_{(0,0)} = I_{(0,0,1)} + I_{(0,1,0)} + I_{(1,0,0)} \\ t\,\frac{1}{2}\left((W_t)^2 - t\right) &= I_{(0)}\,I_{(1,1)} = I_{(0,1,1)} + I_{(1,0,1)} + I_{(1,1,0)} \end{aligned}$$

- Wagner-Platen expansion

$$
\begin{aligned}
S_t \;=\; S_0 \Big(& 1 + a\,t + \sigma\,W_t + a^2\,\frac{t^2}{2} + a\,\sigma\,t\,W_t \\
& + \frac{\sigma^2}{2}\left((W_t)^2 - t\right) + a^3\,\frac{t^3}{6} + a^2 \sigma\,W_t\,\frac{t^2}{2} \\
& + a\,\sigma^2\,\frac{t}{2}\left((W_t)^2 - t\right) + \sigma^3\,\frac{1}{6}\left((W_t)^3 - 3\,t\,W_t\right) \Big) + R_6
\end{aligned}
$$

- **squared Bessel process**

$$dX_t = \nu\, dt + 2\,\sqrt{X_t}\, dW_t$$

for $t \in [0, T]$ with $X_0 > 0$

hierarchical set

$$A = \{\alpha \in \mathcal{M}_1 : \ell(\alpha) \leq 2\}$$

- Wagner-Platen expansion

$$
\begin{aligned}
X_t &= X_0 + (\nu - 1)\, t + 2\sqrt{X_0}\, W_t \\
&\quad + \frac{\nu - 1}{\sqrt{X_0}} \int_0^t s\, dW_s + (W_t)^2 + \tilde{R}
\end{aligned}
$$

2 Introduction to Scenario Simulation

Discrete Time Approximation

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T$$

one-dimensional SDE

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t$$

- **Euler scheme**

  Euler-Maruyama scheme

  Maruyama (1955)

$$\boldsymbol{Y_{n+1} = Y_n + a(\tau_n, Y_n)\,(\tau_{n+1} - \tau_n) + b(\tau_n, Y_n)\left(W_{\tau_{n+1}} - W_{\tau_n}\right)}$$

$$\boldsymbol{Y_0 = X_0,}$$

$$\boldsymbol{Y_n = Y_{\tau_n}}$$

$$\boldsymbol{\Delta_n = \tau_{n+1} - \tau_n}$$

maximum step size

$$\Delta = \max_{n \in \{0,1,\dots,N-1\}} \Delta_n$$

- **equidistant time discretization**

$$\tau_n = n\,\Delta$$

$$\Delta_n \equiv \Delta = \frac{T}{N}$$

- Euler approximation **recursively computed**

- **random increments**

$$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$$

for $n \in \{0, 1, \ldots, N-1\}$

Wiener process

$$W = \{W_t\, t \in [0, T]\}$$

Gaussian distributed

mean

$$E\left(\Delta W_n\right) = 0$$

variance

$$E\left((\Delta W_n)^2\right) = \Delta_n$$

Gaussian pseudo-random numbers generated

- abbreviation

$$f = f(\tau_n, Y_n)$$

Euler scheme

$$Y_{n+1} = Y_n + a\,\Delta_n + b\,\Delta W_n,$$

discrete time approximations

Simulating Geometric Brownian Motion

- **illustrate scenario simulation**

  standard market model as geometric Brownian motions

$$dX_t = a\,X_t\,dt + b\,X_t\,dW_t$$

  for $t \in [0, T]$, $X_0 > 0$

- drift coefficient

$$a(t, x) = a\,x$$

- diffusion coefficient

$$b(t, x) = b\,x$$

- appreciation rate $\quad a$

- volatility $\quad b \neq 0$

- **explicit solution**

$$X_t = X_0 \exp\left(\left(a - \frac{1}{2}\,b^2\right) t + b\,W_t\right)$$

- Wiener process

$$W = \{W_t,\, t \in [0, T]\}$$

Scenario Simulation

- **simulate trajectory of Euler approximation**

  1. initial value $Y_0 = X_0$

  2. proceed recursively

$$Y_{n+1} = Y_n + a\,Y_n\,\Delta_n + b\,Y_n\,\Delta W_n$$

for $n \in \{0, 1, \ldots, N-1\}$

$$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$$

- comparison with explicit solution at time τ_n

$$X_{\tau_n} = X_0 \exp\left(\left(a - \frac{1}{2}\,b^2\right)\tau_n + b\sum_{i=1}^{n} \Delta W_{i-1}\right)$$

for $n \in \{0, 1, \ldots, N-1\}$

Euler approximation

mathematically another new object

different

negative ?

alternative ways ?

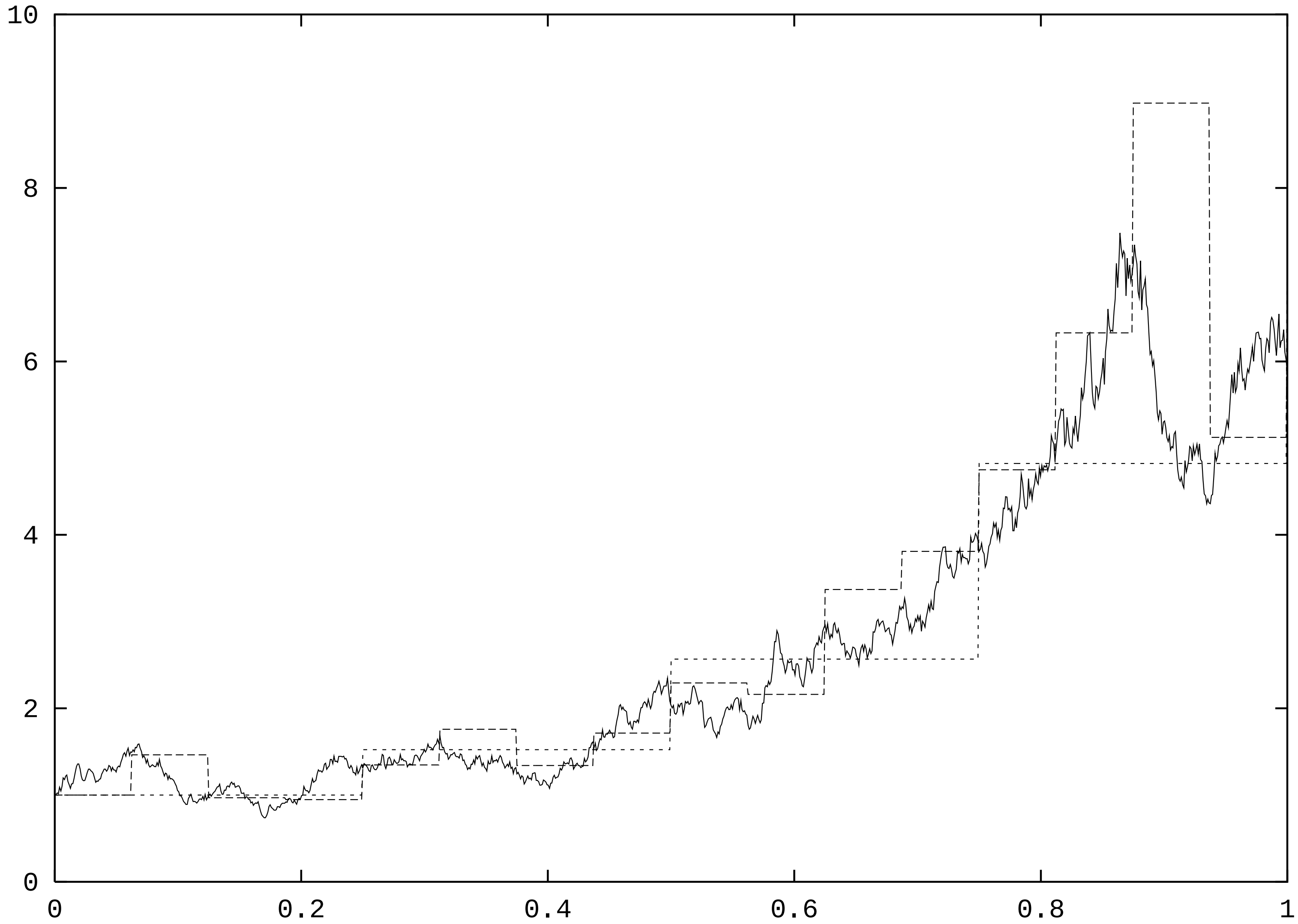

Figure 2.1: Euler approximations for $\mathbf{\Delta = 0.25}$ and $\mathbf{\Delta = 0.0625}$ and exact solution for Black-Scholes SDE.

Strong Approximation

- not specified a **criterion for classification**

- **scenario simulation**
  approximates paths

  testing of **calibration** methods

  statistical **estimators**

  filtering

- **Monte-Carlo simulation**

  approximates **probabilities**

  functionals

  simulates **expectations**

  moments

  prices of contingent claims

  or **risk measures** as Value at Risk

Order of Strong Convergence

- **absolute error criterion**

$$\varepsilon(\Delta) = E\left(\left|X_T - Y_T^{\Delta}\right|\right)$$

A discrete time approximation Y^{Δ} *converges strongly with order* $\gamma > 0$ at time T if there exists a positive constant C, which does not depend on Δ, and a $\delta_0 > 0$ such that

$$\varepsilon(\Delta) = E\left(\left|X_T - Y_T^{\Delta}\right|\right) \leq C\,\Delta^{\gamma}$$

for each $\Delta \in (0, \delta_0)$.

Strong Taylor Schemes

- use **Wagner-Platen expansions**

  appropriate truncation

- **operators**

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,\ell=1}^{d} \sum_{j=1}^{m} b^{k,j}\, b^{\ell,j} \frac{\partial}{\partial x^k\, \partial x^\ell}$$

$$\underline{L}^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} \underline{a}^k \frac{\partial}{\partial x^k}$$

and

$$\boldsymbol{L^j = \underline{L}^j = \sum_{k=1}^{d} b^{k,j} \, \frac{\partial}{\partial x^k}}$$

for $\boldsymbol{j \in \{1, 2, \ldots, m\}}$, where

$$\boldsymbol{\underline{a}^k = a^k - \frac{1}{2} \sum_{j=1}^{m} \underline{L}^j \, b^{k,j}}$$

- **multiple Itô integrals**

$$I_{(j_1,\ldots,j_\ell)} = \int_{\tau_n}^{\tau_{n+1}} \ldots \int_{\tau_n}^{s_2} dW_{s_1}^{j_1} \ldots dW_{s_\ell}^{j_\ell}$$

- **multiple Stratonovich integrals**

$$J_{(j_1,\ldots,j_\ell)} = \int_{\tau_n}^{\tau_{n+1}} \ldots \int_{\tau_n}^{s_2} \circ dW_{s_1}^{j_1} \ldots \circ dW_{s_\ell}^{j_\ell}$$

for $j_1, \ldots, j_\ell \in \{0, 1, \ldots, m\}$, $\ell \in \{1, 2, \ldots\}$
and $n \in \{0, 1, \ldots\}$

$$W_t^0 = t$$

for all $t \in [0, T]$

- **Itô SDE**

$$dX_t = a(t, X_t)\, dt + \sum_{j=1}^{m} b^j(t, X_t)\, dW_t^j$$

- equivalent **Stratonovich SDE**

$$dX_t = \underline{a}(t, X_t)\, dt + \sum_{j=1}^{m} b^j(t, X_t) \circ dW_t^j$$

Euler Scheme

simplest strong Taylor approximation

order of strong convergence $\gamma = 0.5$

- $d = m = 1$

  Euler scheme

$$Y_{n+1} = Y_n + a\,\Delta + b\,\Delta W$$

$$\Delta = \tau_{n+1} - \tau_n$$

$$\Delta W = \Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$$

$N(0, \Delta)$ independent Gaussian distributed

- **multi-dimensional Euler scheme**

  $m = 1$ and $d \in \{1, 2, \ldots\}$

  kth component

$$Y_{n+1}^k = Y_n^k + a^k \, \Delta + b^k \, \Delta W$$

  for $k \in \{1, 2, \ldots, d\}$

  $a = (a^1, \ldots, a^d)^\top$ and $b = (b^1, \ldots, b^d)^\top$

- **general multi-dimensional Euler scheme**

$d, m \in \{1, 2, \ldots\}$

$$Y_{n+1}^k = Y_n^k + a^k \, \Delta + \sum_{j=1}^{m} b^{k,j} \, \Delta W^j$$

$$\Delta W^j = W_{\tau_{n+1}}^j - W_{\tau_n}^j$$

$N(0, \Delta)$ independent Gaussian distributed

ΔW^{j_1} and ΔW^{j_2} independent for $j_1 \neq j_2$

$b = [b^{k,j}]_{k,j=1}^{d,m}$ $d \times m$-matrix

truncated Wagner-Platen expansion

Theorem 2.1 *Suppose that we have initial values* X_0 *and* $Y_0 = Y_0^\Delta$ *such that*

$$E\left(|X_0|^2\right) < \infty$$

and

$$E\left(\left|X_0 - Y_0^\Delta\right|^2\right)^{\frac{1}{2}} \leq K_1\, \Delta^{\frac{1}{2}}.$$

Furthermore, assume the Lipschitz condition

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq K_2\, |x - y|,$$

the linear growth condition

$$|a(t,x)| + |b(t,x)| \leq K_3\, (1 + |x|\,)$$

and

$$|a(s,x)-a(t,x)|+|b(s,x)-b(t,x)| \leq K_4 \ (1+|x|\ )\,|s-t|^{\frac{1}{2}}$$

for all s, $t \in [0,T]$ *and* x, $y \in \Re^d$, *where the constants* $K_1, \ldots, K_4$ *do not depend on* Δ. *Then the Euler approximation* Y^{Δ} *converges with strong order* $\gamma = 0.5$, *that is we have the estimate*

$$E\left(\left|X_T - Y_T^{\Delta}\right|\right) \leq K_5\,\Delta^{\frac{1}{2}},$$

where the constant K_5 *does not depend on* Δ.

A Simulation Example

- **Black-Scholes SDE**

$$dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dW_t$$

- **exact solution**

$$X_T = X_0\, \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma\, W_T \right\}$$

absolute error

$$\varepsilon(\Delta) = E(|X_T - Y_N^\Delta|)$$

default parameters:

$X_0 = 1$, $\mu = 0.06$, $\sigma = 0.2$ and $T = 1$

5000 simulations

fitted line

$$\ln(\varepsilon(\Delta)) = -3.86933 + 0.46739 \ln(\Delta)$$

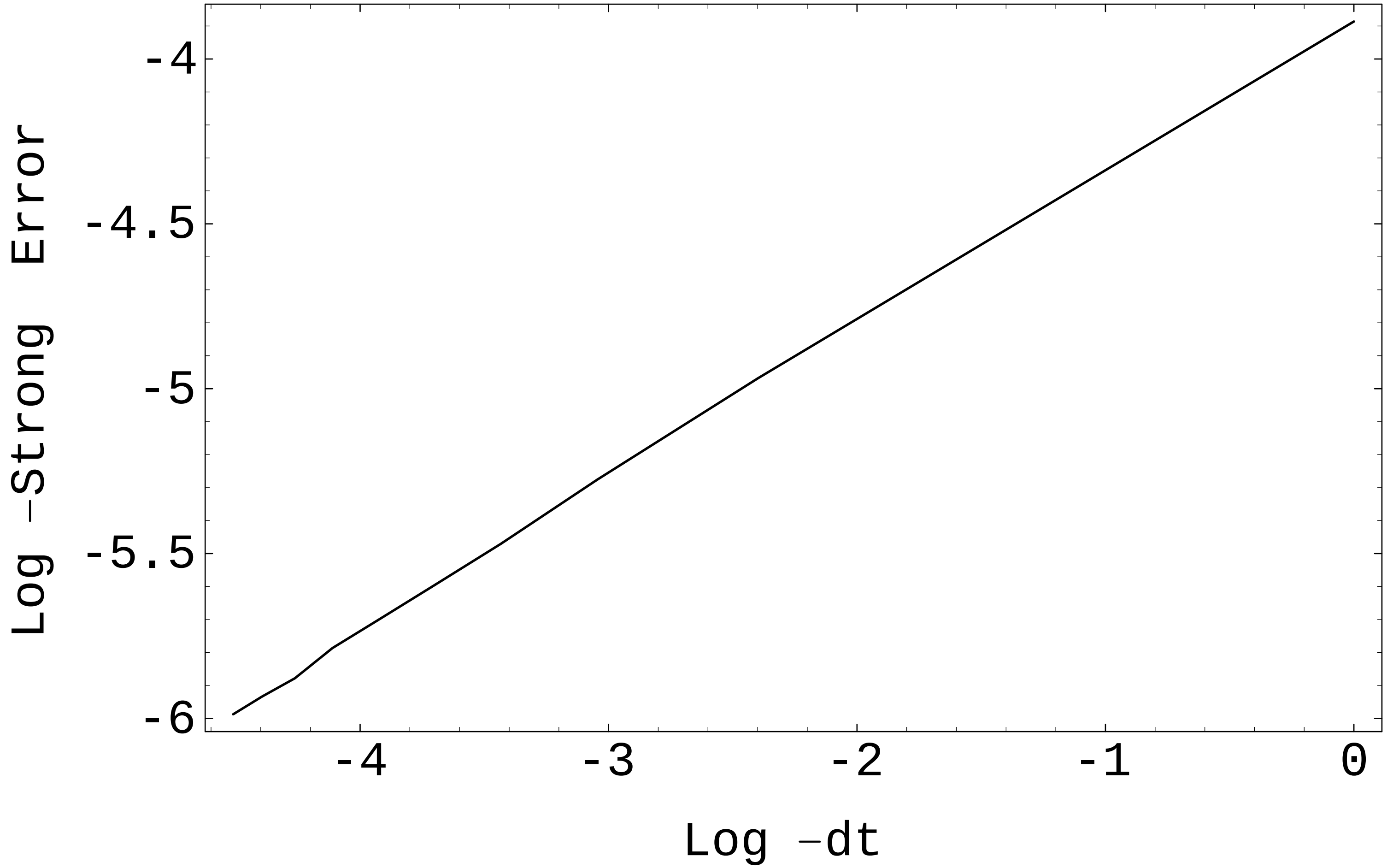

Figure 2.2: Log-log plot of the absolute error $\varepsilon(\Delta)$ for an Euler Scheme against $\ln(\Delta)$.

Milstein Scheme

Milstein (1974)

- **Milstein scheme**

  $d = m = 1$

$$Y_{n+1} = Y_n + a\,\Delta + b\,\Delta W + \frac{1}{2}\,b\,b'\left\{(\Delta W)^2 - \Delta\right\}$$

order $\gamma = 1.0$ of strong convergence

- **multi-dimensional Milstein scheme**

  $m = 1$ and $d \in \{1, 2, \ldots\}$

$$Y_{n+1}^{k} = Y_n^{k} + a^k\,\Delta + b^k\,\Delta W + \left(\sum_{\ell=1}^{d} b^{\ell}\frac{\partial b^k}{\partial x^{\ell}}\right)\frac{1}{2}\left\{(\Delta W)^2 - \Delta\right\}$$

- **general multi-dimensional Milstein scheme**

  $d, m \in \{1, 2, \ldots\}$

  kth component

$$Y_{n+1}^{k} = Y_{n}^{k} + a^{k}\,\Delta + \sum_{j=1}^{m} b^{k,j}\,\Delta W^{j} + \sum_{j_1,j_2=1}^{m} L^{j_1} b^{k,j_2} I_{(j_1,j_2)}$$

  Itô integrals $I_{(j_1,j_2)}$

- alternatively

$$Y_{n+1}^{k} = Y_{n}^{k} + \underline{a}^{k}\,\Delta + \sum_{j=1}^{m} b^{k,j}\,\Delta W^{j} + \sum_{j_1,j_2=1}^{m} \underline{L}^{j_1} b^{k,j_2} J_{(j_1,j_2)}$$

Stratonovich integrals $J_{(j_1,j_2)}$

$j_1 \neq j_2 \quad j_1, j_2 \in \{1, 2, \ldots, m\}$

$$J_{(j_1,j_2)} = I_{(j_1,j_2)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_1} dW_{s_2}^{j_1} \, dW_{s_1}^{j_2}$$

cannot be simply expressed by ΔW^{j_1} and ΔW^{j_2}

$$I_{(j_1,j_1)} = \frac{1}{2} \left\{ (\Delta W^{j_1})^2 - \Delta \right\} \quad \text{and} \quad J_{(j_1,j_1)} = \frac{1}{2} \left(\Delta W^{j_1} \right)^2$$

for $j_1 \in \{1, 2, \ldots, m\}$

Commutative Noise

- **commutativity condition**

$$L^{j_1} b^{k,j_2} = L^{j_2} b^{k,j_1}$$

for all $j_1, j_2 \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, d\}$ and $(t, x) \in [0, T] \times \Re^d$

- satisfied for Black-Scholes SDEs

  additive noise

  single Wiener process

- **Milstein scheme under commutative noise**

$$\boldsymbol{Y^k_{n+1} = Y^k_n + \underline{a}^k\,\Delta + \sum_{j=1}^{m} b^{k,j}\,\Delta W^j + \frac{1}{2}\sum_{j_1,j_2=1}^{m} \underline{L}^{j_1} b^{k,j_2}\,\Delta W^{j_1}\,\Delta W^{j_2}}$$

for $\boldsymbol{k \in \{1, 2, \ldots, d\}}$

no double Wiener integrals

A Black-Scholes Example

- correlated Black-Scholes dynamics

  $$d = m = 2$$

- first risky security

$$\begin{aligned} dX_t^1 &= X_t^1 \left[r\, dt + \theta^1 (\theta^1\, dt + dW_t^1) + \theta^2 (\theta^2\, dt + dW_t^2) \right] \\ &= X_t^1 \left[\left(r + \frac{1}{2} \left((\theta^1)^2 + (\theta^2)^2 \right) \right) dt \right. \\ &\qquad \left. + \theta^1 \circ dW_t^1 + \theta^2 \circ dW_t^2 \right] \end{aligned}$$

- second risky security

$$\begin{aligned} dX_t^2 &= X_t^2 \left[r\,dt + (\theta^1 - \sigma^{1,1})\,(\theta^1\,dt + dW_t^1) \right] \\ &= X_t^2 \left[\left(r + (\theta^1 - \sigma^{1,1}) \left(\frac{1}{2}\,\theta^1 + \frac{1}{2}\,\sigma^{1,1} \right) \right) dt \right. \\ &\qquad \left. + (\theta^1 - \sigma^{1,1}) \circ dW_t^1 \right] \end{aligned}$$

$\Longrightarrow$

$$L^1\, b^{1,2} \;=\; \sum_{k=1}^{2} b^{k,1}\, \frac{\partial}{\partial x^k}\, b^{1,2} = \theta^1\, X_t^1\, \theta^2$$

$$\;=\; \sum_{k=1}^{2} b^{k,2}\, \frac{\partial}{\partial x^k}\, b^{1,1} = L^2\, b^{1,1}$$

and

$$L^1\, b^{2,2} = \sum_{k=1}^{2} b^{k,1}\, \frac{\partial}{\partial x^k}\, b^{2,2} = 0 = \sum_{k=1}^{2} b^{k,2}\, \frac{\partial}{\partial x^k}\, b^{2,1} = L^2\, b^{2,1}$$

$\Longrightarrow$ commutativity condition satisfied

$\Longrightarrow$ Milstein scheme :

$$
\begin{aligned}
Y^1_{n+1} \;=\; & Y^1_n + Y^1_n \biggl(r + \frac{1}{2}\left((\theta^1)^2 + (\theta^2)^2\right)\Delta + \theta^1\,\Delta W^1 + \theta^2\,\Delta W^2 \\
& + \frac{1}{2}\,(\theta^1)^2(\Delta W^1)^2 + \frac{1}{2}\,(\theta^2)^2(\Delta W^2)^2 + \theta^1\,\theta^2\,\Delta W^1\,\Delta W^2 \biggr)
\end{aligned}
$$

$$
\begin{aligned}
Y^2_{n+1} \;=\; & Y^2_n + Y^2_n \biggl(\left(r + \frac{1}{2}\,(\theta^1 - \sigma^{1,1})\,(\theta^1 + \sigma^{1,1}) \right) \Delta \\
& + (\theta^1 - \sigma^{1,1})\,\Delta W^1 + \frac{1}{2}\,(\theta^1 - \sigma^{1,1})^2\,(\Delta W^1)^2 \biggr)
\end{aligned}
$$

(may produce negative trajectories)

A Square Root Process Example

- square root process of dimension ν

$$dX_t = \frac{\nu}{4}\,\eta\left(\frac{1}{\eta} - X_t\right)dt + \sqrt{X_t}\,dW_t^1$$

$X_0 = \frac{1}{\eta}$ for $\nu > 2$

- Milstein

$$Y_{n+1} = Y_n + \frac{\nu}{4}\,\eta\left(\frac{1}{\eta} - Y_n\right)\Delta + \sqrt{Y_n}\,\Delta W + \frac{1}{4}\left(\Delta W^2 - \Delta\right)$$

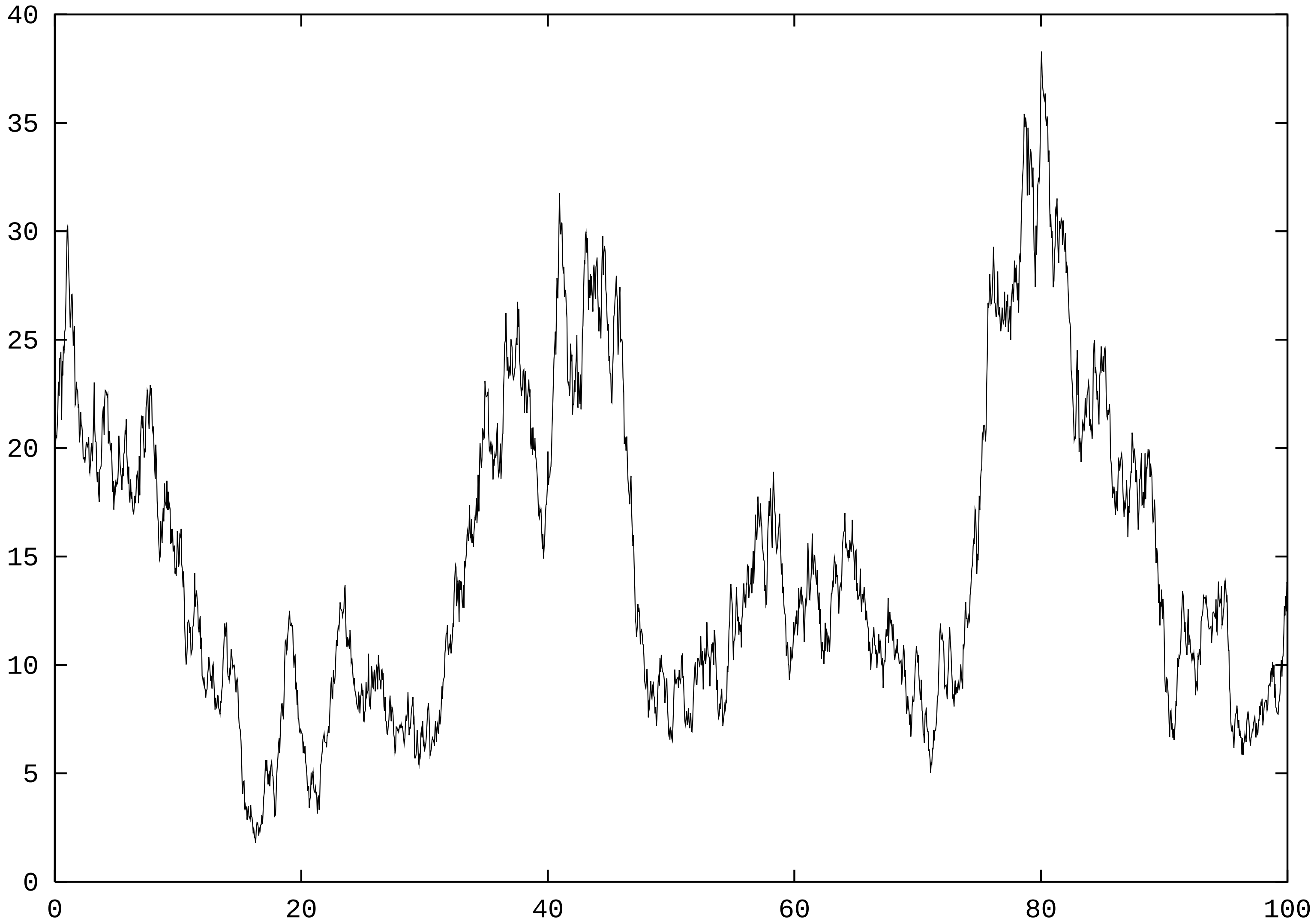

Figure 2.3: Square root process simulated by the Milstein scheme.

Convergence Theorem

$$\boldsymbol{E}\left(\left|\boldsymbol{X_0}\right|^{\mathbf{2}}\right) < \infty$$

$$\boldsymbol{E}\left(\left|\boldsymbol{X_0} - \boldsymbol{Y_0^{\Delta}}\right|^{\mathbf{2}}\right)^{\frac{1}{2}} \leq \boldsymbol{K_1}\,\boldsymbol{\Delta}^{\frac{1}{2}}$$

$$\begin{aligned}
\left|\underline{a}(t,x) - \underline{a}(t,y)\right| &\leq K_2\,|x-y| \\
\left|b^{j_1}(t,x) - b^{j_1}(t,y)\right| &\leq K_2\,|x-y| \\
\left|\underline{L}^{j_1} b^{j_2}(t,x) - \underline{L}^{j_1} b^{j_2}(t,y)\right| &\leq K_2\,|x-y| \\
\\
|\underline{a}(t,x)| + \left|\underline{L}^{j}\,\underline{a}(t,x)\right| &\leq K_3\,(1+|x|) \\
\left|b^{j_1}(t,x)\right| + \left|\underline{L}^{j} b^{j_2}(t,x)\right| &\leq K_3\,(1+|x|) \\
\left|\underline{L}^{j}\,\underline{L}^{j_1} b^{j_2}(t,x)\right| &\leq K_3\,(1+|x|)
\end{aligned}$$

and

$$
\begin{aligned}
\left|\underline{a}(s,x) - \underline{a}(t,x)\right| &\leq K_4\ (1 + |x|\,)\,|s-t|^{\frac{1}{2}} \\
\left|b^{j_1}(s,x) - b^{j_1}(t,x)\right| &\leq K_4\ (1 + |x|\,)\,|s-t|^{\frac{1}{2}} \\
\left|\underline{L}^{j_1} b^{j_2}(s,x) - \underline{L}^{j_1} b^{j_2}(t,x)\right| &\leq K_4\ (1 + |x|\,)\,|s-t|^{\frac{1}{2}}
\end{aligned}
$$

for all $s, t \in [0, T]$, $x, y \in \Re^d$, $j \in \{0, \ldots, m\}$ and $j_1, j_2 \in \{1, 2, \ldots, m\}$. Then the Milstein scheme converges with strong order $\gamma = 1.0$

$$
E\left(\left|X_T - Y_T^{\Delta}\right|\right) \leq K_5\,\Delta.
$$

Kloeden & Pl. (1992b).

A Simulation Study

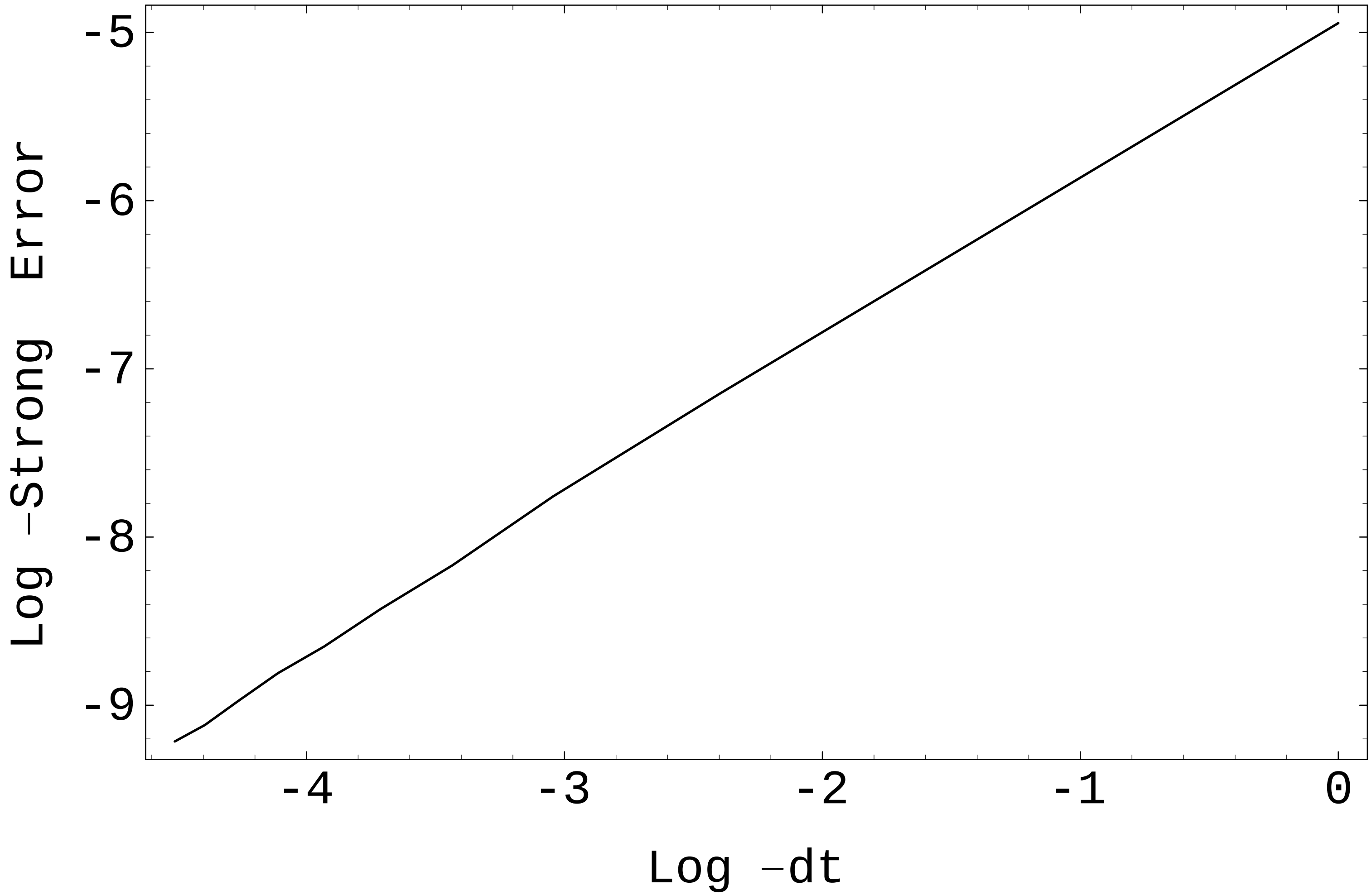

Figure 2.4: Log-log plot of the absolute error against log-step size for a Milstein scheme.

- linear regression

$$\ln(\varepsilon(\Delta)) = -4.91021 + 0.95 \ln(\Delta)$$

Order 1.5 Strong Taylor Scheme

- simulation tasks that require more accurate schemes

  extreme log-returns

- including further multiple stochastic integrals

- **order 1.5 strong Taylor scheme**

  $$d = m = 1$$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + a\,\Delta + b\,\Delta W + \frac{1}{2}\,b\,b'\left\{(\Delta W)^2 - \Delta\right\} \\
& + a'\,b\,\Delta Z + \frac{1}{2}\left(a\,a' + \frac{1}{2}\,b^2\,a''\right)\Delta^2 \\
& + \left(a\,b' + \frac{1}{2}\,b^2\,b''\right)\{\Delta W\,\Delta - \Delta Z\} \\
& + \frac{1}{2}\,b\left(b\,b'' + (b')^2\right)\left\{\frac{1}{3}\,(\Delta W)^2 - \Delta\right\}\Delta W
\end{aligned}
$$

- **double integral**

$$\Delta Z = I_{(1,0)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}\, ds_2$$

Gaussian

$$E(\Delta Z) = 0$$

$$E((\Delta Z)^2) = \frac{1}{3}\Delta^3$$

$$E(\Delta Z\, \Delta W) = \frac{1}{2}\Delta^2$$

- two independent $N(0,1)$

  U_1 and U_2

  $\Longrightarrow$

$$\Delta W = U_1 \sqrt{\Delta}, \qquad \Delta Z = \frac{1}{2}\, \Delta^{\frac{3}{2}} \left(U_1 + \frac{1}{\sqrt{3}}\, U_2 \right)$$

- **triple Wiener integral**

$$I_{(1,1,1)} = \frac{1}{2} \left\{ \frac{1}{3} \, (\Delta W^1)^2 - \Delta \right\} \Delta W^1$$

scaled monic Hermite polynomial

- **multi-dimensional order 1.5 strong Taylor scheme**

  $d \in \{1, 2, \ldots\}$ and $m = 1$

$$\begin{aligned} Y_{n+1}^k &= Y_n^k + a^k \, \Delta + b^k \, \Delta W \\ &\qquad + \frac{1}{2} \, L^1 \, b^k \, \{(\Delta W)^2 - \Delta\} + L^1 \, a^k \, \Delta Z \\ &\qquad + L^0 \, b^k \, \{\Delta W \, \Delta - \Delta Z\} + \frac{1}{2} \, L^0 \, a^k \, \Delta^2 \\ &\qquad + \frac{1}{2} \, L^1 \, L^1 \, b^k \left\{ \frac{1}{3} \, (\Delta W)^2 - \Delta \right\} \Delta W \end{aligned}$$

- **general multi-dimensional order 1.5 strong Taylor scheme**

  $d, m \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1}^k \;=\; & Y_n^k + a^k\,\Delta + \frac{1}{2}\,L^0\,a^k\,\Delta^2 \\
& + \sum_{j=1}^{m} \left(b^{k,j}\,\Delta W^j + L^0\,b^{k,j}\,I_{(0,j)} + L^j\,a^k\,I_{(j,0)}\right) \\
& + \sum_{j_1,j_2=1}^{m} L^{j_1}\,b^{k,j_2}\,I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=1}^{m} L^{j_1}\,L^{j_2}\,b^{k,j_3}\,I_{(j_1,j_2,j_3)}
\end{aligned}
$$

in case of additive noise simplifies considerably

strong order $\gamma = 1.5$

Approximate Multiple Stochastic Integrals

Let ξ_j, $\zeta_{j,1}, \ldots, \zeta_{j,p}$, $\eta_{j,1}, \ldots, \eta_{j,p}$, $\mu_{j,p}$ and $\phi_{j,p}$

be independent $N(0,1)$

for $j, j_1, j_2, j_3 \in \{1, 2, \ldots, m\}$ and some $p \in \{1, 2, \ldots\}$

set

$$I_{(j)} = \Delta W^j = \sqrt{\Delta}\, \xi_j, \qquad I_{(j,0)} = \frac{1}{2} \Delta \left(\sqrt{\Delta}\, \xi_j + a_{j,0} \right)$$

with

$$a_{j,0} = -\frac{\sqrt{2\Delta}}{\pi} \sum_{r=1}^{p} \frac{1}{r} \zeta_{j,r} - 2\sqrt{\Delta\, \varrho_p}\, \mu_{j,p}$$

where

$$\varrho_p = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{r=1}^{p} \frac{1}{r^2}$$

$$I_{(0,j)} = \Delta W^j \, \Delta - I_{(j,0)}, \qquad I_{(j,j)} = \frac{1}{2} \left\{ (\Delta W^j)^2 - \Delta \right\}$$

$$I_{(j,j,j)} = \frac{1}{2} \left\{ \frac{1}{3} \, (\Delta W^j)^2 - \Delta \right\} \Delta W^j$$

$$I^p_{(j_1,j_2)} = \frac{1}{2} \, \Delta \, \xi_{j_1} \, \xi_{j_2} - \frac{1}{2} \sqrt{\Delta} \, (\xi_{j_1} \, a_{j_2,0} - \xi_{j_2} \, a_{j_1,0}) + A^p_{j_1,j_2} \, \Delta$$

Order 2.0 Strong Taylor Scheme

- use Stratonovich-Taylor expansion

$$d = m = 1$$

$$\begin{aligned} Y_{n+1} \;=\; & Y_n + \underline{a}\,\Delta + b\,\Delta W + \frac{1}{2!}\,b\,b'(\Delta W)^2 + b\,\underline{a}'\,\Delta Z \\ & + \frac{1}{2}\,\underline{a}\,\underline{a}'\,\Delta^2 + \underline{a}\,b'\{\Delta W\,\Delta - \Delta Z\} \\ & + \frac{1}{3!}\,b\,(b\,b')'\,(\Delta W)^3 + \frac{1}{4!}\,b\left(b\,(b\,b')'\right)'\,(\Delta W)^4 \\ & + \underline{a}\,(b\,b')'\,J_{(0,1,1)} + b\,(\underline{a}\,b')'\,J_{(1,0,1)} + b\,(b\,\underline{a}')'\,J_{(1,1,0)} \end{aligned}$$

Approximate Multiple Stratonovich Integrals

$$\Delta W = J^p_{(1)} = \sqrt{\Delta}\,\zeta_1$$

$$\Delta Z = J^p_{(1,0)} = \frac{1}{2}\,\Delta\left(\sqrt{\Delta}\,\zeta_1 + a_{1,0}\right)$$

$$J^p_{(1,0,1)} = \frac{1}{3!}\,\Delta^2\zeta_1^2 - \frac{1}{4}\,\Delta\,a_{1,0}^2 + \frac{1}{\pi}\,\Delta^{\frac{3}{2}}\,\zeta_1\,b_1 - \Delta^2\,B^p_{1,1}$$

$$J^p_{(0,1,1)} = \frac{1}{3!}\,\Delta^2\zeta_1^2 - \frac{1}{2\pi}\,\Delta^{\frac{3}{2}}\,\zeta_1\,b_1 + \Delta^2\,B^p_{1,1} - \frac{1}{4}\,\Delta^{\frac{3}{2}}\,a_{1,0}\,\zeta_1 + \Delta^2\,C^p_{1,1}$$

$$J^p_{(1,1,0)} = \frac{1}{3!}\,\Delta^2\zeta_1^2 + \frac{1}{4}\,\Delta\,a_{1,0}^2 - \frac{1}{2\pi}\,\Delta^{\frac{3}{2}}\,\zeta_1\,b_1 + \frac{1}{4}\,\Delta^{\frac{3}{2}}\,a_{1,0}\,\zeta_1 - \Delta^2\,C^p_{1,1}$$

with

$$a_{1,0} = -\frac{1}{\pi}\sqrt{2\Delta}\sum_{r=1}^{p}\frac{1}{r}\,\xi_{1,r} - 2\sqrt{\Delta\varrho_p}\,\mu_{1,p}$$

$$\varrho_p = \frac{1}{12} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^2}$$

$$b_1 = \sqrt{\frac{\Delta}{2}}\sum_{r=1}^{p}\frac{1}{r^2}\,\eta_{1,r} + \sqrt{\Delta\alpha_p}\,\phi_{1,p}$$

$$\alpha_p = \frac{\pi^2}{180} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^4}$$

$$B^p_{1,1} = \frac{1}{4\pi^2}\sum_{r=1}^{p}\frac{1}{r^2}\left(\xi^2_{1,r} + \eta^2_{1,r}\right)$$

and

$$C^{p}_{1,1} = -\frac{1}{2\pi^2} \sum_{\substack{r,l=1 \\ r \neq l}}^{p} \frac{r}{r^2 - l^2} \left(\frac{1}{l}\, \xi_{1,r}\, \xi_{1,\ell} - \frac{l}{r}\, \eta_{1,r}\, \eta_{1,\ell} \right)$$

ζ_1, $\xi_{1,r}$ $\eta_{1,r}$, $\mu_{1,p}$ and $\phi_{1,p} \sim N(0,1)$ i.i.d.

Multi-dimensional Order 2.0 Strong Taylor Scheme

$m = 1$

$$
\begin{aligned}
Y_{n+1}^{k} \;=\; & Y_n^k + \underline{a}^k\,\Delta + b^k\,\Delta W + \frac{1}{2!}\,\underline{L}^1\,b^k\,(\Delta W)^2 + \underline{L}^1\,\underline{a}^k\,\Delta Z \\
& + \frac{1}{2}\,\underline{L}^0\underline{a}^k\,\Delta^2 + \underline{L}^0 b^k\,\{\Delta W\,\Delta - \Delta Z\} \\
& + \frac{1}{3!}\,\underline{L}^1\underline{L}^1 b^k\,(\Delta W)^3 + \frac{1}{4!}\,\underline{L}^1\underline{L}^1\underline{L}^1 b^k\,(\Delta W)^4 \\
& + \underline{L}^0\underline{L}^1 b^k\,J_{(0,1,1)} + \underline{L}^1\underline{L}^0 b^k\,J_{(1,0,1)} + \underline{L}^1\underline{L}^1\underline{a}^k\,J_{(1,1,0)}
\end{aligned}
$$

- **general multi-dimensional order 2.0 strong Taylor scheme**

$$
\begin{aligned}
Y_{n+1}^k &= Y_n^k + \underline{a}^k \Delta + \frac{1}{2} \underline{L}^0 \underline{a}^k \Delta^2 \\
&\quad + \sum_{j=1}^{m} \left(b^{k,j} \Delta W^j + \underline{L}^0 b^{k,j} J_{(0,j)} + \underline{L}^j \underline{a}^k J_{(j,0)} \right) \\
&\quad + \sum_{j_1,j_2=1}^{m} \left(\underline{L}^{j_1} b^{k,j_2} J_{(j_1,j_2)} + \underline{L}^0 \underline{L}^{j_1} b^{k,j_2} J_{(0,j_1,j_2)} \right. \\
&\quad\quad \left. + \underline{L}^{j_1} \underline{L}^0 b^{k,j_2} J_{(j_1,0,j_2)} + \underline{L}^{j_1} \underline{L}^{j_2} \underline{a}^k J_{(j_1,j_2,0)} \right) \\
&\quad + \sum_{j_1,j_2,j_3=1}^{m} \underline{L}^{j_1} \underline{L}^{j_2} b^{k,j_3} J_{(j_1,j_2,j_3)} \\
&\quad + \sum_{j_1,j_2,j_3,j_4=1}^{m} \underline{L}^{j_1} \underline{L}^{j_2} \underline{L}^{j_3} b^{k,j_4} J_{(j_1,j_2,j_3,j_4)}
\end{aligned}
$$

Derivative Free Strong Schemes

- similar to Runge-Kutta schemes for ODEs

Explicit Order 1.0 Strong Schemes

Pl. (1984)

$$d = m = 1$$

$$Y_{n+1} = Y_n + a\,\Delta + b\,\Delta W + \frac{1}{2\sqrt{\Delta}}\,\left\{b(\tau_n, \bar{\Upsilon}_n) - b\right\}\left\{(\Delta W)^2 - \Delta\right\}$$

with supporting value

$$\bar{\Upsilon}_n = Y_n + a\,\Delta + b\,\sqrt{\Delta}$$

- **multi-dimensional Platen scheme**

$m = 1$

$$\begin{aligned} Y_{n+1}^k &= Y_n^k + a^k \, \Delta + b^k \, \Delta W \\ &\qquad + \frac{1}{2\sqrt{\Delta}} \left(b^k(\tau_n, \bar{\Upsilon}_n) - b^k \right) \left((\Delta W)^2 - \Delta \right) \end{aligned}$$

with the vector supporting value

$$\bar{\Upsilon}_n = Y_n + a \, \Delta + b \, \sqrt{\Delta}$$

- general multi-dimensional case

$$\boldsymbol{Y}_{n+1}^{k} = \boldsymbol{Y}_n^k + a^k \, \Delta + \sum_{j=1}^{m} b^{k,j} \, \Delta W^j$$

$$+ \frac{1}{\sqrt{\Delta}} \sum_{j_1, j_2 = 1}^{m} \left(b^{k,j_2} \left(\tau_n, \bar{\Upsilon}_n^{j_1} \right) - b^{k,j_2} \right) I_{(j_1, j_2)}$$

with

$$\bar{\Upsilon}_n^j = Y_n + a \, \Delta + b^j \sqrt{\Delta}$$

for $j \in \{1, 2, \dots\}$

- commutative noise

$$\boldsymbol{Y_{n+1}^k = Y_n^k + \underline{a}^k\,\Delta + \frac{1}{2}\sum_{j=1}^{m}\left(b^{k,j}\left(\tau_n, \bar{\Upsilon}_n\right) + b^{k,j}\right)\Delta W^j}$$

with

$$\boldsymbol{\bar{\Upsilon}_n = Y_n + \underline{a}\,\Delta + \sum_{j=1}^{m} b^j\,\Delta W^j}$$

strong order $\boldsymbol{\gamma = 1.0}$

$$\ln(\varepsilon(\Delta)) = -4.71638 + 0.946112\,\ln(\Delta)$$

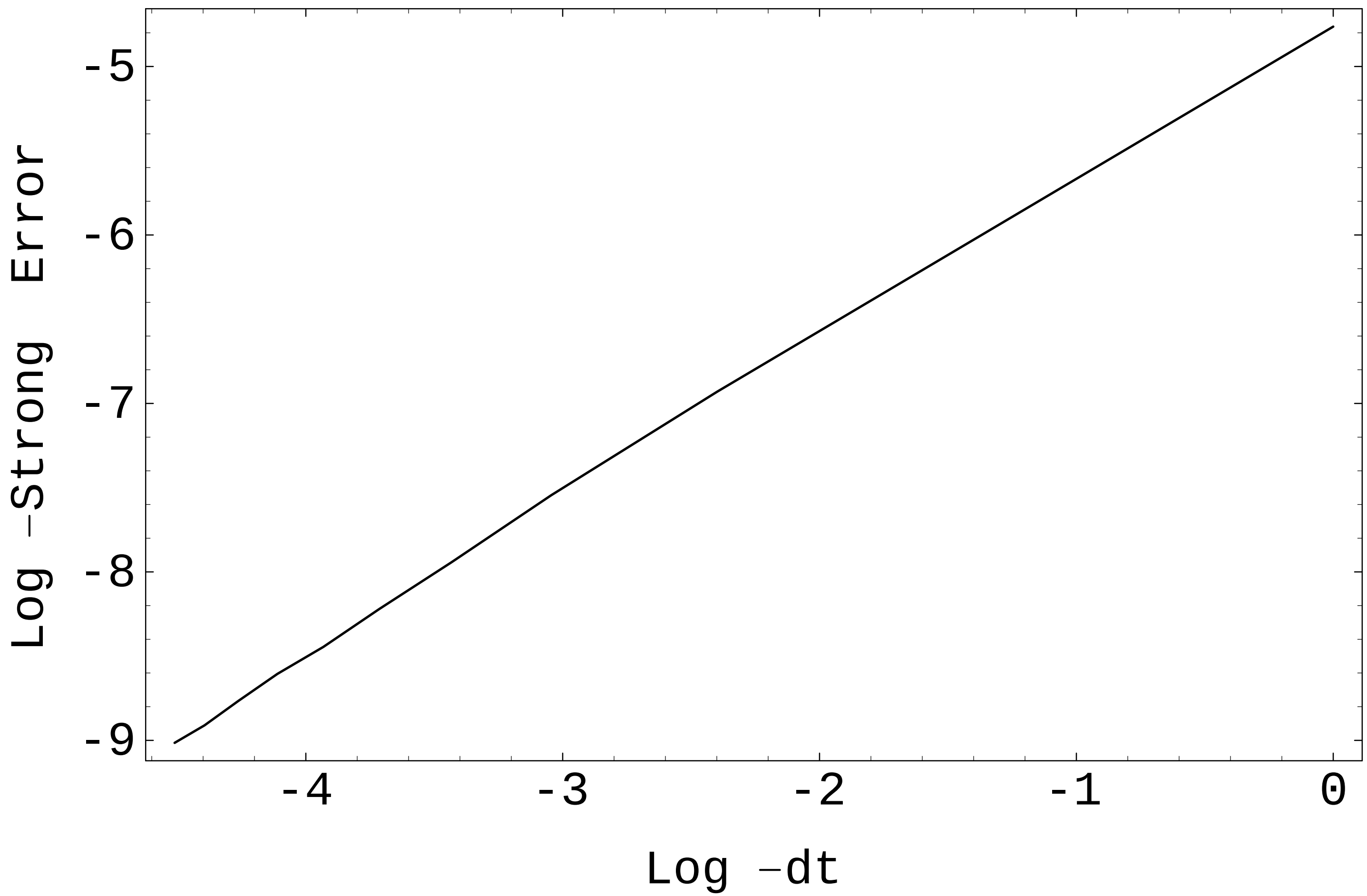

Figure 2.5: Log-log plot of the absolute error for a Platen scheme.

- **two stage Runge-Kutta method $\gamma = 1.0$**

$m = d = 1$

Burrage (1998)

$$Y_{n+1} = Y_n + \left(\underline{a}(Y_n) + 3\,\underline{a}(\bar{Y}_n)\right) \frac{\Delta}{4} + \frac{1}{4} \left(b(Y_n) + 3\,b(\bar{Y}_n)\right) \Delta W$$

with

$$\bar{Y}_n = Y_n + \frac{2}{3}\left(\underline{a}(Y_n)\,\Delta + b(Y_n)\,\Delta W\right)$$

Explicit Order 1.5 Strong Schemes

Pl. (1984)

$\boldsymbol{d = m = 1}$

with

$$\bar{\boldsymbol{\Upsilon}}_{\pm} = \boldsymbol{Y_n + a\,\Delta \pm b\,\sqrt{\Delta}}$$

and

$$\bar{\boldsymbol{\Phi}}_{\pm} = \bar{\boldsymbol{\Upsilon}}_{+} \pm \boldsymbol{b}(\bar{\boldsymbol{\Upsilon}}_{+})\,\sqrt{\boldsymbol{\Delta}}$$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + b\,\Delta W + \frac{1}{2\sqrt{\Delta}}\Big(a(\bar{\Upsilon}_+) - a(\bar{\Upsilon}_-)\Big)\,\Delta Z \\
& + \frac{1}{4}\Big(a(\bar{\Upsilon}_+) + 2a + a(\bar{\Upsilon}_-)\Big)\,\Delta \\
& + \frac{1}{4\sqrt{\Delta}}\Big(b(\bar{\Upsilon}_+) - b(\bar{\Upsilon}_-)\Big)\Big((\Delta W)^2 - \Delta\Big) \\
& + \frac{1}{2\Delta}\Big(b(\bar{\Upsilon}_+) - 2b + b(\bar{\Upsilon}_-)\Big)\Big(\Delta W\,\Delta - \Delta Z\Big) \\
& + \frac{1}{4\Delta}\Big(b(\bar{\Phi}_+) - b(\bar{\Phi}_-) - b(\bar{\Upsilon}_+) + b(\bar{\Upsilon}_-)\Big) \\
& \qquad \times \left(\frac{1}{3}(\Delta W)^2 - \Delta\right)\Delta W
\end{aligned}
$$

- order $\mathbf{1.5}$ Platen scheme

$$
\begin{aligned}
Y_{n+1}^k \;=\; & Y_n^k + a^k\,\Delta + \sum_{j=1}^{m} b^{k,j}\,\Delta W^j \\
& + \frac{1}{2\sqrt{\Delta}} \sum_{j_2=0}^{m} \sum_{j_1=1}^{m} \left(b^{k,j_2}\left(\bar{\Upsilon}_+^{j_1}\right) - b^{k,j_2}\left(\bar{\Upsilon}_-^{j_1}\right) \right) I_{(j_1,j_2)} \\
& + \frac{1}{2\Delta} \sum_{j_2=0}^{m} \sum_{j_1=1}^{m} \left(b^{k,j_2}\left(\bar{\Upsilon}_+^{j_1}\right) - 2b^{k,j_2} + b^{k,j_2}\left(\bar{\Upsilon}_-^{j_1}\right) \right) I_{(0,j_2)} \\
& + \frac{1}{2\Delta} \sum_{j_1,j_2,j_3=1}^{m} \left(b^{k,j_3}\left(\bar{\Phi}_+^{j_1,j_2}\right) - b^{k,j_3}\left(\bar{\Phi}_-^{j_1,j_2}\right) \right. \\
& \qquad \left. - b^{k,j_3}\left(\bar{\Upsilon}_+^{j_1}\right) + b^{k,j_3}\left(\bar{\Upsilon}_-^{j_1}\right) \right) I_{(j_1,j_2,j_3)}
\end{aligned}
$$

with

$$\bar{\Upsilon}^j_{\pm} = Y_n + \frac{1}{m} a \Delta \pm b^j \sqrt{\Delta}$$

and

$$\bar{\Phi}^{j_1,j_2}_{\pm} = \bar{\Upsilon}^{j_1}_{+} \pm b^{j_2} \left(\bar{\Upsilon}^{j_1}_{+} \right) \sqrt{\Delta}$$

where we interpret $b^{k,0}$ as a^k

A Simulation Study

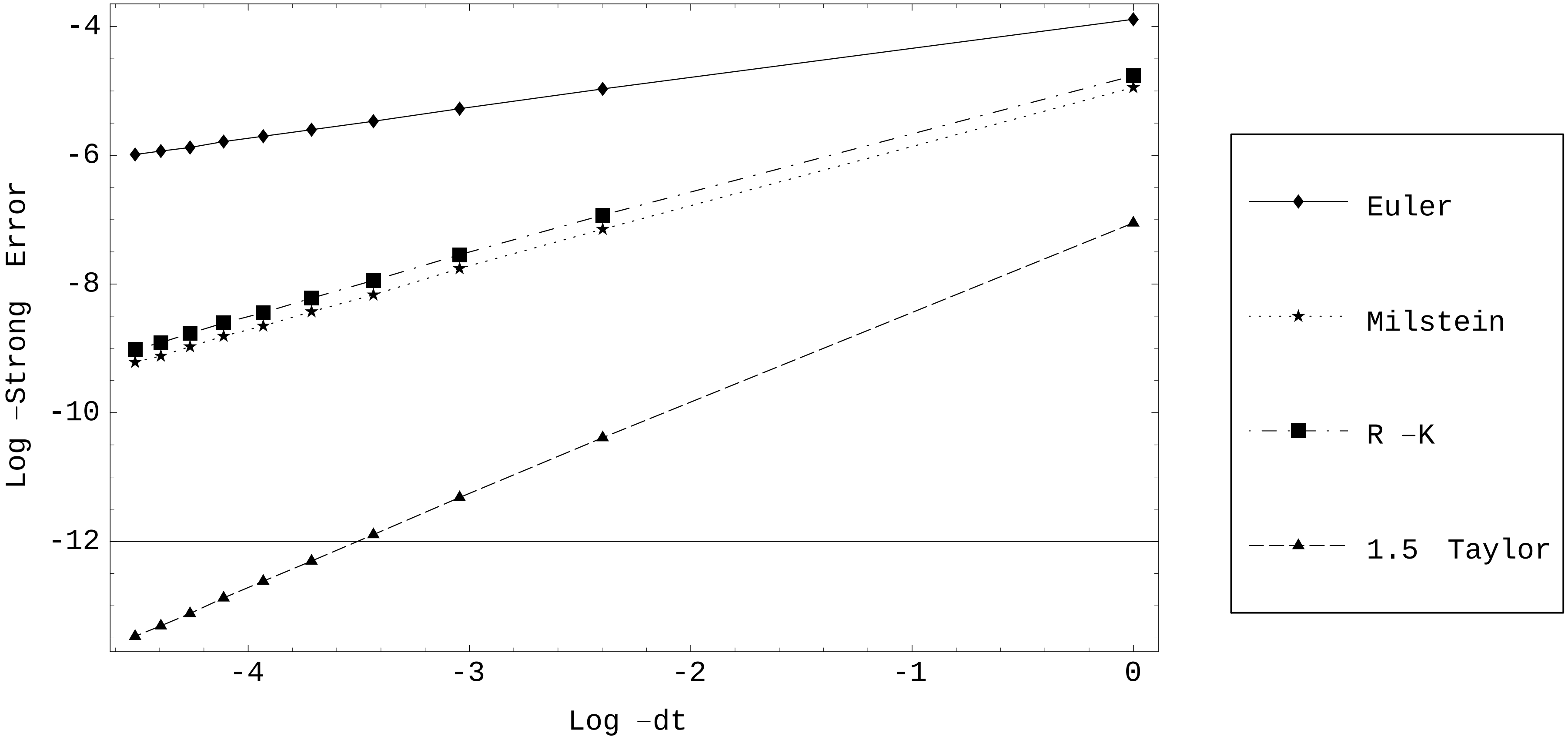

Figure 2.6: Log-log plot of the absolute error for various strong schemes.

Method	CPU Time
Euler	3.5 Seconds
Milstein	4.1 Seconds
1.0 Strong Platen	4.1 Seconds
1.5 Strong Taylor	4.4 Seconds

Table 1: Computational times for various strong methods.

500000 time steps

Numerical Stability

- ability of a scheme to control the propagation of initial and roundoff errors

- numerical stability has higher priority than a potentially high strong order

Deterministic A-Stability

- one-step method

$$Y_{n+1} = Y_n + \Psi(\tau_n, Y_n, Y_{n+1}, \Delta)\,\Delta$$

ordinary differential equation

$$\frac{dx}{dt} = a(t, x)$$

$a(t, x)$ satisfies Lipschitz condition

- **numerically stable**

  if there exist positive constants $\boldsymbol{\Delta_0}$ and $\boldsymbol{M}$ such that

$$|\boldsymbol{Y_n} - \tilde{\boldsymbol{Y}}_{\boldsymbol{n}}| \leq \boldsymbol{M}\, |\boldsymbol{Y_0} - \tilde{\boldsymbol{Y}}_{\boldsymbol{0}}|$$

  $\boldsymbol{n} \in \{\boldsymbol{0}, \boldsymbol{1}, \ldots, \boldsymbol{N}\}$, $\boldsymbol{\Delta} < \boldsymbol{\Delta_0}$
  and any two solutions $\boldsymbol{Y}$, $\tilde{\boldsymbol{Y}}$
  corresponding to the initial values $\boldsymbol{Y_0}$ $\tilde{\boldsymbol{Y}}_{\boldsymbol{0}}$, respectively

- **asymptotically numerically stable**

  if there exist $\boldsymbol{\Delta_0}$ and $\boldsymbol{M}$ such that

$$\lim_{\boldsymbol{n} \to \infty} |\boldsymbol{Y_n} - \tilde{\boldsymbol{Y}}_{\boldsymbol{n}}| \leq \boldsymbol{M}\, |\boldsymbol{Y_0} - \tilde{\boldsymbol{Y}}_{\boldsymbol{0}}|$$

  for any two $\boldsymbol{Y}$, $\tilde{\boldsymbol{Y}}$

- **test equation**

$$\frac{dx_t}{dt} = \lambda \, x_t$$

with $\lambda \in \Re$

numerical scheme in recursive form

$$Y_{n+1} = G(\lambda\,\Delta)\,Y_n$$

- **A-stability region:**

  those $\lambda\,\Delta \in \Re$ for which

$$|G(\lambda\,\Delta)| < 1$$

- explicit Euler scheme

$$Y_{n+1} = Y_n + a(t_n, Y_n)\,\Delta$$

$$Y_{n+1} = (1 + \lambda\,\Delta)\,Y_n$$

A-stability region

open unit interval centered at -1 and ending at 0 since

$$|G(\lambda\,\Delta)| = |1 + \lambda\,\Delta|$$

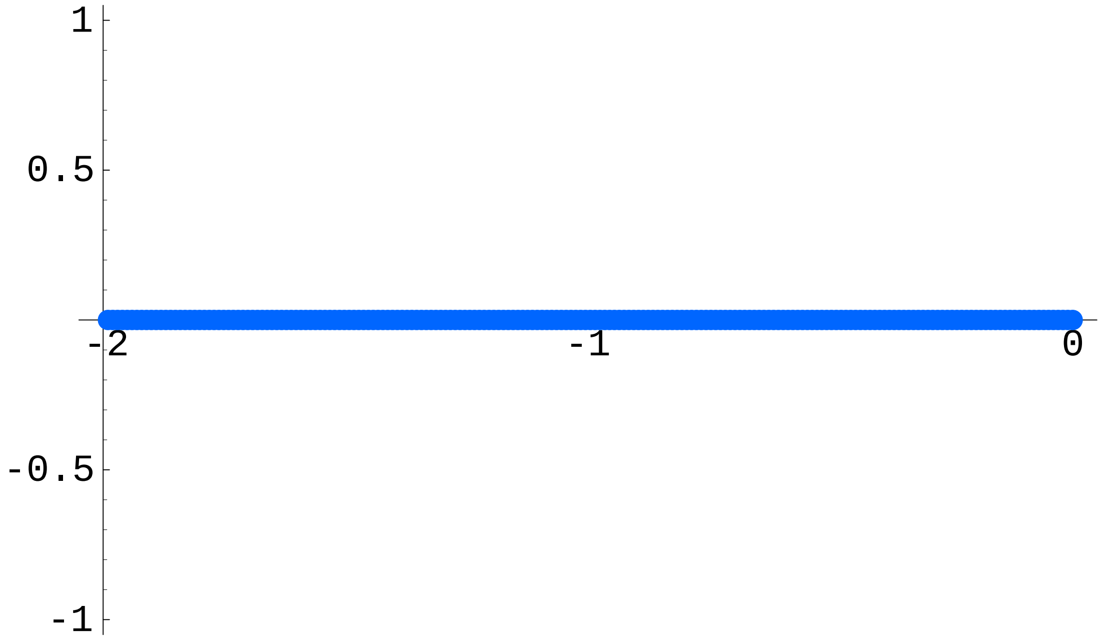

Figure 2.7: Region of A-stability for the deterministic Euler scheme.

- implicit Euler scheme

$$\boldsymbol{Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\,\Delta}$$

$$\boldsymbol{Y_{n+1} = Y_n + \lambda\,\Delta\,Y_{n+1}}$$

$$\boldsymbol{(1 - \lambda\,\Delta)\,Y_{n+1} = Y_n}$$

$$\boldsymbol{G(\lambda\,\Delta) = \frac{1}{1 - \lambda\,\Delta}}$$

$$\boldsymbol{|G(\lambda\,\Delta)| = \frac{1}{|1 - \lambda\,\Delta|}}$$

exterior of an interval with center at $\boldsymbol{1}$ beginning at 0

- scheme is called ***A-stable*** if it covers at least the left half axis

Stochastic A-Stability

- **test equation**

$$dX_t = \lambda\, X_t\, dt + dW_t$$

$\lambda \in \Re$

- discrete time approximation

$$\boldsymbol{Y_{n+1} = G^A(\lambda\,\Delta)\,Y_n + Z_n,}$$

$\boldsymbol{Z_n}$ does not depend on $\boldsymbol{Y_0, Y_1, \ldots, Y_n, Y_{n+1}}$

- **A-stability region**

real numbers $\lambda\Delta$ for which

$$|G^A(\lambda\,\Delta)| < 1$$

If A-stability region covers left half of the real axis

$\Longrightarrow$ then **A-stable**

Drift Implicit Euler Scheme

- **drift implicit Euler scheme**

  strong order $\gamma = 0.5$

  $d = m = 1$

$$Y_{n+1} = Y_n + a\left(\tau_{n+1}, Y_{n+1}\right)\,\Delta + b\,\Delta W$$

- **family of drift implicit Euler schemes**

$$Y_{n+1} = Y_n + \{\alpha\, a\,(\tau_{n+1}, Y_{n+1}) + (1-\alpha)\, a\}\ \Delta + b\,\Delta W$$

degree of implicitness $\alpha \in [0, 1]$

for $\alpha = 0$ explicit Euler scheme

for $\alpha = 1$ drift implicit Euler scheme

- general multi-dimensional

family of drift implicit Euler schemes

$$\begin{aligned} Y_{n+1}^{k} \;&=\; Y_{n}^{k} + \left(\alpha_k a^k\left(\tau_{n+1}, Y_{n+1}\right) + \left(1-\alpha_k\right) a^k\right)\Delta \\ &\quad + \sum_{j=1}^{m} b^{k,j}\,\Delta W^j \end{aligned}$$

$\alpha_k \in [0,1], k \in \{1,2,\ldots,d\}$

- for test equation

$$\boldsymbol{Y_{n+1} = Y_n + \Big(\alpha\,\lambda\,Y_{n+1} + (1-\alpha)\,\lambda\,Y_n\Big)\Delta + \Delta W_n}$$

$$\boldsymbol{Y_{n+1} = G^A(\lambda\,\Delta)\,Y_n + \Delta W_n\,(1-\alpha\,\lambda\,\Delta)^{-1}}$$

transfer function

$$\boldsymbol{G^A(\lambda\,\Delta) = (1-\alpha\,\lambda\,\Delta)^{-1}\,(1+(1-\alpha)\,\lambda\,\Delta)}$$

$\bar{Y}_n$ corresponding discrete time approximation that starts at initial value $\bar{Y}_0$

$$
\begin{aligned}
Y_{n+1} - \bar{Y}_{n+1} &= G^A(\lambda\,\Delta)\,(Y_n - \bar{Y}_n) \\
&= (G^A(\lambda\,\Delta))^n\,(Y_0 - \bar{Y}_0)
\end{aligned}
$$

as long as

$$
|G^A(\lambda\,\Delta)| < 1
$$

the initial error $(Y_0 - \bar{Y}_0)$ is decreased

$$
\Longrightarrow \quad \lambda\,\Delta \in \left(-\frac{2}{1-2\,\alpha}, 0\right)
$$

for $\alpha \geq \frac{1}{2}$ $\quad$ A-stable

Drift Implicit Milstein Scheme

$d = m = 1$

- **Itô version**

$$Y_{n+1} = Y_n + a\,(\tau_{n+1}, Y_{n+1})\;\Delta + b\,\Delta W + \frac{1}{2}\,b\,b'\Big((\Delta W)^2 - \Delta\Big)$$

- **Stratonovich version**

$$Y_{n+1} = Y_n + \underline{a}\,(\tau_{n+1}, Y_{n+1})\;\Delta + b\,\Delta W + \frac{1}{2}\,b\,b'(\Delta W)^2$$

adjusted Stratonovich drift $\underline{a} = a - \frac{1}{2} b\, b'$

both schemes are different

- multi-dimensional case with $d = m \in \{1, 2, \ldots\}$ and **commutative noise**

$$
\begin{aligned}
Y^k_{n+1} \;=\; & Y^k_n + \left\{\alpha_k \, a^k \left(\tau_{n+1}, Y_{n+1}\right) + (1 - \alpha_k)\, a^k\right\} \Delta \\
& + \sum_{j=1}^{m} b^{k,j} \, \Delta W^j \\
& + \frac{1}{2} \sum_{j_1, j_2 = 1}^{m} L^{j_1} b^{k, j_2} \left\{\Delta W^{j_1} \Delta W^{j_2} - 1_{\{j_1 = j_2\}} \Delta\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
Y^k_{n+1} \;=\; & Y^k_n + \left\{\alpha_k \, \underline{a}^k \left(\tau_{n+1}, Y_{n+1}\right) + (1 - \alpha_k)\, \underline{a}^k\right\} \Delta \\
& + \sum_{j=1}^{m} b^{k,j} \, \Delta W^j + \frac{1}{2} \sum_{j_1, j_2 = 1}^{m} L^{j_1} b^{k, j_2} \Delta W^{j_1} \Delta W^{j_2}
\end{aligned}
$$

- **general drift implicit Milstein scheme**

  Itô version

$$
\begin{aligned}
Y^k_{n+1} \;=\; & Y^k_n + \Big(\alpha_k\, a^k\,(\tau_{n+1}, Y_{n+1}) + (1-\alpha_k)\, a^k\Big)\Delta \\
& + \sum_{j=1}^{m} b^{k,j}\, \Delta W^j + \sum_{j_1,j_2=1}^{m} L^{j_1} b^{k,j_2} I_{(j_1,j_2)}
\end{aligned}
$$

Stratonovich version

$$Y^k_{n+1} = Y^k_n + \Big(\alpha_k\, \underline{a}^k\, (\tau_{n+1}, Y_{n+1}) + (1 - \alpha_k)\, \underline{a}^k\Big)\Delta$$
$$+ \sum_{j=1}^{m} b^{k,j}\, \Delta W^j + \sum_{j_1,j_2=1}^{m} L^{j_1} b^{k,j_2} J_{(j_1,j_2)}$$

$\alpha_k \in [0, 1]$ for $k \in \{1,\ldots, d\}$

multiple stochastic integrals $I_{(j_1,j_2)}$ and $J_{(j_1,j_2)}$
approximated

Drift Implicit Order 1.0 Strong Runge-Kutta Scheme

$d = m = 1$

$$
\begin{aligned}
Y_{n+1} &= Y_n + a\left(\tau_{n+1}, Y_{n+1}\right)\,\Delta + b\,\Delta W \\
&\quad + \frac{1}{2\sqrt{\Delta}}\,\left(b\left(\tau_n, \bar{\Upsilon}_n\right) - b\right)\left(\left(\Delta W\right)^2 - \Delta\right)
\end{aligned}
$$

with supporting value

$$
\bar{\Upsilon} = Y_n + a\,\Delta + b\,\sqrt{\Delta}
$$

- **family of drift implicit order 1.0 strong Runge-Kutta schemes**

$$Y_{n+1}^k = Y_n^k + \left(\alpha_k\, a\,(\tau_{n+1}, Y_{n+1}) + (1-\alpha_k)\, a^k\right)\, \Delta$$
$$+ \sum_{j=1}^{m} b^{k,j}\, \Delta W^j$$
$$+ \frac{1}{\sqrt{\Delta}} \sum_{j_1,j_2=1}^{m} \left(b^{k,j_2}\left(\tau_n, \bar{\Upsilon}_n^{j_1}\right) - b^{k,j_2}\right) I_{(j_1,j_2)}$$

with

$$\bar{\Upsilon}_n^j = Y_n + a\,\Delta + b^j\,\sqrt{\Delta}$$

for $j \in \{1, 2, \ldots, m\}$
$\alpha_k \in [0, 1]$ for $k \in \{1, 2, \ldots, d\}$

- for commutative noise

$$
\begin{aligned}
Y_{n+1}^{k} \;=\; & Y_n^k + \left(\alpha_k \, \underline{a}^k \left(\tau_{n+1}, Y_{n+1}\right) + (1-\alpha_k)\; \underline{a}^k\right) \Delta \\
& + \frac{1}{2} \sum_{j=1}^{m} \left(b^{k,j}\left(\tau_n, \bar{\Psi}_n\right) + b^{k,j}\right) \Delta W^j
\end{aligned}
$$

with

$$
\bar{\Psi}_n = Y_n + \underline{a}\,\Delta + \sum_{j=1}^{m} b^j \,\Delta W^j
$$

$\alpha_k \in [0,1]$ for $k \in \{1, 2, \ldots, d\}$

Alternative Implicit Methods

- implicit methods are important

- overcome a range of numerical instabilities

- the above strong schemes do not provide implicit diffusion terms

  $\Longrightarrow$ important limitation

- drift implicit methods are well adapted for small noise and additive noise for relatively large multiplicative noise

  implicit diffusion terms seem unavoidable

- illustration with multiplicative noise

$$dX_t = \sigma\, X_t\, dW_t$$

- explicit strong methods have large errors for not too small time step sizes

- very small time step size may require unrealistic computational time

- cannot apply drift implicit schemes

- **balanced implicit methods**

  Milstein, Pl. & Schurz (1998)

  $m = d = 1$

$$Y_{n+1} = Y_n + a\,\Delta + b\,\Delta W + (Y_n - Y_{n+1})\,C_n$$

  where

$$C_n = c^0(Y_n)\,\Delta + c^1(Y_n)\,|\Delta W|$$

  c^0, c^1 positive, real valued uniformly bounded functions

  strong order $\gamma = 0.5$

- low order strong convergence

  price to pay for numerical stability

- family of specific methods providing a balance between approximating diffusion terms

Simulation Study for the Balanced Method

- Euler scheme

$$\boldsymbol{Y_{n+1} = Y_n + \sigma\, Y_n\, \Delta W}$$

- no simple stochastic counterpart of deterministic implicit Euler method since

$$\boldsymbol{Y_{n+1} = Y_n + \sigma\, Y_{n+1}\, \Delta W}$$

$$\boldsymbol{Y_{n+1} = \frac{Y_n}{1 - \sigma\, \Delta W}}$$

fails because

$$\boldsymbol{E|(1 - \sigma\, \Delta W)^{-1}| = +\infty}$$

- **partially implicit scheme**

$$Y_{n+1} = Y_n + \left(\sigma\, \Delta W + \frac{\sigma^2}{2}\, (\Delta W)^2\right) Y_n - \frac{\sigma^2}{2}\, Y_{n+1}\, \Delta$$

- **balanced implicit method**

$$Y_{n+1} = Y_n + \sigma\, Y_n\, \Delta W + \sigma\, (Y_n - Y_{n+1})\, |\Delta W|$$

$\Longrightarrow$ implicitness also in diffusion term

- explicit solution

$$\boldsymbol{X_T = \exp\left\{\sigma\, W_T - \frac{\sigma^2}{2}\, T\right\} X_0}$$

- absolute error

$$\boldsymbol{\varepsilon_T(\Delta) = E(|X_T - Y_N|)}$$

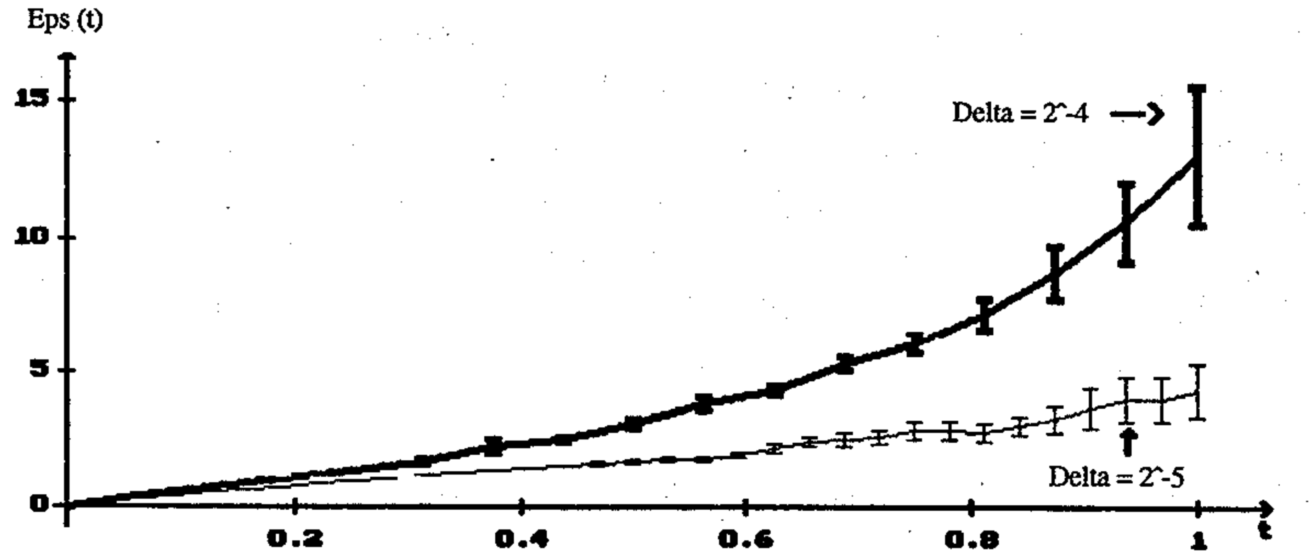

Figure 2.8: Estimated absolute error $\varepsilon_T(\Delta)$ of the Euler method at time T for time step sizes $\Delta = 2^{-4}$ and 2^{-5}.

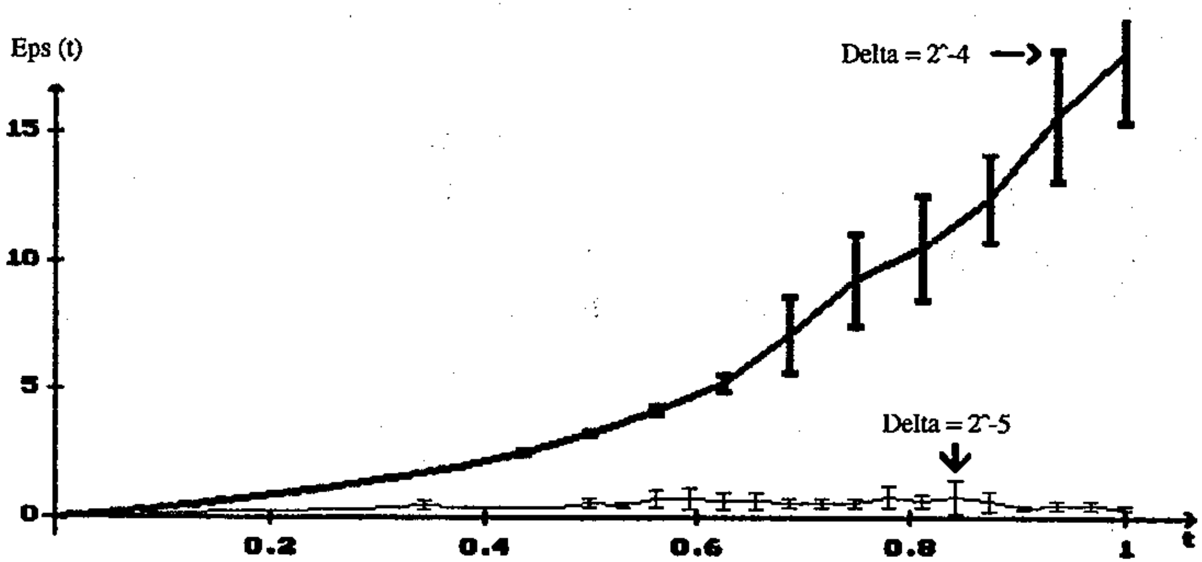

Figure 2.9: Absolute error for the partially implicit method.

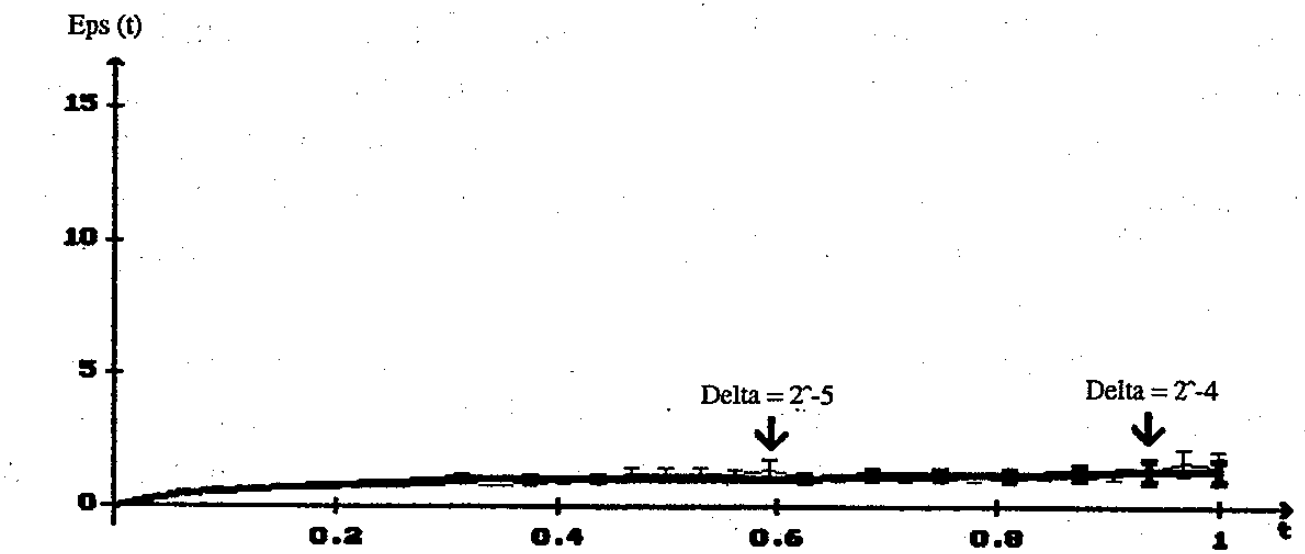

Figure 2.10: Absolute error for the balanced implicit method.

The General Balanced Method and its Convergence

- d-dimensional SDE

$$dX_t = a(t, X_t)\, dt + \sum_{j=1}^{m} b^j(t, X_t)\, dW_t^j$$

- **family of balanced implicit methods**

$$Y_{n+1} = Y_n + a(\tau_n, Y_n)\,\Delta + \sum_{j=1}^{m} b^j(\tau_n, Y_n)\,\Delta W_n^j + C_n(Y_n - Y_{n+1})$$

where

$$C_n = c^0(\tau_n, Y_n)\,\Delta + \sum_{j=1}^{m} c^j(\tau_n, Y_n)\,|\Delta W_n^j|$$

$c^0, c^1, \ldots, c^m$

$d \times d$-matrix-valued uniformly bounded functions

- assume that for any sequence of real (α_i) with $\alpha_0 \in [0, \bar{\alpha}],\ \alpha_1 \geq 0$, $\ldots$, $\alpha_m \geq 0$, where $\bar{\alpha} \geq \Delta$

  matrix

$$M(t, x) = I + \alpha_0\, c^0(t, x) + \sum_{j=1}^{m} a_j\, c^j(t, x)$$

  has an inverse

$$|(M(t, x))^{-1}| \leq K < \infty$$

Theorem (Milstein-Platen-Schurz)

The balanced implicit method converges with strong order $\gamma = 0.5$, that is for all $k \in \{0, 1, \ldots, N\}$ and step size $\Delta = \frac{T}{N}$, $N \in \{1, 2, \ldots\}$ one has

$$
\begin{aligned}
E\left(|X_{\tau_k} - Y_k| \,\middle|\, \mathcal{A}_0\right) &\leq \left(E\left(|X_{\tau_k} - Y_k|^2 \,\middle|\, \mathcal{A}_0\right)\right)^{\frac{1}{2}} \\
&\leq K\left(1 + |X_0|^2\right)^{\frac{1}{2}} \Delta^{\frac{1}{2}},
\end{aligned}
$$

where K does not depend on Δ.

Predictor-Corrector Euler Scheme

- **corrector**

$$Y_{n+1} = Y_n + \Big(\theta\, \bar{a}_\eta(\bar{Y}_{n+1}) + (1-\theta)\, \bar{a}_\eta(Y_n)\Big)\Delta_n + \Big(\eta\, b(\bar{Y}_{n+1}) + (1-\eta)\, b(Y_n)\Big)\, \Delta W_n$$

$\bar{a}_\eta = a - \eta\, b\, b'$

- **predictor**

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\, \Delta_n + b(Y_n)\, \Delta W_n$$

$\theta, \eta \in [0, 1]$ degree of implicitness

3 Monte Carlo Simulation of SDEs

Introduction to Monte Carlo Simulation

- **classical Monte Carlo methods**

  Hammersley & Handscomb (1964)

  Fishman (1996)

- **Monte Carlo methods for SDEs**

  Kloeden & Pl. (1992b)

  Kloeden, Pl. & Schurz (2003)

  Milstein (1995), Glasserman (2004)

Weak Convergence Criterion

- $\tilde{\mathcal{C}}_P(\Re^d, \Re)$ set of all polynomials $g : \Re^d \to \Re$

- SDE

$$dX_t = a(t, X_t)\, dt + \sum_{j=1}^{m} b^j(t, X_t)\, dW_t^j$$

- a discrete time approximation Y^Δ *converges with weak order* $\beta > 0$ to X at time T as $\Delta \to 0$ if for each $g \in \tilde{\mathcal{C}}_P(\Re^d, \Re)$ there exists a constant C_g, which does not depend on Δ and $\Delta_0 \in [0, 1]$ such that

$$\mu(\Delta) = |E(g(X_T)) - E(g(Y_T^\Delta))| \leq C_g\, \Delta^\beta$$

for each $\Delta \in (0, \Delta_0)$

- absolute weak error criterion

Systematic and Statistical Error

- functional

$$\boldsymbol{u = E(g(X_T))}$$

- weak approximations $\boldsymbol{Y^\Delta}$

- **raw Monte Carlo estimate**

$$\boldsymbol{u_{N,\Delta} = \frac{1}{N} \sum_{k=1}^{N} g(Y_T^\Delta(\omega_k))}$$

$\boldsymbol{N}$ independent simulated realizations

$$Y_T^\Delta(\omega_1), Y_T^\Delta(\omega_2), \ldots, Y_T^\Delta(\omega_N)$$

$\omega_k \in \Omega$ for $k \in \{1, 2, \ldots, N\}$

- discrete time weak approximation Y_T^Δ

- **weak error**

$$\hat{\mu}_{N,\Delta} = u_{N,\Delta} - E(g(X_T))$$

- systematic error μ_{sys}
- statistical error μ_{stat}

$$\hat{\mu}_{N,\Delta} = \mu_{\text{sys}} + \mu_{\text{stat}}$$

- **systematic error**

$$
\begin{aligned}
\mu_{\text{sys}} &= E(\hat{\mu}_{N,\Delta}) \\
&= E\left(\frac{1}{N}\sum_{k=1}^{N} g(Y_T^{\Delta}(\omega_k))\right) - E(g(X_T)) \\
&= E(g(Y_T^{\Delta})) - E(g(X_T))
\end{aligned}
$$

$$
\mu(\Delta) = |\mu_{\text{sys}}|
$$

- **statistical error**

  Central Limit Theorem

  asymptotically Gaussian with mean zero and variance

$$\mathbf{Var}(\mu_{\text{stat}}) = \mathbf{Var}(\hat{\mu}_{N,\Delta}) = \frac{1}{N}\,\mathbf{Var}(g(Y_T^{\Delta}))$$

  deviation

$$\text{Dev}(\mu_{\text{stat}}) = \sqrt{\mathbf{Var}(\mu_{\text{stat}})} = \frac{1}{\sqrt{N}}\sqrt{\mathbf{Var}(g(Y_T^{\Delta}))}$$

  decreases at slow rate $\frac{1}{\sqrt{N}}$ as $N \to \infty$

  may need an extremely large number N of sample paths

Confidence Intervals

- statistical error is halved by a fourfold increase in N

- Monte Carlo approach is very general

- high-dimensional functionals

- do usually not know the variance
  of raw Monte Carlo estimates

- form batches

  average of each batch

  approximately Gaussian

  Student t confidence intervals

- length of a confidence interval
  proportional to the square root of the variance

  reformulate the random variable

  same mean but a much smaller variance

- variance reduction techniques

Example of Raw Monte Carlo Simulation

$$u = E(g(X))$$

$$X \sim N(0,1)$$

$$g(X) = \left(\exp\left\{r\Delta + \sigma\sqrt{\Delta}\,X\right\}\right)^2$$

$$r = 0.05,\ \sigma = 0.2,\ \Delta = 1$$

$$u = E\left(\exp\left\{2\left(r\Delta + \sigma\,\sqrt{\Delta}\,X\right)\right\}\right) = \exp\left\{\left(r + \sigma^2\right)2\,\Delta\right\} \approx 1.197$$

• Raw Monte Carlo estimators

$$\hat{u}_N = \frac{1}{N} \sum_{i=1}^{N} \exp\left\{2 \left(r\Delta + \sigma\sqrt{\Delta}\, X(\omega_i)\right)\right\}$$

$N \in \{1, 2, \ldots, 2000\}$

asymptotically normal

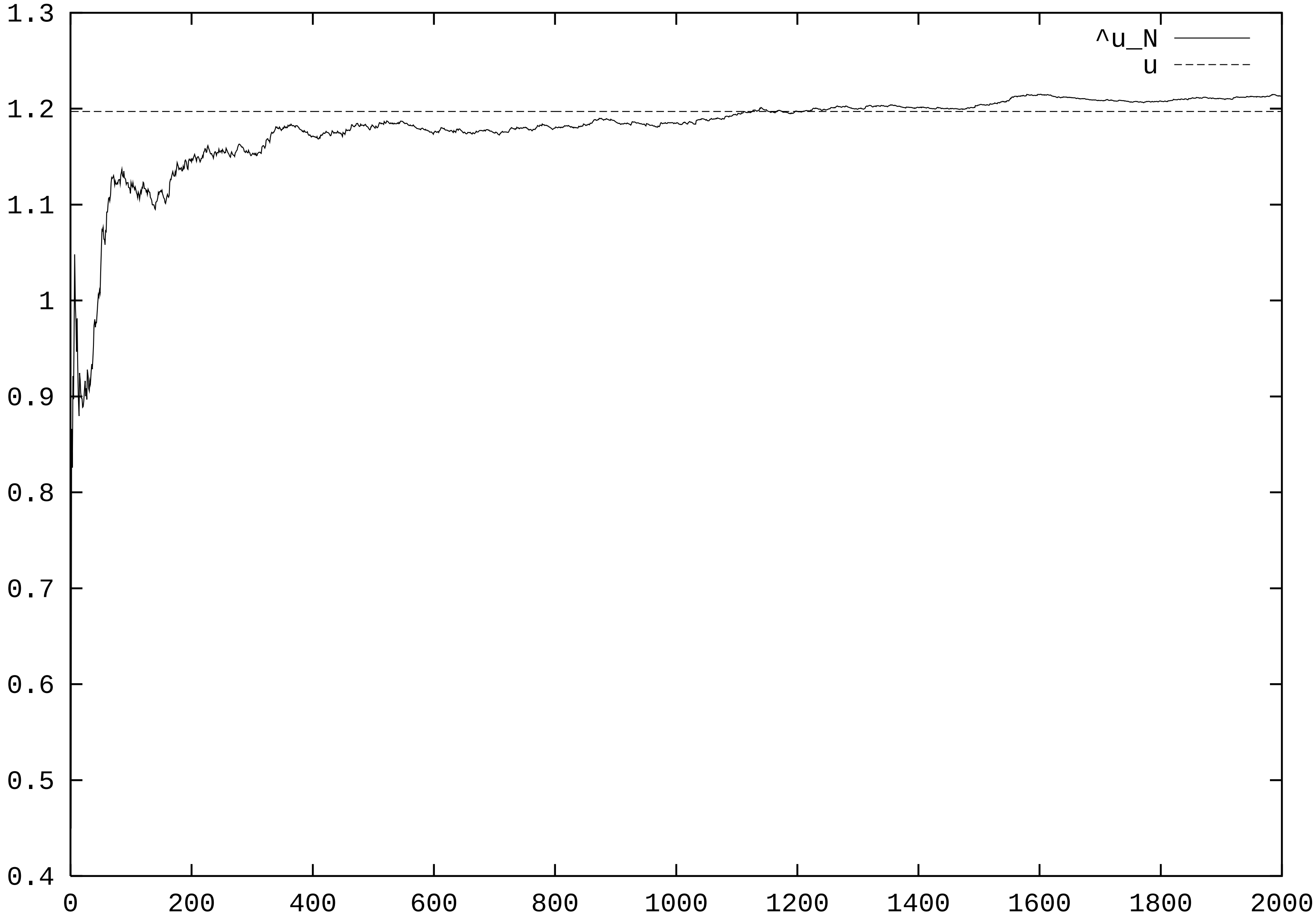

Figure 3.1: Raw Monte Carlo estimates in dependence on the number of simulations.

Normalized Monte Carlo Error

$$\hat{Z}_N = \frac{(\hat{u}_N - u)}{\sqrt{\mathrm{Var}(g(X))}} \sqrt{N} \sim \mathcal{N}(0,1)$$

CLT

$$\mathrm{Var}(g(X)) = \exp\{4\,\Delta\,(r + 2\,\sigma^2)\}\,(1 - \exp\{-4\,\Delta\,\sigma^2\}) \approx 0.25.$$

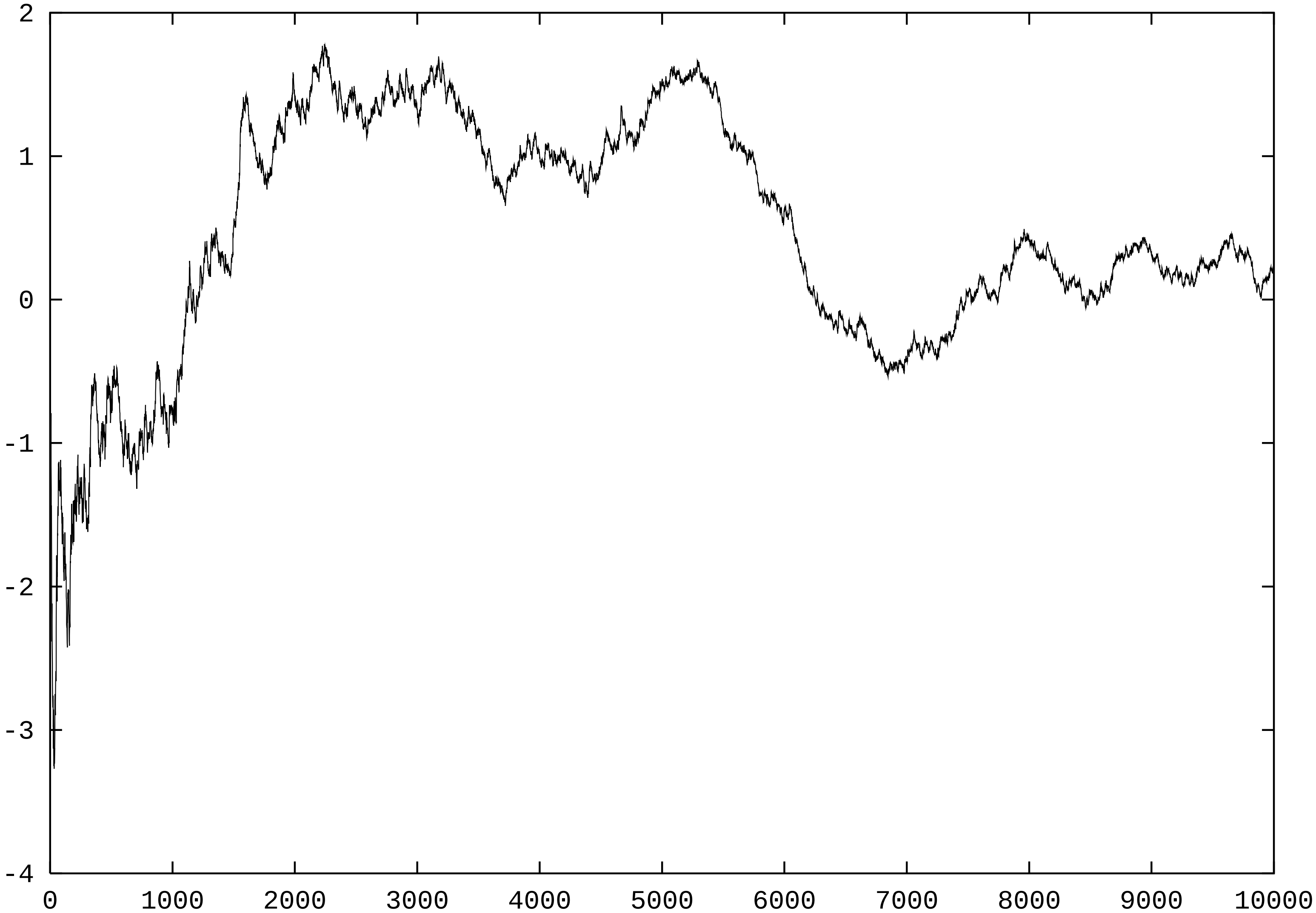

Figure 3.2: Normalized raw Monte Carlo error.

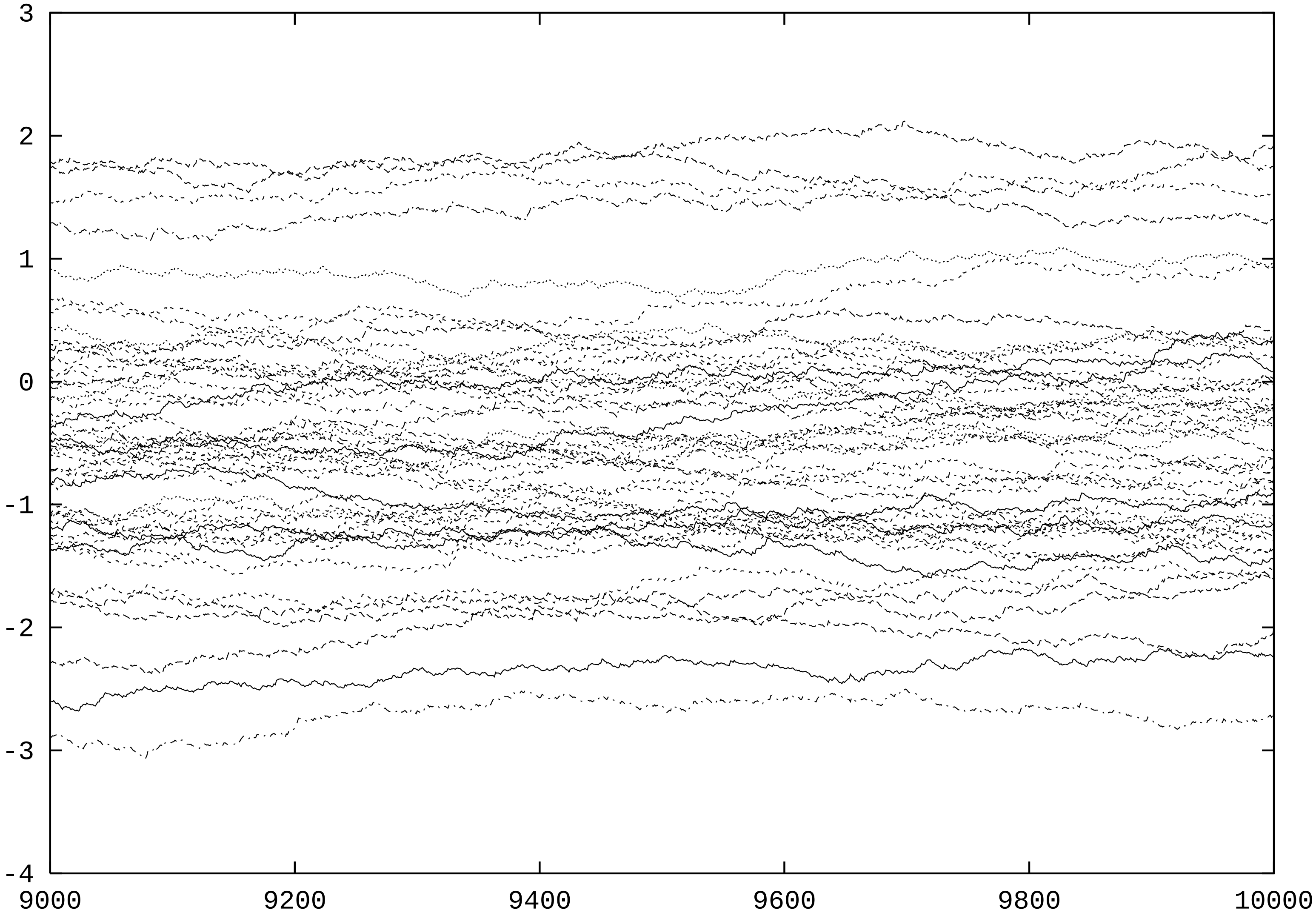

Figure 3.3: Independent realizations of normalized Monte Carlo errors.

Weak Taylor Schemes

Euler and Simplified Weak Euler Scheme

- **Euler scheme**

$$Y^k_{n+1} = Y^k_n + a^k \, \Delta + \sum_{j=1}^{m} b^{k,j} \, \Delta W^j_n$$

$$\Delta W^j_n = W^j_{\tau_{n+1}} - W^j_{\tau_n}$$

truncated Wagner-Platen expansion

weak convergence $\beta = 1.0$

$\Longrightarrow$ weak order $\beta = 1.0$ Taylor scheme

- **simplified weak Euler scheme**

$$Y_{n+1}^k = Y_n^k + a^k \, \Delta + \sum_{j=1}^{M} b^{k,j} \, \Delta \hat{W}_n^j$$

$\Delta \hat{W}_n^j$ independent $\mathcal{A}_{\tau_{n+1}}$-measurable random variables with

$$\left| E\left(\Delta \hat{W}_n^j\right) \right| + \left| E\left(\left(\Delta \hat{W}_n^j\right)^3\right) \right| + \left| E\left(\left(\Delta \hat{W}_n^j\right)^2\right) - \Delta \right| \leq K \, \Delta^2$$

$$\Longrightarrow \quad \beta = 1.0$$

- two-point distributed random variable

$$P\left(\Delta \hat{W}_n^j = \pm\sqrt{\Delta}\right) = \frac{1}{2}$$

- **Simulation Example**

  SDE
$$dX_t = a\,X_t\,dt + b\,X_t\,dW_t$$

  $a = 1.5$, $b = 0.01$ and $T = 1$

  test function $g(X) = x$

  $N = 16,000,000$

  $\Longrightarrow$

  confidence intervals of negligible length

- absolute weak errors
$$\mu(\Delta) = |E(X_T) - E(Y_N)|$$

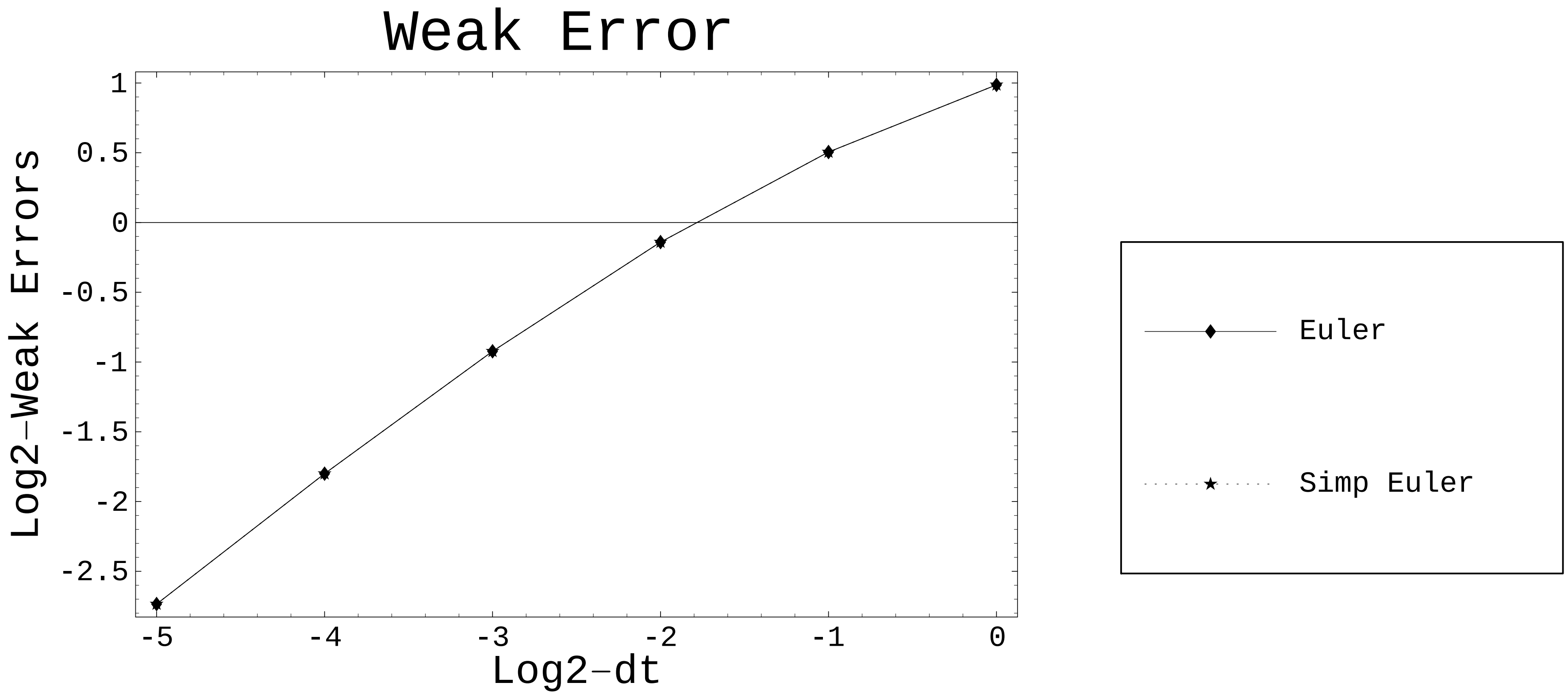

Figure 3.4: Log-log plot of the weak error for an SDE with multiplicative noise for the Euler and simplified Euler schemes.

Weak Order 2.0 Taylor Scheme

further multiple stochastic integrals

- **weak order 2.0 Taylor scheme**
  one-dimensional case $d = m = 1$

$$
\begin{aligned}
Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta W_n + \frac{1}{2}\,b\,b'\Big((\Delta W_n)^2 - \Delta\Big) \\
&\quad + a'\,b\,\Delta Z_n + \frac{1}{2}\left(a\,a' + \frac{1}{2}\,a''b^2\right)\Delta^2 \\
&\quad + \left(a\,b' + \frac{1}{2}\,b''b^2\right)\Big(\Delta W_n\,\Delta - \Delta Z_n\Big)
\end{aligned}
$$

$$
\Delta Z_n = I_{(1,0)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s} dW_z\,ds
$$

generate pair of correlated Gaussian random variables ΔW_n and ΔZ_n

- **simplified weak order 2.0 Taylor scheme**

$$
\begin{aligned}
Y_{n+1} \quad = \quad & Y_n + a\,\Delta + b\,\Delta\hat{W} + \frac{1}{2}\,b\,b'\left(\left(\Delta\hat{W}\right)^2 - \Delta\right) \\
& + \frac{1}{2}\left(a'\,b + a\,b' + \frac{1}{2}\,b''b^2\right)\Delta\hat{W}\,\Delta \\
& + \frac{1}{2}\left(a\,a' + \frac{1}{2}\,a''b^2\right)\Delta^2
\end{aligned}
$$

$$\left|E\left(\Delta\hat{W}\right)\right| + \left|E\left(\left(\Delta\hat{W}\right)^3\right)\right| + \left|E\left(\left(\Delta\hat{W}\right)^5\right)\right|$$

$$+ \left|E\left(\left(\Delta\hat{W}\right)^2\right) - \Delta\right| + \left|E\left(\left(\Delta\hat{W}\right)^4\right) - 3\Delta^2\right| \leq K\,\Delta^3$$

- three-point distributed random variable

$$P\left(\Delta\hat{W} = \pm\sqrt{3\,\Delta}\right) = \frac{1}{6}, \qquad P\left(\Delta\hat{W} = 0\right) = \frac{2}{3}$$

- **weak order 2.0 Taylor scheme with scalar noise**

  $d \in \{1, 2, \ldots\}$ with $m = 1$

$$
\begin{aligned}
Y_{n+1}^{k} \quad &= \quad Y_n^k + a^k \, \Delta + b^k \, \Delta W + \frac{1}{2} \, L^1 b^k \left((\Delta W)^2 - \Delta \right) \\
&\quad + \frac{1}{2} \, L^0 a^k \, \Delta^2 + L^0 b^k \left(\Delta W \, \Delta - \Delta Z \right) + \, L^1 a^k \, \Delta Z
\end{aligned}
$$

- **weak order 2.0 Taylor scheme**

$d, m \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1}^{k} \quad = \quad & Y_{n}^{k} + a^{k}\,\Delta + \frac{1}{2}\,L^{0}a^{k}\,\Delta^{2} \\
& + \sum_{j=1}^{m}\Big(b^{k,j}\,\Delta W^{j} + L^{0}b^{k,j}\,I_{(0,j)} + L^{j}a^{k}\,I_{(j,0)}\Big) \\
& + \sum_{j_1,j_2=1}^{m} L^{j_1}b^{k,j_2}\,I_{(j_1,j_2)}
\end{aligned}
$$

- **simplified weak order 2.0 Taylor scheme**

$$
\begin{aligned}
Y^k_{n+1} &= Y^k_n + a^k\,\Delta + \frac{1}{2}\,L^0 a^k\,\Delta^2 \\
&\quad + \sum_{j=1}^{m} \left(b^{k,j} + \frac{1}{2}\,\Delta\,\left(L^0 b^{k,j} + L^j a^k\right)\right) \Delta\hat{W}^j \\
&\quad + \frac{1}{2} \sum_{j_1,j_2=1}^{m} L^{j_1} b^{k,j_2} \left(\Delta\hat{W}^{j_1}\,\Delta\hat{W}^{j_2} + V_{j_1,j_2}\right)
\end{aligned}
$$

- two-point distributed random variables

$$P\left(V_{j_1,j_2} = \pm\Delta\right) = \frac{1}{2}$$

for $j_2 \in \{1, \ldots, j_1 - 1\}$,

$$V_{j_1,j_1} = -\Delta$$

and

$$V_{j_1,j_2} = -V_{j_2,j_1}$$

for $j_2 \in \{j_1 + 1, \ldots, m\}$ and $j_1 \in \{1, 2, \ldots, m\}$

$\beta = 2.0$

Weak Order 3.0 Taylor Scheme

- **weak order 3.0 Taylor scheme**

  $d, m \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1}^k \;=\;& Y_n^k + a^k\,\Delta + \sum_{j=1}^m b^{k,j}\,\Delta W^j + \sum_{j=0}^m L^j a^k\, I_{(j,0)} \\
&+ \sum_{j_1=0}^m \sum_{j_2=1}^m L^{j_1} b^{k,j_2}\, I_{(j_1,j_2)} + \sum_{j_1,j_2=0}^m L^{j_1} L^{j_2} a^k\, I_{(j_1,j_2,0)} \\
&+ \sum_{j_1,j_2=0}^m \sum_{j_3=1}^m L^{j_1} L^{j_2} b^{k,j_3}\, I_{(j_1,j_2,j_3)}
\end{aligned}
$$

- **simplified weak order 3.0 Taylor scheme**

$m = 1, \quad d = 1$

$$\begin{aligned} Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta\tilde{W} + \frac{1}{2}\,L^1 b\left(\left(\Delta\tilde{W}\right)^2 - \Delta\right) \\ &\quad + L^1 a\,\Delta\tilde{Z} + \frac{1}{2}L^0 a\,\Delta^2 + L^0 b\left(\Delta\tilde{W}\,\Delta - \Delta\tilde{Z}\right) \\ &\quad + \frac{1}{6}\left(L^0L^0 b + L^0L^1 a + L^1L^0 a\right)\,\Delta\tilde{W}\,\Delta^2 \\ &\quad + \frac{1}{6}\left(L^1L^1 a + L^1L^0 b + L^0L^1 b\right)\,\left(\left(\Delta\tilde{W}\right)^2 - \Delta\right)\Delta \\ &\quad + \frac{1}{6}\,L^0L^0 a\,\Delta^3 + \frac{1}{6}\,L^1L^1 b\,\left(\left(\Delta\tilde{W}\right)^2 - 3\Delta\right)\Delta\tilde{W} \end{aligned}$$

$$\Delta \tilde{W} \sim N(0, \Delta), \qquad \Delta \tilde{Z} \sim N\left(0, \frac{1}{3}\, \Delta^3\right)$$

$$E\left(\Delta \tilde{W}\, \Delta \tilde{Z}\right) = \frac{1}{2}\, \Delta^2$$

For $\beta = 3.0$ we can set approximately

$$I_{(1)} = \Delta\tilde{W}^1, \quad I_{(1,0)} \approx \Delta\tilde{Z}, \quad I_{(0,1)} \approx \Delta\Delta\tilde{W} - \Delta\tilde{Z}$$

$$I_{(1,1)} \approx \frac{1}{2}\left(\left(\Delta\tilde{W}\right)^2 - \Delta\right)$$

$$I_{(0,0,1)} \approx I_{(0,1,0)} \approx I_{(1,0,0)} \approx \frac{1}{6}\,\Delta^2\Delta\tilde{W}$$

$$I_{(1,1,0)} \approx I_{(1,0,1)} \approx I_{(0,1,1)} \approx \frac{1}{6}\,\Delta\left(\left(\Delta\tilde{W}\right)^2 - \Delta\right)$$

$$I_{(1,1,1)} \approx \frac{1}{6}\,\Delta\tilde{W}\left(\left(\Delta\tilde{W}\right)^2 - 3\,\Delta\right)$$

where $\Delta\tilde{W}$ and $\Delta\tilde{Z}$

are correlated Gaussian random variables

Hofmann (1994)

$\Longrightarrow$ additive noise, $\beta = 3.0,\ m \in \{1, 2, \ldots\}$

$$I_{(0)} = \Delta, \quad I_{(j)} = \xi_j \sqrt{\Delta}, \quad I_{(0,0)} = \frac{\Delta^2}{2}, \quad I_{(0,0,0)} = \frac{\Delta^3}{6}$$

$$I_{(j,0)} \approx \frac{1}{2} \left(\xi_j + \varphi_j \frac{1}{\sqrt{3\,\Delta}} \right) \Delta^{\frac{3}{2}}, \quad I_{(j,0,0)} \approx I_{(0,j,0)} \approx \frac{\Delta^{\frac{5}{2}}}{6} \xi_j$$

$$I_{(j_1,j_2,0)} \approx \frac{\Delta^2}{6} \left(\xi_{j_1} \xi_{j_2} + \frac{V_{j_1,j_2}}{\Delta} \right)$$

independent four point distributed random variables

$\xi_1, \ldots, \xi_m$

$$P\left(\xi_j = \pm\sqrt{3+\sqrt{6}}\right) = \frac{1}{12+4\sqrt{6}}$$

$$P\left(\xi_j = \pm\sqrt{3-\sqrt{6}}\right) = \frac{1}{12-4\sqrt{6}}$$

independent three point distributed random variables

$\varphi_1, \ldots, \varphi_m$

independent two-point distributed random variables

V_{j_1,j_2}

- **multi-dimensional case**

  $d, m \in \{1, 2, \ldots\}$

  additive noise

  weak order 3.0 Taylor scheme

$$
\begin{aligned}
Y_{n+1}^k \;=\; & Y_n^k + a^k\,\Delta + \sum_{j=1}^m b^{k,j}\,\Delta\tilde{W}^j + \frac{1}{2}\,L^0 a^k\,\Delta^2 + \frac{1}{6}\,L^0 L^0 a^k\,\Delta^3 \\
& + \sum_{j=1}^m \Bigg(L^j a^k\,\Delta\tilde{Z}^j + L^0 b^{k,j}\,\left(\Delta\tilde{W}^j\Delta - \Delta\tilde{Z}^j\right) \\
& \qquad + \frac{1}{6}\left(L^0 L^0 b^{k,j} + L^0 L^j a^k + L^j L^0 a^k\right)\Delta\tilde{W}^j\,\Delta^2 \Bigg) \\
& + \frac{1}{6}\sum_{j_1,j_2=1}^m L^{j_1} L^{j_2} a^k\,\left(\Delta\tilde{W}^{j_1}\Delta\tilde{W}^{j_2} - I_{\{j_1=j_2\}}\,\Delta\right)\Delta
\end{aligned}
$$

$\Delta \tilde{W}^j$ and $\Delta \tilde{Z}^j$

independent pairs of

correlated Gaussian random variables

Weak Order 4.0 Taylor Scheme

Wagner-Platen expansion $\Longrightarrow$

- weak order $\mathbf{4.0}$ Taylor scheme

  $d, m \in \{1, 2, \ldots\}$

$$Y^k_{n+1} = Y^k_n + \sum_{\ell=1}^{4} \sum_{j_1,\ldots,j_\ell=0}^{m} L^{j_1} \cdots L^{j_{\ell-1}} \, b^{k,j_\ell} \, I_{(j_1,\ldots,j_\ell)}$$

  $\beta = 4.0$

  Pl. (1984)

- **simplified weak order 4.0 Taylor scheme for additive noise**

$$\begin{aligned}
Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta\tilde{W} + \frac{1}{2}\,L^0 a\,\Delta^2 + L^1 a\,\Delta\tilde{Z} \\
&\quad + L^0 b\,\left(\Delta\tilde{W}\,\Delta - \Delta\tilde{Z}\right) \\
&\quad + \frac{1}{3!}\left(L^0 L^0 b + L^0 L^1 a\right)\Delta\tilde{W}\,\Delta^2 \\
&\quad + L^1 L^1 a\,\left(2\,\Delta\tilde{W}\,\Delta\tilde{Z} - \frac{5}{6}\,\left(\Delta\tilde{W}\right)^2\,\Delta - \frac{1}{6}\,\Delta^2\right) \\
&\quad + \frac{1}{3!}\,L^0 L^0 a\,\Delta^3 + \frac{1}{4!}\,L^0 L^0 L^0 a\,\Delta^4
\end{aligned}$$

$$+\frac{1}{4!}\left(L^1L^0L^0a+L^0L^1L^0a+L^0L^0L^1a\ +L^0L^0L^0b\right)\Delta\tilde{W}\,\Delta^3$$

$$+\frac{1}{4!}\left(L^1L^1L^0a+L^0L^1L^1a+L^1L^0L^1a\right)\ \left(\left(\Delta\tilde{W}\right)^2-\Delta\right)\Delta^2$$

$$+\frac{1}{4!}\,L^1L^1L^1a\,\Delta\tilde{W}\left(\left(\Delta\tilde{W}\right)^2-3\Delta\right)\Delta$$

$$\Delta\tilde{W}\sim N(0,\Delta),\qquad \Delta\tilde{Z}\sim N(0,\frac{1}{3}\Delta^3)$$

$$E(\Delta\tilde{W}\,\Delta\tilde{Z})=\frac{1}{2}\,\Delta^2$$

A Simulation Study

- **SDE with additive noise**

$$dX_t = a\,X_t\,dt + b\,dW_t$$

$X_0 = 1$, $a = 1.5$, $b = 0.01$ and $T = 1$

$g(X) = x$

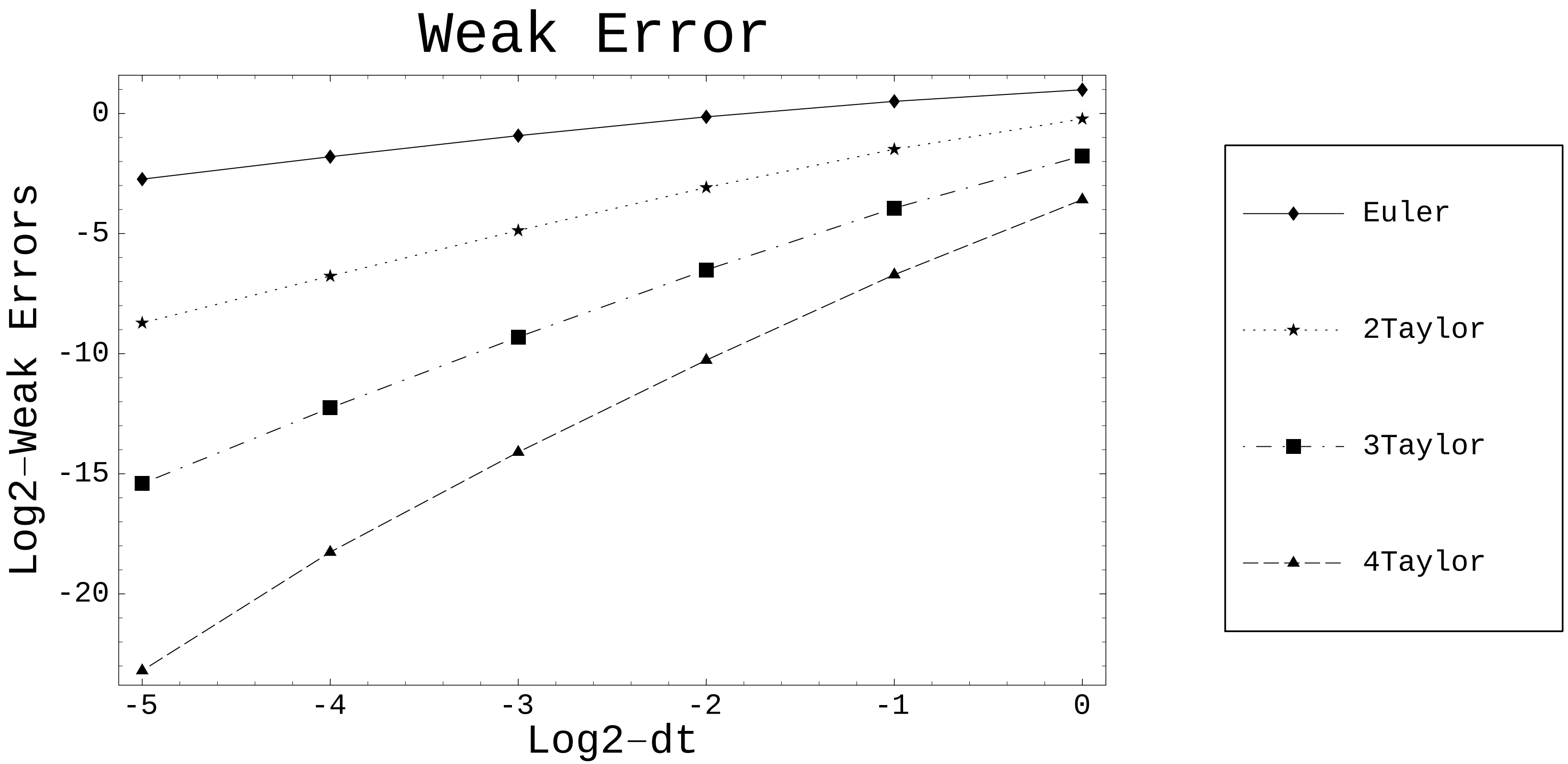

Figure 3.5: Log-log plot of the weak error for an SDE with additive noise using weak Taylor schemes.

- **SDE with multiplicative noise**

$$dX_t = a\, X_t\, dt + b\, X_t\, dW_t$$

$$g(X) = x$$

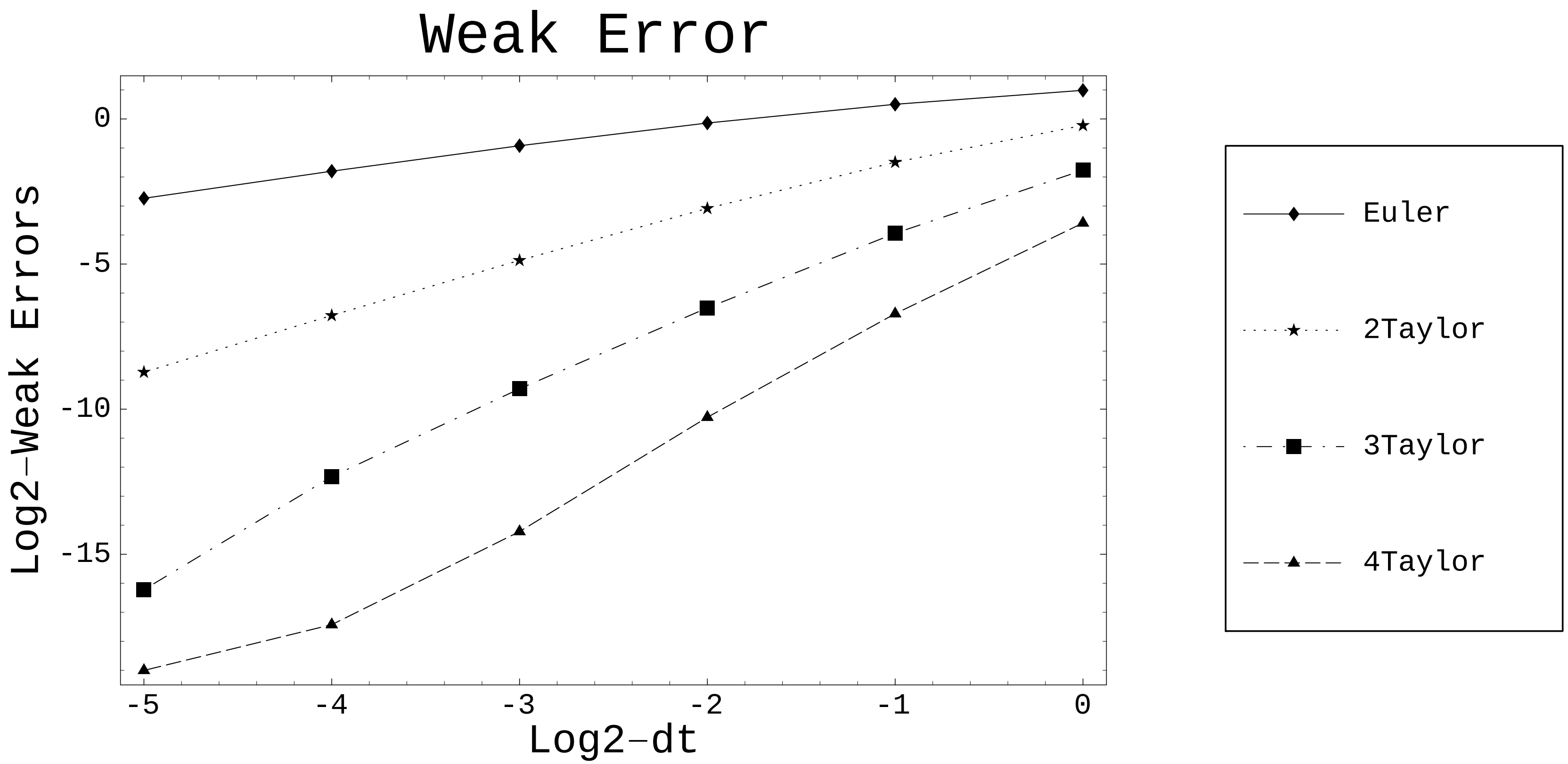

Figure 3.6: Log-log plot of the weak error for an SDE with multiplicative noise using weak Taylor schemes.

Convergence Theorem

- **Wagner-Platen expansion**

- **Itô coefficient functions**

$$f(t,x) \equiv x$$

$$f_\alpha(t,x) = L^{j_1} \cdots L^{j_{\ell-1}}\, b^{j_\ell}(x)$$

$$\alpha = (j_1, \ldots, j_\ell) \in \mathcal{M}_m,\ m \in \{1, 2, \ldots\}$$

- **multiple Itô integral**

$$I_{\alpha,\tau_n,\tau_{n+1}} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_\ell} \cdots \int_{\tau_n}^{s_2} dW_{s_1}^{j_1} \ldots dW_{s_{\ell-1}}^{j_{\ell-1}} dW_{s_\ell}^{j_\ell}$$

$$dW_s^0 = ds$$

- **hierarchical set**

$$\Gamma_\beta = \{\alpha \in \mathcal{M}_m : \, l(\alpha) \leq \beta\}$$

- **time discretization**

$0 = \tau_0 < \tau_1 < \ldots \tau_n < \ldots$

- **weak Taylor scheme of order β**

$$\begin{aligned} Y_{n+1} &= Y_n + \sum_{\alpha \in \Gamma_\beta \setminus \{v\}} f_\alpha \left(\tau_n, Y_n\right) \, I_{\alpha, \tau_n, \tau_{n+1}} \\ &= \sum_{\alpha \in \Gamma_\beta} f_\alpha \left(\tau_n, Y_n\right) \, I_{\alpha, \tau_n, \tau_{n+1}} \end{aligned}$$

$Y_0 = X_0$

Theorem

For some $\beta \in \{1, 2, \ldots\}$ *and autonomous* X *let* Y^{Δ} *be a weak Taylor scheme of order* β*. Suppose that* a *and* b *are Lipschitz continuous with components* a^k*,* $b^{k,j} \in \mathcal{C}_P^{2(\beta+1)}(\Re^d, \Re)$ *for all* $k \in \{1, 2, \ldots, d\}$ *and* $j \in \{0, 1, \ldots, m\}$*, and that the* f_α *satisfy a linear growth bound*

$$|f_\alpha(t, x)| \leq K\ (1 + |x|)\,,$$

for all $\alpha \in \Gamma_\beta$*,* $x \in \Re^d$ *and* $t \in [0, T]$*, where* $K < \infty$*. Then for each* $g \in \mathcal{C}_P^{2(\beta+1)}(\Re^d, \Re)$ *there exists a constant* C_g*, which does not depend on* Δ*, such that*

$$\mu(\Delta) = \left| E\left(g\left(X_T\right)\right) - E\left(g\left(Y_T^{\Delta}\right)\right)\right| \leq C_g\, \Delta^{\beta},$$

that is Y^{Δ} *converges with weak order* β *to* X *at time* T *as* $\Delta \to 0$*.*

Derivative Free Weak Approximations

Explicit Weak Order 2.0 Scheme

- **explicit weak order 2.0 scheme**

  $m = 1$ and $d \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1} \quad = \quad & Y_n + \frac{1}{2}\left(a\left(\bar{\Upsilon}\right) + a\right)\Delta \\
& + \frac{1}{4}\left(b\left(\bar{\Upsilon}^{+}\right) + b\left(\bar{\Upsilon}^{-}\right) + 2b\right)\Delta\hat{W} \\
& + \frac{1}{4}\left(b\left(\bar{\Upsilon}^{+}\right) - b\left(\bar{\Upsilon}^{-}\right)\right)\left(\left(\Delta\hat{W}\right)^{2} - \Delta\right)\Delta^{\frac{1}{2}}
\end{aligned}
$$

with supporting values

$$\bar{\Upsilon} = Y_n + a\,\Delta + b\,\Delta\hat{W}$$

and

$$\bar{\Upsilon}^{\pm} = Y_n + a\,\Delta \pm b\,\sqrt{\Delta}$$

$\Delta\hat{W}$ Gaussian or three-point distributed

$$P\left(\Delta\hat{W} = \pm\sqrt{3\Delta}\right) = \frac{1}{6} \quad \text{and} \quad P\left(\Delta\hat{W} = 0\right) = \frac{2}{3}$$

- **multi-dimensional explicit weak order 2.0 scheme**

$d, m \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + \frac{1}{2}\left(a\left(\bar{\Upsilon}\right) + a\right)\Delta \\
& + \frac{1}{4}\sum_{j=1}^{m}\left[\left(b^j\left(\bar{R}^j_+\right) + b^j\left(\bar{R}^j_-\right) + 2b^j\right)\Delta\hat{W}^j\right. \\
& \left. + \sum_{\substack{r=1\\ r\neq j}}^{m}\left(b^j\left(\bar{U}^r_+\right) + b^j\left(\bar{U}^r_-\right) - 2b^j\right)\Delta\hat{W}^j\,\Delta^{-\frac{1}{2}}\right]
\end{aligned}
$$

$$
+\frac{1}{4}\sum_{j=1}^{m}\Bigg[\left(b^{j}\left(\bar{R}_{+}^{j}\right)-b^{j}\left(\bar{R}_{-}^{j}\right)\right)\left(\left(\Delta\hat{W}^{j}\right)^{2}-\Delta\right)
$$

$$
+\sum_{\substack{r=1\\ r\neq j}}^{m}\left(b^{j}\left(\bar{U}_{+}^{r}\right)-b^{j}\left(\bar{U}_{-}^{r}\right)\right)\left(\Delta\hat{W}^{j}\Delta\hat{W}^{r}+V_{r,j}\right)\Bigg]\Delta^{-\frac{1}{2}}
$$

with supporting values

$$\bar{\Upsilon} = Y_n + a\,\Delta + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j, \qquad \bar{R}^j_{\pm} = Y_n + a\,\Delta \pm b^j\,\sqrt{\Delta}$$

and

$$\bar{U}^j_{\pm} = Y_n \pm b^j\,\sqrt{\Delta}$$

$\Delta\hat{W}^j$ and $V_{r,j}$ as before

- additive noise

$\Longrightarrow$

$$
\begin{aligned}
Y_{n+1} &= Y_n + \frac{1}{2}\left(a\left(Y_n + a\,\Delta + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j\right) + a\right)\Delta \\
&\quad + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j
\end{aligned}
$$

Explicit Weak Order 3.0 Schemes

- **explicit weak order 3.0 scheme**

  $d \in \{1, 2, \ldots\}$ with $m = 1$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + a\,\Delta + b\,\Delta\hat{W} + \frac{1}{2}\left(a_\zeta^+ + a_\zeta^- - \frac{3}{2}\,a - \frac{1}{4}\left(\tilde{a}_\zeta^+ + \tilde{a}_\zeta^-\right)\right)\Delta \\
& + \sqrt{\frac{2}{\Delta}}\left(\frac{1}{\sqrt{2}}\left(a_\zeta^+ - a_\zeta^-\right) - \frac{1}{4}\left(\tilde{a}_\zeta^+ - \tilde{a}_\zeta^-\right)\right)\zeta\,\Delta\hat{Z} \\
& + \frac{1}{6}\left[a\left(Y_n + \left(a + a_\zeta^+\right)\Delta + (\zeta + \varrho)\,b\,\sqrt{\Delta}\right) - a_\zeta^+ - a_\varrho^+ + a\right] \\
& \qquad \times \left[(\zeta + \varrho)\;\Delta\hat{W}\,\sqrt{\Delta} + \Delta + \zeta\,\varrho\left(\left(\Delta\hat{W}\right)^2 - \Delta\right)\right]
\end{aligned}
$$

with

$$a_\phi^{\pm} = a\left(Y_n + a\,\Delta \pm b\,\sqrt{\Delta}\,\phi\right)$$

and

$$\tilde{a}_\phi^{\pm} = a\left(Y_n + 2a\,\Delta \pm b\,\sqrt{2\Delta}\,\phi\right)$$

$$\Delta\hat{W} \sim N(0, \Delta) \quad \text{and} \quad \Delta\hat{Z} \sim N(0, \tfrac{1}{3}\,\Delta^3)$$

$$E(\Delta\hat{W}\,\Delta\hat{Z}) = \frac{1}{2}\Delta^2$$

$$P(\zeta = \pm 1) = P(\varrho = \pm 1) = \frac{1}{2}$$

- **explicit weak order 3.0 scheme for scalar noise**

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + a\,\Delta + b\,\Delta\hat{W} + \frac{1}{2}\,H_a\,\Delta + \frac{1}{\Delta}\,H_b\,\Delta\hat{Z} \\
& + \sqrt{\frac{2}{\Delta}}\,G_a\,\zeta\,\Delta\hat{Z} + \frac{1}{\sqrt{2\Delta}}\,G_b\,\zeta\left(\left(\Delta\hat{W}\right)^2 - \Delta\right) \\
& + \frac{1}{6}\,F_a^{++}\left(\Delta + (\zeta+\varrho)\sqrt{\Delta}\,\Delta\hat{W} + \zeta\varrho\left(\left(\Delta\hat{W}\right)^2 - \Delta\right)\right) \\
& + \frac{1}{24}\left(F_b^{++} + F_b^{-+} + F_b^{+-} + F_b^{--}\right)\Delta\hat{W}
\end{aligned}
$$

$$+\frac{1}{24\sqrt{\Delta}}\left(F_b^{++}-F_b^{-+}+F_b^{+-}-F_b^{--}\right)\left(\left(\Delta\hat{W}\right)^2-\Delta\right)\zeta$$

$$+\frac{1}{24\Delta}\left(F_b^{++}+F_b^{--}-F_b^{-+}-F_b^{+-}\right)\left(\left(\Delta\hat{W}\right)^2-3\right)\Delta\hat{W}\,\zeta\varrho$$

$$+\frac{1}{24\sqrt{\Delta}}\left(F_b^{++}+F_b^{-+}-F_b^{+-}-F_b^{--}\right)\left(\left(\Delta\hat{W}\right)^2-\Delta\right)\varrho$$

with

$$H_g = g^+ + g^- - \frac{3}{2}\, g - \frac{1}{4}\left(\tilde{g}^+ + \tilde{g}^-\right),$$

$$G_g = \frac{1}{\sqrt{2}}\left(g^+ - g^-\right) - \frac{1}{4}\left(\tilde{g}^+ - \tilde{g}^-\right),$$

$$\begin{aligned} F_g^{+\pm} \quad = \quad & g\left(Y_n + \left(a + a^+\right)\Delta + b\,\zeta\,\sqrt{\Delta} \pm b^+\,\varrho\,\sqrt{\Delta}\right) - g^+ \\ & - g\left(Y_n + a\,\Delta \pm b\,\varrho\,\sqrt{\Delta}\right) + g \end{aligned}$$

$$F_g^{-\pm} = g\left(Y_n + (a + a^-)\,\Delta - b\,\zeta\,\sqrt{\Delta} \pm b^-\,\varrho\,\sqrt{\Delta}\right) - g^-$$
$$-g\left(Y_n + a\,\Delta \pm b\,\varrho\,\sqrt{\Delta}\right) + g$$

where

$$g^{\pm} = g\left(Y_n + a\,\Delta \pm b\,\zeta\,\sqrt{\Delta}\right)$$

and

$$\tilde{g}^{\pm} = g\left(Y_n + 2\,a\,\Delta \pm \sqrt{2}\,b\,\zeta\,\sqrt{\Delta}\right)$$

with g being equal to either a or b.

Extrapolation Methods

Weak Order 2.0 Extrapolation

- equidistant time discretizations
- simulate functional

$$E\left(g\left(Y_T^{\Delta}\right)\right)$$

  using Euler scheme

  with step size Δ

- repeat with the double step size 2Δ

$$\Longrightarrow$$

$$E\left(g\left(Y_T^{2\Delta}\right)\right)$$

- combine these two functionals

$\Longrightarrow$

- **weak order 2.0 extrapolation**

$$V_{g,2}^{\Delta}(T) = 2E\left(g\left(Y_T^{\Delta}\right)\right) - E\left(g\left(Y_T^{2\Delta}\right)\right)$$

Talay & Tubaro (1990)

Richardson extrapolation

- **example**

geometric Brownian motion

$$dX_t = a\,X_t\,dt + b\,X_t\,dW_t$$

$X_0 = 0.1$, $a = 1.5$ and $b = 0.01$

Richardson extrapolation $V^{\Delta}_{g,2}(T)$

$g(x) = x$

$\Longrightarrow$ absolute weak error

$$\mu(\Delta) = \left| V^{\Delta}_{g,2}(T) - E\left(g\left(X_T\right)\right) \right|$$

$\beta = 2.0$

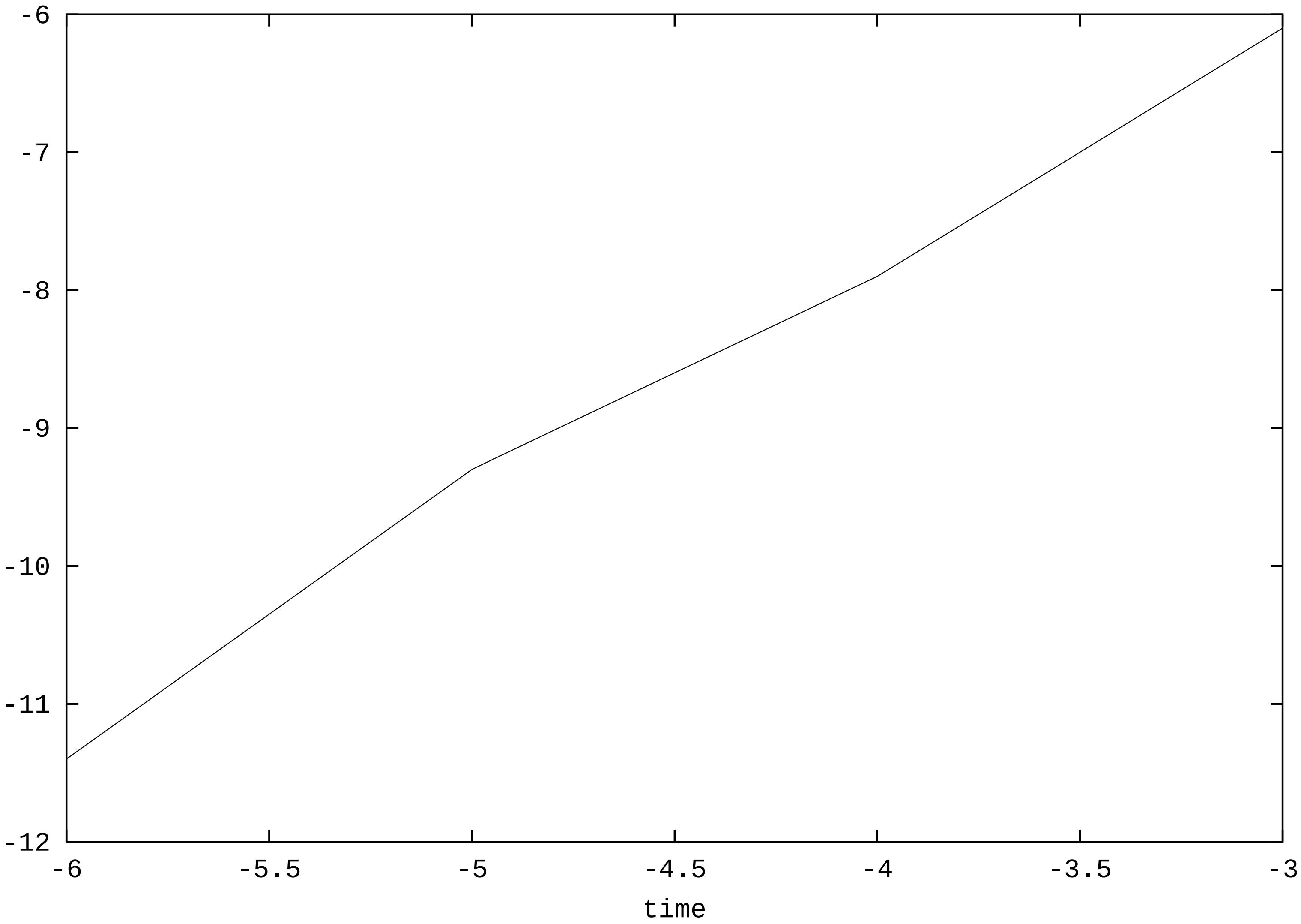

Figure 3.7: Log-log plot for absolute weak error of Richardson extrapolation against step size.

Higher Weak Order Extrapolation

- **weak order 4.0 extrapolation**

$$V_{g,4}^{\Delta}(T) = \frac{1}{21}\left[32\, E\left(g\left(Y_T^{\Delta}\right)\right) - 12\, E\left(g\left(Y_T^{2\Delta}\right)\right) + E\left(g\left(Y_T^{4\Delta}\right)\right)\right]$$

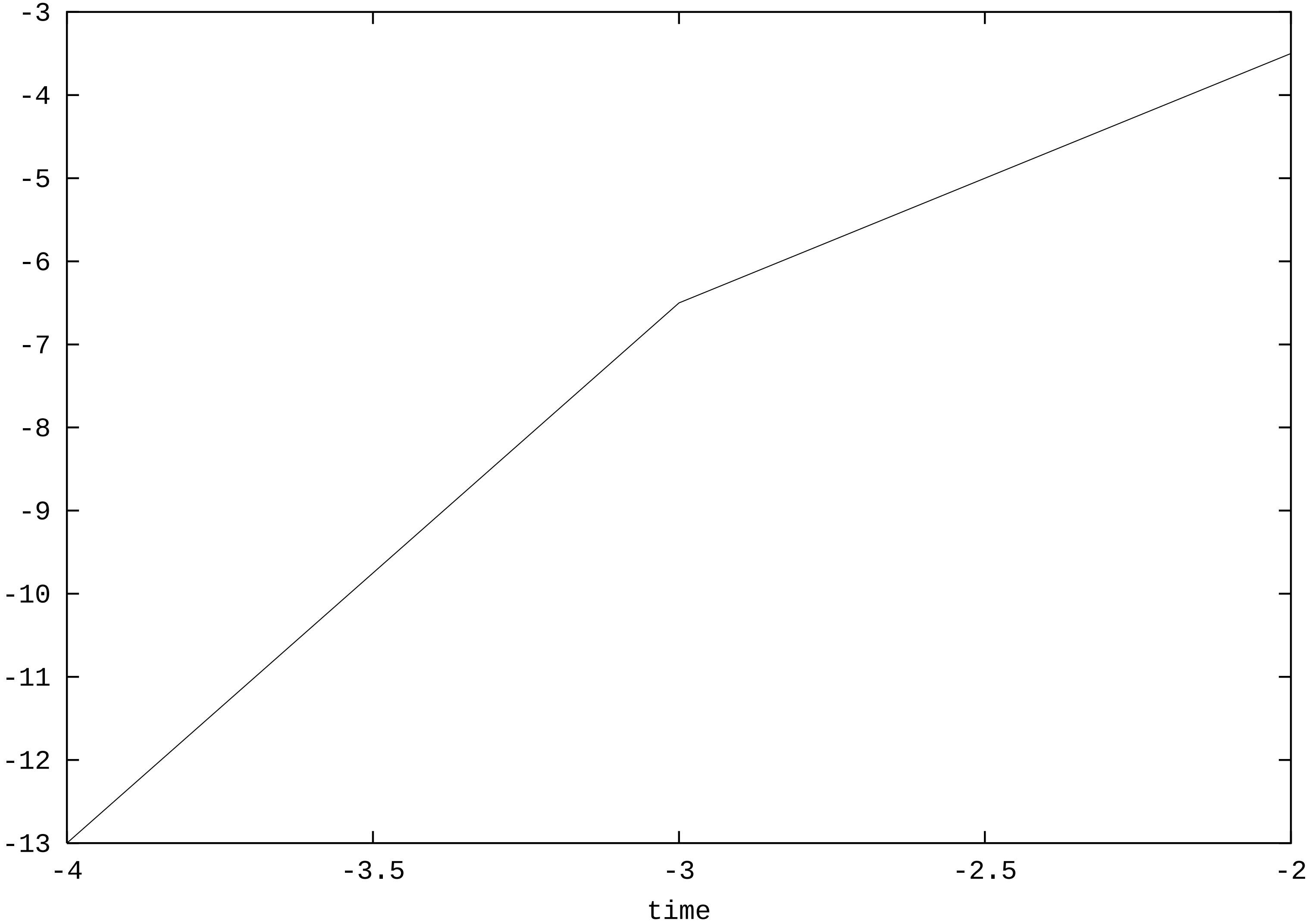

Figure 3.8: Log-log plot for absolute weak error of weak order 4.0 extrapolation.

Implicit and Predictor Corrector Methods

- numerical stability has highest priority

Drift Implicit Euler Scheme

$d, m \in \{1, 2, \ldots\}$

$$Y_{n+1} = Y_n + a\left(\tau_{n+1}, Y_{n+1}\right)\Delta + \sum_{j=1}^{m} b^j\left(\tau_n, Y_n\right)\Delta\hat{W}^j$$

$$P\left(\Delta\hat{W}^j = \pm\sqrt{\Delta}\right) = \frac{1}{2}$$

- **family of drift implicit simplified Euler schemes**

$$
\begin{aligned}
Y_{n+1} \;&=\; Y_n + \Big((1-\alpha)\, a\,(\tau_n, Y_n) + \alpha\, a\,(\tau_{n+1}, Y_{n+1}) \Big) \Delta \\
&\quad + \sum_{j=1}^{m} b^j\,(\tau_n, Y_n)\, \Delta \hat{W}^j
\end{aligned}
$$

α degree of drift implicitness

A-stable for $\alpha \in [0.5, 1]$

region of A-stability

circle of radius $r = (1-2\alpha)^{-1}$ centered at $-r$

Fully Implicit Euler Scheme

simplified schemes allow

implicit diffusion coefficient term

- **fully implicit weak Euler scheme**

$$Y_{n+1} = Y_n + \bar{a}\,(Y_{n+1})\,\Delta + b\,(Y_{n+1})\,\Delta\hat{W}$$

$$\bar{a} = a - b\,b'$$

- **family of implicit weak Euler schemes**

$$\boldsymbol{Y_{n+1} \;=\; Y_n + \Big(\alpha\, \bar{a}_\eta\,(\tau_{n+1}, Y_{n+1}) + (1-\alpha)\, \bar{a}_\eta\,(\tau_n, Y_n)\Big)\Delta}$$

$$\boldsymbol{+ \sum_{j=1}^{m} \Big(\eta\, b^j\,(\tau_{n+1}, Y_{n+1}) + (1-\eta)\, b^j\,(\tau_n, Y_n)\Big)\Delta \hat{W}^j}$$

$$\boldsymbol{\bar{a}_\eta = a - \eta \sum_{j_1, j_2 = 1}^{m} \sum_{k=1}^{d} b^{k, j_1} \frac{\partial b^{j_2}}{\partial x^k}}$$

for $\boldsymbol{\alpha, \eta \in [0, 1]}$

Implicit Weak Order 2.0 Taylor Scheme

$$
\begin{aligned}
Y_{n+1} &= Y_n + a\,(Y_{n+1})\,\Delta + b\,\Delta\hat{W} \\
&\quad - \frac{1}{2}\left(a\,(Y_{n+1})\,a'\,(Y_{n+1}) + \frac{1}{2}\,b^2\,(Y_{n+1})\,a''\,(Y_{n+1})\right)\Delta^2 \\
&\quad + \frac{1}{2}\,b\,b'\left(\left(\Delta\hat{W}\right)^2 - \Delta\right) \\
&\quad + \frac{1}{2}\left(-a'\,b + a\,b' + \frac{1}{2}\,b''b^2\right)\Delta\hat{W}\,\Delta
\end{aligned}
$$

$$
P\left(\Delta\hat{W} = \pm\sqrt{3\Delta}\right) = \frac{1}{6} \quad \text{and} \quad P\left(\Delta\hat{W} = 0\right) = \frac{2}{3}
$$

Milstein (1995)

- **family of implicit weak order 2.0 Taylor schemes**

$$
\begin{aligned}
Y_{n+1} \;=\; Y_n &+ \Big(\alpha\, a\,(\tau_{n+1}, Y_{n+1}) + (1-\alpha)\, a\Big)\Delta \\
&+ \frac{1}{2} \sum_{j_1,j_2=1}^{m} L^{j_1} b^{j_2} \left(\Delta \hat{W}^{j_1} \Delta \hat{W}^{j_2} + V_{j_1,j_2}\right) \\
&+ \sum_{j=1}^{m} \left(b^j + \frac{1}{2}\left(L^0 b^j + (1-2\alpha) L^j a\right)\Delta\right)\Delta \hat{W}^j \\
&+ \frac{1}{2}(1-2\alpha)\Big(\beta\, L^0 a + (1-\beta)\, L^0 a\,(\tau_{n+1}, Y_{n+1})\,\Big)\Delta^2
\end{aligned}
$$

$\alpha = 0.5 \qquad \Longrightarrow$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + \frac{1}{2}\Big(a\,(\tau_{n+1}, Y_{n+1}) + a\Big)\Delta \\
& + \sum_{j=1}^{m} b^j\,\Delta \hat{W}^j + \frac{1}{2}\sum_{j=1}^{m} L^0 b^j\,\Delta \hat{W}^j\,\Delta \\
& + \frac{1}{2}\sum_{j_1,j_2=1}^{m} L^{j_1} b^{j_2}\Big(\Delta \hat{W}^{j_1}\Delta \hat{W}^{j_2} + V_{j_1,j_2}\Big)
\end{aligned}
$$

Implicit Weak Order 2.0 Scheme

$m = 1$

Pl. (1995)

- **implicit weak order 2.0 scheme**

$$
\begin{aligned}
Y_{n+1} \quad &= \quad Y_n + \frac{1}{2}\,(a + a\,(Y_{n+1}))\,\Delta \\
&\quad + \frac{1}{4}\,\left(b\left(\bar{\Upsilon}^+\right) + b\left(\bar{\Upsilon}^-\right) + 2\,b\right)\Delta\hat{W} \\
&\quad + \frac{1}{4}\,\left(b\left(\bar{\Upsilon}^+\right) - b\left(\bar{\Upsilon}^-\right)\right)\left(\left(\Delta\hat{W}\right)^2 - \Delta\right)\Delta^{-\frac{1}{2}}
\end{aligned}
$$

with supporting values

$$\bar{\Upsilon}^{\pm} = Y_n + a\,\Delta \pm b\,\sqrt{\Delta}$$

$\Delta\hat{W}$ can be chosen as before

- **implicit weak order 2.0 scheme**

  $\boldsymbol{d, m \in \{1, 2, \ldots\}}$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + \frac{1}{2}\,(a + a\,(Y_{n+1}))\,\Delta \\
& + \frac{1}{4}\sum_{j=1}^{m}\Bigg[b^j\left(\bar{R}^j_+\right) + b^j\left(\bar{R}^j_-\right) + 2b^j \\
& \qquad + \sum_{\substack{r=1\\ r\neq j}}^{m}\left(b^j\left(\bar{U}^r_+\right) + b^j\left(\bar{U}^r_-\right) - 2b^j\right)\Delta^{-\frac{1}{2}}\Bigg]\Delta\hat{W}^j \\
& + \frac{1}{4}\sum_{j=1}^{m}\Bigg[\left(b^j\left(\bar{R}^j_+\right) - b^j\left(\bar{R}^j_-\right)\right)\left(\left(\Delta\hat{W}^j\right)^2 - \Delta\right) \\
& \quad + \sum_{\substack{r=1\\ r\neq j}}^{m}\left(b^j\left(\bar{U}^r_+\right) - b^j\left(\bar{U}^r_-\right)\right)\left(\Delta\hat{W}^j\Delta\hat{W}^r + V_{r,j}\right)\Bigg]\Delta^{-\frac{1}{2}}
\end{aligned}
$$

supporting values

$$\bar{R}^j_{\pm} = Y_n + a\,\Delta \pm b^j\,\sqrt{\Delta}$$

and

$$\bar{U}^j_{\pm} = Y_n \pm b^j\,\sqrt{\Delta}$$

A-stable

$\beta = 2.0$

Weak Order 1.0 Predictor-Corrector Methods

- **modified trapezoidal method of weak order $\beta = 1.0$**

  corrector

$$Y_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\bar{Y}_{n+1}\right) + a\Big)\Delta + b\,\Delta\hat{W}$$

  predictor

$$\bar{Y}_{n+1} = Y_n + a\,\Delta + b\,\Delta\hat{W}$$

$$P\left(\Delta\hat{W} = \pm\sqrt{\Delta}\right) = \frac{1}{2}$$

- **family of weak order 1.0 predictor-corrector methods**

corrector

$$\begin{aligned} Y_{n+1} \;&=\; Y_n + \Big(\alpha\, \bar{a}_\eta\left(\tau_{n+1}, \bar{Y}_{n+1}\right) + (1-\alpha)\, \bar{a}_\eta\left(\tau_n, Y_n\right)\Big)\Delta \\ &+ \sum_{j=1}^{m} \Big(\eta\, b^j\left(\tau_{n+1}, \bar{Y}_{n+1}\right) + (1-\eta)\, b^j\left(\tau_n, Y_n\right)\Big)\Delta \hat{W}^j \end{aligned}$$

for α, $\eta \in [0,1]$, where

$$\bar{a}_\eta = a - \eta \sum_{j_1, j_2 = 1}^{m} \sum_{k=1}^{d} b^{k, j_1} \frac{\partial b^{j_2}}{\partial x^k}$$

and predictor

$$\bar{Y}_{n+1} = Y_n + a\,\Delta + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j$$

Weak Order 2.0 Predictor-Corrector Methods

- **weak order 2.0 predictor-corrector method**

corrector

$$Y_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\bar{Y}_{n+1}\right) + a\Big)\Delta + \Psi_n$$

with

$$\begin{aligned}\Psi_n \quad &= \quad b\,\Delta\hat{W} + \frac{1}{2}\,b\,b'\left(\left(\Delta\hat{W}\right)^2 - \Delta\right) \\ &\quad + \frac{1}{2}\left(a\,b' + \frac{1}{2}\,b^2\,b''\right)\Delta\hat{W}\,\Delta\end{aligned}$$

and as predictor

$$
\begin{aligned}
\bar{Y}_{n+1} &= Y_n + a\,\Delta + \Psi_n \\
&\quad + \frac{1}{2}\, a'\, b\, \Delta\hat{W}\, \Delta + \frac{1}{2}\left(a\, a' + \frac{1}{2}\, a'' b^2\right) \Delta^2
\end{aligned}
$$

$$
P\left(\Delta\hat{W} = \pm\sqrt{3\Delta}\right) = \frac{1}{6} \quad \text{and} \quad P\left(\Delta\hat{W} = 0\right) = \frac{2}{3}
$$

- general multi-dimensional case

corrector

$$Y_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\tau_{n+1}, \bar{Y}_{n+1}\right) + a\Big)\Delta + \Psi_n$$

where

$$\begin{aligned}\Psi_n &= \sum_{j=1}^{m}\left(b^j + \frac{1}{2}\,L^0 b^j\,\Delta\right)\Delta\hat{W}^j \\ &\quad + \frac{1}{2}\sum_{j_1,j_2=1}^{m} L^{j_1} b^{j_2}\left(\Delta\hat{W}^{j_1}\,\Delta\hat{W}^{j_2} + V_{j_1,j_2}\right)\end{aligned}$$

and predictor

$$\bar{Y}_{n+1} = Y_n + a\,\Delta + \Psi_n + \frac{1}{2}\,L^0 a\,\Delta^2 + \frac{1}{2}\sum_{j=1}^{m} L^j a\,\Delta\hat{W}^j\,\Delta$$

- **derivative free weak order 2.0 predictor-corrector method**

corrector

$$Y_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\bar{Y}_{n+1}\right) + a\Big)\Delta + \phi_n$$

where

$$\begin{aligned}\phi_n \; &= \; \frac{1}{4}\left(b\left(\bar{\Upsilon}^{+}\right) + b\left(\bar{\Upsilon}^{-}\right) + 2\,b\right)\Delta\hat{W} \\ &\quad + \frac{1}{4}\left(b\left(\bar{\Upsilon}^{+}\right) - b\left(\bar{\Upsilon}^{-}\right)\right)\left(\left(\Delta\hat{W}\right)^2 - \Delta\right)\Delta^{-\frac{1}{2}}\end{aligned}$$

with supporting values

$$\bar{\Upsilon}^{\pm} = Y_n + a\,\Delta \pm b\,\sqrt{\Delta}$$

and with predictor

$$\bar{\boldsymbol{Y}}_{\boldsymbol{n+1}} = \boldsymbol{Y}_{\boldsymbol{n}} + \frac{\mathbf{1}}{\mathbf{2}}\Big(\boldsymbol{a}\left(\bar{\boldsymbol{\Upsilon}}\right) + \boldsymbol{a}\Big)\boldsymbol{\Delta} + \boldsymbol{\phi}_{\boldsymbol{n}}$$

with the supporting value

$$\bar{\boldsymbol{\Upsilon}} = \boldsymbol{Y}_{\boldsymbol{n}} + \boldsymbol{a}\,\boldsymbol{\Delta} + \boldsymbol{b}\,\boldsymbol{\Delta}\hat{\boldsymbol{W}}$$

- multi-dimensional generalization

  corrector

$$\boldsymbol{Y_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\bar{Y}_{n+1}\right) + a\Big)\Delta + \phi_n}$$

where

$$
\begin{aligned}
\phi_n \;=\; & \frac{1}{4} \sum_{j=1}^{m} \Bigg[b^j \left(\bar{R}^j_+ \right) + b \left(\bar{R}^j_- \right) + 2\, b^j \\
& + \sum_{\substack{r=1 \\ r \neq j}}^{m} \left(b^j \left(\bar{U}^r_+ \right) + b \left(\bar{U}^r_- \right) - 2\, b^j \right) \Delta^{-\frac{1}{2}} \Bigg] \Delta \hat{W}^j \\
& + \frac{1}{4} \sum_{j=1}^{m} \Bigg[\left(b^j \left(\bar{R}^j_+ \right) - b \left(\bar{R}^j_- \right) \right) \left(\left(\Delta \hat{W} \right)^2 - \Delta \right) \\
& + \sum_{\substack{r=1 \\ r \neq j}}^{m} \left(b^j \left(\bar{U}^r_+ \right) - b \left(\bar{U}^r_- \right) \right) \left(\Delta \hat{W}^j \, \Delta \hat{W}^r + V_{r,j} \right) \Bigg] \Delta^{-\frac{1}{2}}
\end{aligned}
$$

supporting values

$$\bar{\boldsymbol{R}}^{j}_{\pm} = \boldsymbol{Y_n} + \boldsymbol{a}\,\boldsymbol{\Delta} \pm \boldsymbol{b}^{j}\,\sqrt{\boldsymbol{\Delta}} \quad \text{and} \quad \bar{\boldsymbol{U}}^{j}_{\pm} = \boldsymbol{Y_n} \pm \boldsymbol{b}^{j}\,\sqrt{\boldsymbol{\Delta}}$$

predictor

$$\bar{Y}_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\bar{\Upsilon}\right) + a\Big)\Delta + \phi_n$$

supporting value

$$\bar{\Upsilon} = Y_n + a\,\Delta + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j$$

local error

$$Z_{n+1} = \bar{Y}_{n+1} - Y_{n+1}$$

4 Numerical Stability

- roundoff and truncation errors

- propagation of errors

- numerical stability priority over higher order

- **specially designed test equations**

  Hernandez & Spigler (1992, 1993)

  Milstein (1995)

  Kloeden & Pl. (1992b)

  Saito & Mitsui(1993a, 1993b, 1996)

  Hofmann & Pl. (1994, 1996)

  Higham (2000)

- **linear test dynamics**

$$X_t = X_0 \, \exp\left\{(1-\alpha)\,\lambda\, t + \sqrt{\alpha\,|\lambda|}\, W_t\right\}$$

$\alpha, \lambda \in \Re$

$\Longrightarrow$

$$P\left(\lim_{t\to\infty} X_t = 0\right) = 1 \quad \Longleftrightarrow \quad (1-\alpha)\lambda < 0$$

- **linear Itô SDE**

$$dX_t = \left(1 - \frac{3}{2}\,\alpha\right) \lambda\, X_t\, dt + \sqrt{\alpha\, |\lambda|}\, X_t\, dW_t$$

- **corresponding Stratonovich SDE**

$$dX_t = (1 - \alpha)\, \lambda\, X_t\, dt + \sqrt{\alpha\, |\lambda|}\, X_t \circ dW_t$$

- $\alpha = 0$ no randomness

- $\alpha = \frac{2}{3}$ Itô SDE no drift $\Longrightarrow$ martingale

- $\alpha = 1$ Stratonovich SDE no drift

Definition 4.1 $Y = \{Y_t, t \geq 0\}$ *is called* **asymptotically stable** *if*

$$P\left(\lim_{t \to \infty} |Y_t| = 0\right) = 1.$$

impact of perturbations declines asymptotically over time

- **stability region Γ**

  those pairs $(\lambda\Delta, \alpha) \in (-\infty, 0) \times [0, 1)$ for which approximation Y asymptotically stable

- **transfer function**

$$\left| \frac{Y_{n+1}}{Y_n} \right| = G_{n+1}(\lambda \, \Delta, \alpha)$$

Y asymptotically stable $\quad \Longleftrightarrow$

$$E(\ln(G_{n+1}(\lambda \, \Delta, \alpha))) < 0$$

Higham (2000)

- **Euler scheme**

$$\boldsymbol{Y_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n}$$

$$\boldsymbol{G_{n+1}(\lambda\,\Delta, \alpha) = \left|1 + \left(1 - \frac{3}{2}\,\alpha\right)\lambda\,\Delta + \sqrt{|\alpha\,\lambda|}\,\Delta W_n\right|}$$

$$\boldsymbol{\Delta W_n \sim \mathcal{N}(0, \Delta)}$$

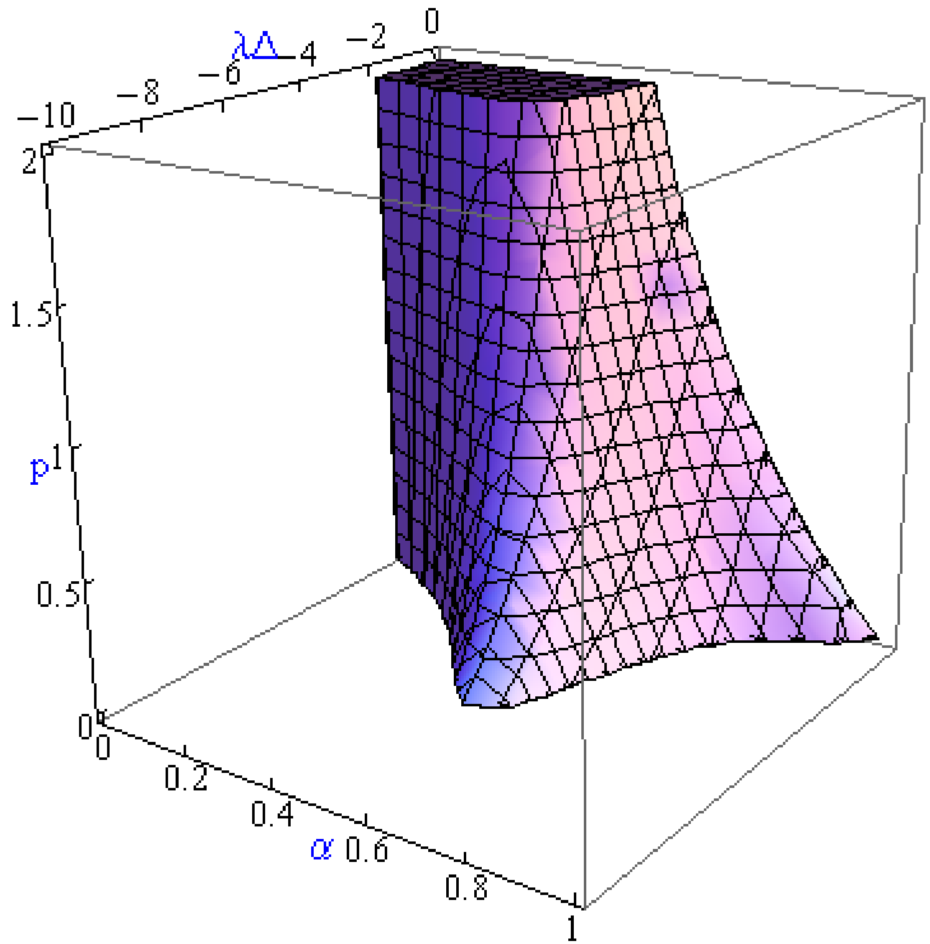

Figure 4.1: A-stability region for the Euler scheme.

- **semi-drift-implicit predictor-corrector Euler method**

$$Y_{n+1} = Y_n + \frac{1}{2}\left(a(\bar{Y}_{n+1}) + a(Y_n)\right)\,\Delta + b(Y_n)\,\Delta W_n$$

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n$$

$$\begin{aligned} G_{n+1}(\lambda\,\Delta, \alpha) \;=\; & \Bigg| 1 + \lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right)\left\{1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right)\right.\right. \\ & \left.\left. + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\} + \sqrt{-\alpha\,\lambda}\,\Delta W_n \Bigg| \end{aligned}$$

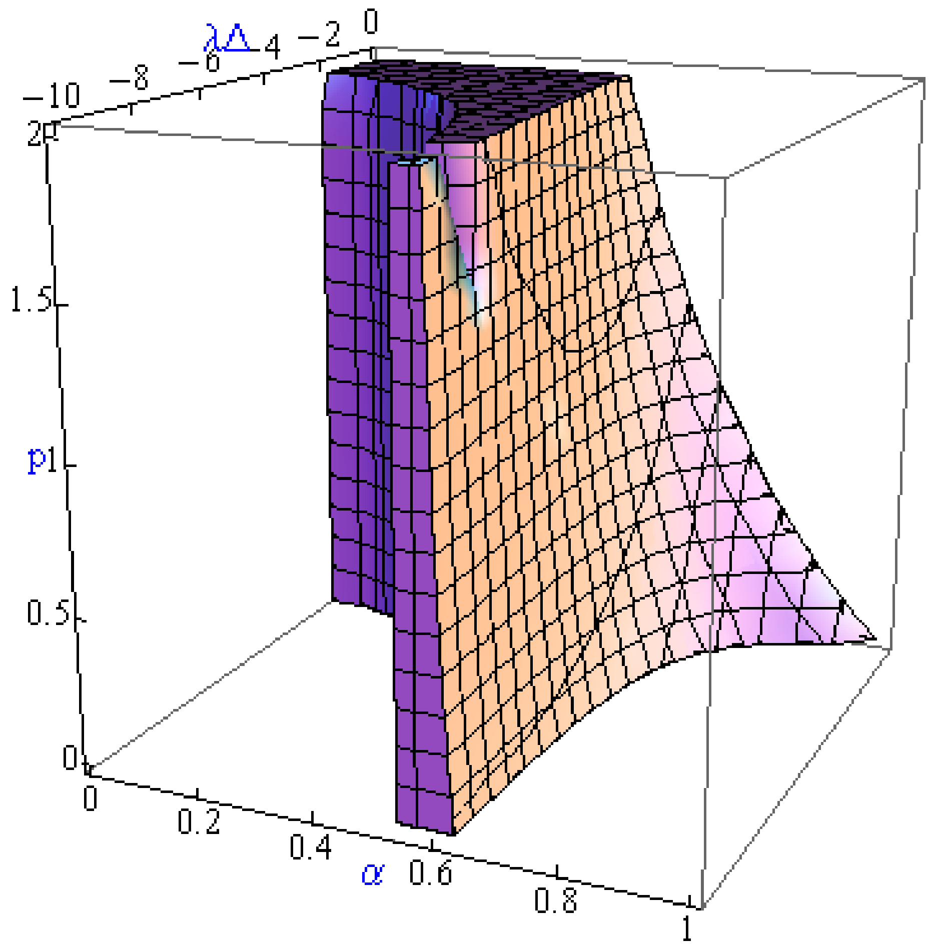

Figure 4.2: A-stability region for semi-drift-implicit predictor-corrector Euler method.

- **drift-implicit predictor-corrector Euler method**

$$Y_{n+1} = Y_n + a(\bar{Y}_{n+1})\,\Delta + b(Y_n)\,\Delta W_n$$

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n$$

$$\begin{aligned} G_{n+1}(\lambda\,\Delta,\alpha) \;=\; & \left|1 + \lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right)\left\{1 + \lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right)\right.\right. \\ & \left.\left. + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right\} + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right| \end{aligned}$$

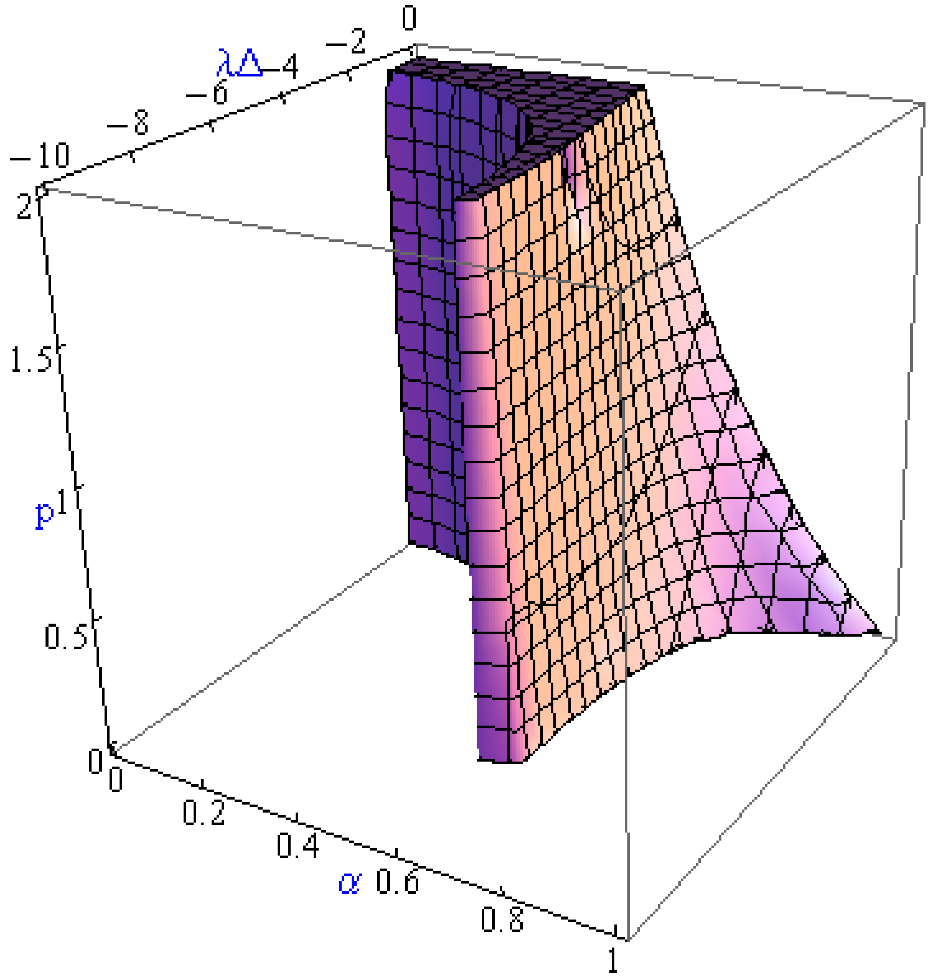

Figure 4.3: A-stability region for drift-implicit predictor-corrector Euler method.

- **semi-implicit diffusion predictor-corrector Euler method**

$$Y_{n+1} = Y_n + \bar{a}_{\frac{1}{2}}(Y_n)\,\Delta + \frac{1}{2}\big(b(\bar{Y}_{n+1}) + b(Y_n)\big)\,\Delta W_n$$

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n$$

$$G_{n+1}(\lambda\,\Delta, \alpha) \;=\; \Bigg| 1 + \lambda\,\Delta\,(1-\alpha) + \sqrt{-\alpha\,\lambda}\,\Delta W_n \times \left\{ 1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\}\Bigg|$$

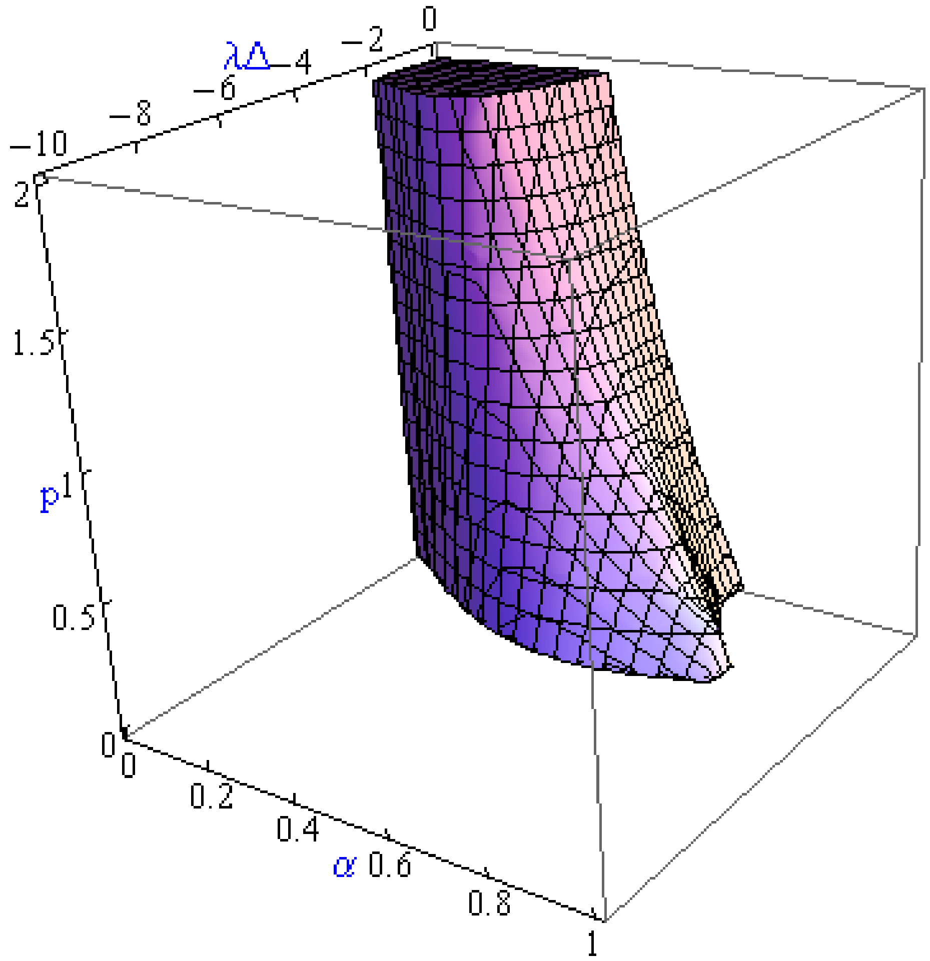

Figure 4.4: A-stability region for the predictor-corrector Euler method with $\boldsymbol{\theta = 0}$ and $\boldsymbol{\eta = \frac{1}{2}}$.

- **symmetric predictor-corrector Euler method**

$$Y_{n+1} = Y_n + \frac{1}{2}\left(\bar{a}_{\frac{1}{2}}(\bar{Y}_{n+1}) + \bar{a}_{\frac{1}{2}}(Y_n)\right)\Delta + \frac{1}{2}\left(b(\bar{Y}_{n+1}) + b(Y_n)\right)\Delta W_n$$

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n$$

$$\begin{aligned} G_{n+1}(\lambda\,\Delta, \alpha) \;=\; & \Bigg| 1 + \lambda\,\Delta\,(1-\alpha)\left\{1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\} \\ & + \sqrt{-\alpha\,\lambda}\,\Delta W_n \left\{1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\}\Bigg| \end{aligned}$$

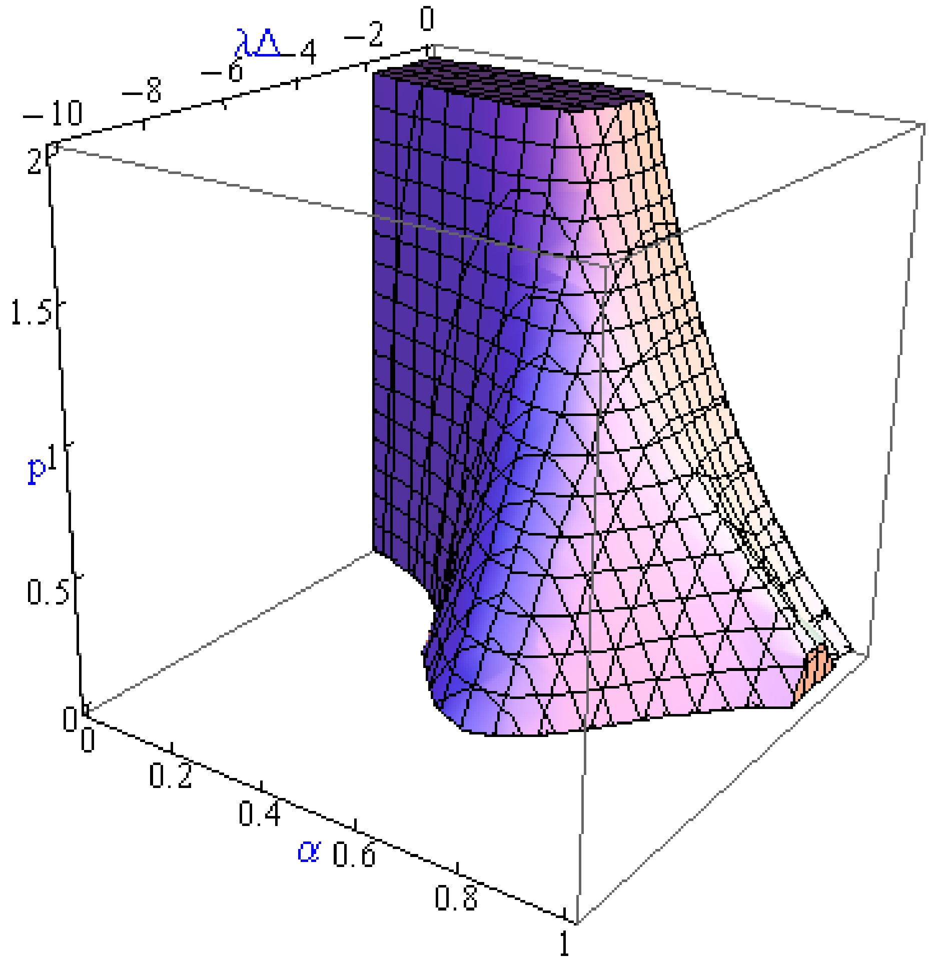

Figure 4.5: A-stability region for the symmetric predictor-corrector Euler method.

$$
\begin{aligned}
G_{n+1}(\lambda\,\Delta,\alpha) &= \Bigg|1+\lambda\,\Delta\left(1-\frac{1}{2}\,\alpha\right)\left\{1+\lambda\,\Delta\left(1-\frac{3}{2}\,\alpha\right)+\sqrt{-\alpha\,\lambda}\,\Delta W_n\right\} \\
&\quad +\sqrt{-\alpha\,\lambda}\,\Delta W_n\left\{1+\lambda\,\Delta\left(1-\frac{3}{2}\,\alpha\right)+\sqrt{-\alpha\,\lambda}\,\Delta W_n\right\}\Bigg| \\
&= \Bigg|1+\left\{1+\lambda\,\Delta\left(1-\frac{3}{2}\,\alpha\right)+\sqrt{-\alpha\,\lambda}\,\Delta W_n\right\} \\
&\qquad \times\left\{\lambda\,\Delta\left(1-\frac{1}{2}\,\alpha\right)+\sqrt{-\alpha\,\lambda}\,\Delta W_n\right\}\Bigg|
\end{aligned}
$$

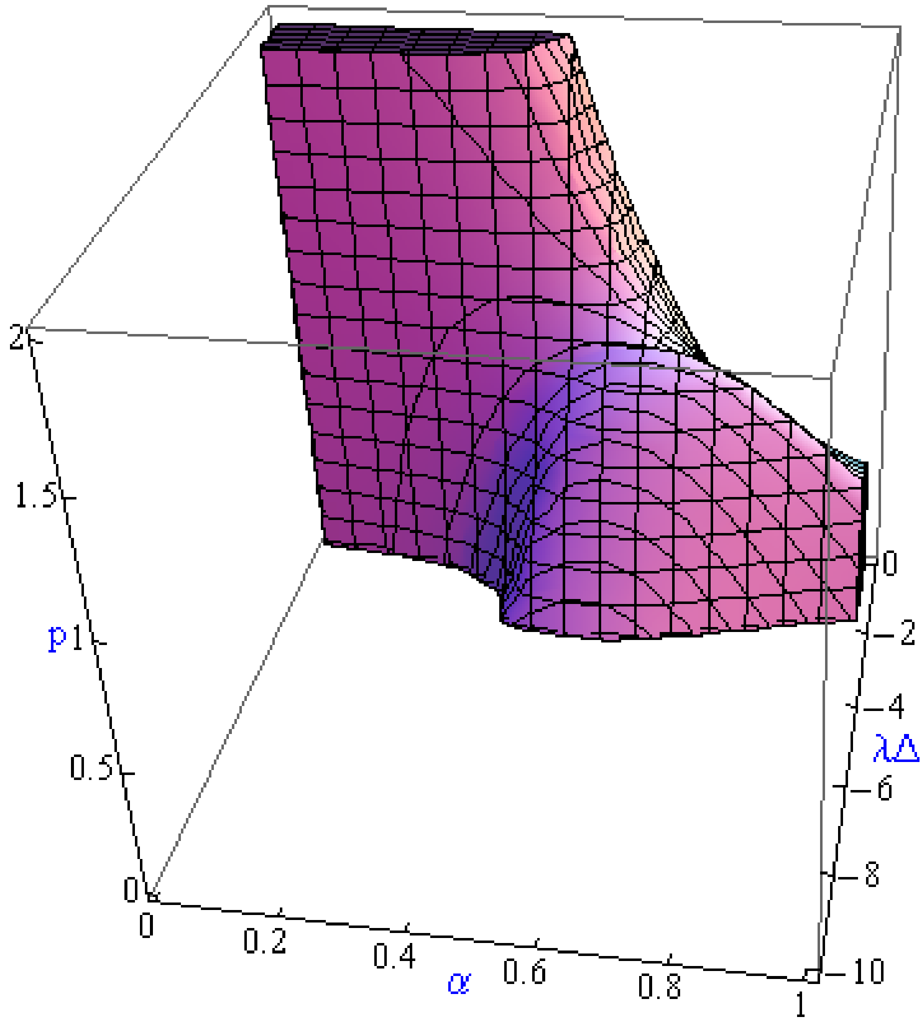

Figure 4.6: A-stability region for fully implicit predictor-corrector Euler method.

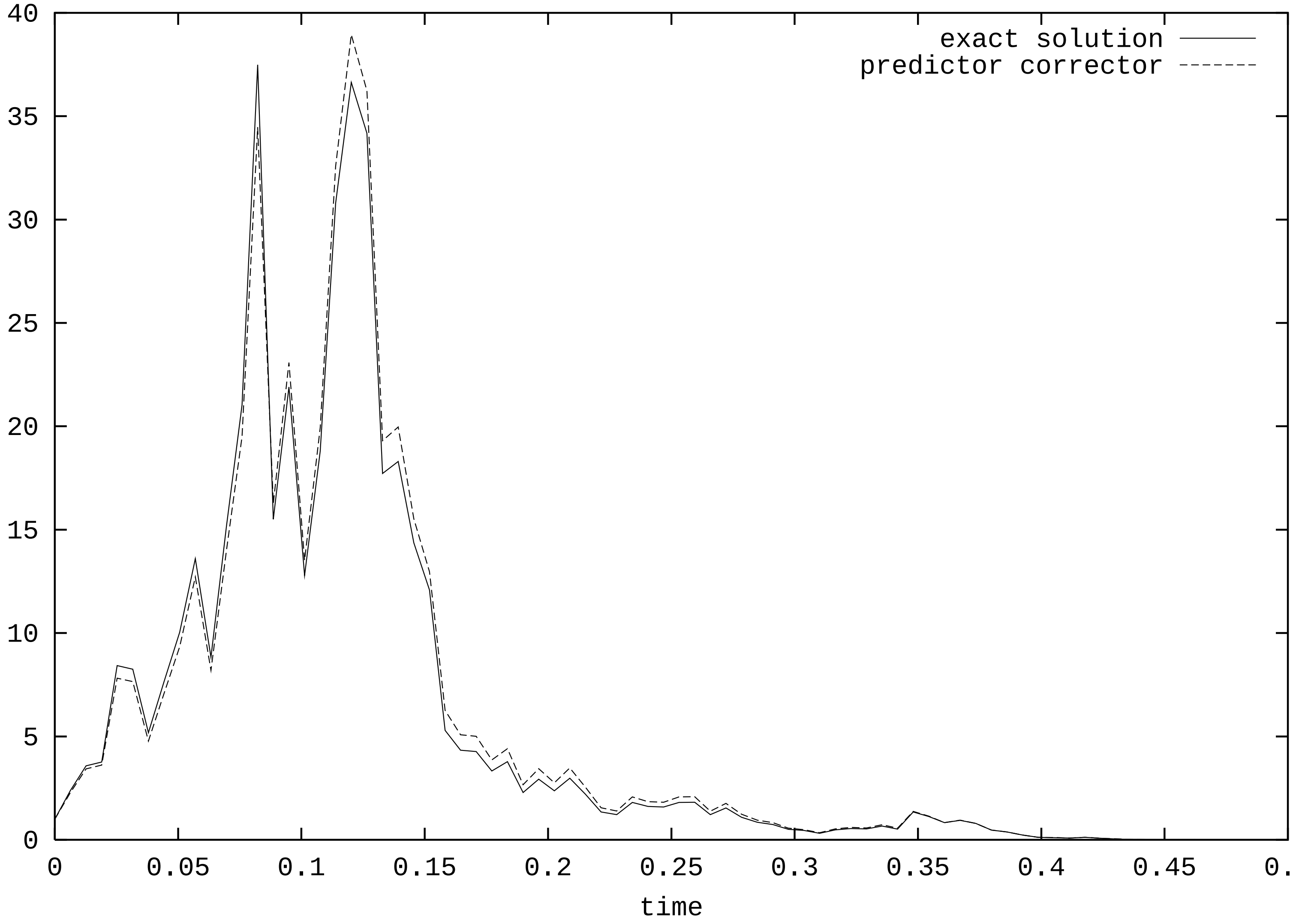

Figure 4.7: Exact solution and approximate solution generated by the symmetric predictor-corrector Euler scheme.

p-Stability

Pl. & Shi (2008)

Definition 4.2 *For* $p > 0$ *a process* $Y = \{Y_t, t > 0\}$ *is called* p-stable *if*

$$\lim_{t \to \infty} E(|Y_t|^p) = 0.$$

For $\alpha \in [0, \frac{1}{1+p/2})$ and $\lambda < 0$ test SDE is p-stable.

- **Stability region** those triplets $(\lambda\Delta, \alpha, p)$ for which Y is p-stable.

For $\lambda \Delta < 0$, $\alpha \in [0, 1)$ and $p > 0$ Y p-stable $\iff$

$$E((G_{n+1}(\lambda \Delta, \alpha))^p) < 1$$

- for $p > 0$

$\Longrightarrow$

$$E(\ln(G_{n+1}(\lambda \Delta, \alpha))) \leq \frac{1}{p} E((G_{n+1}(\lambda \Delta, \alpha))^p - 1) < 0$$

$\Longrightarrow$ asymptotically stable

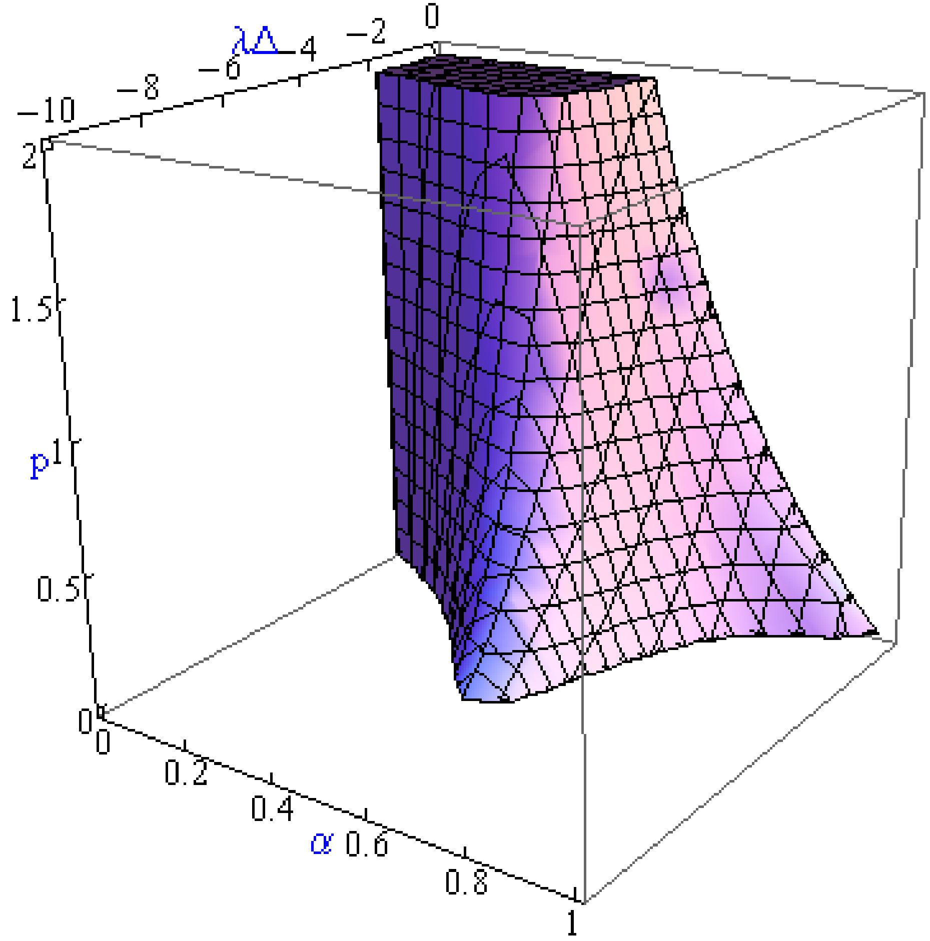

Figure 4.8: Stability region for the Euler scheme.

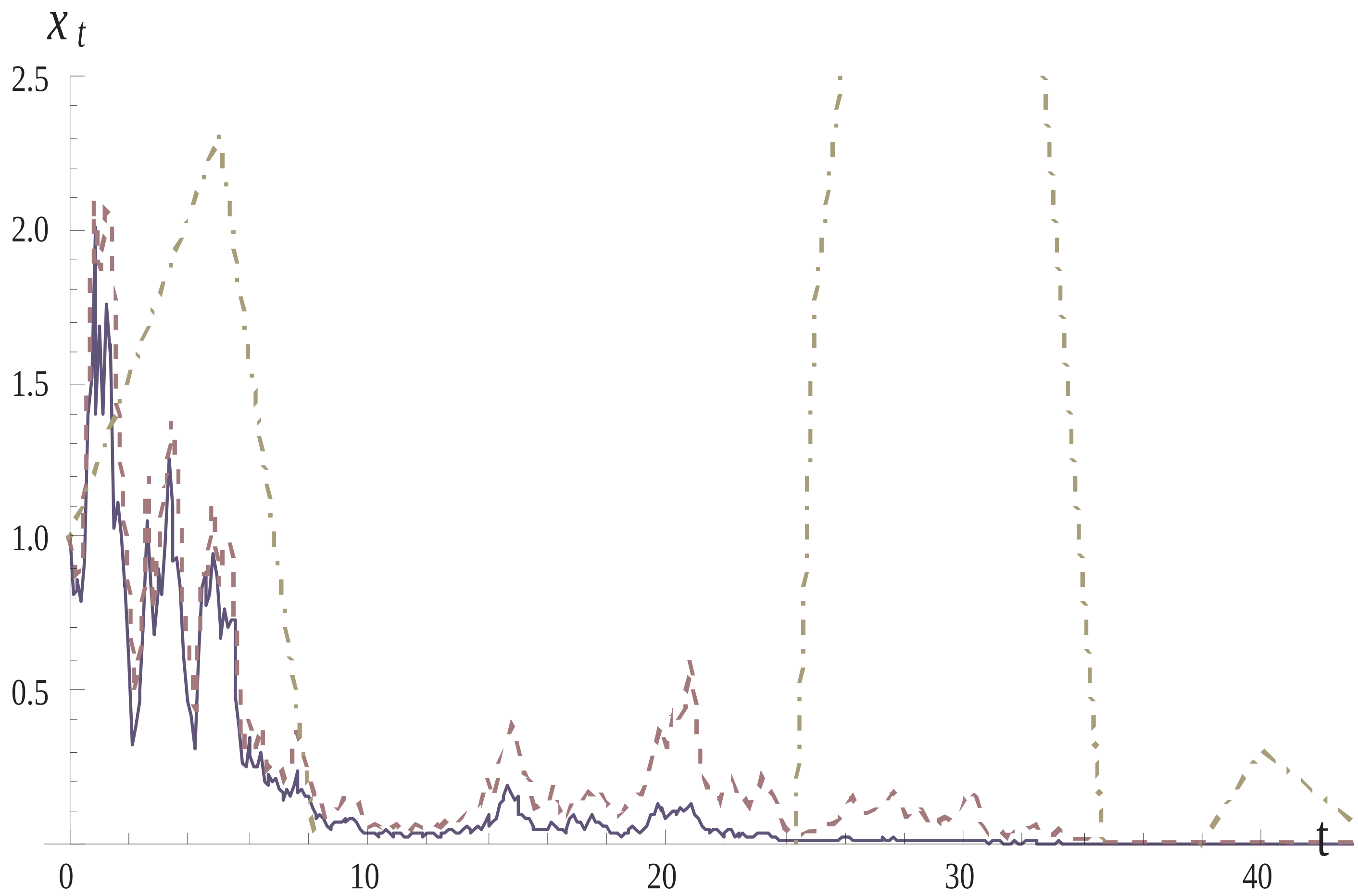

Figure 4.9: Paths of exact solution, Euler scheme with $\mathbf{\Delta = 0.2}$ and Euler scheme with $\mathbf{\Delta = 5}$.

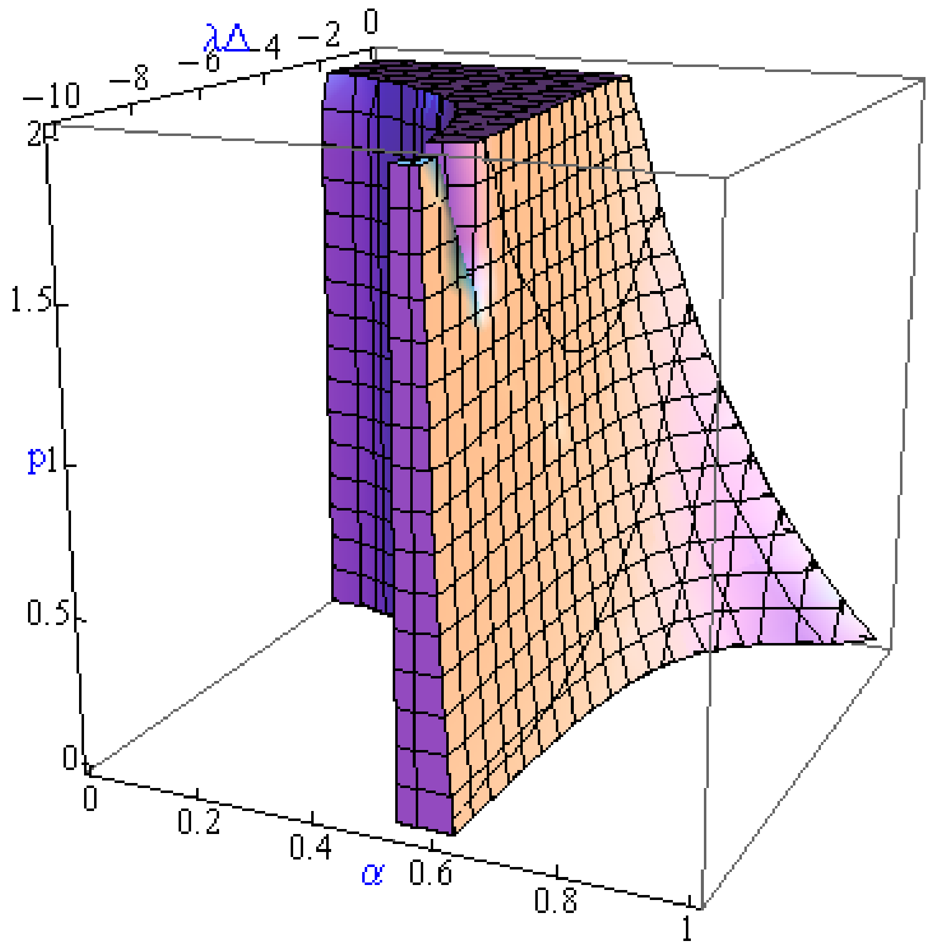

Figure 4.10: Stability region for semi-drift-implicit predictor-corrector Euler method.

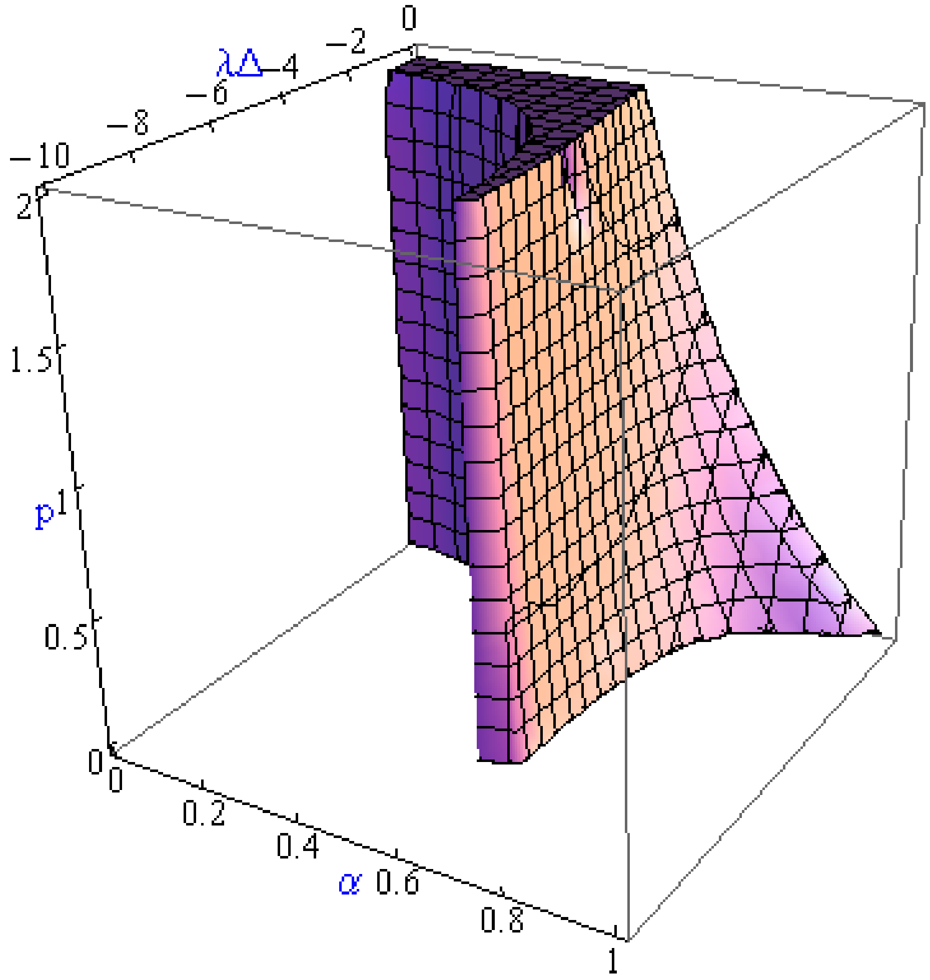

Figure 4.11: Stability region for drift-implicit predictor-corrector Euler method.

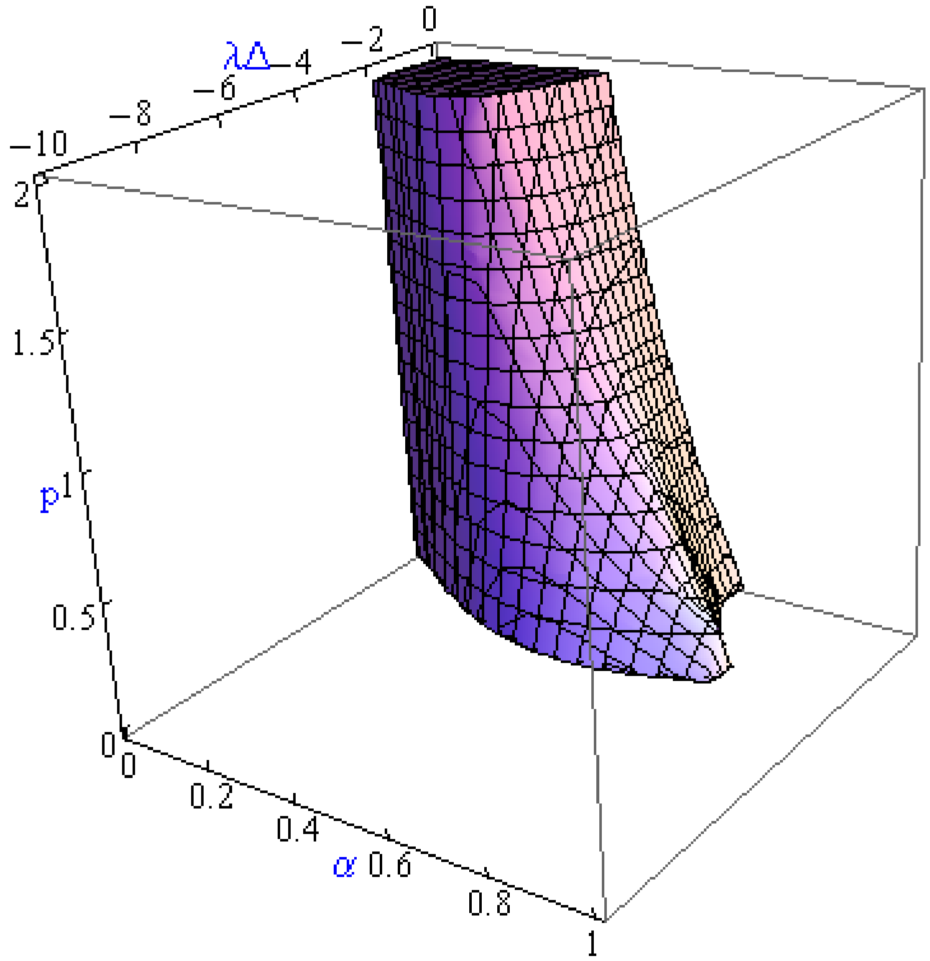

Figure 4.12: Stability region for the predictor-corrector Euler method with $\boldsymbol{\theta = 0}$ and $\boldsymbol{\eta = \frac{1}{2}}$.

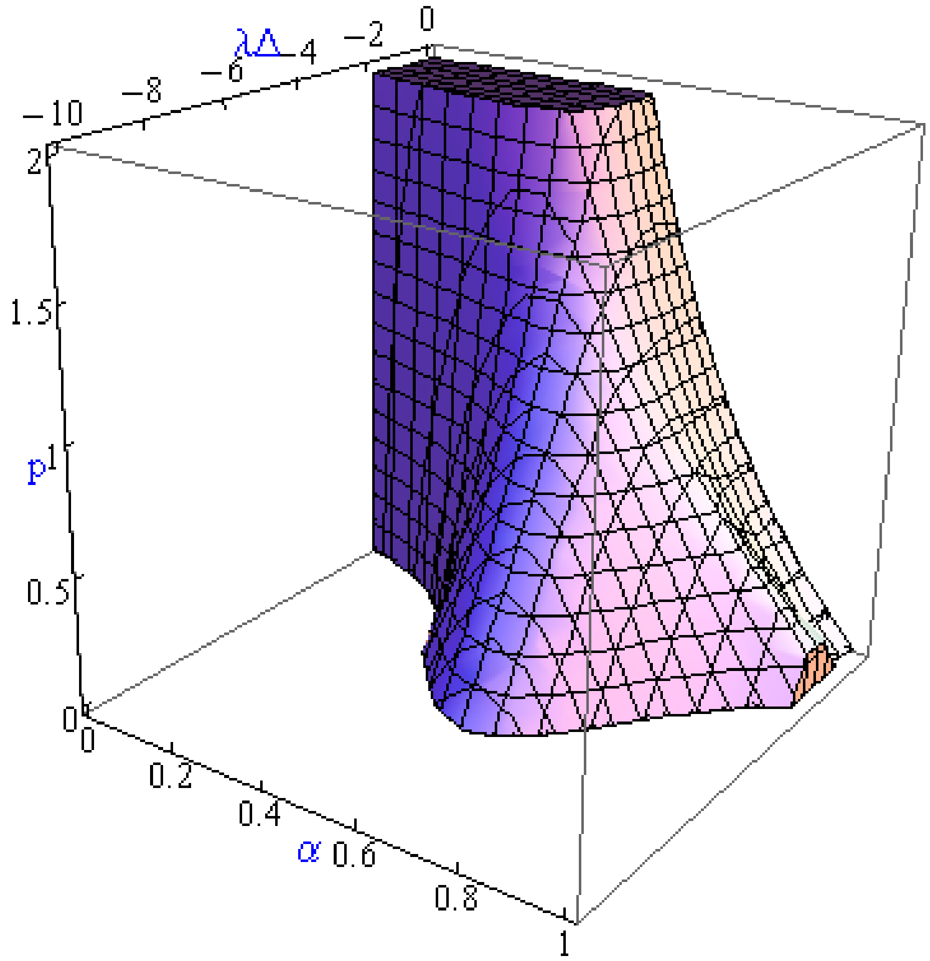

Figure 4.13: Stability region for the symmetric predictor-corrector Euler method.

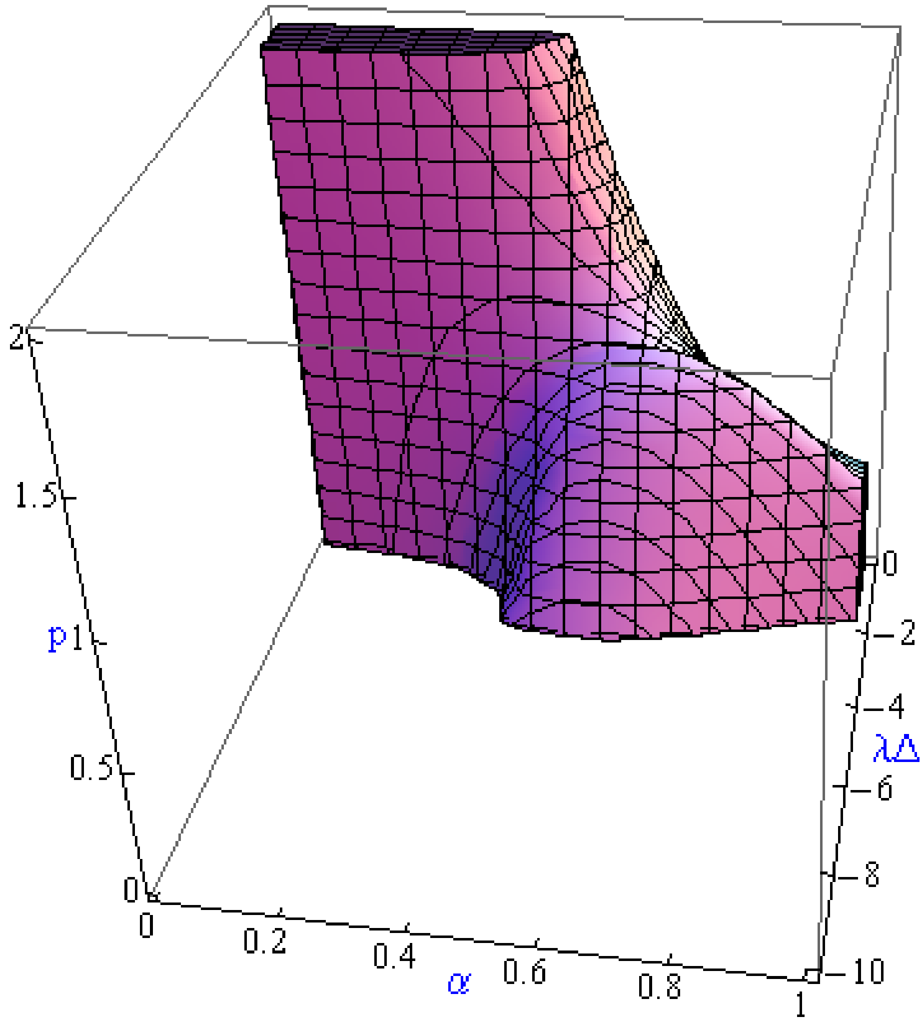

Figure 4.14: Stability region for fully implicit predictor-corrector Euler method.

Stability of Some Implicit Methods

- **semi-drift implicit Euler scheme**

$$Y_{n+1} = Y_n + \frac{1}{2}(a(Y_{n+1}) + a(Y_n))\Delta + b(Y_n)\Delta W_n$$

- **full-drift implicit Euler scheme**

$$Y_{n+1} = Y_n + a(Y_{n+1})\Delta + b(Y_n)\Delta W_n$$

solve algebraic equation

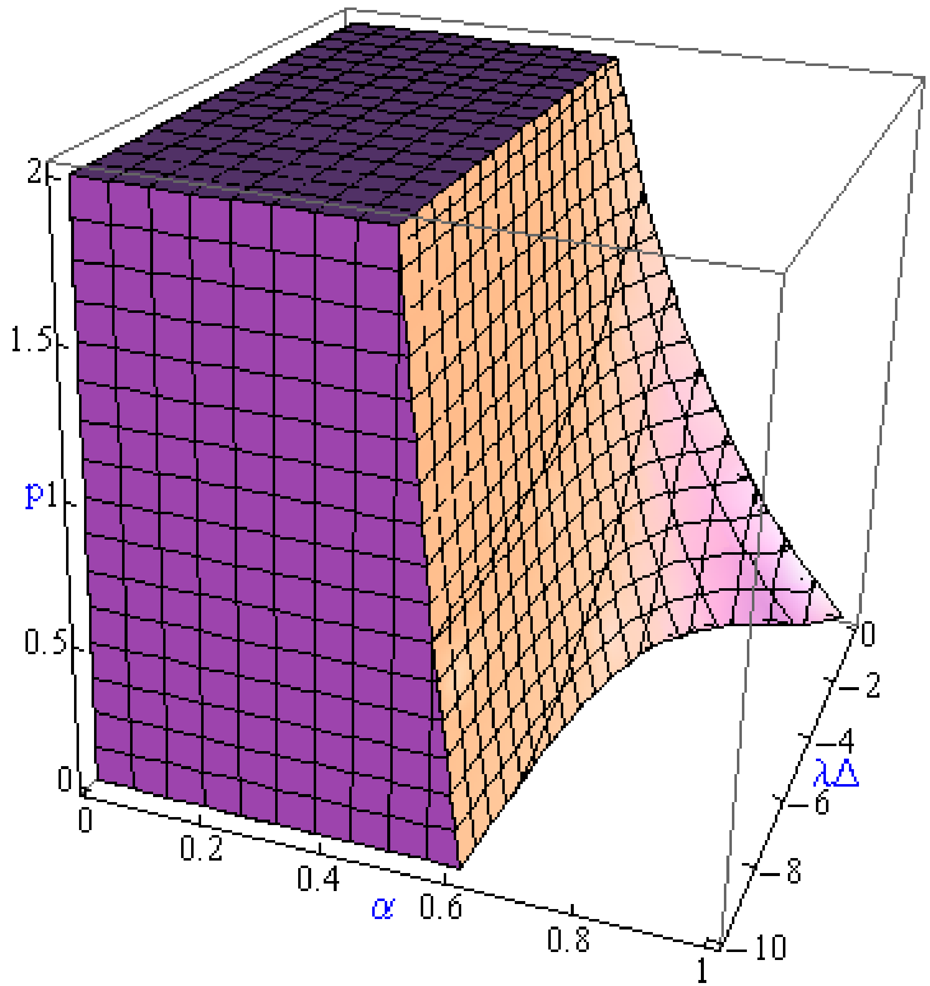

Figure 4.15: Stability region for semi-drift implicit Euler method.

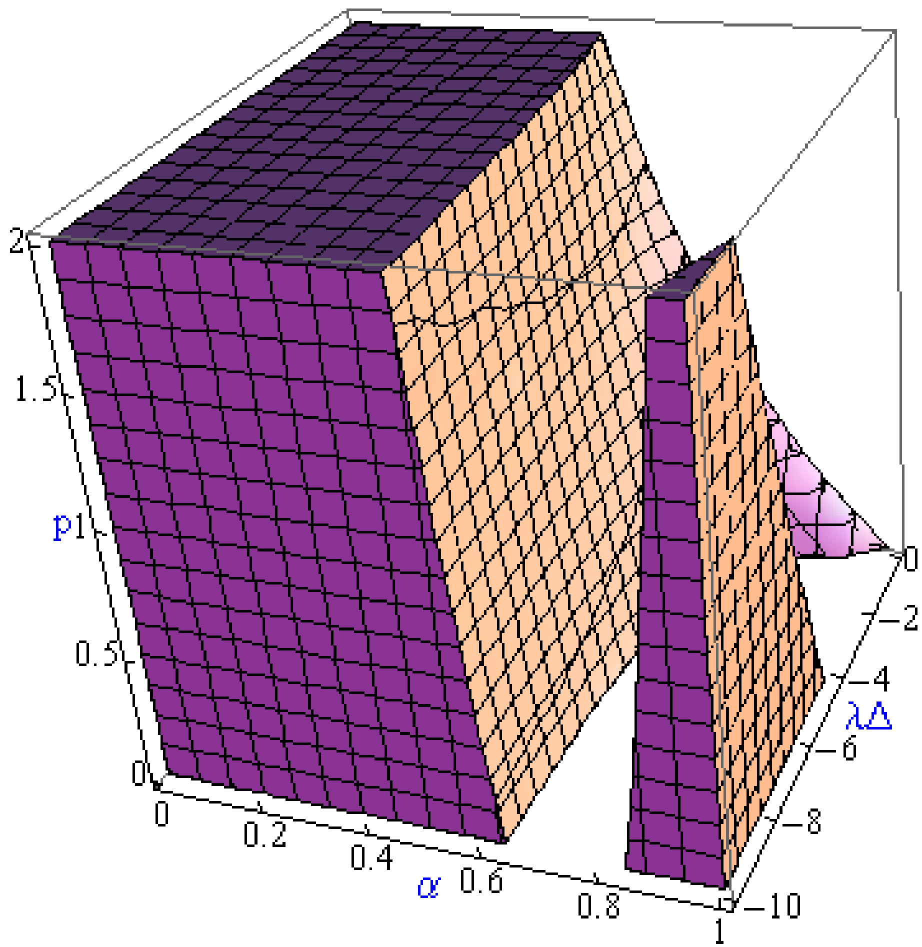

Figure 4.16: Stability region for full-drift implicit Euler method.

- **balanced implicit Euler method**

  Milstein, Pl. & Schurz (1998)

$$Y_{n+1} = Y_n + \left(1 - \frac{3}{2}\,\alpha\right)\lambda Y_n \Delta + \sqrt{\alpha|\lambda|}\,Y_n \Delta W_n + c|\Delta W_n|(Y_n - Y_{n+1})$$

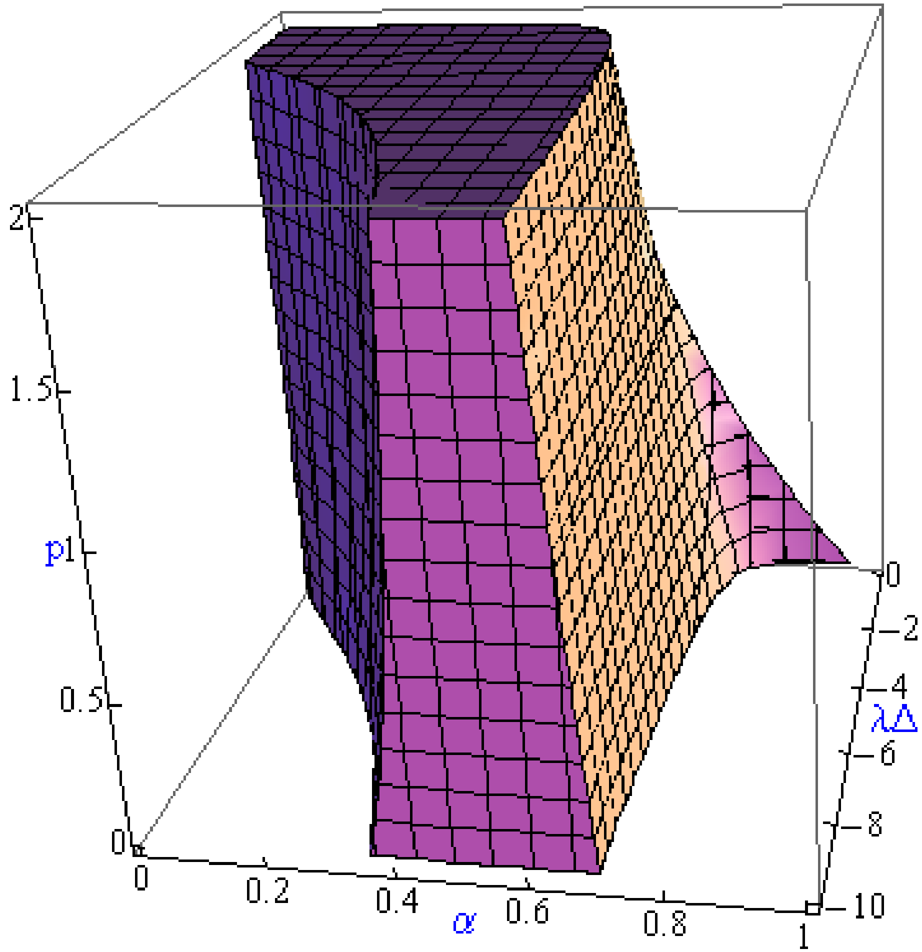

Figure 4.17: Stability region for a balanced implicit Euler method.

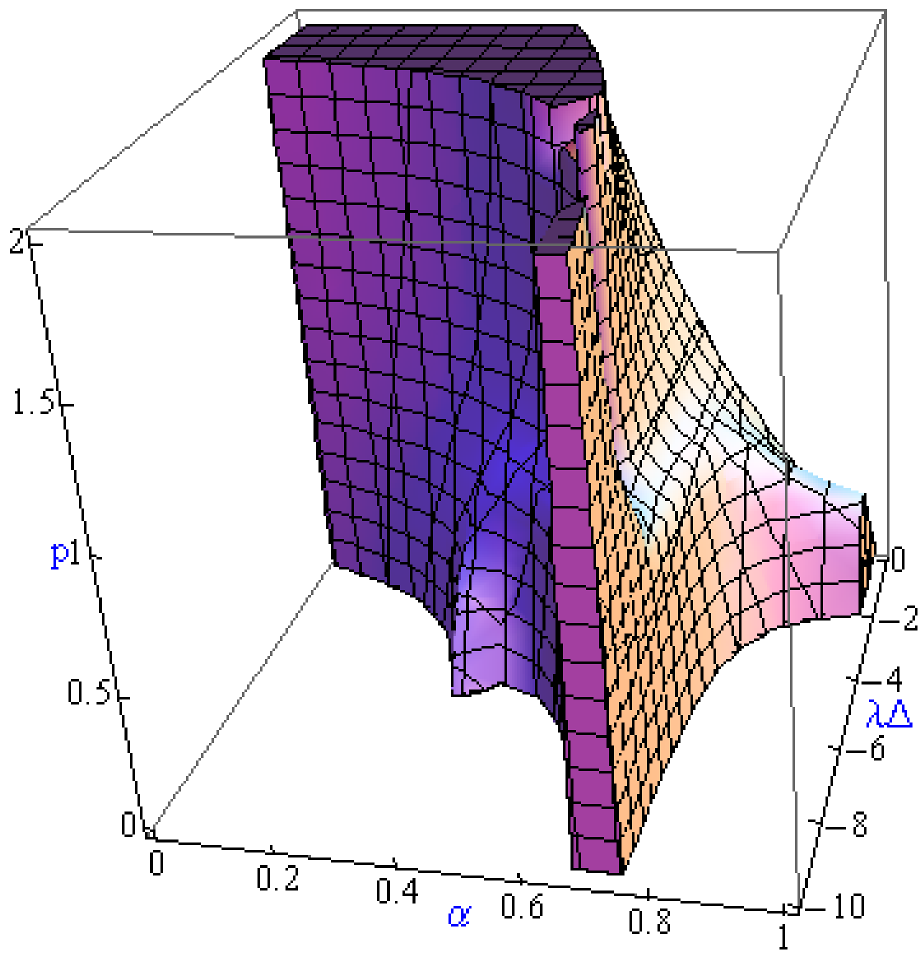

Figure 4.18: Stability region for the simplified symmetric Euler method.

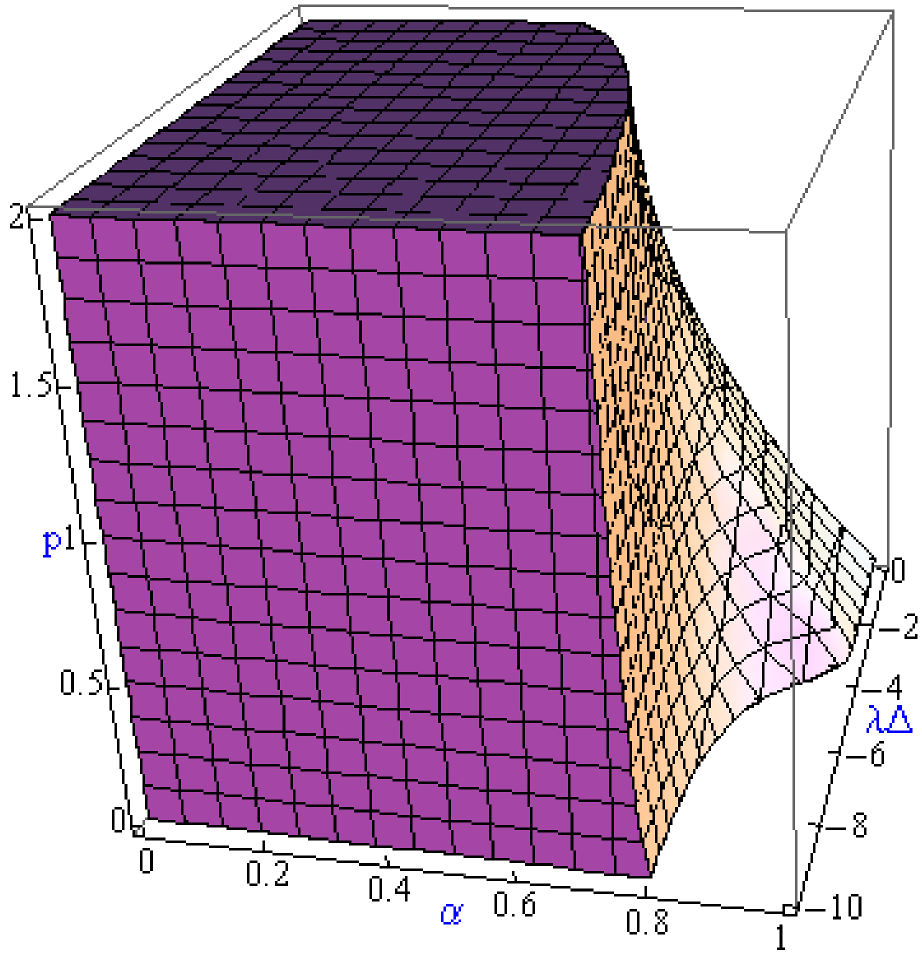

Figure 4.19: Stability region for the simplified symmetric implicit Euler Scheme.

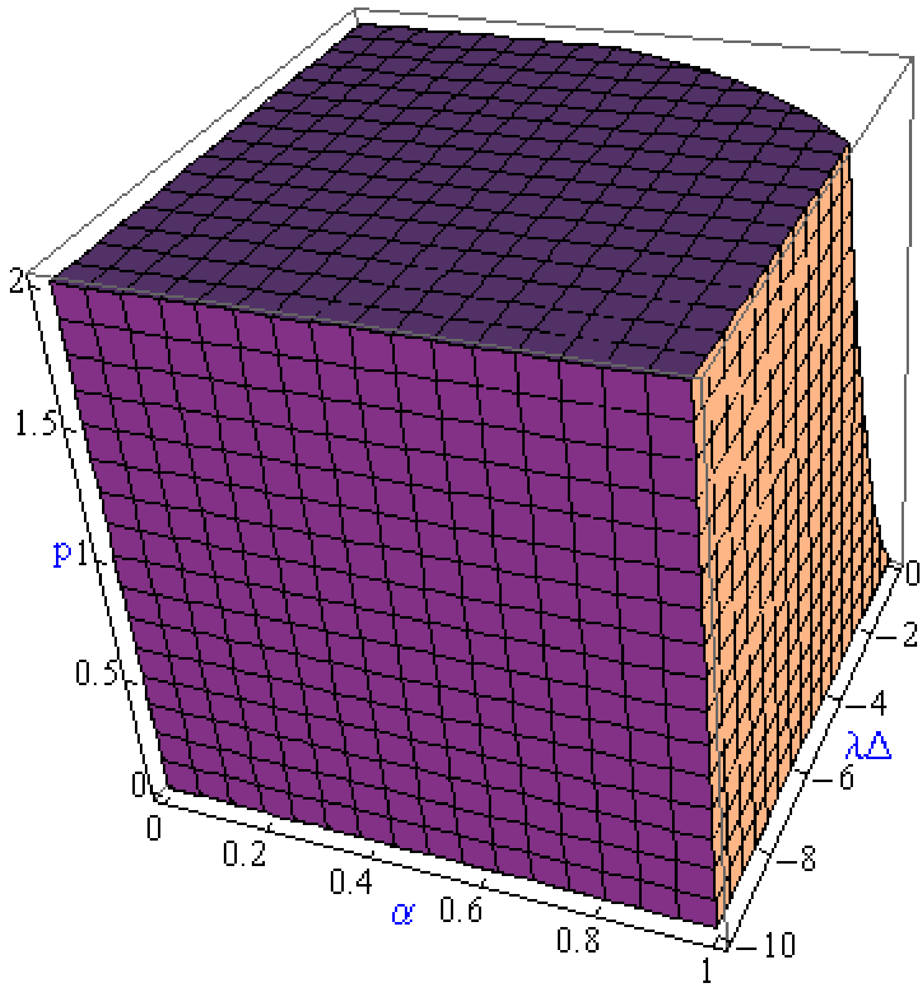

Figure 4.20: Stability region for the simplified fully implicit Euler Scheme.

5 Variance Reduction Techniques

Various Variance Reduction Methods

- **classical**

  Hammersley & Handscomb (1964)
  Ermakov (1975), Boyle (1977)
  Maltz & Hitzl (1979), Rubinstein (1981)
  Ermakov & Mikhailov (1982), Ripley (1983)
  Kalos & Whitlock (1986), Bratley, Fox & Schrage (1987)
  Chang (1987),Wagner (1987)
  Law & Kelton (1991), Ross (1990)

- **stochastic differential equations**

  Boyle (1977)

  Boyle, Broadie & Glasserman (1997)

  Broadie & Glasserman (1997), Fu (1995)

  Grant, Vora & Weeks (1997), Joy, Boyle & Tan (1996)

  Glasserman (2004)

  Longstaff & Schwartz (2001)

  Milstein (1988), Kloeden & Pl. (1992b)

  Hofmann, Pl. & Schweizer (1992), Heath (1995)

  Goldman, Heath, Kentwell & Pl. (1995)

  Fournie, Lasry & Touzi (1997)

  Newton (1994)

Antithetic Variates

- **probability space**

  $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ where $\Omega = \mathcal{C}([t_0, T], \Re^2)$

  $\omega(t) = (\omega_1(t), \omega_2(t))^\top \in \Re^2$ for $t \in [t_0, T]$

- coordinate mappings

  $W_t^1(\omega) = \omega_1(t)$ and $W_t^2(\omega) = \omega_2(t)$

$$dX_t^{t_0,\underline{x}} = a\left(t, X_t^{t_0,\underline{x}}\right)\, dt + b\left(t, X_t^{t_0,\underline{x}}\right)\, dW_t$$

For $\omega \in \Omega$, define $\bar{\omega} \in \Omega$ by $\bar{\omega}(t) = (-\omega_1(t), -\omega_2(t))^\top$, $t \in [t_0, T]$

$$\bar{h}\left(X_T^{t_0,\underline{x}}\right)(\omega) = h\left(X_T^{t_0,\underline{x}}\right)(\bar{\omega})$$

- **unbiased estimator**

$$\hat{h}\left(X_T^{t_0,\underline{x}}\right) = \frac{1}{2}\left(h\left(X_T^{t_0,\underline{x}}\right) + \bar{h}\left(X_T^{t_0,\underline{x}}\right)\right)$$

$\Longrightarrow$

$$\begin{aligned} \mathrm{Var}\left(\hat{h}\left(X_T^{t_0,\underline{x}}\right)\right) &= \frac{1}{4}\left(\mathrm{Var}\left(h\left(X_T^{t_0,\underline{x}}\right)\right) + \mathrm{Var}\left(\bar{h}\left(X_T^{t_0,\underline{x}}\right)\right)\right. \\ &\quad \left. + 2\,\mathrm{Cov}\left(h\left(X_T^{t_0,\underline{x}}\right), \bar{h}\left(X_T^{t_0,\underline{x}}\right)\right)\right) \end{aligned}$$

Stratified Sampling

- **set of events**

$$A_i \subseteq \mathcal{A}_T,\ i \in \{1, 2, \ldots, N\}$$

$$\bigcup_{i=1}^{N} A_i = \Omega, \qquad A_i \cap A_j = \emptyset$$

for $i, j \in \{1, 2, \ldots, N\}$, $P(A_i) = \frac{1}{N}$

$$\mathcal{A} = \sigma\,\{A_i,\ i \in \{1, 2, \ldots, N\}\}$$

- **random variable**

  $$Z : \Omega \to \Re$$

- restriction of Z to A_i

  $$Z_{A_i}(\omega) = Z(\omega) \text{ for } \omega \in A_i$$

- **unbiased estimator for $E(Z)$**

$$\bar{Z} = \frac{1}{N} \sum_{i=1}^{N} Z_{A_i}$$

where Z_{A_i} independent

$$E(\bar{Z}) = \frac{1}{N} \sum_{i=1}^{N} E(Z_{A_i}) = \sum_{i=1}^{N} \int_{A_i} Z \, dP = \int_{\Omega} Z \, dP = E(Z)$$

independence

$\Longrightarrow$

$$
\begin{aligned}
\mathrm{Var}(\bar{Z}) &= \sum_{i=1}^{N} \frac{\mathrm{Var}(Z_{A_i})}{N^2} \\
&= \frac{1}{N}\, E\big(\mathrm{Var}(Z \,\big|\, \mathcal{A})\big) \\
&\leq \frac{1}{N}\, \mathrm{Var}(Z)
\end{aligned}
$$

Example: simplified weak Euler approximation Y^Δ

$\Delta\hat{W}_k$ two-point distributed

two-point variates with N time steps

underlying sample space

$\Longrightarrow$

$$\Omega_N = \{-1, 1\}^{\{0,1,\ldots,N-1\}}$$

'path' for $\hat{W}_k$

given by $\Delta\hat{W}_k(\omega) = \omega_k\sqrt{\Delta}$

probabilities $P_N(\omega) = \frac{1}{2^N}$

only a tiny fraction of these paths can be sampled

A stratified Monte Carlo estimation could consist of exhausting all paths $\omega \in \Omega_N$ up to time $t_{N'}$ and then sampling randomly;

reduces duplicate traversals of the early nodes of the lattice.

Measure Transformation Method

see Kloeden & Pl. (1992b)

- **d-dimensional diffusion**

$$dX_t^{s,x} = a(t, X_t^{s,x})\, dt + \sum_{j=1}^{m} b^j(t, X_t^{s,x})\, dW_t^j$$

on $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$

- **approximate the functional**

$$u(s,x) = E\left(g(X_T^{s,x}) \,\middle|\, \mathcal{A}_s\right)$$

- **Kolmogorov backward equation**

$$L^0\, u(s,x) = 0$$

for $(s,x) \in (0,T) \times \Re^d$ with

$$u(T,y) = g(y)$$

$$L^0 = \frac{\partial}{\partial s} + \sum_{k=1}^{d} a^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,\ell=1}^{d} \sum_{j=1}^{m} b^{k,j}\, b^{\ell,j} \frac{\partial^2}{\partial x^k\, \partial x^\ell}$$

- **Girsanov transformation**

$$\tilde{W}_t^j = W_t^j - \int_0^t d^j\left(z, \tilde{X}_z^{0,x}\right) dz$$

d^j denotes given, rather flexible real-valued function

- **Radon-Nikodym derivative**

$$\frac{d\tilde{P}}{dP} = \frac{\Theta_t}{\Theta_0}$$

• SDE

$$
\begin{aligned}
d\tilde{X}_t^{0,x} &= a\left(t, \tilde{X}_t^{0,x}\right) dt + \sum_{j=1}^{m} b^j \left(t, \tilde{X}_t^{0,x}\right) d\tilde{W}_t^j \\
&= \left(a\left(t, \tilde{X}_t^{0,x}\right) - \sum_{j=1}^{m} b^j \left(t, \tilde{X}_t^{0,x}\right)\, d^j \left(t, \tilde{X}_t^{0,x}\right)\right) dt \\
&\quad + \sum_{j=1}^{m} b^j \left(t, \tilde{X}_t^{0,x}\right) dW_t^j
\end{aligned}
$$

• Radon-Nikodym derivative process

$$
\Theta_t = \Theta_0 + \sum_{j=1}^{m} \int_0^t \Theta_z \, d^j \left(z, \tilde{X}_z^{0,x}\right) dW_z^j
$$

$(\underline{\mathcal{A}}, P)$-martingale

- **diffusion process** $\tilde{X}^{0,x}$ with respect to $\tilde{P}$

  same drift and diffusion coefficients as $X^{s,x}$

$\Longrightarrow$

$$\begin{aligned} E\left(g\left(X_T^{0,x}\right)\right) &= \int_\Omega g\left(X_T^{0,x}\right) dP \\ &= \int_\Omega g\left(\tilde{X}_T^{0,x}\right) d\tilde{P} \\ &= \int_\Omega g\left(\tilde{X}_T^{0,x}\right) \frac{\Theta_T}{\Theta_0}\, dP \\ &= E\left(g\left(\tilde{X}_T^{0,x}\right) \frac{\Theta_T}{\Theta_0}\right) \end{aligned}$$

- **unbiased estimator**

$$g\left(\tilde{X}_T^{0,x}\right)\frac{\Theta_T}{\Theta_0}$$

no particular choice of d^j made so far

- ideally choose d^j in the form

$$d^j(t,x) = -\frac{1}{u(t,x)} \sum_{k=1}^{d} b^{k,j}(t,x) \, \frac{\partial u(t,x)}{\partial x^k}$$

then it can be shown that

$$u\left(t, \tilde{X}_t^{0,x}\right) \Theta_t = u(0,x) \, \Theta_0$$

for all $t \in [0,T]$

$\Longrightarrow$

$$u(0,x) = g\left(\tilde{X}_T^{0,x}\right) \frac{\Theta_T}{\Theta_0}$$

$\Longrightarrow$

$$g\left(\tilde{X}_T^{0,x}\right)\frac{\Theta_T}{\Theta_0}$$

is ***not random***

- guess a function $\bar{u}$

$$d^j(t,x) = -\frac{1}{\bar{u}(t,x)} \sum_{k=1}^{d} b^{k,j}(t,x) \frac{\partial \bar{u}(t,x)}{\partial x^k}$$

- used unbiased estimator

$$g\left(\tilde{X}_T^{0,x}\right) \frac{\Theta_T}{\Theta_0}$$

$$E\left(g\left(\tilde{X}_T^{0,x}\right) \frac{\Theta_T}{\Theta_0}\right) = E\left(g\left(X_T^{0,x}\right)\right)$$

Control Variates and Integral Representations

Clewlow & Carverhill (1992, 1994)

Basic Control Variate Method

- **valuation martingale**

$$M_t = u\left(t, X_t^{t_0,\underline{x}}\right) = E\left(h\left(X_T^{t_0,\underline{x}}\right) \,\Big|\, \mathcal{A}_t\right)$$

construct an accurate and fast estimate of

$$E(h(X_T^{t_0,\underline{x}})) = u(t_0, \underline{x})$$

- **control variate**

  find $\boldsymbol{Y}$ with known mean $\boldsymbol{E(Y)}$

- **unbiased estimator**

$$\boldsymbol{Z = h(X_T^{t_0,\underline{x}}) - \alpha(Y - E(Y))}$$

$\boldsymbol{\alpha \in \Re}$

$$\boldsymbol{E(Z) = E(h(X_T^{t_0,\underline{x}}))}$$

- **known valuation function**

$$\hat{u}\left(t, \hat{X}_t^{t_0,\underline{x}}\right) = E\left(h\left(\hat{X}_T^{t_0,\underline{x}}\right) \,\Big|\, \mathcal{A}_t\right)$$

$\hat{X}^{t_0,\underline{x}}$ approximates $X_T^{t_0,\underline{x}}$

- **unbiased estimator**

$$\begin{aligned}\hat{Z}_T &= h\left(X_T^{t_0,\underline{x}}\right) - \alpha\left(h\left(\hat{X}_T^{t_0,\underline{x}}\right) - E\left(h\left(\hat{X}_T^{t_0,\underline{x}}\right)\right)\right)\\ &= u\left(T, X_T^{t_0,\underline{x}}\right) - \alpha\left(\hat{u}\left(T, \hat{X}_T^{t_0,\underline{x}}\right) - \hat{u}(t_0, \underline{x})\right)\end{aligned}$$

variance can be reduced

$$\mathrm{Var}(\hat{Z}_T) = \mathrm{Var}\left(h\left(X_T^{t_0,\underline{x}}\right)\right) + \alpha^2\,\mathrm{Var}\left(h\left(\hat{X}_T^{t_0,\underline{x}}\right)\right) - 2\,\alpha\,\mathrm{Cov}\left(h\left(X_T^{t_0,\underline{x}}\right), h\left(\hat{X}_T^{t_0,\underline{x}}\right)\right)$$

minimize the variance

$$\alpha_{\min} = \frac{\mathrm{Cov}\left(h\left(X_T^{t_0,\underline{x}}\right), h\left(\hat{X}_T^{t_0,\underline{x}}\right)\right)}{\mathrm{Var}\left(h\left(\hat{X}_T^{t_0,\underline{x}}\right)\right)}$$

Example: Stochastic Volatility

- **SDE**

$$\begin{aligned} dS_t &= \sigma_t\, S_t\, dW_t^1 \\ d\sigma_t &= (\kappa - \sigma_t)\, dt + \xi\, \sigma_t\, dW_t^2 \end{aligned}$$

$$(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$$

- **European call payoff**

$$h(S_T) = (S_T - K)^+$$

- **adjusted price process**

$$
\begin{aligned}
d\hat{S}_t &= \hat{\sigma}_t \hat{S}_t \, dW_t^1 \\
d\hat{\sigma}_t &= (\kappa - \hat{\sigma}_t) \, dt
\end{aligned}
$$

- **adjusted valuation function**

$$
\hat{u}(t, \hat{S}_t, \hat{\sigma}_t) = E\left((\hat{S}_T - K)^+ \,\middle|\, \mathcal{A}_t\right)
$$

- **unbiased variate**

$$
\begin{aligned}
\hat{Z}_T &= (S_T - K)^+ - \alpha((\hat{S}_T - K)^+ - E(\hat{S}_T - K)^+) \\
&= (S_T - K)^+ - \alpha((\hat{S}_T - K)^+ - \hat{u}(t_0, s, \sigma))
\end{aligned}
$$

unbiased variance reduced estimator

Variance Reduction via Integral Representations

Heath & Pl. (2002)

The HP Variance Reduced Estimator

- **SDE**

$$dX_t^{s,x} = a(t, X_t^{s,x})\, dt + \sum_{j=1}^{m} b^j(t, X_t^{s,x})\, dW_t^j$$

- **first exit time**

$$\tau = \inf\{t \geq s : \ (t, X_t^{s,x}) \notin [s, T) \times \Gamma\}$$

- **operators**

$$L^0 f(t,x) = \frac{\partial f(t,x)}{\partial t} + \sum_{i=1}^{d} a^i(t,x)\, \frac{\partial f(t,x)}{\partial x^i}$$

$$+ \frac{1}{2} \sum_{i,k=1}^{d} \sum_{j=1}^{m} b^{i,j}(t,x)\, b^{k,j}(t,x)\, \frac{\partial^2 f(t,x)}{\partial x^i\, \partial x^k}$$

and

$$L^j f(t,x) = \sum_{i=1}^{d} b^{i,j}(t,x)\, \frac{\partial f(t,x)}{\partial x^i}$$

for $(t,x) \in (0,T) \times \Gamma$

- **valuation function**

$$u(t,x) = E\left(h(\tau, X_\tau^{t,x})\right)$$

for $(t,x) \in [0,T] \times \Gamma$

assume

$$M_t = E\left(h\left(\tau, X_\tau^{0,x}\right) \,\big|\, \mathcal{A}_t\right)$$

square integrable $(\underline{\mathcal{A}}, P)$-martingale

martingale representation theorem

$\Longrightarrow$

$$
\begin{aligned}
M_t &= u(t, X^{0,x}_{t\wedge\tau}) \\
&= u(0,x) + \sum_{j=1}^{m} \int_0^{t\wedge\tau} \xi^j_s \, dW^j_s
\end{aligned}
$$

for $t \in [0,T]$

- given an approximation

$$\bar{u} : \; [0, T] \times \Gamma \to \Re \quad \text{to } u$$

$$\bar{u} \in \mathcal{C}^{1,2}([0, T] \times \Gamma)$$

with

$$\bar{M}_t^j = \int_0^{t \wedge \tau} L^j \, \bar{u}(s, X_s^{0,x}) \, dW_s^j$$

square integrable $(\underline{\mathcal{A}}, P)$-martingale

$$\bar{u}(\tau, X_\tau^{0,x}) = u(\tau, X_\tau^{0,x}) = h(\tau, X_\tau^{0,x})$$

$\Longrightarrow$

$$\begin{aligned} \bar{u}(\tau, X_\tau^{0,x}) &= \bar{u}(0, x) + \int_0^\tau L^0 \, \bar{u}(t, X_t^{0,x}) \, dt \\ &\quad + \sum_{j=1}^m \int_0^\tau L^j \, \bar{u}(t, X_t^{0,x}) \, dW_t^j \end{aligned}$$

$\Longrightarrow$

$$
\begin{aligned}
u(0,x) &= E\left(h(\tau, X_\tau^{0,x})\right) \\
&= E\left(\bar{u}(\tau, X_\tau^{0,x})\right) \\
&= \bar{u}(0,x) + E\left(\int_0^\tau L^0\,\bar{u}(t, X_t^{0,x})\,dt\right) \\
&= \bar{u}(0,x) + \int_0^T E\left(1_{\{t<\tau\}} L^0\,\bar{u}(t, X_t^{0,x})\right) dt
\end{aligned}
$$

- **unbiased estimator for** $u(0,x)$

$$
\bar{Z}_\tau = \bar{u}(0,x) + \int_0^\tau L^0\,\bar{u}(t, X_t^{0,x})\,dt
$$

HP estimator

Variance of the HP Estimator

$$\begin{aligned}
\bar{Z}_\tau &= \bar{u}(\tau, X_\tau^{0,x}) - \sum_{j=1}^{m} \int_0^\tau L^j\, \bar{u}(t, X_t^{0,x})\, dW_t^j \\
&= u(\tau, X_\tau^{0,x}) - \sum_{j=1}^{m} \int_0^\tau L^j\, \bar{u}(t, X_t^{0,x})\, dW_t^j \\
&= u(0, x) + \sum_{j=1}^{m} \int_0^\tau \left(\xi_t^j - L^j\, \bar{u}(t, X_t^{0,x})\right) dW_t^j
\end{aligned}$$

$\Longrightarrow$

$$
\begin{aligned}
\mathrm{Var}(\bar{Z}_\tau) &= E\left[\left(\sum_{j=1}^{m} \int_0^{\tau} \left(\xi_t^j - L^j\,\bar{u}(t, X_t^{0,x})\right) dW_t^j\right)^2\right] \\
&= \sum_{j=1}^{m} \int_0^{T} E\left(1_{\{t<\tau\}} \left(\xi_t^j - L^j\,\bar{u}(t, X_t^{0,x})\right)^2\right) dt
\end{aligned}
$$

- **integrands**

$$\xi_t^j = L^j\, u(t, X_t^{0,x})$$

$$\Longrightarrow$$

$$\mathrm{Var}(\bar{Z}_\tau) = \sum_{j=1}^{m} \int_0^T E\left(1_{\{t<\tau\}}\left(\left(L^j\, u - L^j\, \bar{u}\right)(t, X_t^{0,x})\right)^2\right) dt$$

if a good approximation $\bar{u}$ to u can be found,

so that $L^j\, u$ is close to $L^j\, \bar{u}$

variance will be small

- **unbiased estimator**

$$\begin{aligned}\bar{Z}_{\tau,\alpha} &= \bar{u}(\tau, X_\tau^{0,x}) - \alpha \sum_{j=1}^{m} \int_0^\tau L^j \, \bar{u}(t, X_t^{0,x}) \, dW_t^j \\ &= \bar{Z}_\tau + (1-\alpha) \sum_{j=1}^{m} \int_0^\tau L^j \, \bar{u}(t, X_t^{0,x}) \, dW_t^j\end{aligned}$$

An Example for the Heston Model

- **SDEs**

$$
\begin{aligned}
dS_t^{s,x^1} &= u\, S_t^{s,x^1}\, dt + \sqrt{v_t^{s,x^2}}\, S_t^{s,x^1}\, dW_t^1 \\
dv_t^{s,x^2} &= \kappa \left(\theta - v_t^{s,x^2} \right) dt \\
&\quad + \xi \sqrt{v_t^{s,x^2}} \left(\varrho\, dW_t^1 + \sqrt{1-\varrho^2}\, dW_t^2 \right)
\end{aligned}
$$

- equivalent risk neutral martingale measure

$$
\begin{aligned}
dS_t^{s,x^1} &= r\, S_t^{s,x^1}\, dt + \sqrt{v_t^{s,x^2}}\, S_t^{s,x^1}\, d\tilde{W}_t^1 \\
dv_t^{s,x^2} &= \kappa \left(\tilde{\theta} - v_t^{s,x^2}\right) dt \\
&\quad + \xi \sqrt{v_t^{s,x^2}} \left(\varrho\, d\tilde{W}_t^1 + \sqrt{1-\varrho^2}\, d\tilde{W}_t^2\right)
\end{aligned}
$$

for $t \in [s, T]$ and $s \in [0, T]$, where $\tilde{\theta} = \theta - \frac{\xi\, \varrho\, (\mu - r)}{\kappa}$ and

$$
\begin{aligned}
d\tilde{W}_t^1 &= \frac{\mu - r}{\sqrt{v_t}}\, dt + dW_t^1 \\
d\tilde{W}_t^2 &= dW_t^2
\end{aligned}
$$

- **option price**

$$c(0, x) = e^{-rT}\, u(0, x)$$

where

$$u(0, x) = \tilde{E}\left((S_T^{0,x^1} - K)^+\right)$$

- **approximation**

$$\begin{aligned} d\bar{S}_t^{s,x^1} &= r\, \bar{S}_t^{s,x^1}\, dt + \sqrt{\bar{v}_t^{s,x^2}}\, \bar{S}_t^{s,x^1}\, d\tilde{W}_t^1 \\ d\bar{v}_t^{s,x^2} &= \kappa\left(\tilde{\theta} - \bar{v}_t^{s,x^2}\right) dt \end{aligned}$$

explicitly computed

$$\bar{v}_t^{s,x^2} = \tilde{\theta} + (x^2 - \tilde{\theta})\, e^{-\kappa(t-s)}$$

Black-Scholes price

$$\begin{aligned}\bar{u}(t,x) &= \tilde{E}\left((\bar{S}_T^{t,x^1} - K)^+\right) \\ &= e^{r(T-t)}\, BS(x^1, K, r, \bar{\sigma}_t, T-t)\end{aligned}$$

where

$$\begin{aligned}\bar{\sigma}_t &= \sqrt{\frac{1}{T-t}\int_t^T \bar{v}_z^{t,x^2}\, dz} \\ &= \sqrt{\tilde{\theta} - (x^2 - \tilde{\theta})\,\frac{e^{-\kappa(T-t)} - 1}{\kappa(T-t)}}\end{aligned}$$

$$\Longrightarrow$$

$$(L^0 - \bar{L}^0)\, f(t,x) = \xi\, x^2 \left(\varrho\, \frac{\partial^2 f(t,x)}{\partial x^1\, \partial x^2} + \frac{1}{2}\, \xi\, \frac{\partial^2 f(t,x)}{\partial (x^2)^2} \right)$$

$$\begin{aligned}(L^0 - \bar{L}^0)\, \bar{u}(t,x) \quad = \quad & \xi\, x^2\, e^{r(T-t)} \left[\varrho\, \frac{\partial^2 BS(x^1, K, r, \bar{\sigma}_t, T-t)}{\partial x^1\, \partial \bar{\sigma}_t}\, \frac{\partial \bar{\sigma}_t}{\partial x^2} \right. \\ & + \frac{1}{2}\, \xi \left\{ \frac{\partial^2 BS(x^1, K, r, \bar{\sigma}_t, T-t)}{\partial \sigma_t^2} \left(\frac{\partial \bar{\sigma}_t}{\partial x^2} \right)^2 \right. \\ & \left. \left. + \frac{\partial BS(x^1, K, r, \bar{\sigma}_t, T-t)}{\partial \sigma_t}\, \frac{\partial^2 \bar{\sigma}_t}{\partial (x^2)^2} \right\} \right]\end{aligned}$$

$$\frac{\partial BS}{\partial \bar{\sigma}_t}, \quad \frac{\partial^2 BS}{\partial x^1\, \partial \bar{\sigma}_t}, \quad \frac{\partial^2 BS}{\partial \bar{\sigma}_t^2}, \quad \frac{\partial \bar{\sigma}_t}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 \bar{\sigma}_t}{\partial (x^2)^2}$$

can be computed

Monte Carlo Simulation

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = T, \qquad \Delta = \frac{T}{N}$$

- predictor-corrector method of weak order 1.0

$$\begin{aligned} Y_{n+1} &= Y_n + \Big(\alpha\, \hat{a}(\tau_{n+1}, \hat{Y}_{n+1}) + (1-\alpha)\, \hat{a}(\tau_n, Y_n)\Big)\, \Delta \\ &\quad + \sum_{j=1}^{m} \Big(\eta\, b^j(\tau_{n+1}, \hat{Y}_{n+1}) + (1-\eta)\, b^j(\tau_n, Y_n)\Big)\, \Delta W_n^j \end{aligned}$$

for $n \in \{0, 1, \ldots, N-1\}$ with predictor

$$\hat{Y}_{n+1} = Y_n + a(\tau_n, Y_n)\, \Delta + \sum_{j=1}^{m} b^j\, \Delta W_n^j$$

and modified drift coefficient values

$$\hat{a}(\tau_n, Y_n) = a(\tau_n, Y_n) - \eta \sum_{i=1}^{d} \sum_{j=1}^{m} b^{i,j}(\tau_n, Y_n) \frac{\partial b^j(\tau_n, Y_n)}{\partial x^i}$$

$\alpha, \eta \in [0, 1]$ and ΔW_n^j

Gaussian random two-point distributed random variables

Simulation Results

- raw Monte Carlo

  intrinsic value $(S_t^{0,x^1} - K)^+$

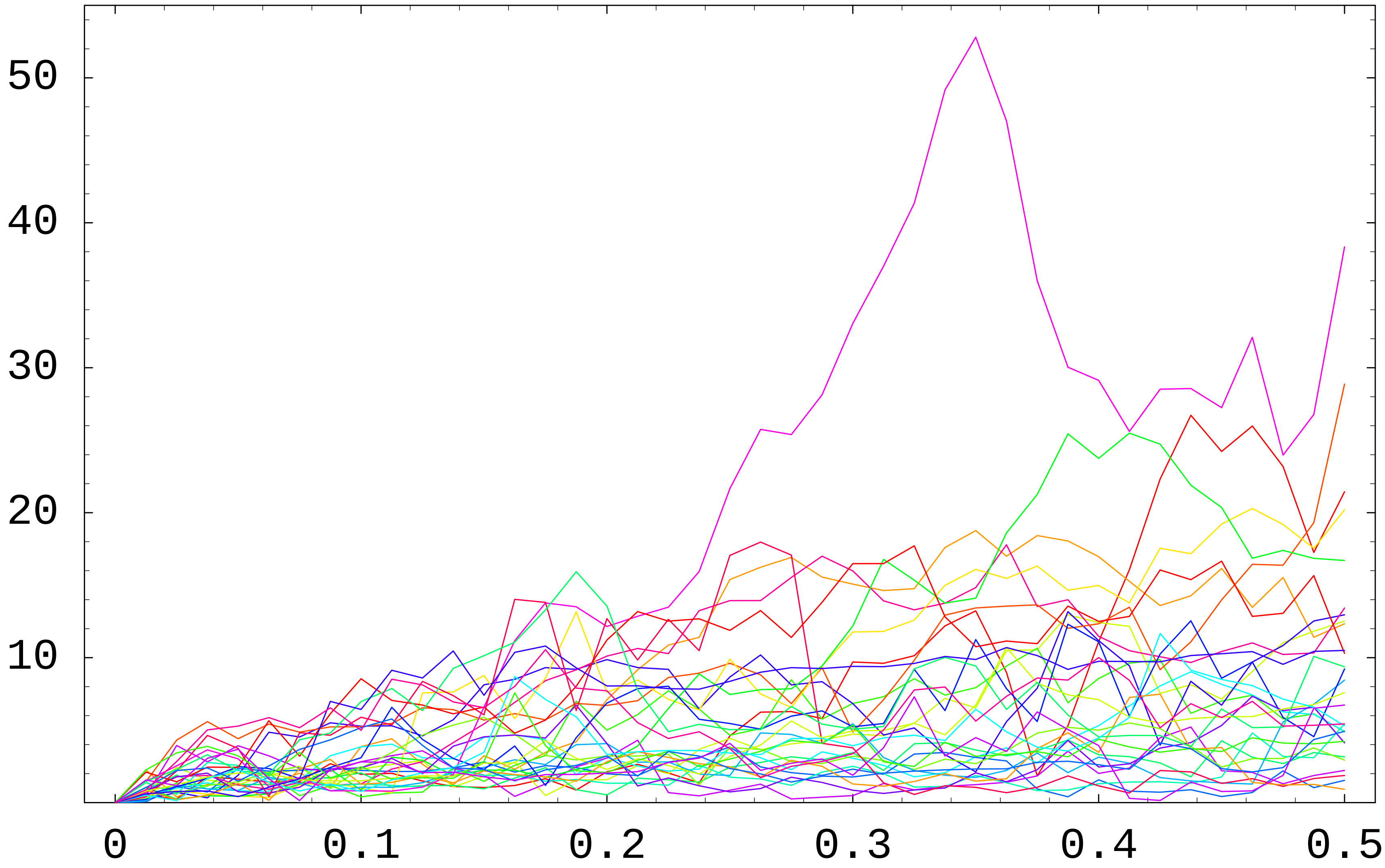

Figure 5.1: Simulated outcomes for the intrinsic value $(S_t^{0,x^1} - K)^+$, $t \in [0, T]$.

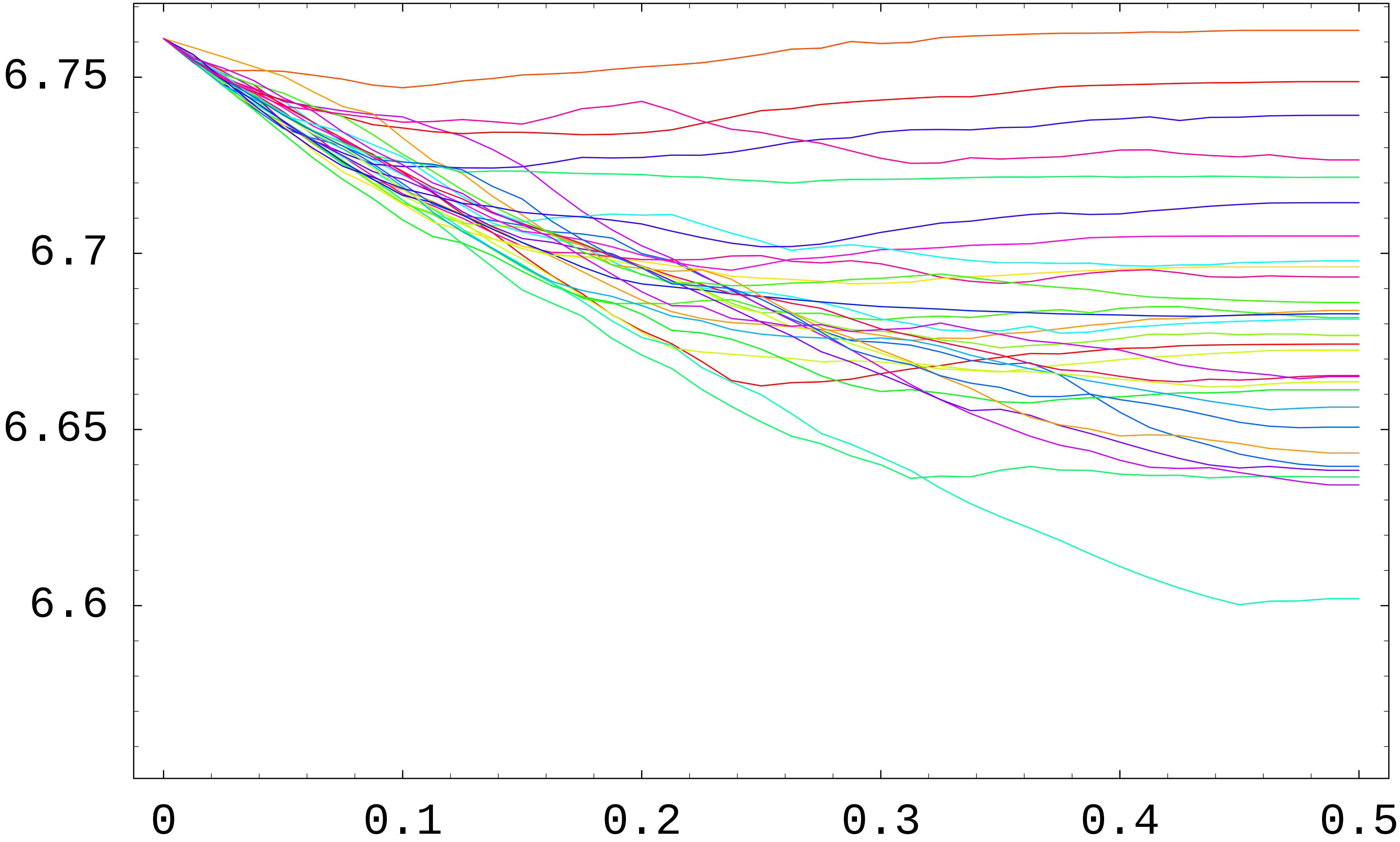

Figure 5.2: Simulated outcomes for the estimator $\bar{Z}_t,\ t \in [0, T]$.

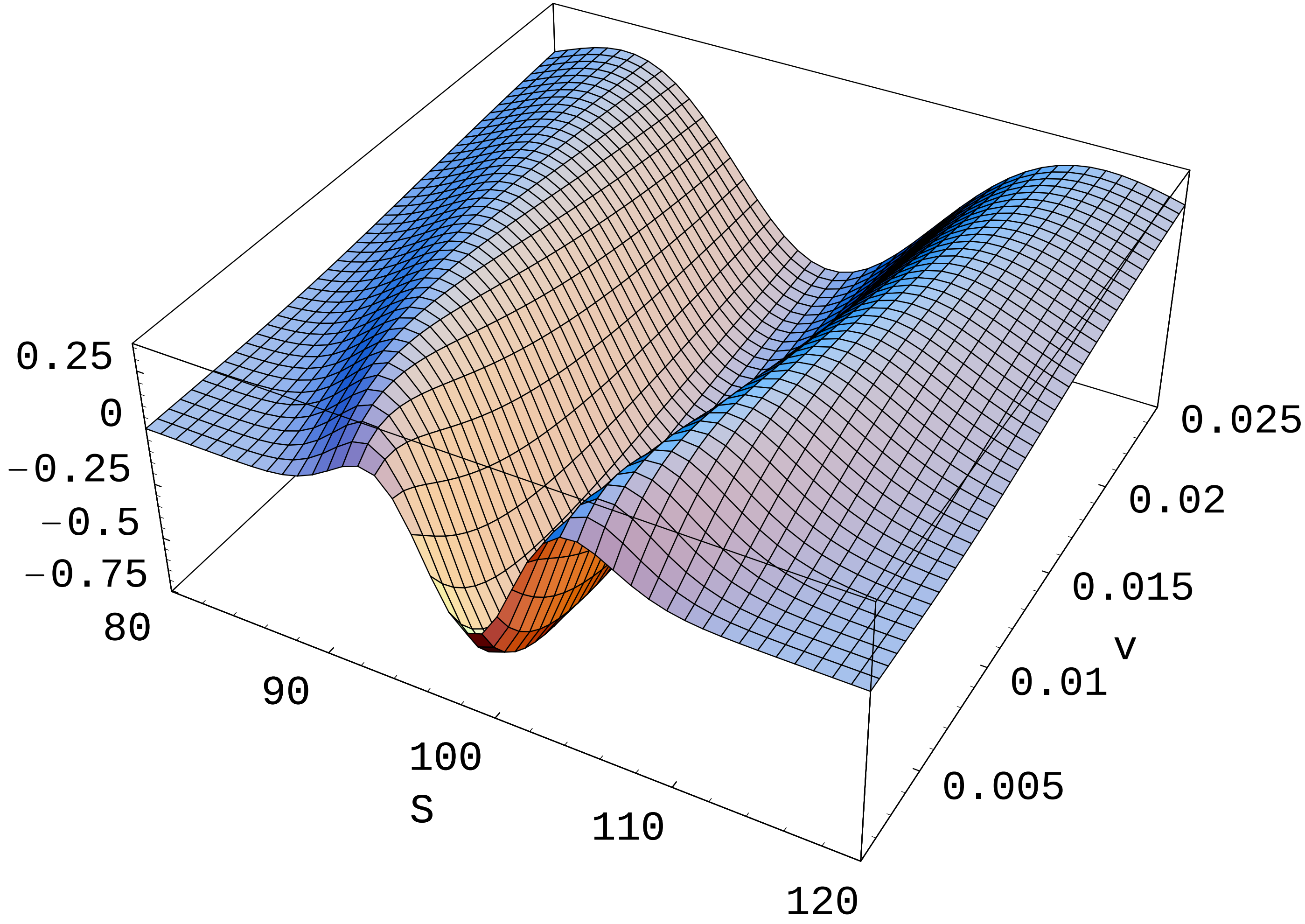

Figure 5.3: Diffusion operator values $(L - \bar{L}^0)\,\bar{u}$ as a function of asset price S and squared volatility v.

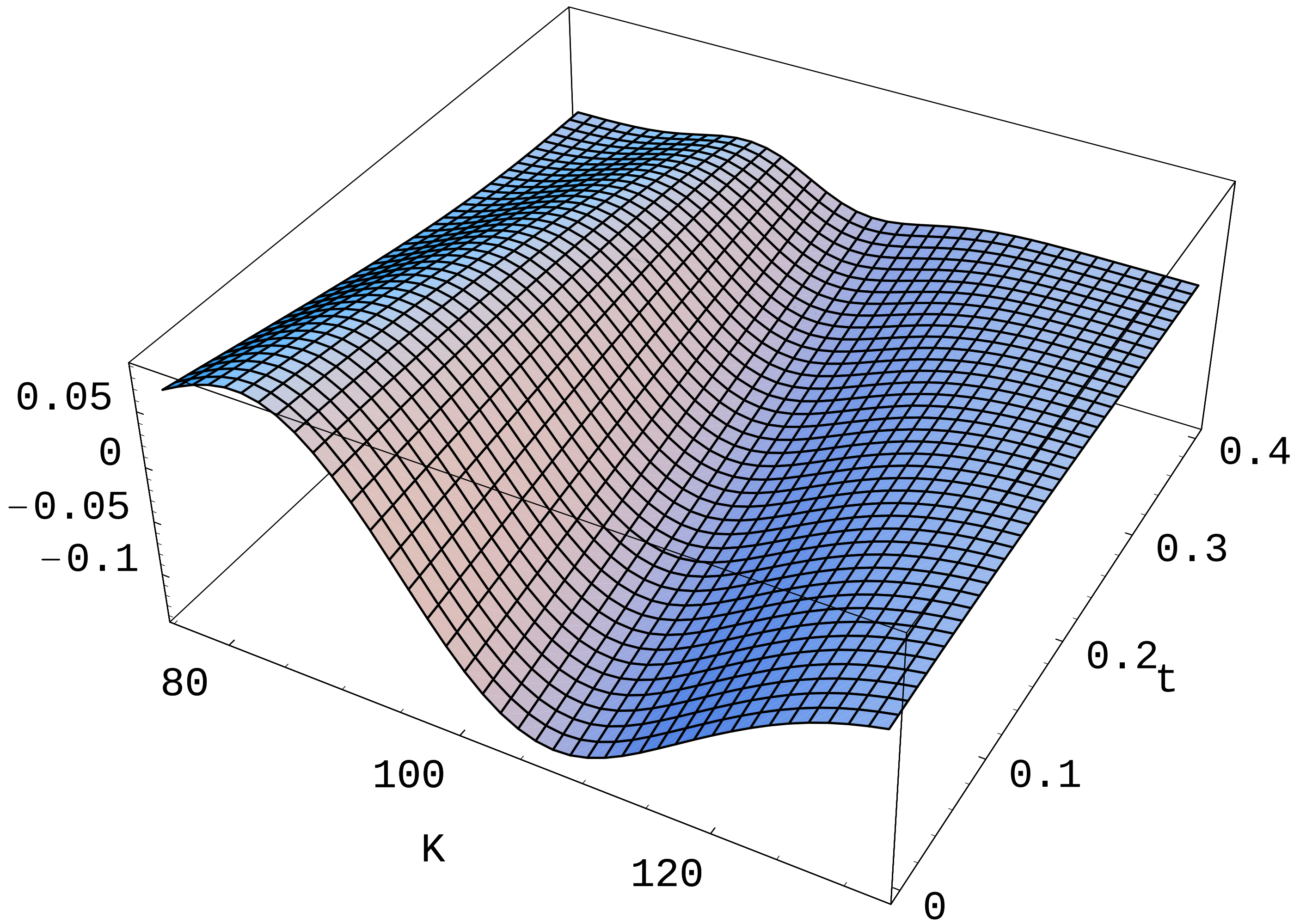

Figure 5.4: Price differences between the Heston and corresponding Black-Scholes model as a function of strike K and time t.

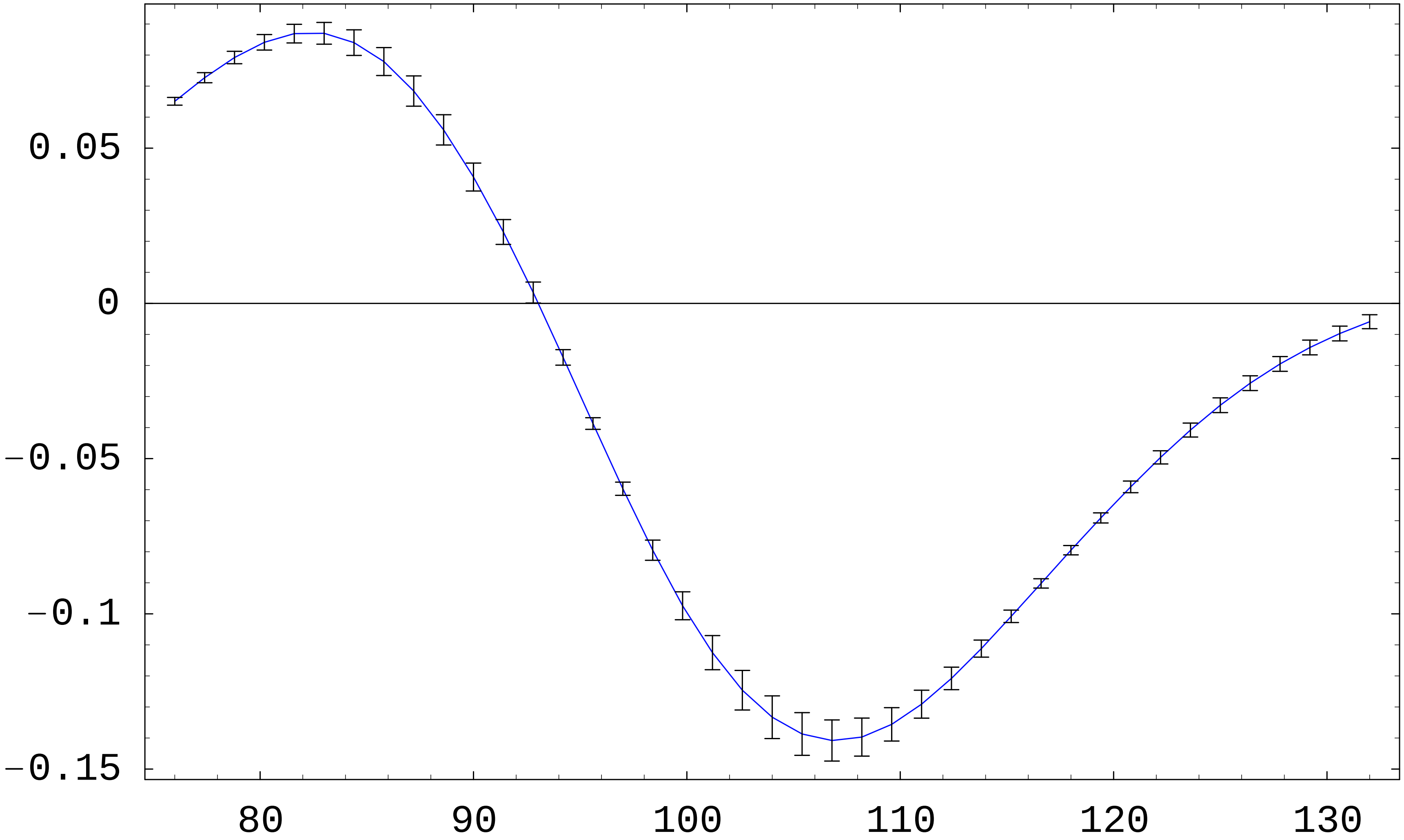

Figure 5.5: Prices and corresponding error bounds as a function of strike $\boldsymbol{K}$.

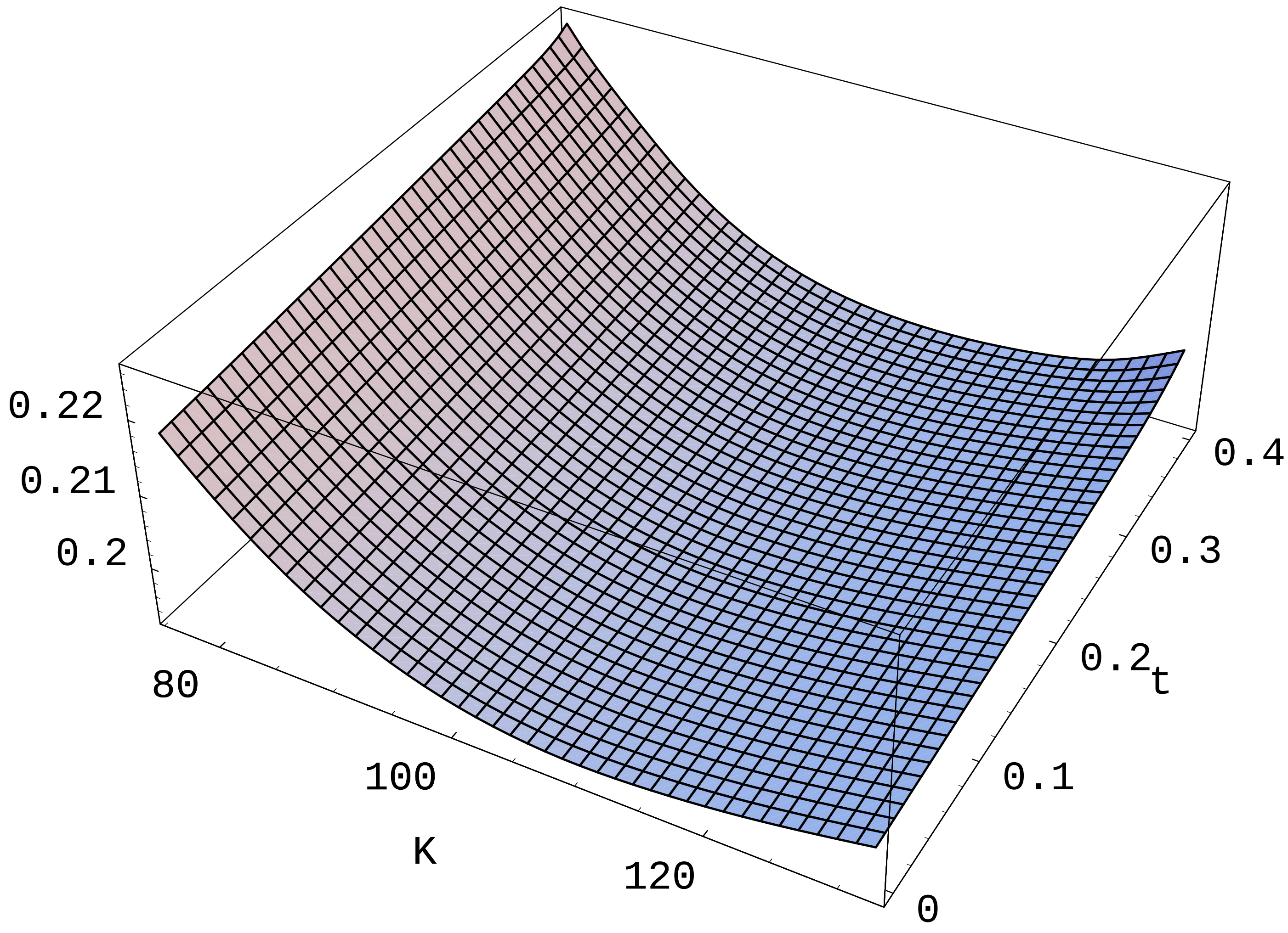

Figure 5.6: Implied volatility term structure for the Heston model.

6 Trees and Markov Chain Approximations

Binomial Option Pricing

Single-Period Binomial Model

- **growth optimal portfolio**

$$S^{\delta_*} = \{S_t^{\delta_*},\, t \in [0,T]\}$$

- **primary security account**

$$S = \{S_t,\, t \in [0,T]\}$$

cum-dividend price

- benchmarked primary security

$$\hat{S}_t = \frac{S_t}{S_t^{\delta_*}}$$

- filtered probability space

$$(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$$

- **benchmarked portfolio**

$$\hat{S}_t^{\delta} = \delta_t^0 + \delta_t^1 \, \hat{S}_t$$

δ_t^0 units in the GOP

δ_t^1 units in the stock

- **single period binomial model**

benchmarked security $\hat{S}_\Delta$ at time $t = \Delta > 0$

$$P\left(\hat{S}_\Delta = (1+u)\,\hat{S}_0\right) = p$$

and

$$P\left(\hat{S}_{\Delta} = (1+d)\,\hat{S}_0\right) = 1 - p$$

for $p \in (0,1)$ and $-d, u \in (0,\infty)$

- upward move

$$u = \frac{\hat{S}_{\Delta} - \hat{S}_0}{\hat{S}_0}$$

- downward move

$$d = \frac{\hat{S}_{\Delta} - \hat{S}_0}{\hat{S}_0}$$

Fair Pricing and Hedging

European call option on benchmarked stock

- **benchmarked payoff**

$$\hat{H}_{\Delta} = (\hat{S}_{\Delta} - \hat{K})^{+}$$

deterministic benchmarked strike $\hat{K}$

from interval $((1 + d)\, \hat{S}_0, (1 + u)\, \hat{S}_0)$

maturity $T = \Delta$

$S_0^{\delta_*} = 1$

- **real world pricing formula**

$\Longrightarrow$

$$
\begin{aligned}
S_0^{\delta_{H_\Delta}} &= S_0^{\delta_*} \, E\left(\hat{H}_\Delta \,\middle|\, \mathcal{A}_0 \right) \\
&= \left((1+u)\, \hat{S}_0 - \hat{K} \right) p
\end{aligned}
$$

- **cum-dividend stock price**

fair price process

$$\Longrightarrow \qquad \hat{S}_0 = E\left(\hat{S}_\Delta \,\middle|\, \mathcal{A}_0\right)$$

$$= p\,(1+u)\,\hat{S}_0 + (1-p)(1+d)\,\hat{S}_0$$

$$= \{p(1+u) + (1-p)\,(1+d)\}\,\hat{S}_0$$

$\Longrightarrow$ condition

$$1 = p\,(1+u) + (1-p)(1+d)$$

$\Longrightarrow$ probability

$$p = \frac{-d}{u-d}$$

- **European call option price**

$$\hat{S}_0^{\delta_{H_\Delta}} = \left((1+u)\,\hat{S}_0 - \hat{K} \right) \frac{-d}{u-d}$$

self-financing

$\Longrightarrow$

$$\hat{S}_0^{\delta_{H_\Delta}} = \delta_0^0 + \delta_0^1\,\hat{S}_0$$

$$\begin{aligned} \hat{S}_\Delta^{\delta_{H_\Delta}} &= \delta_0^0 + \delta_0^1\,\hat{S}_\Delta \\ &= \hat{H}_\Delta = \left(\hat{S}_\Delta - \hat{K} \right)^+ \end{aligned}$$

$$\Longrightarrow$$

- **hedge ratio**

$$\delta_0^1 = \frac{\hat{S}_\Delta^{\delta_{H_\Delta}} - \hat{S}_0^{\delta_{H_\Delta}}}{\hat{S}_\Delta - \hat{S}_0}$$

for upward move $\quad \hat{S}_\Delta = (1+u)\hat{S}_0$

$$\delta_0^1 = \frac{(1+u)\,\hat{S}_0 - \hat{K} - \left((1+u)\,\hat{S}_0 - \hat{K}\right)\,\frac{-d}{u-d}}{u\,\hat{S}_0}$$

$$= \frac{(1+u)\,\hat{S}_0 - \hat{K}}{(u-d)\,\hat{S}_0}$$

for a downward move $\hat{S}_\Delta = (1+d)\,\hat{S}_0$

$$\delta_0^1 = \frac{-\left((1+u)\,\hat{S}_0 - \hat{K}\right)\,\frac{-d}{u-d}}{d\,\hat{S}_0}$$

$$= \frac{(1+u)\,\hat{S}_0 - \hat{K}}{(u-d)\,\hat{S}_0}$$

market is complete, that is, all claims can be hedged,

priced according to real world pricing formula

$\Longrightarrow$ minimal price

- probability p is a real world probability
- refers to an upward move of the benchmarked security
- different to the standard approach
- **variance of ratio**

$$\frac{\hat{S}_\Delta}{\hat{S}_0} = 1 + \eta$$

two-point distributed benchmarked return $\eta \in \{d, u\}$

$$P(\eta = u) = p$$

$$P(\eta = d) = 1 - p$$

- **variance of benchmarked return** η

$$E\left(\left(\frac{\hat{S}_\Delta}{\hat{S}_0}-1\right)^2 \,\middle|\, \mathcal{A}_0\right) = (u-d)^2\, p\,(1-p)$$

$\Longrightarrow$

$$E\left(\left(\frac{\hat{S}_\Delta}{\hat{S}_0}-1\right)^2 \,\middle|\, \mathcal{A}_0\right) = -u\, d$$

- **binomial "volatility"**

$$\sigma_\Delta = \sqrt{\frac{1}{\Delta}\, E\left(\left(\frac{\hat{S}_\Delta}{\hat{S}_0} - 1\right)^2 \,\Big|\, \mathcal{A}_0\right)} = \sqrt{\frac{-u\,d}{\Delta}}$$

- under the BS model option price increases for increasing volatility,

  here not necessarily the case for European call price

$$S_0^{\delta_{H_\Delta}} = \left((1+u)\,\hat{S}_0 - \hat{K}\right) \frac{(\sigma_\Delta)^2\,\Delta}{u\,(u-d)}$$

  substantial freedom
  in u and d,
  not satisfactory do to
  simplicity of the binomial model

Multi-Period Binomial Model

- **benchmarked price of the stock**

  $t = n\Delta$

  upward

$$\hat{S}_{n\Delta} = \hat{S}_{(n-1)\Delta}\,(1+u)$$

  probability

$$p = \frac{-d}{u-d}$$

downward

$$\hat{S}_{n\Delta} = \hat{S}_{(n-1)\Delta}\,(1 + d)$$

probability $\quad 1 - p$

for $n \in \{1, 2, \ldots\}$

- **European call** on $\hat{S}_{2\Delta}$

  three values

$$
\begin{aligned}
\hat{S}_{uu} &= (1+u)^2 \, \hat{S}_0 \\
\hat{S}_{ud} &= \hat{S}_{du} = (1+u)(1+d)\hat{S}_0 \\
\hat{S}_{dd} &= (1+d)^2 \hat{S}_0
\end{aligned}
$$

  binomial tree

- portfolio

$$\hat{S}^{\delta}_{2\Delta} = \delta^0_{2\Delta} + \delta^1_{2\Delta}\,\hat{S}_{2\Delta}$$

at time $t = \Delta$

when $\hat{S}_\Delta = (1+u)\,\hat{S}_0$

$$\hat{S}^{\delta_{H_{2\Delta}}}_{\Delta} = \left((1+u)^2\,\hat{S}_0 - \hat{K}\right)^+ \frac{-d}{u-d} + \left((1+u)(1+d)\,\hat{S}_0 - \hat{K}\right)^+ \frac{u}{u-d}$$

for $\hat{S}_\Delta = (1+d)\hat{S}_0$

$$\hat{S}^{\delta_{H_{2\Delta}}}_{\Delta} = \left((1+u)(1+d)\,\hat{S}_0 - \hat{K}\right)^+ \frac{-d}{u-d} + \left((1+d)^2\,\hat{S}_0 - \hat{K}\right)^+ \frac{u}{u-d}$$

- **fair benchmarked price at time $t = 0$**

$$\hat{S}_0^{\delta_{H_{2\Delta}}} = p\left(\left(\left((1+u)^2\,\hat{S}_0 - \hat{K}\right)^+ \frac{-d}{u-d} + \left((1+u)(1+d)\,\hat{S}_0 - \hat{K}\right)^+ \frac{u}{u-d}\right) + (1-p)\left(\left((1+u)\,(1+d)\,\hat{S}_0 - \hat{K}\right)^+ \frac{-d}{u-d} + \left((1+d)^2\,\hat{S}_0 - \hat{K}\right)^+ \frac{u}{u-d}\right)\right)$$

perfect hedging strategy can be described

- **binomial real world option pricing formula**

  similar to Cox, Ross & Rubinstein (1979) but real world pricing

$$
\begin{aligned}
\hat{S}_0^{\delta_{H_{i\Delta}}} &= E\left(\left(\hat{S}_{i\Delta} - \hat{K}\right)^+ \,\Big|\, \mathcal{A}_0\right) \\
&= \sum_{k=0}^{i} \frac{i\,!}{k\,!\,(i-k)\,!}\, p^k \,(1-p)^{i-k} \\
&\qquad \times \left((1+u)^k (1+d)^{i-k}\, \hat{S}_0 - \hat{K}\right)^+
\end{aligned}
$$

$$= \hat{S}_0 \sum_{k=t_k}^{i} \frac{i\,!}{k\,!\,(i-k)\,!}\, p^k\,(1-p)^{i-k}(1+u)^k(1+d)^{i-k}$$

$$- \hat{K} \sum_{k=t_k}^{i} \frac{i\,!}{k\,!\,(i-k)\,!}\, p^k\,(1-p)^{i-k}$$

t_k first integer k for which $(1+u)^k(1+d)^{i-k}\hat{S}_0 > \hat{K}$

(risk neutral pricing is here avoided)

- **complementary binomial distribution**

$$t_N(t_k, i, p) = \sum_{k=t_k}^{i} \frac{i\,!}{k\,!\,(i-k)\,!}\, p^k\,(1-p)^{i-k}$$

- **binomial option pricing formula**

$$S_0^{\delta_{H_i \Delta}} = S_0\, t_N(t_k, i, 1-p) - \hat{K}\, t_N(t_k, i, p)$$

similar to the Cox-Ross-Rubinstein binomial option pricing formula

similar to the Black-Scholes pricing formula

- hedging strategy

$$\delta_{k\Delta} = (\delta^0_{k\Delta}, \delta^1_{k\Delta})^\top$$

self-financing benchmarked hedge portfolio

$$\hat{S}^{\delta_{H_i\Delta}}_{k\Delta} = \delta^0_{(k-1)\Delta} + \delta^1_{(k-1)\Delta}\,\hat{S}_{k\Delta} = \delta^0_{k\Delta} + \delta^1_{k\Delta}\,\hat{S}_{k\Delta}$$

- hedge ratio

$$\delta^1_{k\Delta} = \sum_{\ell=t_{k\,k}}^{i-k} \frac{(i-k)\,!}{\ell\,!\,(i-k-\ell)\,!}\, p^{(i-k-\ell)}\,(1-p)^{\ell}$$

$t_{k\,k}$ smallest integer ℓ for which

$$(1+u)^{\ell}\,(1+d)^{i-k-\ell}\,\hat{S}_{k\Delta} > \hat{K}$$

$\Longrightarrow$

$$\delta^0_{k\Delta} = \hat{S}^{\delta_{H_{i\Delta}}}_{k\Delta} - \delta^1_{k\Delta}\,\hat{S}_{k\Delta}$$

Approximating the Black-Scholes Price

step size $\Delta \to 0$ limit

converges in a weak sense

- set u and d

$$\ln(1+u) = \sigma\sqrt{\Delta}$$

$$\ln(1+d) = -\sigma\sqrt{\Delta}$$

volatility parameter $\sigma > 0$

$\Longrightarrow$

$$u = \exp\left\{\sigma\sqrt{\Delta}\right\} - 1 \approx \sigma\sqrt{\Delta}$$

$$d = \exp\left\{-\sigma\sqrt{\Delta}\right\} - 1 \approx -\sigma\sqrt{\Delta}$$

for small $\Delta \ll 1 \quad \Longrightarrow$

$$\sigma_{\Delta} = \frac{1}{\sqrt{\Delta}}\sqrt{-u\,d} \approx \sigma$$

- stochastic process

$$Y_t^{\Delta} = \hat{S}_{n\Delta}$$

for $t \in [n\Delta, (n+1)\Delta)$, $n \in \{0, 1, \ldots\}$

$$Y_{n\Delta+\Delta}^{\Delta} = Y_{n\Delta}^{\Delta} + Y_{n\Delta}^{\Delta}\, \eta_n$$

independent random variable η_n

$$P(\eta_n = u) = p = \frac{-d}{u-d}$$

$$P(\eta_n = d) = 1 - p = \frac{u}{u-d}$$

• rewrite approximately

$$\boldsymbol{Y^{\Delta}_{n\Delta+\Delta} \approx Y^{\Delta}_{n\Delta} + Y^{\Delta}_{n\Delta}\, \sigma \sqrt{\Delta}\, \xi_n}$$

independent random variable $\boldsymbol{\xi_n}$

$$\boldsymbol{P(\xi_n = 1) = p = \frac{1 - \exp\{-\sigma \sqrt{\Delta}\}}{\exp\{\sigma \sqrt{\Delta}\} - \exp\{-\sigma \sqrt{\Delta}\}}}$$

$$\boldsymbol{P(\xi_n = -1) = 1 - p}$$

- by

$$E\left(Y^{\Delta}_{n\Delta+\Delta} - Y^{\Delta}_{n\Delta} \,\middle|\, \mathcal{A}_{n\Delta}\right) = 0$$

$$\begin{aligned} E\left(\left(Y^{\Delta}_{n\Delta+\Delta} - Y^{\Delta}_{n\Delta}\right)^2 \,\middle|\, \mathcal{A}_{n\Delta}\right) &= \left(Y^{\Delta}_{n\Delta}\right)^2 \sigma^2_{\Delta}\, \Delta \\ &\approx \left(Y^{\Delta}_{n\Delta}\right)^2 \sigma^2\, \Delta \end{aligned}$$

Y^{Δ} **converges in a weak sense** to

$\hat{X} = \{\hat{X}_t,\ t \in [0, T]\}$ as $\Delta \to 0$, where

$$d\hat{X}_t = \hat{X}_t\, \sigma\, dW_t$$

$\Longrightarrow$

- for any suitable payoff function g

$$\lim_{\Delta \to 0} E\left(g\left(Y^{\Delta}_{n_T \Delta}\right) \,\Big|\, \mathcal{A}_0\right) = E\left(g(\hat{X}_T) \,\big|\, \mathcal{A}_0\right)$$

$\Longrightarrow$

binomial option pricing formula

approximates for $\Delta \to 0$

Black-Scholes pricing formula

$$S_0^{(\delta_{H_T})} = S_0\, N(\hat{d}_1) - \hat{K}\, N(\hat{d}_2)$$

with

$$\hat{d}_1 = \frac{\ln(\frac{S_0}{\hat{K}}) + \frac{1}{2}\,\sigma^2\, T}{\sigma\,\sqrt{T}}$$

and

$$\hat{d}_2 = \hat{d}_1 - \sigma\,\sqrt{T}$$

Numerical Effects of Tree Methods

Option Prices from Binomial Trees

odd and even time steps

number at each time $t = n\Delta$

nodes starting with $j = 1$ from below

- **binomial probability**

$$p_j(n) = \frac{n\,!}{j\,!\,(n-j)\,!}\, p^j\,(1-p)^{n-j}$$

for $n \in \{1, 2, \ldots\}$ and $j \in \{1, 2, \ldots, n\}$ with $p \in (0, 1)$

- **European call or put option**

  setting at maturity the option value equal to the European payoff

  going one step back in time

  obtain at each node the option value

  expectation of the terminal payoff

  using one step transition probabilities

at next step interpret obtained option price as new payoff

further step back in time

one step expectation

backward through the tree

at root of the tree

price of the option

- **difference to Black-Scholes formula**
  using a CRR binomial tree

- **CRR binomial tree**

  $S_0 = 1, K = 1, T = 1, r = 0.05, \sigma = 0.2$

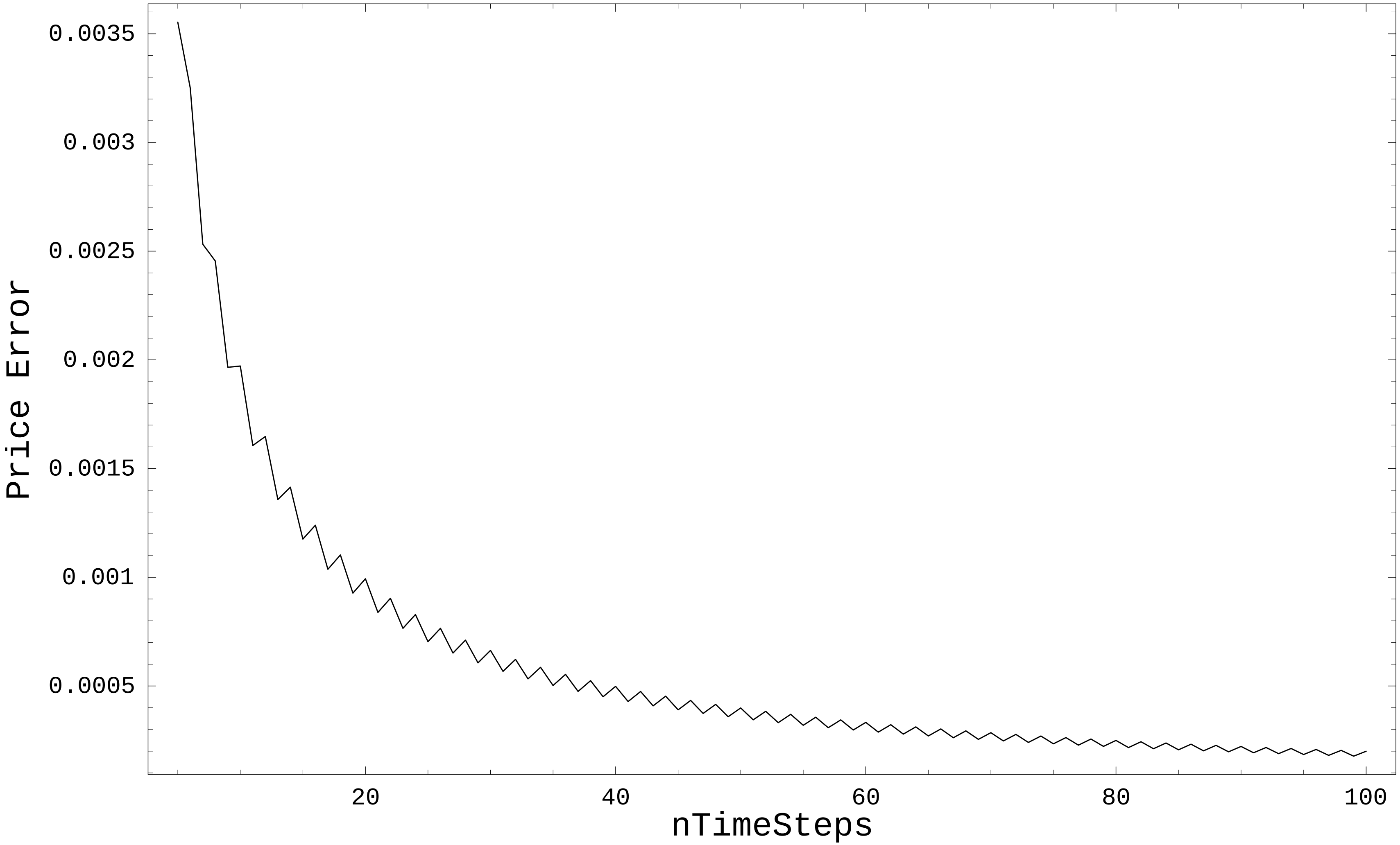

Figure 6.1: Error of at-the-money European call option price for CRR binomial tree in dependence on the number of steps.

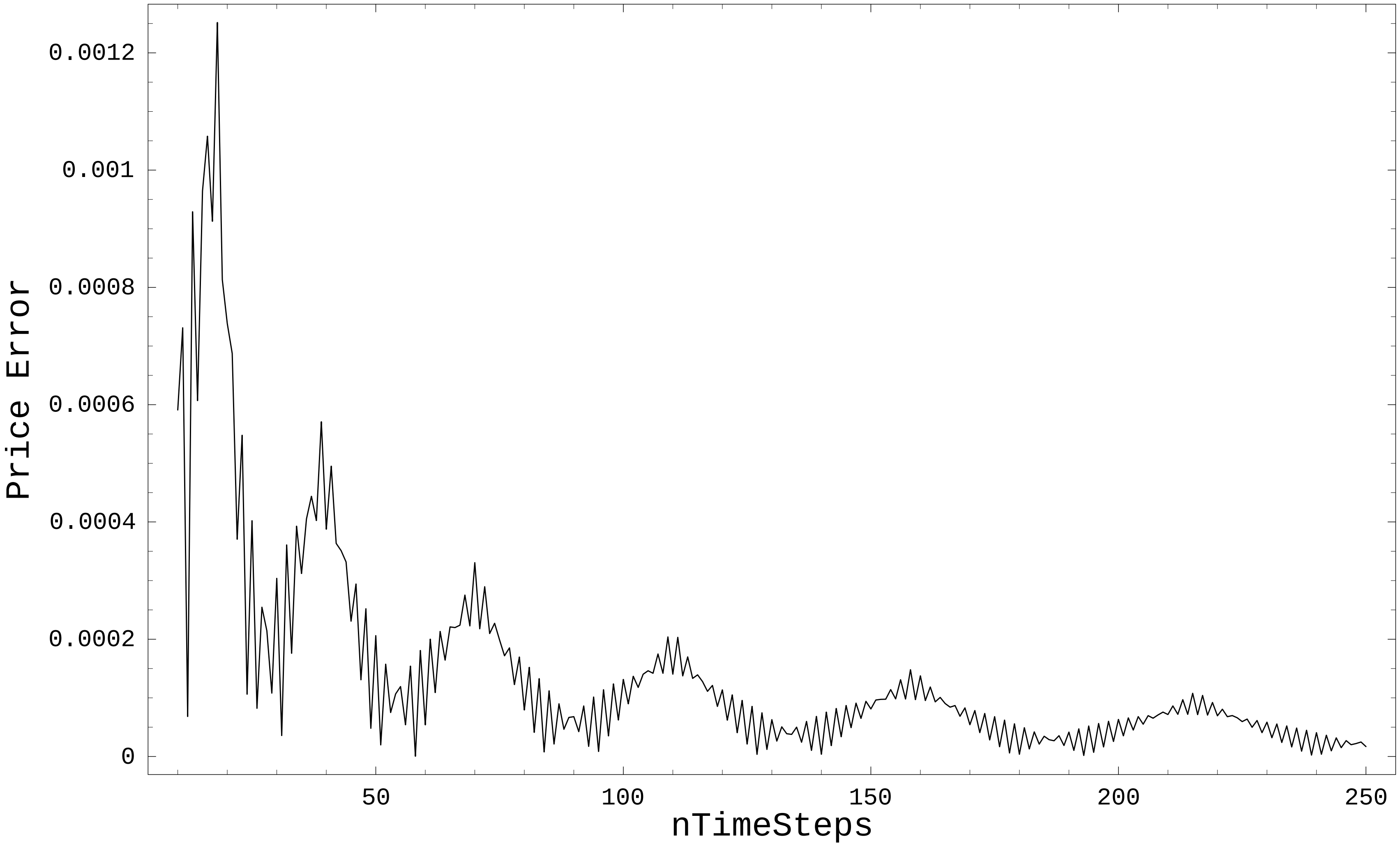

Figure 6.2: Error for out-of-the money CRR binomial European call in dependence on the number of time steps.

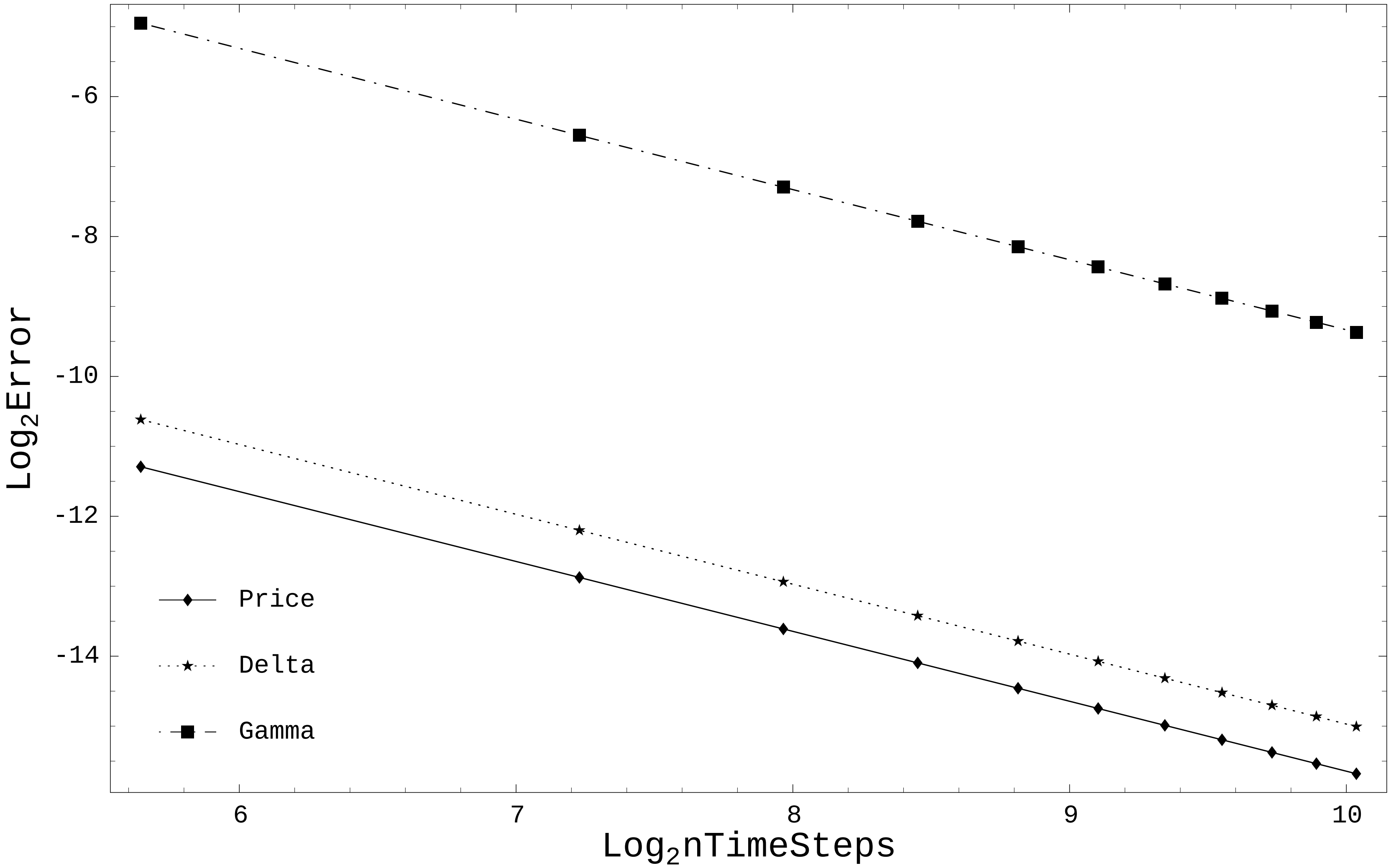

Figure 6.3: Log-log plot of absolute weak error for at-the-money price, delta and gamma of CRR binomial European call.

$$\ln(\mu(\Delta)) \approx -3.91695 - 0.999291 \ln\left(\frac{T}{\Delta}\right)$$

$\beta \approx 1.0$

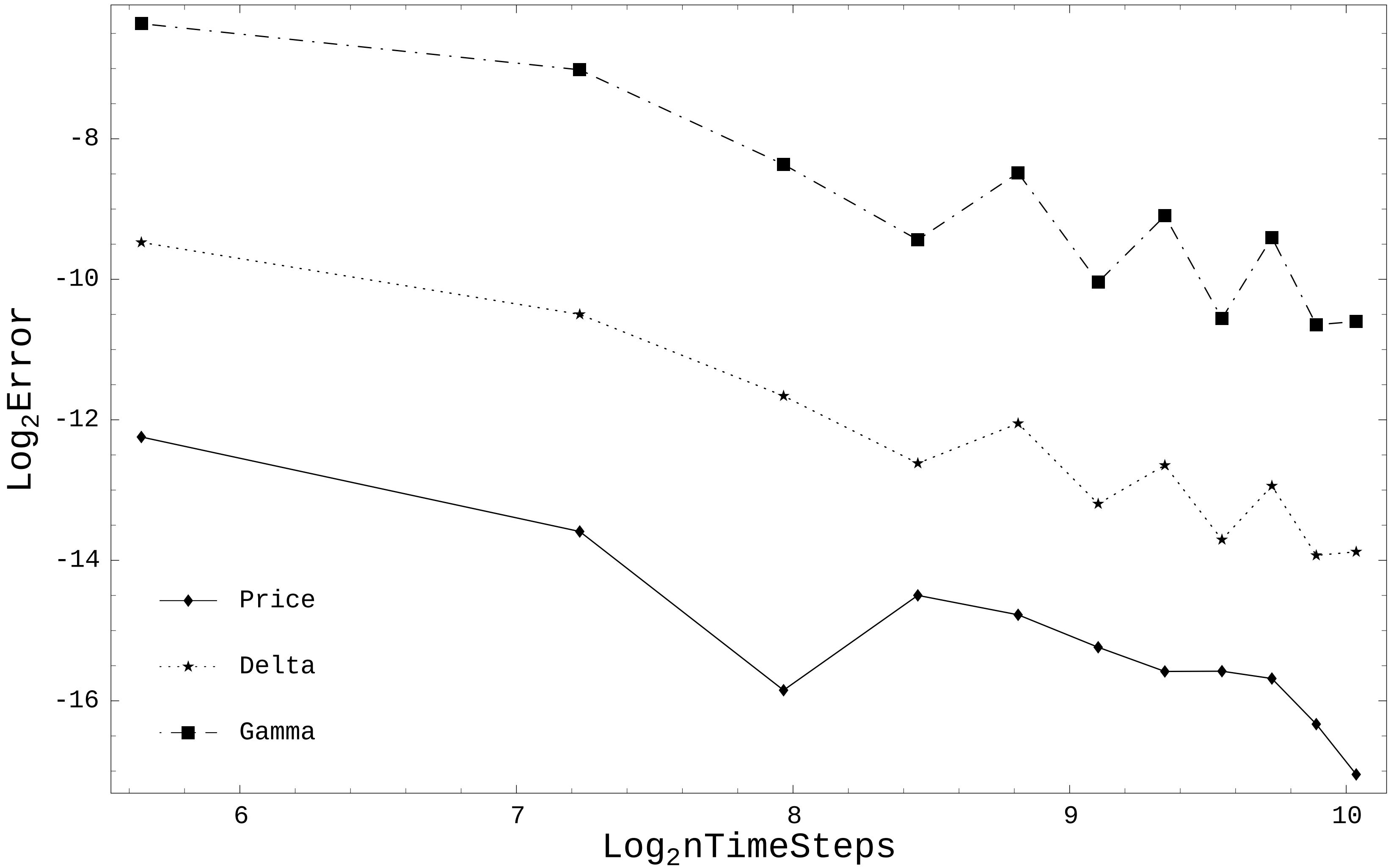

Figure 6.4: Log-log plot of absolute weak error for out-of-the money price, delta and gamma of CRR binomial European call.

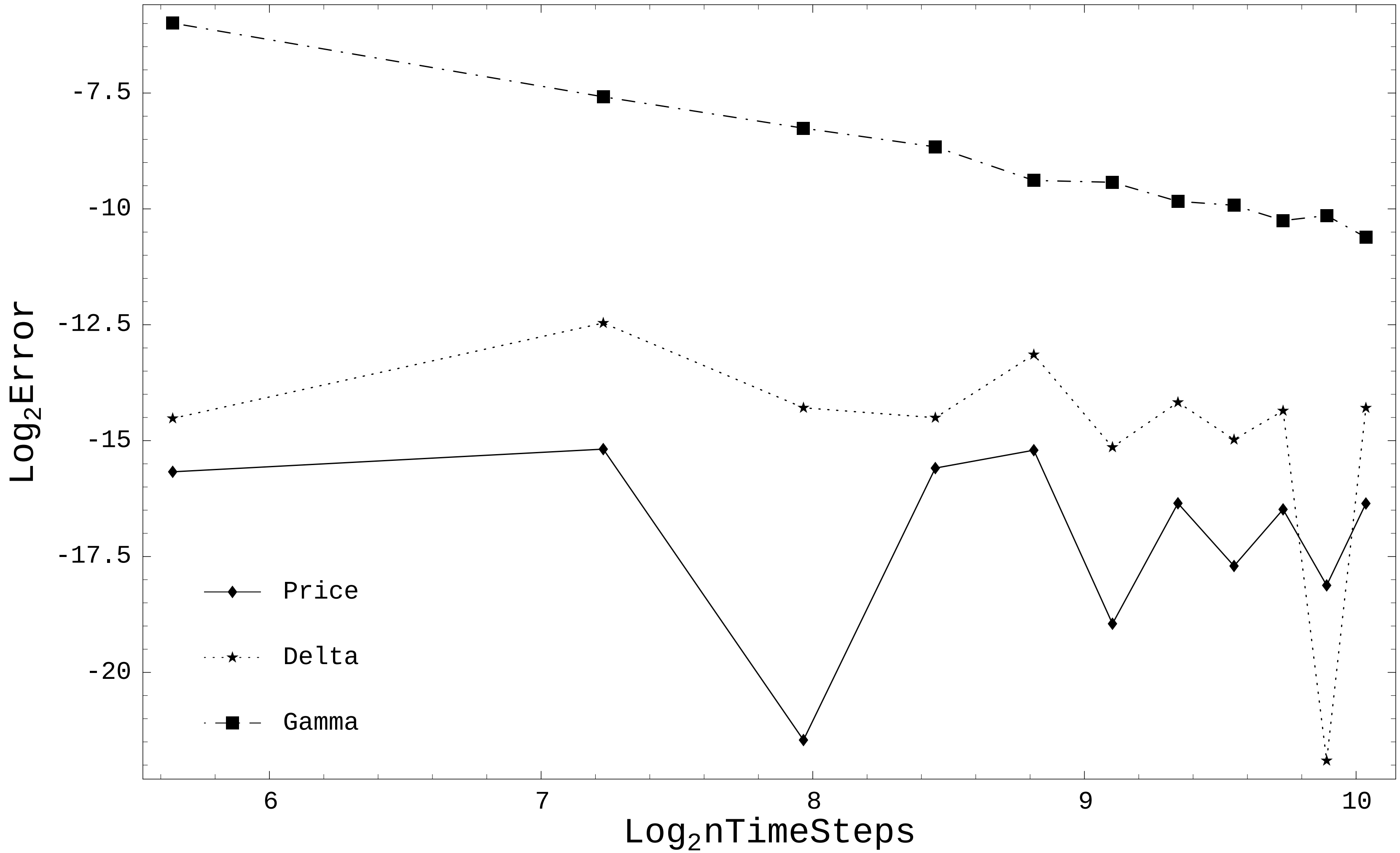

Figure 6.5: Log-log plot of the absolute error for in-the-money price, delta and gamma of CRR binomial European call.

Binomial American Pricing

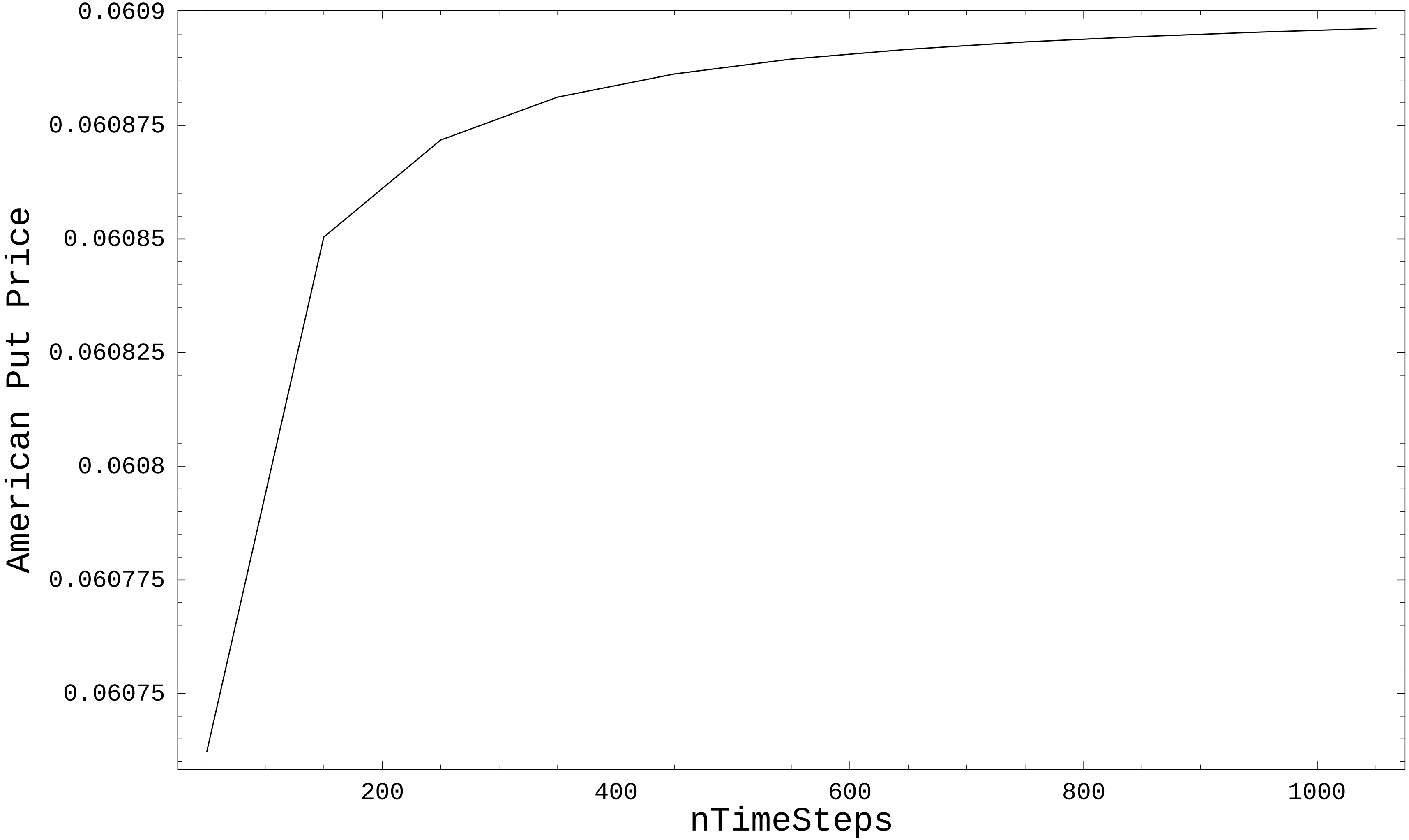

Figure 6.6: CRR binomial American put prices.

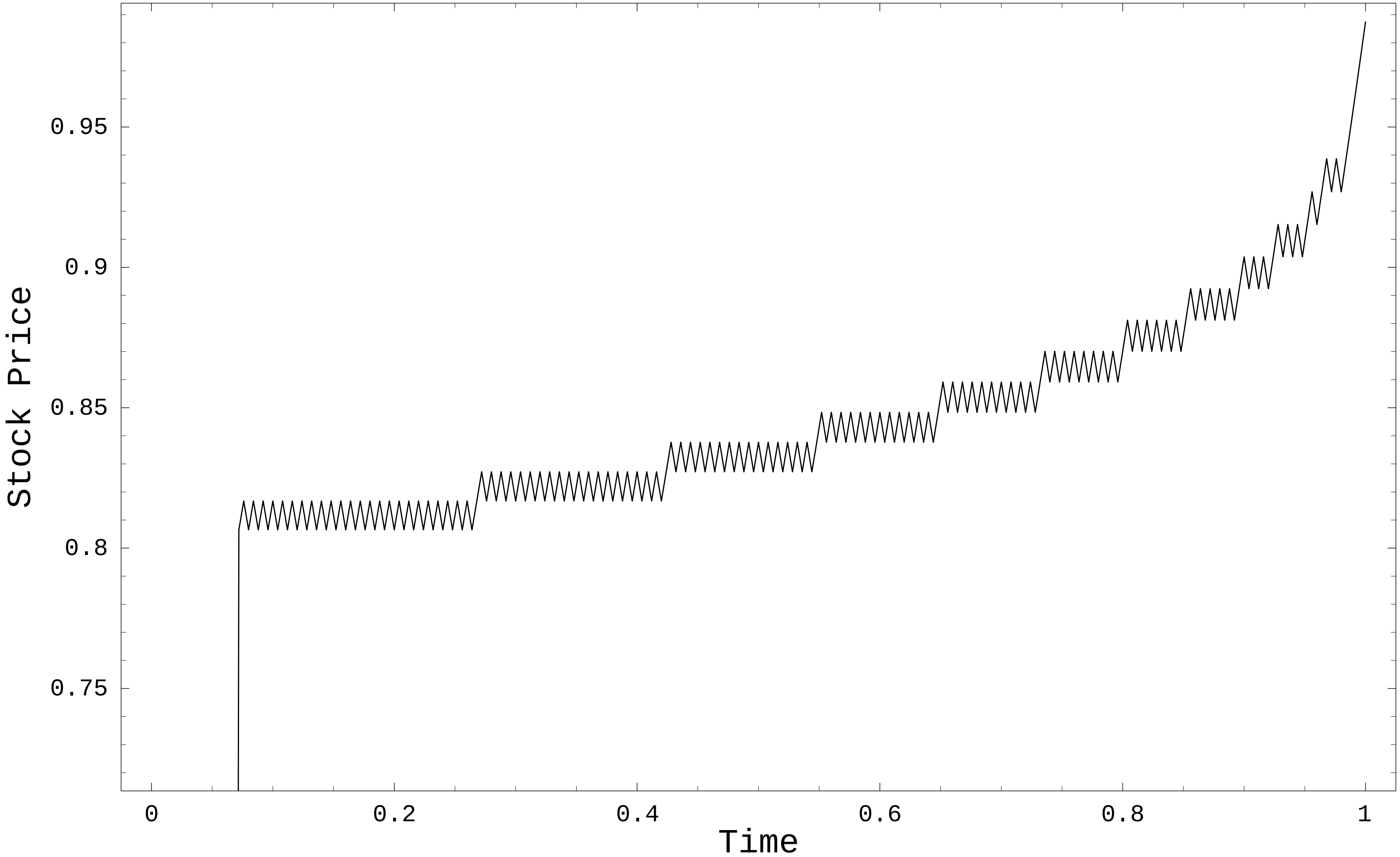

Figure 6.7: Early exercise boundary for American put.

Pricing on a Trinomial Tree

- **next time step**

$S(d+1), Sm$ or $S(u+1), \quad d+1 < m < u+1$

$$
\begin{aligned}
d &= e^{-\lambda\,\sigma\sqrt{\Delta}} - 1 \\
m &= 1 \\
u &= e^{\lambda\,\sigma\sqrt{\Delta}} - 1
\end{aligned}
$$

λ a free parameter

- **Boyle (1988) trinomial probabilities**

$$p_d = \frac{(W - R)(u+1)^2 - (R-1)(u+1)^3}{u^2 (u+2)}$$

$$p_m = 1 - p_d - p_u$$

$$p_u = \frac{(W - R)(u+1) - (R-1)}{u^2 (u+2)}$$

with

$$W = R^2 e^{\sigma^2 \Delta}$$

and

$$R = e^{r\Delta}$$

for $\lambda = 1$ CRR binomial model

- first and second moment of the BS model are matched

$$p_d\,(d+1) + p_m\,m + p_u\,(u+1) = R$$

and

$$p_d\,(d+1)^2 + p_m\,m^2 + p_u\,(u+1)^2 - R^2 = e^{2r\Delta}\,(e^{\sigma^2\Delta} - 1)$$

- **Kamrad & Ritchken (1991) probabilities**

$$
\begin{aligned}
p_d &= \frac{1}{2\,\lambda^2} - \frac{(r - \frac{\sigma^2}{2})\,\sqrt{\Delta}}{2\,\lambda\,\sigma} \\
p_m &= 1 - \frac{1}{\lambda^2} \\
p_u &= \frac{1}{2\,\lambda^2} + \frac{(r - \frac{\sigma^2}{2})\,\sqrt{\Delta}}{2\,\lambda\,\sigma}
\end{aligned}
$$

- matches the first and second moment of the logarithm

$$p_d \ \ln(d+1) + p_u \ \ln(u+1) = \left(r - \frac{\sigma^2}{2}\right) \Delta$$

$$p_d \, (\ln(d+1))^2 + p_u \, (\ln(u+1))^2 \approx \sigma^2 \Delta + \left(r - \frac{\sigma^2}{2}\right)^2 \Delta^2$$

applying Boyle's trinomial tree with $\lambda = \sqrt{3}$

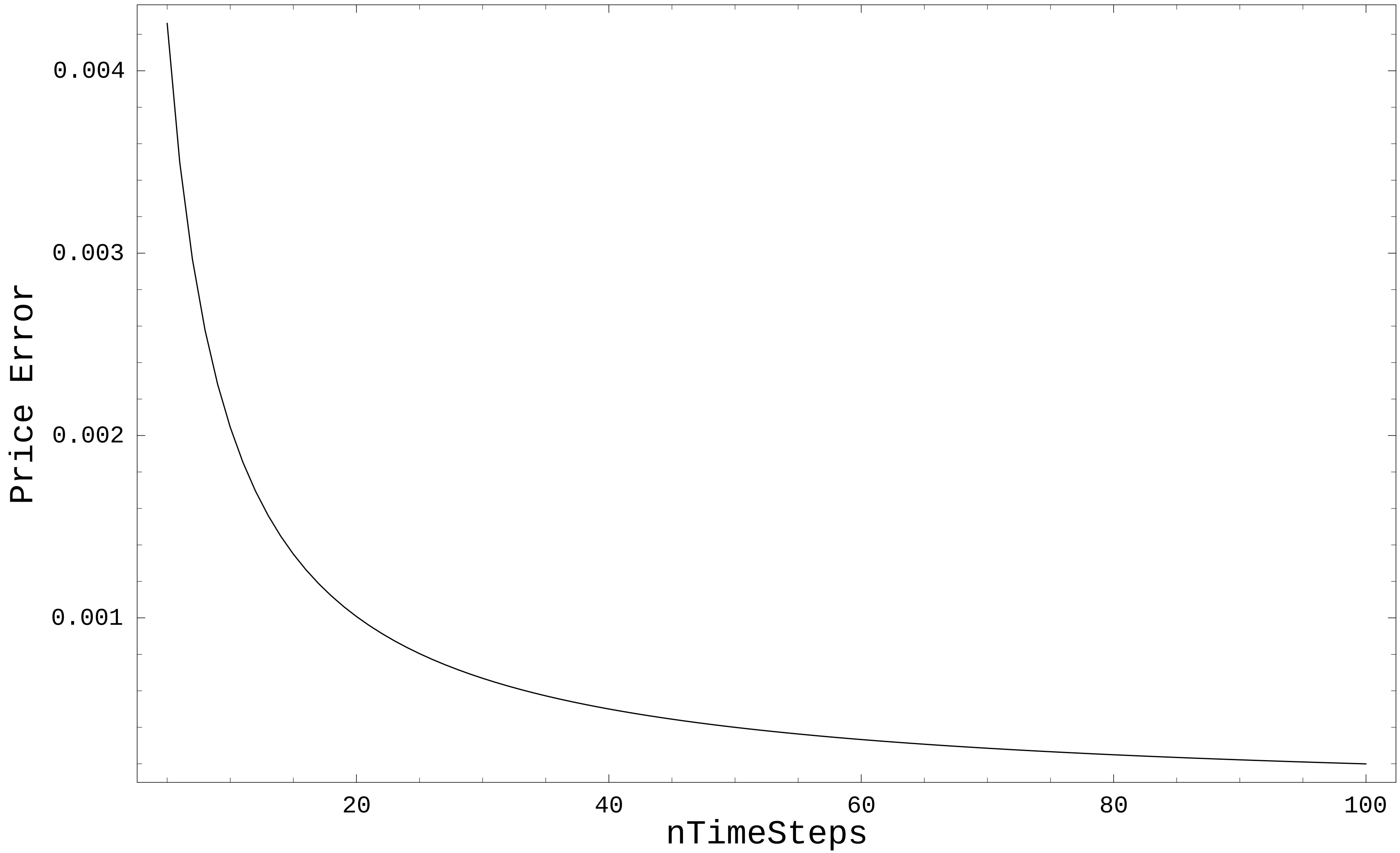

Figure 6.8: Absolute error for Boyle's trinomial at-the-money European call.

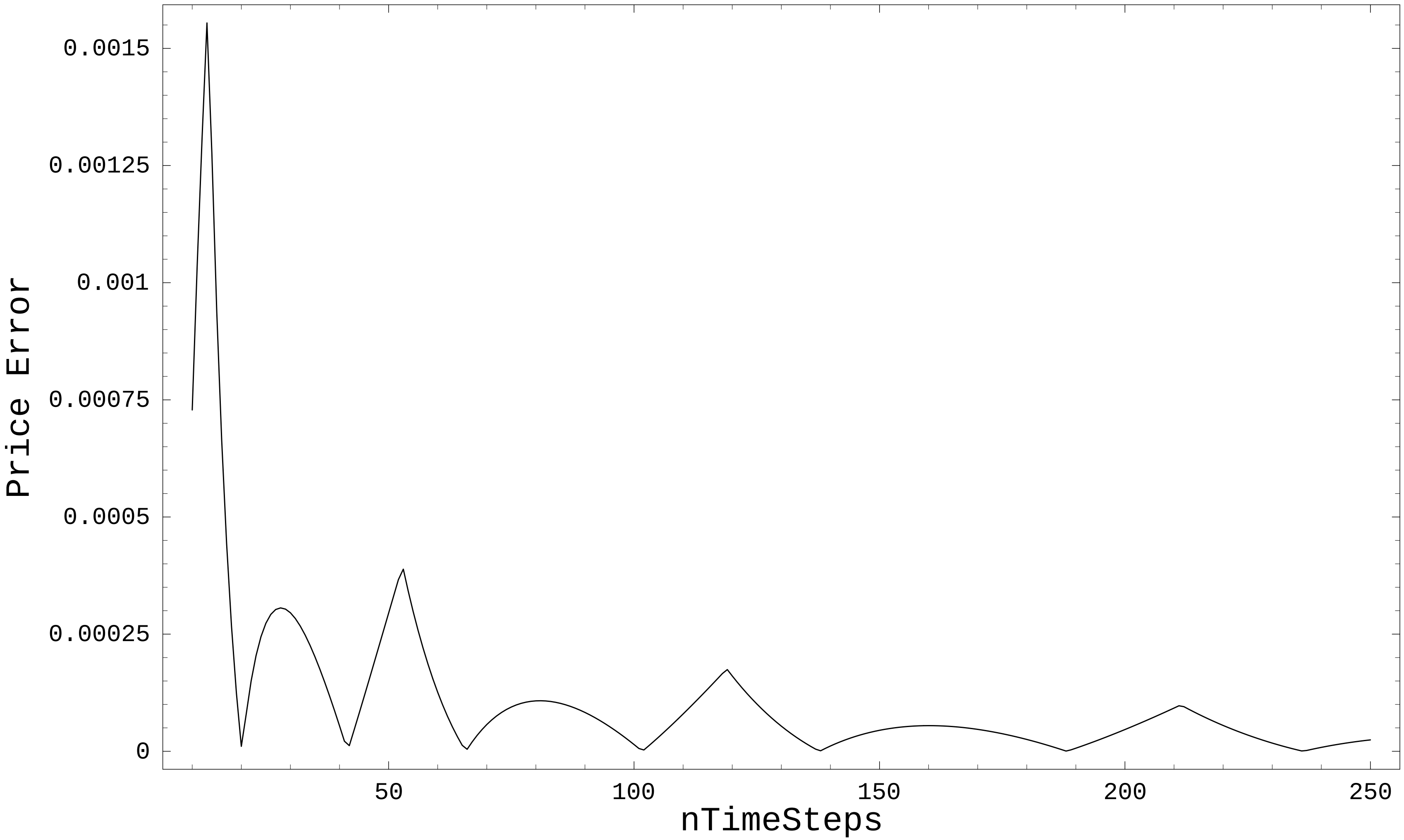

Figure 6.9: Absolute error for out-of-the money Boyle's trinomial European call with $K = 1.1$.

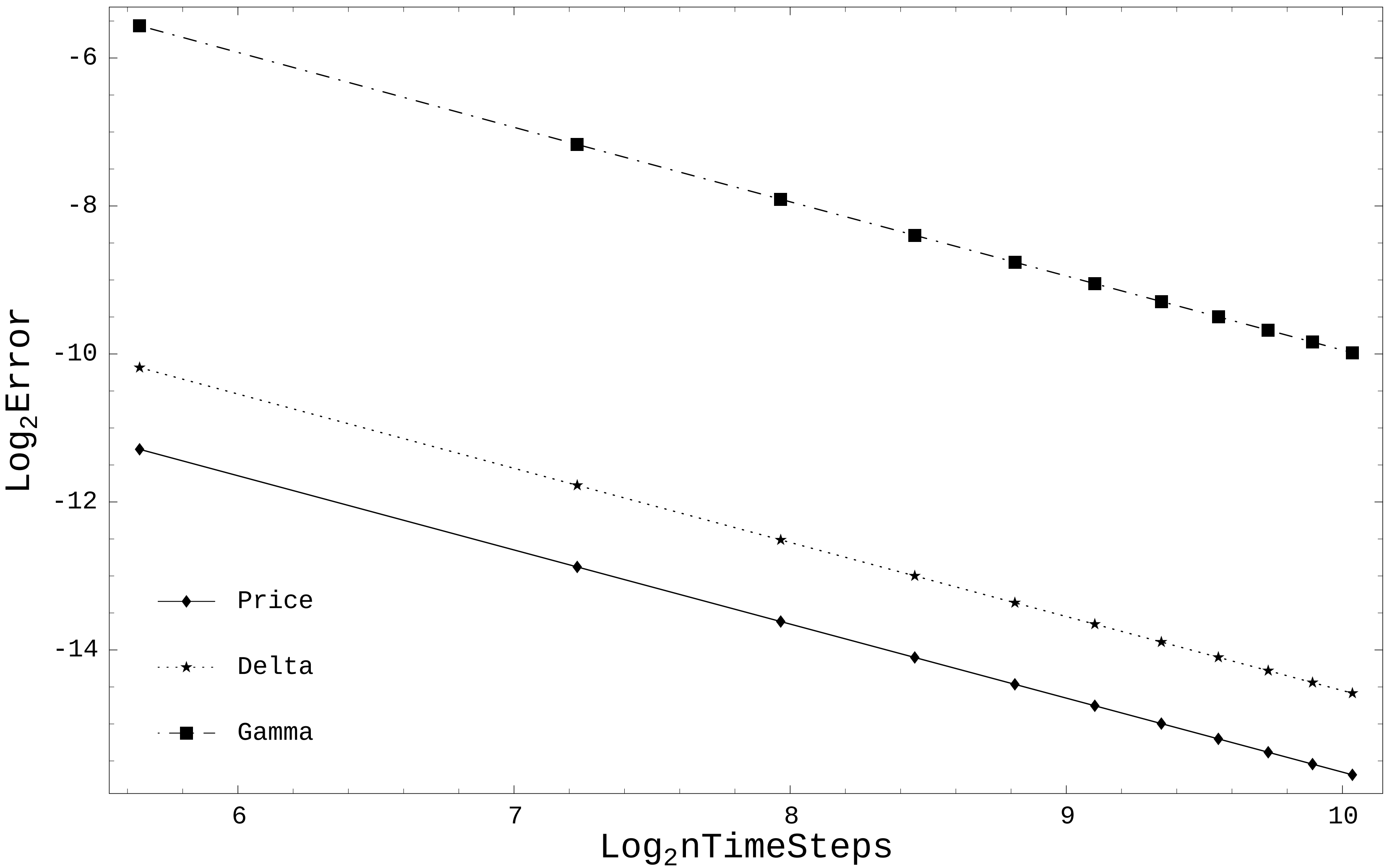

Figure 6.10: Log-log plot of absolute error for at-the-money price, delta and gamma of Boyle's trinomial European call.

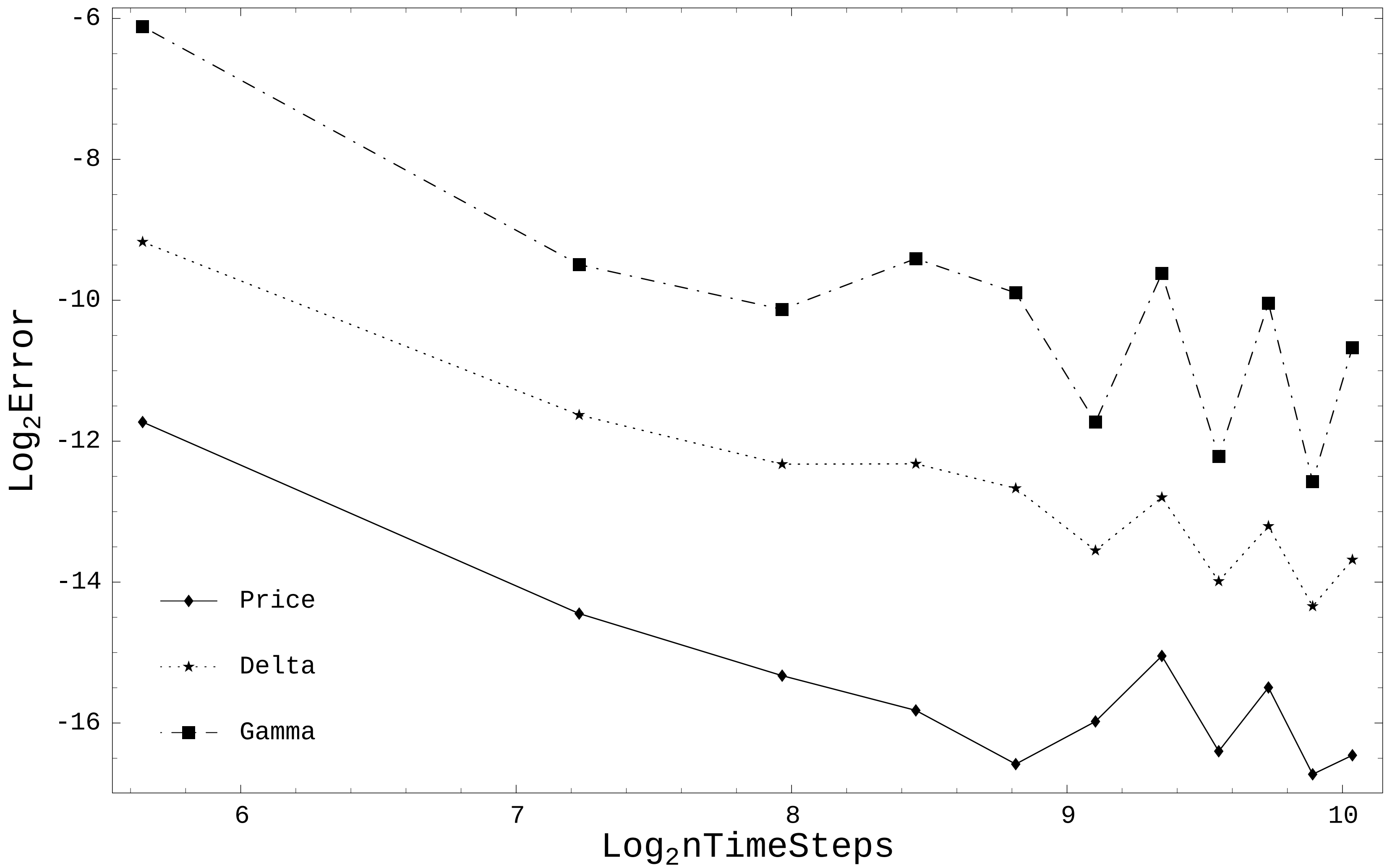

Figure 6.11: Log-log plot of absolute error for out-of-the money price, delta and gamma of Boyle's trinomial European call.

Black-Scholes Modification of a Tree

Broadie & Detemple (1997)

one step before maturity using Black-Scholes formula

$\Longrightarrow$

Black-Scholes modification of a tree

used to price American options

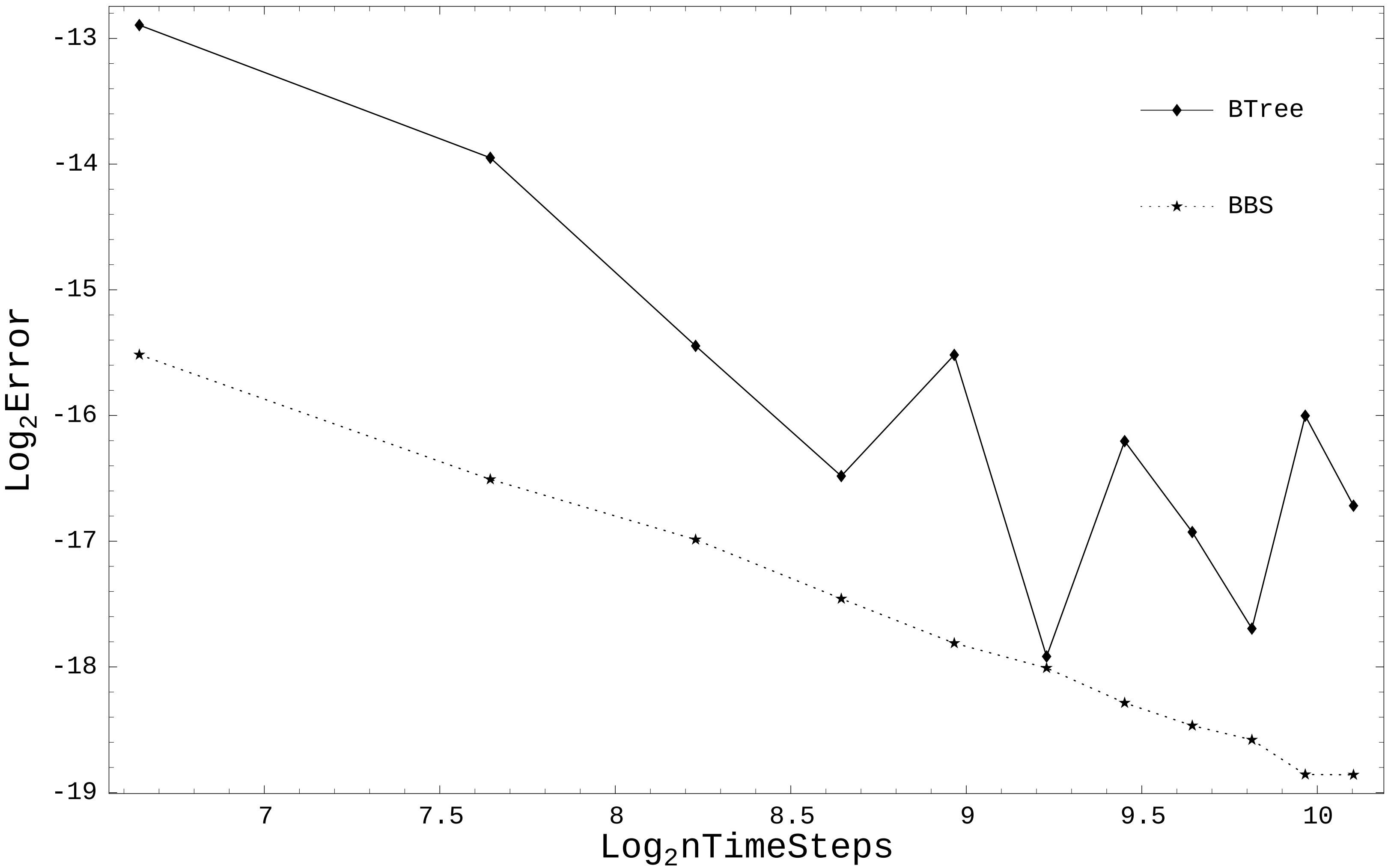

Figure 6.12: Log-log plot of absolute error with and without Black-Scholes modification for CRR binomial out-of-the money European call.

Smooth Payoff

- cubic payoff

$$g(S) = S^3 = \left(S_0 \, \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) T + \sigma \, W_T \right\} \right)^3$$

- theoretical value

$$E(g(S_T)) = (S_0)^3 \, \exp\{(3\,r + 4\,\sigma^2)\,T\}$$

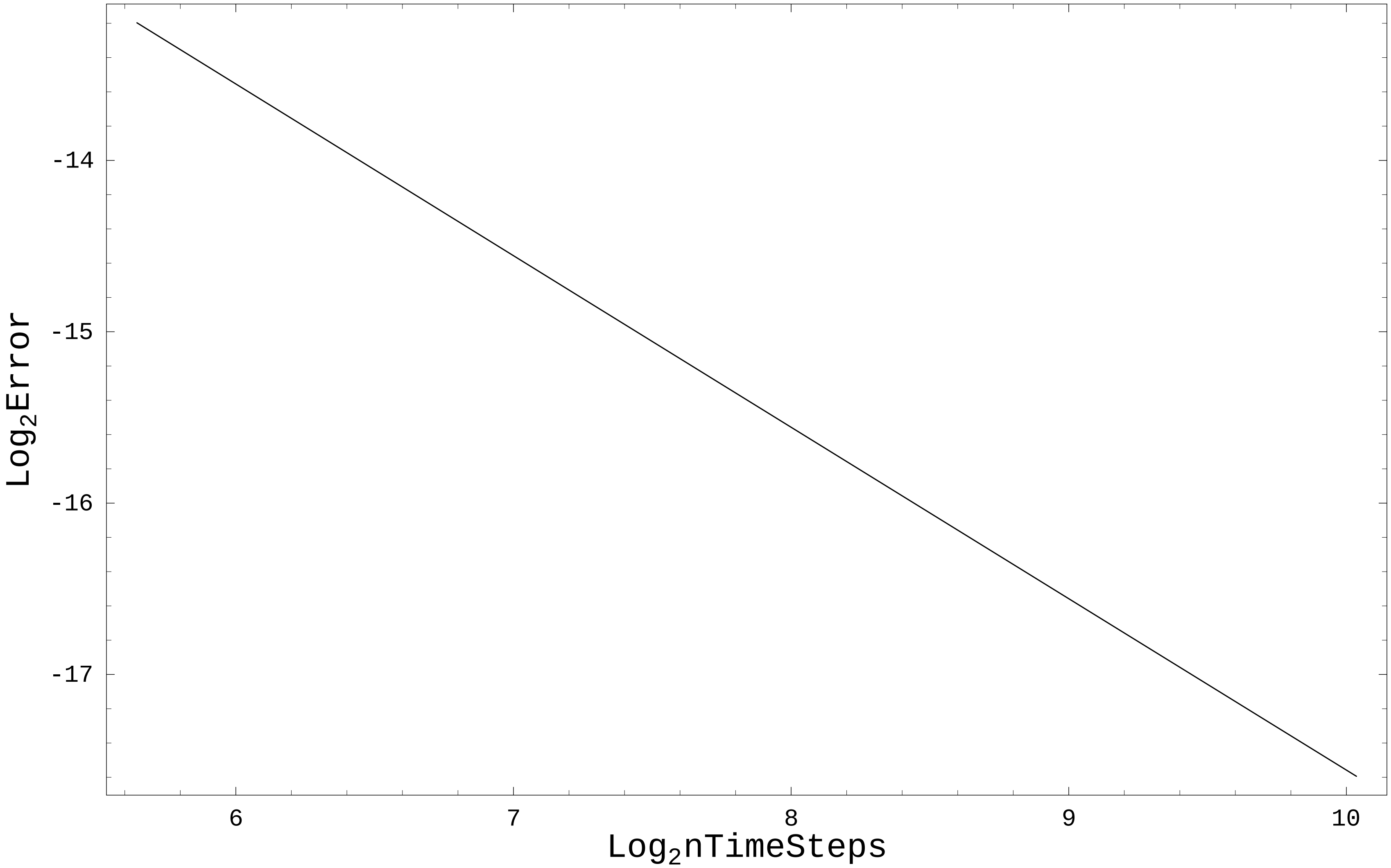

Figure 6.13: Log-log plot for Boyle's trinomial tree with smooth payoff.

- **Heston-Zhou trinomial tree**

$$
\begin{aligned}
m &= \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\Delta\right\} \\
d &= m \exp\left\{-\sigma\sqrt{3\,\Delta}\right\} - 1 \\
u &= m \exp\left\{\sigma\sqrt{3\,\Delta}\right\} - 1
\end{aligned}
$$

with probabilities

$$
p_d = p_u = \frac{1}{6}
$$

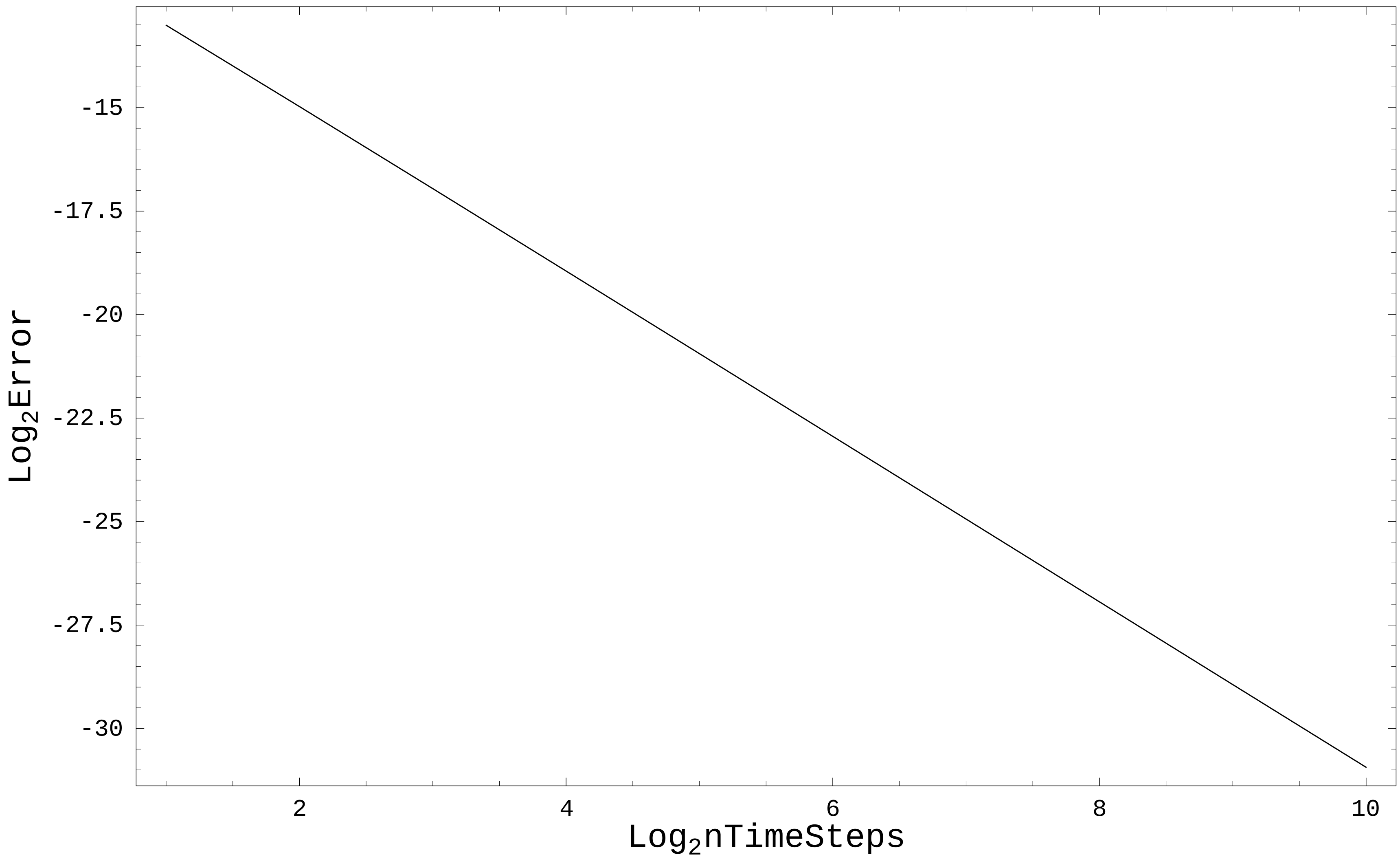

Figure 6.14: Log-log plot for Heston-Zhou trinomial tree with smooth pay-off.

- **calculating European calls**

  nonsmooth payoff experimentally only $\quad \boldsymbol{\beta \approx 1.0}$

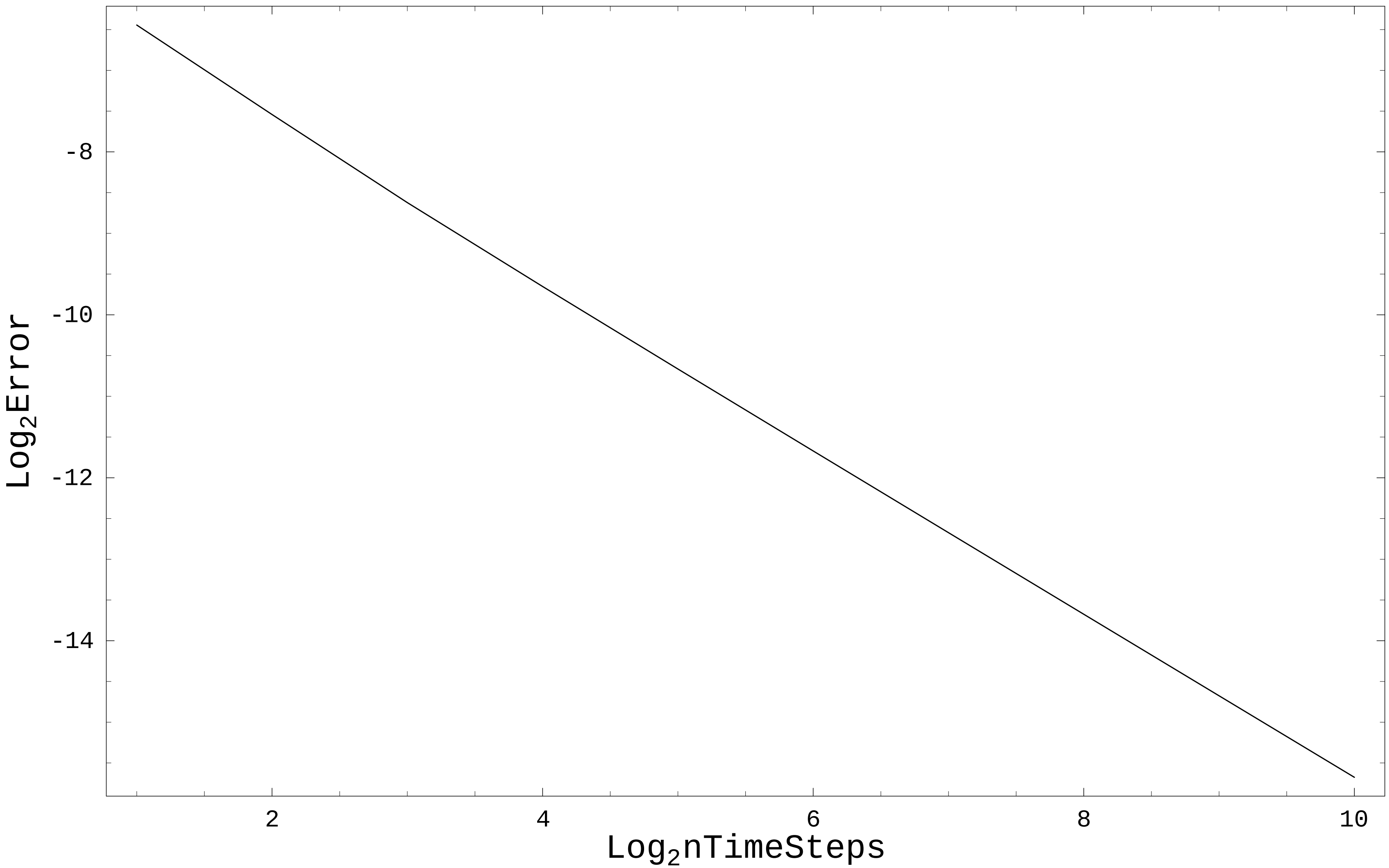

Figure 6.15: Log-log plot for Heston-Zhou trinomial tree for European call.

Extrapolation

- **Richardson extrapolation**

  Kloeden & Pl. (1992b)

  Glasserman (2004)

$$V_{g,2}^{\Delta}(T) = 2\,E\left(g\left(Y_T^{\Delta}\right)\right) - E\left(g\left(Y_T^{2\Delta}\right)\right)$$

  if the leading error coefficients are appropriate

  CRR binomial tree method

  first weak order experimentally

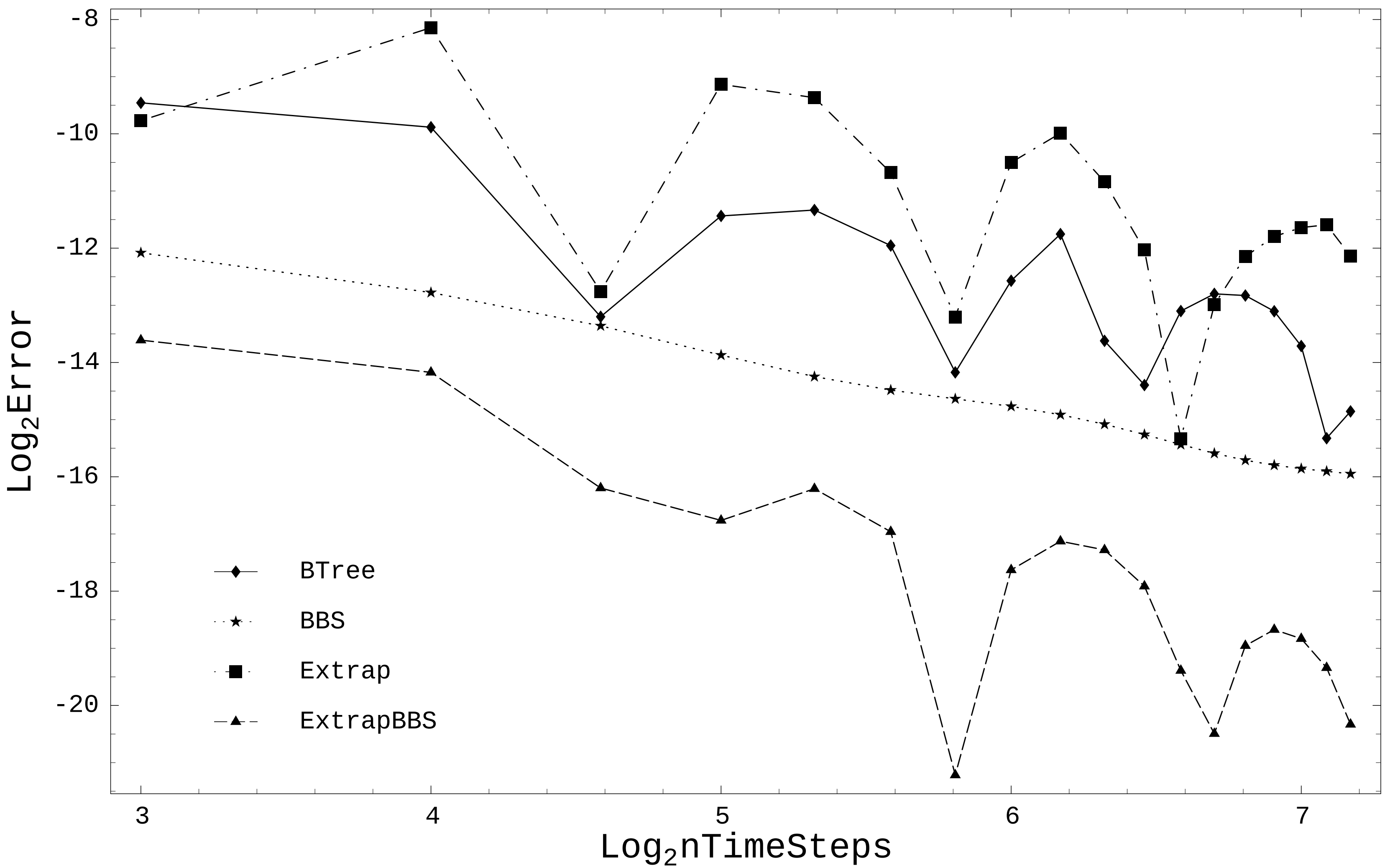

Figure 6.16: Log-log plot for Richardson extrapolation with out-of-the money binomial call payoff and Black-Scholes modification.

- **smooth payoff**

$$g(S) = S^3$$

CRR binomial tree

Richardson extrapolation

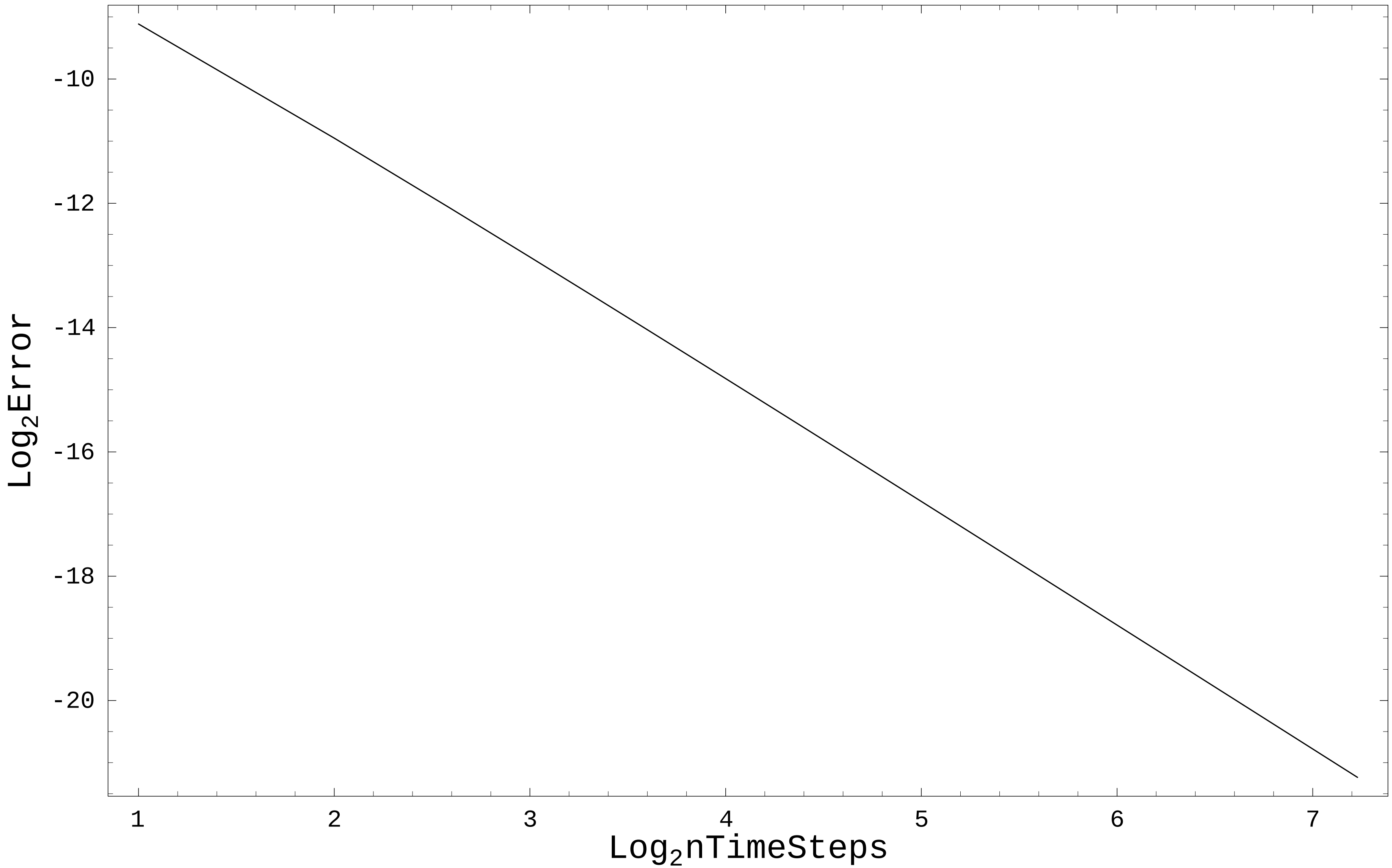

Figure 6.17: Log-log plot for Richardson extrapolation with CRR binomial tree for smooth payoff.

- **extrapolation method**

  Heston & Zhou (2000)

$$V_{g,4}^{\Delta}(T) = \frac{1}{21}\Big(32\, E\left(g\left(Y_T^{\Delta}\right)\right) - 12\, E\left(g\left(Y_T^{2\Delta}\right)\right) + E\left(g\left(Y_T^{4\Delta}\right)\right)\Big)$$

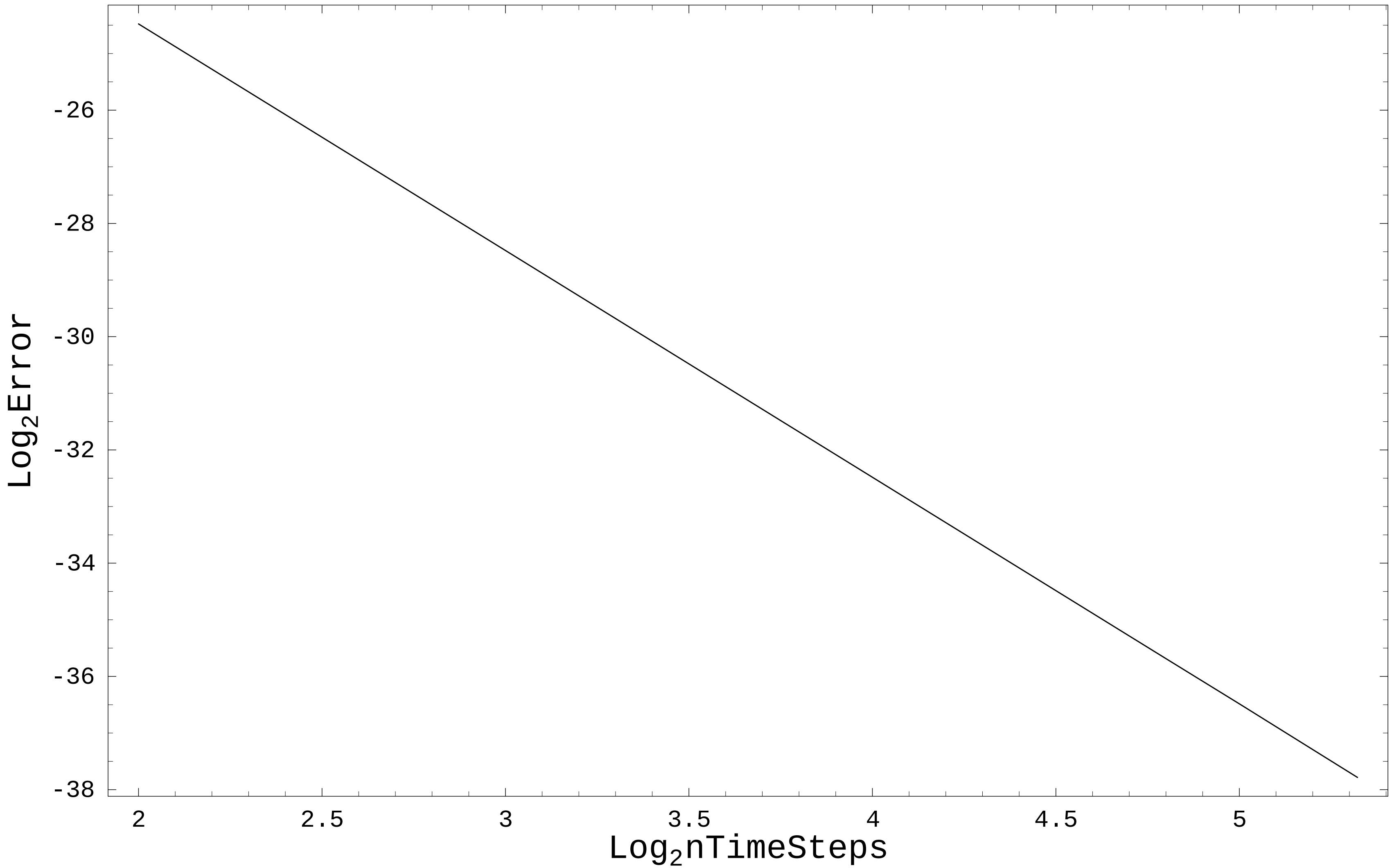

Figure 6.18: Log-log plot for the absolute error of a fourth order extrapolation using the second order Heston-Zhou trinomial tree for smooth payoff.

- **trinomial extrapolation for European call prices**

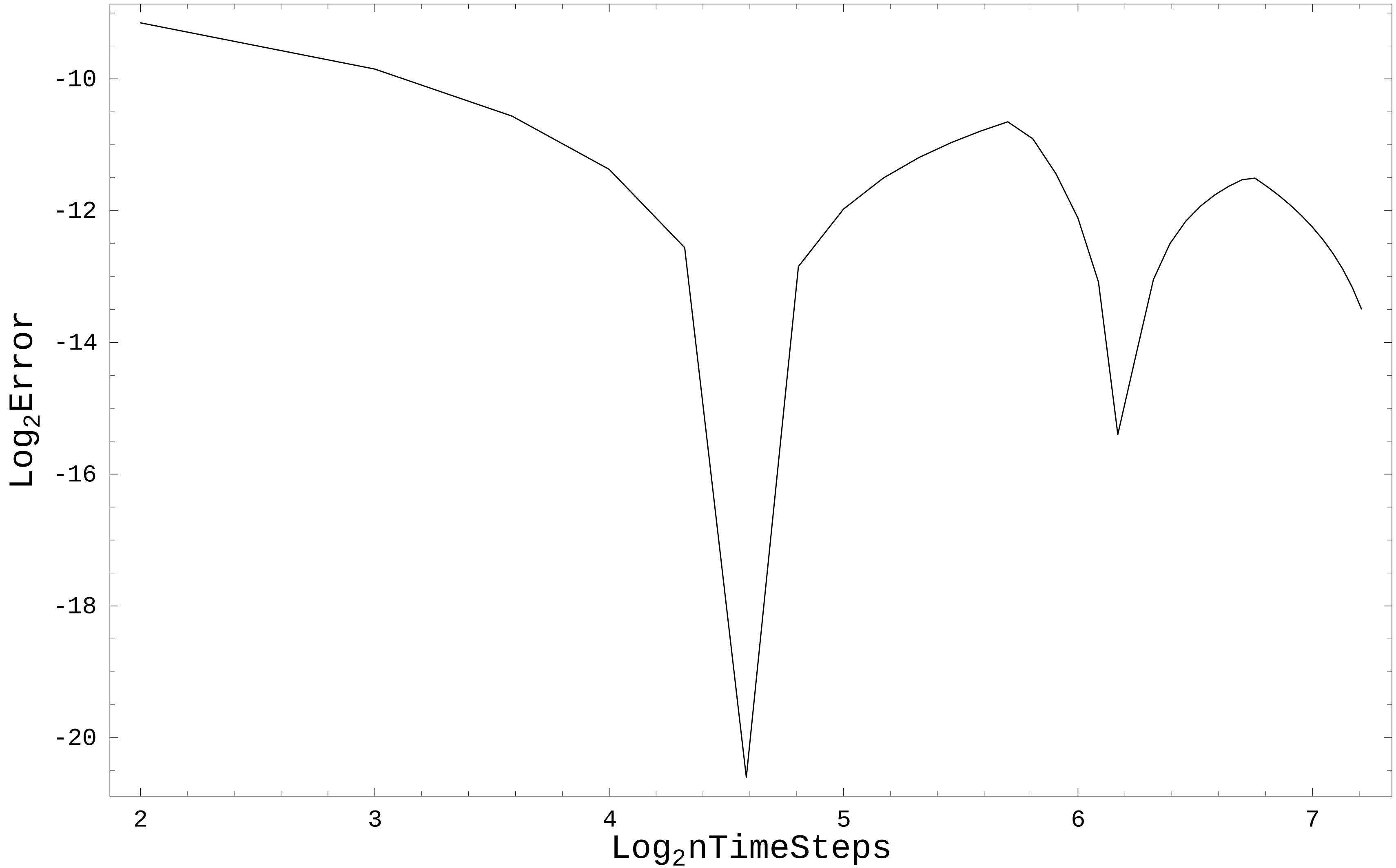

Figure 6.19: Log-log plot for a fourth order extrapolation for out-of-the money Heston-Zhou trinomial European calls.

Efficiency of Simplified Schemes

Bruti-Liberati & Pl. (2004)

random bits generators

Weak Taylor Schemes and Simplified Methods

- SDE

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t$$

- payoff function $g(X_T)$

- equidistant time discretization $\tau_n = n\Delta,\ \Delta = \frac{T}{N}$

- **simplified weak Euler scheme**

$$Y_{n+1} = Y_n + a(\tau_n, Y_n)\,\Delta + b(\tau_n, Y_n)\,\Delta\hat{W}_n$$

$$P(\Delta\hat{W}_n = \pm\sqrt{\Delta}) = \frac{1}{2}$$

first three moments of ΔW_n match

$$\begin{aligned} E(\Delta W_n) &= E(\Delta\hat{W}_n) = 0 \\ E((\Delta W_n)^2) &= E((\Delta\hat{W}_n)^2) = \Delta \\ E((\Delta W_n)^3) &= E((\Delta\hat{W}_n)^3) = 0 \end{aligned}$$

- **weak order 2.0 Taylor scheme**

$$
\begin{aligned}
Y_{n+1} \quad = \quad & Y_n + a\,\Delta + b\,\Delta W_n + \frac{1}{2}\,b'\,b\left\{\left(\Delta W_n^2\right) - \Delta\right\} \\
& + \frac{1}{2}\left(a\,a' + \frac{1}{2}\,a''b^2\right)\Delta^2 \\
& + a'\,b\,\Delta Z_n + \left(a\,b' + \frac{1}{2}\,b''b^2\right)\left\{\Delta W_n\,\Delta - \Delta Z_n\right\}
\end{aligned}
$$

$$
\Delta Z_n = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{z} dW_z\,ds
$$

- **simplified weak order 2.0 Taylor scheme**

$$\begin{aligned} Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta\hat{W}_n + \frac{1}{2}\,b\,b'\left\{\left(\Delta\hat{W}_n\right)^2 - \Delta\right\} \\ &\quad + \frac{1}{2}\left(a\,a' + \frac{1}{2}\,a''b^2\right)\Delta^2 \\ &\quad + \frac{1}{2}\left(a'\,b + a\,b' + \frac{1}{2}\,b''b^2\right)\Delta\hat{W}_n\,\Delta \end{aligned}$$

$$P\left(\Delta\hat{W}_n = \pm\sqrt{3\Delta}\right) = \frac{1}{6}, \qquad P\left(\Delta\hat{W}_n = 0\right) = \frac{2}{3}$$

first five moments are matched

$\beta = 3$ or 4

Kloeden & Pl. (1992b) and Hofmann (1994)

- **numerical stability**

- **fully implicit weak Euler scheme**

$$
\begin{aligned}
Y_{n+1} \quad = \quad & Y_n + \Big\{ a\,(\tau_{n+1}, Y_{n+1}) - b\,(\tau_{n+1}, Y_{n+1}) \\
& \qquad \times \frac{\partial}{\partial y}\, b\,(\tau_{n+1}, Y_{n+1}) \Big\}\, \Delta \\
& + b\,(\tau_{n+1}, Y_{n+1})\, \Delta W_n
\end{aligned}
$$

Makes for certain diffusion coefficients no sense !

- **simplified fully implicit Euler scheme**

$$
\begin{aligned}
Y_{n+1} \quad = \quad & Y_n + \Big\{ a\left(\tau_{n+1}, Y_{n+1}\right) - b\left(\tau_{n+1}, Y_{n+1}\right) \\
& \qquad \times \frac{\partial}{\partial y} b\left(\tau_{n+1}, Y_{n+1}\right) \Big\} \Delta \\
& + b\left(\tau_{n+1}, Y_{n+1}\right) \Delta \hat{W}_n
\end{aligned}
$$

$\beta = 1.0$

Random Bits Generators

- **polar Marsaglia-Bray method**

  coupled with linear congruential random number generator

  Press, Teukolsky, Vetterling & Flannery (2002)

  pair of independent standard Gaussian random variables

- **random bits generator**

  Bruti-Liberati & Pl. (2004)

  generates a single bit 0 or 1 with probability 0.5

  theory of primitive polynomials modulo 2

  every primitive polynomial modulo 2 of order n defines a recurrence relation for obtaining a new bit from the n preceding ones with maximal length

Numerical Results

- SDE

$$dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dW_t$$

$$X_T = X_0\, \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma\, W_T\right\}$$

A Smooth Payoff Function

- simplified methods achieve almost exactly the same accuracy of their Taylor counterparts.

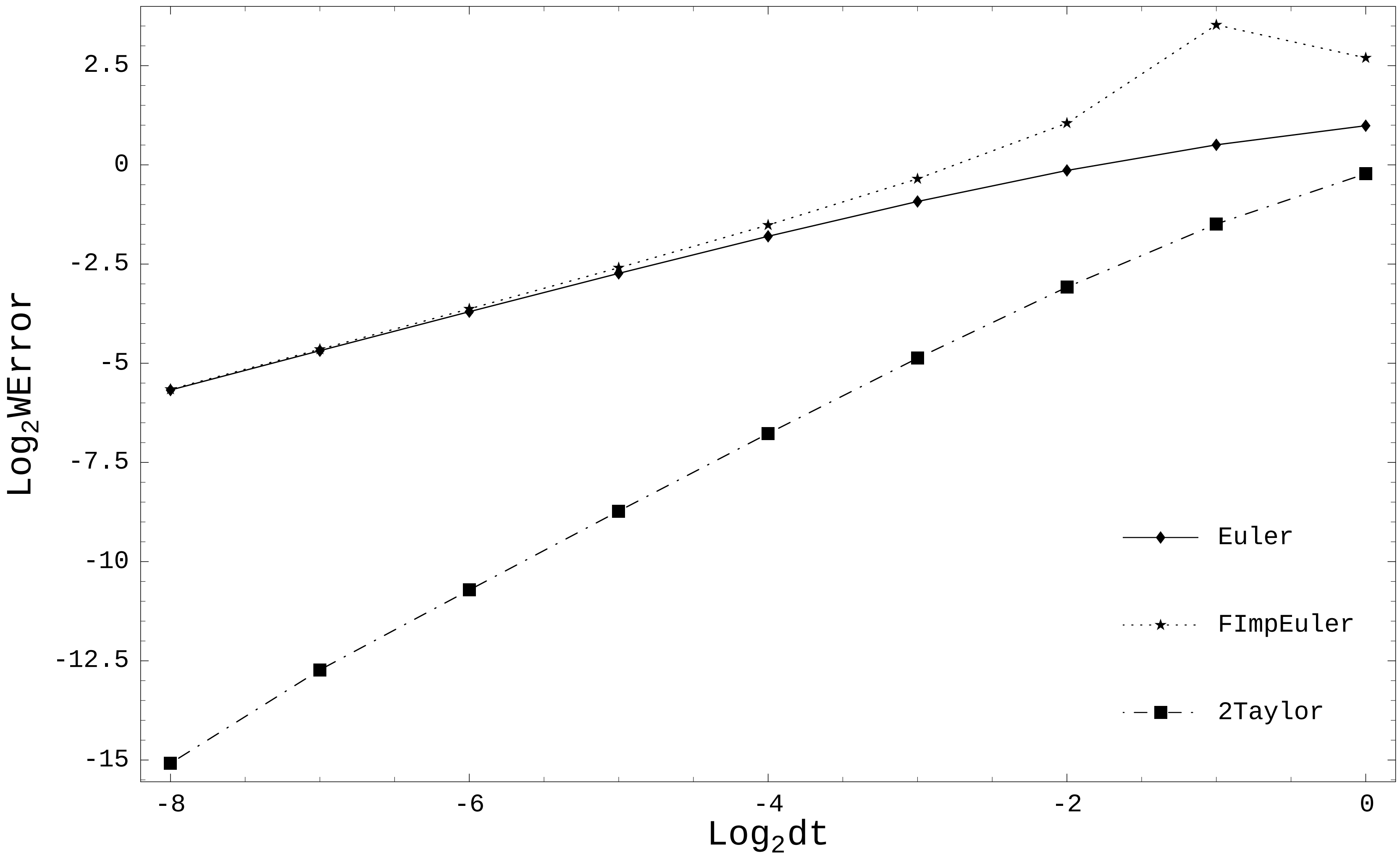

Figure 6.20: Log-log plot of the weak error for the Euler, fully implicit Euler and order **2.0** weak Taylor schemes.

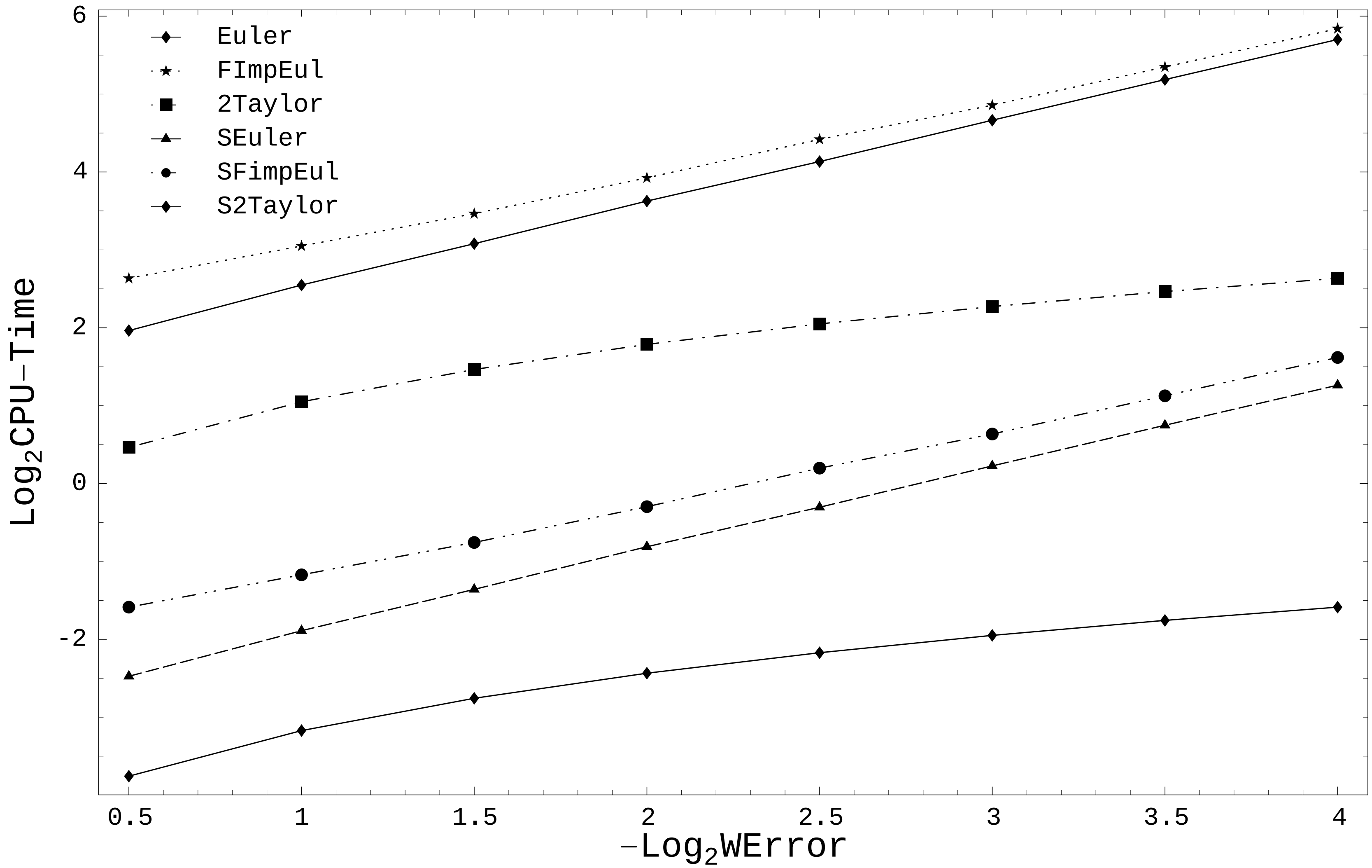

Figure 6.21: Log-log plot of CPU time versus the weak error.

An Option Payoff

- **piecewise differentiable payoff**

$$(X_T - K)^+ = \max(X_T - K, 0)$$

Black-Scholes formula

$$E_\theta(e^{-rT}(X_T - K)^+) = X_0\, N(d_1) - e^{-rT} K\, N(d_2)$$

where

$$d_1 = \frac{\ln(\frac{X_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

$\sigma = 0.2, r = 0.1$

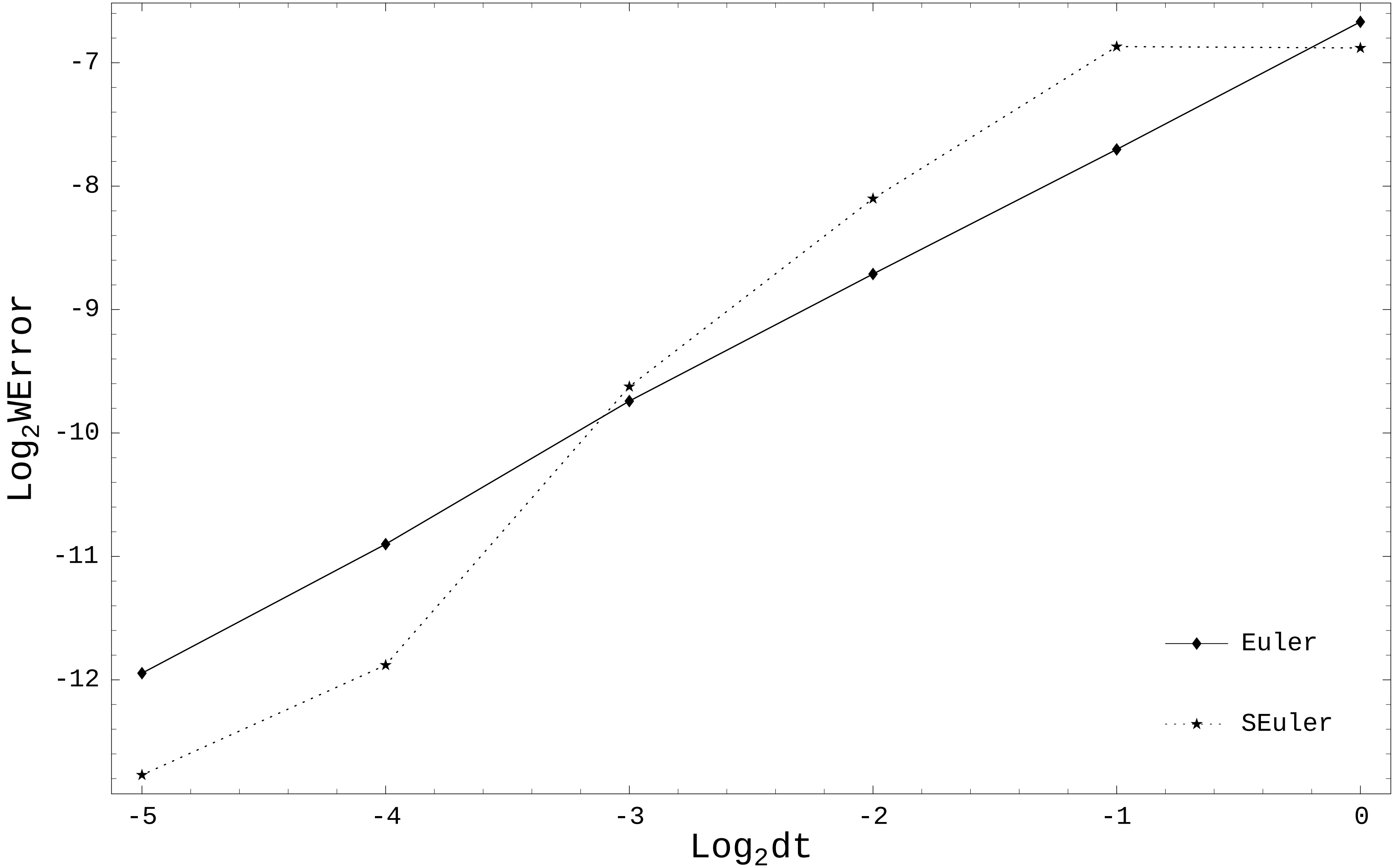

Figure 6.22: Log-log plot of weak error for call option with Euler and simplified Euler scheme.

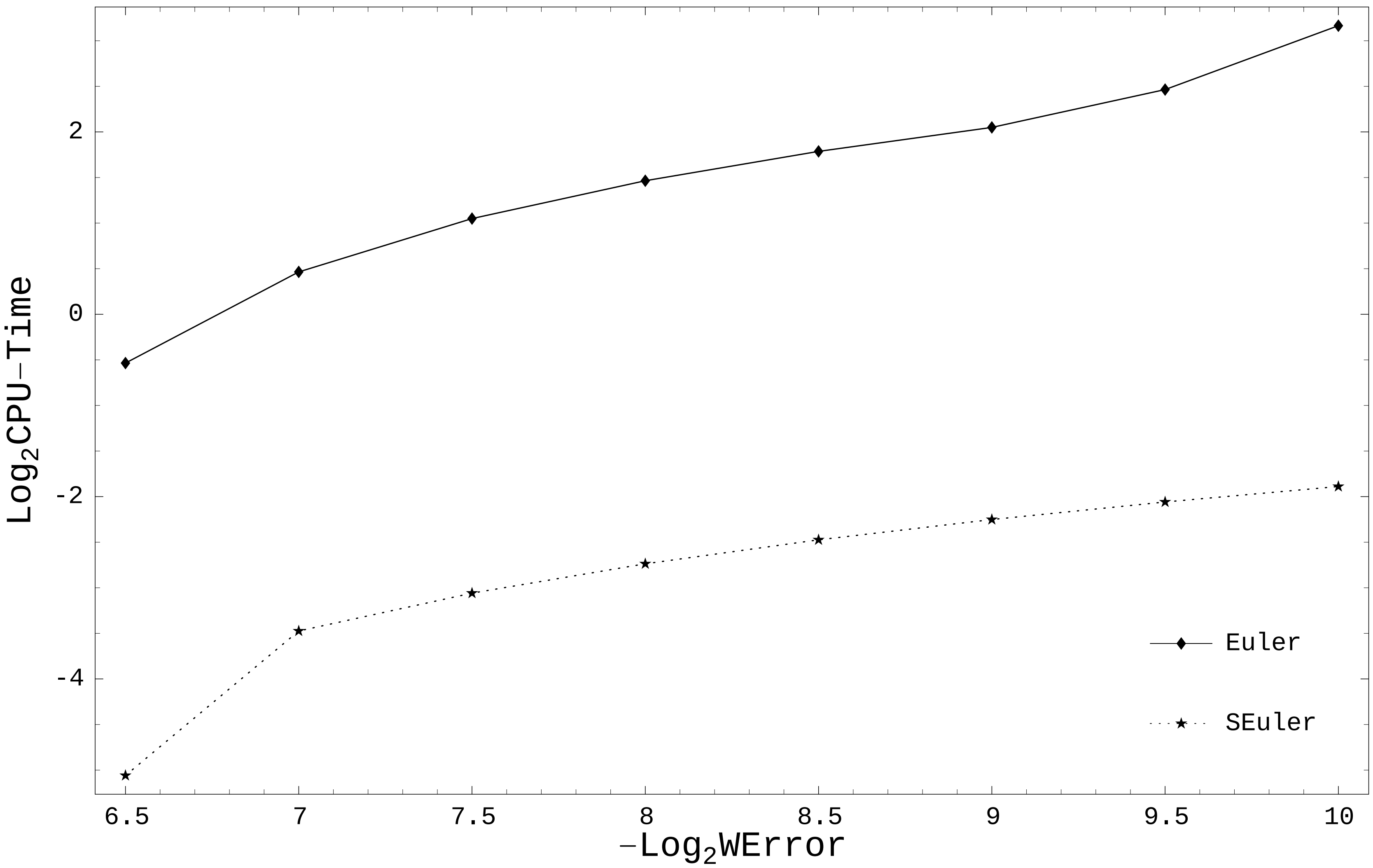

Figure 6.23: Log-log plot of CPU time versus weak error for call option with Euler and simplified Euler scheme.

7 Partial Differential Equation Methods

PDEs and Diffusions

Second Order Parabolic Partial Differential Equations

financial market models

Markovian type

diffusion processes

or jump diffusion processes

- **transition density**

  $p(t, x; s, y)$

  Wiener process

- **Kolmogorov backward equation**

$$\frac{\partial p(t, x; T, y)}{\partial t} + \frac{1}{2} \frac{\partial^2 p(t, x; T, y)}{\partial x^2} = 0$$

for $t \in [0, T]$ and $x, y \in \Re$

- **derivative price**

  expectation of a future payoff

  integrable payoff $g(W_T)$

$$
\begin{aligned}
u(t,x) &= E\left(g(W_T)\,\middle|\,\mathcal{A}_t\right) \\
&= \int_{\Re} g(y)\,p(t,x;T,y)\,dy
\end{aligned}
$$

  when $W_t = x$

- **only integrability of $g(\cdot)\,p(\cdot)$ is required !**

under integrability conditions on $g\frac{\partial P}{\partial t}$ and $g\frac{\partial^2 P}{\partial x^2}$

$\Longrightarrow$

$$\frac{\partial u(t,x)}{\partial t} = \int_{\Re} g(y)\,\frac{\partial p(t,x;T,y)}{\partial t}\,dy$$

and

$$\frac{\partial^2 u(t,x)}{\partial x^2} = \int_{\Re} g(y)\,\frac{\partial^2 p(t,x;T,y)}{\partial x^2}\,dy$$

for $(t,x) \in (0,T) \times \Re$

$\Longrightarrow$

- **Kolmogorov backward PDE**

$$
\begin{aligned}
\frac{\partial u(t,x)}{\partial t} + \frac{1}{2}\,\frac{\partial^2 u(t,x)}{\partial x^2} \\
&= \int_{\Re} g(y) \left(\frac{\partial}{\partial t} + \frac{1}{2}\,\frac{\partial^2}{\partial x^2} \right) p(t,x;T,y)\,dy \\
&= 0
\end{aligned}
$$

for $(t,x) \in (0,T) \times \Re$

Since $\quad p(T, x; T, y) = \delta(y - x)$

Dirac delta measure

$\Longrightarrow$

- **terminal condition**

$$u(T, x) = g(x)$$

for $x \in \Re$

- **Feynman-Kac formula**

pricing function

Kolmogorov backward equation

PDEs naturally arise in finance

rare explicit solution

$\Longrightarrow$ numerical methods

diffusion phenomenon is present in the price formation

$\Longrightarrow$ second order linear parabolic PDEs

heat equation

- **linear PDE**

  if $u_1(t,x)$ and $u_2(t,x)$ are solutions

  then so is

$$c_1 u_1(t,x) + c_2 u_2(t,x)$$

- **second order PDE**

  highest order derivative occurring $\quad \frac{\partial^2 u(t,x)}{\partial x^2}$

Explicit Solutions of PDEs

- heat equation

  satisfies PDE

$$\frac{\partial u(t,x)}{\partial t} + \frac{1}{2}\,\frac{\partial^2 u(t,x)}{\partial x^2} = 0$$

  with terminal condition

$$u(1,x) = \delta(x)$$

$$u(t,x) = \frac{1}{\sqrt{2\,\pi\,t}}\,\exp\left\{-\frac{x^2}{2\,t}\right\}$$

- **exploiting symmetry group methods**

  Craddock & Pl. (2004)

  analytic formulas for transition densities

Generalized Square Root Processes

- **SDE**

$$dX_t = a(X_t)\,dt + \sqrt{2\,X_t}\,dW_t$$

- **transition density**

$$p(s,x;z,y) = p(t,x,y), \quad t = z - s$$

$$\frac{\partial p(t,x,y)}{\partial t} = x\,\frac{\partial^2 p(t,x,y)}{\partial x^2} + a(x)\,\frac{\partial p(t,x,y)}{\partial x}$$

for $t \in (0,\infty)$ with

$$p(0,x,y) = \delta(x-y)$$

for $x, y \in (0,\infty)$

$a(\cdot)$ is a drift function

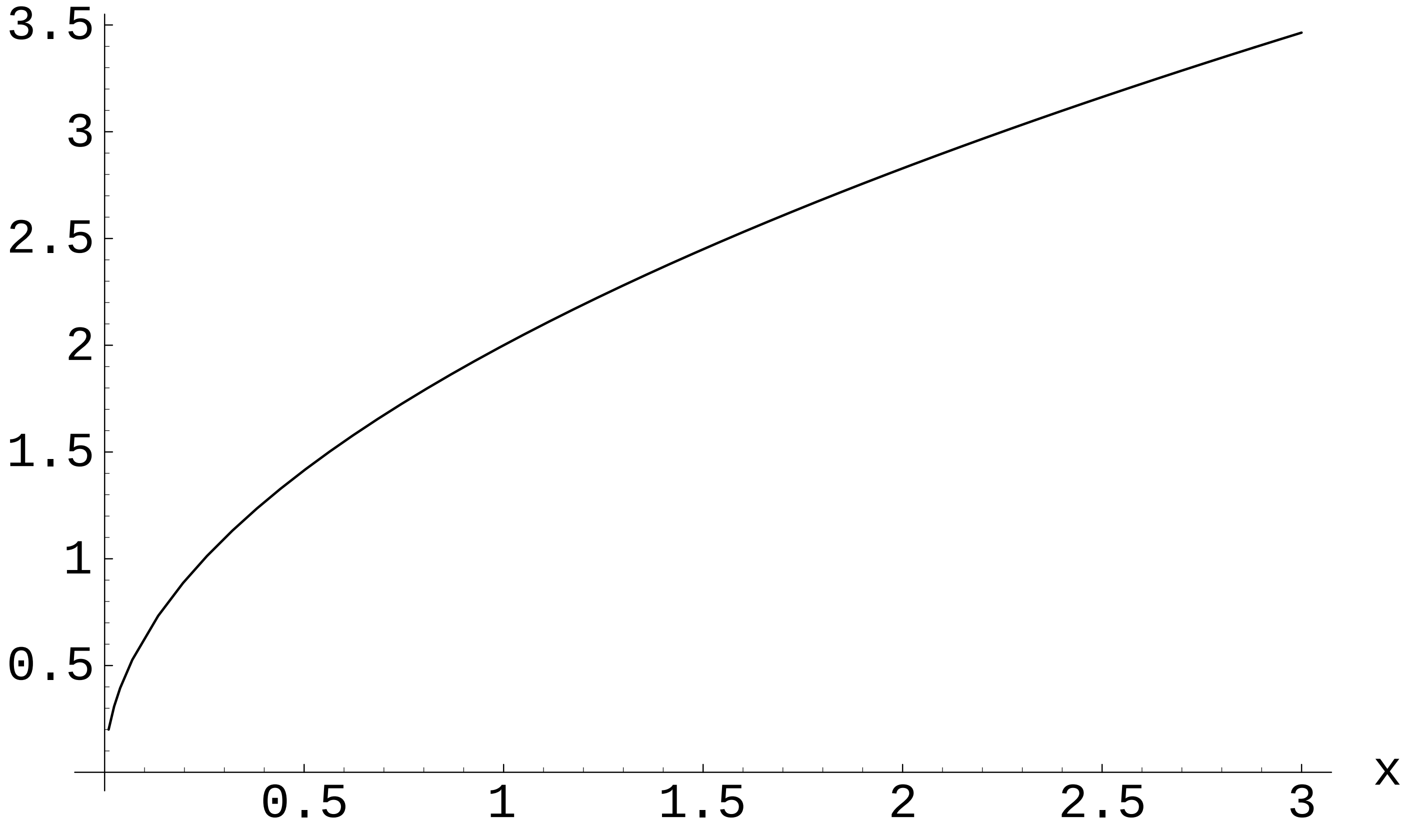

Figure 7.1: Diffusion function of generalized square root process.

(i)

$$a(x) = \alpha > 0$$

squared Bessel process of dimension $\nu = 2\alpha$

$$dX_t = \alpha\, dt + \sqrt{2\, X_t}\, dW_t$$

$$p(t,x,y) = \frac{1}{t} \left(\frac{x}{y}\right)^{\frac{1-\alpha}{2}} \bar{K}_{\alpha-1}\left(\frac{2\sqrt{x\, y}}{t}\right) \exp\left\{-\frac{(x+y)}{t}\right\}$$

$\bar{K}_r$ modified Bessel function of the first kind of order r

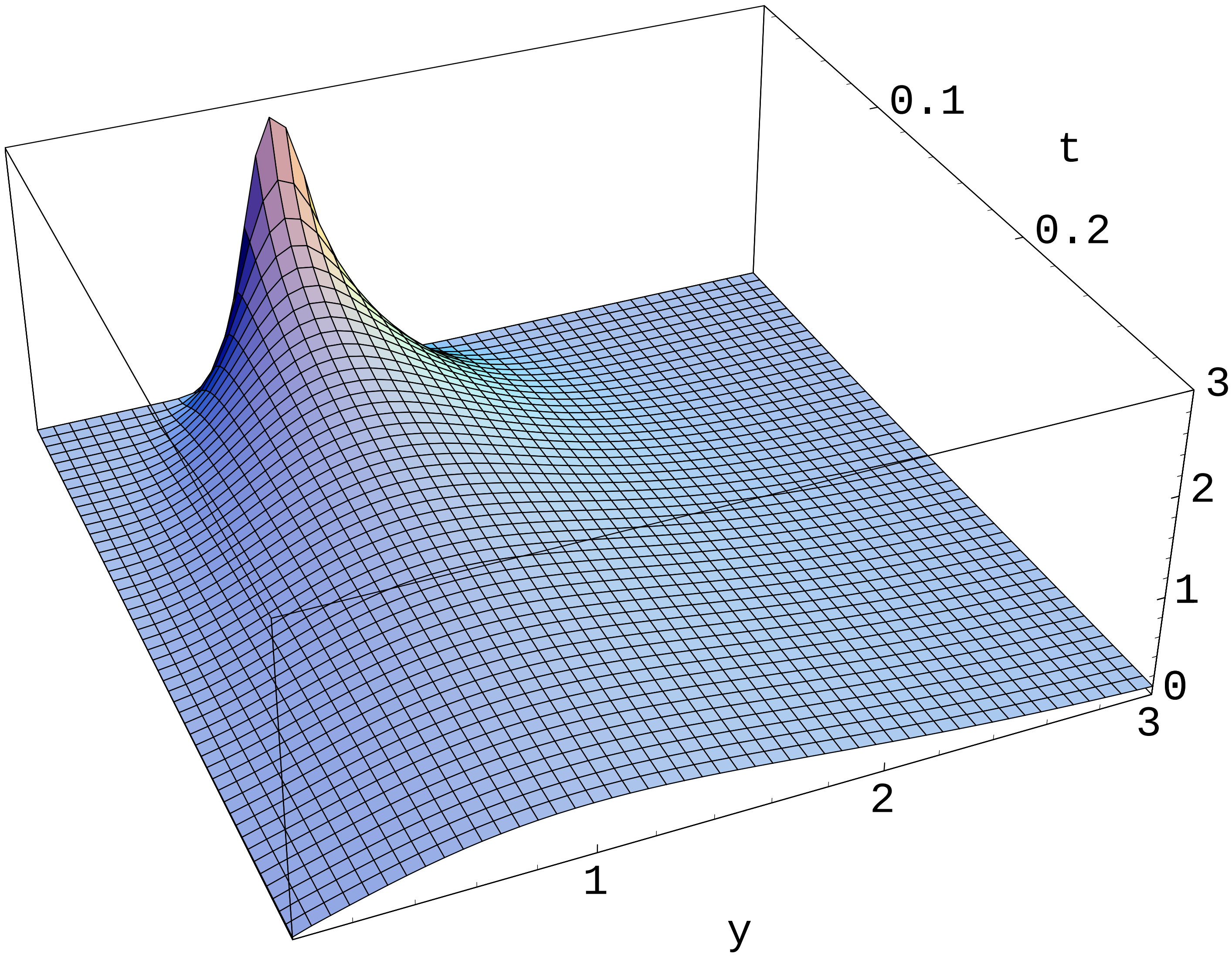

Figure 7.2: Transition density for case (i).

(ii)

$$a(x) = \frac{\mu\, x}{1 + \frac{\mu}{2}\, x} \qquad \text{for } \mu > 0$$

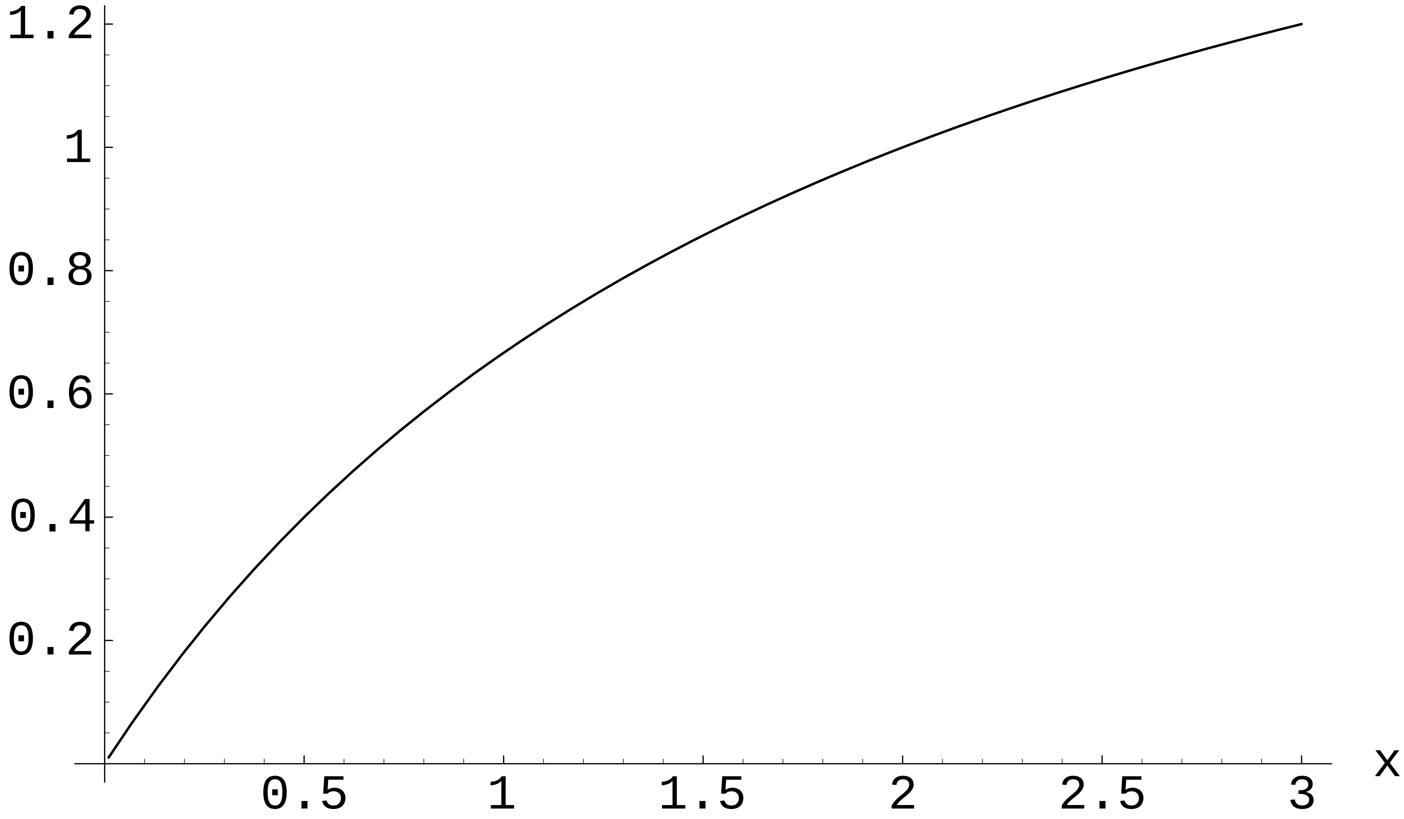

Figure 7.3: Drift function of case (ii).

$$dX_t = \frac{\mu\, X_t}{1 + \frac{\mu}{2}\, X_t}\, dt + \sqrt{2\, X_t}\, dW_t$$

$$p(t,x,y) = \frac{\exp\left\{-\frac{(x+y)}{t}\right\}}{\left(1+\frac{\mu}{2}\,x\right)t}\left[\left(\sqrt{\frac{x}{y}} + \frac{\mu\sqrt{x\,y}}{2}\right)\bar{K}_1\left(\frac{2\sqrt{x\,y}}{t}\right) + t\,\delta(y)\right]$$

$\delta(\cdot)$ Dirac delta function

for $y = 0$ $\quad \frac{\exp\left\{-\frac{x}{t}\right\}}{\left(1+\frac{\mu}{2}\,x\right)}$ probability of absorbtion at zero

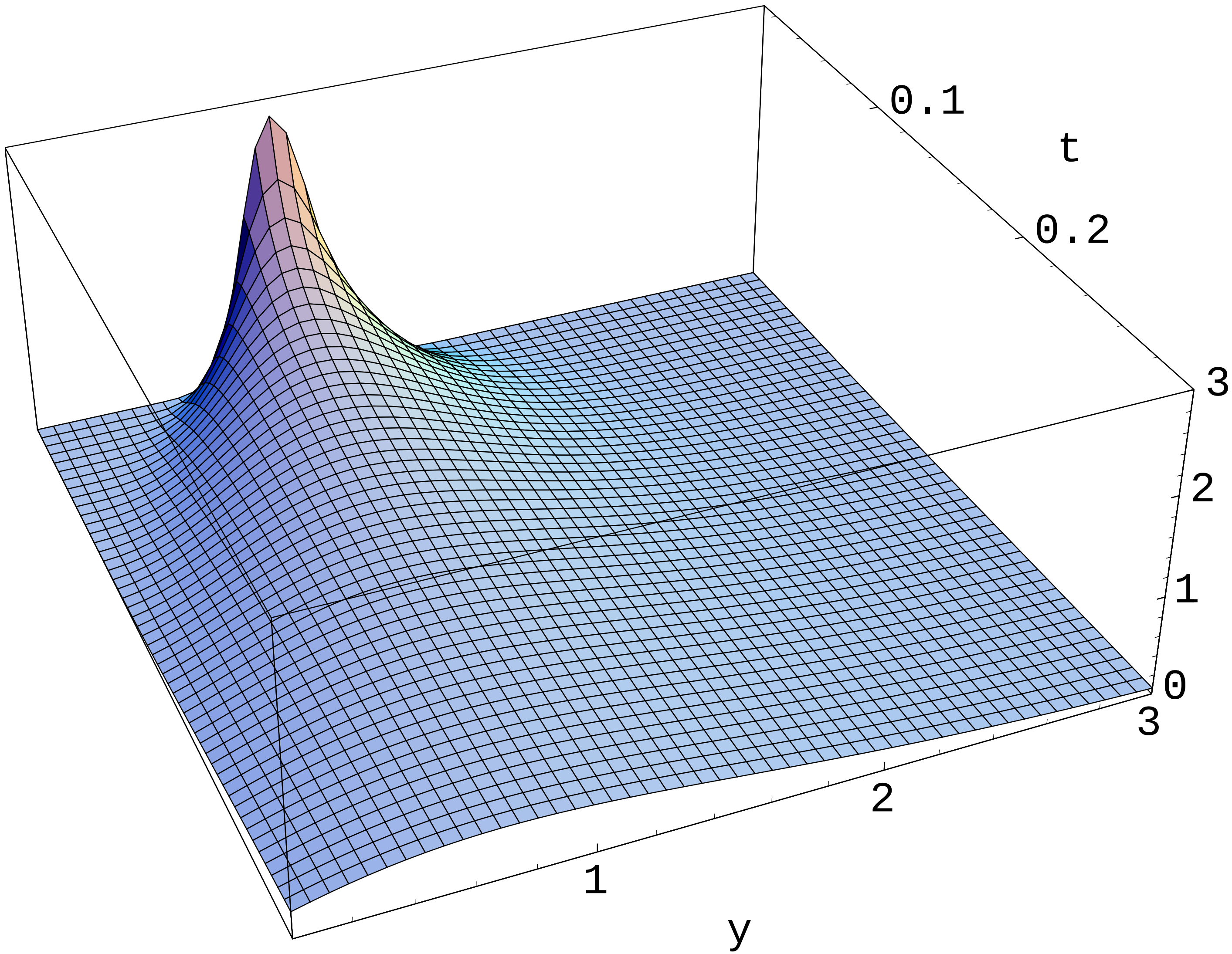

Figure 7.4: Transition density for case (ii).

(iii)

$$a(x) = \frac{1 + 3\sqrt{x}}{2\,(1 + \sqrt{x})}$$

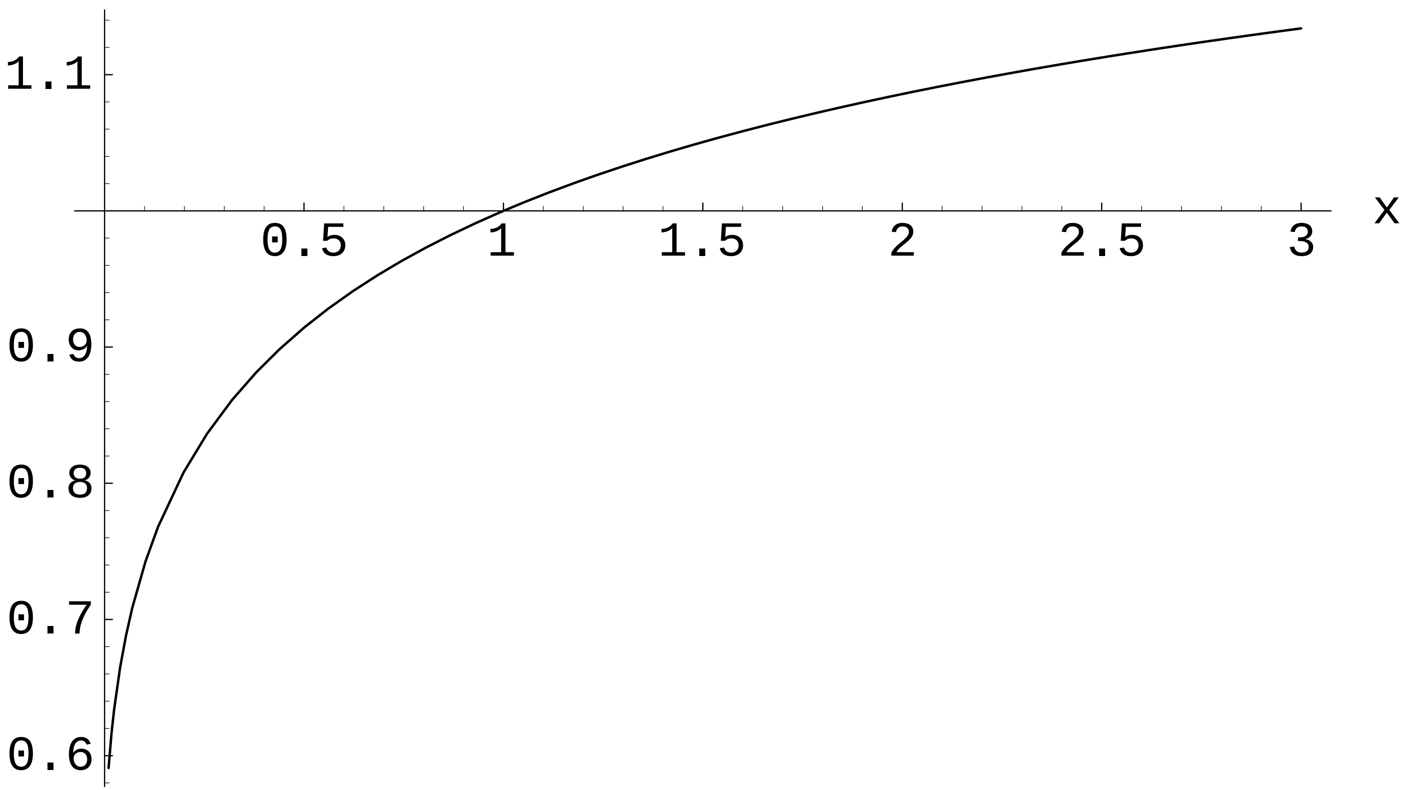

Figure 7.5: Drift function of case (iii).

$$dX_t = \frac{1 + 3\sqrt{X_t}}{2\,(1 + \sqrt{X_t})}\,dt + \sqrt{2\,X_t}\,dW_t$$

$$p(t,x,y) = \frac{\cosh\left(\frac{2\sqrt{x\,y}}{t}\right)}{\sqrt{\pi\,y\,t}\,(1+\sqrt{x})}\left(1 + \sqrt{y}\,\tanh\left(\frac{2\sqrt{x\,y}}{t}\right)\right)\exp\left\{-\frac{(x+y)}{t}\right\}$$

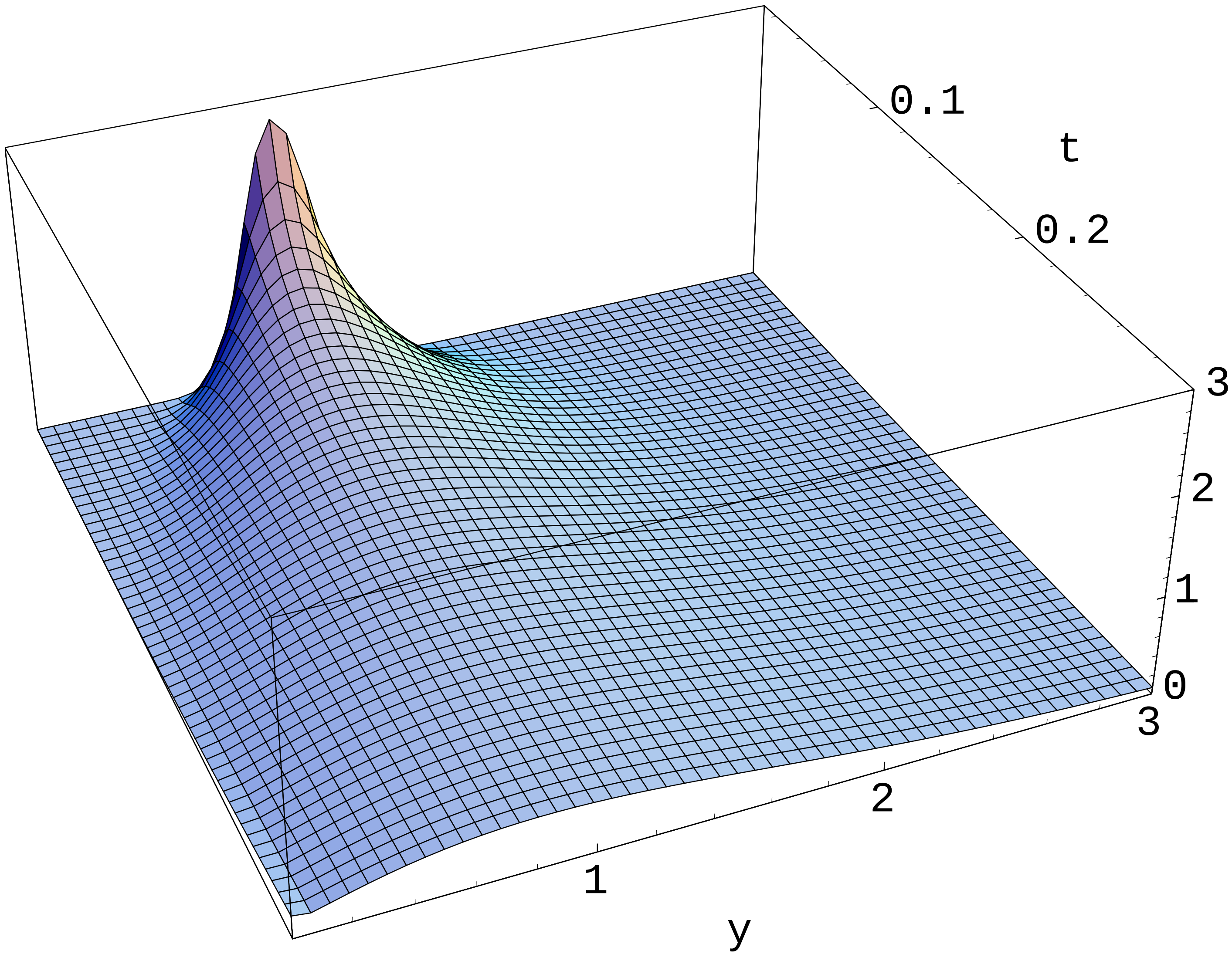

Figure 7.6: Transition density for case (iii).

(iv)

$$a(x) = 1 + \mu \tanh\left(\mu + \frac{1}{2}\mu \ln(x)\right) \quad \text{for } \mu = \frac{1}{2}\sqrt{\frac{5}{2}}$$

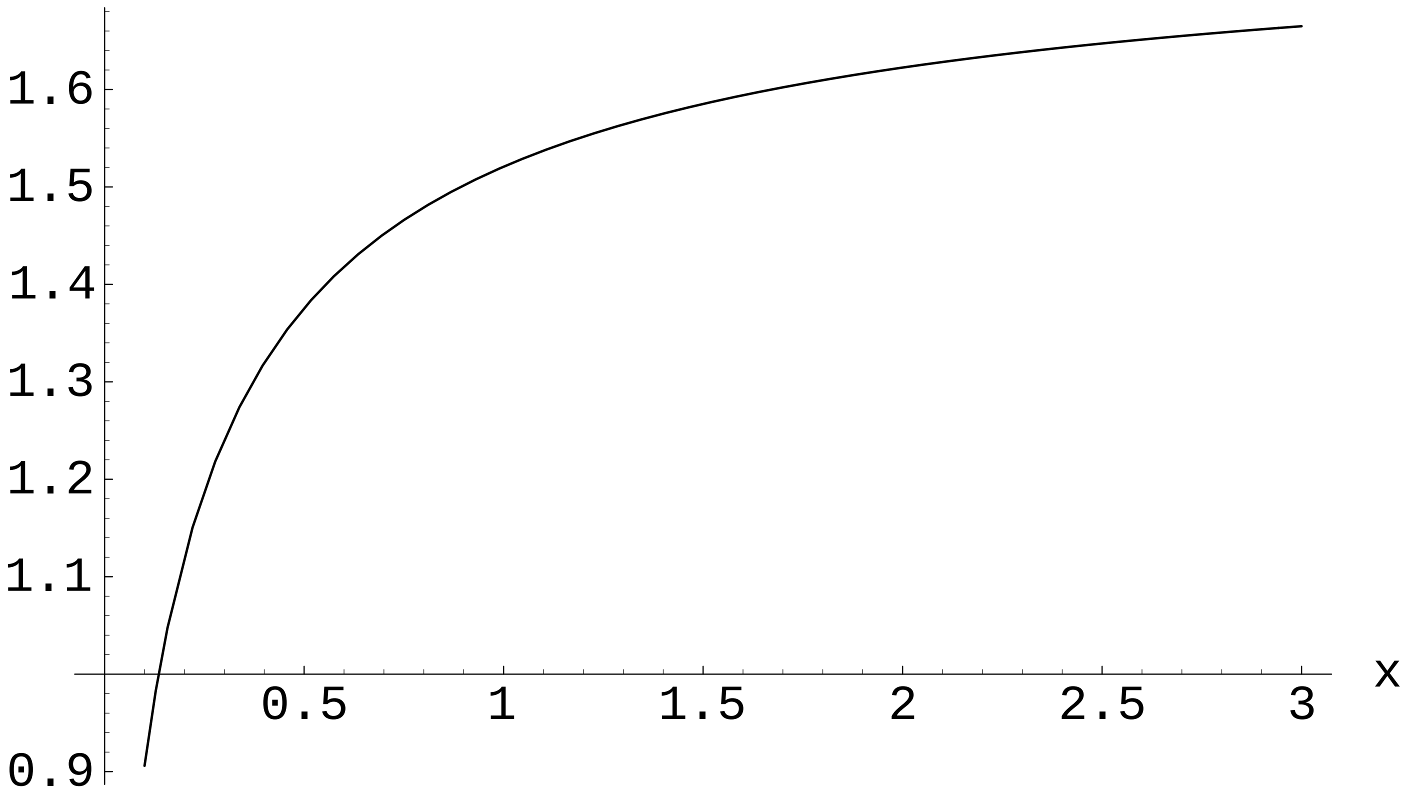

Figure 7.7: Drift function of case (iv).

$$dX_t = \left(1 + \mu \tanh\left(\mu + \frac{1}{2}\,\mu\,\ln(X_t)\right)\right) dt + \sqrt{2\,X_t}\,dW_t$$

$$p(t,x,y) \;=\; \left(\frac{x}{y}\right)^{\frac{\mu}{2}} \left[\bar{K}_{-\mu}\left(\frac{2\sqrt{x\,y}}{t}\right) + e^{2\,\mu}\,y^{\mu}\,\bar{K}_{\mu}\left(\frac{2\sqrt{x\,y}}{t}\right)\right]$$

$$\times \frac{\exp\{-\frac{x+y}{t}\}}{(1+\exp\{2\,\mu\}\,x^{\mu})\,t}$$

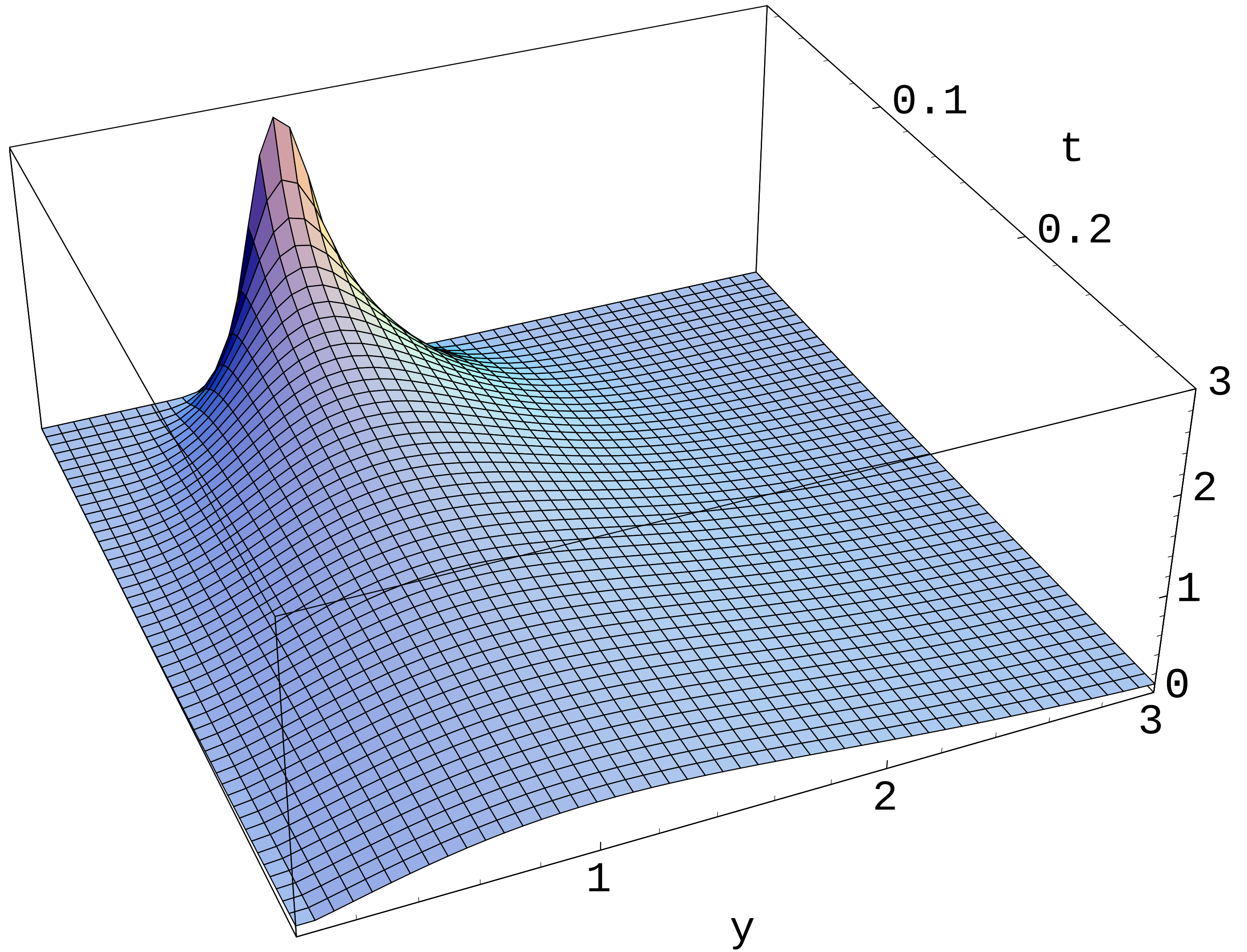

Figure 7.8: Transition density for case (iv).

(v)

$$a(x) = \frac{1}{2} + \sqrt{x}$$

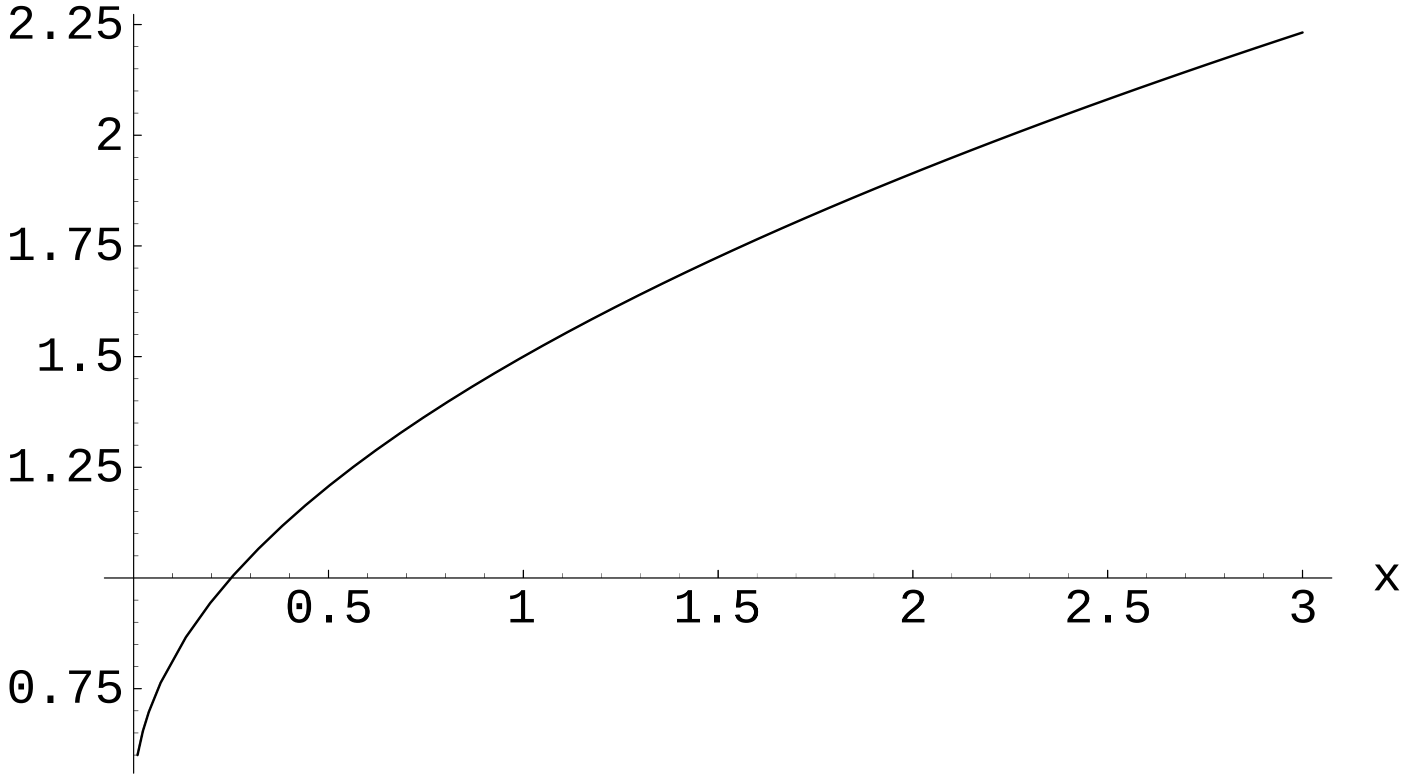

Figure 7.9: Drift function of case (v).

$$dX_t = \left(\frac{1}{2} + \sqrt{X_t} \right) dt + \sqrt{2\, X_t}\, dW_t$$

$$p(t, x, y) = \cosh\left(\frac{(t + 2\sqrt{x})\,\sqrt{y}}{t} \right) \frac{\exp\{-\sqrt{x}\}}{\sqrt{\pi\, y\, t}}\ \exp\left\{ -\frac{(x + y)}{t} - \frac{t}{4} \right\}$$

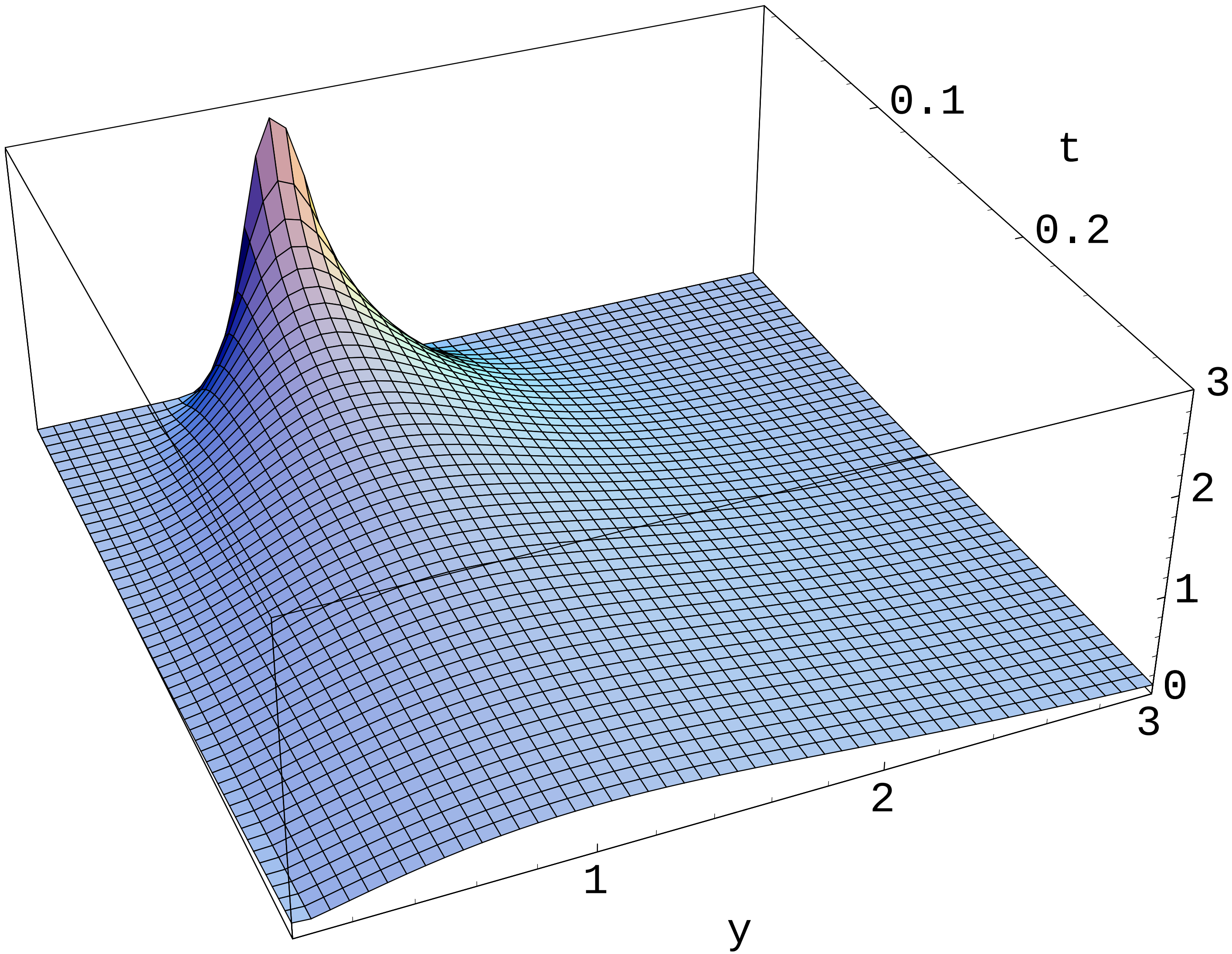

Figure 7.10: Transition density for case (v).

(vi)

$$a(x) = \frac{1}{2} + \sqrt{x}\,\tanh(\sqrt{x})$$

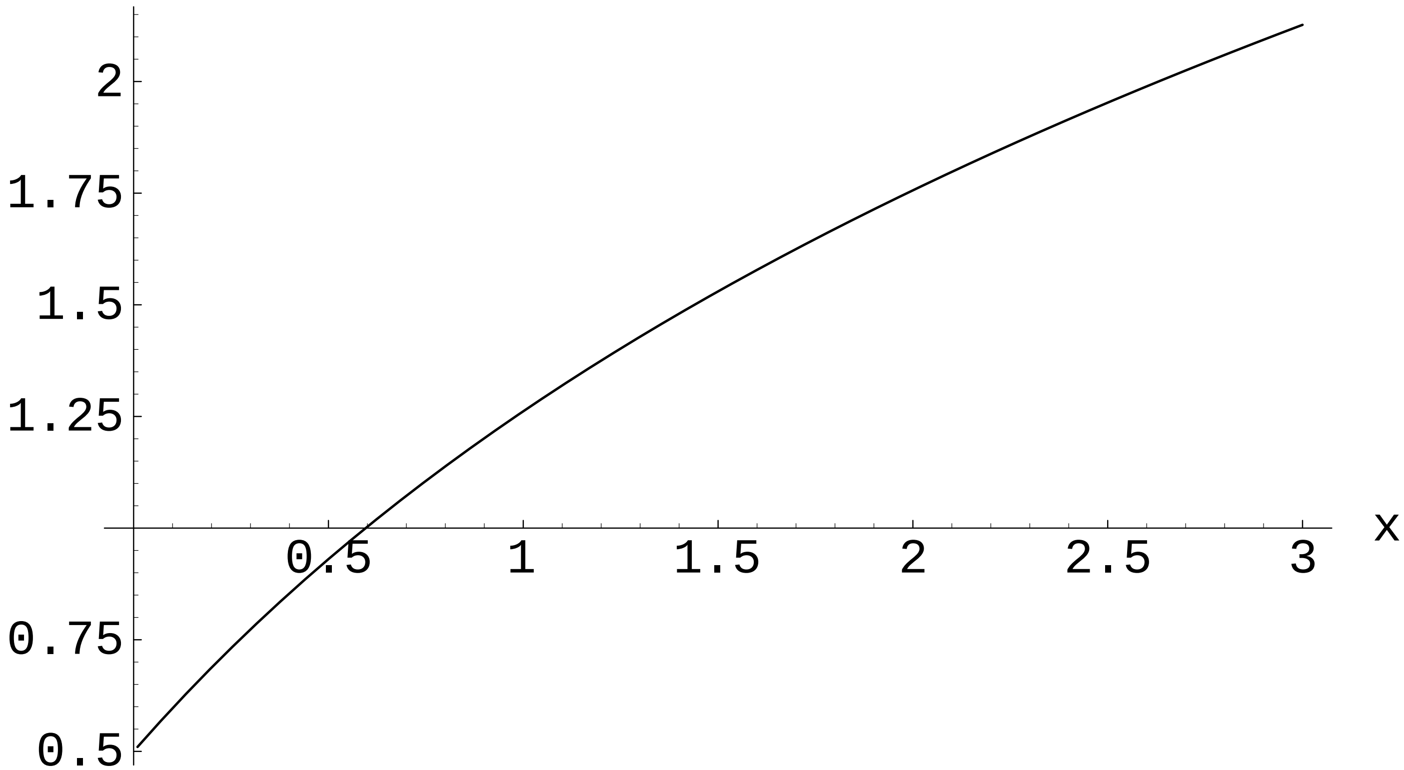

Figure 7.11: Drift function for case (vi).

$$dX_t = \left(\frac{1}{2} + \sqrt{X_t}\,\tanh\left(\sqrt{X_t}\right)\right) dt + \sqrt{2\,X_t}\,dW_t$$

$$p(t,x,y) = \frac{\cosh\left(\frac{2\sqrt{x\,y}}{t}\right)}{\sqrt{\pi\,y\,t}}\,\frac{\cosh(\sqrt{y})}{\cosh(\sqrt{x})}\,\exp\left\{-\frac{(x+y)}{t} - \frac{t}{4}\right\}$$

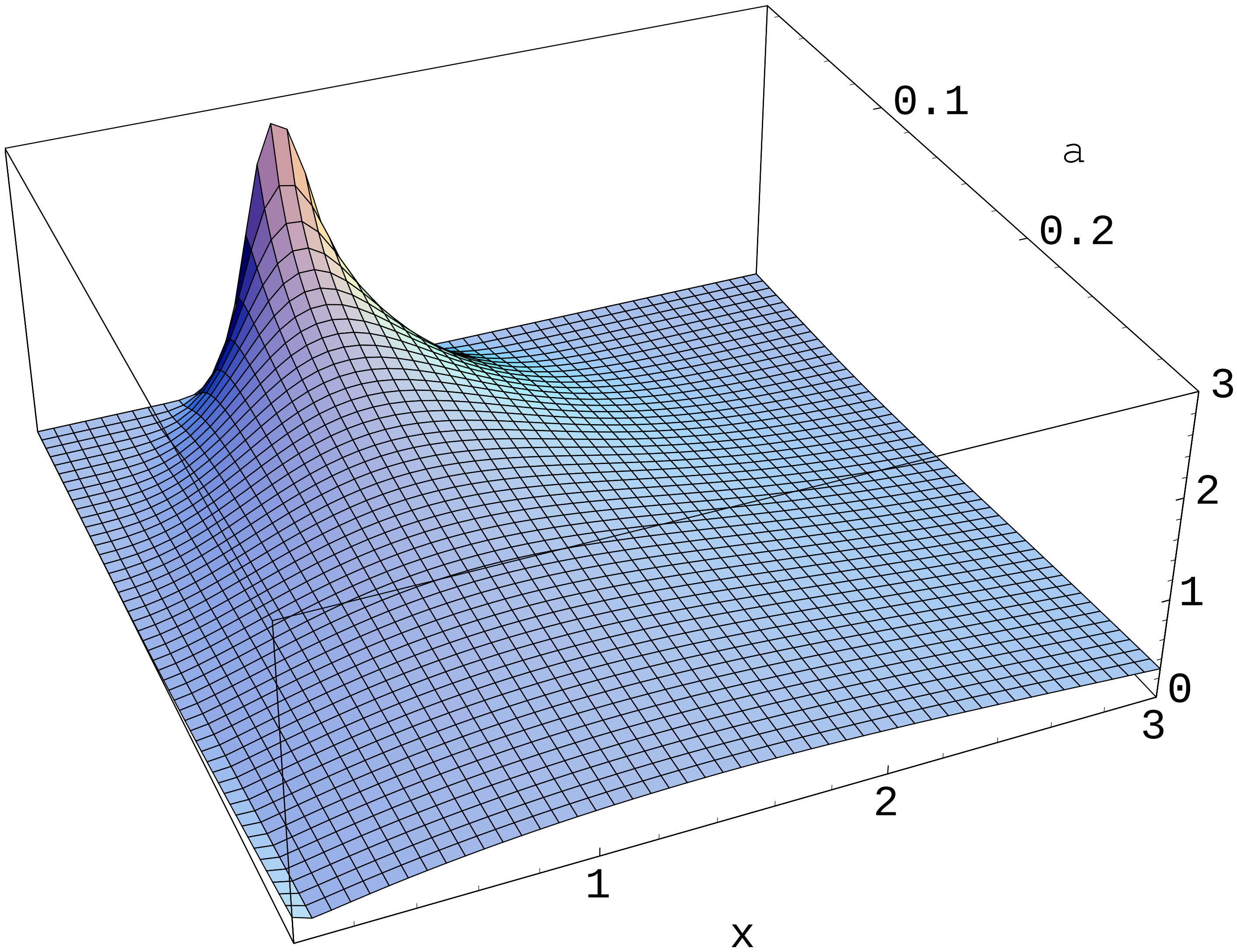

Figure 7.12: Transition density for case (vi).

(vii)

$$a(x) = \frac{1}{2} + \sqrt{x}\,\coth(\sqrt{x})$$

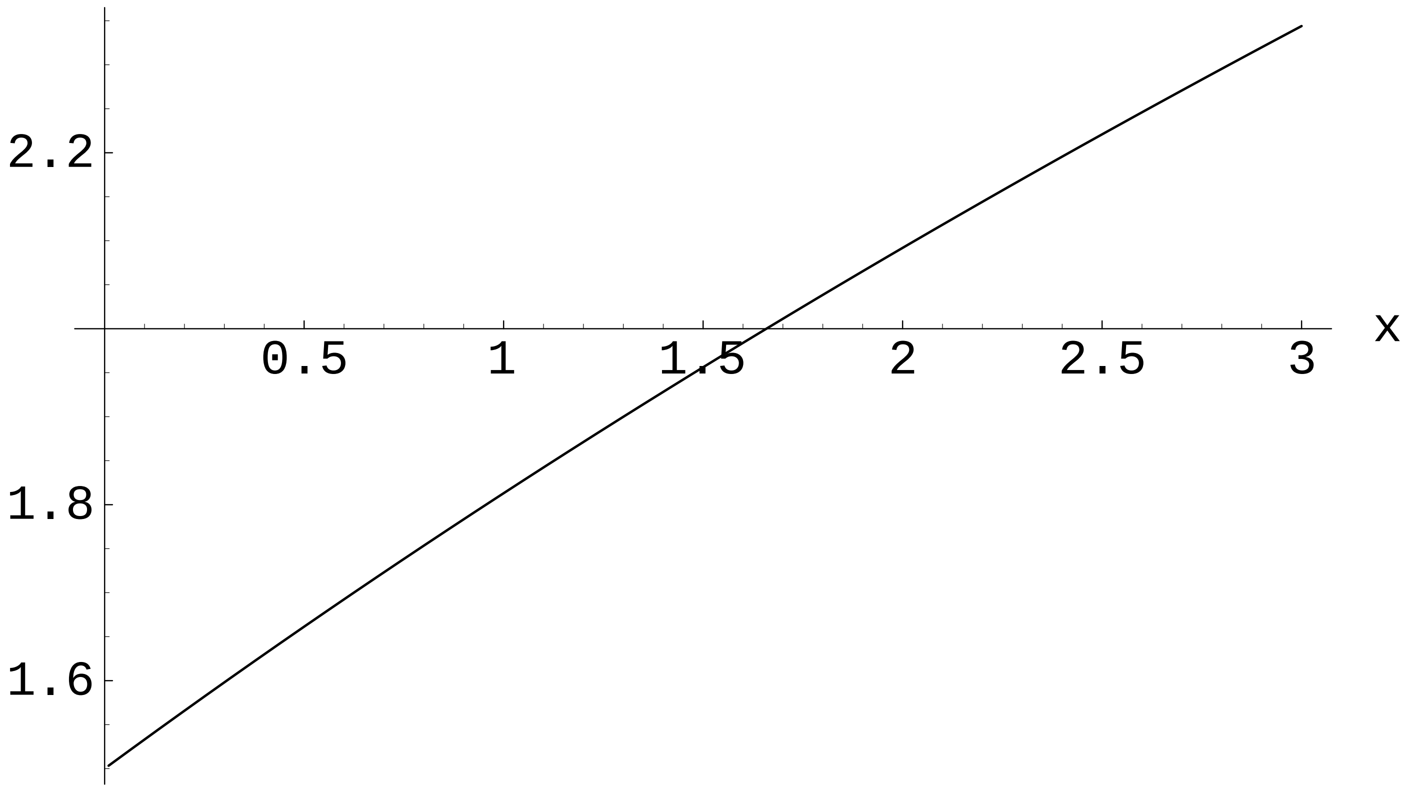

Figure 7.13: Drift function for case (vii).

$$dX_t = \left(\frac{1}{2} + \sqrt{X_t}\,\coth\left(\sqrt{X_t}\right)\right) dt + \sqrt{2\,X_t}\,dW_t$$

$$p(t,x,y) = \frac{\sinh\left(\frac{2\,\sqrt{x\,y}}{t}\right)}{\sqrt{\pi\,y\,t}}\,\frac{\sinh(\sqrt{y})}{\sinh(\sqrt{x})}\,\exp\left\{-\frac{(x+y)}{t} - \frac{t}{4}\right\}$$

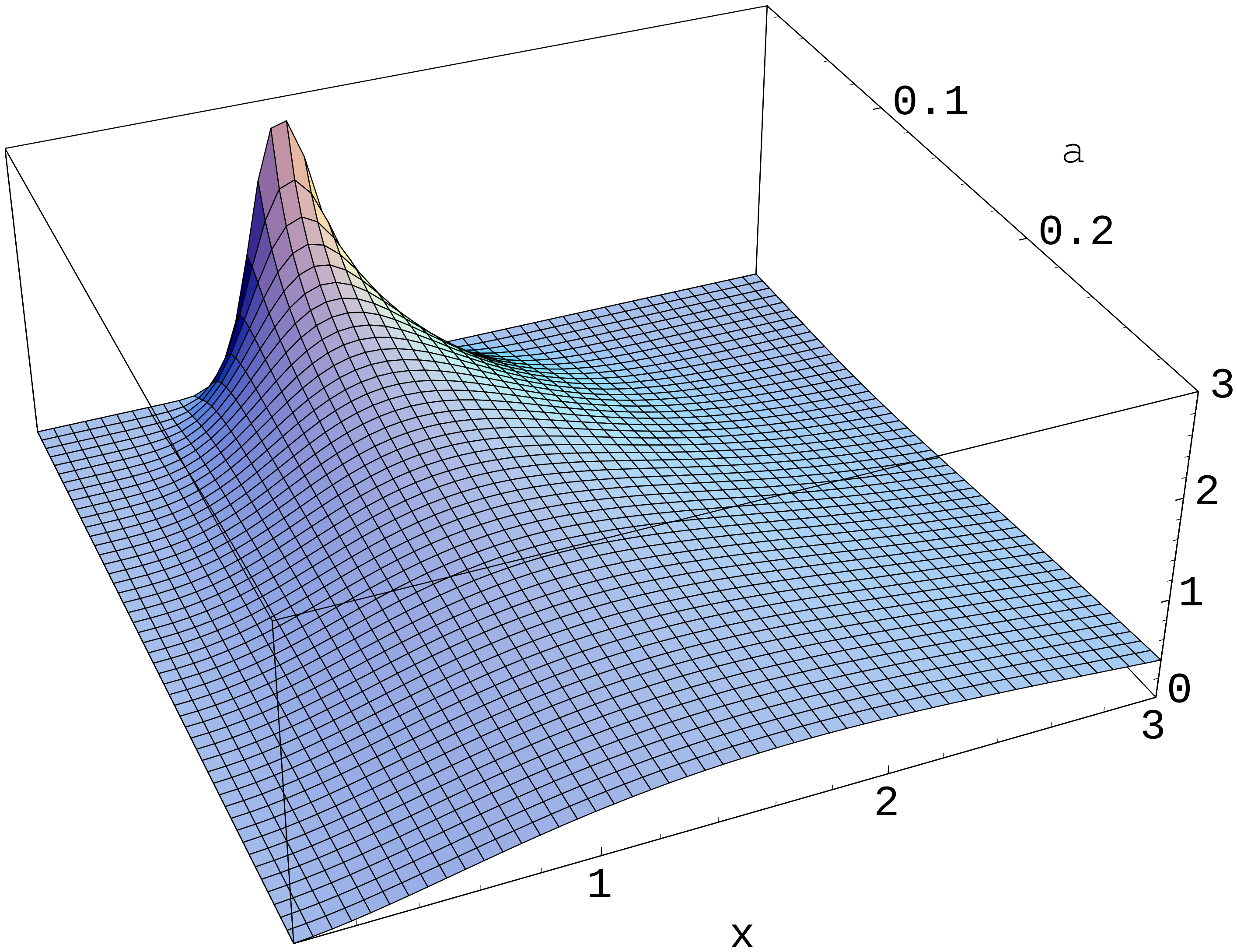

Figure 7.14: Transition density for case (vii).

(viii)

$$a(x) = 1 + \cot(\ln(\sqrt{x}))$$

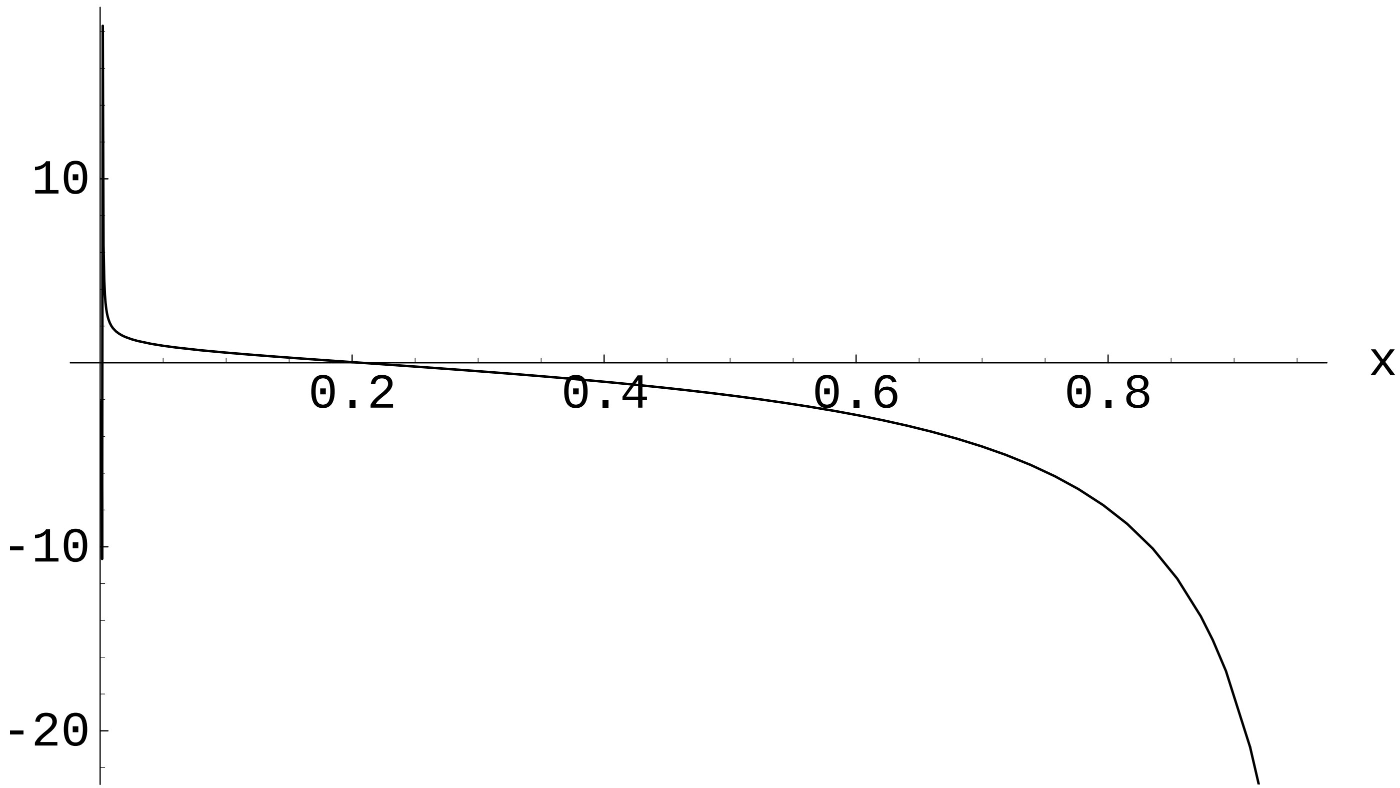

Figure 7.15: Drift function for case (viii).

$$dX_t = \left(1 + \cot\left(\ln\left(\sqrt{X_t}\right)\right)\right) dt + \sqrt{2\,X_t}\,dt$$

$$p(t,x,y) = \frac{\exp\{-\frac{(x+y)}{t}\}}{2\,\imath\,t\sin(\ln(\sqrt{x}))}\left(y^{\frac{\imath}{2}}\,\bar{K}_{\imath}\left(\frac{2\sqrt{x\,y}}{t}\right) - y^{-\frac{\imath}{2}}\,\bar{K}_{-\imath}\left(\frac{2\sqrt{x\,y}}{t}\right)\right)$$

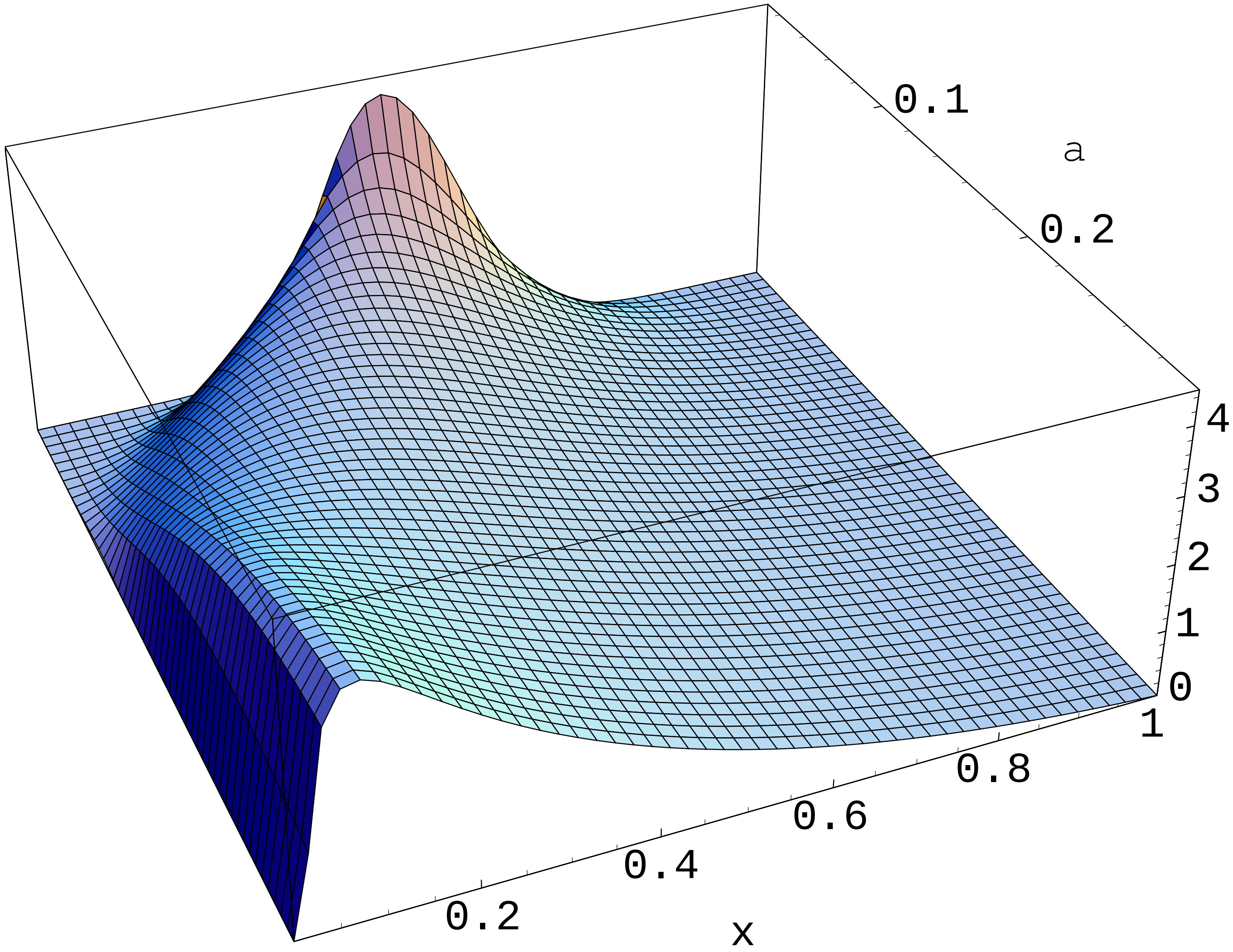

Figure 7.16: Transition density for case (viii).

(ix)

$$a(x) = x \coth\left(\frac{x}{2}\right)$$

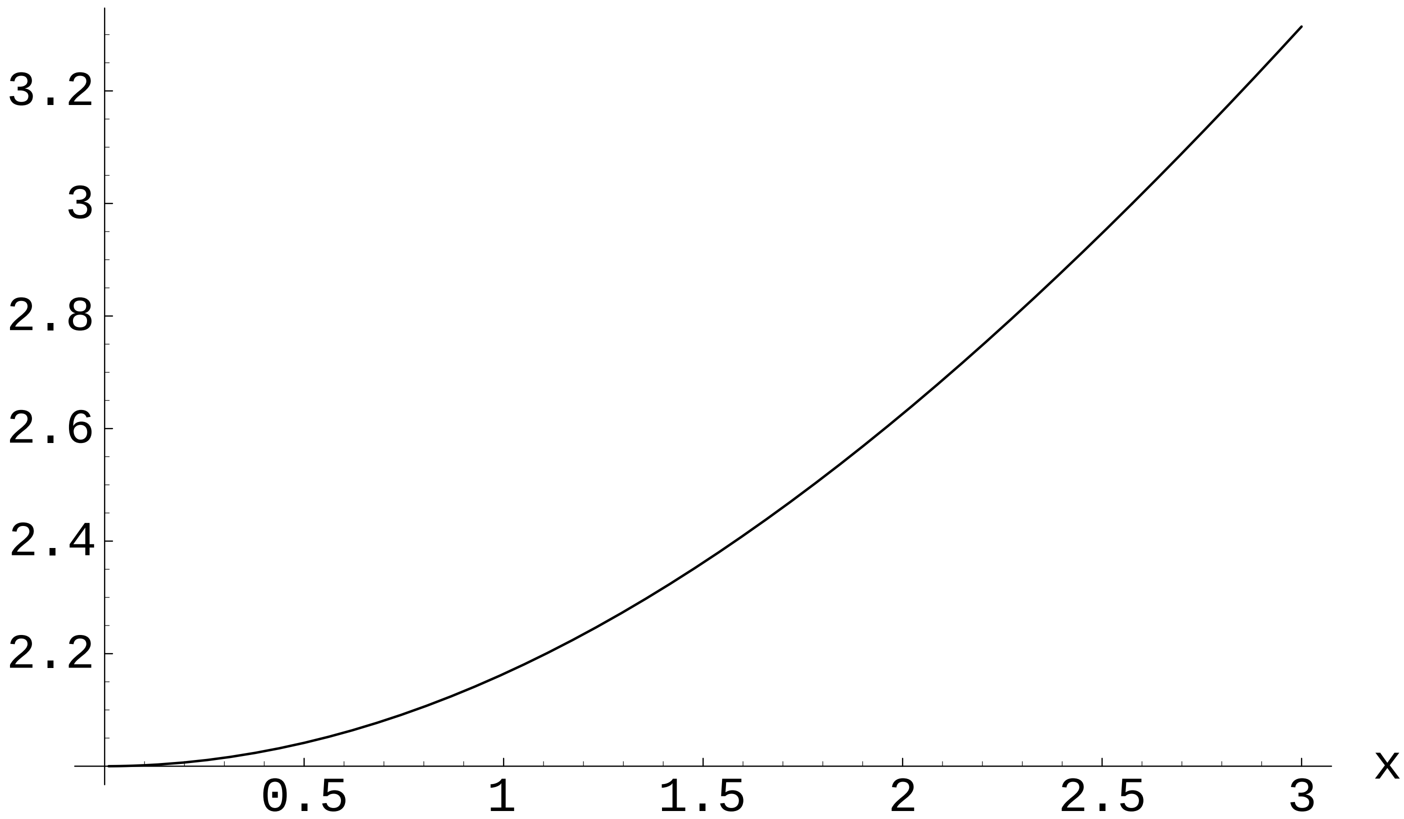

Figure 7.17: Drift function for case (ix).

$$dX_t = X_t \coth\left(\frac{X_t}{2}\right) dt + \sqrt{2\,X_t}\,dW_t$$

$$\begin{aligned} p(t,x,y) \quad &= \quad \frac{\sinh(\frac{y}{2})}{\sinh(\frac{x}{2})} \exp\left\{-\frac{(x+y)}{2\,\tanh(\frac{t}{2})}\right\} \\ &\quad \times \left[\frac{\exp\{\frac{t}{2}\}}{\exp\{t\}-1}\sqrt{\frac{x}{y}}\,\bar{K}_1\left(\frac{\sqrt{x\,y}}{\sinh(\frac{t}{2})}\right) + \delta(y)\right] \end{aligned}$$

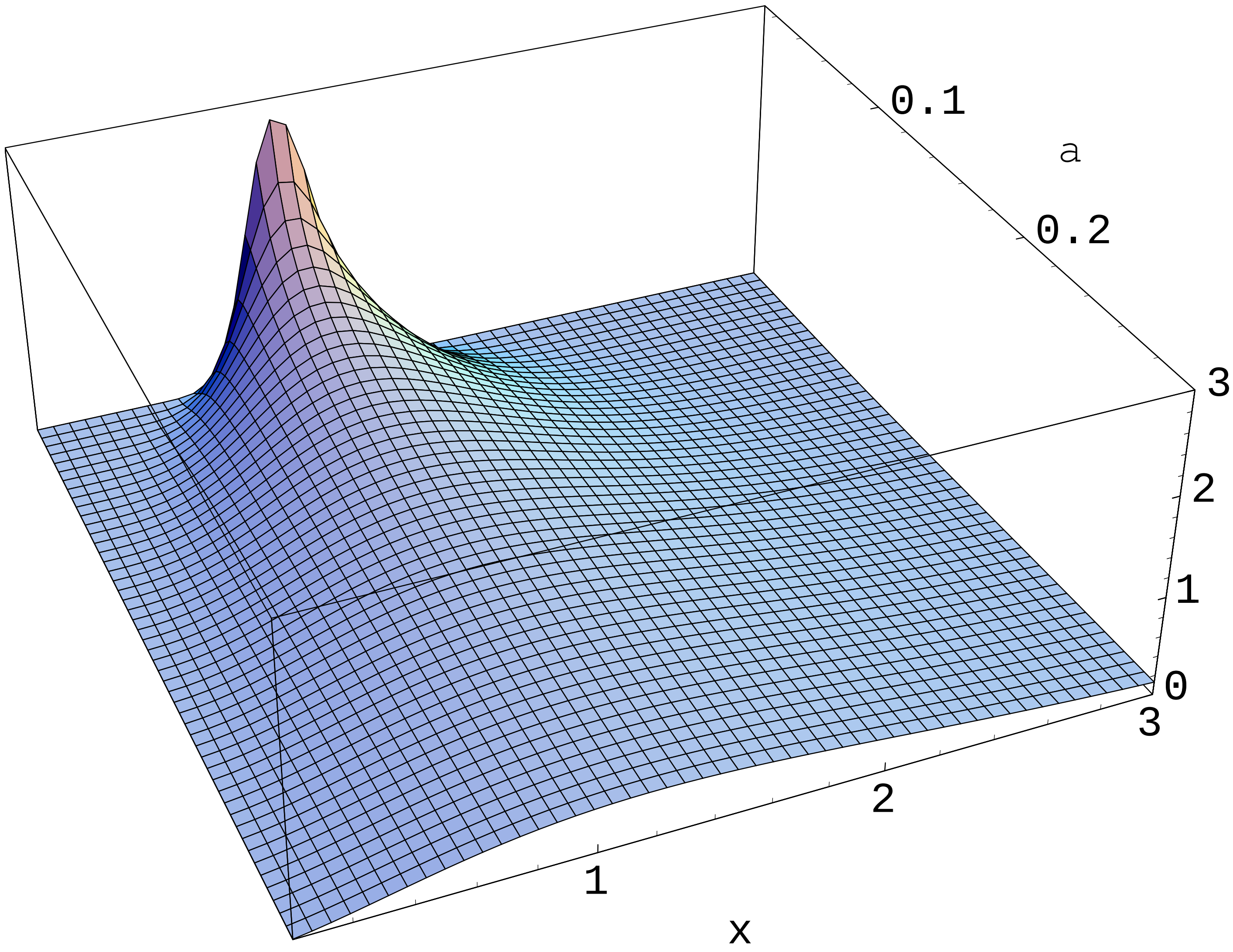

Figure 7.18: Transition density for case (ix).

(x)

$$a(x) = x \ \tanh\left(\frac{x}{2}\right)$$

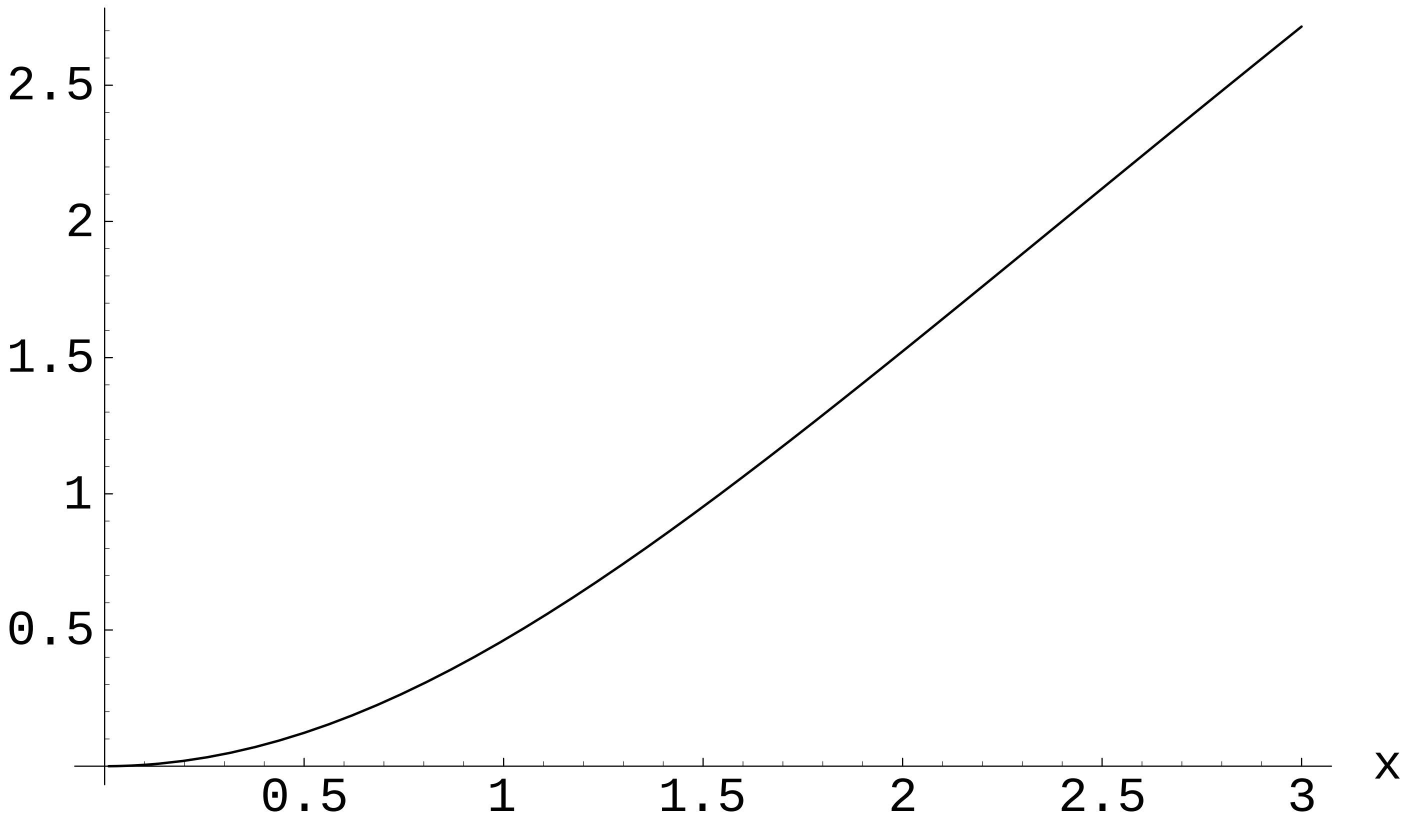

Figure 7.19: Drift function for case (x).

$$dX_t = X_t \tanh\left(\frac{X_t}{2}\right) dt + \sqrt{2\, X_t}\, dW_t$$

$$\begin{aligned} p(t,x,y) \quad &= \quad \frac{\cosh(\frac{y}{2})}{\cosh(\frac{x}{2})} \exp\left\{-\frac{(x+y)}{2\,\tanh(\frac{t}{2})}\right\} \\ &\quad \times \left[\frac{\exp\{\frac{t}{2}\}}{\exp\{t\}-1} \sqrt{\frac{x}{y}}\, \bar{K}_1\left(\frac{\sqrt{x\,y}}{\sinh(\frac{t}{2})}\right) + \delta(y)\right] \end{aligned}$$

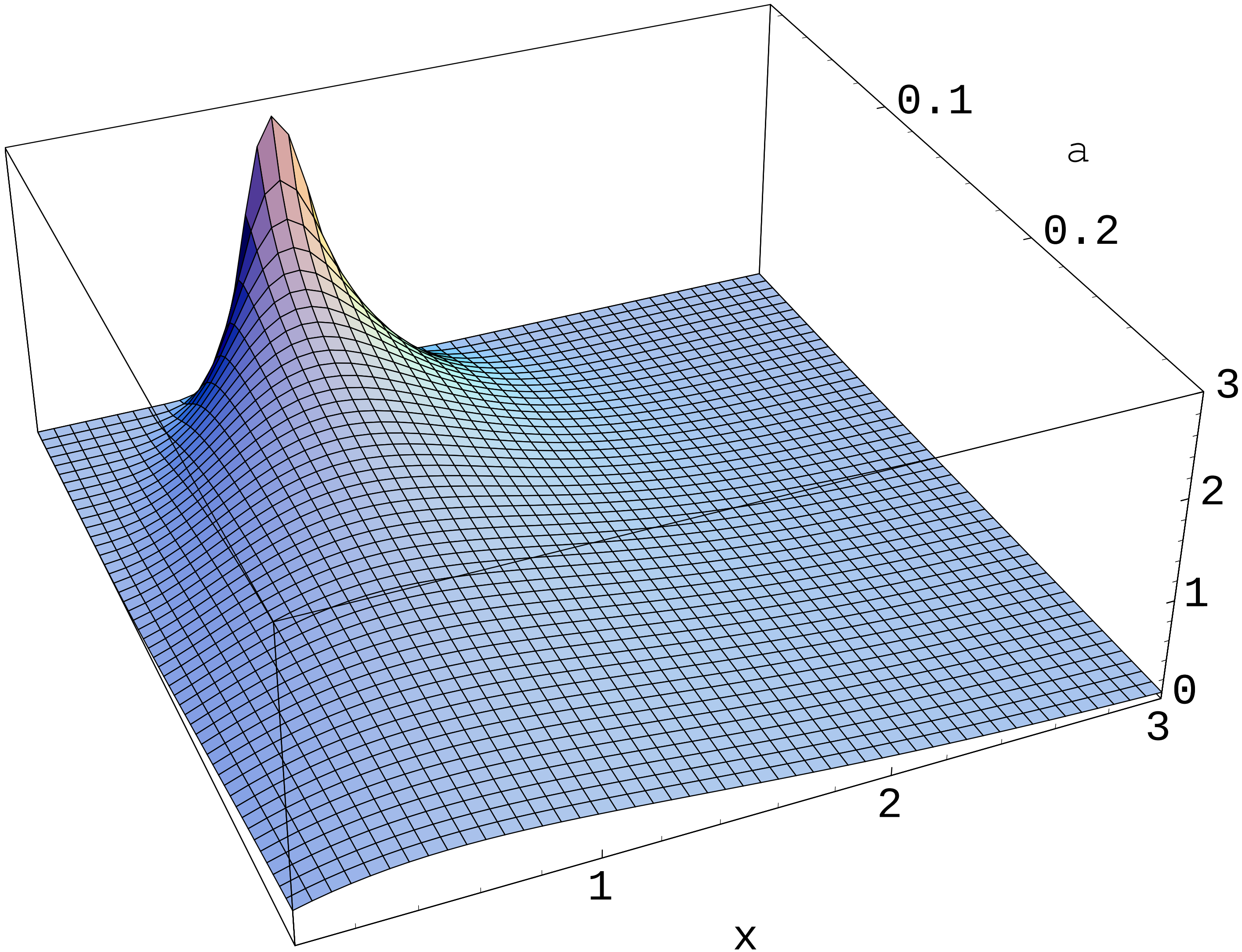

Figure 7.20: Transition density for case (x).

Finite Difference Methods

Derivative Pricing Problem

numerical approximations

finite difference methods

Richtmeyer & Morton (1967)

Smith (1985)

Tavella & Randall (2000)

Shaw (1998)

Wilmott, Dewynne & Howison (1993)

- **risk factor SDE**

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, d\tilde{W}_t$$

for $t \in [0, T]$ with $X_0 > 0$

- **discounted payoff**

$$u(T, X_T) = H(X_T)$$

- **discounted pricing function**

$$u(\cdot, \cdot)$$

PDE

$$\frac{\partial u(t,x)}{\partial t} + a(t,x)\,\frac{\partial u(t,x)}{\partial x} + \frac{1}{2}\,(b(t,x))^2\,\frac{\partial^2 u(t,x)}{\partial x^2} = 0$$

for $t \in [0, T]$ and $x \in [0, \infty)$

terminal condition

$$u(T, x) = H(x)$$

Kolmogorov backward equation

Feynman-Kac formula

calculating conditional expectation

$$u(t, X_t) = \tilde{E}\left(H(X_T) \,\middle|\, \mathcal{A}_t\right)$$

can be also obtained as the fair benchmarked price

Spatial Differences

deterministic Taylor formula

$$\frac{\partial u(t,x)}{\partial x} = \frac{u(t,x+\Delta\, x) - u(t,x-\Delta\, x)}{2\,\Delta\, x} + R_1(t,x)$$

$$\frac{\partial^2 u(t,x)}{\partial x^2} = \frac{\frac{u(t,x+\Delta\, x)-u(t,x)}{\Delta\, x} - \frac{(u(t,x)-u(t,x-\Delta\, x))}{\Delta\, x}}{2\,\Delta\, x} + R_2(t,x)$$

$$= \frac{u(t,x+\Delta\, x) - 2\,u(t,x) + u(t,x-\Delta\, x)}{2\,(\Delta\, x)^2}$$

$$+ R_2(t,x)$$

- **equally spaced grid**

$$\mathcal{X}^1_{\Delta x} = \{x_k = k\,\Delta x : \; k \in \{0, 1, \dots, N\}\}$$

for $N \in \{2, 3, \dots\}$

spatial discretization

$$x_{k+1} - x_k = \Delta\, x$$

writing

$$u_k(t) = u(t, x_k)$$

$\Longrightarrow$

$$\frac{\partial u(t, x_k)}{\partial x} = \frac{u_{k+1}(t) - u_{k-1}(t)}{2\,\Delta\,x} + R_1(t, x_k)$$

$$\frac{\partial^2 u(t, x_k)}{\partial x^2} = \frac{u_{k+1}(t) - 2\,u_k(t) + u_{k-1}(t)}{2\,(\Delta\,x)^2} + R_2(t, x_k)$$

for $k \in \{1, 2, \ldots, N-1\}$

- **interior values** $u_k(t)$ for $k \in \{1, 2, \ldots, N-1\}$

- **boundary values** $u_0(t)$ and $u_N(t)$

- **example**

$$u_0(t) = 2\,u_1(t) - u_2(t)$$

and

$$u_N(t) = 2\,u_{N-1}(t) - u_{N-2}(t)$$

- **approximate system of ODEs**

  vector

$$u(t) = (u_0(t), u_1(t), \ldots, u_N(t))^\top$$

$$\frac{du(t)}{dt} - \frac{A(t)}{(\Delta x)^2}\, u(t) + R(t) = 0$$

$R(t)$ remainder terms

- **matrix**

$$A(t) = [A^{i,j}(t)]_{i,j=0}^{N}$$

$$
\begin{aligned}
A^{0,0}(t) &= A^{0,2}(t) = A^{N,N-2}(t) = A^{N,N}(t) = (\Delta x)^2 \\
A^{0,1}(t) &= A^{N,N-1}(t) = -2\,(\Delta x)^2 \\
A^{k,k}(t) &= -(b(t,x_k))^2 \\
A^{k,k-1}(t) &= -\frac{1}{2}\,(A^{k,k}(t) + a(t,x_k)\,\Delta x) \\
A^{k,k+1}(t) &= -\frac{1}{2}\,(A^{k,k}(t) - a(t,x_k)\,\Delta x) \\
A^{k,j}(t) &= 0
\end{aligned}
$$

for $\boldsymbol{k} \in \{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{N}-\mathbf{1}\}$ and $|\boldsymbol{k}-\boldsymbol{j}| > \mathbf{1}$

$$\boldsymbol{A} = \begin{pmatrix} (\boldsymbol{\Delta x})^2 & -2(\boldsymbol{\Delta x})^2 & (\boldsymbol{\Delta x})^2 & 0 & \ldots & 0 & 0 & 0 \\ \boldsymbol{A}^{1,0} & \boldsymbol{A}^{1,1} & \boldsymbol{A}^{1,2} & 0 & \ldots & 0 & 0 & 0 \\ 0 & \boldsymbol{A}^{2,1} & \boldsymbol{A}^{2,2} & \boldsymbol{A}^{2,3} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \boldsymbol{A}^{N-2,N-2} & \boldsymbol{A}^{N-2,N-1} & 0 \\ 0 & 0 & 0 & 0 & \ldots & \boldsymbol{A}^{N-1,N-2} & \boldsymbol{A}^{N-1,N-1} & \boldsymbol{A}^{N-1,N} \\ 0 & 0 & 0 & 0 & \ldots & (\boldsymbol{\Delta x})^2 & -2(\boldsymbol{\Delta x})^2 & (\boldsymbol{\Delta x})^2 \end{pmatrix}$$

- **first order finite difference approximations**

$$\frac{\partial u(t,x_k)}{\partial x} = \frac{u_{k+1}-u_k}{\Delta\, x} + O(\Delta\, x)$$

$$\frac{\partial u(t,x_k)}{\partial x} = \frac{u_k - u_{k-1}}{\Delta\, x} + O(\Delta\, x)$$

$$\frac{\partial u(t,x_k)}{\partial x} = \frac{u_{k+1}-u_{k-1}}{2\,\Delta\, x} + O((\Delta\, x)^2)$$

$$\frac{\partial u(t,x_k)}{\partial x} = \frac{3\,u_k - 4\,u_{k+1} + u_{k-2}}{2\,\Delta\, x} + O((\Delta\, x)^2)$$

$$\frac{\partial u(t,x_k)}{\partial x} = \frac{-3\,u_k + 4\,u_{k+1} - u_{k-2}}{2\,\Delta\, x} + O((\Delta\, x)^2)$$

- **second order spatial finite differences**

$$\frac{\partial^2 u(t,x_k)}{\partial x^2} = \frac{u_k - 2\,u_{k-1} + u_{k-2}}{(\Delta\,x)^2} + O(\Delta\,x)$$

$$\frac{\partial^2 u(t,x_k)}{\partial x^2} = \frac{u_{k+2} - 2\,u_{k+1} + u_k}{(\Delta\,x)^2} + O(\Delta\,x)$$

$$\frac{\partial^2 u(t,x_k)}{\partial x^2} = \frac{u_{k+1} - 2\,u_k + u_{k-1}}{(\Delta\,x)^2} + O((\Delta\,x)^2)$$

$$\frac{\partial^2 u(t,x_k)}{\partial x^2} = \frac{2\,u_k - 5\,u_{k-1} + 4\,u_{k-2} - u_{k-3}}{(\Delta\,x)^2} + O((\Delta\,x)^2)$$

$$\frac{\partial^2 u(t,x_k)}{\partial x^2} = \frac{-u_{k+3} + 4\,u_{k+2} - 5\,u_{k+1} + 2\,u_k}{(\Delta\,x)^2} + O((\Delta\,x)^2)$$

Time Difference Approximation

$\Longrightarrow$

- **approximate vector ODE**

$$du(t) \approx \frac{A(t)}{(\Delta x)^2}\, u(t)\, dt$$

for $t \in (0, T)$ with terminal condition

$$u(T) = (H(x_0), \ldots, H(x_N))^\top$$

use discrete time approximations for ODEs

- **time discretization** $\tau_n = n\Delta, n \in \{0, 1, \ldots, n_T\}$

Euler scheme

$\Longrightarrow$

$$u(\tau_{n+1}) = u(\tau_n) + A(\tau_n)\, u(\tau_n)\, \frac{\Delta}{(\Delta\, x)^2}$$

for $n \in \{0, 1, \ldots, n_T\}$, $\quad u(T) = u(\tau_{n_T})$

$\Longrightarrow$

$$u_k(T) = u(T, x_k) = H(x_k)$$

for all $k \in \{0, 1, \ldots, N\}$

constitutes a finite difference method

The Theta Method

family of implicit Euler schemes

- **theta method**

$$u(\tau_{n+1}) = u(\tau_n) + \Big(\theta\, A(\tau_{n+1})\, u(\tau_{n+1}) + (1-\theta)\, A(\tau_n)\, u(\tau_n)\Big) \frac{\Delta}{(\Delta x)^2}$$

degree of implicitness θ

$\theta = 0 \quad \Longrightarrow \quad$ Euler or explicit method

interior of a circle

region of A-stability

$\theta = 1 \quad \Longrightarrow$

- **fully implicit method**

$$u(\tau_{n+1}) = u(\tau_n) + A(\tau_{n+1})\, u(\tau_{n+1})\, \frac{\Delta}{(\Delta x)^2}$$

A-stable

unconditionally stable

$$\theta = \tfrac{1}{2} \quad \Longrightarrow$$

- Crank-Nicolson method

$$u(\tau_{n+1}) = u(\tau_n) + \frac{1}{2}\Big(A(\tau_{n+1})\,u(\tau_{n+1}) + A(\tau_n)\,u(\tau_n)\Big)\,\frac{\Delta}{(\Delta x)^2}$$

deterministic trapezodial method

A-stable

unconditionally stable

certain stability issues

- solve at each time step a coupled system of equations

large, sparse linear system of equations

$A(\tau_n) = A$

Crank-Nicolson method

$$u(\tau_{n+1}) = u(\tau_n) + \frac{1}{2}\,A\Big(u(\tau_{n+1}) + u(\tau_n)\Big)\frac{\Delta}{(\Delta x)^2}$$

$\Longrightarrow$

$$\left(I - \frac{1}{2}\,A\,\frac{\Delta}{(\Delta x)^2}\right)u(\tau_{n+1}) = \left(I + \frac{1}{2}\,A\,\frac{\Delta}{(\Delta x)^2}\right)u(\tau_n)$$

With

$$M = I - \frac{1}{2}\, A\, \frac{\Delta}{(\Delta x)^2}$$

and

$$B_n = \left(I + \frac{1}{2}\, A\, \frac{\Delta}{(\Delta x)^2} \right) u(\tau_n)$$

rewrite linear system of equations

$$M\, u(\tau_{n+1}) = B_n$$

needs to invert the matrix M

Sparse Matrix Solvers

- **direct solver**

  tridiagonal solver known as Gaussian elimination method

- **iterative solver**

  accuracy criterion

  flexibility and efficiency

  Barrett et al. (1994)

- **Jacobi method**

  system

$$M\,u = B$$

$M = [M^{i,j}]_{i,j=1}^{N}$, $u = (u^1, \ldots, u^N)^\top$ and $B = (B^1, \ldots, B^N)^\top$

- kth iteration step

$$u_k^i = \frac{1}{M^{i,i}} \left(B^i - \sum_{j \neq i} M^{i,j}\, u_{k-1}^j \right)$$

$u_k = (u_k^1, \ldots, u_k^N)^\top \quad k$th iteration

- **Gauss-Seidel method**

$$u_k^i = \frac{1}{M^{i,i}} \left(B^i - \sum_{j<i} M^{i,j}\, u_k^j - \sum_{j>i} M^{i,j}\, u_{k-1}^j \right)$$

improvements are worked in during the step

- **successive over relaxation method**

$$u_k^i = \alpha\, \bar{u}_k^i + (1-\alpha)\, u_{k-1}^i$$

with

$$\bar{u}_k^i = \frac{1}{M^{i,i}} \left(B^i - \sum_{j<i} M^{i,j}\, u_k^j - \sum_{j>i} M^{i,j}\, u_{k-1}^j \right)$$

$\alpha \in \Re$

averaging a Gauss-Seidel iterate

Predictor-Corrector Methods

- **modified trapezoidal method**

corrector

$$u(\tau_{n+1}) = u(\tau_n) + \frac{1}{2}\Big(A(\tau_{n+1})\,\bar{u}(\tau_{n+1}) + A(\tau_n)\,u(\tau_n)\Big)\,\frac{\Delta}{(\Delta x)^2}$$

predictor

$$\bar{u}(\tau_{n+1}) = u(\tau_n) + A(\tau_n)\,u(\tau_n)\,\frac{\Delta}{(\Delta x)^2}$$

- **predictor-corrector method**

corrector

$$u(\tau_{n+1}) = u(\tau_n) + A(\tau_{n+1})\,\bar{u}(\tau_{n+1})\,\frac{\Delta}{(\Delta x)^2}$$

predictor

$$\bar{u}(\tau_{n+1}) = u(\tau_n) + A(\tau_n)\,u(\tau_n)\,\frac{\Delta}{(\Delta x)^2}$$

does not propagate errors severely

even if $\frac{\Delta}{(\Delta x)^2}$ not extremely small

Boundary Conditions

$$\lim_{S \to 0} c_{T,K}(t, S) = 0$$

$$\lim_{S \to \infty} c_{T,K}(t, S) = S - K \, \exp\{-r \, (T - t)\}$$

boundary conditions have to be translated

into finite difference method

finite interval

$$x_{\min} = x_0 < \ldots < x_N = x_{\max}$$

$x_{\min} = x_0 = 0$ with

$$u_0(t) = 0$$

$x_{\max} = x_N < \infty$ where

$$u_N(t) \approx x_N - K\ \exp\{-r\,(T-t)\}$$

Truncation at the Boundaries

$$X_t = \ln(S_t)$$

$$dX_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t$$

terminal payoff $\quad f(x) \geq 0$

PDE

$$\hat{L}\, u(t,x) = \frac{1}{2}\,\sigma^2\,\frac{\partial^2 u(t,x)}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial u(t,x)}{\partial x} - r\,u(t,x) = 0$$

for $(t,x) \in (0,T) \times \Re$ with

$$u(T,x) = f(x)$$

for $x \in \Re$

- **approximating problem**

  area $\quad [0, T] \times (-\ell, \ell)$

  finite difference method

- Dirichlet conditions

  $u(t, \ell)$ and $u(t, -\ell)$ $\quad$ given values

- Neumann conditions

  $\frac{\partial u(t,\ell)}{\partial x}$ and $\frac{\partial u(t,-\ell)}{\partial x}$

For simplicity

PDE

$$\hat{L}\, u_\ell(t, x) = 0$$

for $(t, x) \in (0, T) \times (-\ell, \ell)$

with Dirichlet boundary conditions

$$u_\ell(t, -\ell) = \underline{u}_\ell(t) \quad \text{and} \quad u_\ell(t, \ell) = \bar{u}_\ell(t)$$

for $t \in [0, T)$

with terminal boundary conditions

$$u_\ell(T, x) = f(x)$$

for $x \in [-\ell, \ell]$

assume European payoff function $f(\cdot)$ is bounded

Lamberton & Lapeyre (2007)

$\Longrightarrow$

Lemma 7.1 *For* $(t, x) \in (0, T) \times \Re$ *one has the limit*

$$\lim_{\ell \to \infty} u_\ell(t, x) = u(t, x).$$

Convergence of Finite Difference Approximation

theta method

$$\begin{aligned} u^{\Delta}(\tau_{n+1}) &= u^{\Delta}(\tau_n) + \Big(\theta\, A(\tau_{n+1})\, u^{\Delta}(\tau_{n+1}) \\ &\quad + (1-\theta)\, A(\tau_n)\, u^{\Delta}(\tau_n)\Big) \frac{\Delta}{(\Delta x)^2} \end{aligned}$$

$$u_k^{\Delta}(\tau_n) = u^{\Delta}(\tau_n, x_k)$$

linearly interpolate in time and space

Assuming uniform boundedness of u and $\frac{\partial u}{\partial x}$

Raviart & Thomas (1983)

$\Longrightarrow$

Lemma 7.2 *For $\theta \in [0, \frac{1}{2})$ it follows as Δ and Δx tend to zero with $\frac{\Delta}{(\Delta x)^2} \to 0$ that*

$$\lim_{\Delta \to 0} u^{\Delta}(t, x) = u(t, x).$$

when Δx gets too small $\quad\Longrightarrow\quad$ unreliable

Lemma 7.3 *For $\theta \in [\frac{1}{2}, 1]$ it follows as Δ and Δx tend to zero that*

$$\lim_{\Delta \to 0} u^{\Delta}(t, x) = u_\ell(t, x)$$

for $(t, x) \in (0, T) \times (-\ell, \ell)$.

ratio $\frac{\Delta}{(\Delta x)^2}$ does not need to go to zero

Crank-Nicolson method $\theta = \frac{1}{2}$

smallest θ

Numerical Results on Finite Difference Methods

Black-Scholes dynamics

$$dS_t = S_t \, (r \, dt + \sigma \, dW_t)$$

zero lowest grid point in S

$S_{\max}$ as upper level

at maturity payoff function

invert the resulting tridiagonal matrix

stability of the explicit theta method

variable

$$\alpha = \left(\frac{\sigma^2 S^2}{(\Delta x)^2} + r \right) \Delta$$

less than one

Wilmott, Dewynne & Howison (1993)

error be not enhanced

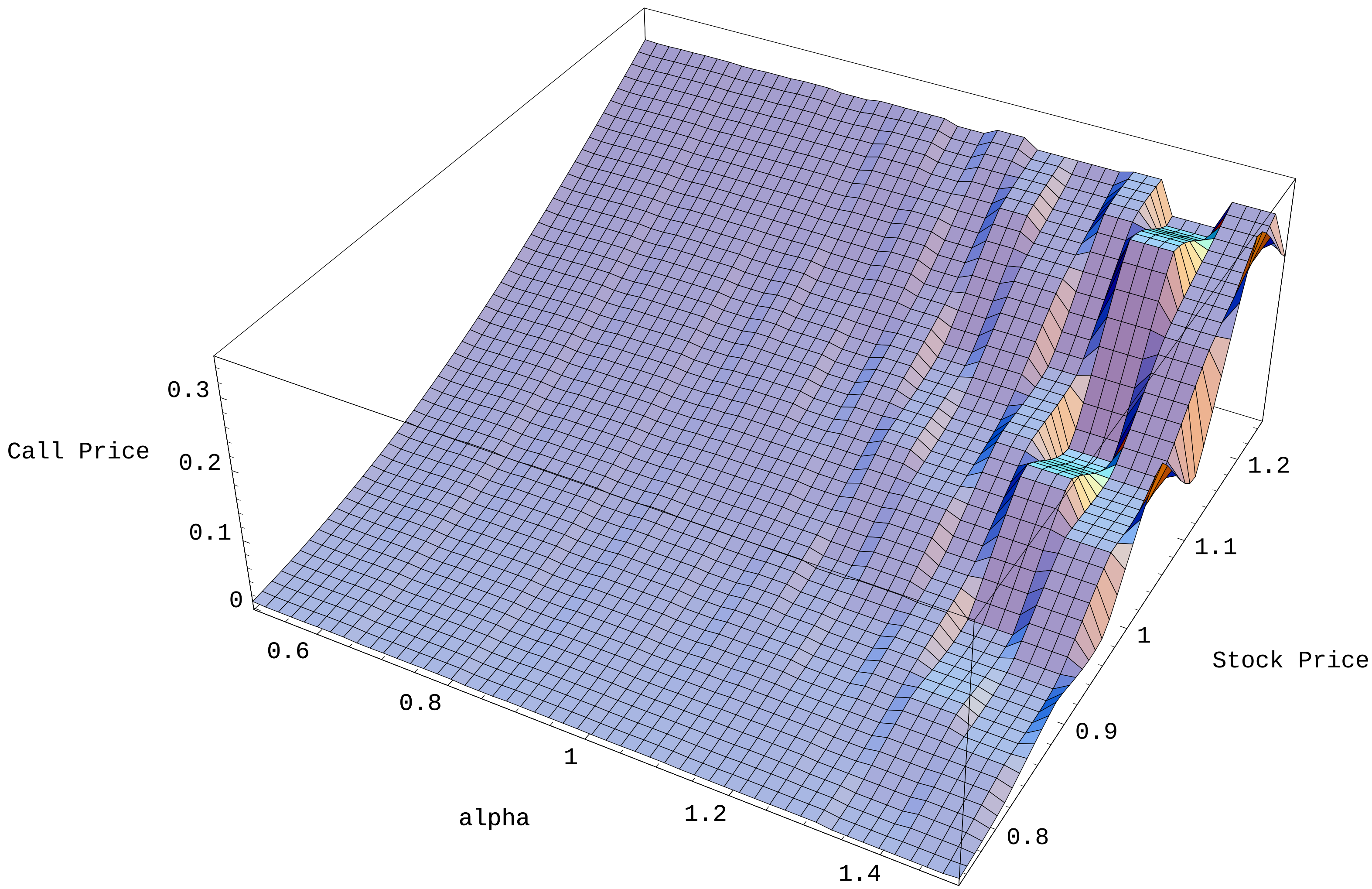

Figure 7.21: Prices in dependence on α and S_0 for the explicit finite difference method.

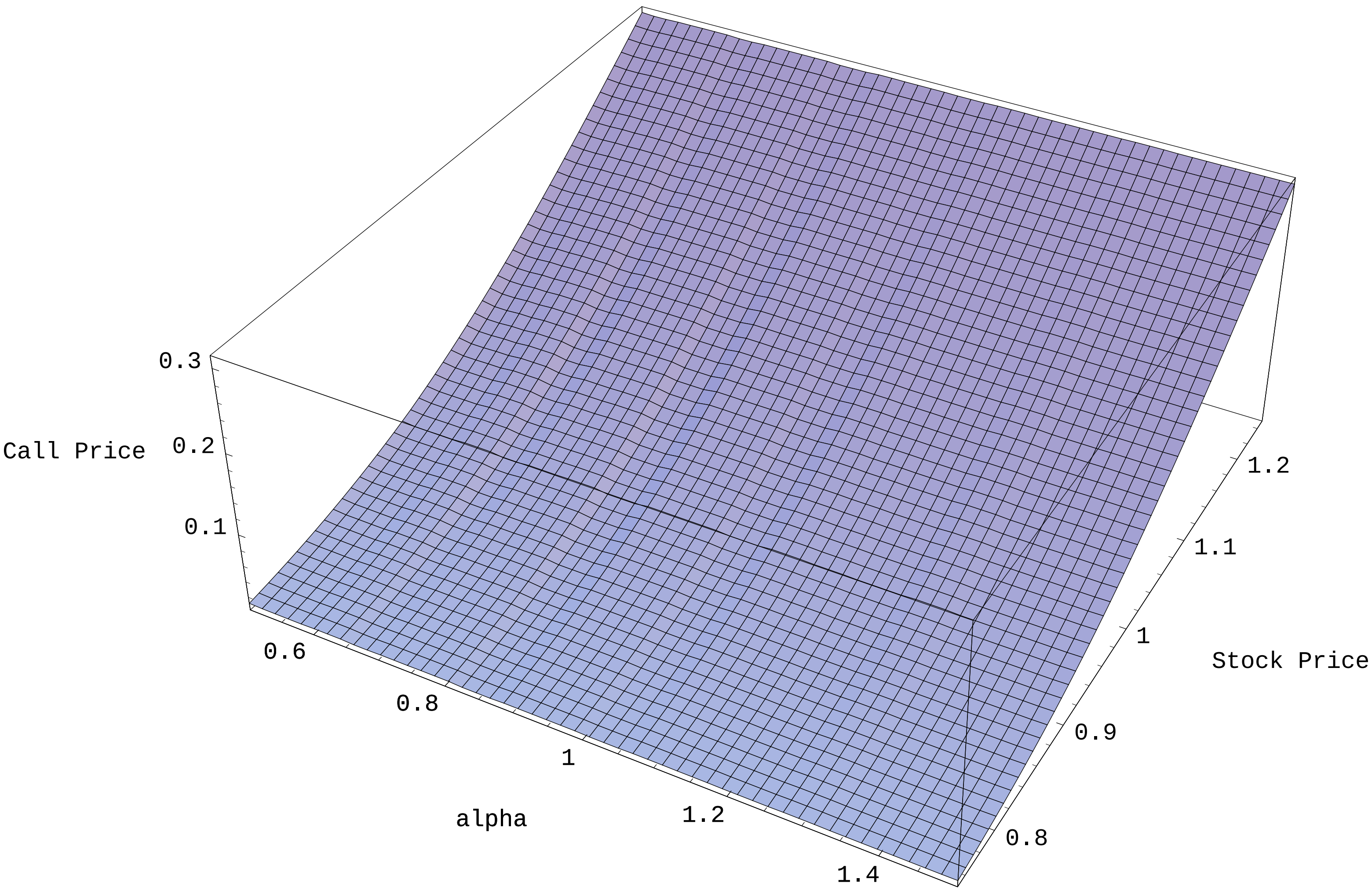

Figure 7.22: Prices in dependence on α for implicit finite difference method.

On the Accuracy of the Finite Difference Methods

for prices   Crank-Nicolson scheme is always superior

symmetric averaging $\Longrightarrow$

higher order of convergence

four digit accuracy   one more digit of precision

four digits of precision

five times more CPU time with explicit method

five digits   Crank-Nicolson   20 times faster

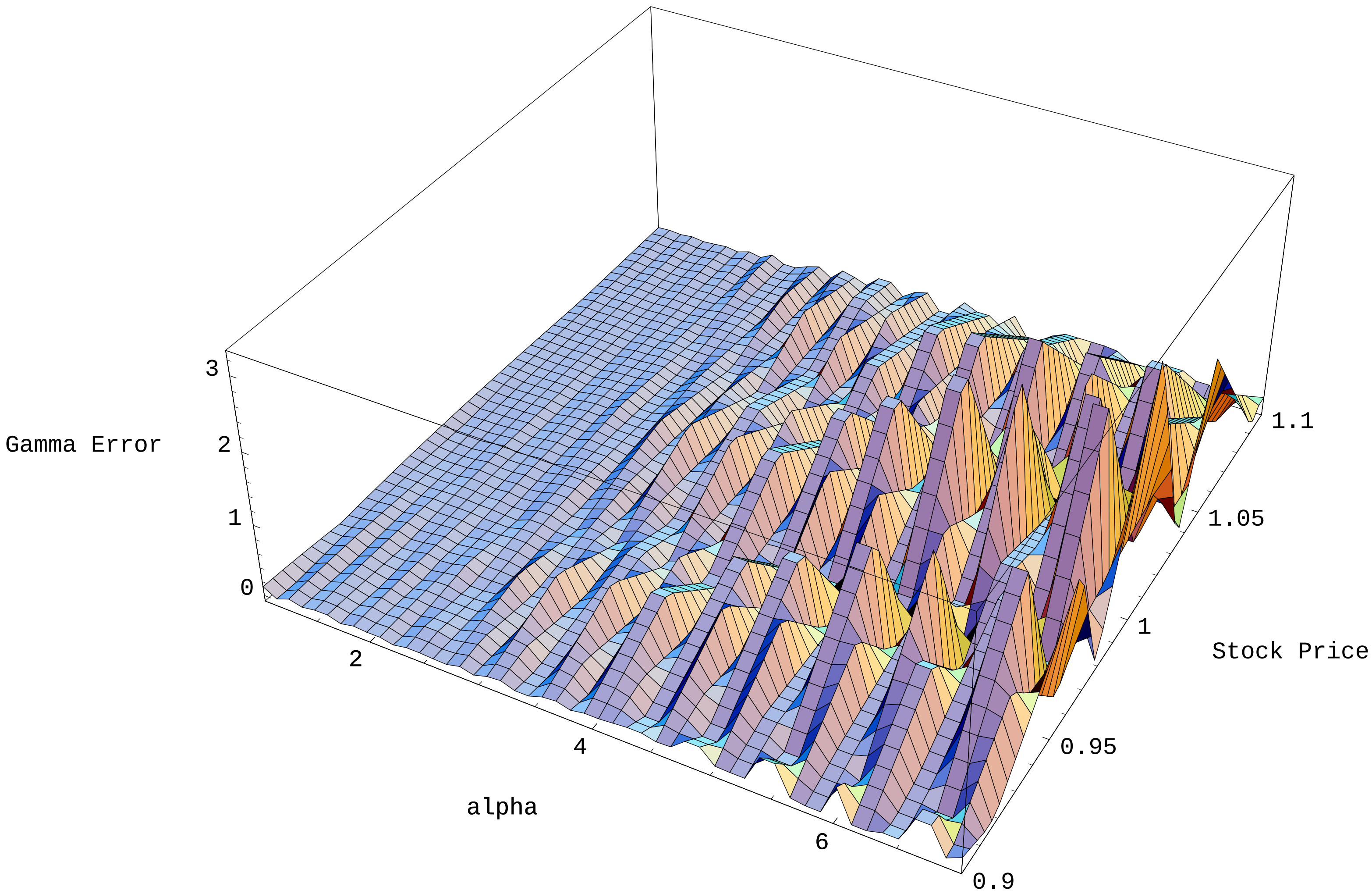

Figure 7.23: Difference between Crank-Nicolson and Black-Scholes gamma.

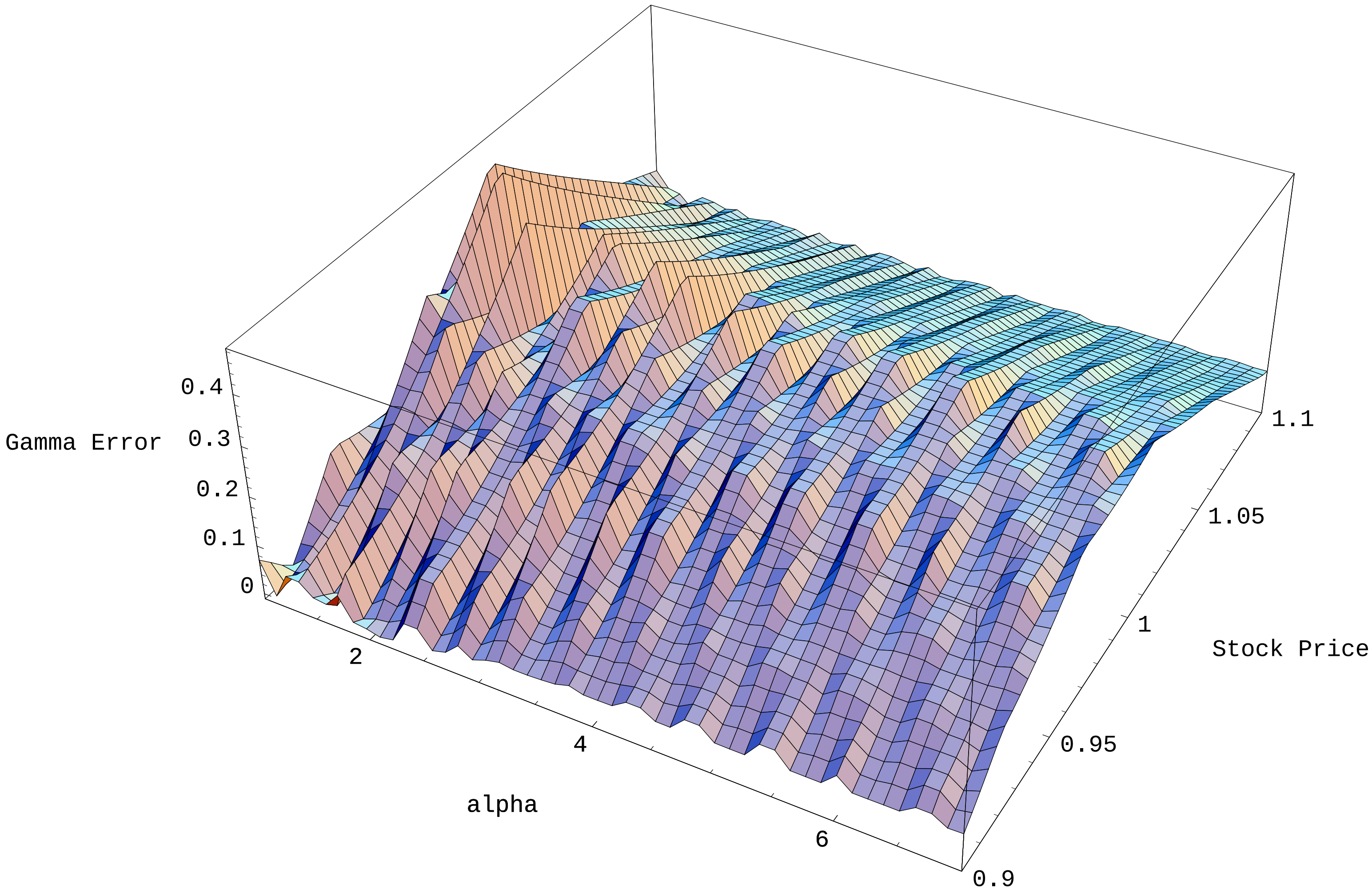

Figure 7.24: Difference between gamma from implicit method and Black-Scholes formula.

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