The Affine Nature of Aggregate Wealth Dynamics

Eckhard Platen
University of Technology Sydney

Joint work with Renata Rendek
Publications:


◊ Platen, E. and Rendek, R. (2012b) **The Affine Nature of Aggregate Wealth Dynamics**.
Research Outline

◊ conjecture normalized aggregate wealth dynamics
⇒ time transformed square root process

◊ Naive Diversification Theorem ⇒ equity index = proxy numéraire portfolio

◊ empirical stylized facts ⇒ falsify models
◊ ⇒ proposed realistic one factor, two component index model
◊ benchmark approach ⇒ realistic model outside classical theory
◊ exact, almost exact simulation ⇒ verify empirical facts, effects of estimation techniques etc.
Empirical Study of World Stock Indices

Index construction

Pl. & Rendek (2008):

![Graph showing stock indices from 28/08/76 to 14/01/04 with labels for WSI104s, EWI104s, DWI104s, and MCI104s.]
Results for log-returns of the EWI104s
Pl. & Rendek (2008)

<table>
<thead>
<tr>
<th></th>
<th>SGH</th>
<th>Student-(t)</th>
<th>NIG</th>
<th>Hyperbolic</th>
<th>VG</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma)</td>
<td>0.98</td>
<td>0.72</td>
<td>0.97</td>
<td>0.96</td>
<td>0.96</td>
</tr>
<tr>
<td>(\bar{\alpha})</td>
<td>0.00</td>
<td>0.97</td>
<td>0.97</td>
<td>0.72</td>
<td></td>
</tr>
<tr>
<td>(\lambda)</td>
<td>-2.16</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\nu)</td>
<td></td>
<td>4.33</td>
<td></td>
<td></td>
<td>1.49</td>
</tr>
<tr>
<td>(\ln(\mathcal{L}^*))</td>
<td>-285796.39</td>
<td>-285796.39</td>
<td>-286448.94</td>
<td>-287152.08</td>
<td>-287499.83</td>
</tr>
<tr>
<td>(L_n)</td>
<td>0.00000004</td>
<td>1305.10</td>
<td>2711.38</td>
<td>3406.88</td>
<td></td>
</tr>
</tbody>
</table>

\(L_n = 0.0000004 < \chi^2_{0.001,1} \approx 0.000002\)
Approximating the Numéraire Portfolio by Naive Diversification
Pl. & Rendek (2012a)

**EWI114**: Equi-weighted index, 2000 constituents, 40 bp. transaction cost

![Graph showing logarithmic growth of indices](image)

**Sharpe Ratio**: 1.29 (EWI), 0.54 (MCI)
Naive Diversification Theorem

In a well-securitized financial market the sequence of benchmarked equi-weighted indices, with fractions given by

$$\pi^j_{\delta_{EWI\ell,t}} = \begin{cases} \frac{1}{\ell} & \text{for } j \in \{1, 2, \ldots, \ell\} \\ 0 & \text{otherwise,} \end{cases}$$

is a sequence of benchmarked approximate numéraire portfolios.
Statistics for the EWI114 with various transaction cost and reallocation terms

<table>
<thead>
<tr>
<th>Transaction cost</th>
<th>0</th>
<th>5</th>
<th>40</th>
<th>80</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reallocation terms</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final value</td>
<td>139338.64</td>
<td>130111.93</td>
<td>80543.07</td>
<td>46555.04</td>
<td>8988.23</td>
</tr>
<tr>
<td>Annualised average return</td>
<td>0.1979</td>
<td>0.1961</td>
<td>0.1834</td>
<td>0.1689</td>
<td>0.1254</td>
</tr>
<tr>
<td>Annualised volatility</td>
<td>0.1135</td>
<td>0.1135</td>
<td>0.1135</td>
<td>0.1135</td>
<td>0.1134</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td><strong>1.4205</strong></td>
<td>1.4046</td>
<td>1.2930</td>
<td>1.1654</td>
<td>0.7822</td>
</tr>
</tbody>
</table>

| Reallocation terms | 2     |       |       |       |       |
| Final value      | 124542.04 | 119369.00 | 88697.63 | 63166.73 | 22808.64 |
| Annualised average return | 0.1949 | 0.1938 | 0.1859 | 0.1770 | 0.1500 |
| Annualised volatility | 0.1134 | 0.1134 | 0.1134 | 0.1134 | 0.1135 |
| Sharpe ratio     | 1.3955 | 1.3856 | 1.3163 | 1.2369 | 0.9987 |

| Reallocation terms | 4     |       |       |       |       |
| Final value      | 111899.82 | 108230.16 | 85698.25 | 65628.82 | 29467.48 |
| Annualised average return | 0.1921 | 0.1912 | 0.1850 | 0.1780 | 0.1568 |
| Annualised volatility | 0.1135 | 0.1135 | 0.1134 | 0.1134 | 0.1134 |
| Sharpe ratio     | 1.3699 | 1.3622 | 1.3080 | 1.2459 | 1.0591 |
The Affine Nature of Aggregate Wealth Dynamics

Object: normalized units of wealth

Total wealth: \( Y_{\tau_i}^\Delta, \tau_i = i\Delta \)

Wealth unit value: \( \sqrt{\Delta} \)

Economic activity: until \( \tau_{i+1} \) "projects" consume \( \eta \Delta \) fraction of wealth; \( \beta \sqrt{\Delta} \) new units generated (branching process) on average

Mean for increment of aggregate wealth: \( \left( \beta - \eta Y_{\tau_i}^\Delta \right) \Delta \)
Assumption 1: Outcomes of "projects" are independent.

Assumption 2: each "project" generates in the period $[\tau_i, \tau_{i+1})$ wealth with variance $\nu^2 \Delta^3_2$
Number of wealth units: \( \frac{Y^\Delta_{\tau_i}}{\sqrt{\Delta}} \)

**Then:** the variance of the increment of the aggregate wealth is \( \nu^2 Y^\Delta_{\tau_i} \Delta \)

for \( \Delta \to 0 \)

\[
Y^\Delta_{\tau_{i+1}} - Y^\Delta_{\tau_i} = \left( \beta - \eta Y^\Delta_{\tau_i} \right) \Delta + \nu \sqrt{Y^\Delta_{\tau_i}} \Delta W_{\tau_i}
\]

\[E(\Delta W_{\tau_i}) = 0, \quad E((\Delta W_{\tau_i})^2) = \Delta\]

conjectures drift and diffusion terms
Week convergence to the square root process: Kleoden & Pl. (1999), Alfonsi (2005), Diop (2003) parameter reduction arises: $\beta = \eta = \nu = 1$

$$dY_{\tau t} = (1 - Y_{\tau t}) d\tau_t + \sqrt{Y_{\tau t}} dW_{\tau t}$$

Quadratic Variation:

$$[Y_{\tau \cdot}]_t = \int_0^t Y_{\tau s} d\tau_s = \int_0^t Y_{\tau s} M_s ds$$

Market Activity:

$$M_t = \frac{d\tau_t}{dt}$$

Integrated Normalized Index:

$$M \int_0^t Y_{\tau s} ds \approx [Y_{\tau \cdot}]_t$$
Quadratic variation and integrated normalized S&P500 monthly data, calendar time

\[ M \approx 0.0178 \]
average long term fit
Market Activity: $M_t = \frac{d\tau_t}{dt}$ from model
Quadratic variation and integrated normalized S&P500 monthly data, $\tau$-time
Stylized Empirical Facts

- falsify potential models, Popper (1959)
- TOTMKWD in 26 currency denominations

about 1000 years of daily data
(i) uncorrelated log-returns

Fig. 2: Average autocorrelation function for log-returns
(ii) correlated absolute log-returns

Fig. 3: Average autocorrelation function for absolute log-returns
(iii) Student-\( t \) distributed log-returns

Fig. 4: Logarithm of empirical density of normalized log-returns with Student-\( t \) density
(iv) volatility clustering

Fig. 5: Estimated volatility
(v) long term exponential growth

Fig. 6: Logarithm of index with trend line
(vi) leverage effect

Fig. 7: Logarithms of normalized index and its volatility
(vii) extreme volatility at major downturns

Fig. 7: Logarithms of normalized index and its volatility
⇒ Discounted Index Model

\[ S_t = A_{\tau_t} (Y_{\tau_t})^q, \]

\[ A_{\tau_t} = A \exp\{a\tau_t\} \]
Normalized index: \((Y_{\tau t})^q = \frac{S_t}{A_{\tau t}}\)

\[
dY_{\tau} = \left(\frac{\delta}{4} - \frac{1}{2} \left( \frac{\Gamma \left(\frac{\delta}{2} + q\right)}{\Gamma \left(\frac{\delta}{2}\right)} \right)^{\frac{1}{q}} Y_{\tau}\right) d\tau + \sqrt{Y_{\tau}} \, dW(\tau)
\]

Long term mean: \(\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (Y)^q_s \, ds = 1 \quad \text{P-a.s}\)
Market activity time: \( d\tau_t = M_t dt \)

Inverse of market activity:

\[
d\left( \frac{1}{M_t} \right) = \left( \frac{\nu}{4} \gamma - \epsilon \frac{1}{M_t} \right) dt + \sqrt{\frac{\gamma}{M_t}} dW_t,
\]

where

\[
dW(\tau_t) = \sqrt{\frac{d\tau_t}{dt}} dW_t = \sqrt{M_t} dW_t
\]

\(*\) only one \( W_t \)

\(*\) two component model
Discounted index SDE:

$$dS_t = S_t (\mu_t dt + \sigma_t dW_t)$$

Expected rate of return:

$$\mu_t = \left( \frac{a}{M_t} - \frac{q}{2} \left( \frac{\Gamma \left( \frac{\delta}{2} + q \right)}{\Gamma \left( \frac{\delta}{2} \right)} \right)^{\frac{1}{q}} + \left( \frac{\delta}{4} q + \frac{1}{2} q(q - 1) \right) \frac{1}{M_t Y_{\tau_t}} \right) M_t$$

Volatility:

$$\sigma_t = q \sqrt{\frac{M_t}{Y_{\tau_t}}}$$

Pl. & Rendek (2012c)
Benchmark Approach

\( \tilde{B}_t \) – benchmark savings account

\[
d\tilde{B}_t = \tilde{B}_t \left( (-\mu_t + \sigma^2_t) dt - \sigma_t dW_t \right)
\]

\[\sigma^2_t \leq \mu_t \Rightarrow \tilde{B}_t \text{ is an } (\mathcal{A}, P)\text{-supermartingale}
\Rightarrow \text{ no strong arbitrage; Pl. (2011)}\]
Assumptions:

**A1.** \( \delta = 2(q + 1) \)

**A2.** \( \frac{q}{2} \left( \frac{\Gamma(2q + 1)}{(q + 1)} \right)^{\frac{1}{q}} \leq a \)
**Pricing:** Real world conditional expectation of the benchmarked payoff $\Rightarrow$ benchmarked derivative price

**Real world pricing formula:**

$$V_t = S_t E\left(\frac{H_T}{S_T}|A_t\right)$$
Fitting the model to TOTMKWD

Step 1: Normalization of Index

\[ A_{\tau_t} \approx A \exp\left\{ \frac{4a\epsilon}{\gamma(\nu-2)} t \right\} \Rightarrow A = 65.21, \quad \frac{4a\epsilon}{\gamma(\nu-2)} \approx 0.048 \]
Step 2: Power $q$: $\delta \approx 4 \Rightarrow q = \frac{\delta}{2} - 1 \approx 1$
Affine nature $\Rightarrow q = 1$

Step 3: Observing Market Activity:

$$\frac{d[\sqrt{Y}]_{\tau_t}}{d\tau_t} = \frac{1}{4} \frac{d\tau_t}{dt} = \frac{M_t}{4}$$

$$\tilde{Q}_{t_i} \approx \frac{[\sqrt{Y}]_{\tau_{t_i+1}} - [\sqrt{Y}]_{\tau_{t_i}}}{t_{i+1} - t_i}$$

$$\tilde{Q}_{t_{i+1}} = \alpha \sqrt{t_{i+1} - t_i} \tilde{Q}_{t_i} + (1 - \alpha \sqrt{t_{i+1} - t_i}) \tilde{Q}_{t_i}, \quad \alpha = 0.92$$
Market activity: $M_t \approx 4\tilde{Q}_t$

$M_0 = 0.0175$
Step 4: Parameters $\gamma$:

$\gamma = 265.12$
Step 5: Parameters $\nu$ and $\epsilon$:

Step 6: Long Term Average Net Growth Rate $a$:

$\nu \approx 4, \epsilon \approx 2.18 \Rightarrow a = 2.55 \Rightarrow$ no strong arbitrage
Calculated Volatility

\[ \sigma_t \approx \sqrt{\frac{4 \tilde{Q}_t}{Y_{\tau t}}} \], average volatility: 11.9\%
$A = 52.09, \epsilon = 2.15, \gamma = 172.3, a = 1.5$

Model applies to proxies of numéraire portfolio
Simulation Study

Step 1: Market activity:

\[
\frac{1}{M_{t_{i+1}}} = \frac{\gamma (1 - e^{-\epsilon(t_{i+1} - t_i)})}{4\epsilon} \left( \chi^2_{3,i} + \left( \sqrt{\frac{4\epsilon e^{-\epsilon(t_{i+1} - t_i)}}{\gamma (1 - e^{-\epsilon(t_{i+1} - t_i)})}} \frac{1}{M_{t_i}} + Z_i \right)^2 \right)
\]
Step 2: \( \tau \)-time:

\[
\tau_{t_{i+1}} - \tau_{t_i} = \int_{t_i}^{t_{i+1}} M_s ds \approx M_{t_i}(t_{i+1} - t_i)
\]
Step 3: Normalized index:

\[ Y_{\tau t_{i+1}} = \frac{1 - e^{-(\tau t_{i+1} - \tau t_i)}}{4} \left( \chi_{3,i}^2 + \left( \frac{4e^{-(\tau t_{i+1} - \tau t_i)}}{1 - e^{-(\tau t_{i+1} - \tau t_i)}} Y_{\tau t_i} + Z_i \right)^2 \right) \]
Model recovers stylized empirical facts:

Model is difficult to falsify: Popper (1934)

1. Uncorrelated returns
2. Correlated absolute returns
### 3. Student-\( t \) distributed returns

<table>
<thead>
<tr>
<th>Simulation</th>
<th>Student-( t )</th>
<th>NIG</th>
<th>Hyperbolic</th>
<th>VG</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.008934</td>
<td>37.474149</td>
<td>102.719638</td>
<td>131.240780</td>
<td>4.012850</td>
</tr>
<tr>
<td>2</td>
<td>11.485226</td>
<td>11.175028</td>
<td>96.457136</td>
<td>132.916256</td>
<td>3.450916</td>
</tr>
<tr>
<td>3</td>
<td>0.000000</td>
<td>100.928524</td>
<td>244.190151</td>
<td>294.719960</td>
<td>2.734148</td>
</tr>
<tr>
<td>4</td>
<td>9.002421</td>
<td>35.759464</td>
<td>347.060676</td>
<td>331.014904</td>
<td>2.579009</td>
</tr>
<tr>
<td>5</td>
<td>8.767003</td>
<td>11.551178</td>
<td>121.190482</td>
<td>144.084964</td>
<td>3.170449</td>
</tr>
<tr>
<td>6</td>
<td>0.401429</td>
<td>60.570898</td>
<td>205.788160</td>
<td>252.591737</td>
<td>3.432435</td>
</tr>
<tr>
<td>7</td>
<td>12.239056</td>
<td>4.354888</td>
<td>46.411554</td>
<td>78.273485</td>
<td>3.957696</td>
</tr>
<tr>
<td>8</td>
<td>1.693411</td>
<td>23.910523</td>
<td>94.408789</td>
<td>130.623174</td>
<td>3.849691</td>
</tr>
<tr>
<td>9</td>
<td>1.232454</td>
<td>47.830407</td>
<td>202.073144</td>
<td>237.168411</td>
<td>3.236322</td>
</tr>
<tr>
<td>10</td>
<td>0.000000</td>
<td>43.037206</td>
<td>128.807757</td>
<td>162.582353</td>
<td>3.774957</td>
</tr>
<tr>
<td>11</td>
<td>0.433645</td>
<td>47.782681</td>
<td>172.736397</td>
<td>208.847632</td>
<td>3.431803</td>
</tr>
<tr>
<td>12</td>
<td>0.000000</td>
<td>56.019354</td>
<td>146.077121</td>
<td>185.624888</td>
<td>3.899403</td>
</tr>
<tr>
<td>13</td>
<td>7.137154</td>
<td>48.219756</td>
<td>579.922931</td>
<td>477.383441</td>
<td>2.293363</td>
</tr>
<tr>
<td>14</td>
<td>5.873948</td>
<td>16.515390</td>
<td>107.770531</td>
<td>135.508299</td>
<td>3.388307</td>
</tr>
<tr>
<td>15</td>
<td>0.000000</td>
<td>54.718046</td>
<td>184.112794</td>
<td>217.304105</td>
<td>3.402049</td>
</tr>
<tr>
<td>16</td>
<td>6.982560</td>
<td>3.991610</td>
<td>29.192198</td>
<td>47.105125</td>
<td>4.268740</td>
</tr>
<tr>
<td>17</td>
<td>2.966916</td>
<td>22.914863</td>
<td>108.513143</td>
<td>138.044416</td>
<td>3.553629</td>
</tr>
<tr>
<td>18</td>
<td>0.000000</td>
<td>52.066364</td>
<td>129.790856</td>
<td>160.373085</td>
<td>3.959605</td>
</tr>
<tr>
<td>19</td>
<td>0.006909</td>
<td>39.568695</td>
<td>111.398645</td>
<td>143.914350</td>
<td>3.982892</td>
</tr>
<tr>
<td>20</td>
<td>0.000001</td>
<td>56.845664</td>
<td>169.915512</td>
<td>211.260626</td>
<td>3.651091</td>
</tr>
<tr>
<td>21</td>
<td>1.1674578</td>
<td>17.681088</td>
<td>61.710576</td>
<td>90.679738</td>
<td>4.265834</td>
</tr>
<tr>
<td>22</td>
<td>14.010840</td>
<td>3.279722</td>
<td>47.433693</td>
<td>73.789313</td>
<td>3.770825</td>
</tr>
<tr>
<td>23</td>
<td>11.198940</td>
<td>12.074044</td>
<td>114.888817</td>
<td>143.800553</td>
<td>3.257146</td>
</tr>
<tr>
<td>24</td>
<td>0.455557</td>
<td>27.676102</td>
<td>86.841704</td>
<td>114.452947</td>
<td>4.006528</td>
</tr>
</tbody>
</table>
4. Volatility clustering
5. Long term exponential growth

\[ 4.16 + 0.05t \]
6. Leverage effect
7. Extreme volatility at major market downturns
Conclusions:

- equity index model: 3 initial parameters, 3 structural parameters and 1 Wiener process (nondiversifiable uncertainty)

- model recovers 7 stylized empirical facts

- long dated derivative pricing under benchmark approach

- leads outside classical theory