NUMERICAL METHODS OF FINANCE

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4 Stochastic Expansions

Stochastic Taylor Expansions

Deterministic Taylor Formula

- ordinary differential equation

$$\frac{d}{dt} X_t = a(X_t)$$

- integral form

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)\, ds$$

deterministic chain rule

$\Longrightarrow$

$$\frac{d}{dt} f(X_t) = a(X_t) \frac{\partial}{\partial x} f(X_t)$$

- operator

$$L = a \frac{\partial}{\partial x}$$

$\Longrightarrow$

integral equation

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} L\, f(X_s)\, ds$$

- special case

  $f(x) \equiv x$

  $Lf = a, LLf = (L)^2 f = La, \ldots$

- apply to $f = a$

$\Longrightarrow$

$$
\begin{aligned}
\boldsymbol{X}_t &= \boldsymbol{X}_{t_0} + \int_{t_0}^{t} \left(\boldsymbol{a}(\boldsymbol{X}_{t_0}) + \int_{t_0}^{s} \boldsymbol{L}\, \boldsymbol{a}(\boldsymbol{X}_z)\, \boldsymbol{dz} \right) \boldsymbol{ds} \\
&= \boldsymbol{X}_{t_0} + \boldsymbol{a}(\boldsymbol{X}_{t_0}) \int_{t_0}^{t} \boldsymbol{ds} + \int_{t_0}^{t} \int_{t_0}^{s} \boldsymbol{L}\, \boldsymbol{a}(\boldsymbol{X}_z)\, \boldsymbol{dz}\, \boldsymbol{ds}
\end{aligned}
$$

$\Longrightarrow$

nontrivial Taylor expansion

- apply $\boldsymbol{f = L a}$

$\Longrightarrow$

$$\boldsymbol{X_t = X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + L\, a(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dz\, ds + R_1}$$

remainder term

$$\boldsymbol{R_1 = \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} (L)^2 a(X_u)\, du\, dz\, ds}$$

- continuing

$\Longrightarrow$

classical deterministic Taylor formula

$$f(X_t) \;=\; f(X_{t_0}) + \sum_{l=1}^{r} \frac{(t-t_0)^\ell}{l!} (L)^\ell f(X_{t_0})$$

$$+ \int_{t_0}^{t} \cdots \int_{t_0}^{s_2} (L)^{r+1} f(X_{s_1})\, ds_1 \ldots ds_{r+1}$$

for $t \in [t_0, T]$ and $r \in \mathcal{N}$

Wagner-Platen Expansion

Wagner & Platen (1978), Platen (1982b),

Platen & Wagner (1982) and Kloeden & Platen (1992)

- SDE

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)\, ds + \int_{t_0}^{t} b(X_s)\, dW_s$$

- Itô formula

$$
\begin{aligned}
f(X_t) &= f(X_{t_0}) \\
&\quad + \int_{t_0}^{t} \left(a(X_s)\,\frac{\partial}{\partial x} f(X_s) + \frac{1}{2}\, b^2(X_s)\,\frac{\partial^2}{\partial x^2} f(X_s) \right) ds \\
&\quad + \int_{t_0}^{t} b(X_s)\,\frac{\partial}{\partial x} f(X_s)\, dW_s \\
&= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s)\, ds + \int_{t_0}^{t} L^1 f(X_s)\, dW_s
\end{aligned}
$$

operators

$$L^0 = a\,\frac{\partial}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2}{\partial x^2}$$

and

$$L^1 = b\,\frac{\partial}{\partial x}$$

$$f(x) \equiv x \quad \Longrightarrow \quad L^0 f = a \text{ and } L^1 f = b$$

- apply Itô formula to $f = a$ and $f = b$

$$
\begin{aligned}
X_t &= X_{t_0} \\
&\quad + \int_{t_0}^{t} \left(a(X_{t_0}) + \int_{t_0}^{s} L^0 a(X_z)\, dz + \int_{t_0}^{s} L^1 a(X_z)\, dW_z \right) ds \\
&\quad + \int_{t_0}^{t} \left(b(X_{t_0}) + \int_{t_0}^{s} L^0 b(X_z)\, dz + \int_{t_0}^{s} L^1 b(X_z)\, dW_z \right) dW_s \\
&= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + R_2
\end{aligned}
$$

remainder term

$$R_2 = \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(X_z)\,dz\,ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(X_z)\,dW_z\,ds$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z)\,dz\,dW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(X_z)\,dW_z\,dW_s$$

- apply Itô formula to $f = L^1 b$

$\Longrightarrow$

$$X_t = X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + L^1 b(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dW_z \, dW_s + R_3$$

remainder term

$$
\begin{aligned}
R_3 \;=\; & \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(X_z)\,dz\,ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(X_z)\,dW_z\,ds \\
& + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z)\,dz\,dW_s \\
& + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^0 L^1 b(X_u)\,du\,dW_z\,dW_s \\
& + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^1 L^1 b(X_u)\,dW_u\,dW_z\,dW_s
\end{aligned}
$$

example for Wagner-Platen expansion

- multiple Itô integrals

$$\int_{t_0}^{t} ds = t - t_0,$$

$$\int_{t_0}^{t} dW_s = W_t - W_{t_0},$$

$$\int_{t_0}^{t}\int_{t_0}^{s} dW_z\, dW_s = \frac{1}{2}\Big((W_t - W_{t_0})^2 - (t - t_0)\Big)$$

- remainder term R_3

  consisting of next following multiple Itô integrals

  with nonconstant integrands

Generalized Wagner-Platen Expansion

- expanding with respect to process $X = \{X_t,\ t \in [0, T]\}$

- Itô formula

$$f(t, X_t) = f(t_0, X_{t_0}) + \int_{t_0}^{t} \frac{\partial}{\partial t} f(s, X_s)\, ds + \int_{t_0}^{t} \frac{\partial}{\partial x} f(s, X_s)\, dX_s + \frac{1}{2} \int_{t_0}^{t} \frac{\partial^2}{\partial x^2} f(s, X_s)\, d[X]_s$$

quadratic variation

$$[X]_t = [X]_{t_0} + \int_{t_0}^{t} b^2 (X_s)\, ds$$

- expand further

$$f(t, X_t) = f(t_0, X_{t_0})$$

$$+ \int_{t_0}^{t} \left\{ \frac{\partial}{\partial t} f(t_0, X_{t_0}) + \int_{t_0}^{s} \frac{\partial^2}{\partial t^2} f(z, X_z)\, dz \right.$$

$$\left. + \int_{t_0}^{s} \frac{\partial^2}{\partial x\, \partial t} f(z, X_z)\, dX_z + \int_{t_0}^{s} \frac{1}{2} \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial t} f(z, X_z)\, d[X]_z \right\} ds$$

$$
\begin{aligned}
&+\int_{t_0}^{t}\left\{\frac{\partial}{\partial x}f(t_0,X_{t_0})+\int_{t_0}^{s}\frac{\partial^2}{\partial t\,\partial x}f(z,X_z)\,dz\right.\\
&\qquad\left.+\int_{t_0}^{s}\frac{\partial^2}{\partial x^2}f(z,X_z)\,dX_z+\int_{t_0}^{s}\frac{1}{2}\,\frac{\partial^3}{\partial x^3}f(z,X_z)\,d[X]_z\right\}dX_s\\
&+\frac{1}{2}\int_{t_0}^{t}\left\{\frac{\partial^2}{\partial x^2}f(t_0,X_{t_0})+\int_{t_0}^{s}\frac{\partial^3}{\partial t\,\partial x^2}f(z,X_z)\,dz\right.\\
&\qquad\left.+\int_{t_0}^{s}\frac{\partial^3}{\partial x^3}f(z,X_z)\,dX_z+\int_{t_0}^{s}\frac{1}{2}\,\frac{\partial^4}{\partial x^4}f(z,X_z)\,d[X]_z\right\}d[X]_s
\end{aligned}
$$

$$
\begin{aligned}
=\;& f(t_0,X_{t_0})+\frac{\partial}{\partial t}f(t_0,X_{t_0})\,(t-t_0)+\frac{\partial}{\partial x}f(t_0,X_{t_0})\,(X_t-X_{t_0})\\
&+\frac{1}{2}\,\frac{\partial^2}{\partial x^2}f(t_0,X_{t_0})\,\left([X]_t-[X]_{t_0}\right)+R_f(t_0,t)
\end{aligned}
$$

$$
\begin{aligned}
f(t, X_t) \;=\; & f(t_0, X_{t_0}) + \frac{\partial}{\partial t} f(t_0, X_{t_0})\,(t - t_0) \\
& + \frac{\partial}{\partial x} f(t_0, X_{t_0})\,(X_t - X_{t_0}) \\
& + \frac{1}{2}\,\frac{\partial^2}{\partial x^2} f(t_0, X_{t_0})\;([X]_t - [X]_{t_0}) \\
& + \frac{\partial^2}{\partial x\,\partial t} f(t_0, X_{t_0})\;\int_{t_0}^{t}\int_{t_0}^{s} dX_z\, ds \\
& + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\,\frac{\partial}{\partial t} f(t_0, X_{t_0})\;\int_{t_0}^{t}\int_{t_0}^{s} d[X]_z\, ds \\
& + \frac{\partial^2}{\partial t\,\partial x} f(t_0, X_{t_0})\;\int_{t_0}^{t}\int_{t_0}^{s} dz\, dX_s
\end{aligned}
$$

$$+\frac{\partial^2}{\partial x^2} f(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dX_z \, dX_s$$

$$+\frac{1}{2} \frac{\partial^3}{\partial x^3} f(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} d[X]_z \, dX_s$$

$$+\frac{1}{2} \frac{\partial^4}{\partial t \, \partial x^3} f(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dz \, d[X]_s$$

$$+\frac{1}{2} \frac{\partial^3}{\partial x^3} f(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dX_z \, d[X]_s$$

$$+\frac{1}{4} \frac{\partial^4}{\partial x^4} f(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} d[X]_z \, d[X]_s + \bar{R}_f(t_0, t)$$

Expansions with Jumps

- counting process $N = \{N_t,\, t \in [0, T]\}$

- for right-continuous process Z

  jump size

$$\Delta\, Z_t = Z_t - Z_{t-}$$

$\Longrightarrow$

$$N_t = \sum_{s \in (0,t]} \Delta\, N_s$$

- measurable function

$f : \Re \to \Re$

$\Longrightarrow$

$$f(N_t) = f(N_0) + \sum_{s \in (0,t]} \Delta f(N_s)$$

consistent with Itô formula

$\Longrightarrow$

$$f(N_t) = f(N_0) + \int_{(0,t]} \Big(f(N_{s-} + 1) - f(N_{s-}) \Big) \, dN_s$$

- measurable function

$$\tilde{\Delta} f(N) = f(N+1) - f(N)$$

$\Longrightarrow$

$$\begin{aligned} f(N_t) &= f(N_0) + \int_{(0,t]} \tilde{\Delta} f(N_{s-})\, dN_s \\ &= f(N_0) + \int_{(0,t]} \tilde{\Delta} f(N_0)\, dN_s \\ &\quad + \int_{(0,t]} \int_{(0,s_2)} \tilde{\Delta} \left(\tilde{\Delta} f(N_{s_1-}) \right) dN_{s_1}\, dN_{s_2} \end{aligned}$$

$\Longrightarrow$

multiple stochastic integrals with respect to N

Engel (1982)

Platen (1984)

Studer (2001)

$$\int_{(0,t]} dN_s = N_t$$

$$\int_{(0,t]} \int_{(0,s_1)} dN_{s_1}\, dN_{s_2} = \frac{1}{2\,!}\, N_t\, (N_t - 1),$$

$$\int_{(0,t]} \int_{(0,s_1)} \int_{(0,s_2)} dN_{s_1}\, dN_{s_2}\, dN_{s_3} = \frac{1}{3\,!}\, N_t\, (N_t - 1)\, (N_t - 2),$$

$$\int_{(0,t]} \int_{(0,s_1)} \cdots \int_{(0,s_n)} dN_{s_1}\, \ldots\, dN_{s_{n-1}}\, dN_{s_n} = \begin{cases} \binom{N_t}{n} & \text{for } N_t \geq n \\ 0 & \text{otherwise} \end{cases}$$

for $t \in [0, T]$, where

$$\binom{i}{n} = \frac{i\,(i-1)\,(i-2)\ldots(i-n+1)}{1 \cdot 2 \cdot \ldots \cdot n} = \frac{i\,!}{n\,!\,(i-n)\,!}$$

for $i \geq n$

- **stochastic Taylor expansion**

$$f(N_t) = f(N_0) + \int_{(0,t]} \tilde{\Delta} f(N_0)\, dN_s + \int_{(0,t]} \int_{(0,s_2)} \tilde{\Delta}\left(\tilde{\Delta} f(N_0)\right) dN_{s_1}\, dN_{s_2} + \bar{R}_3(t)$$

with

$$\bar{R}_3(t) = \int_{(0,t]} \int_{(0,s_2)} \int_{(0,s_3)} \tilde{\Delta}\left(\tilde{\Delta}\left(\tilde{\Delta} f(N_{s_1-})\right)\right) dN_{s_1}\, dN_{s_2}\, dN_{s_3}$$

$\Longrightarrow$

$$f(N_t) = f(N_0) + \tilde{\Delta} f(N_0) \binom{N_t}{1} + \tilde{\Delta}\left(\tilde{\Delta} f(N_0)\right) \binom{N_t}{2} + \bar{R}_3(t)$$

$$\tilde{\Delta} f(N_0) = \tilde{\Delta} f(0) = f(1) - f(0)$$

$$\tilde{\Delta}\left(\tilde{\Delta} f(N_0)\right) = f(2) - 2 f(1) + f(0)$$

$\Longrightarrow$

$$f(N_t) = f(0) + (f(1) - f(0)) N_t + (f(2) - 2 f(1) + f(0)) \frac{1}{2} N_t (N_t - 1) + \bar{R}_3(t)$$

for $N_t \leq 3$ $\quad \bar{R}_3(t)$ equals zero

Measuring Market Risk for Linear Portfolios

Primary Securities

$$S_t = \left(S_t^{(0)}, \dots, S_t^{(d)}\right)^\top$$

$$dS_t^{(j)} = S_t^{(j)} \left\{ r_t \, dt + \sum_{k=1}^{d} b_t^{j,k} \left(\theta_t^k \, dt + dW_t^k \right) \right\}$$

$j \in \{0, 1, \dots, d\}$

invertible volatility matrix

$$b_t = [b_t^{j,k}]_{j,k=1}^d$$

market price for risk

$$\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)^\top = b_t^{-1}\,(a_t - r_t\,\mathbf{1})$$

savings account

$$S_t^{(0)} = \exp\left\{\int_0^t r_s\,ds\right\}$$

Strategies and Portfolios

- strategy

$$\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \ldots, \delta_t^d)^\top,\ t \in [0, T]\}$$

- portfolio

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j\, S_t^{(j)} = \delta_t^\top\, S_t$$

- self-financing

$$dS_t^\delta = \sum_{j=0}^{d} \delta_t^j\, dS_t^{(j)} = \delta_t^\top\, dS_t$$

- jth fraction

$$\pi_\delta^j(t) = \delta_t^j \, \frac{S_t^{(j)}}{S_t^\delta},$$

$$\sum_{j=0}^{d} \pi_\delta^j(t) = 1$$

- portfolio SDE

$$dS_t^\delta = S_t^\delta \left(r_t\, dt + \sum_{k=1}^{d} \beta_\delta^k(t)\, (\theta_t^k\, dt + dW_t^k) \right)$$

- portfolio volatility

$$\beta_\delta^k(t) = \sum_{j=0}^{d} \pi_\delta^j(t)\, b_t^{j,k}$$

$\Longrightarrow$

$$d\ln\left(S_t^\delta\right) = (r_t + g_\delta(t))\, dt + \sum_{k=1}^{d} \beta_\delta^k(t)\, dW_t^k$$

- portfolio net growth rate

$$g_\delta(t) = \sum_{k=1}^{d} \beta_\delta^k(t) \left(\theta_t^k - \frac{1}{2} \beta_\delta^k(t) \right)$$

- growth optimal portfolio

$$dS_t^{\delta_*} = S_t^{\delta_*} \left(r_t \, dt + \sum_{k=1}^{d} \theta_t^k \, (\theta_t^k \, dt + dW_t^k) \right)$$

General and Specific Market Risk

- each institution obliged to put aside sufficient capital as a buffer for large losses

  $\Longrightarrow$ **regulatory capital**

  investment bank, managed fund, insurance company or similar business

- **market risk** - risk of losing money from adverse movements of financial markets

  $\Longrightarrow$ *specific* and *general market risk*

  Basle(1996a, 1996b)

  Platen & Stahl (2003)

- **general market risk**

  risk exposure of the portfolio against the equity market as a whole

- **specific market risk**

  risk of holding an individual security within a portfolio which is not covered by general market risk

- **broadly based index**

  required by regulations for measuring general market risk

  take world stock index $\Longrightarrow$ **general market risk**

- **particular portfolio** - S_t^δ

  corresponding **benchmarked portfolio**

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}$$

$\Longrightarrow$ **specific risk**

efficient and natural separation of market risk

into general and specific market risk

Coherent Risk Measures

Artzner, Delbaen, Eber & Heath (1997)

Föllmer & Schiedt (2002)

A mapping ϱ is a *coherent risk measure* if

(i) If $X \geq 0$, then $\varrho(X) \leq 0$;

(ii) $\varrho(X_1 + X_2) \leq \varrho(X_1) + \varrho(X_2)$;

(ii) $\varrho(\lambda X) = \lambda \varrho(X)$ for $\lambda \geq 0$;

(iv) $\varrho(a + X) = \varrho(X) - a$ for each constant $a \in \Re$.

(i) - natural negative sign for positive movements

(ii) - *subadditivity*

(iii) - *positive homogeneity*

(iv) - *translation invariance*

- there exist also convex risk measures

Value at Risk

Basle (1996a, 1996b)

- strictly increasing distribution function F_X

- quantile function

$$F_X^{-1}(\alpha) = \inf\{x \in \Re :\, F_X(x) > \alpha\}$$

- α-quantile

$$q_\alpha = F_X^{-1}(\alpha)$$

Value at Risk of a random return

$$X = \frac{S^\delta_{t+\Delta} - S^\delta_t}{S^\delta_t}$$

for a portfolio S^δ_t and a given level $\alpha \in (0, 1)$ is

$$\mathrm{VaR}_\alpha(X) = -F^{-1}_X(\alpha)\, S^\delta_t$$

$\Longrightarrow$

$$\mathrm{VaR}_\alpha(X) = -q_\alpha\, S^\delta_t$$

Use Wagner-Platen expansion to approximate X

- Gaussian return

  mean $\mu = 0$

  variance v

  $\tilde{q}_\alpha$ denotes the α-quantile of a standard Gaussian random variable

  $\Longrightarrow$

$$\mathrm{VaR}_\alpha(X) = -\tilde{q}_\alpha \sqrt{v}\, S^\delta_t$$

  $\tilde{q}_{0.05} = -1.645, \ \tilde{q}_{0.01} = -2.326$

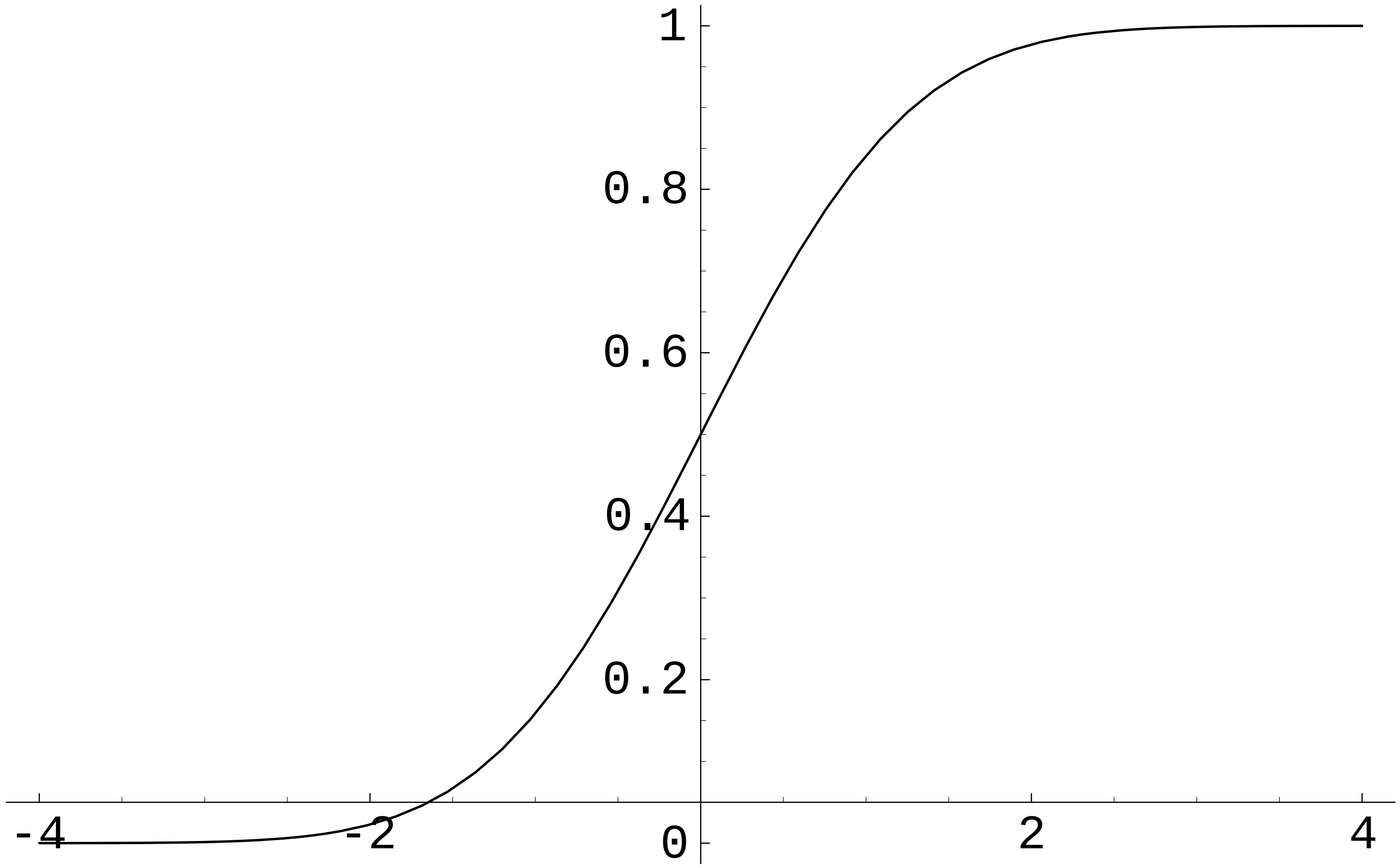


Figure 4.1: Gaussian distribution function.

- **VaR is *not* a coherent risk measure** under general circumstances

  Artzner, Delbaen, Eber & Heath (1997)

- random return linear combination

  of underlying risk factors $X_1, \ldots, X_d$

  elliptical joint distribution

  density is constant on ellipsoids

  $\Longrightarrow$

  $\mathbf{VaR}_\alpha(X)$ is a **coherent risk measure**

  Embrechts, McNeal & Straumann (1999)

- in general, **subadditivity** can be violated

  risk capital assigned to a diversified position may be bigger than the sum of the risk capitals allocated

- for option $\mathbf{VaR}_{\boldsymbol{\alpha}}(\boldsymbol{S}^{\delta}(\boldsymbol{T}))$ can be misleading

- in practice if portfolio is diversified
  and market model is realistic

  subadditivity is likely

  $\Longrightarrow$ $\quad \mathbf{VaR}_{\boldsymbol{\alpha}}$ coherent risk measure

VaR for Conditionally Gaussian Random Variables

- conditionally Gaussian random variable

  stochastic but independent variance v

  $\Longrightarrow$ mixture of normals

- if $\frac{1}{v}$ is χ^2-distributed with ν degrees of freedom

  $\Longrightarrow$ X is Student t with ν degrees of freedom

- if v is gamma distributed

  $\Longrightarrow$ X is variance gamma

- if X is a linear combination of returns

$$X = \sum_{i=1}^{d} \pi_\delta^i \, R_i$$

with

$$R_i = Z_i \sqrt{v}$$

Z_i - standard Gaussian

v - independent random variance

$\Longrightarrow$ $R_1, \ldots, R_d$ is elliptical

$\Longrightarrow$ **coherent risk measure**

- **correlated conditionally Gaussian** returns

covariance matrix

$$D = \left[d^{j,\ell}\right]_{j,\ell=1}^{d}$$

$$D = b\, b^\top$$

Cholesky decomposition

$$b = \left[b^{j,\ell}\right]_{j,\ell=1}^{d}$$

invertible matrix

- **return vector**

$$\boldsymbol{R} = (\boldsymbol{R_1}, \ldots, \boldsymbol{R_d})^\top = \sqrt{\boldsymbol{v}}\, \boldsymbol{b}\, \boldsymbol{Z}$$

correlated according to $\boldsymbol{D}$

$\boldsymbol{R_1}, \ldots, \boldsymbol{R_d}$ is elliptical

$\Longrightarrow$ representation

$$\boldsymbol{X} = \boldsymbol{\pi}_{\boldsymbol{\delta}}^\top \boldsymbol{R} = |\boldsymbol{\pi}_{\boldsymbol{\delta}}^\top \boldsymbol{b}|\, \boldsymbol{\xi}$$

weight vector

$$\boldsymbol{\pi}_{\boldsymbol{\delta}} = \left(\boldsymbol{\pi}_{\boldsymbol{\delta}}^{\boldsymbol{1}}, \ldots, \boldsymbol{\pi}_{\boldsymbol{\delta}}^{\boldsymbol{d}}\right)^\top \in \Re^{\boldsymbol{d}}$$

$\boldsymbol{\xi}$ normal mixture distributed scalar random variable with variance $c^2 \Delta$

For Student t distributed multivariate log-returns
ξ is Student t distributed

- $\mathbf{VaR}_\alpha(X)$ is in this case a **coherent risk measure**

$\Longrightarrow$ internal risk measure for capital allocation

significantly simplifies the VaR calculation

$$\mathbf{VaR}_\alpha(X) \approx -\sqrt{\pi_\delta^\top D\, \pi_\delta}\, c\, \sqrt{\Delta}\, \tilde{t}_\alpha S_t^\delta$$

$\tilde{t}_\alpha$ - α-quantile of standardized normal mixture distribution

S_t^δ - present face value of the portfolio

$\tilde{q}_\alpha$ - standard Gaussian α-quantile

- **event factor**

$$\tilde{\varphi}_\alpha = \frac{\tilde{t}_\alpha}{\tilde{q}_\alpha}$$

Platen & Stahl (2003)

$\Longrightarrow$

$$\mathrm{VaR}_\alpha(X) = -\sqrt{\pi_\delta^\top D\, \pi_\delta}\; c \sqrt{\Delta}\; \tilde{q}_\alpha\, \tilde{\varphi}_\alpha\, S_t^\delta$$

- **event risk**

  Basle(1996a, 1996b)

  consider **Student t** distributed log-returns with $\nu \in \{2, 3, 4, 5, 10\}$

  $\Longrightarrow$ event factor $\tilde{\varphi}_\alpha$

ν	∞	10	5	4	3	2
$\tilde{\varphi}_{0.01}$	1	1.06	1.11	1.12	1.14	1.16

Table 1: Event factor $\tilde{\varphi}_\alpha$ in dependence on degrees of freedom ν.

- extensive **historical simulations**

  Gibson (2001)

  $\Longrightarrow$ average event factor $\tilde{\varphi}_{\mathbf{0.01}} \approx \mathbf{1.12}$

  typical portfolios of US institutions

  coincides exactly with the event factor that is obtained for the **Student t distribution with four degrees** of freedom

  $\Longrightarrow$ supports alternative market model

Internal Models

- *internal models* for calculating **regulatory capital**

  Basle(1996a, 1996b)

  reduces the required *regulatory capital*

- if the portfolio of an institution is not linear
  but forms a **diversified portfolio**

  $\Longrightarrow$ approximately GOP

  Fergusson & Platen (2006)

  Student t distributed

Wagner-Platen Expansion for a BS Portfolio

Studer (2001)

Giannelli & Primbs (2001)

Black-Scholes dynamics of $S^{(1)}$

$$
\begin{aligned}
\Delta\, S_t^{(\tilde{\delta})} &= \int_t^{t+\Delta} \tilde{\delta}_s^{(0)}\, dS_s^{(0)} + \int_t^{t+\Delta} \tilde{\delta}_s^{(1)}\, dS_s^{(1)} \\
&= \int_t^{t+\Delta} \tilde{\delta}_s^{(0)}\, S_s^{(0)}\, r_s\, ds + \int_t^{t+\Delta} \tilde{\delta}_s^{(1)}\, S_s^{(1)} \\
&\qquad \times \left(\left(r_s + b_s^{1,1}\, \theta_s^1 \right) ds + b_s^{1,1}\, dW_s^1 \right)
\end{aligned}
$$

$$
\begin{aligned}
\approx\quad & \left(\tilde{\delta}^{(0)}_t\, S^{(0)}_t\, r_t + \tilde{\delta}^{(1)}_t\, S^{(1)}_t\left(r_t + b^{1,1}_t\,\theta^1_t\right)\right)\Delta \\
& + \tilde{\delta}^{(1)}_t\, S^{(1)}_t\, b^{1,1}_t\,\Delta W^1_t \\
& + \frac{\partial}{\partial S^{(1)}}\left[\tilde{\delta}^{(1)}_t\, S^{(1)}_t\, b^{1,1}_t\right]\tilde{\delta}^{(1)}_t\, S^{(1)}_t\, b^{1,1}_t\,\frac{1}{2}\,\left[(\Delta W^1_t)^2 - \Delta\right] \\
=\quad & \overline{\Delta\, S^{(\tilde{\delta})}_t}
\end{aligned}
$$

$$
\Delta W^1_t = W^1_{t+\Delta} - W^1_t
$$

Multiple Stochastic Integrals

- d-dimensional Itô equation

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)\, ds + \sum_{j=1}^{m} \int_{t_0}^{t} b^j(s, X_s)\, dW_s^j$$

- equivalent Stratonovich equation

$$X_t = X_{t_0} + \int_{t_0}^{t} \underline{a}(s, X_s)\, ds + \sum_{j=1}^{m} \int_{t_0}^{t} b^j(s, X_s) \circ dW_s^j$$

$$\underline{a}^i = a^i - \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{d} b^{k,j} \frac{\partial b^{i,j}}{\partial x^k}$$

Multi-indices

$$\alpha = (j_1, j_2, \ldots, j_\ell)$$

where

$$j_i \in \{0, 1, \ldots, m\}$$

for $i \in \{1, 2, \ldots, \ell\}$ and $m \in \mathcal{N}$ is a *multi-index* of *length*

$$\ell = \ell(\alpha) \in \mathcal{N}$$

- m - number of components of Wiener process

  v - multi-index of length zero

$$\ell(v) = 0$$

  $\ell((0,1)) = 2,\ \ell((0,1,0)) = 3$

- $n(\alpha)$ - number of components of a multi-index α which equal 0

  $n((1,0,1)) = 1, n((0,1,0)) = 2, n((0,0)) = 2$

- **set of all multi-indices**

$$\mathcal{M}_m = \Big\{(j_1, j_2, \ldots, j_\ell) : j_i \in \{0, 1, \ldots, m\},$$
$$i \in \{1, 2, \ldots, \ell\}, \text{ for } \ell \in \mathcal{N}\Big\} \cup \{v\}$$

- Given $\alpha \in \mathcal{M}_m$ with $\ell(\alpha) \geq 1$,

  $-\alpha$ or $\alpha-$ obtained by deleting

  the first or the last component of α, respectively.

  $-(1, 0) = (0)$, $\quad (1, 0)- = (1)$ and

  $-(0, 1, 1) = (1, 1)$, $\quad (0, 1, 1)- = (0, 1)$

- for $\alpha = (j_1, j_2, \ldots, j_k)$, $\bar{\alpha} = (\bar{j}_1, \bar{j}_2, \ldots, \bar{j}_\ell)$

concatenation operation $*$

$$\alpha * \bar{\alpha} = (j_1, j_2, \ldots, j_k, \bar{j}_1, \bar{j}_2, \ldots, \bar{j}_\ell)$$

for $\alpha = (0, 1, 2)$ and $\bar{\alpha} = (1, 3)$

$\alpha * \bar{\alpha} = (0, 1, 2, 1, 3)$ and $\bar{\alpha} * \alpha = (1, 3, 0, 1, 2)$

Multiple Itô Integrals

Let ϱ and τ be two stopping times

$$0 \leq \varrho \leq \tau \leq T$$

multi-index $\alpha = (j_1, j_2, \ldots, j_\ell) \in \mathcal{M}_m$

$f \in \mathcal{H}_\alpha$

- **multiple Itô integral**

$$I_\alpha[f_\cdot]_{\varrho,\tau} = \begin{cases} f_\tau & \text{for} \quad \ell = 0 \\ \int_\varrho^\tau I_{\alpha-}[f_\cdot]_{\varrho,s}\, ds & \text{for} \quad \ell \geq 1 \ \text{ and } \ j_\ell = 0 \\ \int_\varrho^\tau I_{\alpha-}[f_\cdot]_{\varrho,s}\, dW_s^{j_\ell} & \text{for} \quad \ell \geq 1 \ \text{ and } \ j_\ell \geq 1 \end{cases}$$

- in the case $f_t = 1$

  abbreviate $I_{\alpha,\tau}$ by I_α

Relationships between Multiple Itô Integrals

- write

$$I_{\alpha,t} = I_\alpha[1]_{0,t}$$

$$W_t^0 = t$$

- **relationship**

  for $\alpha = (j_1, \ldots, j_i, j_{i+1}, \ldots, j_\ell)$

$$W_t^j \, I_{\alpha,t} = \sum_{i=0}^{\ell} I_{(j_1,\ldots,j_i,j,j_{i+1},\ldots,j_\ell),t} + \sum_{i=1}^{\ell} \mathbf{1}_{\{j_i=j\neq 0\}} \, I_{(j_1,\ldots,j_{i-1},0,j_{i+1},\ldots,j_\ell),t}$$

Hermite Polynomials and Multiple Itô Integrals

Kloeden & Platen (1999)

- **Hermite polynomials**

$$H_n(x) = (-1)^n \, e^{x^2} \, \frac{d^n}{dx^n} \, \exp\{-x^2\}$$

generating function

$$\exp\{2\, z\, x - z^2\} = \sum_{n=0}^{\infty} H_n(x) \, \frac{z^n}{n\,!}$$

- **monic Hermite polynomials**

$$\tilde{H}_n(t,x) = \left(\frac{t}{2}\right)^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2t}}\right)$$

- **scaled monic Hermite polynomials**

$$h_n(t,x) = \frac{1}{n\,!}\,\tilde{H}_n(t,x)$$

for $n \in \{0, 1, \ldots\}$, $x \in \Re$, $t \in (0, \infty)$

$\Longrightarrow$ **Hermite polynomials**

$$H_n(x) = n!\sum_{j=0}^{[\frac{n}{2}]} \frac{(-1)^j\,(2\,x)^{n-2j}}{j!\,(n-2j)!}$$

for $n \in \{0, 1, \dots\}$ and $x \in \Re$

$[y]$ - largest integer not greater than y

$\Longrightarrow$ **scaled monic Hermite polynomials**

$$h_n(t,x) = \sum_{j=0}^{[\frac{n}{2}]} \frac{(-1)^j\,x^{n-2j}}{j!\,(n-2j)!}\,\left(\frac{t}{2}\right)^j$$

with properties

$$\frac{\partial}{\partial x}\, h_n(t,x) = h_{n-1}(t,x)$$

and

$$\frac{\partial}{\partial t}\, h_n(t,x) = -\frac{1}{2}\,\frac{\partial^2}{\partial x^2}\, h_n(t,x)$$

for $x \in \Re$, $t \in (0,\infty)$ and $n \in \{1,2,\ldots\}$

- For $\alpha = (j_1, \ldots, j_\ell) = (0, \ldots, 0)$
  with $j_i = 0$ for $i \in \{1, 2, \ldots, \ell\}$

$$I_{\alpha,t} = \int_0^t \ldots \int_0^{s_2} ds_1 \ldots ds_\ell = \frac{t^\ell}{\ell\,!}$$

- For $\alpha = (j_1, \ldots, j_\ell) = (j, \ldots, j)$

  with $j_i = j \in \{1, 2, \ldots, m\}$ for $i \in \{1, 2, \ldots, \ell\}$

$$I_{\alpha,t} = \int_0^t \ldots \int_0^{s_2} dW_{s_1}^j \ldots dW_{s_\ell}^j = h_\ell(t, W_t^j)$$

$$
\begin{aligned}
I_{(j),t} &= W_t^j \\
I_{(j,j),t} &= \frac{1}{2}\left(\left(W_t^j\right)^2 - t\right) \\
I_{(j,j,j),t} &= \frac{1}{3\,!}\left(\left(W_t^j\right)^3 - 3\,t\,W_t^j\right) \\
I_{(j,j,j,j),t} &= \frac{1}{4\,!}\left(\left(W_t^j\right)^4 - 6\,t\,(W_t^j)^2 + 3\,t^2\right)
\end{aligned}
$$

$$
I_{(j,j,j,j,j),t} = \frac{1}{5\,!}\left(\left(W_t^j\right)^5 - 10\,t\,(W_t^j)^3 + 15\,t^2\,W_t^j\right)
$$

for $t \in [0, \infty)$, $j \in \{1, 2, \ldots, m\}$

Multiple Stratonovich Integrals

stopping times

$$0 \leq \varrho \leq \tau \leq T$$

$$J_\alpha[g(\cdot, X_\cdot)]_{\varrho,\tau} = \begin{cases} g(\tau, X_\tau) & \text{for} \quad \ell = 0 \\ \int_\varrho^\tau J_{\alpha-}[g(\cdot, X_\cdot)]_{\varrho,s}\, ds & \text{for} \quad \ell \geq 1,\ j_\ell = 0 \\ \int_\varrho^\tau J_{\alpha-}[g(\cdot, X_\cdot)]_{\varrho,s} \circ dW_s^{j_\ell} & \text{for} \quad \ell \geq 1,\ j_\ell \geq 1 \end{cases}$$

Relationships between Multiple Stratonovich Integrals

$$J_{\alpha,t} = J_\alpha[1]_{0,t}$$

$$W_t^0 = t$$

Let $j_1, \ldots, j_\ell \in \{0, 1, \ldots, m\}$ and $\alpha = (j_1, \ldots, j_\ell) \in \mathcal{M}_m$ where $\ell \in \mathcal{N}$. Then

$$W_t^j \, J_{\alpha,t} = \sum_{i=0}^{\ell} J_{(j_1,\ldots,j_i,j,j_{i+1},\ldots,j_\ell),t}$$

for all $t \in [0, T]$.

Coefficient Functions

- **operators**

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,l=1}^{d} \sum_{j=1}^{m} b^{k,j} b^{l,j} \frac{\partial^2}{\partial x^k \partial x^\ell}$$

$$L^j = \sum_{k=1}^{d} b^{k,j} \frac{\partial}{\partial x^k}$$

$j \in \{1, 2, \ldots, m\}$

- **Itô coefficient function**

$$f_\alpha = \begin{cases} f & \text{for} \quad l = 0 \\ L^{j_1} f_{-\alpha} & \text{for} \quad l \geq 1 \end{cases}$$

multi-index $\alpha = (j_1, \ldots, j_\ell)$

function $f : [0, T] \times \Re^d \to \Re$

for $f(t, x) = x$

$f_{(0)} = a$, $f_{(1)} = b$, $f_{(1,1)} = b\, b'$ and $f_{(0,1)} = a\, b' + \frac{1}{2}\, b^2 b''$

Stratonovich Coefficient Functions

- **operators**

$$\underline{L}^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} \underline{a}^k \frac{\partial}{\partial x^k}$$

$$\underline{L}^j = L^j = \sum_{k=1}^{d} b^{k,j} \frac{\partial}{\partial x^k}$$

for $j \in \{1, 2, \ldots, m\}$

$$\underline{a} = a - \frac{1}{2} \sum_{j=1}^{m} \underline{L}^j b^j$$

- **Stratonovich coefficient function**

$$\underline{f}_\alpha = \begin{cases} f & \text{for} \quad l = 0 \\ \underline{L}^{j_1} \underline{f}_{-\alpha} & \text{for} \quad l \geq 1 \end{cases}$$

for $\alpha = (j_1, \ldots, j_\ell)$

- **Examples for Stratonovich coefficient functions**

$$\underline{f}_{(0)} = \underline{a}, \quad \underline{f}_{(j_1)} = b^{j_1}, \quad \underline{f}_{(0,0)} = \underline{a}\,\underline{a}',$$

$$\underline{f}_{(0,j_1)} = \underline{a}\,b^{j_1\,\prime}, \quad \underline{f}_{(j_1,0)} = \underline{a}'b^{j_1}, \quad \underline{f}_{(j_1,j_2)} = b^{j_1}b^{j_2\,\prime},$$

$$\underline{f}_{(0,0,0)} = \underline{a}\left(\underline{a}\,\underline{a}'' + (\underline{a}')^2\right), \quad \underline{f}_{(0,0,j_1)} = \underline{a}\left(\underline{a}\,b^{j_1\,\prime\prime} + \underline{a}'b^{j_1\,\prime}\right),$$

$$\underline{f}_{(0,j_1,0)} = \underline{a}\left(\underline{a}''b^{j_1} + \underline{a}'b^{j_1\,\prime}\right), \quad \underline{f}_{(j_1,0,0)} = b^{j_1}\left(\underline{a}\,\underline{a}'' + (\underline{a}')^2\right),$$

$$\underline{f}_{(0,j_1,j_2)} = \underline{a}\left(b^{j_1}b^{j_2\prime\prime} + b^{j_1\prime}b^{j_2\prime}\right), \quad \underline{f}_{(j_1,0,j_2)} = b^{j_1}\left(\underline{a}\,b^{j_1\prime\prime} + \underline{a}^{\prime}b^{j_1\prime}\right),$$

$$\underline{f}_{(j_1,j_2,0)} = b^{j_1}\left(\underline{a}^{\prime\prime}b^{j_2} + \underline{a}^{\prime}b^{j_2\prime}\right), \quad \underline{f}_{(j_1,j_2,j_3)} = b^{j_1}\left(b^{j_2}b^{j_3\prime\prime} + b^{j_2\prime}b^{j_3\prime}\right),$$

where $j_1, j_2, j_3 \in \{1, 2, \ldots, m\}$

Hierarchical Sets

We call a subset $A \subset \mathcal{M}_m$ a *hierarchical set* if

$$A \neq \emptyset$$

$$\sup_{\alpha \in A} \ell(\alpha) < \infty$$

and

$$-\alpha \in A \quad \text{for each} \quad \alpha \in A \setminus \{v\}.$$

Examples:

$\{v\}, \{v, (0), (1)\}, \{v, (0), (1), (1, 1)\}$ are hierarchical sets

Remainder Sets

For a given hierarchical set A

$$\mathcal{B}(A) = \{\alpha \in \mathcal{M}_m \setminus A : -\alpha \in A\}.$$

Remainder set consists of all of the next following multi-indices with respect to the given hierarchical set.

Examples:

When $m = 1$ then

$$\mathcal{B}(\{v\}) = \{(0), (1)\}$$

and

$$\mathcal{B}(\{v, (0), (1)\}) = \{(0,0), (0,1), (1,0), (1,1)\}$$

General Stochastic Taylor Expansion

Wagner-Platen Expansion

Theorem 4.1 (Wagner-Platen)

Let ϱ *and* τ *be two stopping times with*

$$0 \leq \varrho(\omega) \leq \tau(\omega) \leq T,$$

a.s., $A \subset \mathcal{M}_m$ *a hierarchical set*

and $f : [0, T] \times \Re^d \to \Re$ *a sufficiently smooth function, then we have*

$$f(\tau, X_\tau) = \sum_{\alpha \in A} I_\alpha \left[f_\alpha(\varrho, X_\varrho) \right]_{\varrho, \tau} + \sum_{\alpha \in \mathcal{B}(A)} I_\alpha \left[f_\alpha(\cdot, X_\cdot) \right]_{\varrho, \tau}.$$

Example

$d = m = 1$

for function $f(t, x) \equiv x$,

times $\varrho = 0$, $\tau = t$

hierarchical set $\quad A = \{\alpha \in \mathcal{M}_m : \ell(\alpha) \leq 3\}$

drift $\quad a(t, x) = a(x)$

diffusion coefficient $\quad b(t, x) = b(x)$

$$dX_t = a\,(X_t)\,dt + b(X_t)\,dW_t$$

$\Longrightarrow$

Wagner-Platen expansion in the form:

$$
\begin{aligned}
X_t \;=\; & X_0 + a\, I_{(0)} + b\, I_{(1)} + \left(a\, a' + \frac{1}{2}\, b^2 a''\right) I_{(0,0)} \\
& + \left(a\, b' + \frac{1}{2}\, b^2 b''\right) I_{(0,1)} + b\, a' I_{(1,0)} + b\, b' I_{(1,1)} \\
& + \Bigg[a \left(a\, a'' + (a')^2 + b\, b' a'' + \frac{1}{2}\, b^2 a'''\right) + \frac{1}{2}\, b^2 \big(a\, a''' + 3\, a' a'' \\
& \quad + \left((b')^2 + b\, b''\right) a'' + 2\, b\, b' a'''\big) + \frac{1}{4}\, b^4 a^{(4)} \Bigg] I_{(0,0,0)} \\
& + \Bigg[a \left(a' b' + a\, b'' + b\, b' b'' + \frac{1}{2}\, b^2 b'''\right) + \frac{1}{2}\, b^2 \Big(a'' b' + 2\, a' b'' \\
& \quad + a\, b''' + \left((b')^2 + b\, b''\right) b'' + 2\, b\, b' b''' + \frac{1}{2}\, b^2 b^{(4)}\Big) \Bigg] I_{(0,0,1)}
\end{aligned}
$$

$$+ \left[a\left(b'a' + b\,a''\right) + \frac{1}{2}\,b^2\left(b''a' + 2\,b'a'' + b\,a'''\right)\right] I_{(0,1,0)}$$

$$+ \left[a\left((b')^2 + b\,b''\right) + \frac{1}{2}\,b^2\left(b''b' + 2\,b\,b'' + b\,b'''\right)\right] I_{(0,1,1)}$$

$$+ b\left(a\,a'' + (a')^2 + b\,b'a'' + \frac{1}{2}\,b^2 a'''\right) I_{(1,0,0)}$$

$$+ b\left(a\,b'' + a'b' + b\,b'b'' + \frac{1}{2}\,b^2 b'''\right) I_{(1,0,1)}$$

$$+ b\left(a'b' + a''b\right) I_{(1,1,0)} + b\left((b')^2 + b\,b''\right) I_{(1,1,1)} + R_6$$

Examples for Wagner Platen Expansions

- **Vasicek interest rate model**

$$dr_t = \gamma\,(\bar{r} - r_t)\,dt + \beta\,dW_t$$

for $t \in [0, T]$, $r_0 \geq 0$

- hierarchical set

$$A = \{\alpha \in \mathcal{M}_1 : \ell(\alpha) \leq 3\}$$

$$\begin{aligned} r_t &= r_0 + \gamma\,(\bar{r} - r_0)\,t + \beta\,W_t - \gamma^2\,(\bar{r} - r_0)\,\frac{t^2}{2} \\ &\quad - \beta\,\gamma \int_0^t W_s\,ds + \gamma^3\,(\bar{r} - r_0)\,\frac{t^3}{6} \\ &\quad + \beta\,\gamma^2 \int_0^t \int_0^{s_2} W_{s_1}\,ds_1\,ds_2 + R_6 \end{aligned}$$

- **Black-Scholes dynamics**

$$dS_t = S_t\,(a\,dt + \sigma\,dW_t)$$

for $t \in [0, T]$, $S_0 \geq 0$

- Wagner-Platen expansion

hierarchical set

$$A = \{\alpha \in \mathcal{M}_1 : \ell(\alpha) \leq 3\}$$

is given by

$$
\begin{aligned}
S_t \;=\; S_0 \Bigg(& 1 + a\,t + \sigma\,W_t + a^2\,\frac{t^2}{2} + a\,\sigma\,(I_{(0,1)} + I_{(1,0)}) \\
& + \sigma^2\,\frac{1}{2}\left((W_t)^2 - t\right) + a^3\,\frac{t^3}{6} \\
& + a^2\,\sigma\,(I_{(0,0,1)} + I_{(0,1,0)} + I_{(1,0,0)}) \\
& + a\,\sigma^2\,(I_{(0,1,1)} + I_{(1,0,1)} + I_{(1,1,0)}) \\
& + \sigma^3\,\frac{1}{6}\left((W_t)^3 - 3\,t\,W_t\right)\Bigg) \\
& + R_6
\end{aligned}
$$

- relationship

$$\begin{aligned} t\,W_t &= I_{(0)}\,I_{(1)} = I_{(0,1)} + I_{(1,0)} \\ W_t\,\frac{t^2}{2} &= I_{(1)}\,I_{(0,0)} = I_{(0,0,1)} + I_{(0,1,0)} + I_{(1,0,0)} \\ t\,\frac{1}{2}\left((W_t)^2 - t\right) &= I_{(0)}\,I_{(1,1)} = I_{(0,1,1)} + I_{(1,0,1)} + I_{(1,1,0)} \end{aligned}$$

- Wagner-Platen expansion

$$
\begin{aligned}
S_t \;=\; & S_0 \Big(1 + a\,t + \sigma\,W_t + a^2\,\frac{t^2}{2} + a\,\sigma\,t\,W_t \\
& + \frac{\sigma^2}{2}\left((W_t)^2 - t\right) + a^3\,\frac{t^3}{6} + a^2\sigma\,W_t\,\frac{t^2}{2} \\
& + a\,\sigma^2\,\frac{t}{2}\left((W_t)^2 - t\right) + \sigma^3\,\frac{1}{6}\left((W_t)^3 - 3\,t\,W_t\right)\Big) + R_6
\end{aligned}
$$

- **squared Bessel process**

$$dX_t = \nu \, dt + 2 \sqrt{X_t} \, dW_t$$

for $t \in [0, T]$ with $X_0 > 0$

hierarchical set

$$A = \{\alpha \in \mathcal{M}_1 : \ell(\alpha) \leq 2\}$$

- Wagner-Platen expansion

$$\begin{aligned} X_t &= X_0 + (\nu - 1)\, t + 2\sqrt{X_0}\, W_t \\ &\quad + \frac{\nu - 1}{\sqrt{X_0}} \int_0^t s\, dW_s + (W_t)^2 + \tilde{R} \end{aligned}$$

Stratonovich-Taylor Expansion

(Kloeden-Platen) ϱ and τ stopping times

$0 \leq \varrho \leq \tau \leq T,\ f : [0, T] \times \Re^d \to \Re$ and

$A \subset \mathcal{M}_m$ hierarchical set, then

$$f(\tau, X_\tau) = \sum_{\alpha \in A} J_\alpha \left[\underline{f}_\alpha (\varrho, X_\varrho) \right]_{\varrho,\tau} + \sum_{\alpha \in \mathcal{B}(A)} J_\alpha \left[\underline{f}_\alpha (\cdot, X_\cdot) \right]_{\varrho,\tau}$$

- example Stratonovich-Taylor expansion

$$
\begin{aligned}
X_t \;=\; & X_0 + \underline{a}\, J_{(0)} + b\, J_{(1)} + \underline{a}\, \underline{a}'\, J_{(0,0)} + \underline{a}\, b'\, J_{(0,1)} + b\, \underline{a}' J_{(1,0)} \\
& + b\, b' J_{(1,1)} + \underline{a} \left(\underline{a}\, \underline{a}'' + (\underline{a}')^2 \right) J_{(0,0,0)} + \underline{a} \left(\underline{a}\, b'' + \underline{a}' b' \right) J_{(0,0,1)} \\
& + \underline{a} \left(\underline{a}'' b + \underline{a}' b' \right) J_{(0,1,0)} + b \left(\underline{a}\, \underline{a}'' + (\underline{a}')^2 \right) J_{(1,0,0)} \\
& + \underline{a} \left(b\, b'' + (b')^2 \right) J_{(0,1,1)} + b \left(\underline{a}\, b'' + \underline{a}' b' \right) J_{(1,0,1)} \\
& + b \left(\underline{a}'' b + \underline{a}' b' \right) J_{(1,1,0)} + b \left(b\, b'' + (b')^2 \right) J_{(1,1,1)} + R_7
\end{aligned}
$$

Moments of Multiple Itô Integrals

First Moments

$\alpha \in \mathcal{M}_m \setminus \{v\}$ with $\ell(\alpha) \neq n(\alpha)$

$$E\left(I_\alpha \left[f_\cdot \right]_{\varrho,\tau} \,\middle|\, \mathcal{A}_\varrho \right) = 0$$

Estimates of Higher Moments

$\alpha \in \mathcal{M}_m$

$$\left(E \left(\left| I_\alpha \left[g_\cdot \right]_{\varrho,\tau} \right|^{2q} \,\Big|\, \mathcal{A}_\varrho \right) \right)^{\frac{1}{q}}$$

$$\leq \left(2\,(2\,q - 1)\, e^T \right)^{\ell(\alpha) - n(\alpha)} (\tau - \varrho)^{\ell(\alpha) + n(\alpha)} R_8,$$

where

$$R_8 = \left(E \left(\sup_{\varrho \leq s \leq \tau} |g_s|^{2q} \,\Big|\, \mathcal{A}_\varrho \right) \right)^{\frac{1}{q}}$$

- **truncated Wagner-Platen expansions**

$$X_k(t) = \sum_{\alpha \in \Lambda_k} I_\alpha \left[f_\alpha(0, X_0) \right]_{0,t}$$

hierarchical set

$$\Lambda_k = \{\alpha \in \mathcal{M}_m : \ell(\alpha) + n(\alpha) \leq k\}$$

under appropriate assumptions

$$X_t \overset{\text{a.s.}}{=} \lim_{k \to \infty} X_k(t) \overset{\text{a.s.}}{=} \sum_{\alpha \in \mathcal{M}_m} I_\alpha \left[f_\alpha(0, X_0) \right]_{0,t}$$

Exercises of Chapter 4

4.1 Use the Wagner-Platen expansion with time increment $h > 0$ and Wiener process increment $W_{t_0+h} - W_{t_0}$ in the expansion part to expand the increment $X_{t_0+h} - X_{t_0}$ of a geometric Brownian motion at time t_0, where

$$dX_t = a\,X_t\,dt + b\,X_t\,dW_t.$$

4.2 Expand the geometric Brownian motion from Exercise 4.1 such that all double integrals appear in the expansion part.

4.3 For multi-indices $\alpha = (0,0,0,0)$, $(1,0,2)$, $(0,2,0,1)$ determine $-\alpha$, $\alpha-$, $\ell(\alpha)$ and $n(\alpha)$.

4.4 Write out in full the multiple Itô stochastic integrals $I_{(0,0),t}$, $I_{(1,0),t}$, $I_{(1,1),t}$ and $I_{(1,2),t}$.

4.5 Express the multiple Itô integral $I_{(0,1)}$ in terms of $I_{(1)}$, $I_{(0)}$ and $I_{(1,0)}$.

4.6 Verify that $I_{(1,0),\Delta}$ is Gaussian distributed with

$$E(I_{(1,0),\Delta}) = 0, \quad E\left((I_{(1,0),\Delta})^2\right) = \frac{\Delta^3}{3},$$

$$E(I_{(1,0),\Delta}\, I_{(1),\Delta}) = \frac{\Delta^2}{2}.$$

4.7 For the case $d = m = 1$ determine the Itô coefficient functions $f_{(1,0)}$ and $f_{(1,1,1)}$.

4.8 Which of the following sets of multi-indices are not hierarchical sets: $\emptyset$, $\{(1)\}$, $\{v,(1)\}$, $\{v,(0),(0,1)\}$, $\{v,(0),(1),(0,1)\}$?

4.9 Determine the remainder sets that correspond to the hierarchical sets $\{v,(1)\}$ and $\{v,(0),(1),(0,1)\}$.

4.10 Determine the truncated Wagner-Platen expansion at time $t = 0$ for the solution of the Itô SDE

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t$$

using the hierarchical set $\mathcal{A} = \{v, (0), (1), (1, 1)\}$.

4.11 In the notation of Sect. 4.2 where the component -1 denotes in a multi-index α a jump term, determine for $\alpha = (1, 0, -1)$ its length $\ell(\alpha)$, its number of zeros and its number of time integrations.

4.12 For the multi-index $\alpha = (1, 0, -1)$ derive in the notation of Sect. 4.2 $-\alpha$.

5 Introduction to Scenario Simulation

Discrete Time Approximation

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T$$

one-dimensional SDE

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t$$

- **Euler scheme**

  Euler-Maruyama scheme

  Maruyama (1955)

$$Y_{n+1} = Y_n + a(\tau_n, Y_n)\,(\tau_{n+1} - \tau_n) + b(\tau_n, Y_n)\,\left(W_{\tau_{n+1}} - W_{\tau_n}\right)$$

$$Y_0 = X_0,$$

$$Y_n = Y_{\tau_n}$$

$$\Delta_n = \tau_{n+1} - \tau_n$$

maximum step size

$$\Delta = \max_{n \in \{0,1,\dots,N-1\}} \Delta_n$$

- **equidistant time discretization**

$$\tau_n = n\,\Delta$$

$$\Delta_n \equiv \Delta = \tfrac{T}{N}$$

- Euler approximation **recursively computed**

- **random increments**

$$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$$

for $n \in \{0, 1, \ldots, N-1\}$

Wiener process

$$W = \{W_t\, t \in [0, T]\}$$

Gaussian distributed

mean

$$E\left(\Delta W_n\right) = 0$$

variance

$$E\left((\Delta W_n)^2\right) = \Delta_n$$

Gaussian pseudo-random numbers generated

- abbreviation

$$f = f(\tau_n, Y_n)$$

Euler scheme

$$Y_{n+1} = Y_n + a\,\Delta_n + b\,\Delta W_n,$$

discrete time approximations

Interpolation

- **discrete time approximation**

  stochastic process on $[0, T]$

  piecewise constant interpolation

$$Y_t = Y_{n_t}$$

for $t \in [0, T]$

$$n_t = \max\{n \in \{0, 1, \ldots, N\} : \tau_n \leq t\}$$

largest integer n for which τ_n does not exceed t

- **linear interpolation**

$$Y_t = Y_{n_t} + \frac{t - \tau_{n_t}}{\tau_{n_t+1} - \tau_{n_t}} \left(Y_{n_t+1} - Y_{n_t} \right)$$

Simulating Geometric Brownian Motion

- **illustrate scenario simulation**

  standard market model as geometric Brownian motions

$$dX_t = a\, X_t\, dt + b\, X_t\, dW_t$$

  for $t \in [0, T]$, $X_0 > 0$

- drift coefficient

$$a(t, x) = a\, x$$

- diffusion coefficient

$$b(t, x) = b\, x$$

- appreciation rate $\quad a$

- volatility $\quad b \neq 0$

- **explicit solution**

$$X_t = X_0 \exp\left(\left(a - \frac{1}{2}\,b^2\right) t + b\,W_t\right)$$

- Wiener process

$$W = \{W_t,\, t \in [0, T]\}$$

Scenario Simulation

- **simulate trajectory of Euler approximation**

  1. initial value $Y_0 = X_0$

  2. proceed recursively

$$Y_{n+1} = Y_n + a\, Y_n\, \Delta_n + b\, Y_n\, \Delta W_n$$

for $n \in \{0, 1, \ldots, N-1\}$

$$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$$

- comparison with explicit solution at time τ_n

$$X_{\tau_n} = X_0 \exp\left(\left(a - \frac{1}{2}\, b^2\right) \tau_n + b \sum_{i=1}^{n} \Delta W_{i-1}\right)$$

for $n \in \{0, 1, \ldots, N-1\}$

Euler approximation

mathematically another new object

different

negative ?

alternative ways ?

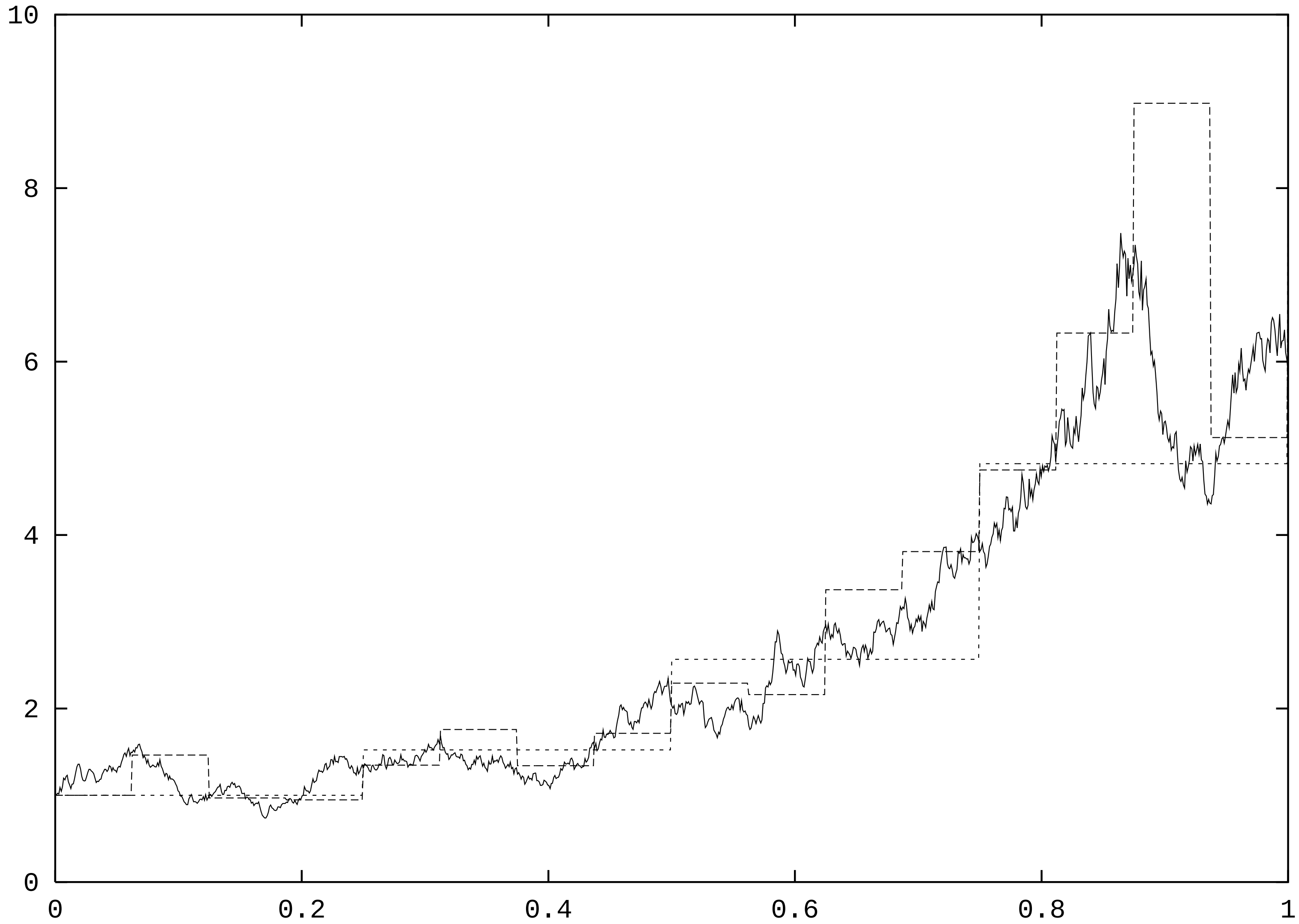


Figure 5.1: Euler approximations for $\boldsymbol{\Delta = 0.25}$ and $\boldsymbol{\Delta = 0.0625}$ and exact solution for Black-Scholes SDE.

Strong Approximation

- not specified a **criterion for classification**

- **scenario simulation**
  approximates paths

  testing of **calibration** methods

  statistical **estimators**

  filtering

- **Monte-Carlo simulation**

  approximates **probabilities**

  functionals

  simulates **expectations**

  moments

  prices of contingent claims

  or **risk measures** as Value at Risk

Order of Strong Convergence

- **absolute error criterion**

$$\varepsilon(\Delta) = E\left(\left|X_T - Y_T^{\Delta}\right|\right)$$

A discrete time approximation Y^{Δ} *converges strongly with order* $\gamma > 0$ at time T if there exists a positive constant C, which does not depend on Δ, and a $\delta_0 > 0$ such that

$$\varepsilon(\Delta) = E\left(\left|X_T - Y_T^{\Delta}\right|\right) \leq C\,\Delta^{\gamma}$$

for each $\Delta \in (0, \delta_0)$.

Strong Taylor Schemes

- use **Wagner-Platen expansions**

  appropriate truncation

- **operators**

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,\ell=1}^{d} \sum_{j=1}^{m} b^{k,j}\, b^{\ell,j} \frac{\partial}{\partial x^k\, \partial x^\ell}$$

$$\underline{L}^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} \underline{a}^k \frac{\partial}{\partial x^k}$$

and

$$L^j = \underline{L}^j = \sum_{k=1}^{d} b^{k,j} \, \frac{\partial}{\partial x^k}$$

for $j \in \{1, 2, \ldots, m\}$, where

$$\underline{a}^k = a^k - \frac{1}{2} \sum_{j=1}^{m} \underline{L}^j \, b^{k,j}$$

- **multiple Itô integrals**

$$I_{(j_1,\ldots,j_\ell)} = \int_{\tau_n}^{\tau_{n+1}} \ldots \int_{\tau_n}^{s_2} dW_{s_1}^{j_1} \ldots dW_{s_\ell}^{j_\ell}$$

- **multiple Stratonovich integrals**

$$J_{(j_1,\ldots,j_\ell)} = \int_{\tau_n}^{\tau_{n+1}} \ldots \int_{\tau_n}^{s_2} \circ dW_{s_1}^{j_1} \ldots \circ dW_{s_\ell}^{j_\ell}$$

for $j_1, \ldots, j_\ell \in \{0, 1, \ldots, m\}$, $\ell \in \{1, 2, \ldots\}$
and $n \in \{0, 1, \ldots\}$

$$W_t^0 = t$$

for all $t \in [0, T]$

- **Itô SDE**

$$dX_t = a(t, X_t)\, dt + \sum_{j=1}^{m} b^j(t, X_t)\, dW_t^j$$

- equivalent **Stratonovich SDE**

$$dX_t = \underline{a}(t, X_t)\, dt + \sum_{j=1}^{m} b^j(t, X_t) \circ dW_t^j$$

Euler Scheme

simplest strong Taylor approximation

order of strong convergence $\gamma = 0.5$

- $d = m = 1$

  Euler scheme

$$Y_{n+1} = Y_n + a\,\Delta + b\,\Delta W$$

$$\Delta = \tau_{n+1} - \tau_n$$

$$\Delta W = \Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$$

$N(0, \Delta)$ independent Gaussian distributed

- **multi-dimensional Euler scheme**

  $m = 1$ and $d \in \{1, 2, \ldots\}$

  kth component

$$Y_{n+1}^{k} = Y_{n}^{k} + a^{k}\,\Delta + b^{k}\,\Delta W$$

  for $k \in \{1, 2, \ldots, d\}$

  $a = (a^{1}, \ldots, a^{d})^{\top}$ and $b = (b^{1}, \ldots, b^{d})^{\top}$

- **general multi-dimensional Euler scheme**

$d, m \in \{1, 2, \ldots\}$

$$Y_{n+1}^k = Y_n^k + a^k \, \Delta + \sum_{j=1}^m b^{k,j} \, \Delta W^j$$

$$\Delta W^j = W_{\tau_{n+1}}^j - W_{\tau_n}^j$$

$N(0, \Delta)$ independent Gaussian distributed

ΔW^{j_1} and ΔW^{j_2} independent for $j_1 \neq j_2$

$b = [b^{k,j}]_{k,j=1}^{d,m}$ $d \times m$-matrix

truncated Wagner-Platen expansion

Theorem 5.1 *Suppose that we have initial values* X_0 *and* $Y_0 = Y_0^\Delta$ *such that*

$$E\left(|X_0|^2\right) < \infty$$

and

$$E\left(\left|X_0 - Y_0^\Delta\right|^2\right)^{\frac{1}{2}} \leq K_1\,\Delta^{\frac{1}{2}}.$$

Furthermore, assume the Lipschitz condition

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq K_2\,|x - y|,$$

the linear growth condition

$$|a(t,x)| + |b(t,x)| \leq K_3\ (1 + |x|\,)$$

and

$$|a(s,x) - a(t,x)| + |b(s,x) - b(t,x)| \leq K_4 \; (1 + |x|\,)\, |s - t|^{\frac{1}{2}}$$

for all $s, t \in [0, T]$ *and* $x, y \in \Re^d$*, where the constants* $K_1, \ldots, K_4$ *do not depend on* Δ*. Then the Euler approximation* Y^{Δ} *converges with strong order* $\gamma = 0.5$*, that is we have the estimate*

$$E\left(\left|X_T - Y_T^{\Delta}\right|\right) \leq K_5\, \Delta^{\frac{1}{2}},$$

where the constant K_5 *does not depend on* Δ*.*

A Simulation Example

- **Black-Scholes SDE**

$$dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dW_t$$

- **exact solution**

$$X_T = X_0\, \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma\, W_T \right\}$$

absolute error

$$\varepsilon(\Delta) = E(|X_T - Y_N^{\Delta}|)$$

default parameters:

$X_0 = 1$, $\mu = 0.06$, $\sigma = 0.2$ and $T = 1$

5000 simulations

fitted line

$$\ln(\varepsilon(\Delta)) = -3.86933 + 0.46739 \ln(\Delta)$$

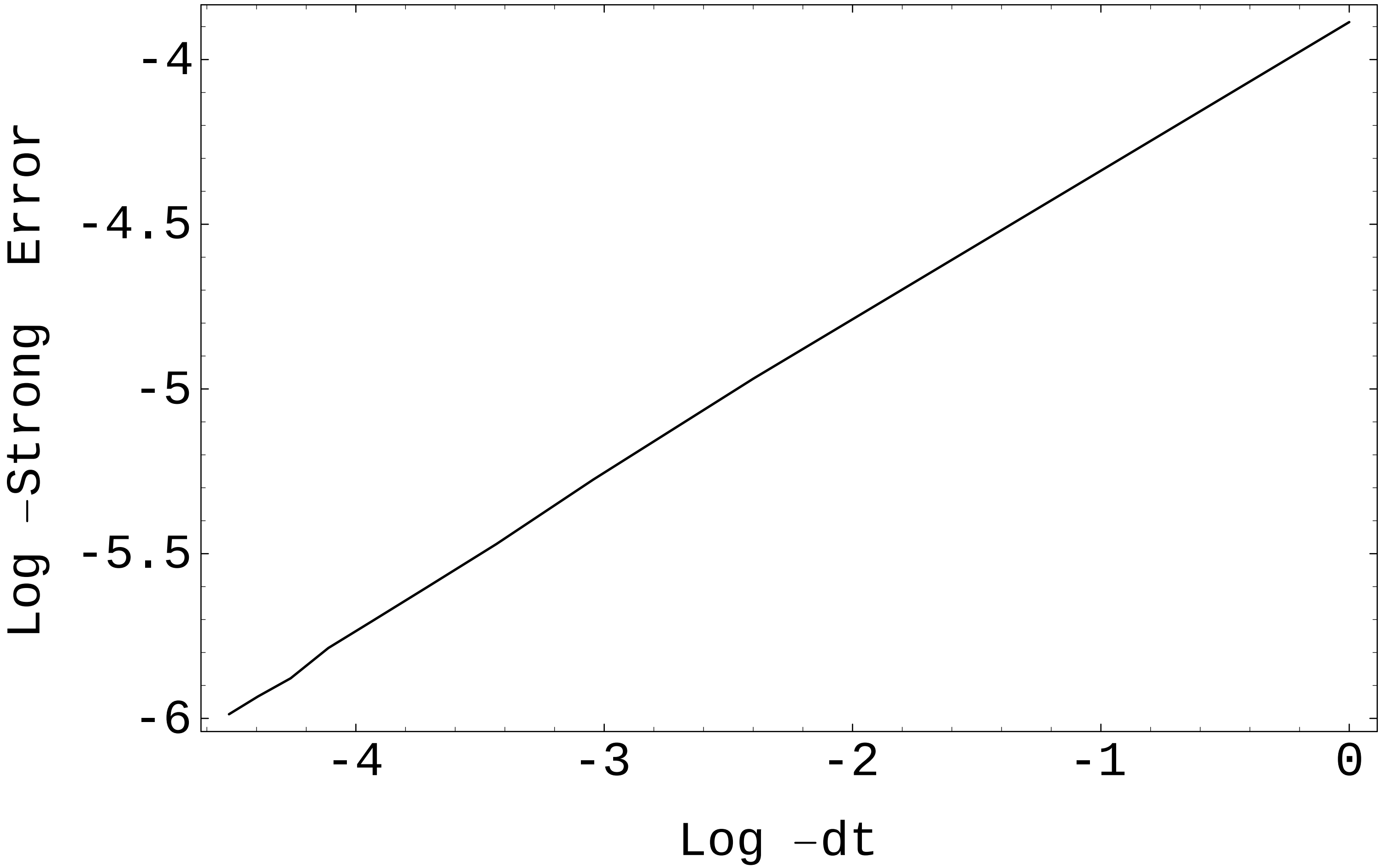


Figure 5.2: Log-log plot of the absolute error $\varepsilon(\Delta)$ for an Euler Scheme against $\ln(\Delta)$.

Milstein Scheme

Milstein (1974)

- **Milstein scheme**

  $d = m = 1$

$$Y_{n+1} = Y_n + a\,\Delta + b\,\Delta W + \frac{1}{2}\,b\,b'\,\left\{(\Delta W)^2 - \Delta\right\}$$

order $\gamma = 1.0$ of strong convergence

- **multi-dimensional Milstein scheme**

  $m = 1$ and $d \in \{1, 2, \ldots\}$

$$Y_{n+1}^k = Y_n^k + a^k \Delta + b^k \Delta W + \left(\sum_{\ell=1}^{d} b^\ell \frac{\partial b^k}{\partial x^\ell} \right) \frac{1}{2} \left\{ (\Delta W)^2 - \Delta \right\}$$

- **general multi-dimensional Milstein scheme**

  $d, m \in \{1, 2, \ldots\}$

  kth component

$$Y^k_{n+1} = Y^k_n + a^k \Delta + \sum_{j=1}^{m} b^{k,j} \Delta W^j + \sum_{j_1, j_2 = 1}^{m} L^{j_1} b^{k, j_2} I_{(j_1, j_2)}$$

  Itô integrals $I_{(j_1, j_2)}$

- alternatively

$$Y^k_{n+1} = Y^k_n + \underline{a}^k \Delta + \sum_{j=1}^{m} b^{k,j} \Delta W^j + \sum_{j_1, j_2 = 1}^{m} \underline{L}^{j_1} b^{k, j_2} J_{(j_1, j_2)}$$

Stratonovich integrals $J_{(j_1,j_2)}$

$j_1 \neq j_2 \quad j_1, j_2 \in \{1, 2, \ldots, m\}$

$$J_{(j_1,j_2)} = I_{(j_1,j_2)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_1} dW_{s_2}^{j_1} \, dW_{s_1}^{j_2}$$

cannot be simply expressed by ΔW^{j_1} and ΔW^{j_2}

$$I_{(j_1,j_1)} = \frac{1}{2} \left\{ (\Delta W^{j_1})^2 - \Delta \right\} \quad \text{and} \quad J_{(j_1,j_1)} = \frac{1}{2} \left(\Delta W^{j_1} \right)^2$$

for $j_1 \in \{1, 2, \ldots, m\}$

Commutative Noise

- **commutativity condition**

$$L^{j_1} b^{k,j_2} = L^{j_2} b^{k,j_1}$$

  for all $j_1, j_2 \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, d\}$ and $(t, x) \in [0, T] \times \Re^d$

- satisfied for Black-Scholes SDEs

  additive noise

  single Wiener process

- **Milstein scheme under commutative noise**

$$Y_{n+1}^k = Y_n^k + \underline{a}^k \Delta + \sum_{j=1}^{m} b^{k,j} \Delta W^j + \frac{1}{2} \sum_{j_1, j_2 = 1}^{m} \underline{L}^{j_1} b^{k,j_2} \Delta W^{j_1} \Delta W^{j_2}$$

for $k \in \{1, 2, \ldots, d\}$

no double Wiener integrals

A Black-Scholes Example

- correlated Black-Scholes dynamics

  $$d = m = 2$$

- first risky security

$$\begin{aligned} dX_t^1 &= X_t^1 \left[r\, dt + \theta^1 (\theta^1\, dt + dW_t^1) + \theta^2 (\theta^2\, dt + dW_t^2) \right] \\ &= X_t^1 \left[\left(r + \frac{1}{2} \left((\theta^1)^2 + (\theta^2)^2 \right) \right) dt \right. \\ &\qquad \left. + \theta^1 \circ dW_t^1 + \theta^2 \circ dW_t^2 \right] \end{aligned}$$

- second risky security

$$
\begin{aligned}
dX_t^2 &= X_t^2 \left[r\,dt + (\theta^1 - \sigma^{1,1})\,(\theta^1\,dt + dW_t^1) \right] \\
&= X_t^2 \left[\left(r + (\theta^1 - \sigma^{1,1}) \left(\frac{1}{2}\,\theta^1 + \frac{1}{2}\,\sigma^{1,1} \right) \right) dt \right. \\
&\qquad \left. + (\theta^1 - \sigma^{1,1}) \circ dW_t^1 \right]
\end{aligned}
$$

$\Longrightarrow$

$$L^1\, b^{1,2} \;=\; \sum_{k=1}^{2} b^{k,1}\, \frac{\partial}{\partial x^k}\, b^{1,2} = \theta^1\, X_t^1\, \theta^2$$

$$= \;\sum_{k=1}^{2} b^{k,2}\, \frac{\partial}{\partial x^k}\, b^{1,1} = L^2\, b^{1,1}$$

and

$$L^1\, b^{2,2} = \sum_{k=1}^{2} b^{k,1}\, \frac{\partial}{\partial x^k}\, b^{2,2} = 0 = \sum_{k=1}^{2} b^{k,2}\, \frac{\partial}{\partial x^k}\, b^{2,1} = L^2\, b^{2,1}$$

$\Longrightarrow$ commutativity condition satisfied

$\Longrightarrow$ Milstein scheme :

$$
\begin{aligned}
Y^1_{n+1} \;&=\; Y^1_n + Y^1_n \biggl(r + \frac{1}{2}\left((\theta^1)^2 + (\theta^2)^2\right)\Delta + \theta^1\,\Delta W^1 + \theta^2\,\Delta W^2 \\
&\qquad + \frac{1}{2}\,(\theta^1)^2 (\Delta W^1)^2 + \;\frac{1}{2}\,(\theta^2)^2 (\Delta W^2)^2 + \theta^1\,\theta^2\,\Delta W^1\,\Delta W^2 \biggr)
\end{aligned}
$$

$$
\begin{aligned}
Y^2_{n+1} \;&=\; Y^2_n + Y^2_n \biggl(\biggl(r + \frac{1}{2}\,(\theta^1 - \sigma^{1,1})\,(\theta^1 + \sigma^{1,1}) \biggr) \Delta \\
&\qquad + (\theta^1 - \sigma^{1,1})\,\Delta W^1 + \;\frac{1}{2}\,(\theta^1 - \sigma^{1,1})^2\,(\Delta W^1)^2 \biggr)
\end{aligned}
$$

(may produce negative trajectories)

A Square Root Process Example

- square root process of dimension ν

$$dX_t = \frac{\nu}{4}\,\eta\left(\frac{1}{\eta} - X_t\right)dt + \sqrt{X_t}\,dW_t^1$$

$X_0 = \frac{1}{\eta}$ for $\nu > 2$

- Milstein

$$Y_{n+1} = Y_n + \frac{\nu}{4}\,\eta\left(\frac{1}{\eta} - Y_n\right)\Delta + \sqrt{Y_n}\,\Delta W + \frac{1}{4}\left(\Delta W^2 - \Delta\right)$$

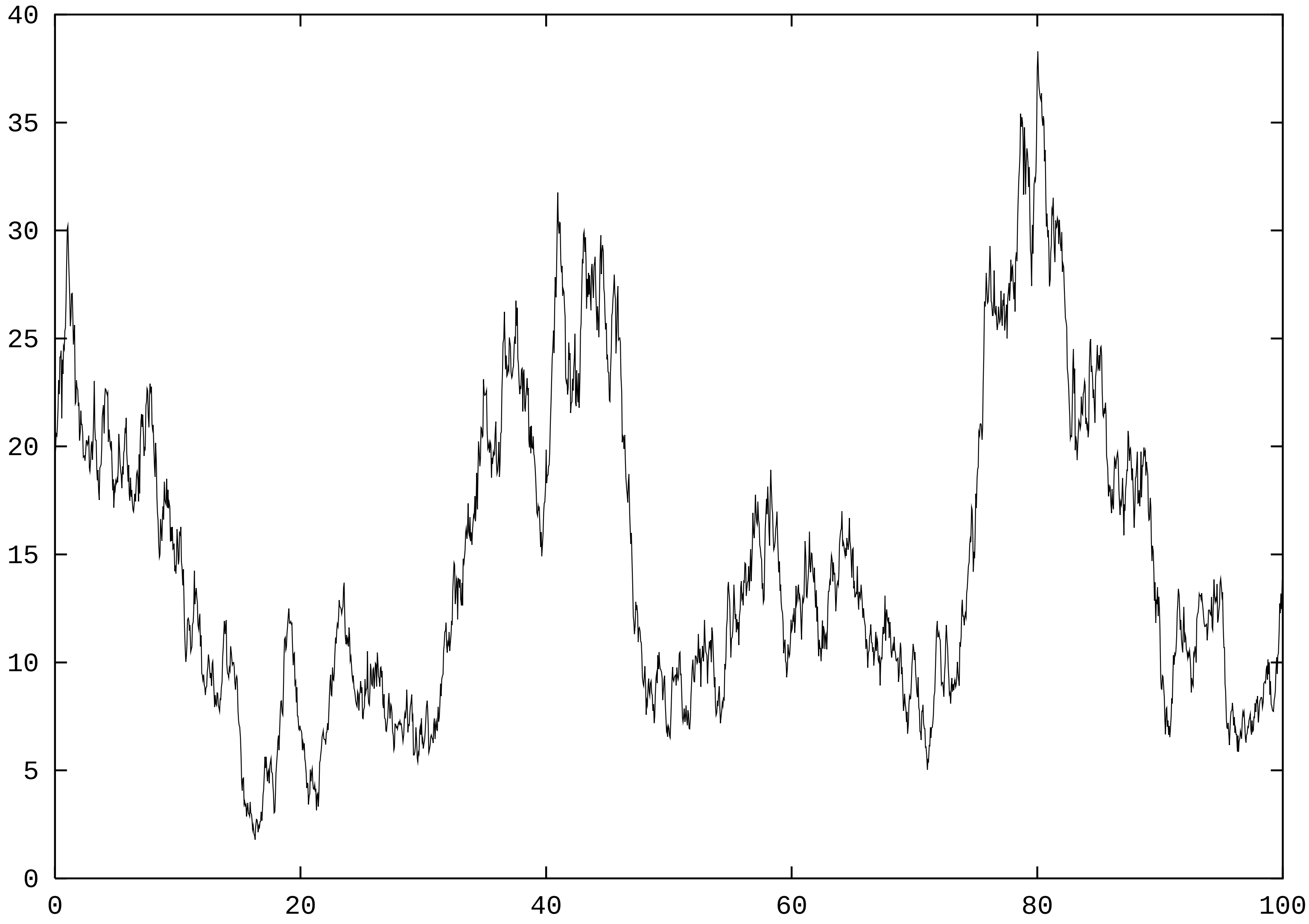


Figure 5.3: Square root process simulated by the Milstein scheme.

Approximate Double Wiener Integrals

For $j_1 \neq j_2$ with $j_1, j_2 \in \{1, 2, \ldots, m\}$ approximate

$J_{(j_1,j_2)} = I_{(j_1,j_2)}$ by

$$J^p_{(j_1,j_2)} = \Delta \left(\frac{1}{2} \xi_{j_1} \xi_{j_2} + \sqrt{\varrho_p}\, \left(\mu_{j_1,p}\, \xi_{j_2} - \mu_{j_2,p}\, \xi_{j_1} \right) \right)$$

$$+ \frac{\Delta}{2\pi} \sum_{r=1}^{p} \frac{1}{r} \left(\zeta_{j_1,r} \left(\sqrt{2}\, \xi_{j_2} + \eta_{j_2,r} \right) - \zeta_{j_2,r} \left(\sqrt{2}\, \xi_{j_1} + \eta_{j_1,r} \right) \right)$$

where

$$\varrho_p = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{r=1}^{p} \frac{1}{r^2}$$

ξ_j, $\mu_{j,p}$, $\eta_{j,r}$ and $\zeta_{j,r}$

independent $N(0,1)$

$$\xi_j = \frac{1}{\sqrt{\Delta}}\,\Delta W^j$$

for $j \in \{1,2,\ldots,m\}$, $r \in \{1,\ldots p\}$ and $p \in \{1,2,\ldots\}$ if

$$p = p(\Delta) \geq \frac{K}{\Delta}$$

$\Longrightarrow$

$$\gamma = 1.0$$

Convergence Theorem

$$E\left(|X_0|^2\right) < \infty$$

$$E\left(\left|X_0 - Y_0^{\Delta}\right|^2\right)^{\frac{1}{2}} \leq K_1\,\Delta^{\frac{1}{2}}$$

$$
\begin{aligned}
|\underline{a}(t,x) - \underline{a}(t,y)| &\leq K_2\,|x-y| \\
\left|b^{j_1}(t,x) - b^{j_1}(t,y)\right| &\leq K_2\,|x-y| \\
\left|\underline{L}^{j_1} b^{j_2}(t,x) - \underline{L}^{j_1} b^{j_2}(t,y)\right| &\leq K_2\,|x-y| \\
|\underline{a}(t,x)| + \left|\underline{L}^{j}\,\underline{a}(t,x)\right| &\leq K_3\,\left(1+|x|\right) \\
\left|b^{j_1}(t,x)\right| + \left|\underline{L}^{j} b^{j_2}(t,x)\right| &\leq K_3\,\left(1+|x|\right) \\
\left|\underline{L}^{j}\,\underline{L}^{j_1} b^{j_2}(t,x)\right| &\leq K_3\,\left(1+|x|\right)
\end{aligned}
$$

and

$$
\begin{aligned}
|\underline{a}(s,x) - \underline{a}(t,x)| &\leq K_4 \; (1 + |x|\,)\,|s-t|^{\frac{1}{2}} \\
\left| b^{j_1}(s,x) - b^{j_1}(t,x) \right| &\leq K_4 \; (1 + |x|\,)\,|s-t|^{\frac{1}{2}} \\
\left| \underline{L}^{j_1} b^{j_2}(s,x) - \underline{L}^{j_1} b^{j_2}(t,x) \right| &\leq K_4 \; (1 + |x|\,)\,|s-t|^{\frac{1}{2}}
\end{aligned}
$$

for all s, $t \in [0, T]$, x, $y \in \Re^d$, $j \in \{0, \ldots, m\}$ and j_1, $j_2 \in \{1, 2, \ldots, m\}$. Then the Milstein scheme converges with strong order $\gamma = 1.0$

$$E\left(\left|X_T - Y_T^\Delta\right|\right) \leq K_5\,\Delta.$$

Kloeden & Platen (1999).

A Simulation Study

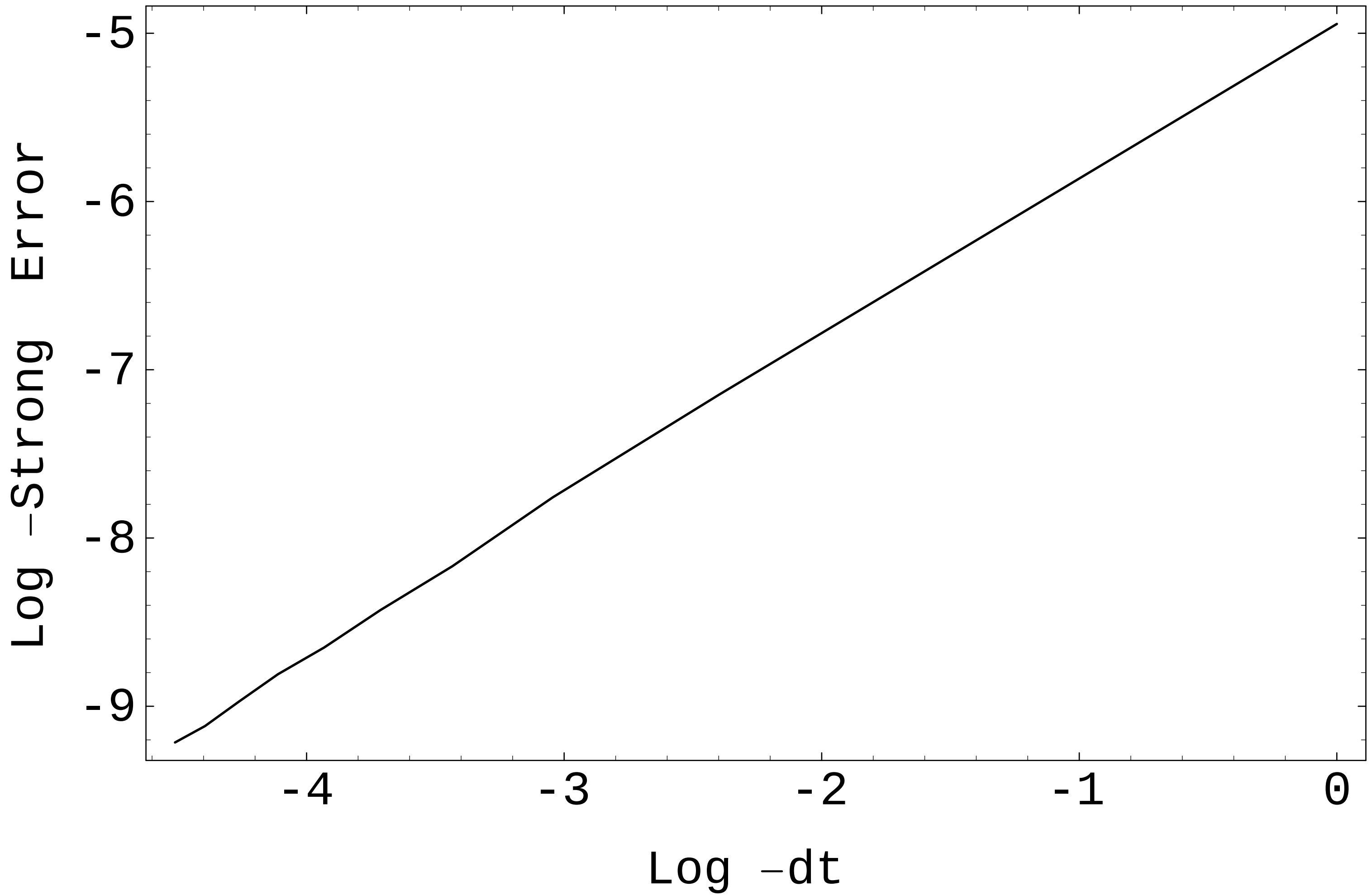


Figure 5.4: Log-log plot of the absolute error against log-step size for a Milstein scheme.

- linear regression

$$\ln(\varepsilon(\Delta)) = -4.91021 + 0.95 \ \ln(\Delta)$$

Order 1.5 Strong Taylor Scheme

- simulation tasks that require more accurate schemes

  extreme log-returns

- including further multiple stochastic integrals

- **order 1.5 strong Taylor scheme**

  $d = m = 1$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + a\,\Delta + b\,\Delta W + \frac{1}{2}\,b\,b'\left\{(\Delta W)^2 - \Delta\right\} \\
& + a'\,b\,\Delta Z + \frac{1}{2}\left(a\,a' + \frac{1}{2}\,b^2\,a''\right)\Delta^2 \\
& + \left(a\,b' + \frac{1}{2}\,b^2\,b''\right)\{\Delta W\,\Delta - \Delta Z\} \\
& + \frac{1}{2}\,b\left(b\,b'' + (b')^2\right)\left\{\frac{1}{3}\,(\Delta W)^2 - \Delta\right\}\Delta W
\end{aligned}
$$

- **double integral**

$$\Delta Z = I_{(1,0)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}\, ds_2$$

Gaussian

$$E(\Delta Z) = 0$$

$$E((\Delta Z)^2) = \frac{1}{3}\Delta^3$$

$$E(\Delta Z\,\Delta W) = \frac{1}{2}\Delta^2$$

- two independent $N(0,1)$

  U_1 and U_2

  $\Longrightarrow$

$$\Delta W = U_1 \sqrt{\Delta}, \qquad \Delta Z = \frac{1}{2} \Delta^{\frac{3}{2}} \left(U_1 + \frac{1}{\sqrt{3}} U_2 \right)$$

- **triple Wiener integral**

$$I_{(1,1,1)} = \frac{1}{2} \left\{ \frac{1}{3} (\Delta W^1)^2 - \Delta \right\} \Delta W^1$$

  scaled monic Hermite polynomial

- **multi-dimensional order 1.5 strong Taylor scheme**

  $d \in \{1, 2, \ldots\}$ and $m = 1$

$$
\begin{aligned}
Y^k_{n+1} \;=\; & Y^k_n + a^k\,\Delta + b^k\,\Delta W \\
& + \frac{1}{2}\,L^1\,b^k\,\{(\Delta W)^2 - \Delta\} + L^1\,a^k\,\Delta Z \\
& + L^0\,b^k\,\{\Delta W\,\Delta - \Delta Z\} + \frac{1}{2}\,L^0\,a^k\,\Delta^2 \\
& + \frac{1}{2}\,L^1\,L^1\,b^k\left\{\frac{1}{3}\,(\Delta W)^2 - \Delta\right\}\Delta W
\end{aligned}
$$

- **general multi-dimensional order 1.5 strong Taylor scheme**

  $d, m \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1}^k &= Y_n^k + a^k \Delta + \frac{1}{2} L^0 a^k \Delta^2 \\
&\quad + \sum_{j=1}^m \left(b^{k,j} \Delta W^j + L^0 b^{k,j} I_{(0,j)} + L^j a^k I_{(j,0)}\right) \\
&\quad + \sum_{j_1,j_2=1}^m L^{j_1} b^{k,j_2} I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=1}^m L^{j_1} L^{j_2} b^{k,j_3} I_{(j_1,j_2,j_3)}
\end{aligned}
$$

in case of additive noise simplifies considerably

strong order $\gamma = 1.5$

Approximate Multiple Stochastic Integrals

Let $\xi_j, \zeta_{j,1}, \ldots, \zeta_{j,p}, \eta_{j,1}, \ldots, \eta_{j,p}, \mu_{j,p}$ and $\phi_{j,p}$

be independent $N(0,1)$

for $j, j_1, j_2, j_3 \in \{1, 2, \ldots, m\}$ and some $p \in \{1, 2, \ldots\}$

set

$$I_{(j)} = \Delta W^j = \sqrt{\Delta}\,\xi_j, \qquad I_{(j,0)} = \frac{1}{2}\,\Delta\left(\sqrt{\Delta}\,\xi_j + a_{j,0}\right)$$

with

$$a_{j,0} = -\frac{\sqrt{2\Delta}}{\pi}\sum_{r=1}^{p}\frac{1}{r}\,\zeta_{j,r} - 2\sqrt{\Delta\,\varrho_p}\,\mu_{j,p}$$

where

$$\varrho_p = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{r=1}^{p} \frac{1}{r^2}$$

$$I_{(0,j)} = \Delta W^j \, \Delta - I_{(j,0)}, \qquad I_{(j,j)} = \frac{1}{2} \left\{ (\Delta W^j)^2 - \Delta \right\}$$

$$I_{(j,j,j)} = \frac{1}{2} \left\{ \frac{1}{3} \, (\Delta W^j)^2 - \Delta \right\} \Delta W^j$$

$$I^p_{(j_1,j_2)} = \frac{1}{2} \, \Delta \, \xi_{j_1} \, \xi_{j_2} - \frac{1}{2} \sqrt{\Delta} \, (\xi_{j_1} \, a_{j_2,0} - \xi_{j_2} \, a_{j_1,0}) + A^p_{j_1,j_2} \, \Delta$$

Order 2.0 Strong Taylor Scheme

- use Stratonovich-Taylor expansion

  $d = m = 1$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + \underline{a}\,\Delta + b\,\Delta W + \frac{1}{2!}\,b\,b'(\Delta W)^2 + b\,\underline{a}'\,\Delta Z \\
& + \frac{1}{2}\,\underline{a}\,\underline{a}'\,\Delta^2 + \underline{a}\,b'\{\Delta W\,\Delta - \Delta Z\} \\
& + \frac{1}{3!}\,b\left(b\,b'\right)'\,(\Delta W)^3 + \frac{1}{4!}\,b\left(b\left(b\,b'\right)'\right)'\,(\Delta W)^4 \\
& + \underline{a}\left(b\,b'\right)'\,J_{(0,1,1)} + b\left(\underline{a}\,b'\right)'\,J_{(1,0,1)} + b\left(b\,\underline{a}'\right)'\,J_{(1,1,0)}
\end{aligned}
$$

Approximate Multiple Stratonovich Integrals

$$\Delta W = J^p_{(1)} = \sqrt{\Delta}\,\zeta_1$$

$$\Delta Z = J^p_{(1,0)} = \frac{1}{2}\,\Delta\left(\sqrt{\Delta}\,\zeta_1 + a_{1,0}\right)$$

$$J^p_{(1,0,1)} = \frac{1}{3!}\,\Delta^2\zeta_1^2 - \frac{1}{4}\,\Delta\,a_{1,0}^2 + \frac{1}{\pi}\,\Delta^{\frac{3}{2}}\,\zeta_1\,b_1 - \Delta^2\,B^p_{1,1}$$

$$J^p_{(0,1,1)} = \frac{1}{3!}\,\Delta^2\zeta_1^2 - \frac{1}{2\pi}\,\Delta^{\frac{3}{2}}\,\zeta_1\,b_1 + \Delta^2\,B^p_{1,1} - \frac{1}{4}\,\Delta^{\frac{3}{2}}\,a_{1,0}\,\zeta_1 + \Delta^2\,C^p_{1,1}$$

$$J^p_{(1,1,0)} = \frac{1}{3!}\,\Delta^2\zeta_1^2 + \frac{1}{4}\,\Delta\,a_{1,0}^2 - \frac{1}{2\pi}\,\Delta^{\frac{3}{2}}\,\zeta_1\,b_1 + \frac{1}{4}\,\Delta^{\frac{3}{2}}\,a_{1,0}\,\zeta_1 - \Delta^2\,C^p_{1,1}$$

with

$$a_{1,0} = -\frac{1}{\pi}\sqrt{2\Delta}\sum_{r=1}^{p}\frac{1}{r}\,\xi_{1,r} - 2\sqrt{\Delta\varrho_p}\,\mu_{1,p}$$

$$\varrho_p = \frac{1}{12} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^2}$$

$$b_1 = \sqrt{\frac{\Delta}{2}}\sum_{r=1}^{p}\frac{1}{r^2}\,\eta_{1,r} + \sqrt{\Delta\alpha_p}\,\phi_{1,p}$$

$$\alpha_p = \frac{\pi^2}{180} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^4}$$

$$B_{1,1}^{p} = \frac{1}{4\pi^2}\sum_{r=1}^{p}\frac{1}{r^2}\left(\xi_{1,r}^2 + \eta_{1,r}^2\right)$$

and

$$C_{1,1}^{p} = -\frac{1}{2\pi^2} \sum_{\substack{r,l=1 \\ r \neq l}}^{p} \frac{r}{r^2 - l^2} \left(\frac{1}{l} \, \xi_{1,r} \, \xi_{1,\ell} - \frac{l}{r} \, \eta_{1,r} \, \eta_{1,\ell} \right)$$

ζ_1, $\xi_{1,r}$ $\eta_{1,r}$, $\mu_{1,p}$ and $\phi_{1,p} \sim N(0,1)$ i.i.d.

Multi-dimensional Order 2.0 Strong Taylor Scheme

$m = 1$

$$
\begin{aligned}
Y^k_{n+1} \;=\; & Y^k_n + \underline{a}^k\,\Delta + b^k\,\Delta W + \frac{1}{2!}\,\underline{L}^1\,b^k\,(\Delta W)^2 + \underline{L}^1\,\underline{a}^k\,\Delta Z \\
& + \frac{1}{2}\,\underline{L}^0\underline{a}^k\,\Delta^2 + \underline{L}^0 b^k\,\{\Delta W\,\Delta - \Delta Z\} \\
& + \frac{1}{3!}\,\underline{L}^1\underline{L}^1 b^k\,(\Delta W)^3 + \frac{1}{4!}\,\underline{L}^1\underline{L}^1\underline{L}^1 b^k\,(\Delta W)^4 \\
& + \underline{L}^0\underline{L}^1 b^k\,J_{(0,1,1)} + \underline{L}^1\underline{L}^0 b^k\,J_{(1,0,1)} + \underline{L}^1\underline{L}^1\underline{a}^k\,J_{(1,1,0)}
\end{aligned}
$$

- **general multi-dimensional order 2.0 strong Taylor scheme**

$$
\begin{aligned}
Y_{n+1}^k &= Y_n^k + \underline{a}^k \Delta + \frac{1}{2} \underline{L}^0 \underline{a}^k \Delta^2 \\
&\quad + \sum_{j=1}^{m} \left(b^{k,j} \Delta W^j + \underline{L}^0 b^{k,j} J_{(0,j)} + \underline{L}^j \underline{a}^k J_{(j,0)} \right) \\
&\quad + \sum_{j_1,j_2=1}^{m} \Big(\underline{L}^{j_1} b^{k,j_2} J_{(j_1,j_2)} + \underline{L}^0 \underline{L}^{j_1} b^{k,j_2} J_{(0,j_1,j_2)} \\
&\qquad + \underline{L}^{j_1} \underline{L}^0 b^{k,j_2} J_{(j_1,0,j_2)} + \underline{L}^{j_1} \underline{L}^{j_2} \underline{a}^k J_{(j_1,j_2,0)} \Big) \\
&\quad + \sum_{j_1,j_2,j_3=1}^{m} \underline{L}^{j_1} \underline{L}^{j_2} b^{k,j_3} J_{(j_1,j_2,j_3)} \\
&\quad + \sum_{j_1,j_2,j_3,j_4=1}^{m} \underline{L}^{j_1} \underline{L}^{j_2} \underline{L}^{j_3} b^{k,j_4} J_{(j_1,j_2,j_3,j_4)}
\end{aligned}
$$

Convergence Theorem

- hierarchical set

$$\mathcal{A}_\gamma = \left\{ \alpha \in \mathcal{M} : \ell(\alpha) + n(\alpha) \leq 2\,\gamma \ \text{ or } \ \ell(\alpha) = n(\alpha) = \gamma + \frac{1}{2} \right\}$$

- **order γ strong Taylor approximation**

$$\begin{aligned} Y_t^{\Delta} &= Y_{n_t}^{\Delta} + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{v\}} I_\alpha \left[f_\alpha \left(\tau_{n_t}, Y_{n_t}^{\Delta} \right) \right]_{\tau_{n_t}, t} \\ &= \sum_{\alpha \in \mathcal{A}_\gamma} I_\alpha \left[f_\alpha \left(\tau_{n_t}, Y_{n_t}^{\Delta} \right) \right]_{\tau_{n_t}, t} \end{aligned}$$

for $\gamma \in \{0.5, 1.0, 1.5, 2.0, \ldots\}$

stochastic interpolation

Theorem

$\boldsymbol{Y}^{\boldsymbol{\Delta}} = \{\boldsymbol{Y}_{\boldsymbol{t}}^{\boldsymbol{\Delta}}, \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{T}]\}$ *order* $\boldsymbol{\gamma}$ *strong Taylor approximation*

$\gamma \in \{\mathbf{0.5}, \mathbf{1.0}, \mathbf{1.5}, \mathbf{2.0}, \ldots\}$

$$|\boldsymbol{f}_{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{x}) - \boldsymbol{f}_{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{y})| \leq \boldsymbol{K}_{\mathbf{1}} \, |\boldsymbol{x} - \boldsymbol{y}|$$

for all $\boldsymbol{\alpha} \in \boldsymbol{\mathcal{A}}_{\gamma}, \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{T}]$ *and* $\boldsymbol{x}, \boldsymbol{y} \in \Re^{\boldsymbol{d}}$,

$$\boldsymbol{f}_{-\boldsymbol{\alpha}} \in \boldsymbol{C}^{\mathbf{1},\mathbf{2}} \quad \textit{and} \quad \boldsymbol{f}_{\boldsymbol{\alpha}} \in \boldsymbol{\mathcal{H}}_{\boldsymbol{\alpha}}$$

for all $\boldsymbol{\alpha} \in \boldsymbol{\mathcal{A}}_{\gamma} \cup \boldsymbol{\mathcal{B}}\,(\boldsymbol{\mathcal{A}}_{\gamma})$ *and*

$$|\boldsymbol{f}_{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{x})| \leq \boldsymbol{K}_{\mathbf{2}} \; (\mathbf{1} + |\boldsymbol{x}|)$$

for all $\boldsymbol{\alpha} \in \boldsymbol{\mathcal{A}}_\gamma \cup \boldsymbol{\mathcal{B}}\,(\boldsymbol{\mathcal{A}}_\gamma)$, $\boldsymbol{t} \in [\mathbf{0}, \boldsymbol{T}]$ *and* $\boldsymbol{x} \in \Re^{\boldsymbol{d}}$. *Then*

$$\boldsymbol{E}\left(\sup_{0\le t\le T}\left|\boldsymbol{X}_t - \boldsymbol{Y}_t^{\boldsymbol{\Delta}}\right|^2 \left|\boldsymbol{\mathcal{A}}_0\right.\right)$$

$$\le\; \boldsymbol{K}_3\;\left(\mathbf{1} + |\boldsymbol{X}_0|^2\right)\;\boldsymbol{\Delta}^{2\gamma} + \boldsymbol{K}_4\;\left|\boldsymbol{X}_0 - \boldsymbol{Y}_0^{\boldsymbol{\Delta}}\right|^2,$$

where $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, $\boldsymbol{K}_3$, *and* $\boldsymbol{K}_4$ *do not depend on* $\boldsymbol{\Delta}$.

Derivative Free Strong Schemes

- similar to Runge-Kutta schemes for ODEs

Explicit Order 1.0 Strong Schemes

Platen (1984)

$d = m = 1$

$$Y_{n+1} = Y_n + a\,\Delta + b\,\Delta W + \frac{1}{2\sqrt{\Delta}}\,\left\{b(\tau_n, \bar{\Upsilon}_n) - b\right\}\left\{(\Delta W)^2 - \Delta\right\}$$

with supporting value

$$\bar{\Upsilon}_n = Y_n + a\,\Delta + b\,\sqrt{\Delta}$$

- **multi-dimensional Platen scheme**

$m = 1$

$$Y^k_{n+1} = Y^k_n + a^k\,\Delta + b^k\,\Delta W + \frac{1}{2\sqrt{\Delta}}\left(b^k(\tau_n, \bar{\Upsilon}_n) - b^k\right)\left((\Delta W)^2 - \Delta\right)$$

with the vector supporting value

$$\bar{\Upsilon}_n = Y_n + a\,\Delta + b\,\sqrt{\Delta}$$

- general multi-dimensional case

$$Y_{n+1}^k = Y_n^k + a^k \Delta + \sum_{j=1}^{m} b^{k,j} \Delta W^j$$

$$+ \frac{1}{\sqrt{\Delta}} \sum_{j_1, j_2 = 1}^{m} \left(b^{k,j_2} \left(\tau_n, \bar{\Upsilon}_n^{j_1} \right) - b^{k,j_2} \right) I_{(j_1, j_2)}$$

with

$$\bar{\Upsilon}_n^j = Y_n + a \Delta + b^j \sqrt{\Delta}$$

for $j \in \{1, 2, \ldots\}$

- commutative noise

$$\boldsymbol{Y}_{n+1}^{k} = \boldsymbol{Y}_n^k + \underline{\boldsymbol{a}}^k \, \boldsymbol{\Delta} + \frac{1}{2} \sum_{j=1}^{m} \left(\boldsymbol{b}^{k,j} \left(\boldsymbol{\tau}_n, \bar{\boldsymbol{\Upsilon}}_n \right) + \boldsymbol{b}^{k,j} \right) \boldsymbol{\Delta W}^j$$

with

$$\bar{\boldsymbol{\Upsilon}}_n = \boldsymbol{Y}_n + \underline{\boldsymbol{a}} \, \boldsymbol{\Delta} + \sum_{j=1}^{m} \boldsymbol{b}^j \, \boldsymbol{\Delta W}^j$$

strong order $\boldsymbol{\gamma = 1.0}$

$$\ln(\varepsilon(\Delta)) = -4.71638 + 0.946112 \ln(\Delta)$$

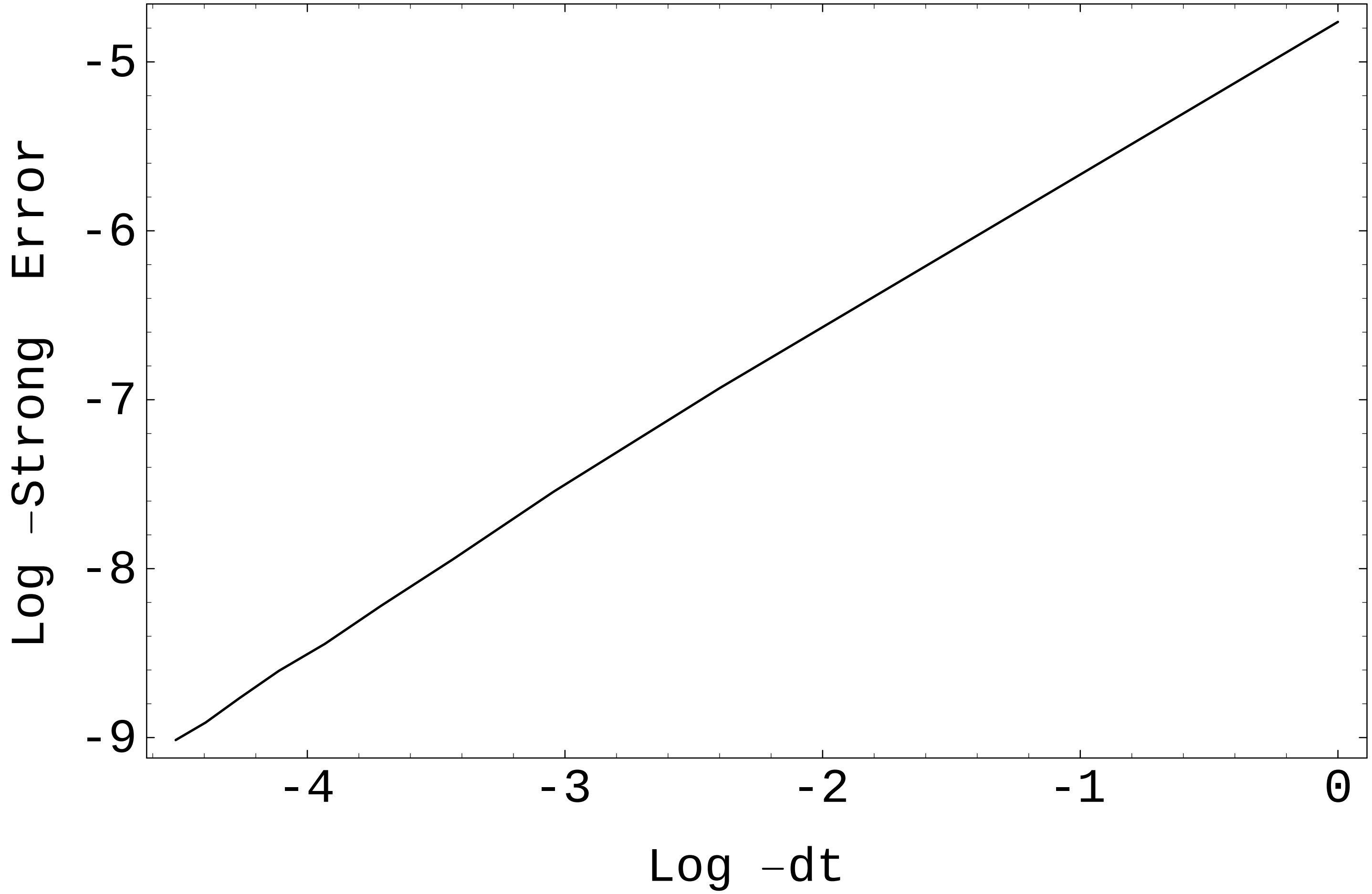


Figure 5.5: Log-log plot of the absolute error for a Platen scheme.

- **two stage Runge-Kutta method** $\gamma = 1.0$

  $m = d = 1$

  Burrage (1998)

$$
\begin{aligned}
Y_{n+1} \;&=\; Y_n + \left(\underline{a}(Y_n) + 3\,\underline{a}(\bar{Y}_n)\right)\, \frac{\Delta}{4} \\
&\quad + \frac{1}{4}\,\left(b(Y_n) + 3\,b(\bar{Y}_n)\right)\, \Delta W
\end{aligned}
$$

with

$$
\bar{Y}_n = Y_n + \frac{2}{3}\,(\underline{a}(Y_n)\,\Delta + b(Y_n)\,\Delta W)
$$

Explicit Order 1.5 Strong Schemes

Platen (1984)

$d = m = 1$

with

$$\bar{\Upsilon}_{\pm} = Y_n + a\,\Delta \pm b\,\sqrt{\Delta}$$

and

$$\bar{\Phi}_{\pm} = \bar{\Upsilon}_{+} \pm b(\bar{\Upsilon}_{+})\,\sqrt{\Delta}$$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + b\,\Delta W + \frac{1}{2\sqrt{\Delta}}\Big(a(\bar{\Upsilon}_+) - a(\bar{\Upsilon}_-)\Big)\,\Delta Z \\
& + \frac{1}{4}\Big(a(\bar{\Upsilon}_+) + 2a + a(\bar{\Upsilon}_-)\Big)\,\Delta \\
& + \frac{1}{4\sqrt{\Delta}}\Big(b(\bar{\Upsilon}_+) - b(\bar{\Upsilon}_-)\Big)\Big((\Delta W)^2 - \Delta\Big) \\
& + \frac{1}{2\Delta}\Big(b(\bar{\Upsilon}_+) - 2b + b(\bar{\Upsilon}_-)\Big)\Big(\Delta W\,\Delta - \Delta Z\Big) \\
& + \frac{1}{4\Delta}\Big(b(\bar{\Phi}_+) - b(\bar{\Phi}_-) - b(\bar{\Upsilon}_+) + b(\bar{\Upsilon}_-)\Big) \\
& \qquad \times \left(\frac{1}{3}(\Delta W)^2 - \Delta\right)\Delta W
\end{aligned}
$$

- order $\mathbf{1.5}$ Platen scheme

$$
\begin{aligned}
Y_{n+1}^k &= Y_n^k + a^k\,\Delta + \sum_{j=1}^m b^{k,j}\,\Delta W^j \\
&\quad + \frac{1}{2\sqrt{\Delta}} \sum_{j_2=0}^m \sum_{j_1=1}^m \left(b^{k,j_2}\left(\bar{\Upsilon}_+^{j_1}\right) - b^{k,j_2}\left(\bar{\Upsilon}_-^{j_1}\right)\right) I_{(j_1,j_2)} \\
&\quad + \frac{1}{2\Delta} \sum_{j_2=0}^m \sum_{j_1=1}^m \left(b^{k,j_2}\left(\bar{\Upsilon}_+^{j_1}\right) - 2b^{k,j_2} + b^{k,j_2}\left(\bar{\Upsilon}_-^{j_1}\right)\right) I_{(0,j_2)} \\
&\quad + \frac{1}{2\Delta} \sum_{j_1,j_2,j_3=1}^m \left(b^{k,j_3}\left(\bar{\Phi}_+^{j_1,j_2}\right) - b^{k,j_3}\left(\bar{\Phi}_-^{j_1,j_2}\right)\right. \\
&\qquad \left. - b^{k,j_3}\left(\bar{\Upsilon}_+^{j_1}\right) + b^{k,j_3}\left(\bar{\Upsilon}_-^{j_1}\right)\right) I_{(j_1,j_2,j_3)}
\end{aligned}
$$

with

$$\bar{\Upsilon}^j_{\pm} = Y_n + \frac{1}{m}\, a\, \Delta \pm b^j \sqrt{\Delta}$$

and

$$\bar{\Phi}^{j_1,j_2}_{\pm} = \bar{\Upsilon}^{j_1}_{+} \pm b^{j_2}\left(\bar{\Upsilon}^{j_1}_{+}\right) \sqrt{\Delta}$$

where we interpret $b^{k,0}$ as a^k

A Simulation Study

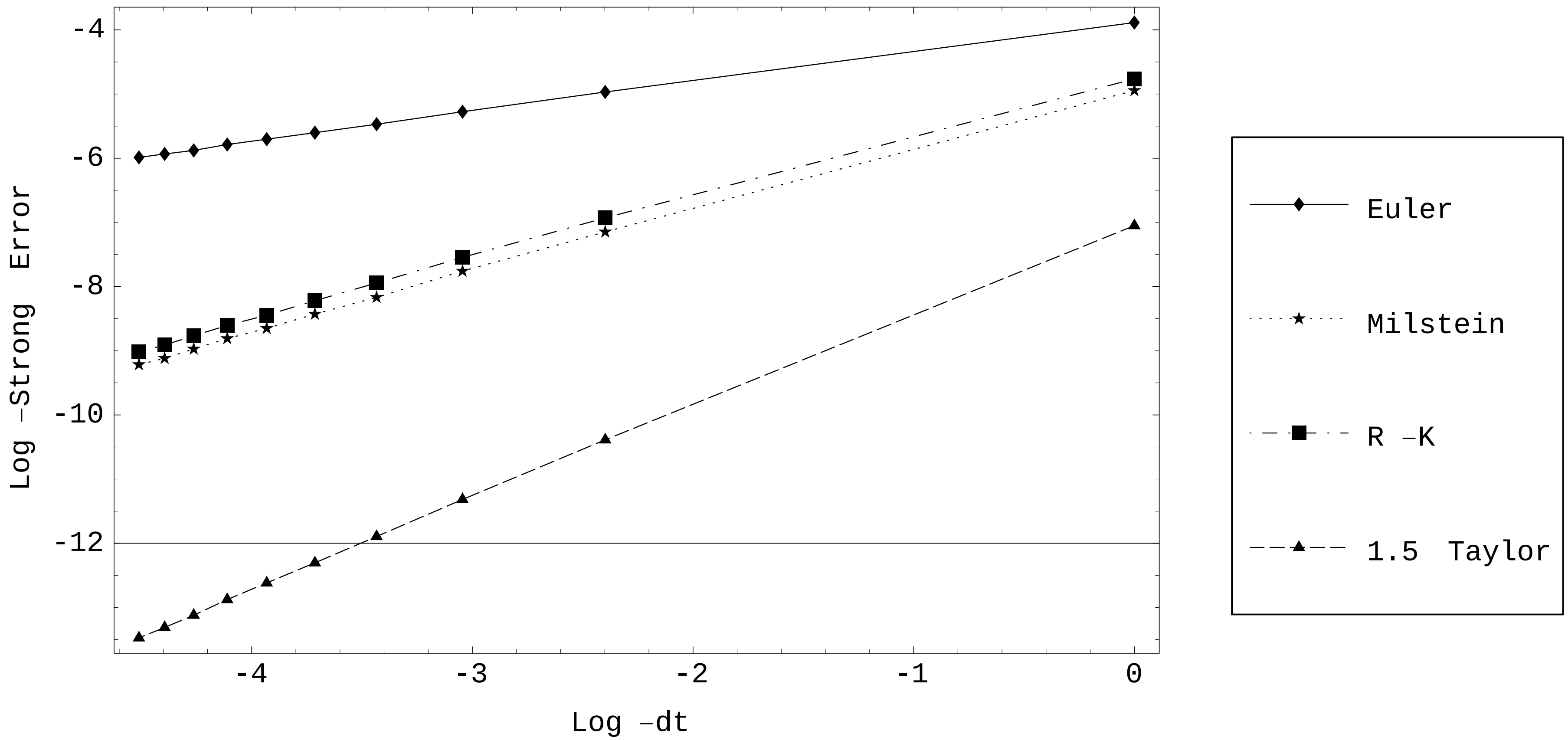


Figure 5.6: Log-log plot of the absolute error for various strong schemes.

Method	CPU Time
Euler	$\mathbf{3.5}$ Seconds
Milstein	$\mathbf{4.1}$ Seconds
1.0 Strong Platen	$\mathbf{4.1}$ Seconds
1.5 Strong Taylor	$\mathbf{4.4}$ Seconds

Table 2: Computational times for various strong methods.

$\mathbf{500000}$ time steps

Numerical Stability

- ability of a scheme to control the propagation of initial and roundoff errors

- numerical stability has higher priority than a potentially high strong order

Deterministic A-Stability

- one-step method

$$Y_{n+1} = Y_n + \Psi(\tau_n, Y_n, Y_{n+1}, \Delta)\,\Delta$$

ordinary differential equation

$$\frac{dx}{dt} = a(t, x)$$

$a(t, x)$ satisfies Lipschitz condition

- **numerically stable**

  if there exist positive constants $\boldsymbol{\Delta_0}$ and $\boldsymbol{M}$ such that

$$|\boldsymbol{Y_n} - \tilde{\boldsymbol{Y}}_{\boldsymbol{n}}| \leq \boldsymbol{M}\,|\boldsymbol{Y_0} - \tilde{\boldsymbol{Y}}_{\boldsymbol{0}}|$$

  $\boldsymbol{n \in \{0, 1, \ldots, N\}}$, $\boldsymbol{\Delta < \Delta_0}$
  and any two solutions $\boldsymbol{Y}$, $\tilde{\boldsymbol{Y}}$
  corresponding to the initial values $\boldsymbol{Y_0}$ $\tilde{\boldsymbol{Y}}_{\boldsymbol{0}}$, respectively

- **asymptotically numerically stable**

  if there exist $\boldsymbol{\Delta_0}$ and $\boldsymbol{M}$ such that

$$\lim_{n\to\infty} |\boldsymbol{Y_n} - \tilde{\boldsymbol{Y}}_{\boldsymbol{n}}| \leq \boldsymbol{M}\,|\boldsymbol{Y_0} - \tilde{\boldsymbol{Y}}_{\boldsymbol{0}}|$$

  for any two $\boldsymbol{Y}$, $\tilde{\boldsymbol{Y}}$

- **test equation**

$$\frac{dx_t}{dt} = \lambda\, x_t$$

with $\lambda \in \Re$

numerical scheme in recursive form

$$Y_{n+1} = G(\lambda\,\Delta)\,Y_n$$

- ***A*-stability region:**

  those $\lambda\,\Delta \in \Re$ for which

$$|G(\lambda\,\Delta)| < 1$$

- explicit Euler scheme

$$Y_{n+1} = Y_n + a(t_n, Y_n)\,\Delta$$

$$Y_{n+1} = (1 + \lambda\,\Delta)\,Y_n$$

A-stability region

open unit interval centered at -1 and ending at 0 since

$$|G(\lambda\,\Delta)| = |1 + \lambda\,\Delta|$$

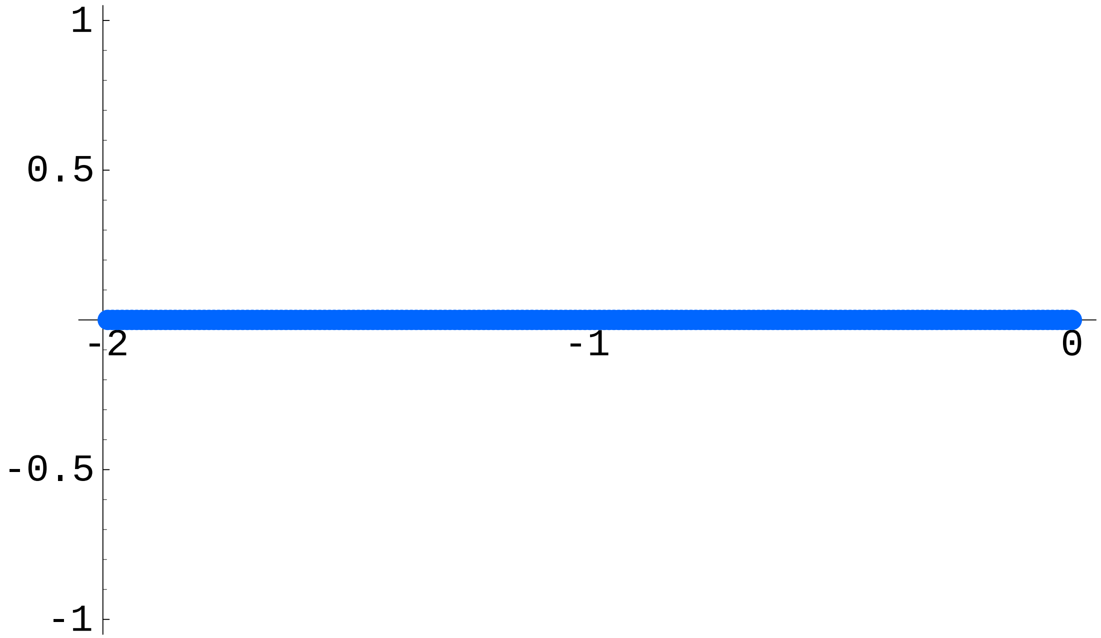


Figure 5.7: Region of A-stability for the deterministic Euler scheme.

- implicit Euler scheme

$$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\,\Delta$$

$$Y_{n+1} = Y_n + \lambda\,\Delta\,Y_{n+1}$$

$$(1 - \lambda\,\Delta)\,Y_{n+1} = Y_n$$

$$G(\lambda\,\Delta) = \frac{1}{1 - \lambda\,\Delta}$$

$$|G(\lambda\,\Delta)| = \frac{1}{|1 - \lambda\,\Delta|}$$

exterior of an interval with center at $\mathbf{1}$ beginning at 0

- scheme is called ***A-stable*** if it covers at least the left half axis

Stochastic A-Stability

- **test equation**

$$dX_t = \lambda \, X_t \, dt + dW_t$$

$\lambda \in \Re$

- discrete time approximation

$$Y_{n+1} = G^A(\lambda\,\Delta)\,Y_n + Z_n,$$

Z_n does not depend on $Y_0, Y_1, \ldots, Y_n, Y_{n+1}$

- **A-stability region**

  real numbers $\lambda\Delta$ for which

$$|G^A(\lambda\,\Delta)| < 1$$

  If A-stability region covers left half of the real axis

  $\Longrightarrow$ then **A-stable**

Drift Implicit Euler Scheme

- **drift implicit Euler scheme**

  strong order $\gamma = 0.5$

  $d = m = 1$

$$Y_{n+1} = Y_n + a\,(\tau_{n+1}, Y_{n+1})\;\Delta + b\,\Delta W$$

- **family of drift implicit Euler schemes**

$$Y_{n+1} = Y_n + \{\alpha\, a\,(\tau_{n+1}, Y_{n+1}) + (1-\alpha)\, a\}\ \Delta + b\, \Delta W$$

degree of implicitness $\alpha \in [0, 1]$

for $\alpha = 0$ explicit Euler scheme

for $\alpha = 1$ drift implicit Euler scheme

- general multi-dimensional

  family of drift implicit Euler schemes

$$Y^k_{n+1} = Y^k_n + \left(\alpha_k a^k\,(\tau_{n+1}, Y_{n+1}) + (1-\alpha_k)\,a^k\right)\Delta + \sum_{j=1}^{m} b^{k,j}\,\Delta W^j$$

$\alpha_k \in [0,1], k \in \{1,2,\dots,d\}$

- for test equation

$$Y_{n+1} = Y_n + \Big(\alpha\,\lambda\,Y_{n+1} + (1-\alpha)\,\lambda\,Y_n\Big)\Delta + \Delta W_n$$

$$Y_{n+1} = G^A(\lambda\,\Delta)\,Y_n + \Delta W_n\,(1-\alpha\,\lambda\,\Delta)^{-1}$$

transfer function

$$G^A(\lambda\,\Delta) = (1-\alpha\,\lambda\,\Delta)^{-1}\,(1+(1-\alpha)\,\lambda\,\Delta)$$

$\bar{Y}_n$ corresponding discrete time approximation that starts at initial value $\bar{Y}_0$

$$
\begin{aligned}
Y_{n+1} - \bar{Y}_{n+1} &= G^A(\lambda\,\Delta)\,(Y_n - \bar{Y}_n) \\
&= (G^A(\lambda\,\Delta))^n\,(Y_0 - \bar{Y}_0)
\end{aligned}
$$

as long as

$$
|G^A(\lambda\,\Delta)| < 1
$$

the initial error $(Y_0 - \bar{Y}_0)$ is decreased

$$
\Longrightarrow \quad \lambda\,\Delta \in \left(-\frac{2}{1 - 2\,\alpha}, 0\right)
$$

for $\alpha \geq \frac{1}{2}$ $\quad$ A-stable

Drift Implicit Milstein Scheme

$d = m = 1$

- **Itô version**

$$Y_{n+1} = Y_n + a\,(\tau_{n+1}, Y_{n+1})\;\Delta + b\,\Delta W + \frac{1}{2}\,b\,b'\Big((\Delta W)^2 - \Delta\Big)$$

- **Stratonovich version**

$$Y_{n+1} = Y_n + \underline{a}\,(\tau_{n+1}, Y_{n+1})\ \Delta + b\,\Delta W + \frac{1}{2}\,b\,b'(\Delta W)^2$$

adjusted Stratonovich drift $\underline{a} = a - \frac{1}{2} b\, b'$

both schemes are different

- multi-dimensional case with $d = m \in \{1, 2, \ldots\}$ and **commutative noise**

$$
\begin{aligned}
Y^k_{n+1} \;=\; & Y^k_n + \left\{\alpha_k\, a^k\, (\tau_{n+1}, Y_{n+1}) + (1-\alpha_k)\, a^k\right\}\, \Delta \\
& + \sum_{j=1}^{m} b^{k,j}\, \Delta W^j \\
& + \frac{1}{2} \sum_{j_1, j_2 = 1}^{m} L^{j_1} b^{k,j_2} \left\{\Delta W^{j_1} \Delta W^{j_2} - 1_{\{j_1 = j_2\}} \Delta\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
Y^k_{n+1} \;=\; & Y^k_n + \left\{\alpha_k\, \underline{a}^k\, (\tau_{n+1}, Y_{n+1}) + (1-\alpha_k)\, \underline{a}^k\right\}\, \Delta \\
& + \sum_{j=1}^{m} b^{k,j}\, \Delta W^j + \frac{1}{2} \sum_{j_1, j_2 = 1}^{m} L^{j_1} b^{k,j_2}\, \Delta W^{j_1} \Delta W^{j_2}
\end{aligned}
$$

- **general drift implicit Milstein scheme**

  Itô version

$$\begin{aligned} Y^k_{n+1} \;&=\; Y^k_n + \Big(\alpha_k\, a^k\,(\tau_{n+1}, Y_{n+1}) + (1-\alpha_k)\, a^k\Big)\Delta \\ &\quad + \sum_{j=1}^{m} b^{k,j}\,\Delta W^j + \sum_{j_1,j_2=1}^{m} L^{j_1} b^{k,j_2} I_{(j_1,j_2)} \end{aligned}$$

Stratonovich version

$$\begin{aligned} Y^k_{n+1} &= Y^k_n + \Big(\alpha_k \, \underline{a}^k \, (\tau_{n+1}, Y_{n+1}) + (1 - \alpha_k) \, \underline{a}^k\Big) \Delta \\ &\quad + \sum_{j=1}^{m} b^{k,j} \, \Delta W^j + \sum_{j_1, j_2 = 1}^{m} L^{j_1} b^{k,j_2} J_{(j_1, j_2)} \end{aligned}$$

$\alpha_k \in [0, 1]$ for $k \in \{1, . . ., d\}$

multiple stochastic integrals $I_{(j_1, j_2)}$ and $J_{(j_1, j_2)}$
approximated

Drift Implicit Order 1.0 Strong Runge-Kutta Scheme

$d = m = 1$

$$Y_{n+1} = Y_n + a\,(\tau_{n+1}, Y_{n+1})\,\Delta + b\,\Delta W + \frac{1}{2\sqrt{\Delta}}\,\big(b\,(\tau_n, \bar{\Upsilon}_n) - b\big)\,\big((\Delta W)^2 - \Delta\big)$$

with supporting value

$$\bar{\Upsilon} = Y_n + a\,\Delta + b\,\sqrt{\Delta}$$

- **family of drift implicit order 1.0 strong Runge-Kutta schemes**

$$
\begin{aligned}
Y_{n+1}^k &= Y_n^k + \left(\alpha_k\, a\left(\tau_{n+1}, Y_{n+1}\right) + (1-\alpha_k)\, a^k\right)\, \Delta \\
&\quad + \sum_{j=1}^{m} b^{k,j}\, \Delta W^j \\
&\quad + \frac{1}{\sqrt{\Delta}} \sum_{j_1,j_2=1}^{m} \left(b^{k,j_2}\left(\tau_n, \bar{\Upsilon}_n^{j_1}\right) - b^{k,j_2}\right) I_{(j_1,j_2)}
\end{aligned}
$$

with

$$
\bar{\Upsilon}_n^j = Y_n + a\,\Delta + b^j\,\sqrt{\Delta}
$$

for $j \in \{1, 2, \ldots, m\}$
$\alpha_k \in [0, 1]$ for $k \in \{1, 2, \ldots, d\}$

- for commutative noise

$$
\begin{aligned}
Y_{n+1}^k &= Y_n^k + \left(\alpha_k \, \underline{a}^k \left(\tau_{n+1}, Y_{n+1}\right) + \left(1 - \alpha_k\right) \, \underline{a}^k\right) \Delta \\
&\quad + \frac{1}{2} \sum_{j=1}^{m} \left(b^{k,j} \left(\tau_n, \bar{\Psi}_n\right) + b^{k,j}\right) \Delta W^j
\end{aligned}
$$

with

$$
\bar{\Psi}_n = Y_n + \underline{a} \, \Delta + \sum_{j=1}^{m} b^j \, \Delta W^j
$$

$\alpha_k \in [0, 1]$ for $k \in \{1, 2, \ldots, d\}$

Alternative Implicit Methods

- implicit methods are important

- overcome a range of numerical instabilities
- the above strong schemes do not provide implicit diffusion terms

  $\Longrightarrow$ important limitation

- drift implicit methods are well adapted for small noise and additive noise for relatively large multiplicative noise

  implicit diffusion terms seem unavoidable

- illustration with multiplicative noise

$$dX_t = \sigma\, X_t\, dW_t$$

- explicit strong methods have large errors for not too small time step sizes

- very small time step size may require unrealistic computational time

- cannot apply drift implicit schemes

- **balanced implicit methods**

  Milstein, Platen & Schurz (1998)

  $m = d = 1$

$$Y_{n+1} = Y_n + a\,\Delta + b\,\Delta W + (Y_n - Y_{n+1})\,C_n$$

  where

$$C_n = c^0(Y_n)\,\Delta + c^1(Y_n)\,|\Delta W|$$

  c^0, c^1 positive, real valued uniformly bounded functions

  strong order $\gamma = 0.5$

- low order strong convergence

  price to pay for numerical stability

- family of specific methods providing a balance between approximating diffusion terms

Simulation Study for the Balanced Method

- Euler scheme

$$Y_{n+1} = Y_n + \sigma\, Y_n\, \Delta W$$

- no simple stochastic counterpart of deterministic implicit Euler method since

$$Y_{n+1} = Y_n + \sigma\, Y_{n+1}\, \Delta W$$

$$Y_{n+1} = \frac{Y_n}{1 - \sigma\, \Delta W}$$

fails because

$$E|(1 - \sigma\, \Delta W)^{-1}| = +\infty$$

- **partially implicit scheme**

$$Y_{n+1} = Y_n + \left(\sigma\,\Delta W + \frac{\sigma^2}{2}\,(\Delta W)^2\right) Y_n - \frac{\sigma^2}{2}\,Y_{n+1}\,\Delta$$

- **balanced implicit method**

$$Y_{n+1} = Y_n + \sigma\,Y_n\,\Delta W + \sigma\,(Y_n - Y_{n+1})\,|\Delta W|$$

$\Longrightarrow$ implicitness also in diffusion term

- explicit solution

$$X_T = \exp\left\{\sigma\, W_T - \frac{\sigma^2}{2}\, T\right\} X_0$$

- absolute error

$$\varepsilon_T(\Delta) = E(|X_T - Y_N|)$$

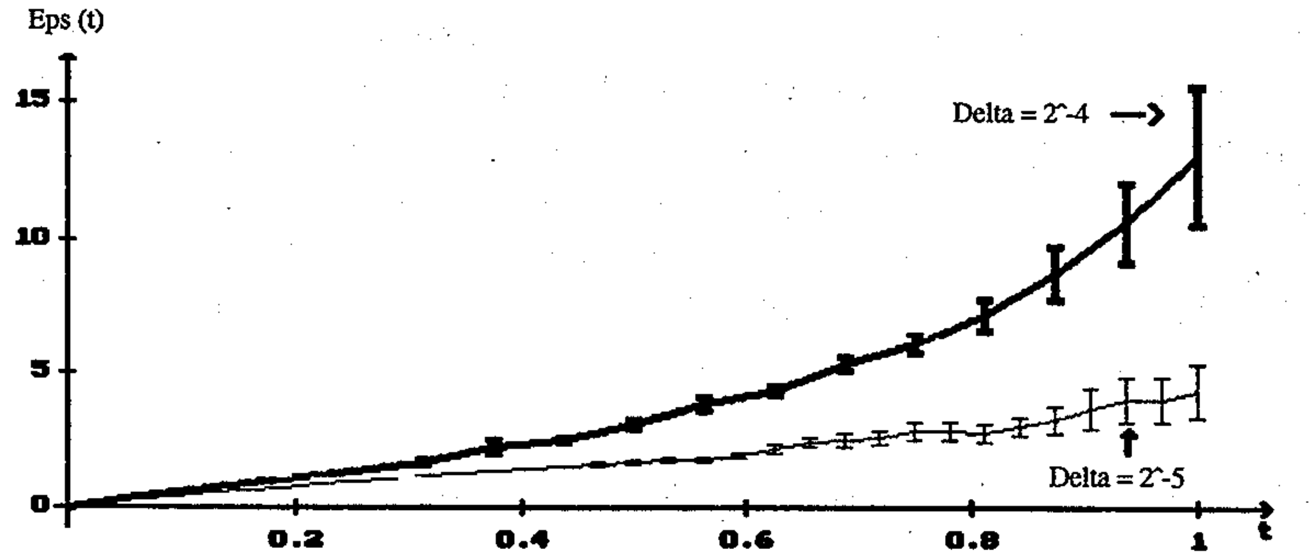


Figure 5.8: Estimated absolute error $\varepsilon_T(\Delta)$ of the Euler method at time T for time step sizes $\Delta = 2^{-4}$ and 2^{-5}.

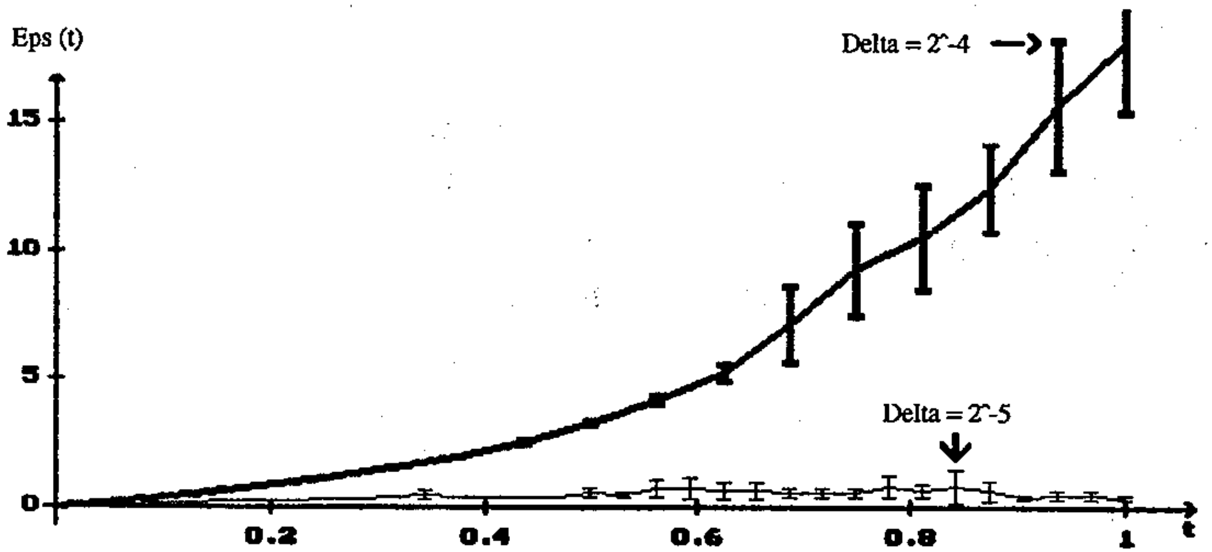


Figure 5.9: Absolute error for the partially implicit method.

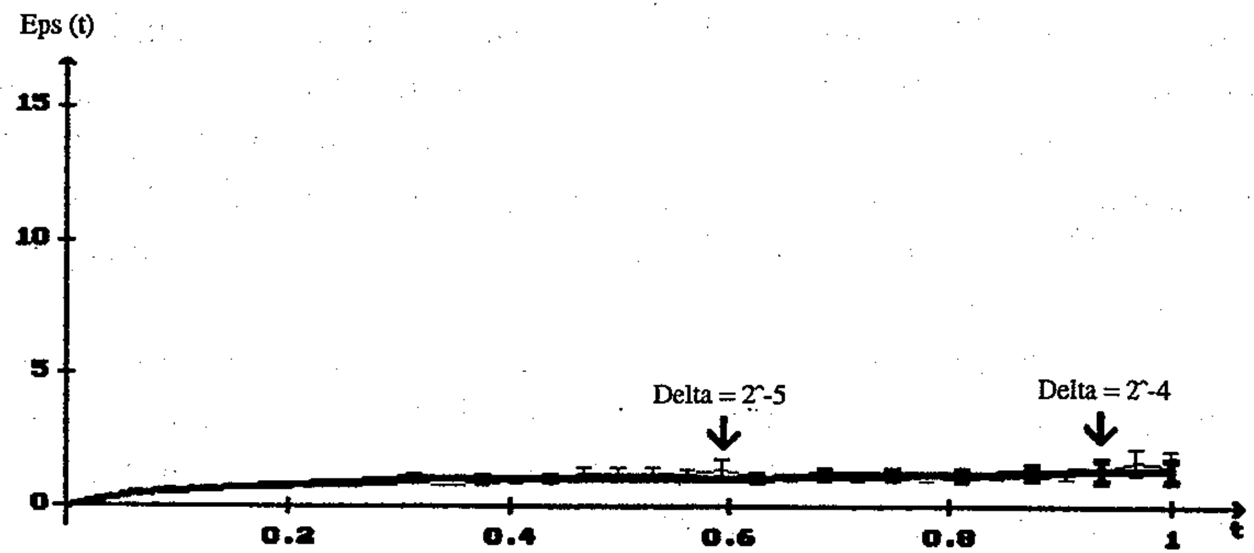


Figure 5.10: Absolute error for the balanced implicit method.

The General Balanced Method and its Convergence

- d-dimensional SDE

$$dX_t = a(t, X_t)\,dt + \sum_{j=1}^{m} b^j(t, X_t)\,dW_t^j$$

- **family of balanced implicit methods**

$$Y_{n+1} = Y_n + a(\tau_n, Y_n)\,\Delta + \sum_{j=1}^{m} b^j(\tau_n, Y_n)\,\Delta W_n^j + C_n(Y_n - Y_{n+1})$$

where

$$C_n = c^0(\tau_n, Y_n)\,\Delta + \sum_{j=1}^{m} c^j(\tau_n, Y_n)\,|\Delta W_n^j|$$

$c^0, c^1, \ldots, c^m$

$d \times d$-matrix-valued uniformly bounded functions

- assume that for any sequence of real (α_i) with $\alpha_0 \in [0, \bar{\alpha}]$, $\alpha_1 \geq 0, \ldots,$ $\alpha_m \geq 0$, where $\bar{\alpha} \geq \Delta$

  matrix

$$M(t, x) = I + \alpha_0\, c^0(t, x) + \sum_{j=1}^{m} a_j\, c^j(t, x)$$

  has an inverse

$$|(M(t, x))^{-1}| \leq K < \infty$$

Theorem (Milstein-Platen-Schurz)

The balanced implicit method converges with strong order $\gamma = 0.5$, that is for all $k \in \{0, 1, \ldots, N\}$ and step size $\Delta = \frac{T}{N}$, $N \in \{1, 2, \ldots\}$ one has

$$
\begin{aligned}
E\left(|X_{\tau_k} - Y_k| \,\middle|\, \mathcal{A}_0\right) &\leq \left(E\left(|X_{\tau_k} - Y_k|^2 \,\middle|\, \mathcal{A}_0\right)\right)^{\frac{1}{2}} \\
&\leq K\left(1 + |X_0|^2\right)^{\frac{1}{2}} \Delta^{\frac{1}{2}},
\end{aligned}
$$

where K does not depend on Δ.

Predictor-Corrector Euler Scheme

- **corrector**

$$Y_{n+1} = Y_n + \Big(\theta\,\bar{a}_\eta(\bar{Y}_{n+1}) + (1-\theta)\,\bar{a}_\eta(Y_n)\Big)\Delta_n + \Big(\eta\,b(\bar{Y}_{n+1}) + (1-\eta)\,b(Y_n)\Big)\,\Delta W_n$$

$$\bar{a}_\eta = a - \eta\,b\,b'$$

- **predictor**

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta_n + b(Y_n)\,\Delta W_n$$

$\theta, \eta \in [0, 1]$ degree of implicitness

Strong Approximation of SDEs with Jumps

Some Jump Diffusions in Finance

- **Merton model** Merton (1976)

$$dS_t = S_{t-} \left(a\, dt + \sigma\, dW_t + dY_t\right)$$

- **compound Poisson process**

$$Y_t = \sum_{k=1}^{N_t} \xi_k$$

with

$$dY_t = \xi_{N_t}\, dN_t$$

- **Poisson process**

  $N = \{N_t,\, t \in [0,T]\}$ with intensity $\lambda > 0$

  jump sizes

  i.i.d. random variables $\xi_1, \xi_2, \ldots$

- **explicit solution**

$$S_t = S_0 \, \exp\left\{\left(a - \frac{1}{2}\,\sigma^2\right) t + \sigma\, W_t\right\} \prod_{k=1}^{N_t} (\xi_k + 1)$$

- compound Poisson process

  jump size

$$\Delta Y_{\tau_k} = Y_{\tau_k} - Y_{\tau_k -} = \xi_k$$

- asset price jump size

$$\Delta S_{\tau_k} = S_{\tau_k} - S_{\tau_k -} = S_{\tau_k -}\,\xi_k$$

$\Longrightarrow$

$$S_{\tau_k} = S_{\tau_k -}\,(\xi_k + 1)$$

$\Longrightarrow$ extension of the Black-Scholes model

- for $\xi_1 = -1$

  default of the stock

- for $\xi_k = \delta - 1, k \in \{1, 2, \ldots\}$ with $\delta \in [0, 1]$

  δ recovery rate of a stock

- **scalar jump diffusion**

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t + c(t-, X_{t-})\, dN_t$$

at a jump time τ

$$X_\tau = X_{\tau-} + c(\tau-, X_{\tau-})$$

- **multi-dimensional jump diffusion**

interacting factors $X_t^1, \dots, X_t^d$

independent standard Wiener processes $W^1, \dots, W^m$

n Poisson processes with intensities $\lambda^1(t, X_t), \ldots, \lambda^n(t, X_t)$

SDE

$$dX_t^i = a^i(t, X_t)\, dt + \sum_{k=1}^{m} b^{i,k}(t, X_t)\, dW_t^k + \sum_{\ell=1}^{n} c^{i,\ell}(t-, X_{t-})\, dN_t^\ell$$

$i \in \{1, 2, \ldots, d\}$

credit risk, operational risk and insurance modeling

Random Jump Size

- **Poisson jump measure** $p^{\ell}_{\varphi_\ell}(\cdot, \cdot)$

- **mark set** $\mathcal{E} = \Re \backslash \{0\}$

$$\varphi_\ell(\mathcal{E}) < \infty$$

$$\ell \in \{1, 2, \ldots, d\}$$

jumps arrive at time $\tau_1 < \tau_2 < \ldots$ with marks $v_1, v_2, \ldots$

$$\sum_{k=1}^{N_t^\ell} c^{i,\ell}(v_k) \quad \text{as} \quad \int_0^t \int_{\mathcal{E}} c^{i,\ell}(v)\, p_{\varphi_\ell}^\ell(dv, dt)$$

$$\int_0^t \int_{\mathcal{E}} c^{i,\ell}(v) \, \left(p_{\varphi_\ell}^\ell(dv, dt) - \varphi_\ell(dv)\, dt\right)$$

$(\underline{\mathcal{A}}, P)$-martingale

- **jump diffusion SDE**

$$dX_t^i = a^i(t, X_t)\, dt + \sum_{k=1}^{m} b^{i,k}(t, X_t)\, dW_t^k$$

$$+ \sum_{\ell=1}^{n} \int_{\mathcal{E}} c^{i,\ell}(v, t-, X_{t-})\, p_{\varphi_\ell}^{\ell}(dv, dt)$$

$$X_\tau^i = X_{\tau-}^i + c^{i,\ell}(v, \tau-, X_{\tau-}).$$

$(\underline{\mathcal{A}}, P)$-martingale measure

$$q_{\varphi_\ell}^{\ell}(dv, dt) = p_{\varphi_\ell}^{\ell}(dv, dt) - \varphi_\ell(dv)\, dt$$

- Lévy process models

  Barndorff-Nielsen (1998)
  Madan & Seneta (1990)

- Affine jump-diffusions

  Duffie, Pan & Singleton (2000)

- interest rate term structure

  Björk, Kabanov & Runggaldier (1997)
  Glasserman & Merener (2003)

Discrete Time Approximation with Jump Times

- discrete time approximation of jump diffusions

  Platen (1982a, 1984)
  Platen & Rebolledo (1985)
  Maghsoodi & Harris (1987)
  Mikulevicius & Platen (1988)
  Maghsoodi (1996, 1998)
  Kubilius & Platen (2002)
  Glasserman & Merener (2003)
  Bruti-Liberati & Platen (2007)

- **jump adapted time discretization**

  $\{t_i\}$ with $t_0 < t_1 < \ldots$

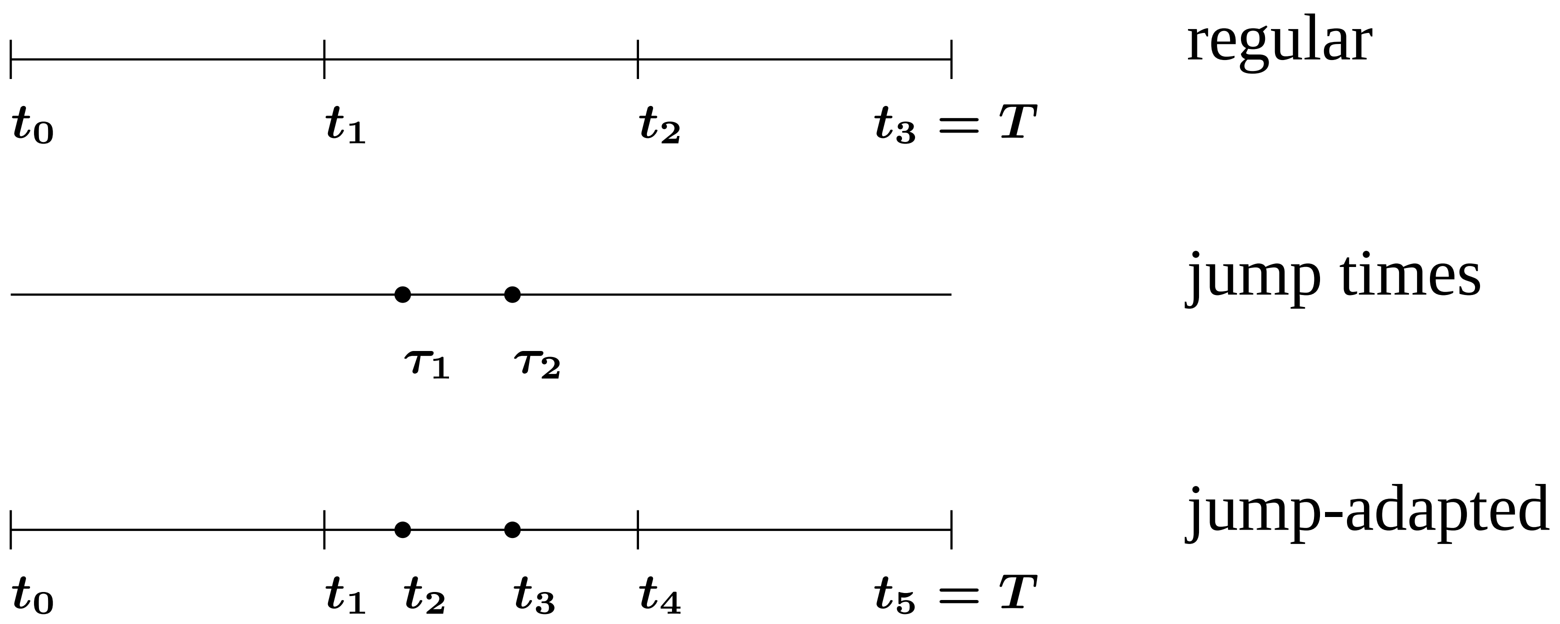
  superposition of the random jump times $\tau_1, \tau_2, \ldots$ of the Poisson processes $p^{\ell}_{\varphi_\ell}(\mathcal{E}, [0, \cdot])$ and a deterministic grid

  can be precomputed

- maximum step size is $\Delta > 0$

  $t_{i+1} - t_i \leq \Delta$ a.s

Jump-Adapted Time Discretization



Jump-Adapted Strong Approximations

jump-adapted time discretization

$\Downarrow$

jump times included in time discretization

- **jump-adapted Euler scheme**

$$Y_{t_{n+1}-} = Y_{t_n} + a(Y_{t_n})\Delta_{t_n} + b(Y_{t_n})\Delta W_{t_n}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1}-} + c(Y_{t_{n+1}-})\,\Delta p_n$$

- $\gamma = 0.5$

Euler Scheme

$d = m = n = 1$

$$Y_{t_{i+1}-} = Y_{t_i} + a\,(t_{i+1} - t_i) + b\left(W_{t_{i+1}} - W_{t_i}\right)$$

jump part

$$Y_{t_{i+1}} = Y_{t_{i+1}-} + \int_{\mathcal{E}} c(v, t_{i+1}-, Y_{t_{i+1}-})\, p_{\varphi}(dv, \{t_{i+1}\})$$

- **multi-dimensional**

  Euler scheme

$$Y^i_{t_{i+1}-} = Y^i_{t_i} + a^i\,(t_{i+1} - t_i) + \sum_{k=1}^{m} b^{i,k}\left(W^k_{t_{i+1}} - W^k_{t_i}\right)$$

with

$$\boldsymbol{Y}^{i}_{t_{i+1}} = \boldsymbol{Y}^{i}_{t_{i+1}-} + \sum_{\ell=1}^{n} \int_{\mathcal{E}} c^{i,\ell}(v, t_{i+1}-, Y_{t_{i+1}-})\, p^{\ell}_{\varphi_\ell}(dv, \{t_{i+1}\})$$

for $i \in \{1, 2, \ldots, d\}$

Platen (1982a)

$\gamma = 0.5$

Order 1.0 Taylor Scheme

$$
\begin{aligned}
Y^i_{t_{i+1}-} &= Y^i_{t_i} + a^i\,(t_{i+1} - t_i) + \sum_{k=1}^{m} b^{i,k}\left(W^k_{t_{i+1}} - W^k_{t_i}\right) \\
&\quad + \sum_{j_1,j_2=1}^{m} L^{j_1}\, b^{i,j_2}\, I_{(j_1,j_2)_{t_i,t_{i+1}}}
\end{aligned}
$$

with

$$
Y^i_{t_{i+1}} = Y^i_{t_{i+1}-} + \sum_{\ell=1}^{n} \int_{\mathcal{E}} c^{i,\ell}(v, t_{i+1}-, Y_{t_{i+1}-})\, p^{\ell}_{\varphi_\ell}(dv, \{t_{i+1}\})
$$

for $i \in \{1, 2, \ldots, d\}$

- if noise is commutative $\Longrightarrow$ scheme simplifies

  Platen (1982a)

  $\boldsymbol{\gamma = 1.0}$

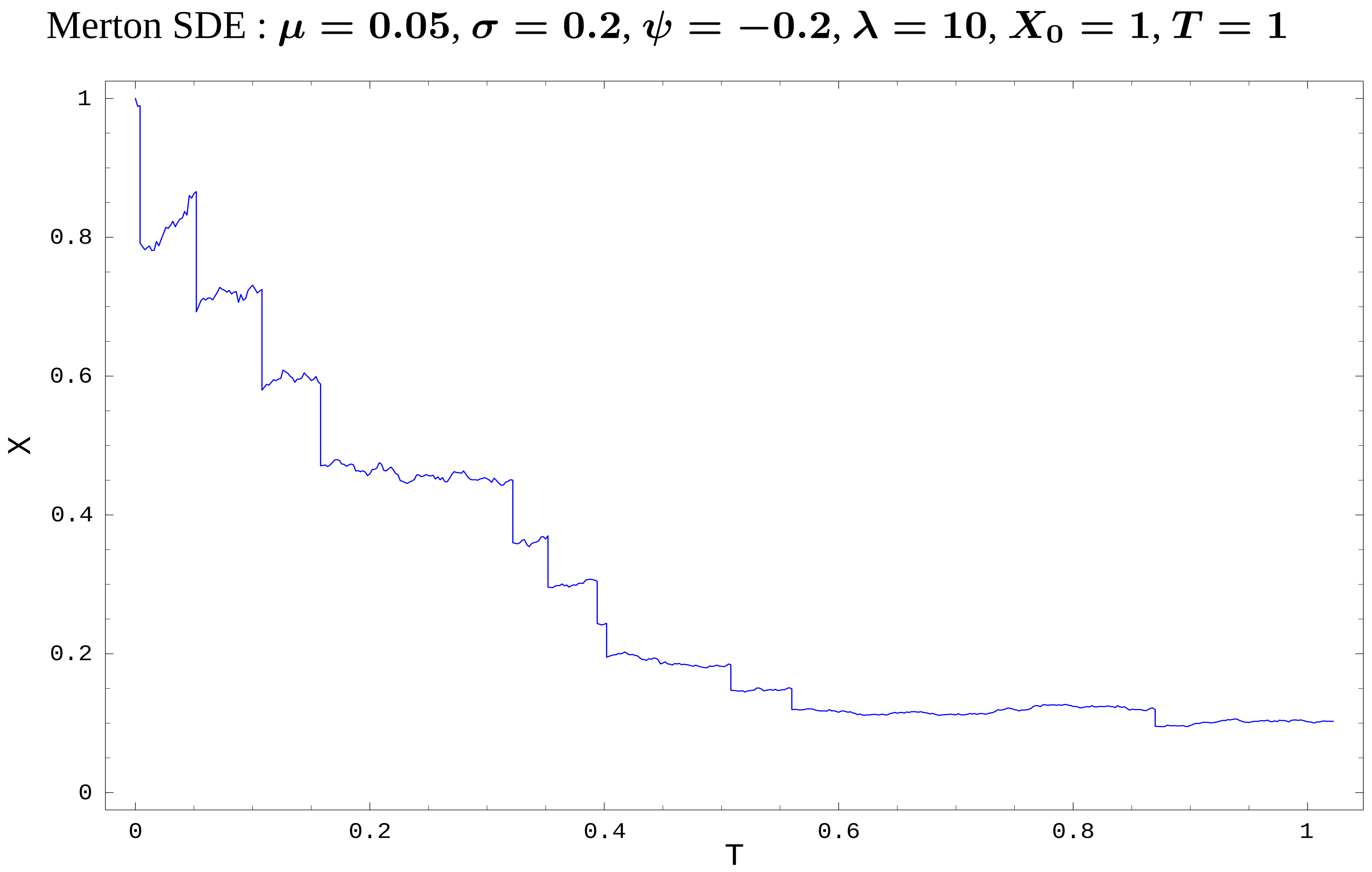


Figure 5.11: Plot of a jump-diffusion path.

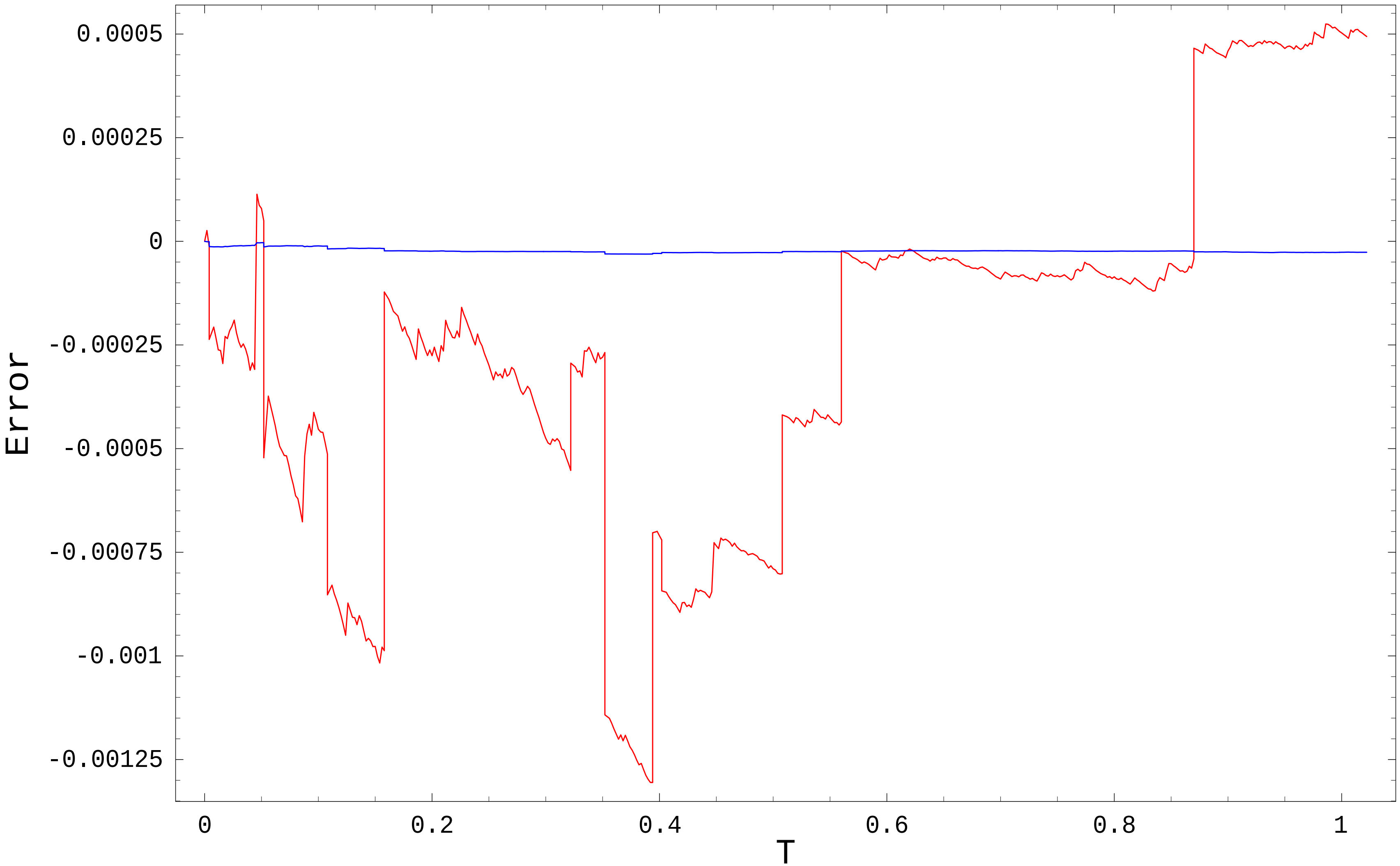


Figure 5.12: Plot of the strong error for Euler(red) and **1.0** Taylor(blue) scheme.

Merton SDE : $\boldsymbol{\mu = -0.05, \sigma = 0.1, \lambda = 1, X_0 = 1, T = 0.5}$

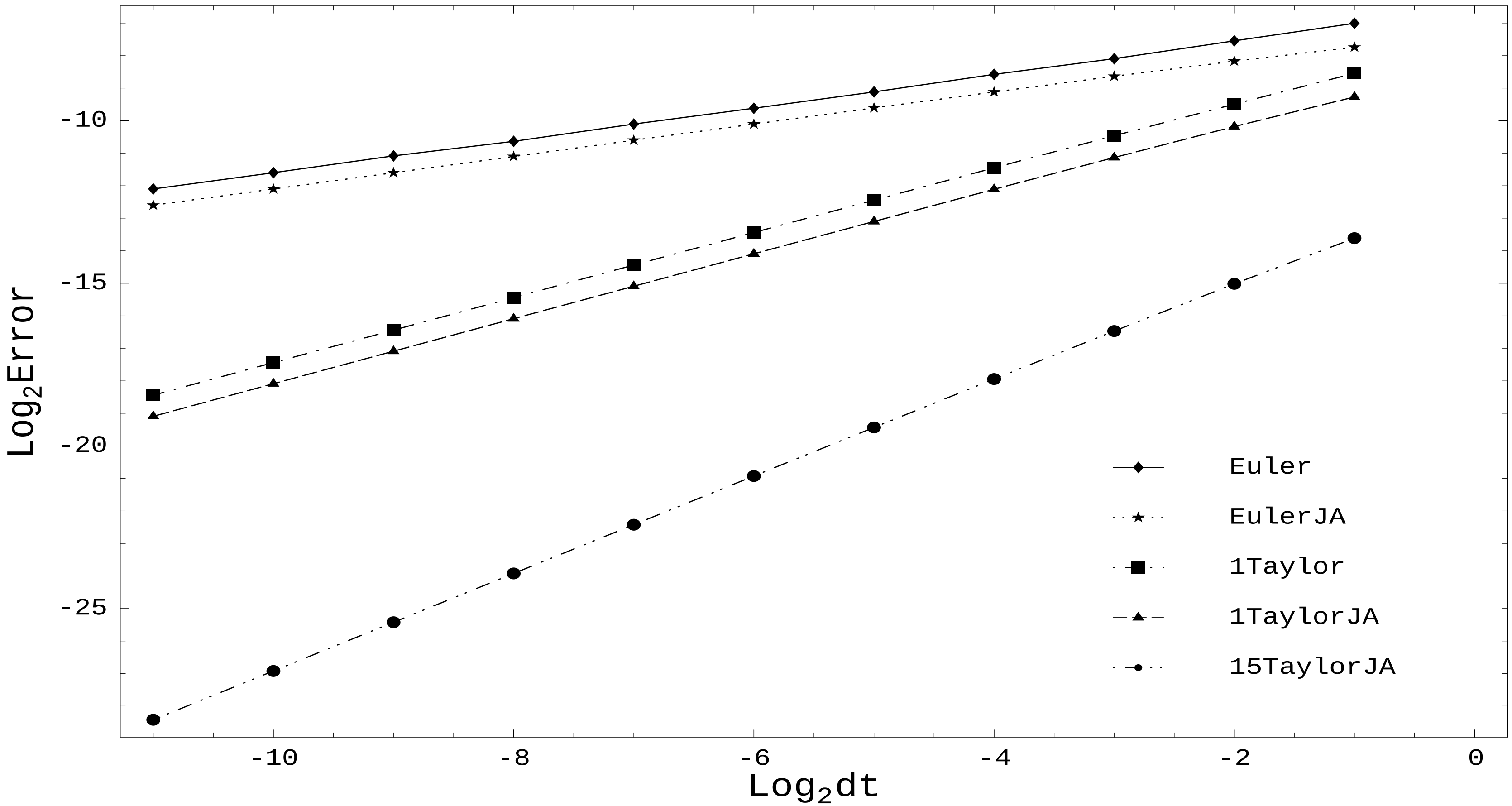


Figure 5.13: Log-log plot of strong error versus time step size.

Exercises of Chapter 5, 6 and 7

5.1 Write down for the linear SDE

$$dX_t = (\mu\, X_t + \eta)\, dt + \gamma\, X_t\, dW_t$$

with $X_0 = 1$ the Euler scheme and the Milstein scheme.

5.2 Determine for the Vasicek short rate model

$$dr_t = \gamma(\bar{r} - r_t)\, dt + \beta\, dW_t$$

the Euler and Milstein schemes. What are the differences between the resulting two schemes?

5.3 Write down for the SDE in Exercise 5.1 the order 1.5 strong Taylor scheme.

5.4 Derive for the Black-Scholes SDE

$$dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dW_t$$

with $X_0 = 1$ the explicit order 1.0 strong scheme.

11 Monte Carlo Simulation of SDEs

Introduction to Monte Carlo Simulation

- **classical Monte Carlo methods**

  Hammersley & Handscomb (1964)

  Fishman (1996)

- **Monte Carlo methods for SDEs**

  Kloeden & Platen (1999)

  Kloeden, Platen & Schurz (2003)

  Milstein (1995), Glasserman (2004)

Weak Convergence Criterion

- $\tilde{\mathcal{C}}_P(\Re^d, \Re)$ set of all polynomials $g : \Re^d \to \Re$

- SDE

$$dX_t = a(t, X_t)\, dt + \sum_{j=1}^{m} b^j(t, X_t)\, dW_t^j$$

- a discrete time approximation Y^Δ *converges with weak order* $\beta > 0$ to X at time T as $\Delta \to 0$ if for each $g \in \tilde{\mathcal{C}}_P(\Re^d, \Re)$ there exists a constant C_g, which does not depend on Δ and $\Delta_0 \in [0, 1]$ such that

$$\mu(\Delta) = |E(g(X_T)) - E(g(Y_T^\Delta))| \leq C_g\, \Delta^\beta$$

for each $\Delta \in (0, \Delta_0)$

- absolute weak error criterion

Systematic and Statistical Error

- functional

$$u = E(g(X_T))$$

- weak approximations Y^{Δ}

- **raw Monte Carlo estimate**

$$u_{N,\Delta} = \frac{1}{N} \sum_{k=1}^{N} g(Y_T^{\Delta}(\omega_k))$$

N independent simulated realizations

$$Y_T^{\Delta}(\omega_1), Y_T^{\Delta}(\omega_2), \ldots, Y_T^{\Delta}(\omega_N)$$

$\omega_k \in \Omega$ for $k \in \{1, 2, \ldots, N\}$

- discrete time weak approximation Y_T^{Δ}

- **weak error**

$$\hat{\mu}_{N,\Delta} = u_{N,\Delta} - E(g(X_T))$$

- systematic error μ_{sys}
- statistical error μ_{stat}

$$\hat{\mu}_{N,\Delta} = \mu_{\text{sys}} + \mu_{\text{stat}}$$

- **systematic error**

$$\begin{aligned}\mu_{\rm sys} &= E(\hat{\mu}_{N,\Delta}) \\ &= E\left(\frac{1}{N}\sum_{k=1}^{N} g(Y_T^{\Delta}(\omega_k))\right) - E(g(X_T)) \\ &= E(g(Y_T^{\Delta})) - E(g(X_T))\end{aligned}$$

$$\mu(\Delta) = |\mu_{\rm sys}|$$

- **statistical error**

  Central Limit Theorem

  asymptotically Gaussian with mean zero and variance

$$\mathbf{Var}(\mu_{\text{stat}}) = \mathbf{Var}(\hat{\mu}_{N,\Delta}) = \frac{1}{N}\,\mathbf{Var}(g(Y_T^{\Delta}))$$

  deviation

$$\text{Dev}(\mu_{\text{stat}}) = \sqrt{\mathbf{Var}(\mu_{\text{stat}})} = \frac{1}{\sqrt{N}}\,\sqrt{\mathbf{Var}(g(Y_T^{\Delta}))}$$

  decreases at slow rate $\frac{1}{\sqrt{N}}$ as $N \to \infty$

  may need an extremely large number N of sample paths

Confidence Intervals

- statistical error is halved by a fourfold increase in N

- Monte Carlo approach is very general

- high-dimensional functionals

- do usually not know the variance
  of raw Monte Carlo estimates

- form batches

  average of each batch

  approximately Gaussian

  Student t confidence intervals

- length of a confidence interval
  proportional to the square root of the variance

  reformulate the random variable

  same mean but a much smaller variance

- variance reduction techniques

Example of Raw Monte Carlo Simulation

$$u = E(g(X))$$

$$X \sim N(0,1)$$

$$g(X) = \left(\exp\left\{r\Delta + \sigma\sqrt{\Delta}\,X\right\}\right)^2$$

$$r = 0.05,\ \sigma = 0.2,\ \Delta = 1$$

$$u = E\left(\exp\left\{2\left(r\Delta + \sigma\,\sqrt{\Delta}\,X\right)\right\}\right) = \exp\left\{\left(r + \sigma^2\right)2\,\Delta\right\} \approx 1.197$$

- Raw Monte Carlo estimators

$$\hat{u}_N = \frac{1}{N} \sum_{i=1}^{N} \exp\left\{2\ \left(r\Delta + \sigma\sqrt{\Delta}\, X(\omega_i)\right)\right\}$$

$N \in \{1, 2, \ldots, 2000\}$

asymptotically normal

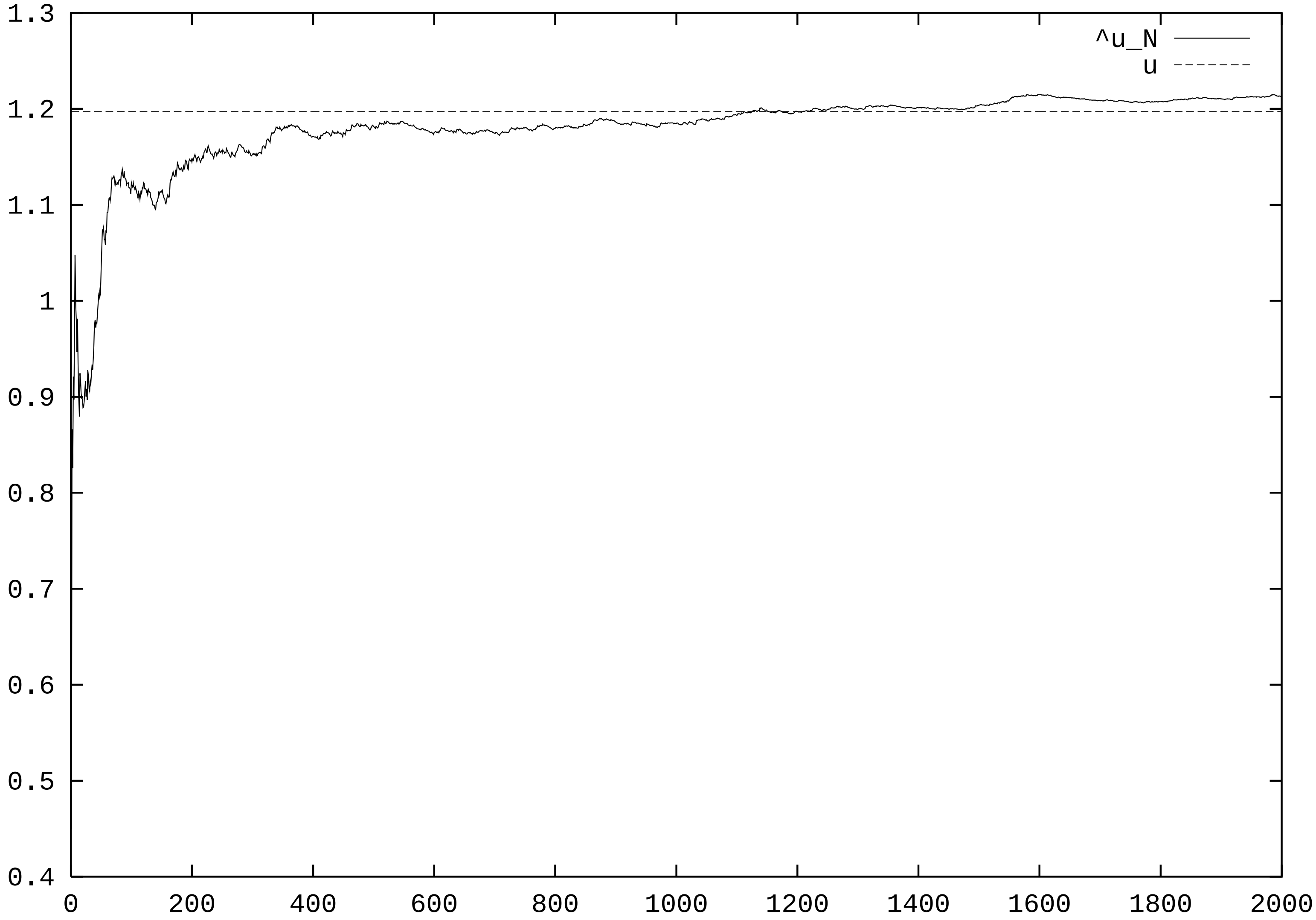


Figure 11.1: Raw Monte Carlo estimates in dependence on the number of simulations.

Normalized Monte Carlo Error

$$\hat{Z}_N = \frac{(\hat{u}_N - u)}{\sqrt{\mathrm{Var}(g(X))}} \sqrt{N} \sim \mathcal{N}(0,1)$$

CLT

$$\mathrm{Var}(g(X)) = \exp\{4\,\Delta\,(r + 2\,\sigma^2)\}\,(1 - \exp\{-4\,\Delta\,\sigma^2\}) \approx 0.25.$$

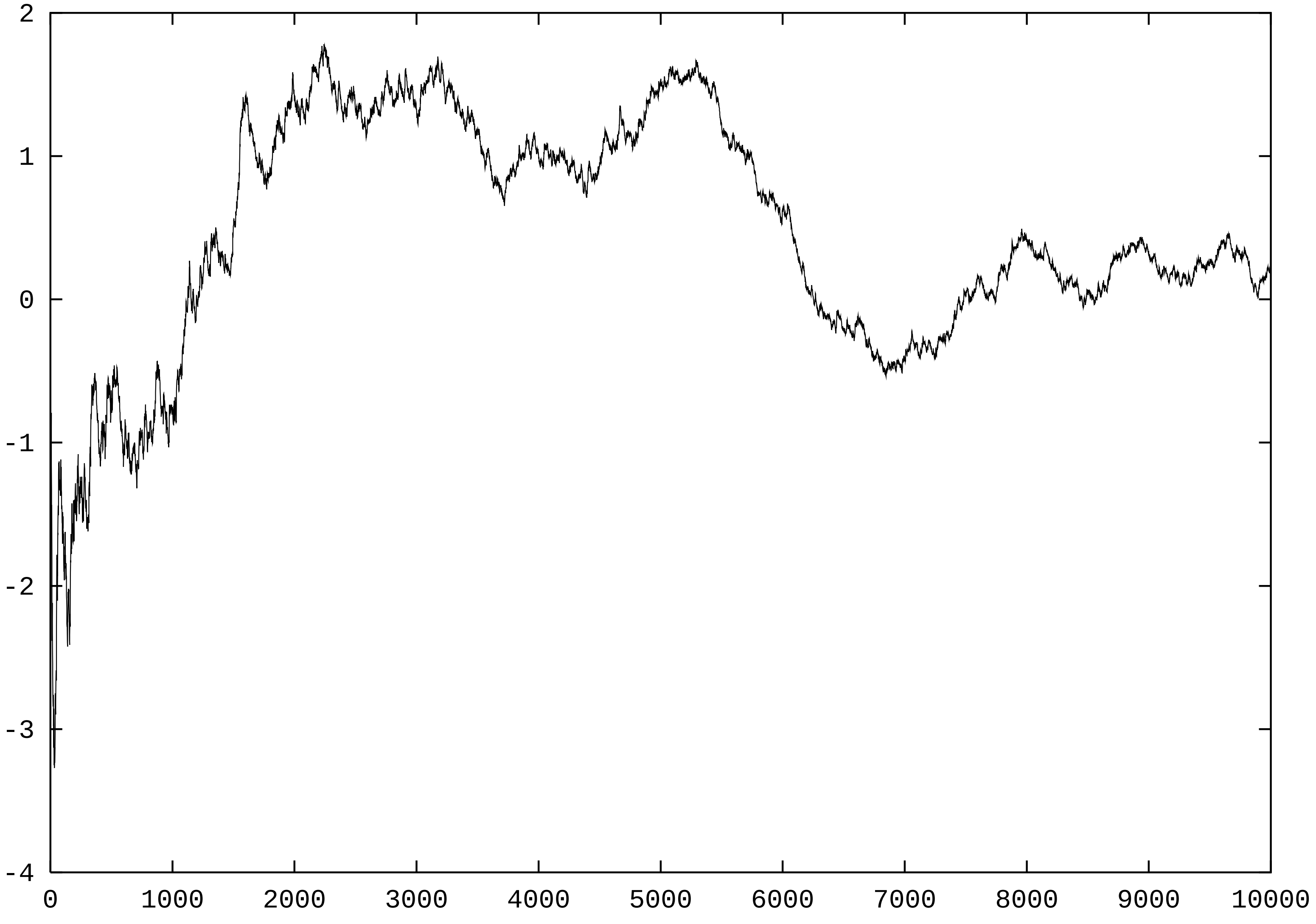


Figure 11.2: Normalized raw Monte Carlo error.

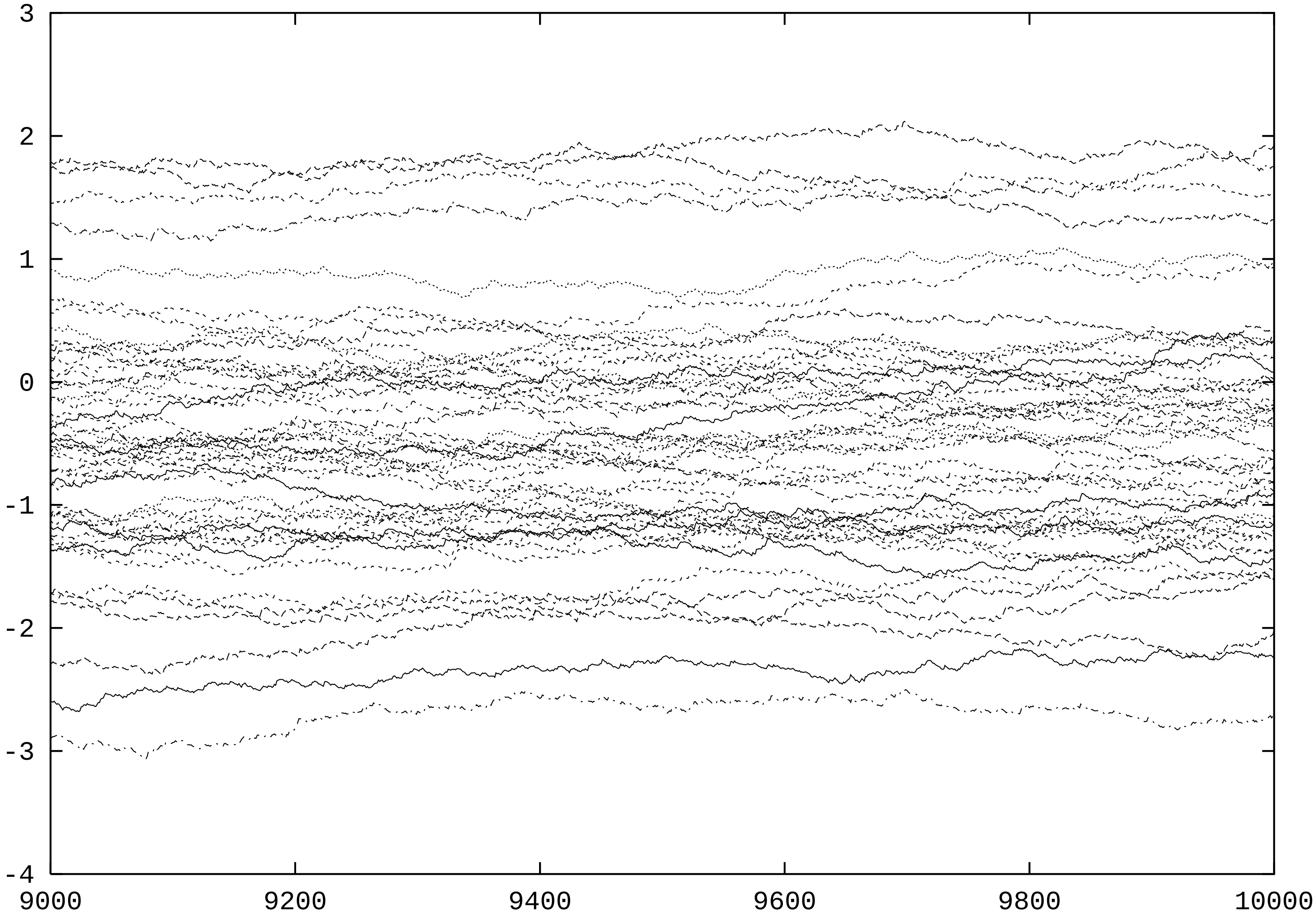


Figure 11.3: Independent realizations of normalized Monte Carlo errors.

Weak Taylor Schemes

Euler and Simplified Weak Euler Scheme

- **Euler scheme**

$$Y^k_{n+1} = Y^k_n + a^k\,\Delta + \sum_{j=1}^{m} b^{k,j}\,\Delta W^j_n$$

$$\Delta W^j_n = W^j_{\tau_{n+1}} - W^j_{\tau_n}$$

truncated Wagner-Platen expansion

weak convergence $\beta = 1.0$

$\Longrightarrow$ weak order $\beta = 1.0$ Taylor scheme

- **simplified weak Euler scheme**

$$Y^k_{n+1} = Y^k_n + a^k\,\Delta + \sum_{j=1}^{M} b^{k,j}\,\Delta\hat{W}^j_n$$

$\Delta\hat{W}^j_n$ independent $\mathcal{A}_{\tau_{n+1}}$-measurable random variables with

$$\left|E\left(\Delta\hat{W}^j_n\right)\right| + \left|E\left(\left(\Delta\hat{W}^j_n\right)^3\right)\right| + \left|E\left(\left(\Delta\hat{W}^j_n\right)^2\right) - \Delta\right| \leq K\,\Delta^2$$

$\Longrightarrow \quad \beta = 1.0$

- two-point distributed random variable

$$P\left(\Delta\hat{W}^j_n = \pm\sqrt{\Delta}\right) = \frac{1}{2}$$

- **Simulation Example**

  SDE

$$dX_t = a\, X_t\, dt + b\, X_t\, dW_t$$

  $a = 1.5$, $b = 0.01$ and $T = 1$

  test function $g(X) = x$

  $N = 16,000,000$

  $\Longrightarrow$

  confidence intervals of negligible length

- absolute weak errors

$$\mu(\Delta) = |E(X_T) - E(Y_N)|$$

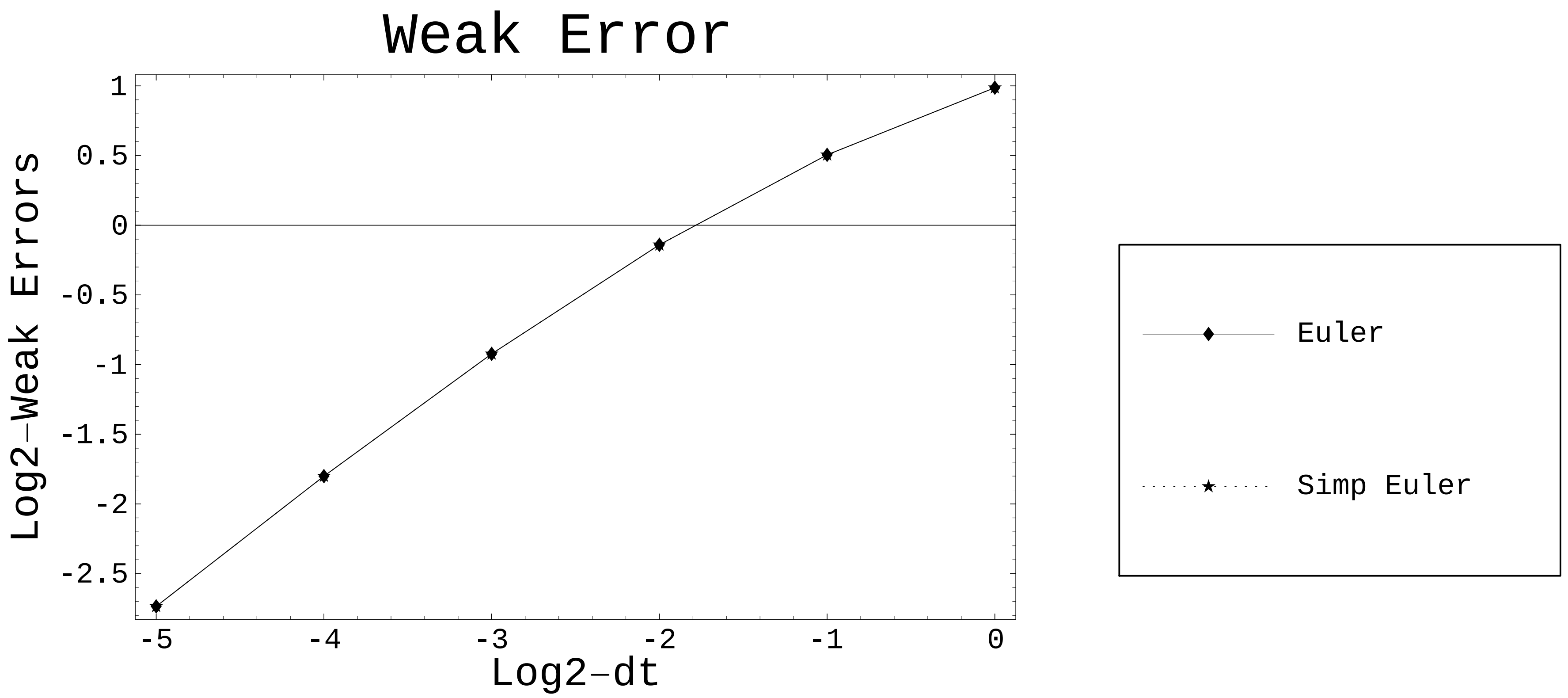


Figure 11.4: Log-log plot of the weak error for an SDE with multiplicative noise for the Euler and simplified Euler schemes.

Weak Order 2.0 Taylor Scheme

further multiple stochastic integrals

- **weak order 2.0 Taylor scheme**
  one-dimensional case $d = m = 1$

$$
\begin{aligned}
Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta W_n + \frac{1}{2}\,b\,b'\Big((\Delta W_n)^2 - \Delta\Big) \\
&\quad + a'\,b\,\Delta Z_n + \frac{1}{2}\left(a\,a' + \frac{1}{2}\,a''b^2\right)\Delta^2 \\
&\quad + \left(a\,b' + \frac{1}{2}\,b''b^2\right)\Big(\Delta W_n\,\Delta - \Delta Z_n\Big)
\end{aligned}
$$

$$
\Delta Z_n = I_{(1,0)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s} dW_z\,ds
$$

generate pair of correlated Gaussian random variables ΔW_n and ΔZ_n

- **simplified weak order 2.0 Taylor scheme**

$$
\begin{aligned}
Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta\hat{W} + \frac{1}{2}\,b\,b'\left(\left(\Delta\hat{W}\right)^2 - \Delta\right) \\
&\quad + \frac{1}{2}\left(a'\,b + a\,b' + \frac{1}{2}\,b''b^2\right)\Delta\hat{W}\,\Delta \\
&\quad + \frac{1}{2}\left(a\,a' + \frac{1}{2}\,a''b^2\right)\Delta^2
\end{aligned}
$$

$$\left|E\left(\Delta\hat{W}\right)\right| + \left|E\left(\left(\Delta\hat{W}\right)^3\right)\right| + \left|E\left(\left(\Delta\hat{W}\right)^5\right)\right|$$

$$+ \left|E\left(\left(\Delta\hat{W}\right)^2\right) - \Delta\right| + \left|E\left(\left(\Delta\hat{W}\right)^4\right) - 3\Delta^2\right| \leq K\,\Delta^3$$

- three-point distributed random variable

$$P\left(\Delta\hat{W} = \pm\sqrt{3\,\Delta}\right) = \frac{1}{6}, \qquad P\left(\Delta\hat{W} = 0\right) = \frac{2}{3}$$

- **weak order 2.0 Taylor scheme with scalar noise**

  $d \in \{1, 2, \ldots\}$ with $m = 1$

$$Y^k_{n+1} = Y^k_n + a^k \Delta + b^k \Delta W + \frac{1}{2} L^1 b^k \Big((\Delta W)^2 - \Delta \Big) + \frac{1}{2} L^0 a^k \Delta^2 + L^0 b^k \big(\Delta W \Delta - \Delta Z \big) + L^1 a^k \Delta Z$$

- **weak order 2.0 Taylor scheme**

  $d, m \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1}^k &= Y_n^k + a^k \, \Delta + \frac{1}{2} \, L^0 a^k \, \Delta^2 \\
&+ \sum_{j=1}^{m} \Big(b^{k,j} \Delta W^j + L^0 b^{k,j} \, I_{(0,j)} + L^j a^k \, I_{(j,0)} \Big) \\
&+ \sum_{j_1, j_2 = 1}^{m} L^{j_1} b^{k, j_2} \, I_{(j_1, j_2)}
\end{aligned}
$$

- **simplified weak order 2.0 Taylor scheme**

$$
\begin{aligned}
Y_{n+1}^k &= Y_n^k + a^k\,\Delta + \frac{1}{2}\,L^0 a^k\,\Delta^2 \\
&\quad + \sum_{j=1}^{m}\left(b^{k,j} + \frac{1}{2}\,\Delta\,\left(L^0 b^{k,j} + L^j a^k\right)\right)\Delta\hat{W}^j \\
&\quad + \frac{1}{2}\sum_{j_1,j_2=1}^{m} L^{j_1} b^{k,j_2}\left(\Delta\hat{W}^{j_1}\Delta\hat{W}^{j_2} + V_{j_1,j_2}\right)
\end{aligned}
$$

- two-point distributed random variables

$$P\left(V_{j_1,j_2} = \pm\Delta\right) = \frac{1}{2}$$

for $j_2 \in \{1, \ldots, j_1 - 1\}$,

$$V_{j_1,j_1} = -\Delta$$

and

$$V_{j_1,j_2} = -V_{j_2,j_1}$$

for $j_2 \in \{j_1 + 1, \ldots, m\}$ and $j_1 \in \{1, 2, \ldots, m\}$

$\beta = 2.0$

Weak Order 3.0 Taylor Scheme

- **weak order 3.0 Taylor scheme**

  $d, m \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1}^k &= Y_n^k + a^k \Delta + \sum_{j=1}^m b^{k,j} \Delta W^j + \sum_{j=0}^m L^j a^k I_{(j,0)} \\
&\quad + \sum_{j_1=0}^m \sum_{j_2=1}^m L^{j_1} b^{k,j_2} I_{(j_1,j_2)} + \sum_{j_1,j_2=0}^m L^{j_1} L^{j_2} a^k I_{(j_1,j_2,0)} \\
&\quad + \sum_{j_1,j_2=0}^m \sum_{j_3=1}^m L^{j_1} L^{j_2} b^{k,j_3} I_{(j_1,j_2,j_3)}
\end{aligned}
$$

- **simplified weak order 3.0 Taylor scheme**

$$m = 1, \quad d = 1$$

$$\begin{aligned} Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta\tilde{W} + \frac{1}{2}\,L^1 b\left(\left(\Delta\tilde{W}\right)^2 - \Delta\right) \\ &+ L^1 a\,\Delta\tilde{Z} + \frac{1}{2}L^0 a\,\Delta^2 + L^0 b\left(\Delta\tilde{W}\,\Delta - \Delta\tilde{Z}\right) \\ &+ \frac{1}{6}\left(L^0L^0 b + L^0L^1 a + L^1L^0 a\right)\,\Delta\tilde{W}\,\Delta^2 \\ &+ \frac{1}{6}\left(L^1L^1 a + L^1L^0 b + L^0L^1 b\right)\,\left(\left(\Delta\tilde{W}\right)^2 - \Delta\right)\Delta \\ &+ \frac{1}{6}\,L^0L^0 a\,\Delta^3 + \frac{1}{6}\,L^1L^1 b\,\left(\left(\Delta\tilde{W}\right)^2 - 3\Delta\right)\Delta\tilde{W} \end{aligned}$$

$$\Delta\tilde{W} \sim N(0, \Delta), \qquad \Delta\tilde{Z} \sim N\left(0, \frac{1}{3}\,\Delta^3\right)$$

$$E\left(\Delta\tilde{W}\,\Delta\tilde{Z}\right) = \frac{1}{2}\,\Delta^2$$

For $\beta = 3.0$ we can set approximately

$$I_{(1)} = \Delta \tilde{W}^1, \quad I_{(1,0)} \approx \Delta \tilde{Z}, \quad I_{(0,1)} \approx \Delta \Delta \tilde{W} - \Delta \tilde{Z}$$

$$I_{(1,1)} \approx \frac{1}{2} \left(\left(\Delta \tilde{W} \right)^2 - \Delta \right)$$

$$I_{(0,0,1)} \approx I_{(0,1,0)} \approx I_{(1,0,0)} \approx \frac{1}{6} \Delta^2 \Delta \tilde{W}$$

$$I_{(1,1,0)} \approx I_{(1,0,1)} \approx I_{(0,1,1)} \approx \frac{1}{6} \Delta \left(\left(\Delta \tilde{W} \right)^2 - \Delta \right)$$

$$I_{(1,1,1)} \approx \frac{1}{6} \Delta \tilde{W} \left(\left(\Delta \tilde{W} \right)^2 - 3 \Delta \right)$$

where $\Delta \tilde{W}$ and $\Delta \tilde{Z}$

are correlated Gaussian random variables

Hofmann (1994)

$\Longrightarrow$ additive noise, $\beta = 3.0,\ m \in \{1, 2, \ldots\}$

$$I_{(0)} = \Delta, \quad I_{(j)} = \xi_j \sqrt{\Delta}, \quad I_{(0,0)} = \frac{\Delta^2}{2}, \quad I_{(0,0,0)} = \frac{\Delta^3}{6}$$

$$I_{(j,0)} \approx \frac{1}{2}\left(\xi_j + \varphi_j \frac{1}{\sqrt{3\,\Delta}}\right)\Delta^{\frac{3}{2}}, \quad I_{(j,0,0)} \approx I_{(0,j,0)} \approx \frac{\Delta^{\frac{5}{2}}}{6}\,\xi_j$$

$$I_{(j_1,j_2,0)} \approx \frac{\Delta^2}{6}\left(\xi_{j_1}\,\xi_{j_2} + \frac{V_{j_1,j_2}}{\Delta}\right)$$

independent four point distributed random variables

$\xi_1, \ldots, \xi_m$

$$P\left(\xi_j = \pm\sqrt{3+\sqrt{6}}\right) = \frac{1}{12+4\sqrt{6}}$$

$$P\left(\xi_j = \pm\sqrt{3-\sqrt{6}}\right) = \frac{1}{12-4\sqrt{6}}$$

independent three point distributed random variables

$\varphi_1, \ldots, \varphi_m$

independent two-point distributed random variables

V_{j_1,j_2}

- **multi-dimensional case**

  $d, m \in \{1, 2, \ldots\}$

  additive noise

  weak order 3.0 Taylor scheme

$$
\begin{aligned}
Y_{n+1}^{k} \;=\; & Y_n^k + a^k\,\Delta + \sum_{j=1}^{m} b^{k,j}\,\Delta\tilde{W}^j + \frac{1}{2}\,L^0 a^k\,\Delta^2 + \frac{1}{6}\,L^0 L^0 a^k\,\Delta^3 \\
& + \sum_{j=1}^{m}\Big(L^j a^k\,\Delta\tilde{Z}^j + L^0 b^{k,j}\,\left(\Delta\tilde{W}^j\Delta - \Delta\tilde{Z}^j\right) \\
& \qquad + \frac{1}{6}\left(L^0 L^0 b^{k,j} + L^0 L^j a^k + L^j L^0 a^k\right)\Delta\tilde{W}^j\,\Delta^2\Big) \\
& + \frac{1}{6}\sum_{j_1,j_2=1}^{m} L^{j_1} L^{j_2} a^k\,\left(\Delta\tilde{W}^{j_1}\Delta\tilde{W}^{j_2} - I_{\{j_1=j_2\}}\,\Delta\right)\Delta
\end{aligned}
$$

$\Delta\tilde{W}^j$ and $\Delta\tilde{Z}^j$

independent pairs of

correlated Gaussian random variables

Weak Order 4.0 Taylor Scheme

Wagner-Platen expansion $\Longrightarrow$

- weak order $\mathbf{4.0}$ Taylor scheme

  $d, m \in \{1, 2, \ldots\}$

$$Y_{n+1}^k = Y_n^k + \sum_{\ell=1}^{4} \sum_{j_1,\ldots,j_\ell=0}^{m} L^{j_1} \cdots L^{j_{\ell-1}} \, b^{k,j_\ell} \, I_{(j_1,\ldots,j_\ell)}$$

  $\beta = 4.0$

  Platen (1984)

- **simplified weak order 4.0 Taylor scheme for additive noise**

$$
\begin{aligned}
Y_{n+1} \quad = \quad & Y_n + a\,\Delta + b\,\Delta\tilde{W} + \frac{1}{2}\,L^0 a\,\Delta^2 + L^1 a\,\Delta\tilde{Z} \\
& + L^0 b\,\left(\Delta\tilde{W}\,\Delta - \Delta\tilde{Z}\right) \\
& + \frac{1}{3!}\left(L^0 L^0 b + L^0 L^1 a\right)\Delta\tilde{W}\,\Delta^2 \\
& + L^1 L^1 a\,\left(2\,\Delta\tilde{W}\,\Delta\tilde{Z} - \frac{5}{6}\,\left(\Delta\tilde{W}\right)^2\,\Delta - \frac{1}{6}\,\Delta^2\right) \\
& + \frac{1}{3!}\,L^0 L^0 a\,\Delta^3 + \frac{1}{4!}\,L^0 L^0 L^0 a\,\Delta^4
\end{aligned}
$$

$$+\frac{1}{4!}\left(L^1L^0L^0a+L^0L^1L^0a+L^0L^0L^1a\ +L^0L^0L^0b\right)\Delta\tilde{W}\,\Delta^3$$

$$+\frac{1}{4!}\left(L^1L^1L^0a+L^0L^1L^1a+L^1L^0L^1a\right)\left(\left(\Delta\tilde{W}\right)^2-\Delta\right)\Delta^2$$

$$+\frac{1}{4!}\,L^1L^1L^1a\,\Delta\tilde{W}\left(\left(\Delta\tilde{W}\right)^2-3\Delta\right)\Delta$$

$$\Delta\tilde{W}\sim N(0,\Delta),\qquad \Delta\tilde{Z}\sim N(0,\frac{1}{3}\Delta^3)$$

$$E(\Delta\tilde{W}\,\Delta\tilde{Z})=\frac{1}{2}\,\Delta^2$$

A Simulation Study

- **SDE with additive noise**

$$dX_t = a\, X_t\, dt + b\, dW_t$$

$X_0 = 1, a = 1.5, b = 0.01$ and $T = 1$

$g(X) = x$

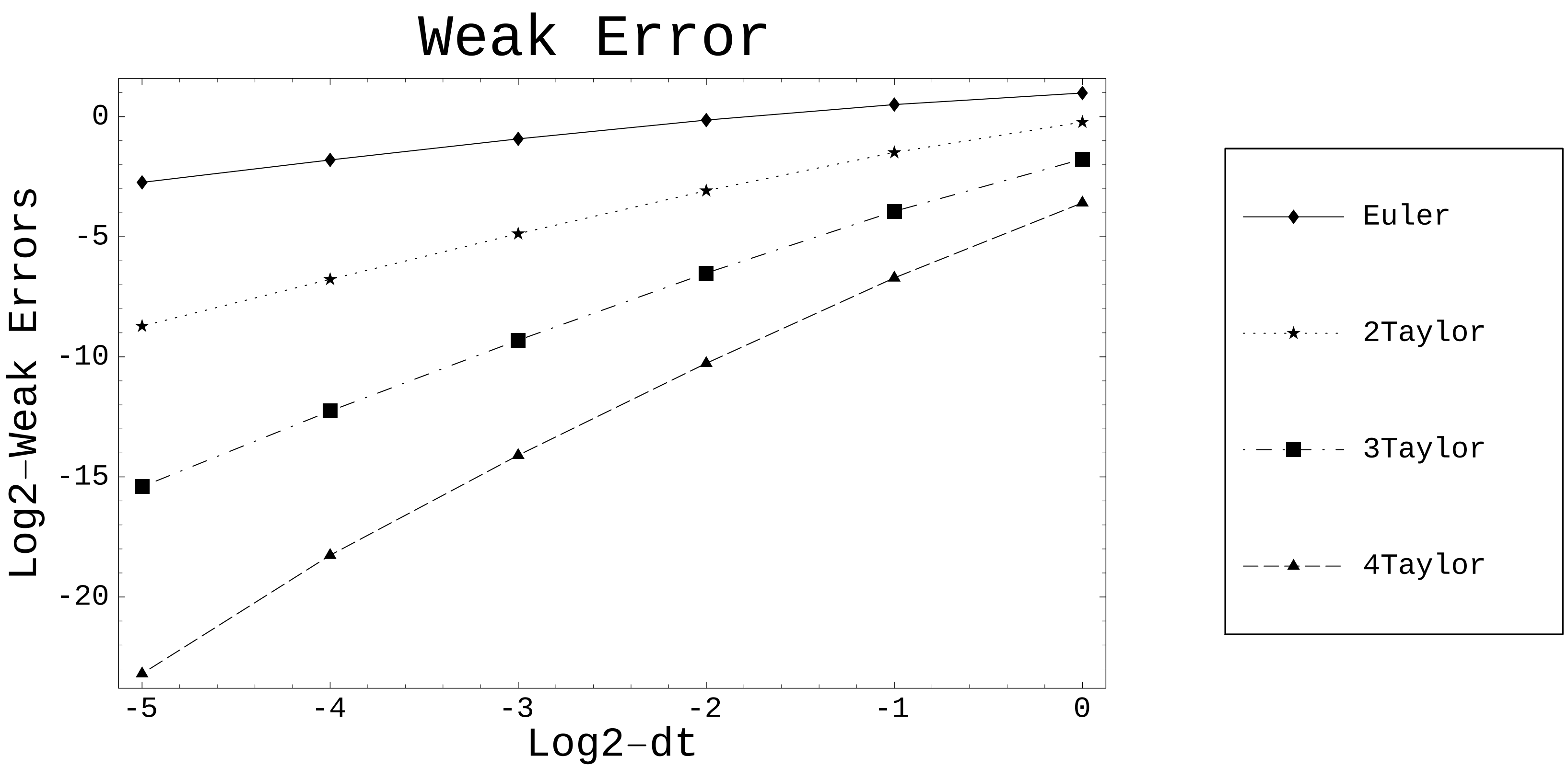


Figure 11.5: Log-log plot of the weak error for an SDE with additive noise using weak Taylor schemes.

- **SDE with multiplicative noise**

$$dX_t = a\, X_t\, dt + b\, X_t\, dW_t$$

$$g(X) = x$$

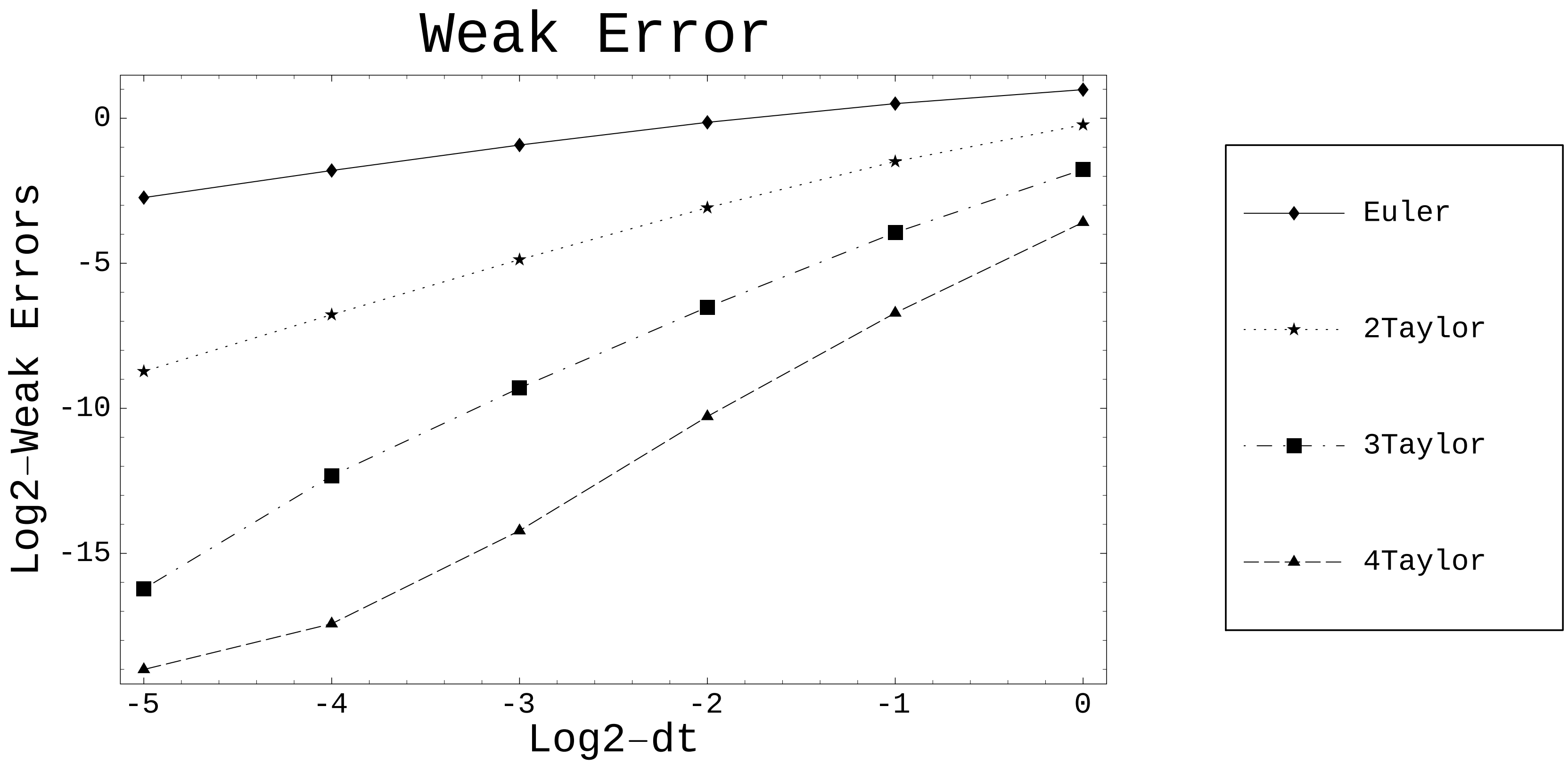


Figure 11.6: Log-log plot of the weak error for an SDE with multiplicative noise using weak Taylor schemes.

Convergence Theorem

- **Wagner-Platen expansion**

- **Itô coefficient functions**

$$f(t,x) \equiv x$$

$$f_\alpha(t,x) = L^{j_1} \cdots L^{j_{\ell-1}}\, b^{j_\ell}(x)$$

$$\alpha = (j_1, \ldots, j_\ell) \in \mathcal{M}_m,\ m \in \{1, 2, \ldots\}$$

- **multiple Itô integral**

$$I_{\alpha,\tau_n,\tau_{n+1}} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_\ell} \cdots \int_{\tau_n}^{s_2} dW_{s_1}^{j_1} \ldots dW_{s_{\ell-1}}^{j_{\ell-1}} dW_{s_\ell}^{j_\ell}$$

$$dW_s^0 = ds$$

- **hierarchical set**

$$\Gamma_\beta = \{\alpha \in \mathcal{M}_m : l(\alpha) \le \beta\}$$

- **time discretization**

$$0 = \tau_0 < \tau_1 < \ldots \tau_n < \ldots$$

- **weak Taylor scheme of order β**

$$\begin{aligned} Y_{n+1} &= Y_n + \sum_{\alpha \in \Gamma_\beta \setminus \{v\}} f_\alpha\left(\tau_n, Y_n\right) I_{\alpha,\tau_n,\tau_{n+1}} \\ &= \sum_{\alpha \in \Gamma_\beta} f_\alpha\left(\tau_n, Y_n\right) I_{\alpha,\tau_n,\tau_{n+1}} \end{aligned}$$

$$Y_0 = X_0$$

Theorem

For some $\beta \in \{1, 2, \ldots\}$ and autonomous X let Y^{Δ} be a weak Taylor scheme of order β. Suppose that a and b are Lipschitz continuous with components a^k, $b^{k,j}$ $\in \mathcal{C}_P^{2(\beta+1)}\left(\Re^d, \Re\right)$ for all $k \in \{1, 2, \ldots, d\}$ and $j \in \{0, 1, \ldots, m\}$, and that the f_α satisfy a linear growth bound

$$|f_\alpha(t, x)| \leq K \; (1 + |x|) \,,$$

for all $\alpha \in \Gamma_\beta$, $x \in \Re^d$ and $t \in [0, T]$, where $K < \infty$. Then for each $g \in \mathcal{C}_P^{2(\beta+1)}\left(\Re^d, \Re\right)$ there exists a constant C_g, which does not depend on Δ, such that

$$\mu(\Delta) = \left| E\left(g\left(X_T\right)\right) - E\left(g\left(Y_T^{\Delta}\right)\right)\right| \leq C_g \, \Delta^{\beta},$$

that is Y^{Δ} converges with weak order β to X at time T as $\Delta \to 0$.

Derivative Free Weak Approximations

Explicit Weak Order 2.0 Scheme

- **explicit weak order 2.0 scheme**

$m = 1$ and $d \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1} \quad = \quad & Y_n + \frac{1}{2}\left(a\left(\bar{\Upsilon}\right) + a\right)\Delta \\
& + \frac{1}{4}\left(b\left(\bar{\Upsilon}^{+}\right) + b\left(\bar{\Upsilon}^{-}\right) + 2b\right)\Delta\hat{W} \\
& + \frac{1}{4}\left(b\left(\bar{\Upsilon}^{+}\right) - b\left(\bar{\Upsilon}^{-}\right)\right)\left(\left(\Delta\hat{W}\right)^2 - \Delta\right)\Delta^{\frac{1}{2}}
\end{aligned}
$$

with supporting values

$$\bar{\Upsilon} = Y_n + a\,\Delta + b\,\Delta\hat{W}$$

and

$$\bar{\Upsilon}^{\pm} = Y_n + a\,\Delta \pm b\,\sqrt{\Delta}$$

$\Delta\hat{W}$ Gaussian or three-point distributed

$$P\left(\Delta\hat{W} = \pm\sqrt{3\Delta}\right) = \frac{1}{6} \quad \text{and} \quad P\left(\Delta\hat{W} = 0\right) = \frac{2}{3}$$

- **multi-dimensional explicit weak order 2.0 scheme**

$d, m \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + \frac{1}{2}\left(a\left(\bar{\Upsilon}\right) + a\right)\Delta \\
& + \frac{1}{4}\sum_{j=1}^{m}\Bigg[\left(b^j\left(\bar{R}^j_+\right) + b^j\left(\bar{R}^j_-\right) + 2b^j\right)\Delta\hat{W}^j \\
& + \sum_{\substack{r=1\\ r\neq j}}^{m}\left(b^j\left(\bar{U}^r_+\right) + b^j\left(\bar{U}^r_-\right) - 2b^j\right)\Delta\hat{W}^j\,\Delta^{-\frac{1}{2}}\Bigg]
\end{aligned}
$$

$$+ \frac{1}{4} \sum_{j=1}^{m} \Bigg[\left(b^j \left(\bar{R}_+^j \right) - b^j \left(\bar{R}_-^j \right) \right) \left(\left(\Delta \hat{W}^j \right)^2 - \Delta \right)$$

$$+ \sum_{\substack{r=1 \\ r \neq j}}^{m} \left(b^j \left(\bar{U}_+^r \right) - b^j \left(\bar{U}_-^r \right) \right) \left(\Delta \hat{W}^j \Delta \hat{W}^r + V_{r,j} \right) \Bigg] \Delta^{-\frac{1}{2}}$$

with supporting values

$$\bar{\Upsilon} = Y_n + a\,\Delta + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j, \qquad \bar{R}^j_{\pm} = Y_n + a\,\Delta \pm b^j\,\sqrt{\Delta}$$

and

$$\bar{U}^j_{\pm} = Y_n \pm b^j\,\sqrt{\Delta}$$

$\Delta\hat{W}^j$ and $V_{r,j}$ as before

- additive noise

$\Longrightarrow$

$$\boldsymbol{Y_{n+1}} \;=\; \boldsymbol{Y_n} + \frac{1}{2}\left(\boldsymbol{a}\left(\boldsymbol{Y_n} + \boldsymbol{a}\,\boldsymbol{\Delta} + \sum_{j=1}^{m} \boldsymbol{b^j}\,\boldsymbol{\Delta}\hat{\boldsymbol{W}}^{\boldsymbol{j}}\right) + \boldsymbol{a}\right)\boldsymbol{\Delta}$$

$$+ \sum_{j=1}^{m} \boldsymbol{b^j}\,\boldsymbol{\Delta}\hat{\boldsymbol{W}}^{\boldsymbol{j}}$$

Explicit Weak Order 3.0 Schemes

- **explicit weak order 3.0 scheme**

  $d \in \{1, 2, \ldots\}$ with $m = 1$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + a\,\Delta + b\,\Delta\hat{W} + \frac{1}{2}\left(a_\zeta^+ + a_\zeta^- - \frac{3}{2}\,a - \frac{1}{4}\left(\tilde{a}_\zeta^+ + \tilde{a}_\zeta^-\right)\right)\Delta \\
& + \sqrt{\frac{2}{\Delta}}\left(\frac{1}{\sqrt{2}}\left(a_\zeta^+ - a_\zeta^-\right) - \frac{1}{4}\left(\tilde{a}_\zeta^+ - \tilde{a}_\zeta^-\right)\right)\zeta\,\Delta\hat{Z} \\
& + \frac{1}{6}\left[a\left(Y_n + \left(a + a_\zeta^+\right)\Delta + (\zeta + \varrho)\,b\,\sqrt{\Delta}\right) - a_\zeta^+ - a_\varrho^+ + a\right] \\
& \qquad \times \left[(\zeta + \varrho)\;\Delta\hat{W}\,\sqrt{\Delta} + \Delta + \zeta\,\varrho\left(\left(\Delta\hat{W}\right)^2 - \Delta\right)\right]
\end{aligned}
$$

with

$$a_\phi^{\pm} = a\left(Y_n + a\,\Delta \pm b\,\sqrt{\Delta}\,\phi\right)$$

and

$$\tilde{a}_\phi^{\pm} = a\left(Y_n + 2a\,\Delta \pm b\,\sqrt{2\Delta}\,\phi\right)$$

$\Delta\hat{W} \sim N(0, \Delta)$ and $\Delta\hat{Z} \sim N(0, \frac{1}{3}\,\Delta^3)$

$$E(\Delta\hat{W}\,\Delta\hat{Z}) = \frac{1}{2}\Delta^2$$

$$P(\zeta = \pm 1) = P(\varrho = \pm 1) = \frac{1}{2}$$

- **explicit weak order 3.0 scheme for scalar noise**

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + a\,\Delta + b\,\Delta\hat{W} + \frac{1}{2}\,H_a\,\Delta + \frac{1}{\Delta}\,H_b\,\Delta\hat{Z} \\
& + \sqrt{\frac{2}{\Delta}}\,G_a\,\zeta\,\Delta\hat{Z} + \frac{1}{\sqrt{2\Delta}}\,G_b\,\zeta\left(\left(\Delta\hat{W}\right)^2 - \Delta\right) \\
& + \frac{1}{6}\,F_a^{++}\left(\Delta + (\zeta + \varrho)\sqrt{\Delta}\,\Delta\hat{W} + \zeta\varrho\left(\left(\Delta\hat{W}\right)^2 - \Delta\right)\right) \\
& + \frac{1}{24}\left(F_b^{++} + F_b^{-+} + F_b^{+-} + F_b^{--}\right)\Delta\hat{W}
\end{aligned}
$$

$$+\frac{1}{24\sqrt{\Delta}}\left(F_b^{++}-F_b^{-+}+F_b^{+-}-F_b^{--}\right)\left(\left(\Delta\hat{W}\right)^2-\Delta\right)\zeta$$

$$+\frac{1}{24\Delta}\left(F_b^{++}+F_b^{--}-F_b^{-+}-F_b^{+-}\right)\left(\left(\Delta\hat{W}\right)^2-3\right)\Delta\hat{W}\,\zeta\varrho$$

$$+\frac{1}{24\sqrt{\Delta}}\left(F_b^{++}+F_b^{-+}-F_b^{+-}-F_b^{--}\right)\left(\left(\Delta\hat{W}\right)^2-\Delta\right)\varrho$$

with

$$H_g = g^+ + g^- - \frac{3}{2}\, g - \frac{1}{4}\left(\tilde{g}^+ + \tilde{g}^-\right),$$

$$G_g = \frac{1}{\sqrt{2}}\left(g^+ - g^-\right) - \frac{1}{4}\left(\tilde{g}^+ - \tilde{g}^-\right),$$

$$\begin{aligned} F_g^{+\pm} &= g\left(Y_n + \left(a + a^+\right)\Delta + b\,\zeta\,\sqrt{\Delta} \pm b^+\,\varrho\,\sqrt{\Delta}\right) - g^+ \\ &\quad - g\left(Y_n + a\,\Delta \pm b\,\varrho\,\sqrt{\Delta}\right) + g \end{aligned}$$

$$\begin{aligned} F_g^{-\pm} &= g\left(Y_n + \left(a + a^-\right)\Delta - b\,\zeta\,\sqrt{\Delta} \pm b^-\,\varrho\,\sqrt{\Delta}\right) - g^- \\ &\quad - g\left(Y_n + a\,\Delta \pm b\,\varrho\,\sqrt{\Delta}\right) + g \end{aligned}$$

where

$$\boldsymbol{g}^{\pm} = \boldsymbol{g}\left(\boldsymbol{Y}_n + \boldsymbol{a}\,\boldsymbol{\Delta} \pm \boldsymbol{b}\,\boldsymbol{\zeta}\,\sqrt{\boldsymbol{\Delta}}\right)$$

and

$$\tilde{\boldsymbol{g}}^{\pm} = \boldsymbol{g}\left(\boldsymbol{Y}_n + \boldsymbol{2}\,\boldsymbol{a}\,\boldsymbol{\Delta} \pm \sqrt{\boldsymbol{2}}\,\boldsymbol{b}\,\boldsymbol{\zeta}\,\sqrt{\boldsymbol{\Delta}}\right)$$

with $\boldsymbol{g}$ being equal to either $\boldsymbol{a}$ or $\boldsymbol{b}$.

Extrapolation Methods

Weak Order 2.0 Extrapolation

- equidistant time discretizations

- simulate functional

$$\boldsymbol{E}\left(\boldsymbol{g}\left(\boldsymbol{Y}_{\boldsymbol{T}}^{\boldsymbol{\Delta}}\right)\right)$$

  using Euler scheme

  with step size $\boldsymbol{\Delta}$

- repeat with the double step size $\mathbf{2\Delta}$

$\Longrightarrow$

$$\boldsymbol{E}\left(\boldsymbol{g}\left(\boldsymbol{Y}_{\boldsymbol{T}}^{\mathbf{2}\boldsymbol{\Delta}}\right)\right)$$

- combine these two functionals

$\Longrightarrow$

- **weak order 2.0 extrapolation**

$$V_{g,2}^{\Delta}(T) = 2E\left(g\left(Y_T^{\Delta}\right)\right) - E\left(g\left(Y_T^{2\Delta}\right)\right)$$

Talay & Tubaro (1990)

Richardson extrapolation

- **example**

geometric Brownian motion

$$dX_t = a\,X_t\,dt + b\,X_t\,dW_t$$

$X_0 = 0.1$, $a = 1.5$ and $b = 0.01$

Richardson extrapolation $V^{\Delta}_{g,2}(T)$

$g(x) = x$

$\Longrightarrow$ absolute weak error

$$\mu(\Delta) = \left| V^{\Delta}_{g,2}(T) - E\left(g\left(X_T\right)\right) \right|$$

$\beta = 2.0$

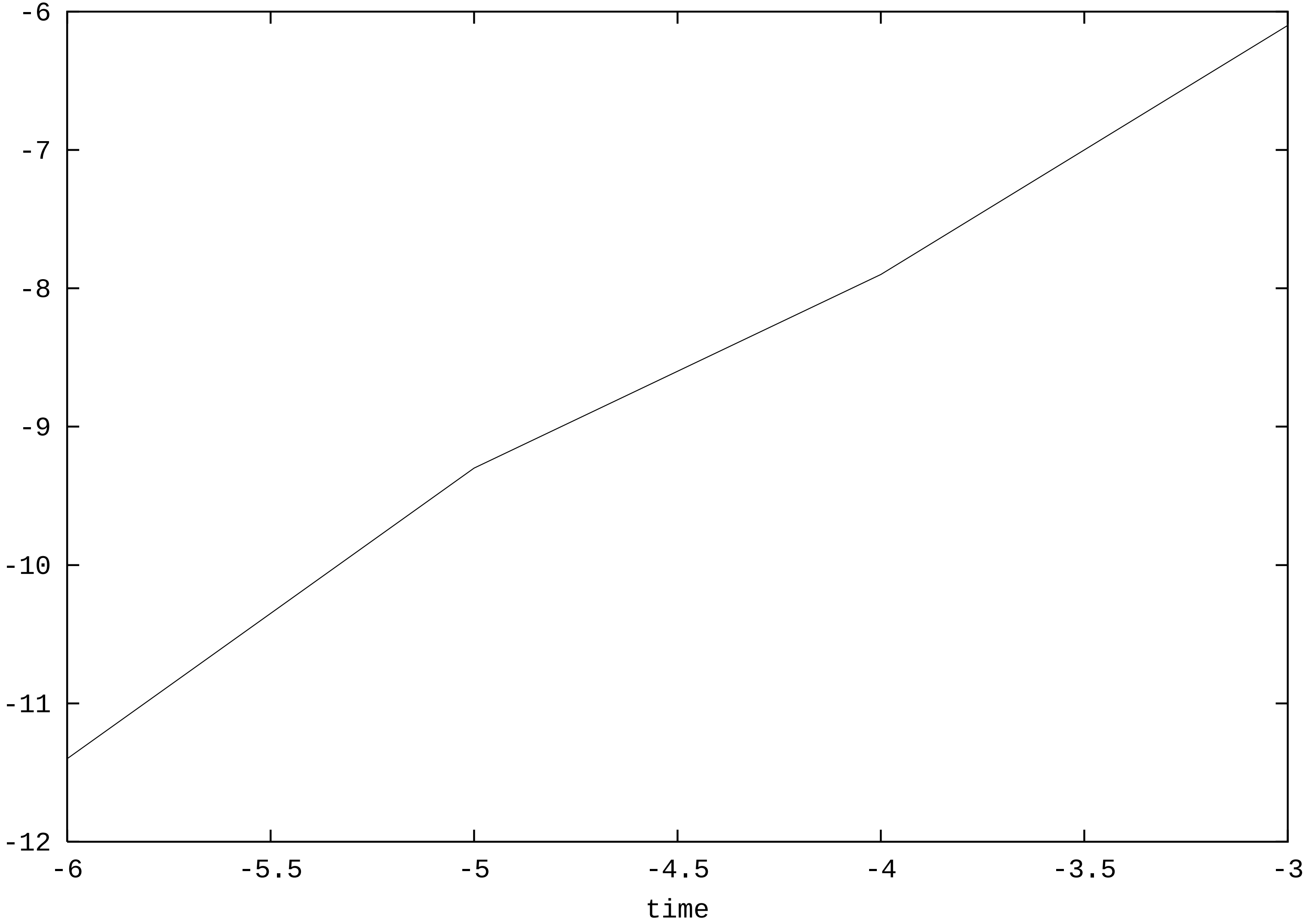


Figure 11.7: Log-log plot for absolute weak error of Richardson extrapolation against step size.

Higher Weak Order Extrapolation

- **weak order 4.0 extrapolation**

$$V^{\Delta}_{g,4}(T) = \frac{1}{21}\left[32\, E\left(g\left(Y^{\Delta}_{T}\right)\right) - 12\, E\left(g\left(Y^{2\Delta}_{T}\right)\right) + E\left(g\left(Y^{4\Delta}_{T}\right)\right)\right]$$

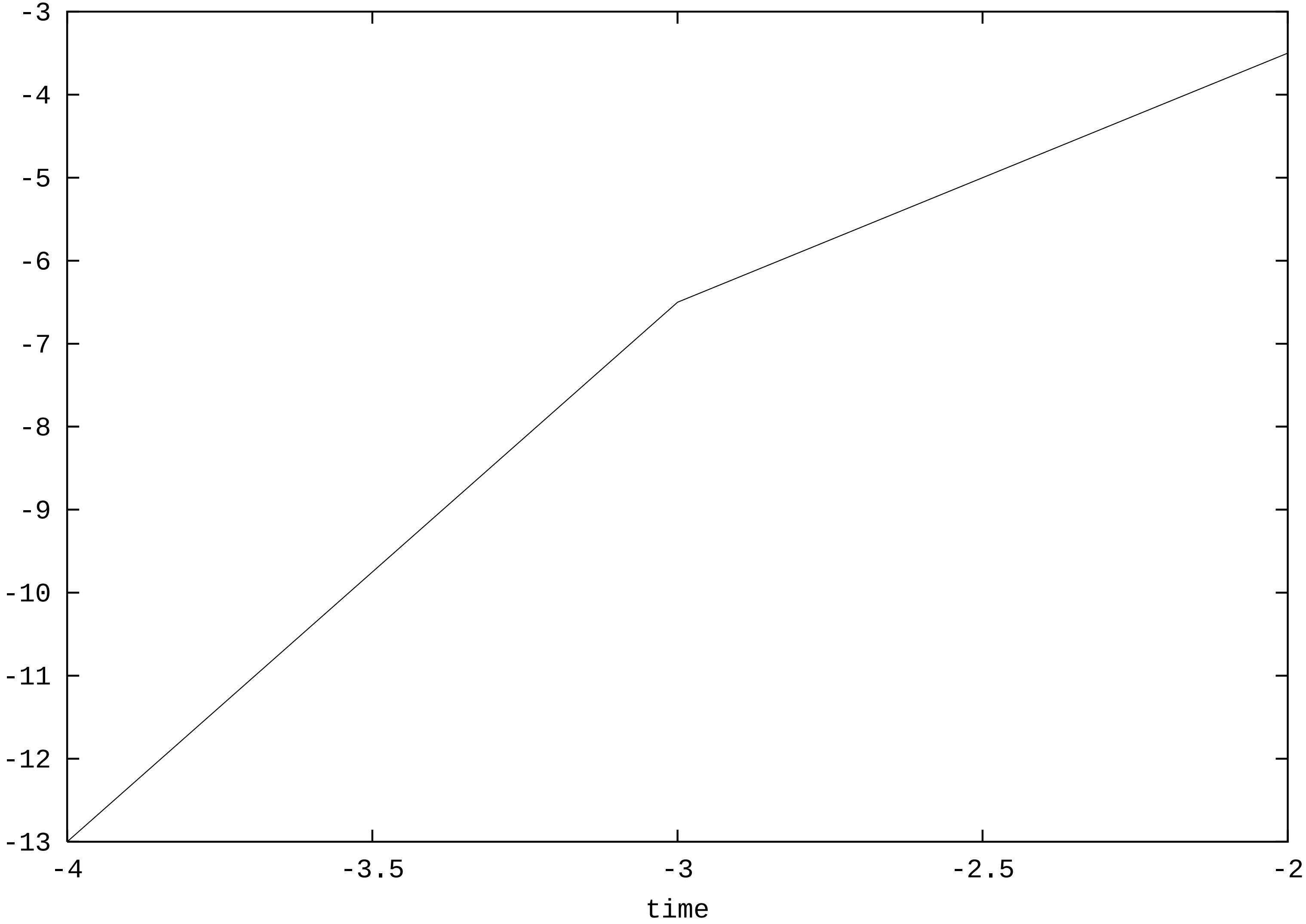


Figure 11.8: Log-log plot for absolute weak error of weak order 4.0 extrapolation.

Implicit and Predictor Corrector Methods

- numerical stability has highest priority

Drift Implicit Euler Scheme

$d, m \in \{1, 2, \ldots\}$

$$Y_{n+1} = Y_n + a\left(\tau_{n+1}, Y_{n+1}\right)\Delta + \sum_{j=1}^{m} b^j\left(\tau_n, Y_n\right)\Delta\hat{W}^j$$

$$P\left(\Delta\hat{W}^j = \pm\sqrt{\Delta}\right) = \frac{1}{2}$$

- **family of drift implicit simplified Euler schemes**

$$
\begin{aligned}
Y_{n+1} &= Y_n + \Big((1-\alpha)\, a\,(\tau_n, Y_n) + \alpha\, a\,(\tau_{n+1}, Y_{n+1})\, \Big)\Delta \\
&\quad + \sum_{j=1}^{m} b^j\,(\tau_n, Y_n)\,\Delta \hat{W}^j
\end{aligned}
$$

α degree of drift implicitness

A-stable for $\alpha \in [0.5, 1]$

region of A-stability

circle of radius $r = (1-2\alpha)^{-1}$ centered at $-r$

Fully Implicit Euler Scheme

simplified schemes allow

implicit diffusion coefficient term

- **fully implicit weak Euler scheme**

$$Y_{n+1} = Y_n + \bar{a}\,(Y_{n+1})\,\Delta + b\,(Y_{n+1})\,\Delta\hat{W}$$

$$\bar{a} = a - b\,b'$$

- **family of implicit weak Euler schemes**

$$\begin{aligned} Y_{n+1} &= Y_n + \Big(\alpha\, \bar{a}_\eta\, (\tau_{n+1}, Y_{n+1}) + (1-\alpha)\, \bar{a}_\eta\, (\tau_n, Y_n)\Big)\Delta \\ &\quad + \sum_{j=1}^{m} \Big(\eta\, b^j\, (\tau_{n+1}, Y_{n+1}) + (1-\eta)\, b^j\, (\tau_n, Y_n)\Big)\Delta \hat{W}^j \end{aligned}$$

$$\bar{a}_\eta = a - \eta \sum_{j_1, j_2 = 1}^{m} \sum_{k=1}^{d} b^{k, j_1} \frac{\partial b^{j_2}}{\partial x^k}$$

for $\alpha, \eta \in [0, 1]$

Implicit Weak Order 2.0 Taylor Scheme

$$
\begin{aligned}
Y_{n+1} \quad = \quad & Y_n + a\left(Y_{n+1}\right)\Delta + b\,\Delta\hat{W} \\
& - \frac{1}{2}\left(a\left(Y_{n+1}\right)a'\left(Y_{n+1}\right) + \frac{1}{2}\,b^2\left(Y_{n+1}\right)a''\left(Y_{n+1}\right)\right)\Delta^2 \\
& + \frac{1}{2}\,b\,b'\left(\left(\Delta\hat{W}\right)^2 - \Delta\right) \\
& + \frac{1}{2}\left(-a'\,b + a\,b' + \frac{1}{2}\,b''b^2\right)\Delta\hat{W}\,\Delta
\end{aligned}
$$

$$
P\left(\Delta\hat{W} = \pm\sqrt{3\Delta}\right) = \frac{1}{6} \quad \text{and} \quad P\left(\Delta\hat{W} = 0\right) = \frac{2}{3}
$$

Milstein (1995)

- **family of implicit weak order 2.0 Taylor schemes**

$$
\begin{aligned}
Y_{n+1} &= Y_n + \Big(\alpha\, a\,(\tau_{n+1}, Y_{n+1}) + (1-\alpha)\, a\Big)\Delta \\
&\quad + \frac{1}{2} \sum_{j_1, j_2 = 1}^{m} L^{j_1} b^{j_2} \left(\Delta \hat{W}^{j_1} \Delta \hat{W}^{j_2} + V_{j_1, j_2}\right) \\
&\quad + \sum_{j=1}^{m} \left(b^j + \frac{1}{2}\left(L^0 b^j + (1-2\alpha) L^j a\right)\Delta\right)\Delta \hat{W}^j \\
&\quad + \frac{1}{2}(1-2\alpha)\Big(\beta\, L^0 a + (1-\beta)\, L^0 a\,(\tau_{n+1}, Y_{n+1})\Big)\Delta^2
\end{aligned}
$$

$\alpha = 0.5 \qquad \Longrightarrow$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + \frac{1}{2}\Big(a\,(\tau_{n+1}, Y_{n+1}) + a\Big)\Delta \\
& + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j + \frac{1}{2}\sum_{j=1}^{m} L^0 b^j\,\Delta\hat{W}^j\,\Delta \\
& + \frac{1}{2}\sum_{j_1,j_2=1}^{m} L^{j_1} b^{j_2}\Big(\Delta\hat{W}^{j_1}\Delta\hat{W}^{j_2} + V_{j_1,j_2}\Big)
\end{aligned}
$$

Implicit Weak Order 2.0 Scheme

$m = 1$

Platen (1995)

- **implicit weak order 2.0 scheme**

$$
\begin{aligned}
Y_{n+1} &= Y_n + \frac{1}{2}\,(a + a\,(Y_{n+1}))\,\Delta \\
&+ \frac{1}{4}\,\left(b\left(\bar{\Upsilon}^{+}\right) + b\left(\bar{\Upsilon}^{-}\right) + 2\,b\right)\Delta\hat{W} \\
&+ \frac{1}{4}\,\left(b\left(\bar{\Upsilon}^{+}\right) - b\left(\bar{\Upsilon}^{-}\right)\right)\left(\left(\Delta\hat{W}\right)^{2} - \Delta\right)\Delta^{-\frac{1}{2}}
\end{aligned}
$$

with supporting values

$$\bar{\Upsilon}^{\pm} = Y_n + a\,\Delta \pm b\sqrt{\Delta}$$

$\Delta\hat{W}$ can be chosen as before

- **implicit weak order 2.0 scheme**

  $d, m \in \{1, 2, \ldots\}$

$$
\begin{aligned}
Y_{n+1} \;=\; & Y_n + \frac{1}{2}\,\left(a + a\left(Y_{n+1}\right)\right)\Delta \\
& + \frac{1}{4}\sum_{j=1}^{m}\Bigg[b^j\left(\bar{R}^j_+\right) + b^j\left(\bar{R}^j_-\right) + 2b^j \\
& \quad + \sum_{\substack{r=1\\ r\neq j}}^{m}\left(b^j\left(\bar{U}^r_+\right) + b^j\left(\bar{U}^r_-\right) - 2b^j\right)\Delta^{-\frac{1}{2}}\Bigg]\Delta\hat{W}^j \\
& + \frac{1}{4}\sum_{j=1}^{m}\Bigg[\left(b^j\left(\bar{R}^j_+\right) - b^j\left(\bar{R}^j_-\right)\right)\left(\left(\Delta\hat{W}^j\right)^2 - \Delta\right) \\
& \quad + \sum_{\substack{r=1\\ r\neq j}}^{m}\left(b^j\left(\bar{U}^r_+\right) - b^j\left(\bar{U}^r_-\right)\right)\left(\Delta\hat{W}^j\Delta\hat{W}^r + V_{r,j}\right)\Bigg]\Delta^{-\frac{1}{2}}
\end{aligned}
$$

supporting values

$$\bar{R}^j_{\pm} = Y_n + a\,\Delta \pm b^j\,\sqrt{\Delta}$$

and

$$\bar{U}^j_{\pm} = Y_n \pm b^j\,\sqrt{\Delta}$$

A-stable

$\beta = 2.0$

Weak Order 1.0 Predictor-Corrector Methods

- **modified trapezoidal method of weak order $\beta = 1.0$**

corrector

$$Y_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\bar{Y}_{n+1}\right) + a\Big)\Delta + b\,\Delta\hat{W}$$

predictor

$$\bar{Y}_{n+1} = Y_n + a\,\Delta + b\,\Delta\hat{W}$$

$$P\left(\Delta\hat{W} = \pm\sqrt{\Delta}\right) = \frac{1}{2}$$

- **family of weak order 1.0 predictor-corrector methods**

corrector

$$\begin{aligned} Y_{n+1} &= Y_n + \Big(\alpha\, \bar{a}_\eta\left(\tau_{n+1}, \bar{Y}_{n+1}\right) + (1-\alpha)\, \bar{a}_\eta\left(\tau_n, Y_n\right)\Big)\Delta \\ &\quad + \sum_{j=1}^{m} \Big(\eta\, b^j\left(\tau_{n+1}, \bar{Y}_{n+1}\right) + (1-\eta)\, b^j\left(\tau_n, Y_n\right)\Big)\Delta \hat{W}^j \end{aligned}$$

for $\alpha, \eta \in [0, 1]$, where

$$\bar{a}_\eta = a - \eta \sum_{j_1, j_2 = 1}^{m} \sum_{k=1}^{d} b^{k, j_1} \frac{\partial b^{j_2}}{\partial x^k}$$

and predictor

$$\bar{Y}_{n+1} = Y_n + a\,\Delta + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j$$

Weak Order 2.0 Predictor-Corrector Methods

- **weak order 2.0 predictor-corrector method**

  corrector

$$Y_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\bar{Y}_{n+1}\right) + a\Big)\Delta + \Psi_n$$

with

$$\begin{aligned}\Psi_n &= b\,\Delta\hat{W} + \frac{1}{2}\,b\,b'\left(\left(\Delta\hat{W}\right)^2 - \Delta\right)\\ &\quad + \frac{1}{2}\left(a\,b' + \frac{1}{2}\,b^2\,b''\right)\Delta\hat{W}\,\Delta\end{aligned}$$

and as predictor

$$\bar{Y}_{n+1} = Y_n + a\,\Delta + \Psi_n + \frac{1}{2}\,a'\,b\,\Delta\hat{W}\,\Delta + \frac{1}{2}\left(a\,a' + \frac{1}{2}\,a''b^2\right)\Delta^2$$

$$P\left(\Delta\hat{W} = \pm\sqrt{3\Delta}\right) = \frac{1}{6} \quad \text{and} \quad P\left(\Delta\hat{W} = 0\right) = \frac{2}{3}$$

- general multi-dimensional case

corrector

$$\boldsymbol{Y}_{n+1} = \boldsymbol{Y}_n + \frac{1}{2}\Big(\boldsymbol{a}\left(\tau_{n+1}, \bar{\boldsymbol{Y}}_{n+1}\right) + \boldsymbol{a}\Big)\Delta + \boldsymbol{\Psi}_n$$

where

$$\boldsymbol{\Psi}_n = \sum_{j=1}^{m}\left(\boldsymbol{b}^j + \frac{1}{2} L^0 \boldsymbol{b}^j \Delta\right)\Delta\hat{W}^j + \frac{1}{2}\sum_{j_1,j_2=1}^{m} L^{j_1}\boldsymbol{b}^{j_2}\left(\Delta\hat{W}^{j_1}\,\Delta\hat{W}^{j_2} + V_{j_1,j_2}\right)$$

and predictor

$$\bar{\boldsymbol{Y}}_{n+1} = \boldsymbol{Y}_n + \boldsymbol{a}\,\Delta + \boldsymbol{\Psi}_n + \frac{1}{2} L^0 \boldsymbol{a}\,\Delta^2 + \frac{1}{2}\sum_{j=1}^{m} L^j \boldsymbol{a}\,\Delta\hat{W}^j\,\Delta$$

- **derivative free weak order 2.0 predictor-corrector method**

corrector

$$Y_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\bar{Y}_{n+1}\right) + a\Big)\Delta + \phi_n$$

where

$$\begin{aligned}\phi_n \;=\; & \frac{1}{4}\left(b\left(\bar{\Upsilon}^{+}\right) + b\left(\bar{\Upsilon}^{-}\right) + 2\,b\right)\Delta\hat{W} \\ & + \frac{1}{4}\left(b\left(\bar{\Upsilon}^{+}\right) - b\left(\bar{\Upsilon}^{-}\right)\right)\left(\left(\Delta\hat{W}\right)^2 - \Delta\right)\Delta^{-\frac{1}{2}}\end{aligned}$$

with supporting values

$$\bar{\Upsilon}^{\pm} = Y_n + a\,\Delta \pm b\sqrt{\Delta}$$

and with predictor

$$\bar{Y}_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\bar{\Upsilon}\right) + a\Big)\Delta + \phi_n$$

with the supporting value

$$\bar{\Upsilon} = Y_n + a\,\Delta + b\,\Delta\hat{W}$$

- multi-dimensional generalization

  corrector

$$\boldsymbol{Y}_{n+1} = \boldsymbol{Y}_n + \frac{1}{2}\Big(\boldsymbol{a}\left(\bar{\boldsymbol{Y}}_{n+1}\right) + \boldsymbol{a}\Big)\boldsymbol{\Delta} + \boldsymbol{\phi}_n$$

where

$$\begin{aligned}
\phi_n \;=\; & \frac{1}{4}\sum_{j=1}^{m}\Bigg[b^j\left(\bar{R}^j_+\right)+b\left(\bar{R}^j_-\right)+2\,b^j \\
& +\sum_{\substack{r=1\\ r\neq j}}^{m}\left(b^j\left(\bar{U}^r_+\right)+b\left(\bar{U}^r_-\right)-2\,b^j\right)\Delta^{-\frac{1}{2}}\Bigg]\Delta\hat{W}^j \\
& +\frac{1}{4}\sum_{j=1}^{m}\Bigg[\left(b^j\left(\bar{R}^j_+\right)-b\left(\bar{R}^j_-\right)\right)\left(\left(\Delta\hat{W}\right)^2-\Delta\right) \\
& +\sum_{\substack{r=1\\ r\neq j}}^{m}\left(b^j\left(\bar{U}^r_+\right)-b\left(\bar{U}^r_-\right)\right)\left(\Delta\hat{W}^j\,\Delta\hat{W}^r+V_{r,j}\right)\Bigg]\Delta^{-\frac{1}{2}}
\end{aligned}$$

supporting values

$$\bar{R}^j_{\pm} = Y_n + a\,\Delta \pm b^j\,\sqrt{\Delta} \quad \text{and} \quad \bar{U}^j_{\pm} = Y_n \pm b^j\,\sqrt{\Delta}$$

predictor

$$\bar{Y}_{n+1} = Y_n + \frac{1}{2}\Big(a\left(\bar{\Upsilon}\right) + a\Big)\Delta + \phi_n$$

supporting value

$$\bar{\Upsilon} = Y_n + a\,\Delta + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j$$

local error

$$Z_{n+1} = \bar{Y}_{n+1} - Y_{n+1}$$

Weak Approximation with Jumps

Platen (1982a)

Mikulevicius & Platen (1988)

Kubilius & Platen (2002)

Bruti-Liberati & Platen (2006)

Jump Diffusion

- SDE

$$X_t = x + \int_0^t a(X_s)\,ds + \int_0^t b(X_s)\,dW_s + \int_0^t \int_{\mathcal{E}} c(v, X_s)\,q_\varphi(dv, ds)$$

mark set $\boldsymbol{\mathcal{E}} = \Re \backslash \{0\}$

- jump martingale measure

$$\boldsymbol{q_\varphi(dv, ds) = p_\varphi(dv, ds) - \varphi(dv)\, ds}$$

intensity measure $\boldsymbol{\varphi(dv) ds}$

$$\boldsymbol{\varphi(\mathcal{E}) < \infty}$$

- **simulation of functionals**

$$u(s, y) = E\left(g(X_T^{s,y}) \,\middle|\, \mathcal{A}_s\right)$$

- **Kolmogorov backward equation**

$$\frac{\partial}{\partial s}\, u(s,y) + \sum_{i=1}^{d} a^i(y)\, \frac{\partial}{\partial y^i}\, u(s,y)$$

$$+ \frac{1}{2} \sum_{i,r=1}^{d} \sum_{j=1}^{m} b^{i,j}(y)\, b^{r,j}(y)\, \frac{\partial^2}{\partial y^i\, \partial y^r}\, u(s,y)$$

$$+ \int_{\mathcal{E}} \Bigg(u(s, y + c(v,y)) - u(s,y)$$

$$- \sum_{i=1}^{d} c^i(v,y)\, \frac{\partial}{\partial y^i}\, u(s,y) \Bigg)\, \varphi(dv) = 0$$

for all $(s, y) \in (0, T) \times \Re^d$ with

$$u(T, y) = g(y)$$

for $y \in \Re^d$

Jump Adapted Time Discretization

- absolute weak error criterion

$$\mu(\Delta) = \left| E(g(X_T^{0,y})) - E(g(Y_T)) \right| \leq K\,\Delta^{\beta}$$

- Poisson process

$$p_\varphi = \{p_\varphi(\mathcal{E}, [0,t]),\ t \in [0,T]\}$$

finite intensity $\quad \varphi(\mathcal{E}) < \infty$

generates a sequence of jump times

- jump adapted time discretization with maximum step size $\Delta > 0$

$$0 = \tau_0 < \tau_1 < \ldots < \tau_{n_T} = T$$

including all jump times of p_φ

$$n_t = \max\{n \in \{0, 1, \ldots\} : \, t_n \leq t\}$$

$$\max_{n \in \{1,\ldots,n_T\}} (\tau_n - \tau_{n-1}) \leq \Delta$$

Jump Adapted Weak Euler Scheme

$$Y_{\tau_{n+1}-} = Y_{\tau_n} + a(Y_{\tau_n})\,(\tau_{n+1} - \tau_n) + b(Y_{\tau_n})\,(W_{\tau_{n+1}} - W_{\tau_n})$$

$$- \int_{\mathcal{E}} c(v, Y_{\tau_n})\,\varphi(du)\,(\tau_{n+1} - \tau_n),$$

$$Y_{\tau_{n+1}} = Y_{\tau_{n+1}-} + \int_{\mathcal{E}} c(v, Y_{\tau_{n+1}-})\,p_\varphi(dv, \{\tau_{n+1}\})$$

for $n \in \{0, 1, \ldots, n_T - 1\}$ and $Y_0 = x$

$\beta = 1$

simplified weak Euler scheme can be used

to approximate diffusion part

Second Order Weak Schemes

$d = m = 1$

$$\begin{aligned}
Y_{\tau_{n+1}-} \quad = \quad & Y_{\tau_n} + \tilde{a}(Y_{\tau_n})\,(\tau_{n+1} - \tau_n) + b(Y_{\tau_n})\,\Delta W \\
& + b(Y_{\tau_n})\,b'(Y_{\tau_n})\,\frac{1}{2}\left((\Delta W)^2 - (\tau_{n+1} - \tau_n)\right) \\
& + b(Y_{\tau_n})\,\tilde{a}'(Y_{\tau_n})\Delta Z \\
& + \left(\tilde{a}(Y_{\tau_n})\,b'(Y_{\tau_n}) + \frac{1}{2}\,b(Y_{\tau_n})^2\,b''(Y_{\tau_n})\right) \\
& \qquad \times \left(\Delta W\,(\tau_{n+1} - \tau_n) - \Delta Z\right) \\
& + \left(\tilde{a}(Y_{\tau_n})\,\tilde{a}'(Y_{\tau_n}) + \frac{1}{2}\,b(Y_{\tau_n})^2\,\tilde{a}''(Y_{\tau_n})\right)\frac{(\tau_{n+1} - \tau_n)^2}{2}
\end{aligned}$$

$$Y_{\tau_{n+1}} \quad = \quad Y_{\tau_{n+1}-} + \int_{\mathcal{E}} c(v, Y_{\tau_{n+1}-})\,p_\varphi(dv, \{\tau_{n+1}\})$$

for $n \in \{0, 1, \ldots, n_T - 1\}$

with $Y_0 = x$, $\tilde{a} = a - \int_{\mathcal{E}} c\, d\varphi$

$$\Delta W = \int_{\tau_n}^{\tau_{n+1}} dW_s \sim N(0, \tau_{n+1} - \tau_n)$$

$$\Delta Z = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}\, ds_2 \sim N\left(0, \frac{(\tau_{n+1} - \tau_n)^3}{3}\right)$$

$$E(\Delta W\, \Delta Z) = \frac{(\tau_{n+1} - \tau_n)^2}{2}$$

Jump Adapted Weak Taylor Approximations

- Wagner-Platen expansion

  hierarchical set

$$A_\beta = \{\alpha \in \mathcal{M}_m : l(\alpha) \leq \beta\}$$

- **jump adapted weak Taylor approximation of order β**

$$Y_{\tau_{n+1}-} = \sum_{\alpha \in \mathcal{A}_\beta} I_\alpha(f_\alpha(Y_{\tau_n}))_{\tau_n,\tau_{n+1}}$$

$$Y_{\tau_{n+1}} = Y_{\tau_{n+1}-} + \int_{\mathcal{E}} c(v, Y_{\tau_{n+1}-})\, p_\varphi(dv, \{\tau_{n+1}\})$$

$n \in \{0, 1, \ldots, n_T - 1\}$, with $Y_0 = x$

$f(x) = x$

Theorem (Mikulevicius-Platen)

Let be given a function $g \in \mathcal{C}_P^{2(\beta+1)}(\Re^d, \Re)$*, a time discretization* $\{\tau_n\}_{n\in\{0,1,\ldots\}}$ *with maximum step size* $\Delta > 0$ *and a corresponding jump adapted weak Taylor approximation* Y *of order* β*.*

(i) a*,* b *and* c *are* $2(\beta+1)$*-times continuously differentiable, where the derivatives are uniformly bounded.*

(ii) *For all* $\alpha \in \mathcal{M}_m$ *with* $l(\alpha) \leq \beta$ *and* $y \in \Re^d$ *it holds* $|f_\alpha(y)| \leq K\,(1 + |y|)$*.*

Then the jump adapted weak Taylor approximation Y *of order* β *converges with weak order* β*, which means that*

$$|E(g(X_T)) - E(g(Y_T))| \leq C\,\Delta^\beta,$$

where C *is a constant not depending on* Δ*.*

Exercises of Chapter 11

11.1 Verify that the two point distributed random variable $\Delta\hat{W}$ with

$$P(\Delta\hat{W} = \pm\sqrt{\Delta}) = \frac{1}{2}$$

satisfies the moment conditions

$$\left|E\left(\Delta\hat{W}\right)\right| + \left|E\left(\left(\Delta\hat{W}\right)^3\right)\right| + \left|E\left(\left(\Delta\hat{W}\right)^2\right) - \Delta\right| \leq K\,\Delta^2.$$

11.2 Prove that the three point distributed random variable $\Delta\tilde{W}$ with

$$P\left(\Delta\tilde{W} = \pm\sqrt{3\,\Delta}\right) = \frac{1}{6} \quad \text{and} \quad P\left(\Delta\tilde{W} = 0\right) = \frac{2}{3}$$

satisfies the moment conditions

$$\left|E\left(\Delta\tilde{W}\right)\right| + \left|E\left(\left(\Delta\tilde{W}\right)^3\right)\right| + \left|E\left(\left(\Delta\tilde{W}\right)^5\right)\right| +$$

$$+\left|E\left(\left(\Delta\tilde{W}\right)^2\right) - \Delta\right| + \left|E\left(\left(\Delta\tilde{W}\right)^4\right) - 3\,\Delta^2\right| \leq K\,\Delta^3.$$

11.3 For the Black-Scholes model with SDE

$$dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dW_t$$

for $t \in [0, T]$ with $X_0 = 0$ and W a standard Wiener process write down a simplified Euler scheme. Which weak order of convergence does this scheme achieve ?

11.4 For the BS model in Exercise 11.3 describe a simplified weak order 2.0 method.

11.5 For the BS model in Exercise 11.3 construct a drift implicit Euler scheme.

11.6 For the BS model in Exercise 11.3 describe a fully implicit Euler scheme.

14 Numerical Stability

- roundoff and truncation errors

- propagation of errors

- numerical stability priority over higher order

- **specially designed test equations**

  Hernandez & Spigler (1992, 1993)

  Milstein (1995)

  Kloeden & Pl. (1999)

  Saito & Mitsui(1993a, 1993b, 1996)

  Hofmann & Pl. (1994, 1996)

  Higham (2000)

- **linear test dynamics**

$$X_t = X_0 \exp\left\{(1-\alpha)\,\lambda\,t + \sqrt{\alpha\,|\lambda|}\,W_t\right\}$$

$\alpha, \lambda \in \Re$

$\Longrightarrow$

$$P\left(\lim_{t\to\infty} X_t = 0\right) = 1 \quad \Longleftrightarrow \quad (1-\alpha)\lambda < 0$$

- **linear Itô SDE**

$$dX_t = \left(1 - \frac{3}{2}\,\alpha\right)\lambda\, X_t\, dt + \sqrt{\alpha\,|\lambda|}\, X_t\, dW_t$$

- **corresponding Stratonovich SDE**

$$dX_t = (1 - \alpha)\,\lambda\, X_t\, dt + \sqrt{\alpha\,|\lambda|}\, X_t \circ dW_t$$

- $\alpha = 0$ no randomness

- $\alpha = \frac{2}{3}$ Itô SDE no drift $\Longrightarrow$ martingale

- $\alpha = 1$ Stratonovich SDE no drift

Definition 14.1 $Y = \{Y_t, t \geq 0\}$ *is called* **asymptotically stable** *if*

$$P\left(\lim_{t\to\infty} |Y_t| = 0\right) = 1.$$

impact of perturbations declines asymptotically over time

- **stability region Γ**

  those pairs $(\lambda\Delta, \alpha) \in (-\infty, 0) \times [0, 1)$ for which approximation Y asymptotically stable

- **transfer function**

$$\left|\frac{Y_{n+1}}{Y_n}\right| = G_{n+1}(\lambda\,\Delta, \alpha)$$

Y asymptotically stable $\Longleftrightarrow$

$$E(\ln(G_{n+1}(\lambda\,\Delta, \alpha))) < 0$$

Higham (2000)

- **Euler scheme**

$$Y_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n$$

$$G_{n+1}(\lambda\,\Delta, \alpha) = \left| 1 + \left(1 - \frac{3}{2}\,\alpha\right)\lambda\,\Delta + \sqrt{|\alpha\,\lambda|}\,\Delta W_n \right|$$

$$\Delta W_n \sim \mathcal{N}(0, \Delta)$$

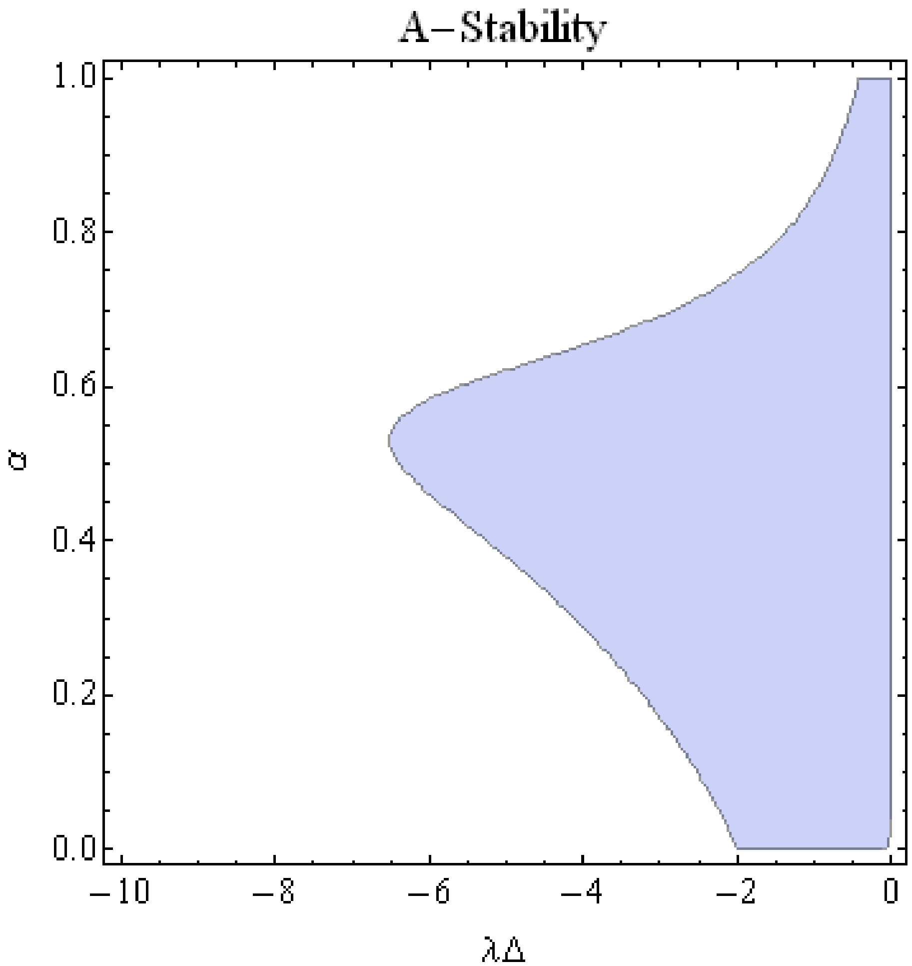


Figure 14.1: A-stability region for the Euler scheme.

- **semi-drift-implicit predictor-corrector Euler method**

$$Y_{n+1} = Y_n + \frac{1}{2}\left(a(\bar{Y}_{n+1}) + a(Y_n)\right)\,\Delta + b(Y_n)\,\Delta W_n$$

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n$$

$$\begin{aligned} G_{n+1}(\lambda\,\Delta, \alpha) \;=\; & \left|1 + \lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right)\left\{1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right)\right.\right.\right. \\ & \left.\left.\left. + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\} + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right| \end{aligned}$$

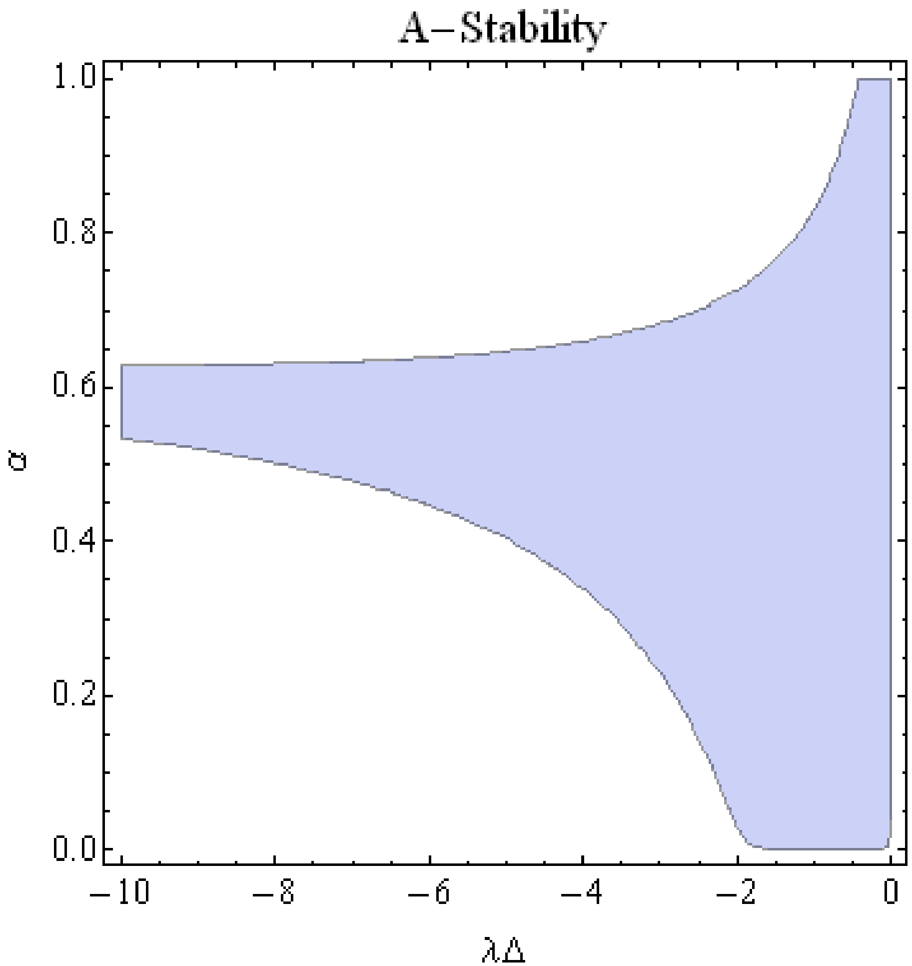


Figure 14.2: A-stability region for semi-drift-implicit predictor-corrector Euler method.

- **drift-implicit predictor-corrector Euler method**

$$Y_{n+1} = Y_n + a(\bar{Y}_{n+1})\,\Delta + b(Y_n)\,\Delta W_n$$

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n$$

$$\begin{aligned} G_{n+1}(\lambda\,\Delta, \alpha) \;=\; & \Bigg|1 + \lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right)\Bigg\{1 + \lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) \\ & + \sqrt{-\alpha\,\lambda}\,\Delta W_n\Bigg\} + \sqrt{-\alpha\,\lambda}\,\Delta W_n\Bigg| \end{aligned}$$

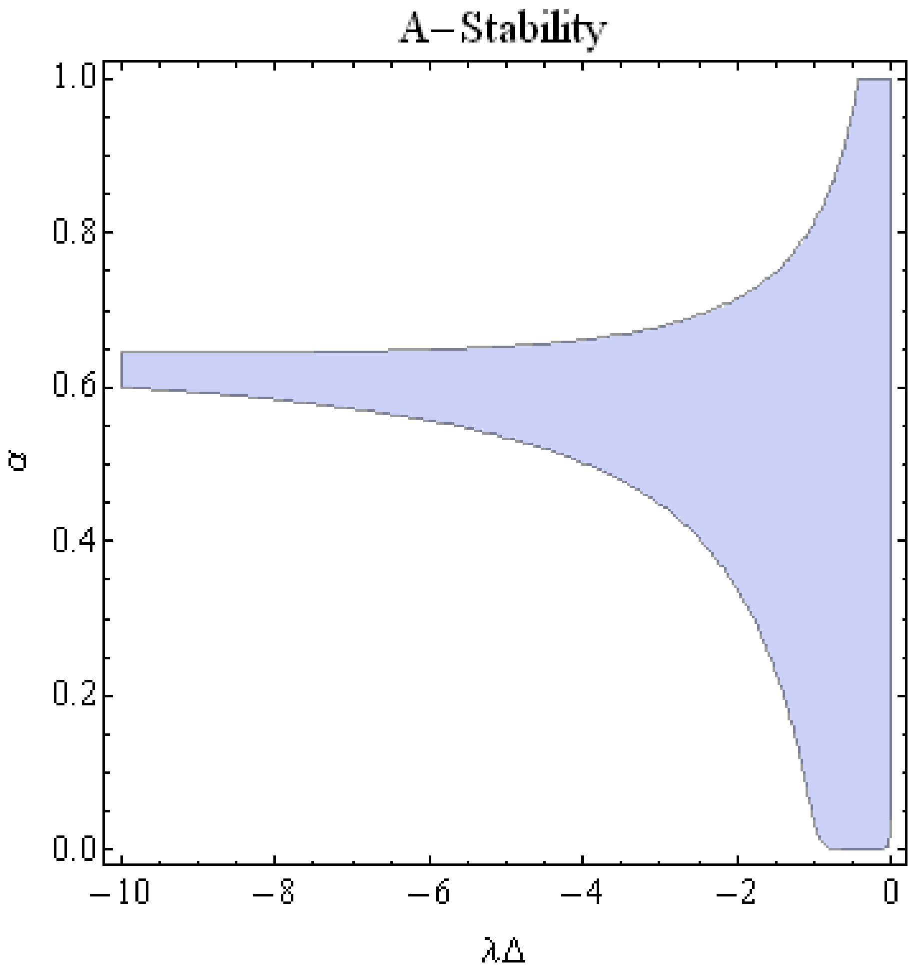


Figure 14.3: A-stability region for drift-implicit predictor-corrector Euler method.

- **semi-implicit diffusion predictor-corrector Euler method**

$$Y_{n+1} = Y_n + \bar{a}_{\frac{1}{2}}(Y_n)\,\Delta + \frac{1}{2}\big(b(\bar{Y}_{n+1}) + b(Y_n)\big)\,\Delta W_n$$

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n$$

$$\begin{aligned} G_{n+1}(\lambda\,\Delta, \alpha) \;=\; & \Big|1 + \lambda\,\Delta\,(1-\alpha) + \sqrt{-\alpha\,\lambda}\,\Delta W_n \\ & \times \left\{1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\}\Big| \end{aligned}$$

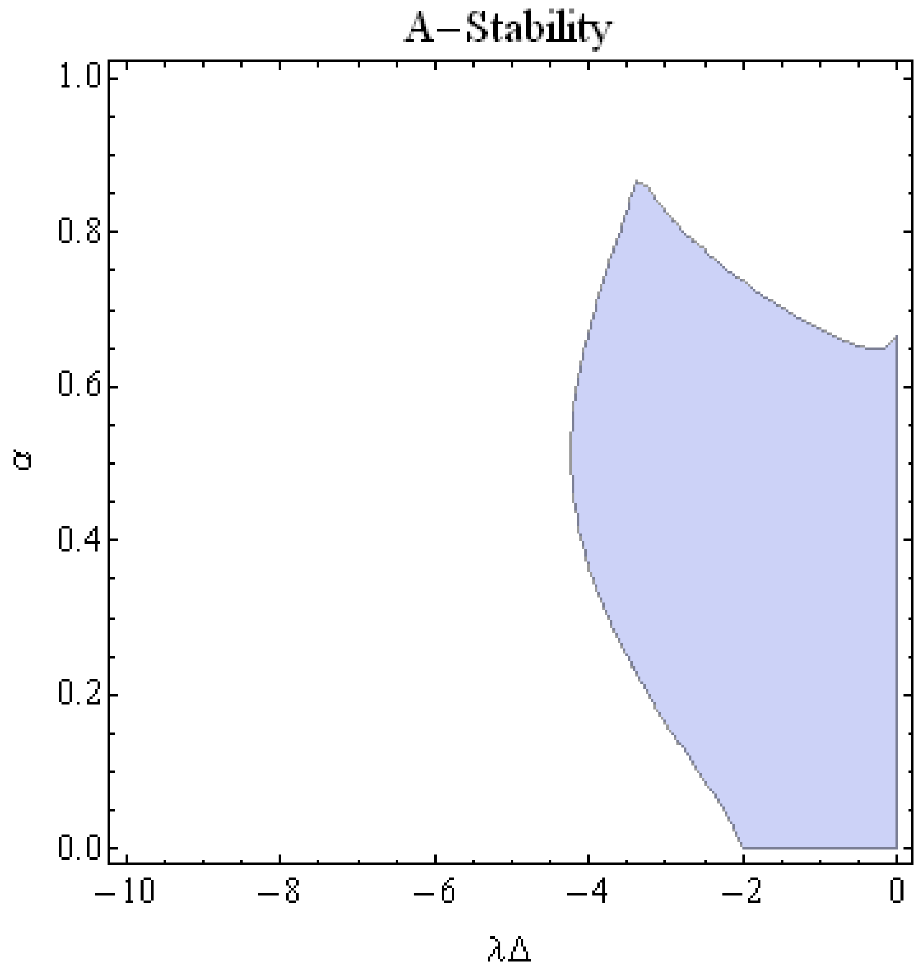


Figure 14.4: A-stability region for the predictor-corrector Euler method with $\boldsymbol{\theta = 0}$ and $\boldsymbol{\eta = \frac{1}{2}}$.

- **symmetric predictor-corrector Euler method**

$$Y_{n+1} = Y_n + \frac{1}{2}\left(\bar{a}_{\frac{1}{2}}(\bar{Y}_{n+1}) + \bar{a}_{\frac{1}{2}}(Y_n)\right)\Delta + \frac{1}{2}\left(b(\bar{Y}_{n+1}) + b(Y_n)\right)\Delta W_n$$

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n$$

$$\begin{aligned} G_{n+1}(\lambda\,\Delta,\alpha) \;=\; & \Bigg|1 + \lambda\,\Delta\,(1-\alpha)\left\{1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\} \\ & + \sqrt{-\alpha\,\lambda}\,\Delta W_n\left\{1 + \frac{1}{2}\left(\lambda\,\Delta\left(1 - \frac{3}{2}\,\alpha\right) + \sqrt{-\alpha\,\lambda}\,\Delta W_n\right)\right\}\Bigg| \end{aligned}$$

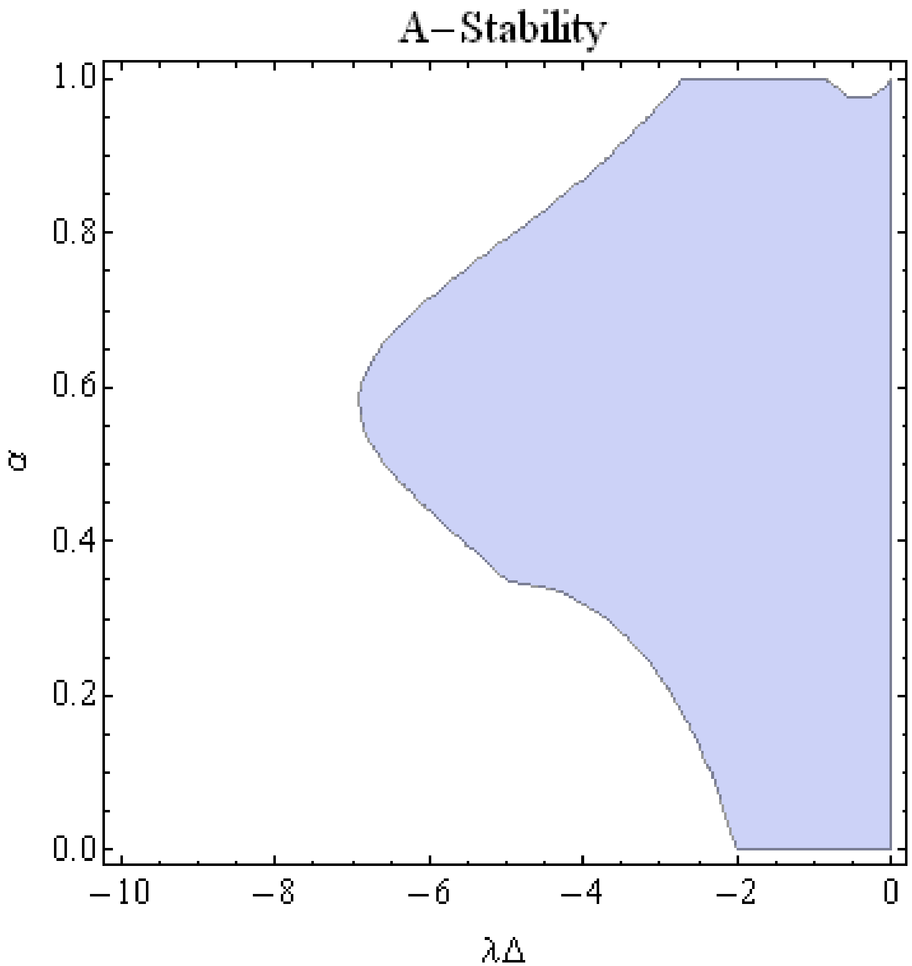


Figure 14.5: A-stability region for the symmetric predictor-corrector Euler method.

$$
\begin{aligned}
G_{n+1}(\lambda\,\Delta,\alpha) &= \Bigg|1+\lambda\,\Delta\left(1-\frac{1}{2}\,\alpha\right)\left\{1+\lambda\,\Delta\left(1-\frac{3}{2}\,\alpha\right)+\sqrt{-\alpha\,\lambda}\,\Delta W_n\right\} \\
&\quad +\sqrt{-\alpha\,\lambda}\,\Delta W_n\left\{1+\lambda\,\Delta\left(1-\frac{3}{2}\,\alpha\right)+\sqrt{-\alpha\,\lambda}\,\Delta W_n\right\}\Bigg| \\
&= \Bigg|1+\left\{1+\lambda\,\Delta\left(1-\frac{3}{2}\,\alpha\right)+\sqrt{-\alpha\,\lambda}\,\Delta W_n\right\} \\
&\quad \times\left\{\lambda\,\Delta\left(1-\frac{1}{2}\,\alpha\right)+\sqrt{-\alpha\,\lambda}\,\Delta W_n\right\}\Bigg|
\end{aligned}
$$

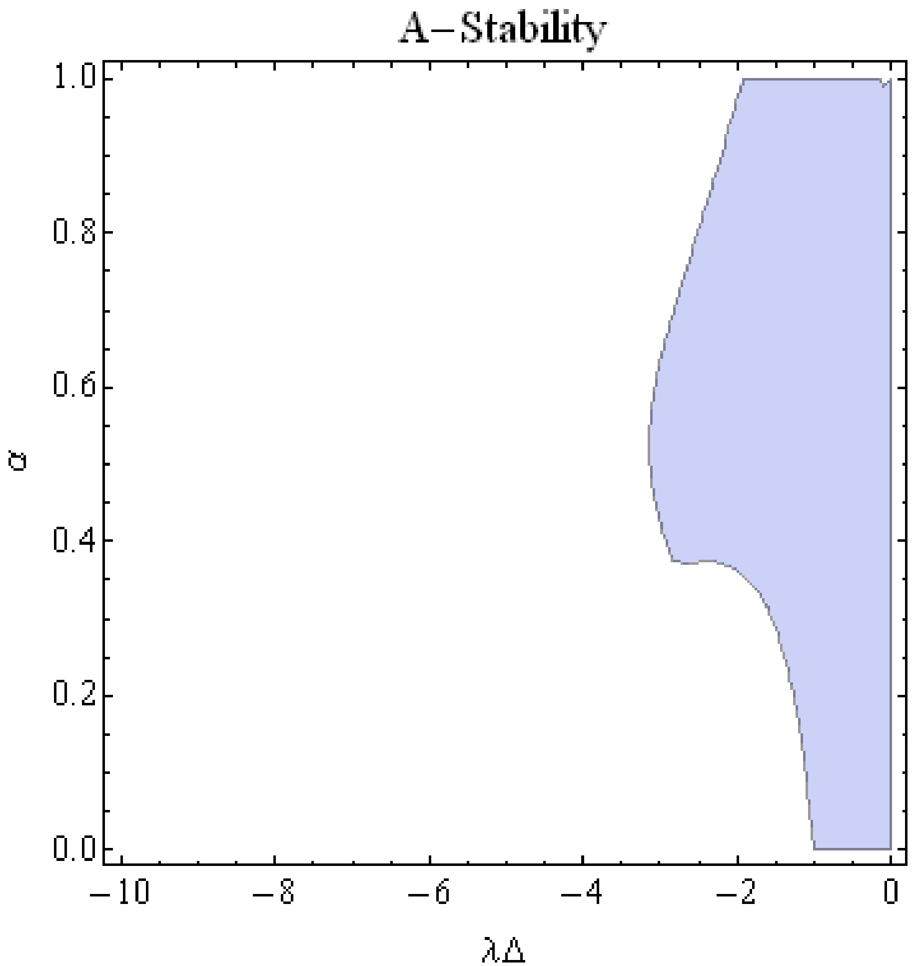


Figure 14.6: A-stability region for fully implicit predictor-corrector Euler method.

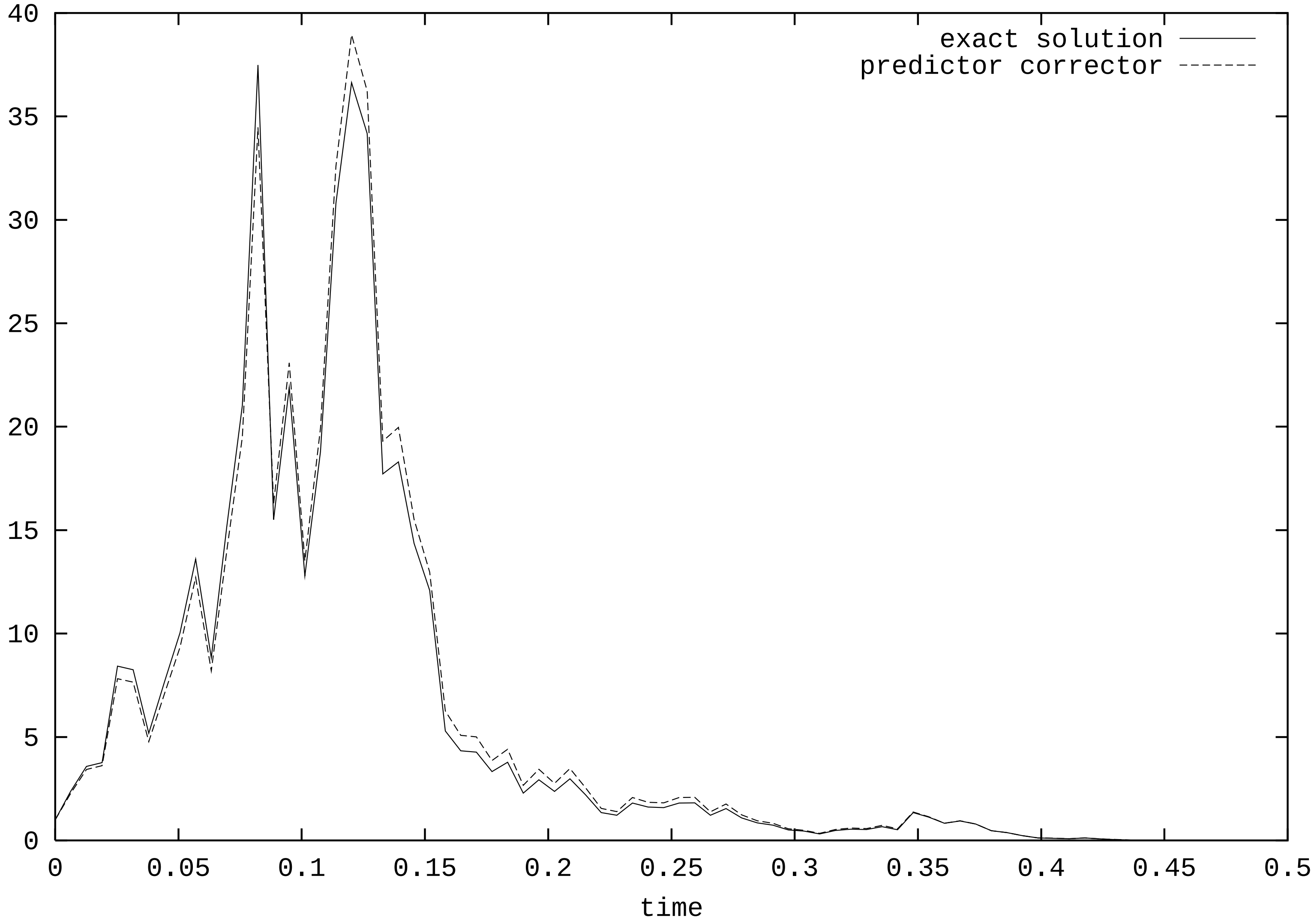


Figure 14.7: Exact solution and approximate solution generated by the symmetric predictor-corrector Euler scheme.

p-Stability

Pl. & Shi (2008)

Definition 14.2 *For* $p > 0$ *a process* $Y = \{Y_t, t > 0\}$ *is called* p-stable *if*

$$\lim_{t\to\infty} E(|Y_t|^p) = 0.$$

For $\alpha \in [0, \frac{1}{1+p/2})$ and $\lambda < 0$ test SDE is p-stable.

- **Stability region** those triplets $(\boldsymbol{\lambda\Delta}, \boldsymbol{\alpha}, \boldsymbol{p})$ for which $\boldsymbol{Y}$ is $\boldsymbol{p}$-stable.

For $\boldsymbol{\lambda\,\Delta < 0, \alpha \in [0,1)}$ and $\boldsymbol{p > 0}$ $\boldsymbol{Y}$ $\boldsymbol{p}$-stable $\Longleftrightarrow$

$$\boldsymbol{E((G_{n+1}(\lambda\,\Delta, \alpha))^p) < 1}$$

- for $\boldsymbol{p > 0}$

  $\Longrightarrow$

$$\boldsymbol{E(\ln(G_{n+1}(\lambda\,\Delta, \alpha))) \leq \frac{1}{p}\,E((G_{n+1}(\lambda\,\Delta, \alpha))^p - 1) < 0}$$

$\Longrightarrow$ asymptotically stable

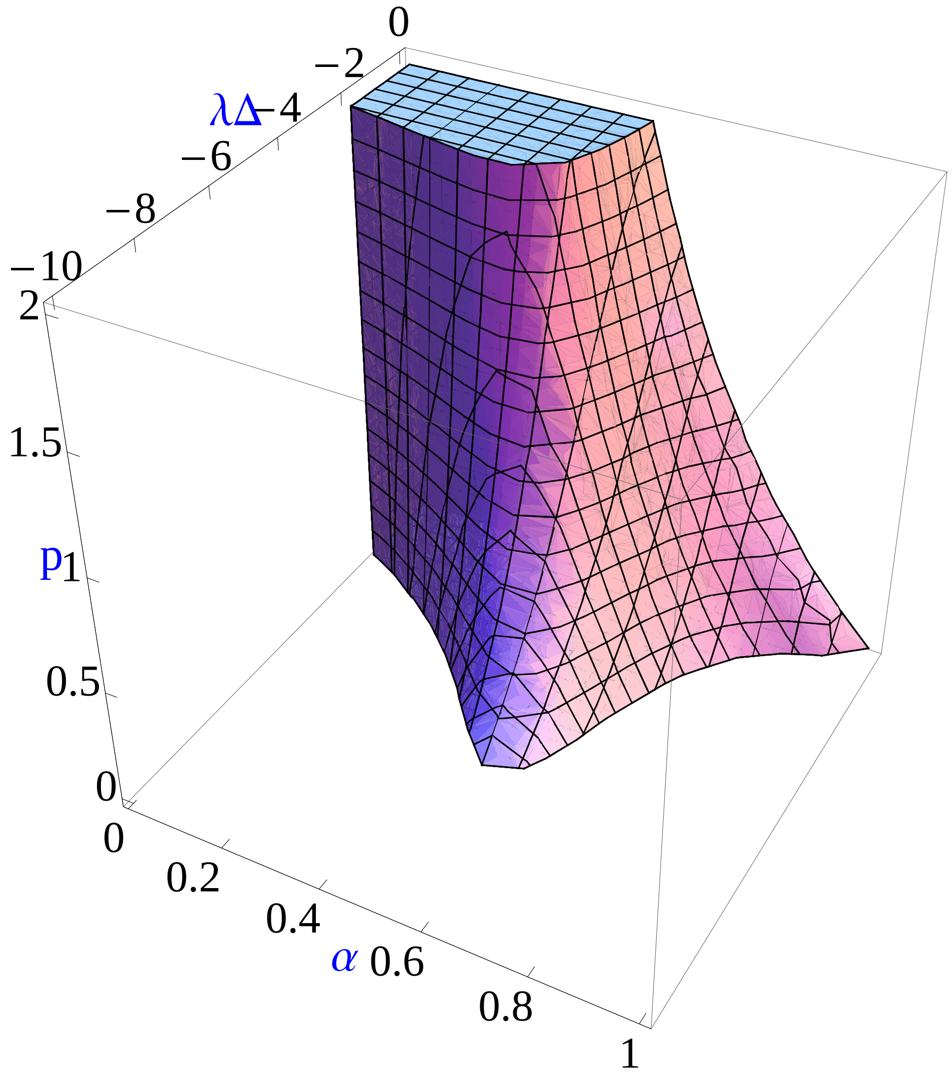


Figure 14.8: Stability region for the Euler scheme.

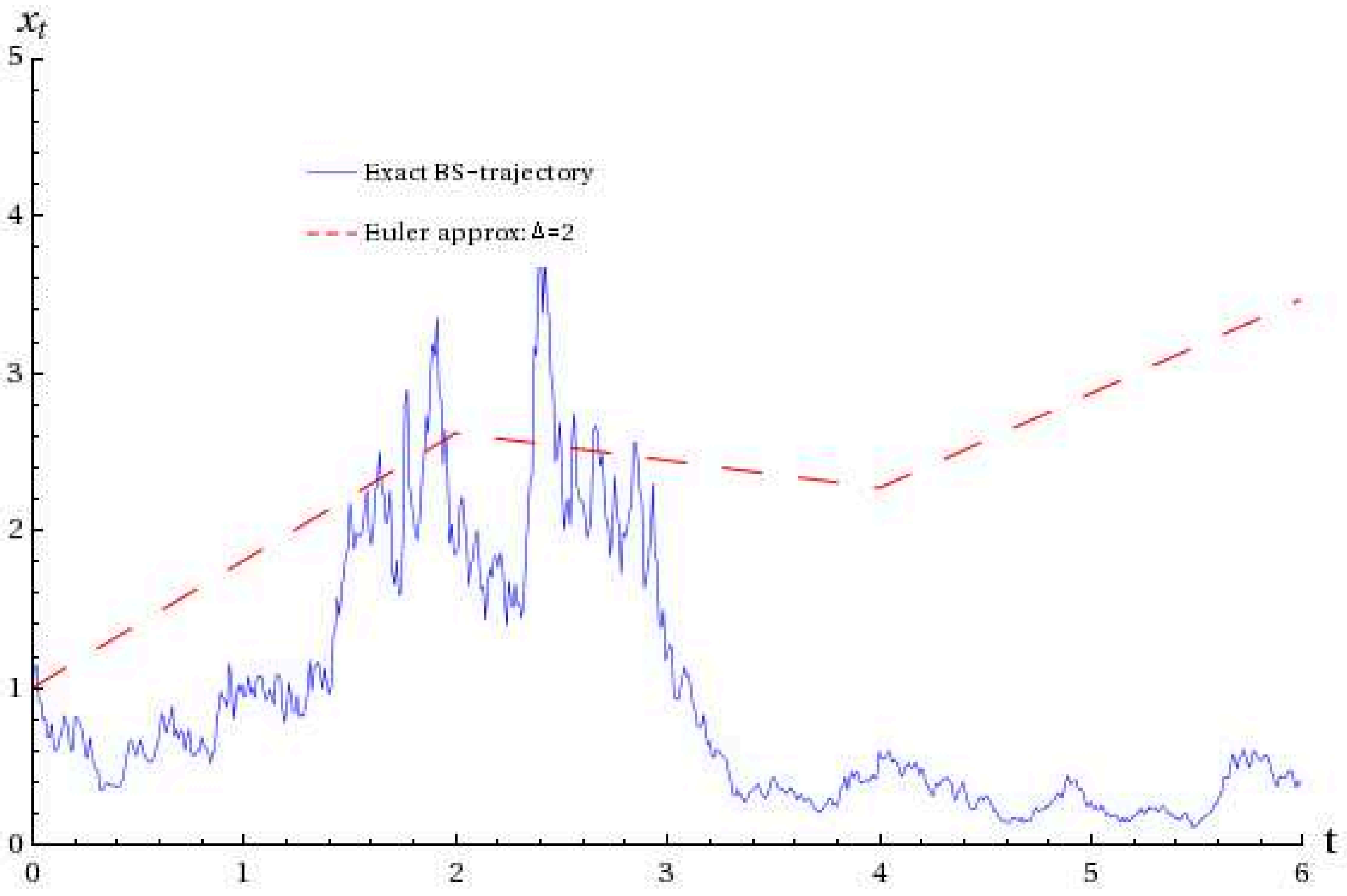


Figure 14.9: Paths of exact solution, Euler scheme with $\mathbf{\Delta = 0.2}$ and Euler scheme with $\mathbf{\Delta = 5}$.

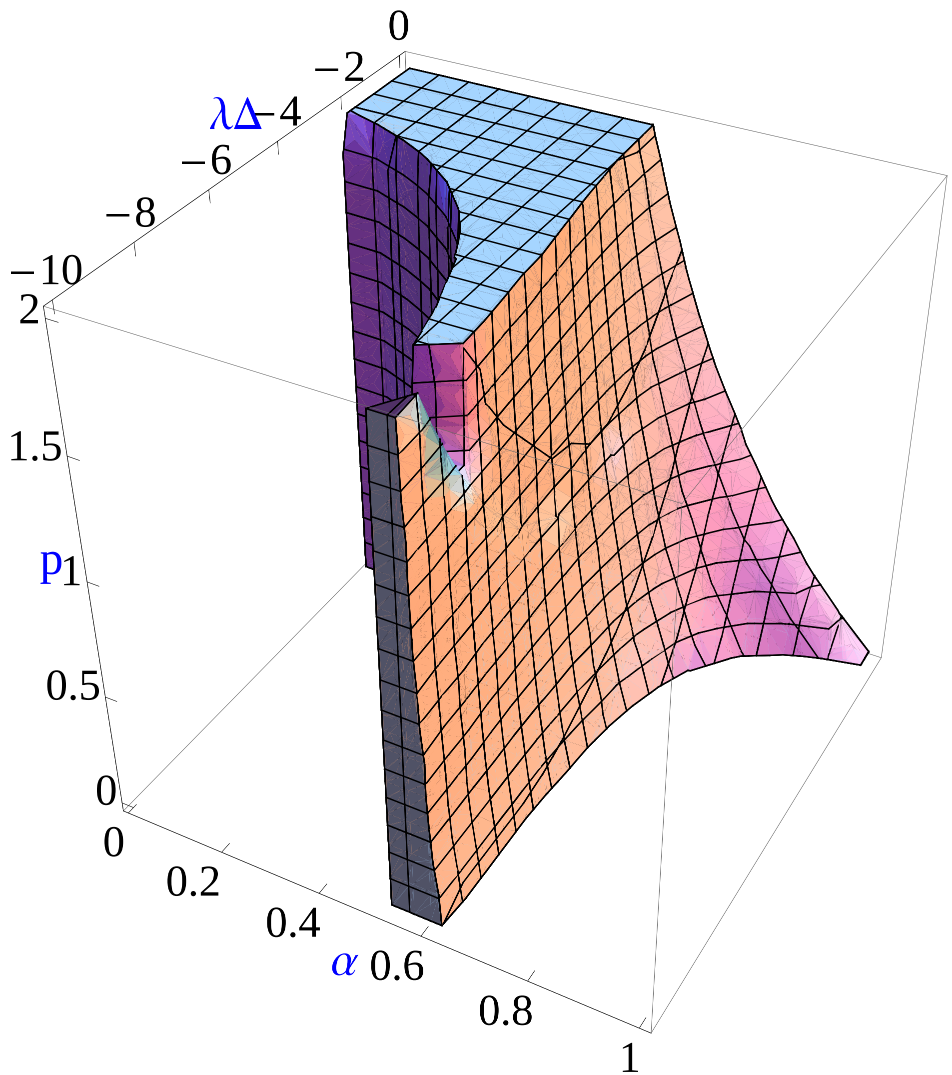


Figure 14.10: Stability region for semi-drift-implicit predictor-corrector Euler method.

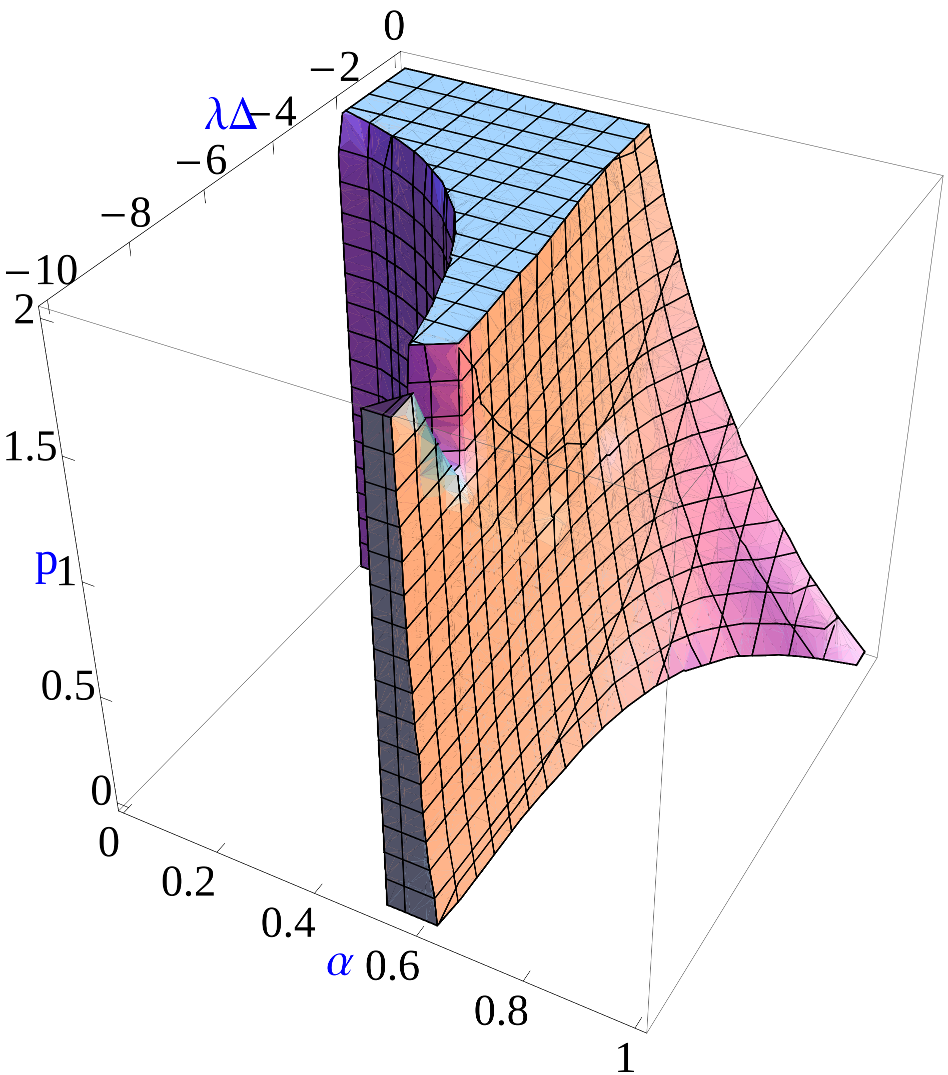


Figure 14.11: Stability region for drift-implicit predictor-corrector Euler method.

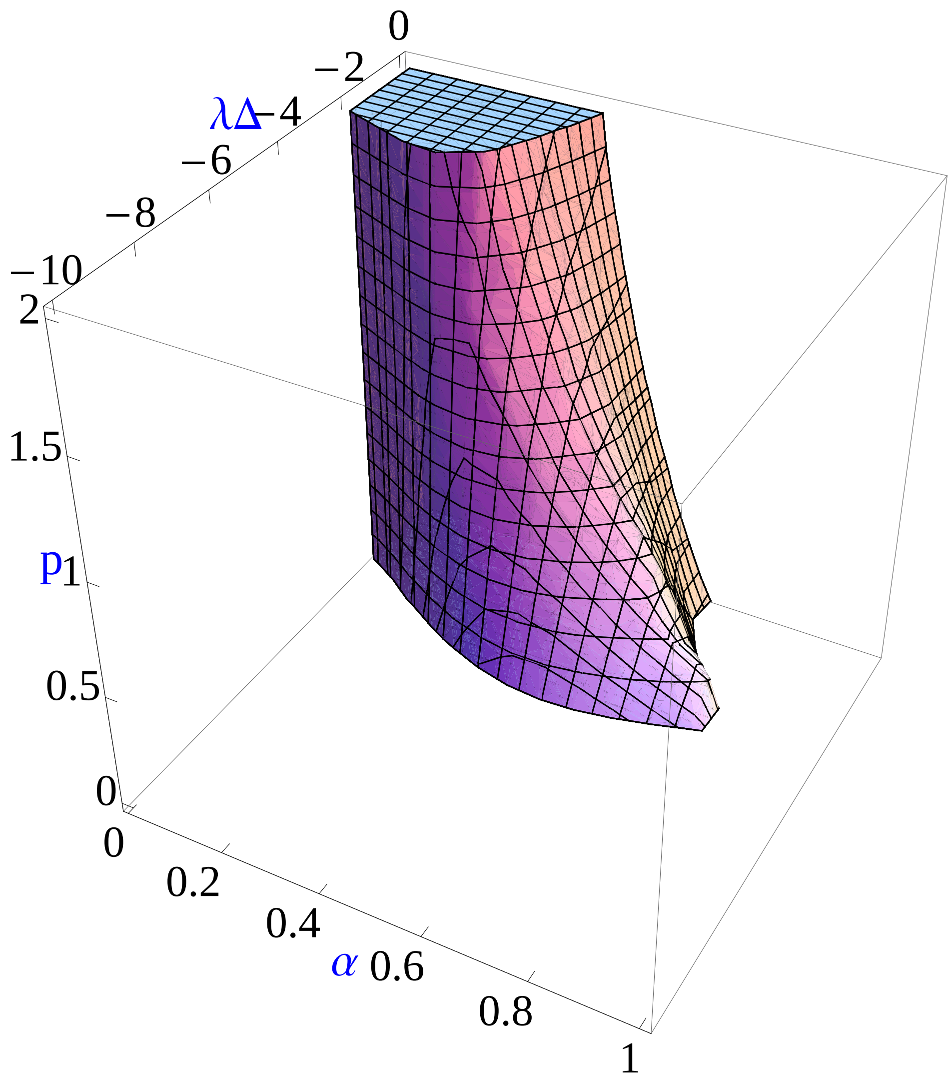


Figure 14.12: Stability region for the predictor-corrector Euler method with $\boldsymbol{\theta = 0}$ and $\boldsymbol{\eta = \frac{1}{2}}$.

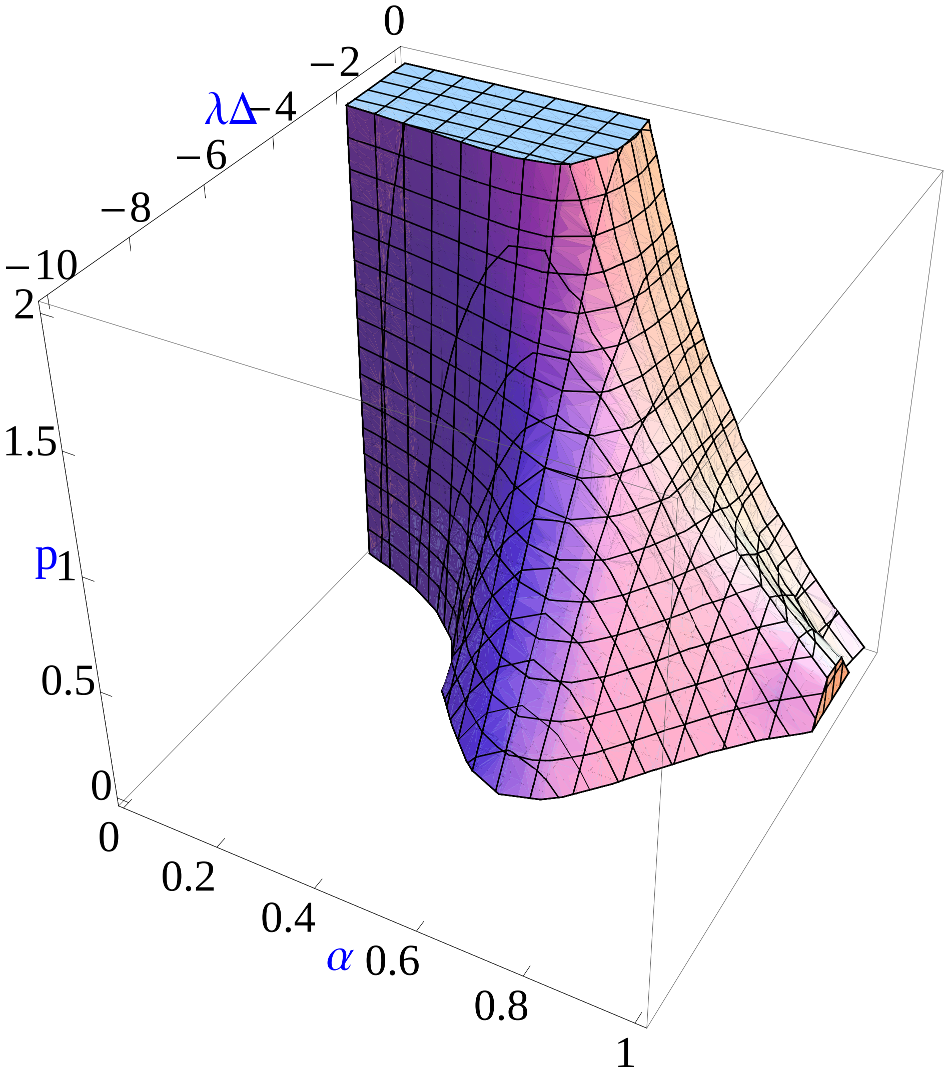


Figure 14.13: Stability region for the symmetric predictor-corrector Euler method.

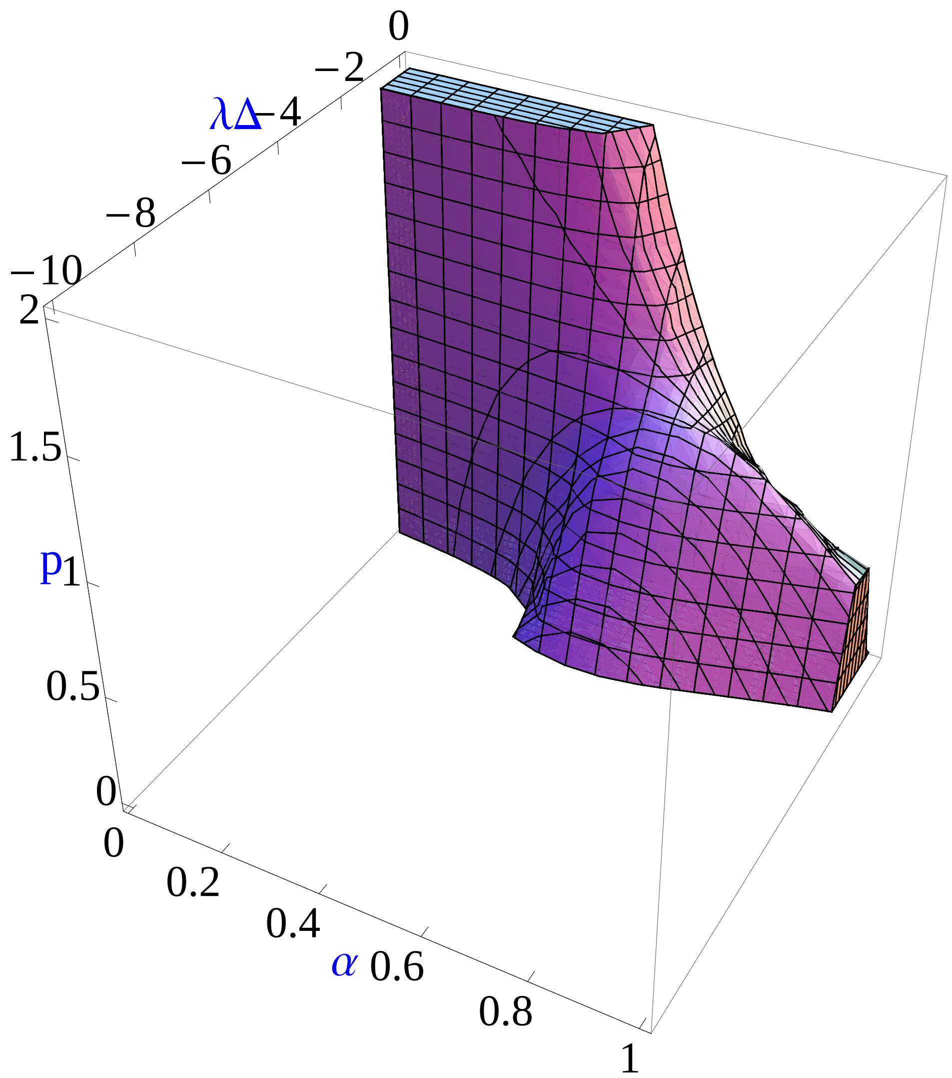


Figure 14.14: Stability region for fully implicit predictor-corrector Euler method.

Stability of Some Implicit Methods

- **semi-drift implicit Euler scheme**

$$Y_{n+1} = Y_n + \frac{1}{2}(a(Y_{n+1}) + a(Y_n))\Delta + b(Y_n)\Delta W_n$$

- **full-drift implicit Euler scheme**

$$Y_{n+1} = Y_n + a(Y_{n+1})\Delta + b(Y_n)\Delta W_n$$

solve algebraic equation

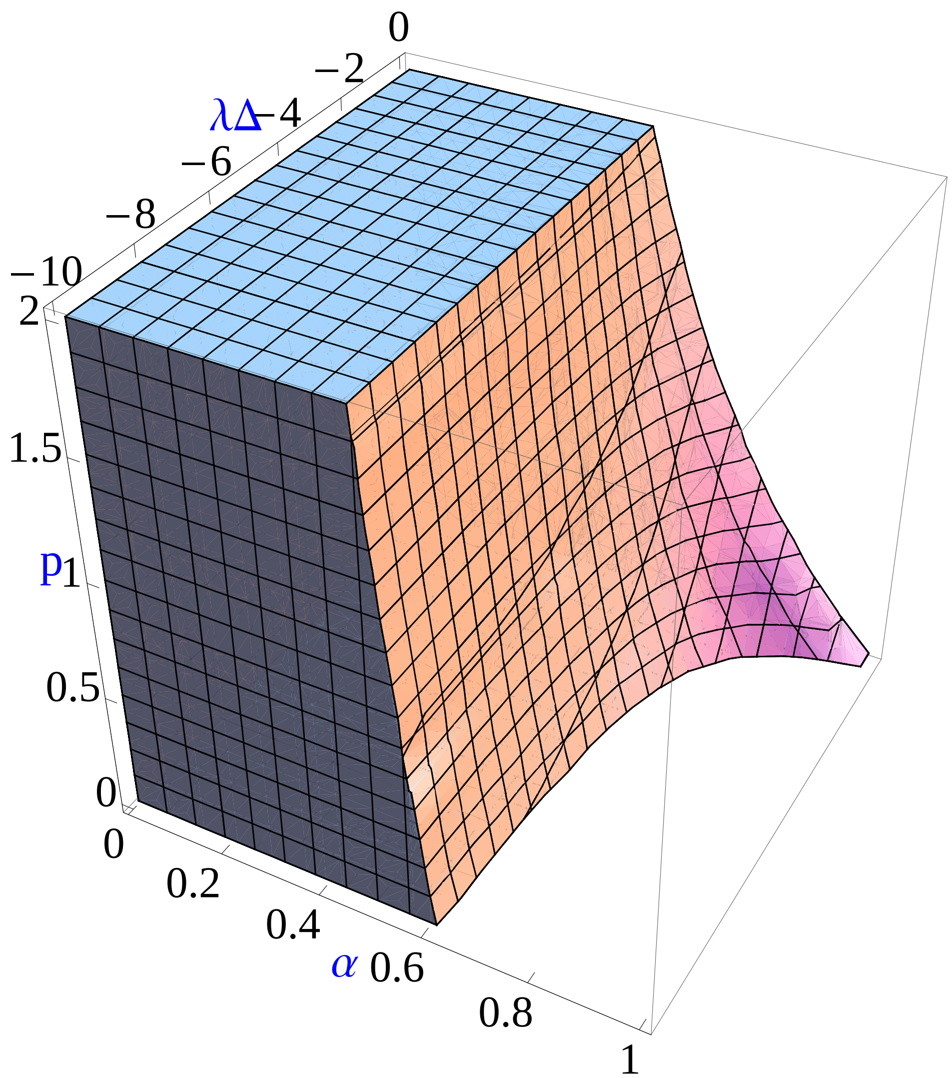


Figure 14.15: Stability region for semi-drift implicit Euler method.

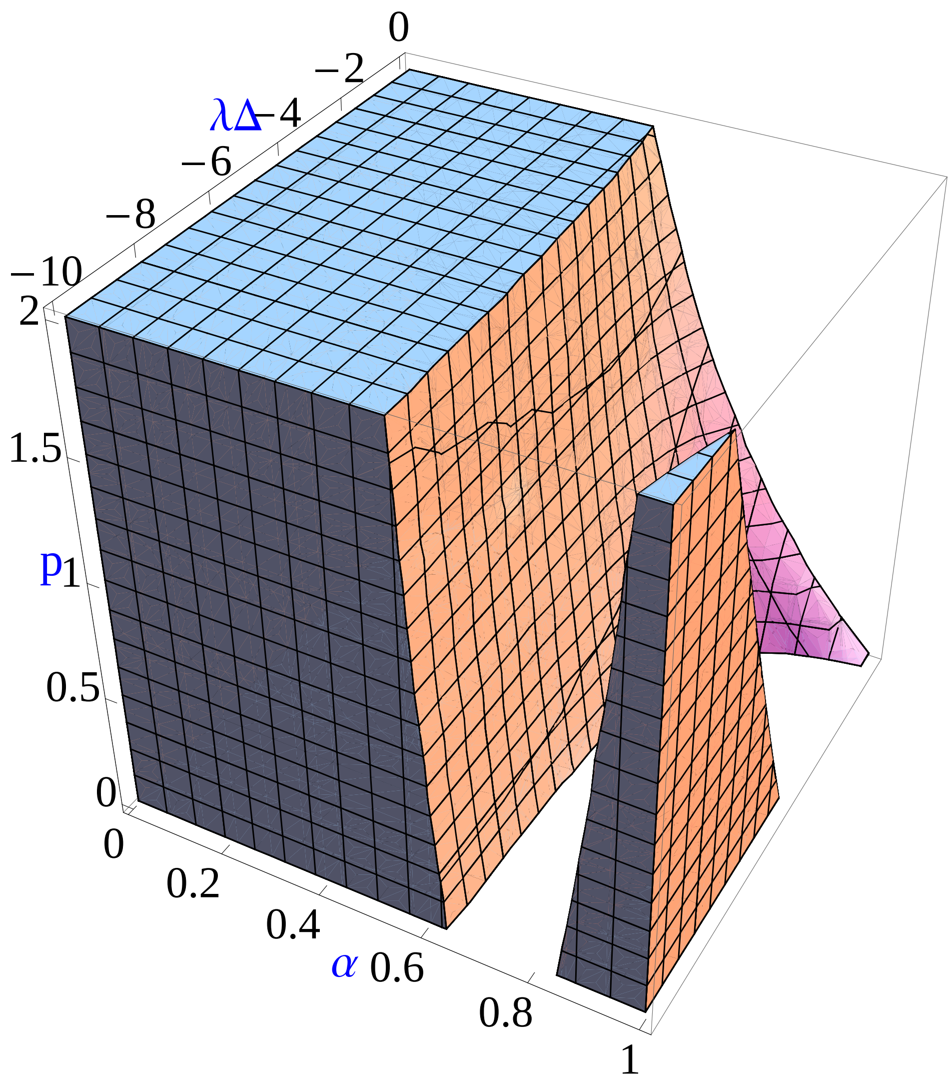


Figure 14.16: Stability region for full-drift implicit Euler method.

- **balanced implicit Euler method**

  Milstein, Pl. & Schurz (1998)

$$Y_{n+1} = Y_n + \left(1 - \frac{3}{2}\,\alpha\right)\lambda Y_n \Delta + \sqrt{\alpha|\lambda|}\,Y_n \Delta W_n + c|\Delta W_n|(Y_n - Y_{n+1})$$

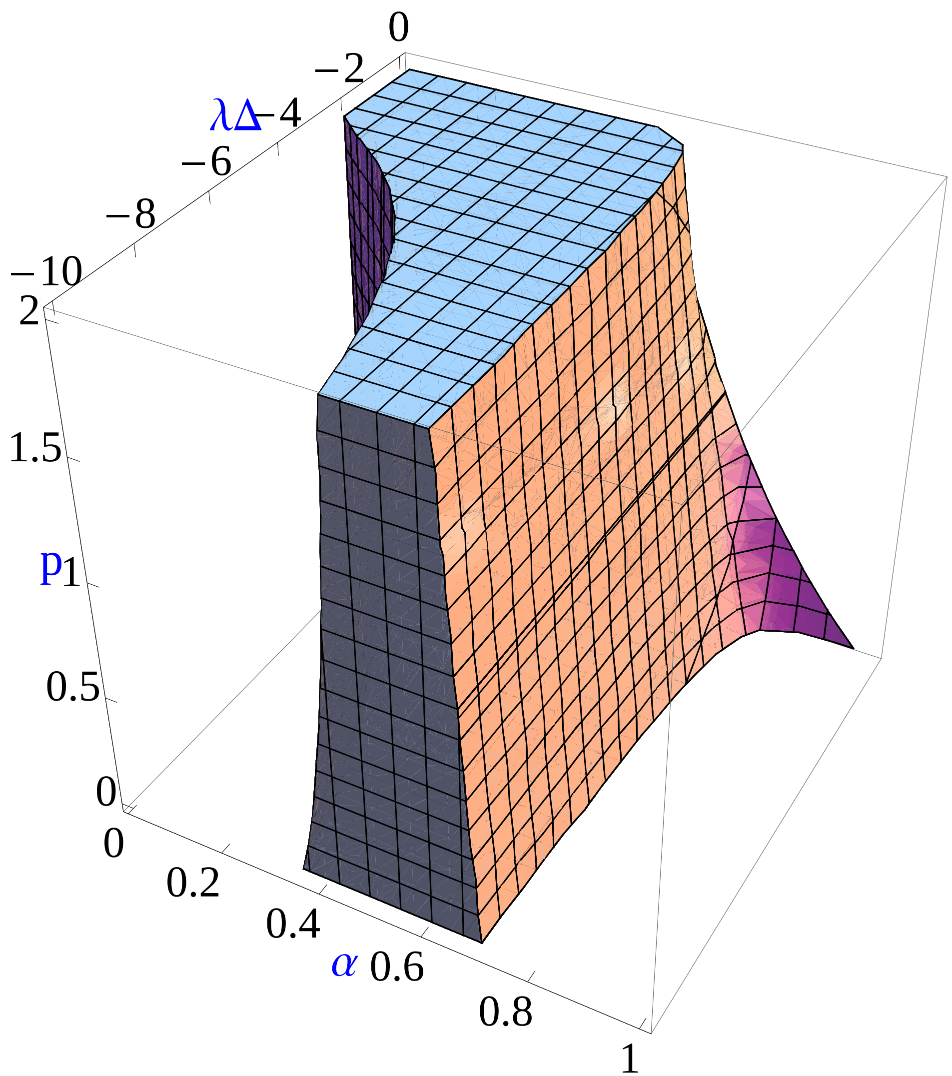


Figure 14.17: Stability region for a balanced implicit Euler method.

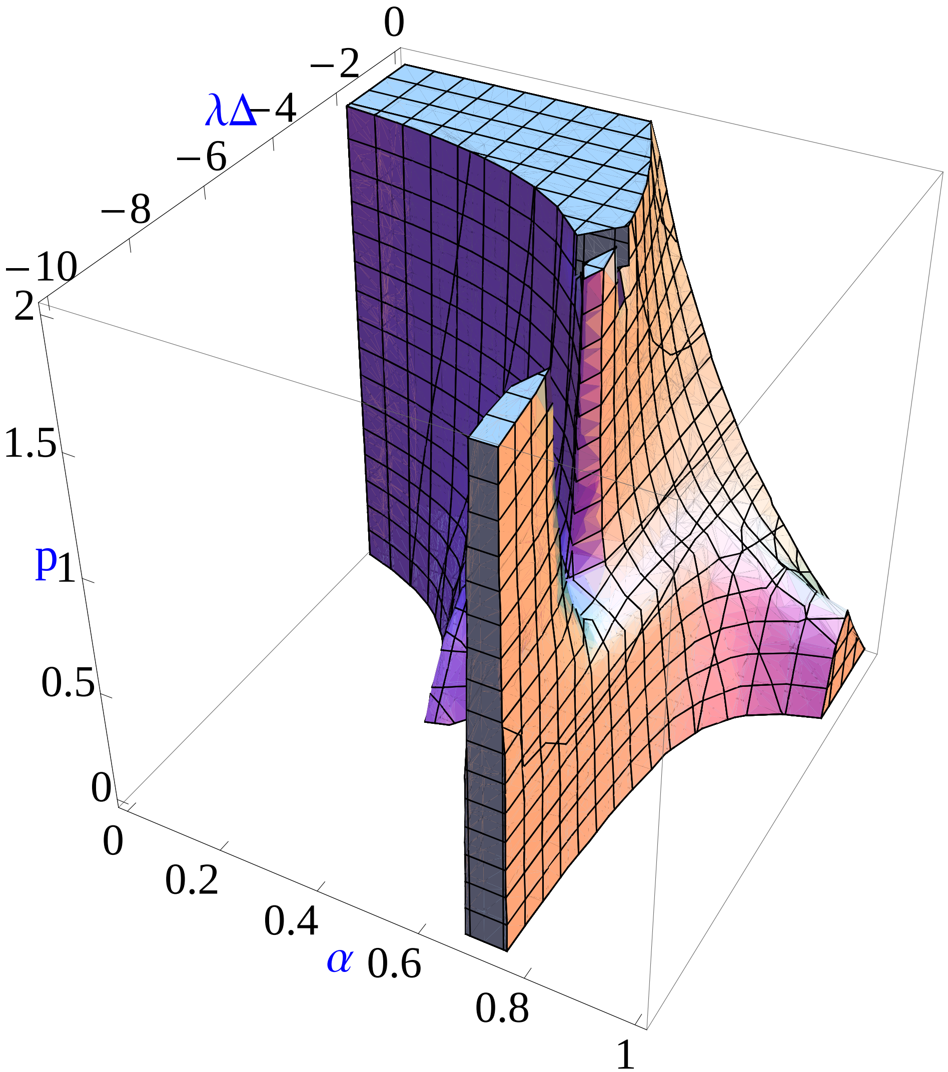


Figure 14.18: Stability region for the simplified symmetric Euler method.

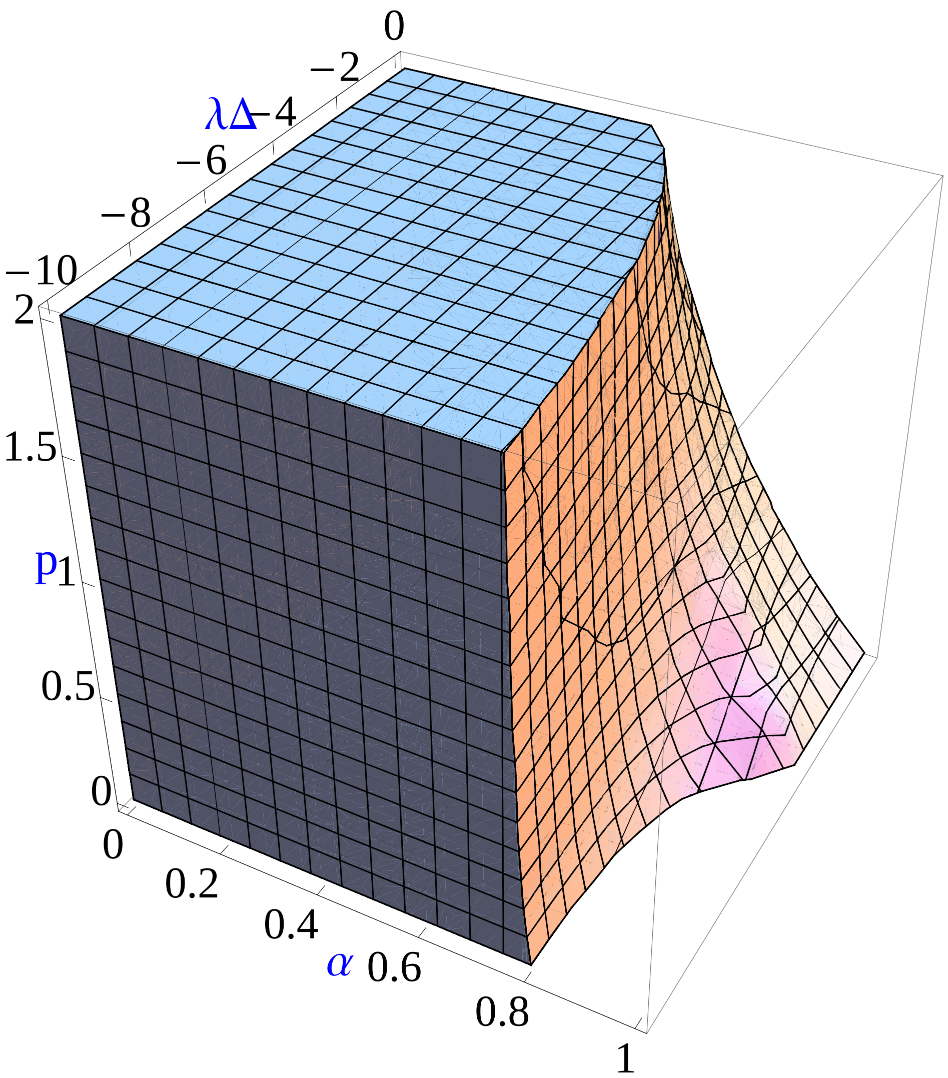


Figure 14.19: Stability region for the simplified symmetric implicit Euler Scheme.

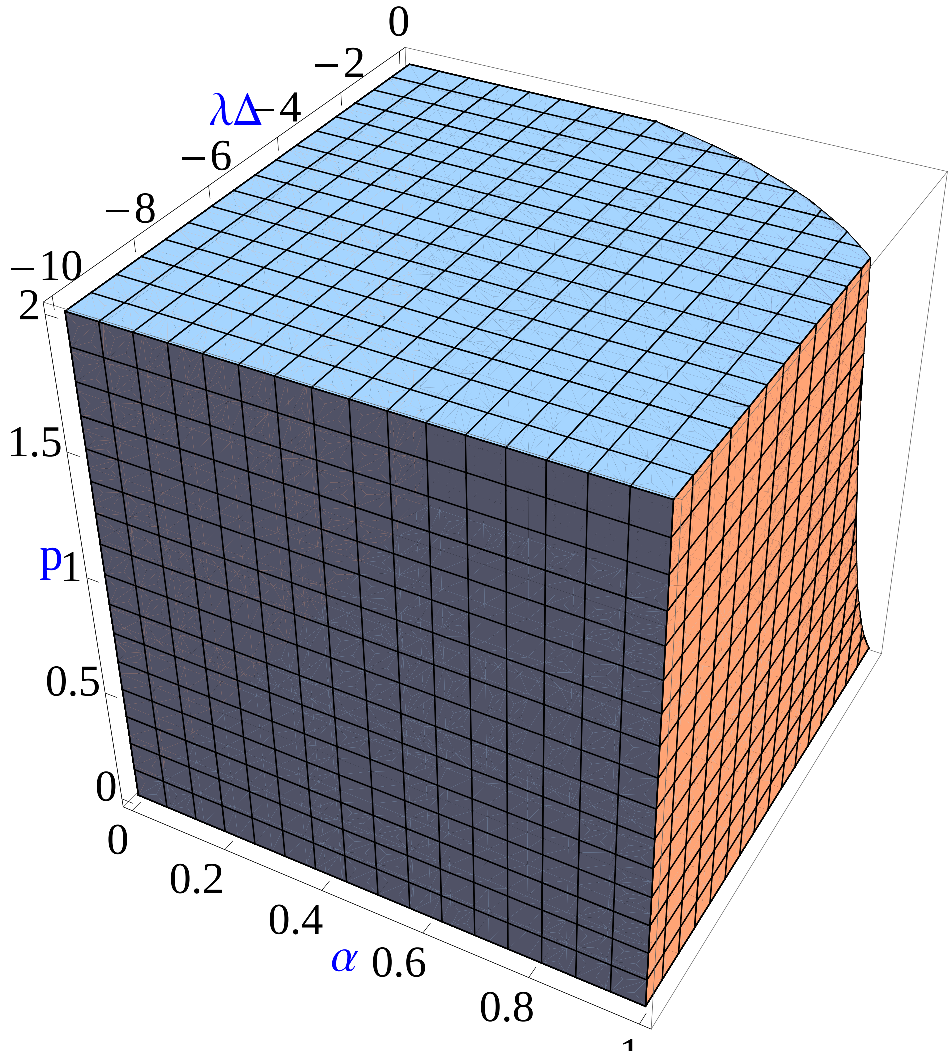


Figure 14.20: Stability region for the simplified fully implicit Euler Scheme.

16 Variance Reduction Techniques

Various Variance Reduction Methods

- **classical**

  Hammersley & Handscomb (1964)
  Ermakov (1975), Boyle (1977)
  Maltz & Hitzl (1979), Rubinstein (1981)
  Ermakov & Mikhailov (1982), Ripley (1983)
  Kalos & Whitlock (1986), Bratley, Fox & Schrage (1987)
  Chang (1987),Wagner (1987)
  Law & Kelton (1991), Ross (1990)

- **stochastic differential equations**

  Boyle (1977)

  Boyle, Broadie & Glasserman (1997)

  Broadie & Glasserman (1997b), Fu (1995)

  Grant, Vora & Weeks (1997), Joy, Boyle & Tan (1996)

  Glasserman (2004)

  Longstaff & Schwartz (2001)

  Milstein (1988), Kloeden & Platen (1999)

  Hofmann, Platen & Schweizer (1992), Heath (1995)

  Goldman, Heath, Kentwell & Platen (1995)

  Fournie, Lasry & Touzi (1997)

  Newton (1994)

Antithetic Variates

- **probability space**

  $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ where $\Omega = \mathcal{C}([t_0, T], \Re^2)$

  $\omega(t) = (\omega_1(t), \omega_2(t))^\top \in \Re^2$ for $t \in [t_0, T]$

- coordinate mappings

  $W_t^1(\omega) = \omega_1(t)$ and $W_t^2(\omega) = \omega_2(t)$

$$dX_t^{t_0,\underline{x}} = a\left(t, X_t^{t_0,\underline{x}}\right) dt + b\left(t, X_t^{t_0,\underline{x}}\right) dW_t$$

For $\omega \in \Omega$, define $\bar{\omega} \in \Omega$ by $\bar{\omega}(t) = (-\omega_1(t), -\omega_2(t))^\top,\ t \in [t_0, T]$

$$\bar{h}\left(X_T^{t_0,\underline{x}}\right)(\omega) = h\left(X_T^{t_0,\underline{x}}\right)(\bar{\omega})$$

- **unbiased estimator**

$$\hat{h}\left(X_T^{t_0,\underline{x}}\right) = \frac{1}{2}\left(h\left(X_T^{t_0,\underline{x}}\right) + \bar{h}\left(X_T^{t_0,\underline{x}}\right)\right)$$

$\Longrightarrow$

$$\begin{aligned} \mathrm{Var}\left(\hat{h}\left(X_T^{t_0,\underline{x}}\right)\right) &= \frac{1}{4}\left(\mathrm{Var}\left(h\left(X_T^{t_0,\underline{x}}\right)\right) + \mathrm{Var}\left(\bar{h}\left(X_T^{t_0,\underline{x}}\right)\right)\right. \\ &\quad \left. + 2\,\mathrm{Cov}\left(h\left(X_T^{t_0,\underline{x}}\right), \bar{h}\left(X_T^{t_0,\underline{x}}\right)\right)\right) \end{aligned}$$

Stratified Sampling

- **set of events**

  $A_i \subseteq \mathcal{A}_T,\ i \in \{1, 2, \ldots, N\}$

$$\bigcup_{i=1}^{N} A_i = \Omega, \qquad A_i \cap A_j = \emptyset$$

  for $i, j \in \{1, 2, \ldots, N\}$, $P(A_i) = \frac{1}{N}$

  $\mathcal{A} = \sigma\{A_i,\ i \in \{1, 2, \ldots, N\}\}$

- **random variable**

  $$Z : \Omega \to \Re$$

- restriction of Z to A_i

  $$Z_{A_i}(\omega) = Z(\omega) \text{ for } \omega \in A_i$$

- **unbiased estimator for $E(Z)$**

$$\bar{Z} = \frac{1}{N} \sum_{i=1}^{N} Z_{A_i}$$

where Z_{A_i} independent

$$E(\bar{Z}) = \frac{1}{N} \sum_{i=1}^{N} E(Z_{A_i}) = \sum_{i=1}^{N} \int_{A_i} Z \, dP = \int_{\Omega} Z \, dP = E(Z)$$

independence

$\Longrightarrow$

$$
\begin{aligned}
\mathbf{Var}(\bar{Z}) &= \sum_{i=1}^{N} \frac{\mathbf{Var}(Z_{A_i})}{N^2} \\
&= \frac{1}{N}\, E(\mathbf{Var}(Z \,|\, \mathcal{A})) \\
&\leq \frac{1}{N}\, \mathbf{Var}(Z)
\end{aligned}
$$

Example: simplified weak Euler approximation Y^{Δ}

$\Delta \hat{W}_k$ two-point distributed

two-point variates with N time steps

underlying sample space

$\Longrightarrow$

$$\Omega_N = \{-1, 1\}^{\{0,1,\ldots,N-1\}}$$

'path' for $\hat{W}_k$

given by $\Delta \hat{W}_k(\omega) = \omega_k \sqrt{\Delta}$

probabilities $P_N(\omega) = \frac{1}{2^N}$

only a tiny fraction of these paths can be sampled

A stratified Monte Carlo estimation could consist of exhausting all paths $\omega \in \Omega_N$ up to time $t_{N'}$ and then sampling randomly;

reduces duplicate traversals of the early nodes of the lattice.

Measure Transformation Method

see Kloeden & Platen (1999)

- **d-dimensional diffusion**

$$dX_t^{s,x} = a(t, X_t^{s,x})\, dt + \sum_{j=1}^{m} b^j(t, X_t^{s,x})\, dW_t^j$$

on $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$

- **approximate the functional**

$$u(s,x) = E\left(g(X_T^{s,x}) \,\middle|\, \mathcal{A}_s\right)$$

- **Kolmogorov backward equation**

$$L^0\, u(s,x) = 0$$

for $(s,x) \in (0,T) \times \Re^d$ with

$$u(T,y) = g(y)$$

$$L^0 = \frac{\partial}{\partial s} + \sum_{k=1}^{d} a^k \, \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,\ell=1}^{d} \sum_{j=1}^{m} b^{k,j}\, b^{\ell,j}\, \frac{\partial^2}{\partial x^k\, \partial x^\ell}$$

- **Girsanov transformation**

$$\tilde{W}_t^j = W_t^j - \int_0^t d^j\left(z, \tilde{X}_z^{0,x}\right) dz$$

d^j denotes given, rather flexible real-valued function

- **Radon-Nikodym derivative**

$$\frac{d\tilde{P}}{dP} = \frac{\Theta_t}{\Theta_0}$$

- SDE

$$
\begin{aligned}
d\tilde{X}_t^{0,x} &= a\left(t, \tilde{X}_t^{0,x}\right) dt + \sum_{j=1}^{m} b^j\left(t, \tilde{X}_t^{0,x}\right) d\tilde{W}_t^j \\
&= \left(a\left(t, \tilde{X}_t^{0,x}\right) - \sum_{j=1}^{m} b^j\left(t, \tilde{X}_t^{0,x}\right)\, d^j\left(t, \tilde{X}_t^{0,x}\right)\right) dt \\
&\quad + \sum_{j=1}^{m} b^j\left(t, \tilde{X}_t^{0,x}\right) dW_t^j
\end{aligned}
$$

- Radon-Nikodym derivative process

$$
\Theta_t = \Theta_0 + \sum_{j=1}^{m} \int_0^t \Theta_z\, d^j\left(z, \tilde{X}_z^{0,x}\right) dW_z^j
$$

$(\underline{\mathcal{A}}, P)$-martingale

- **diffusion process** $\tilde{X}^{0,x}$ with respect to $\tilde{P}$

  same drift and diffusion coefficients as $X^{s,x}$

  $\Longrightarrow$

$$
\begin{aligned}
E\left(g\left(X_T^{0,x}\right)\right) &= \int_\Omega g\left(X_T^{0,x}\right) dP \\
&= \int_\Omega g\left(\tilde{X}_T^{0,x}\right) d\tilde{P} \\
&= \int_\Omega g\left(\tilde{X}_T^{0,x}\right) \frac{\Theta_T}{\Theta_0}\, dP \\
&= E\left(g\left(\tilde{X}_T^{0,x}\right) \frac{\Theta_T}{\Theta_0}\right)
\end{aligned}
$$

- **unbiased estimator**

$$g\left(\tilde{X}_T^{0,x}\right)\frac{\Theta_T}{\Theta_0}$$

no particular choice of d^j made so far

- ideally choose d^j in the form

$$d^j(t,x) = -\frac{1}{u(t,x)} \sum_{k=1}^{d} b^{k,j}(t,x) \frac{\partial u(t,x)}{\partial x^k}$$

then it can be shown that

$$u\left(t, \tilde{X}_t^{0,x}\right) \Theta_t = u(0,x)\, \Theta_0$$

for all $t \in [0,T]$

$\Longrightarrow$

$$u(0,x) = g\left(\tilde{X}_T^{0,x}\right) \frac{\Theta_T}{\Theta_0}$$

$\Longrightarrow$

$$g\left(\tilde{X}_T^{0,x}\right)\frac{\Theta_T}{\Theta_0}$$

is ***not random***

- guess a function $\bar{u}$

$$d^j(t, x) = -\frac{1}{\bar{u}(t, x)} \sum_{k=1}^{d} b^{k,j}(t, x) \frac{\partial \bar{u}(t, x)}{\partial x^k}$$

- used unbiased estimator

$$g\left(\tilde{X}_T^{0,x}\right) \frac{\Theta_T}{\Theta_0}$$

$$E\left(g\left(\tilde{X}_T^{0,x}\right) \frac{\Theta_T}{\Theta_0}\right) = E\left(g\left(X_T^{0,x}\right)\right)$$

Control Variates and Integral Representations

Clewlow & Carverhill (1992, 1994)

Basic Control Variate Method

- **valuation martingale**

$$M_t = u\left(t, X_t^{t_0,\underline{x}}\right) = E\left(h\left(X_T^{t_0,\underline{x}}\right) \,\Big|\, \mathcal{A}_t\right)$$

construct an accurate and fast estimate of

$$E(h(X_T^{t_0,\underline{x}})) = u(t_0, \underline{x})$$

- **control variate**

  find $\boldsymbol{Y}$ with known mean $\boldsymbol{E}(\boldsymbol{Y})$

- **unbiased estimator**

$$\boldsymbol{Z} = \boldsymbol{h}(\boldsymbol{X}_T^{t_0,\underline{\boldsymbol{x}}}) - \boldsymbol{\alpha}(\boldsymbol{Y} - \boldsymbol{E}(\boldsymbol{Y}))$$

$\boldsymbol{\alpha} \in \Re$

$$\boldsymbol{E}(\boldsymbol{Z}) = \boldsymbol{E}(\boldsymbol{h}(\boldsymbol{X}_T^{t_0,\underline{\boldsymbol{x}}}))$$

- **known valuation function**

$$\hat{u}\left(t, \hat{X}_t^{t_0,\underline{x}}\right) = E\left(h\left(\hat{X}_T^{t_0,\underline{x}}\right) \,\Big|\, \mathcal{A}_t\right)$$

$\hat{X}^{t_0,\underline{x}}$ approximates $X_T^{t_0,\underline{x}}$

- **unbiased estimator**

$$\begin{aligned}\hat{Z}_T &= h\left(X_T^{t_0,\underline{x}}\right) - \alpha\left(h\left(\hat{X}_T^{t_0,\underline{x}}\right) - E\left(h\left(\hat{X}_T^{t_0,\underline{x}}\right)\right)\right) \\ &= u\left(T, X_T^{t_0,\underline{x}}\right) - \alpha\left(\hat{u}\left(T, \hat{X}_T^{t_0,\underline{x}}\right) - \hat{u}(t_0,\underline{x})\right)\end{aligned}$$

variance can be reduced

$$\mathbf{Var}(\hat{Z}_T) = \mathbf{Var}\left(h\left(X_T^{t_0,\underline{x}}\right)\right) + \alpha^2\,\mathbf{Var}\left(h\left(\hat{X}_T^{t_0,\underline{x}}\right)\right) - 2\,\alpha\,\mathrm{Cov}\left(h\left(X_T^{t_0,\underline{x}}\right), h\left(\hat{X}_T^{t_0,\underline{x}}\right)\right)$$

minimize the variance

$$\alpha_{\min} = \frac{\mathrm{Cov}\left(h\left(X_T^{t_0,\underline{x}}\right), h\left(\hat{X}_T^{t_0,\underline{x}}\right)\right)}{\mathbf{Var}\left(h\left(\hat{X}_T^{t_0,\underline{x}}\right)\right)}$$

Example: Stochastic Volatility

- **SDE**

$$\begin{aligned} dS_t &= \sigma_t \, S_t \, dW_t^1 \\ d\sigma_t &= (\kappa - \sigma_t) \, dt + \xi \, \sigma_t \, dW_t^2 \end{aligned}$$

$$(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$$

- **European call payoff**

$$h(S_T) = (S_T - K)^+$$

- **adjusted price process**

$$
\begin{aligned}
d\hat{S}_t &= \hat{\sigma}_t \hat{S}_t \, dW_t^1 \\
d\hat{\sigma}_t &= (\kappa - \hat{\sigma}_t) \, dt
\end{aligned}
$$

- **adjusted valuation function**

$$
\hat{u}(t, \hat{S}_t, \hat{\sigma}_t) = E\left((\hat{S}_T - K)^+ \,\middle|\, \mathcal{A}_t \right)
$$

- **unbiased variate**

$$
\begin{aligned}
\hat{Z}_T &= (S_T - K)^+ - \alpha((\hat{S}_T - K)^+ - E(\hat{S}_T - K)^+) \\
&= (S_T - K)^+ - \alpha((\hat{S}_T - K)^+ - \hat{u}(t_0, s, \sigma))
\end{aligned}
$$

unbiased variance reduced estimator

Variance Reduction via Integral Representations

Heath & Platen (2002)

The HP Variance Reduced Estimator

- **SDE**

$$dX_t^{s,x} = a(t, X_t^{s,x})\, dt + \sum_{j=1}^{m} b^j(t, X_t^{s,x})\, dW_t^j$$

- **first exit time**

$$\tau = \inf\{t \geq s :\ (t, X_t^{s,x}) \notin [s, T) \times \Gamma\}$$

- **operators**

$$L^0 f(t,x) = \frac{\partial f(t,x)}{\partial t} + \sum_{i=1}^{d} a^i(t,x) \frac{\partial f(t,x)}{\partial x^i} + \frac{1}{2} \sum_{i,k=1}^{d} \sum_{j=1}^{m} b^{i,j}(t,x)\, b^{k,j}(t,x) \frac{\partial^2 f(t,x)}{\partial x^i \, \partial x^k}$$

and

$$L^j f(t,x) = \sum_{i=1}^{d} b^{i,j}(t,x) \frac{\partial f(t,x)}{\partial x^i}$$

for $(t,x) \in (0,T) \times \Gamma$

- **valuation function**

$$u(t,x) = E\left(h(\tau, X_\tau^{t,x})\right)$$

for $(t,x) \in [0,T] \times \Gamma$

assume

$$M_t = E\left(h\left(\tau, X_\tau^{0,x}\right) \,\middle|\, \mathcal{A}_t\right)$$

square integrable $(\underline{\mathcal{A}}, P)$-martingale

martingale representation theorem

$\Longrightarrow$

$$
\begin{aligned}
M_t &= u(t, X_{t\wedge\tau}^{0,x}) \\
&= u(0,x) + \sum_{j=1}^{m} \int_0^{t\wedge\tau} \xi_s^j \, dW_s^j
\end{aligned}
$$

for $t \in [0, T]$

- given an approximation

$$\bar{u} : [0, T] \times \Gamma \to \Re \quad \text{to } u$$

$$\bar{u} \in \mathcal{C}^{1,2}([0, T] \times \Gamma)$$

with

$$\bar{M}_t^j = \int_0^{t \wedge \tau} L^j \, \bar{u}(s, X_s^{0,x}) \, dW_s^j$$

square integrable $(\underline{\mathcal{A}}, P)$-martingale

$$\bar{u}(\tau, X_\tau^{0,x}) = u(\tau, X_\tau^{0,x}) = h(\tau, X_\tau^{0,x})$$

$\Longrightarrow$

$$\begin{aligned}\bar{u}(\tau, X_\tau^{0,x}) &= \bar{u}(0, x) + \int_0^\tau L^0\, \bar{u}(t, X_t^{0,x})\, dt \\ &\quad + \sum_{j=1}^m \int_0^\tau L^j\, \bar{u}(t, X_t^{0,x})\, dW_t^j\end{aligned}$$

$\Longrightarrow$

$$
\begin{aligned}
u(0,x) &= E\left(h(\tau, X_\tau^{0,x})\right) \\
&= E\left(\bar{u}(\tau, X_\tau^{0,x})\right) \\
&= \bar{u}(0,x) + E\left(\int_0^\tau L^0\, \bar{u}(t, X_t^{0,x})\, dt\right) \\
&= \bar{u}(0,x) + \int_0^T E\left(1_{\{t<\tau\}} L^0\, \bar{u}(t, X_t^{0,x})\right) dt
\end{aligned}
$$

- **unbiased estimator for** $u(0,x)$

$$
\bar{Z}_\tau = \bar{u}(0,x) + \int_0^\tau L^0\, \bar{u}(t, X_t^{0,x})\, dt
$$

HP estimator

Variance of the HP Estimator

$$
\begin{aligned}
\bar{Z}_\tau &= \bar{u}(\tau, X_\tau^{0,x}) - \sum_{j=1}^{m} \int_0^\tau L^j\, \bar{u}(t, X_t^{0,x})\, dW_t^j \\
&= u(\tau, X_\tau^{0,x}) - \sum_{j=1}^{m} \int_0^\tau L^j\, \bar{u}(t, X_t^{0,x})\, dW_t^j \\
&= u(0, x) + \sum_{j=1}^{m} \int_0^\tau \left(\xi_t^j - L^j\, \bar{u}(t, X_t^{0,x}) \right) dW_t^j
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
\mathbf{Var}(\bar{Z}_{\tau}) &= E\left[\left(\sum_{j=1}^{m}\int_{0}^{\tau}\left(\xi_t^j - L^j\,\bar{u}(t,X_t^{0,x})\right)dW_t^j\right)^2\right] \\
&= \sum_{j=1}^{m}\int_{0}^{T} E\left(1_{\{t<\tau\}}\left(\xi_t^j - L^j\,\bar{u}(t,X_t^{0,x})\right)^2\right)dt
\end{aligned}
$$

- **integrands**

$$\xi_t^j = L^j\, u(t, X_t^{0,x})$$

$$\Longrightarrow$$

$$\mathrm{Var}(\bar{Z}_\tau) = \sum_{j=1}^{m} \int_0^T E\left(1_{\{t<\tau\}} \left(\left(L^j\, u - L^j\, \bar{u}\right)(t, X_t^{0,x})\right)^2\right) dt$$

if a good approximation $\bar{u}$ to u can be found,

so that $L^j\, u$ is close to $L^j\, \bar{u}$

variance will be small

- **unbiased estimator**

$$\bar{Z}_{\tau,\alpha} = \bar{u}(\tau, X_\tau^{0,x}) - \alpha \sum_{j=1}^{m} \int_0^\tau L^j \, \bar{u}(t, X_t^{0,x}) \, dW_t^j$$

$$= \bar{Z}_\tau + (1-\alpha) \sum_{j=1}^{m} \int_0^\tau L^j \, \bar{u}(t, X_t^{0,x}) \, dW_t^j$$

An Example for the Heston Model

- **SDEs**

$$
\begin{aligned}
dS_t^{s,x^1} &= u\, S_t^{s,x^1}\, dt + \sqrt{v_t^{s,x^2}}\, S_t^{s,x^1}\, dW_t^1 \\
dv_t^{s,x^2} &= \kappa \left(\theta - v_t^{s,x^2}\right) dt \\
&\quad + \xi \sqrt{v_t^{s,x^2}} \left(\varrho\, dW_t^1 + \sqrt{1-\varrho^2}\, dW_t^2\right)
\end{aligned}
$$

- equivalent risk neutral martingale measure

$$\begin{aligned} dS_t^{s,x^1} &= r\, S_t^{s,x^1}\, dt + \sqrt{v_t^{s,x^2}}\, S_t^{s,x^1}\, d\tilde{W}_t^1 \\ dv_t^{s,x^2} &= \kappa \left(\tilde{\theta} - v_t^{s,x^2}\right) dt \\ &\quad + \xi \sqrt{v_t^{s,x^2}} \left(\varrho\, d\tilde{W}_t^1 + \sqrt{1-\varrho^2}\, d\tilde{W}_t^2\right) \end{aligned}$$

for $t \in [s, T]$ and $s \in [0, T]$, where $\tilde{\theta} = \theta - \frac{\xi\, \varrho\, (\mu - r)}{\kappa}$ and

$$\begin{aligned} d\tilde{W}_t^1 &= \frac{\mu - r}{\sqrt{v_t}}\, dt + dW_t^1 \\ d\tilde{W}_t^2 &= dW_t^2 \end{aligned}$$

- **option price**

$$c(0,x) = e^{-rT}\, u(0,x)$$

where

$$u(0,x) = \tilde{E}\left((S_T^{0,x^1} - K)^+\right)$$

- **approximation**

$$d\bar{S}_t^{s,x^1} = r\,\bar{S}_t^{s,x^1}\,dt + \sqrt{\bar{v}_t^{s,x^2}}\,\bar{S}_t^{s,x^1}\,d\tilde{W}_t^1$$

$$d\bar{v}_t^{s,x^2} = \kappa\left(\tilde{\theta} - \bar{v}_t^{s,x^2}\right)dt$$

explicitly computed

$$\bar{v}_t^{s,x^2} = \tilde{\theta} + (x^2 - \tilde{\theta})\,e^{-\kappa(t-s)}$$

Black-Scholes price

$$\begin{aligned}\bar{u}(t,x) &= \tilde{E}\left((\bar{S}_T^{t,x^1} - K)^+\right)\\ &= e^{r(T-t)}\,BS(x^1, K, r, \bar{\sigma}_t, T-t)\end{aligned}$$

where

$$\begin{aligned}\bar{\sigma}_t &= \sqrt{\frac{1}{T-t}\int_t^T \bar{v}_z^{t,x^2}\,dz}\\ &= \sqrt{\tilde{\theta} - (x^2 - \tilde{\theta})\,\frac{e^{-\kappa(T-t)} - 1}{\kappa(T-t)}}\end{aligned}$$

$\Longrightarrow$

$$(L^0 - \bar{L}^0)\, f(t,x) = \xi\, x^2 \left(\varrho\, \frac{\partial^2 f(t,x)}{\partial x^1\, \partial x^2} + \frac{1}{2}\, \xi\, \frac{\partial^2 f(t,x)}{\partial (x^2)^2} \right)$$

$$\begin{aligned}
(L^0 - \bar{L}^0)\, \bar{u}(t,x) \quad = \quad & \xi\, x^2\, e^{r(T-t)} \left[\varrho\, \frac{\partial^2 BS(x^1, K, r, \bar{\sigma}_t, T-t)}{\partial x^1\, \partial \bar{\sigma}_t}\, \frac{\partial \bar{\sigma}_t}{\partial x^2} \right. \\
& + \frac{1}{2}\, \xi \left\{ \frac{\partial^2 BS(x^1, K, r, \bar{\sigma}_t, T-t)}{\partial \sigma_t^2} \left(\frac{\partial \bar{\sigma}_t}{\partial x^2} \right)^2 \right. \\
& \left. \left. + \frac{\partial BS(x^1, K, r, \bar{\sigma}_t, T-t)}{\partial \sigma_t}\, \frac{\partial^2 \bar{\sigma}_t}{\partial (x^2)^2} \right\} \right]
\end{aligned}$$

$$\frac{\partial BS}{\partial \bar{\sigma}_t}, \quad \frac{\partial^2 BS}{\partial x^1\, \partial \bar{\sigma}_t}, \quad \frac{\partial^2 BS}{\partial \bar{\sigma}_t^2}, \quad \frac{\partial \bar{\sigma}_t}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 \bar{\sigma}_t}{\partial (x^2)^2}$$

can be computed

Monte Carlo Simulation

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = T, \qquad \Delta = \frac{T}{N}$$

- predictor-corrector method of weak order 1.0

$$\begin{aligned} Y_{n+1} &= Y_n + \Big(\alpha\, \hat{a}(\tau_{n+1}, \hat{Y}_{n+1}) + (1-\alpha)\, \hat{a}(\tau_n, Y_n)\Big)\, \Delta \\ &\quad + \sum_{j=1}^{m} \Big(\eta\, b^j(\tau_{n+1}, \hat{Y}_{n+1}) + (1-\eta)\, b^j(\tau_n, Y_n)\Big)\, \Delta W_n^j \end{aligned}$$

for $n \in \{0, 1, \ldots, N-1\}$ with predictor

$$\hat{Y}_{n+1} = Y_n + a(\tau_n, Y_n)\, \Delta + \sum_{j=1}^{m} b^j\, \Delta W_n^j$$

and modified drift coefficient values

$$\hat{a}(\tau_n, Y_n) = a(\tau_n, Y_n) - \eta \sum_{i=1}^{d} \sum_{j=1}^{m} b^{i,j}(\tau_n, Y_n) \, \frac{\partial b^j(\tau_n, Y_n)}{\partial x^i}$$

$\alpha, \eta \in [0, 1]$ and ΔW_n^j

Gaussian random two-point distributed random variables

Simulation Results

- raw Monte Carlo

  intrinsic value $(S_t^{0,x^1} - K)^+$

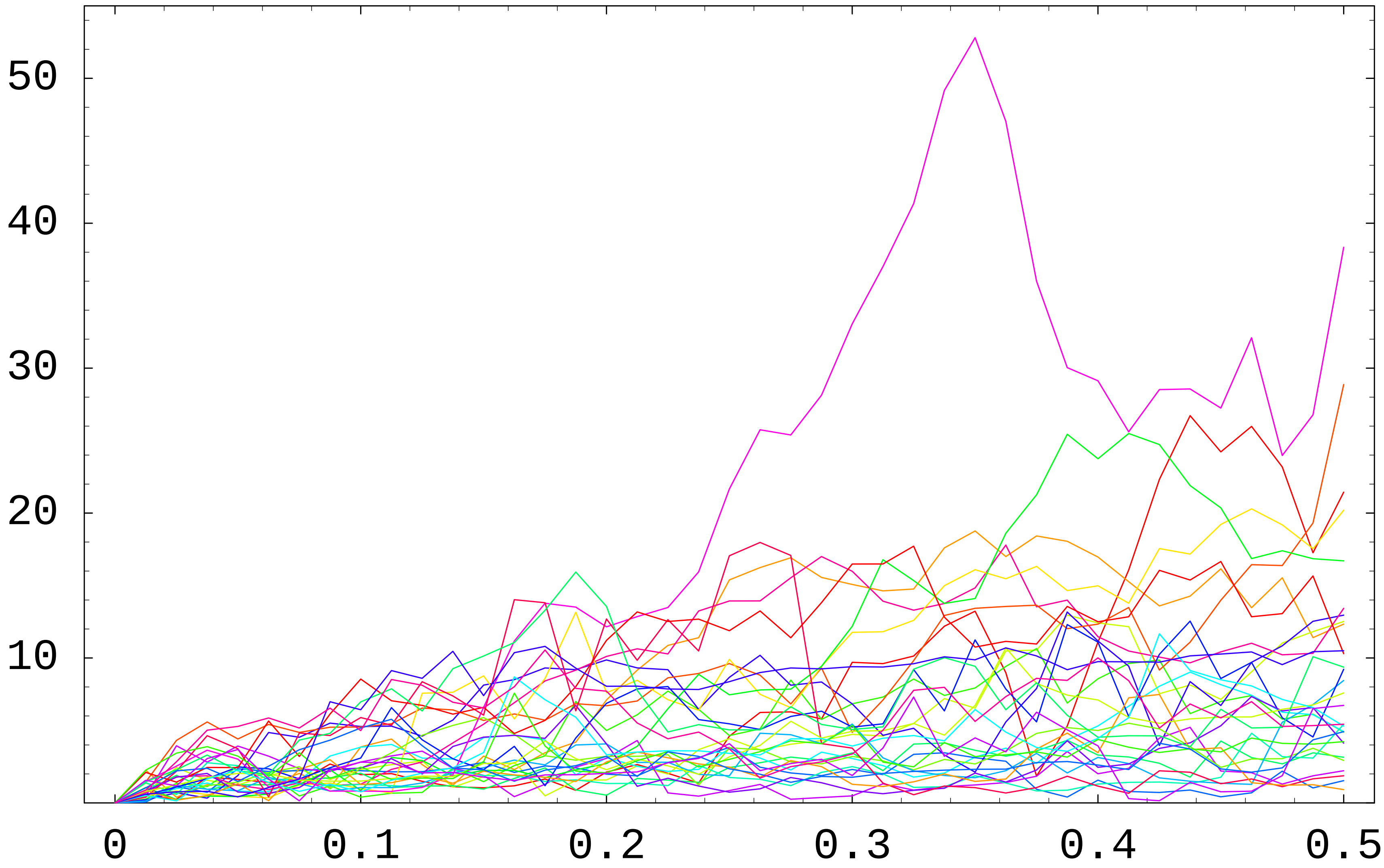


Figure 16.1: Simulated outcomes for the intrinsic value $(S_t^{0,x^1} - K)^+$, $t \in [0, T]$.

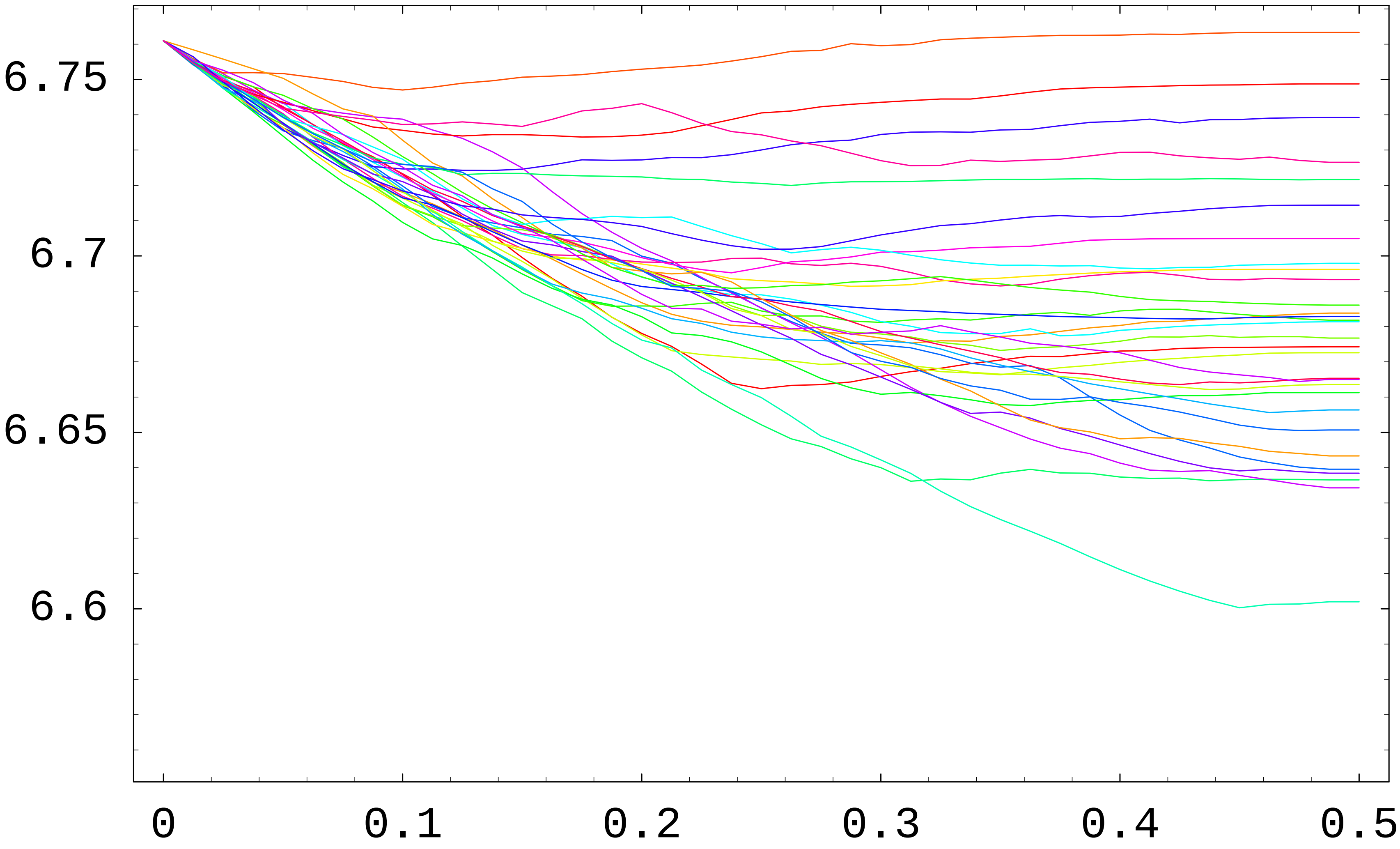


Figure 16.2: Simulated outcomes for the estimator $\bar{Z}_t,\ t \in [0, T]$.

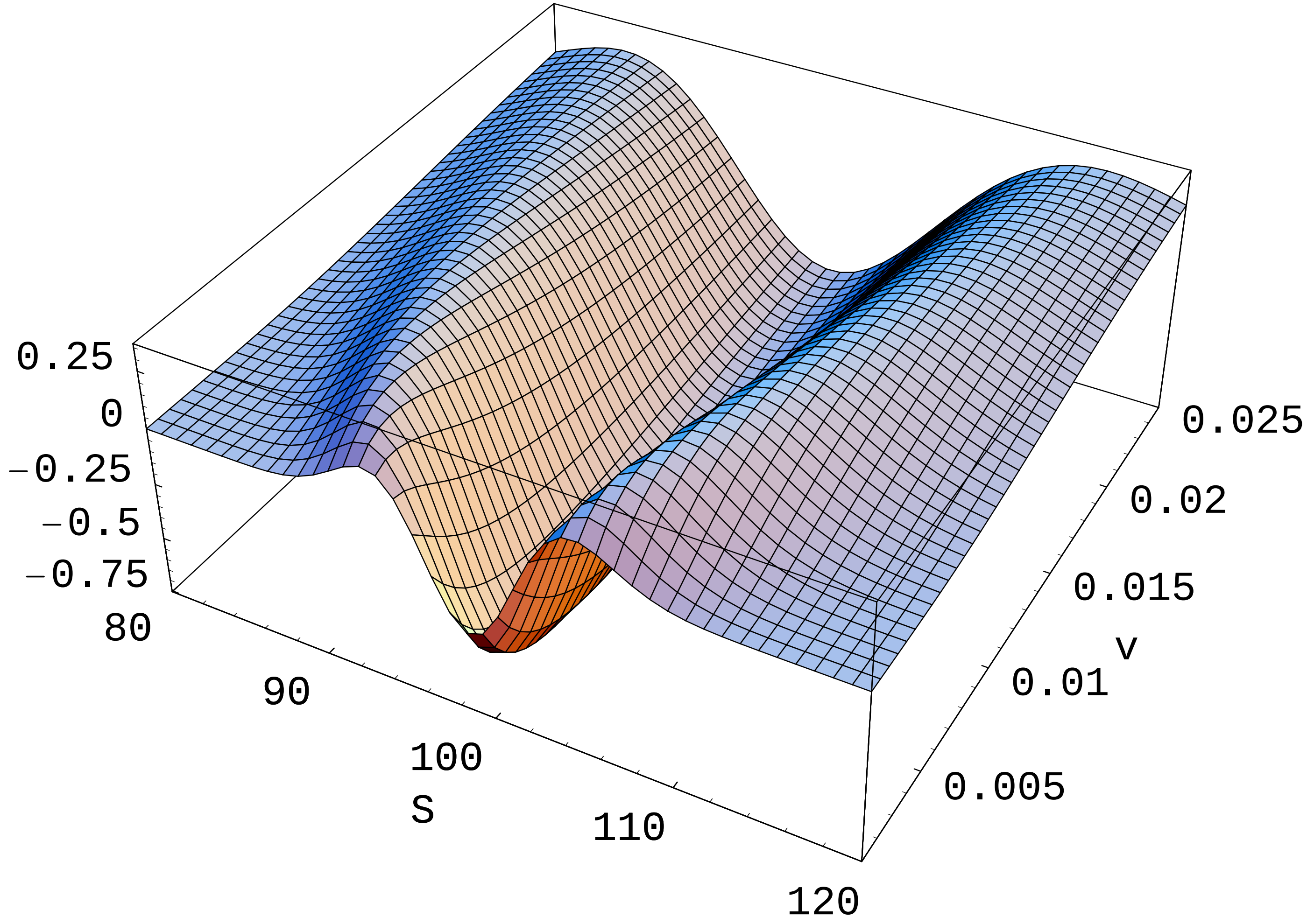


Figure 16.3: Diffusion operator values $(\boldsymbol{L} - \bar{\boldsymbol{L}}^0)\,\bar{\boldsymbol{u}}$ as a function of asset price $\boldsymbol{S}$ and squared volatility $\boldsymbol{v}$.

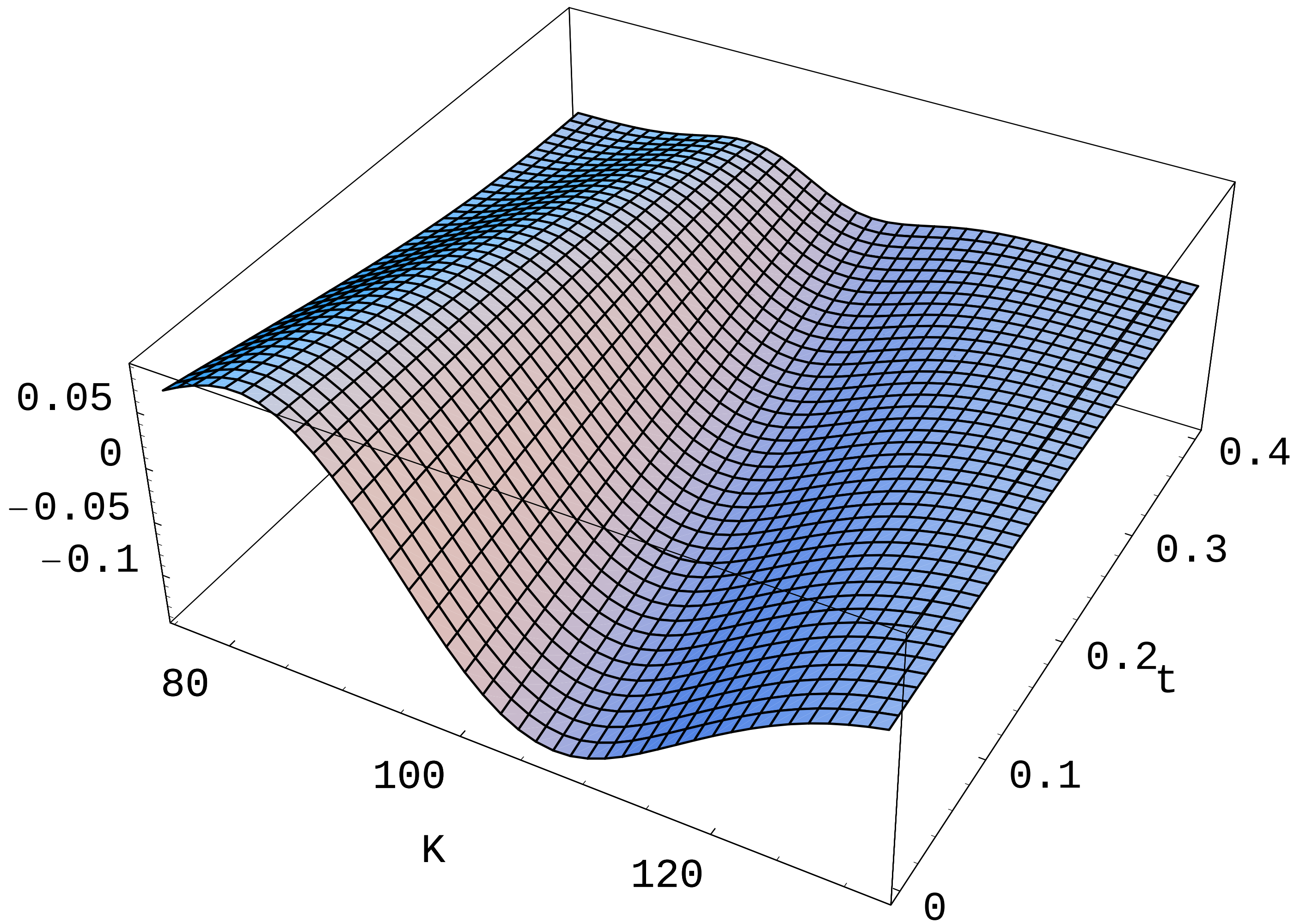


Figure 16.4: Price differences between the Heston and corresponding Black-Scholes model as a function of strike K and time t.

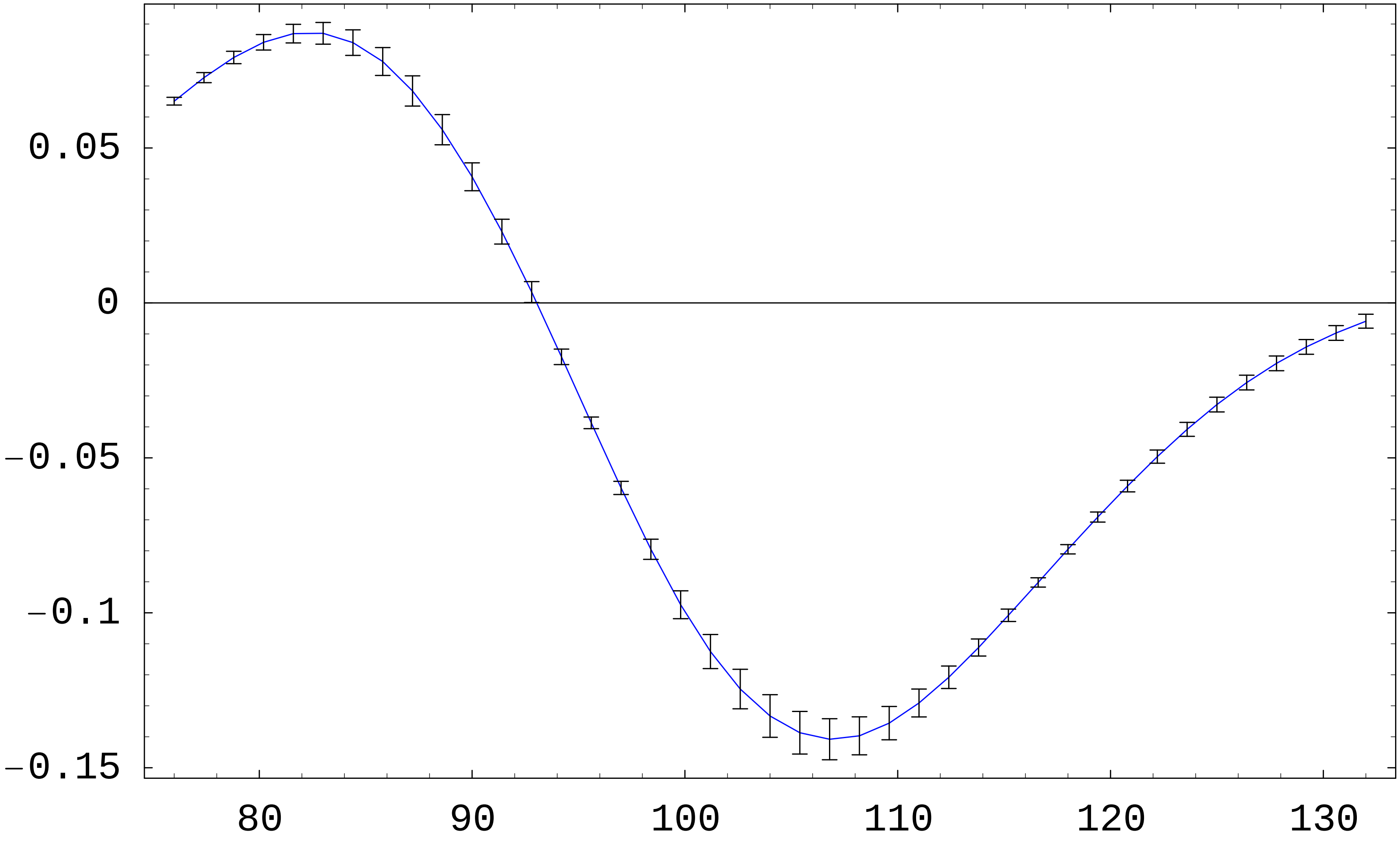


Figure 16.5: Prices and corresponding error bounds as a function of strike K.

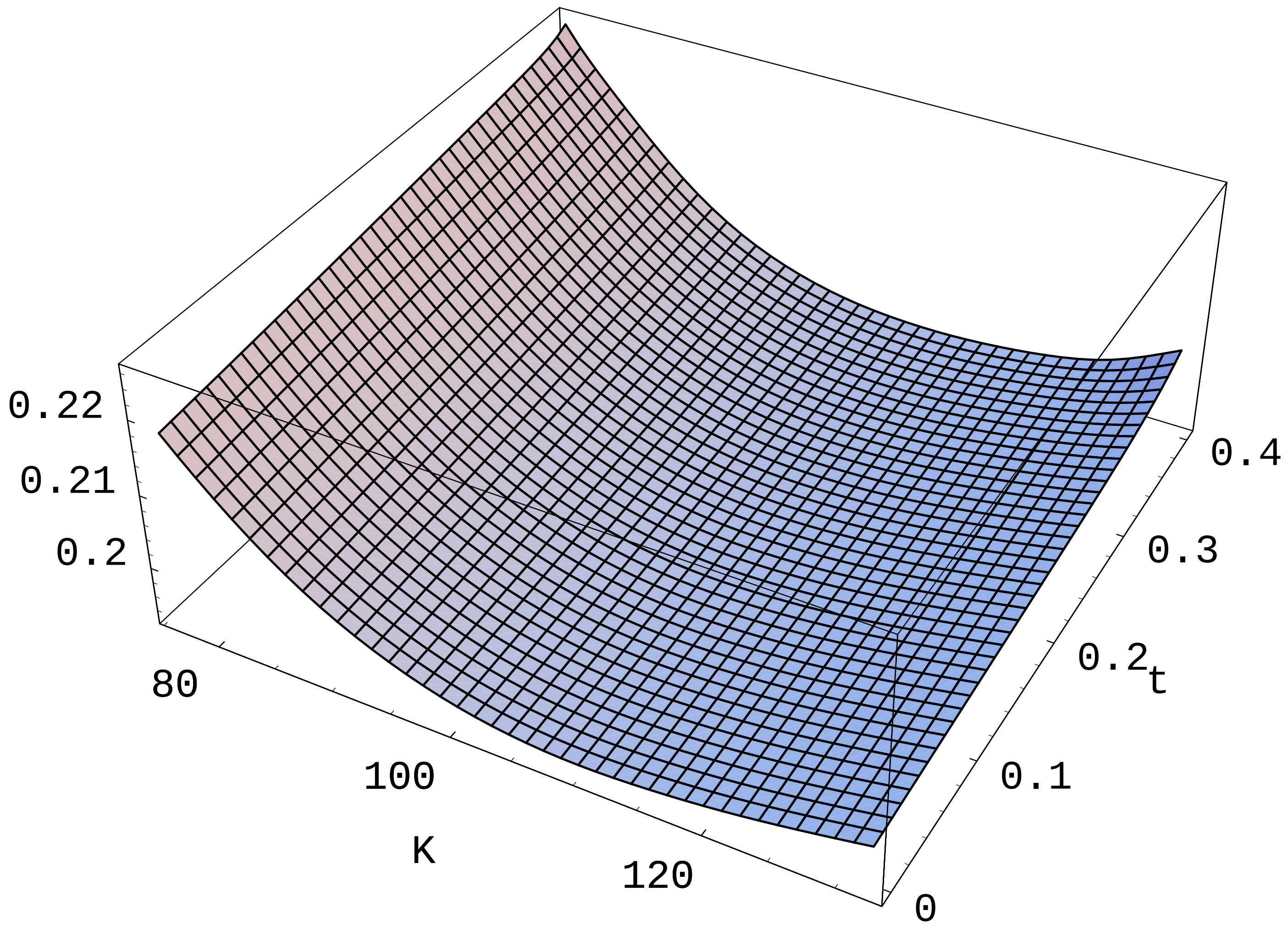


Figure 16.6: Implied volatility term structure for the Heston model.

Exercises of Chapter 16

16.1 Construct an antithetic variance reduction method, where you have to simulate the expectation $E(1 + Z + \frac{1}{2}(Z)^2)$ for a standard Gaussian random variable $Z \sim N(0,1)$. For this purpose combine the raw Monte Carlo estimate

$$V_N^+ = \frac{1}{N} \sum_{k=1}^{N} \left(1 + Z(\omega_k) + \frac{1}{2} \left(Z(\omega_k)\right)^2\right)$$

that uses outcomes of Z with the antithetic one

$$V_N^- = \frac{1}{N} \sum_{k=1}^{N} \left(1 - Z(\omega_k) + \frac{1}{2} \left(-Z(\omega_k)\right)^2\right)$$

that uses in parallel the same outcomes with a negative sign. Demonstrate the variance reduction that can be achieved for the estimator

$$\hat{V}_N = \frac{1}{2} \left(V_N^+ + V_N^-\right)$$

instead of using only V_N^+. Is $\hat{V}_N$ an unbiased estimator?

16.2 For calculating the expectation $E((1 + Z + \frac{1}{2}\,(Z)^2))$ use the control variate

$$V_N^* = \frac{1}{N} \sum_{k=1}^{N} (1 + Z(\omega_k))\,.$$

Analyze the variance reduction that can be achieved by the estimate

$$\tilde{V}_N = V_N^+ + \alpha(\gamma - V_N^*)$$

for $\alpha \in \Re$. For which choice of $\gamma \in \Re$ one obtains an unbiased estimate? For which $\alpha \in \Re$ one achieves the minimum variance ?

16.3 Check for the measure transformation variance reduction method and $g(z) = z^2$ that $g(\tilde{X}_T^{0,x})\,\frac{\theta_T}{\theta_0}$ is non-random and equals

$$u(t, x) = E\left(g\left(X_T^{0,x}\right) \,\middle|\, \mathcal{A}_t\right)$$

for $t = 0$, assuming the linear SDEs

$$dX_T^{0,x} = \alpha\, X_T^{0,x}\, dt + \beta\, X_T^{0,x}\, dW_t$$

and

$$d\tilde{X}_T^{0,x} = \alpha\, \tilde{X}_T^{0,x}\, dt + \beta\, \tilde{X}_T^{0,x}\, d\tilde{W}_t$$

for $t \in [0, T]$ with $X_T^{0,x} = \tilde{X}_T^{0,x} = x_0$, where

$$d\tilde{W}_t = dW_t - d(t, \tilde{X}_t^{0,x})\, dt$$

and

$$d\theta_t = \theta_t\, d(t, \tilde{X}_t^{0,x})\, dW_t$$

with

$$d(t, \tilde{X}_t^{0,x}) = -\frac{\beta\, \tilde{X}_t^{0,x}}{u(t, \tilde{X}_t^{0,x})}\, \frac{\partial u(t, \tilde{X}_t^{0,x})}{\partial x}.$$

16.4 Formulate a drift implicit simplified weak Euler scheme for the two-factor Heston model

$$
\begin{aligned}
dX_t &= (r_d - r_f)\, X_t + k\,\sqrt{v_t}\, X_t\, dW^1_t \\
dv_t &= \kappa\,(v_t - \bar{v})\, dt + p\,\sqrt{v_t}\left(\varrho\, dW^1_t + \sqrt{1-\varrho^2}\, dW^2_t\right),
\end{aligned}
$$

for $t \in [0, T]$ with $t \in [0, T^*]$ $X_0 > 0$ and $v_0 > 0$, where W^1 and W^2 are independent standard Wiener processes.

16.5 Consider the Heston model of Exercise 16.4 for $\varrho = 0$. Make also the diffusion term in the X component of the simplified weak Euler scheme implicit and correct appropriately the drift term in the resulting scheme.

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