The Affine Nature of Aggregate Wealth Dynamics

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Joint work with Renata Rendek

Publications:

◇ Platen, E. and Rendek, R. (2012a) **Approximating the Numéraire Portfolio by Naive Diversification**. Journal of Asset Management, 13(1), 34-50.

◇ Platen, E. and Rendek, R. (2008) **Empirical Evidence on Student-t Log-Returns of Diversified World Stock Indices**, Journal of Statistical Theory and Practice, Vol. 2, No. 2, 233-251.

◇ Platen, E. and Rendek, R. (2012b) **The Affine Nature of Aggregate Wealth Dynamics**.

Research Outline

$\diamond$ conjecture normalized aggregate wealth dynamics
$\Rightarrow$ time transformed square root process

$\diamond$ Naive Diversification Theorem $\Rightarrow$ equity index = proxy numéraire portfolio

$\diamond$ empirical stylized facts $\Rightarrow$ falsify models

$\diamond \Rightarrow$ proposed realistic one factor, two component index model

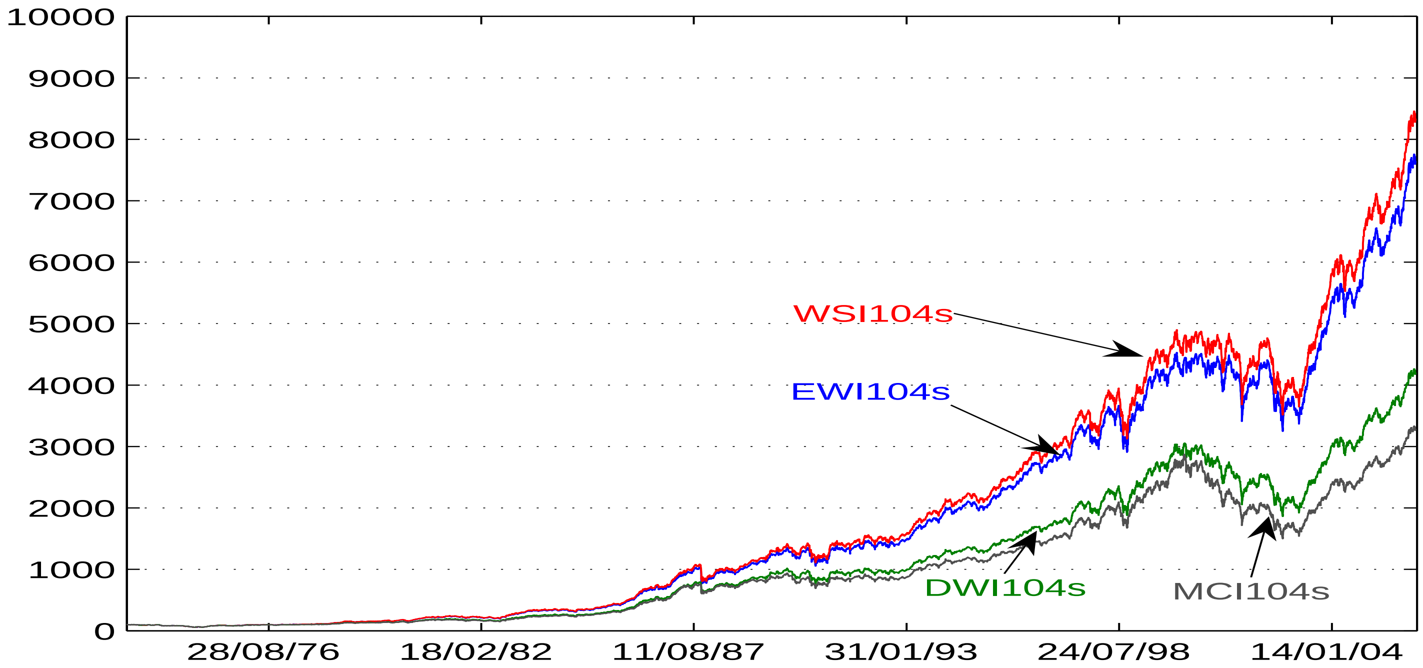
$\diamond$ benchmark approach $\Rightarrow$ realistic model outside classical theory

$\diamond$ exact, almost exact simulation $\Rightarrow$ verify empirical facts, effects of estimation techniques etc.

Empirical Study of World Stock Indices

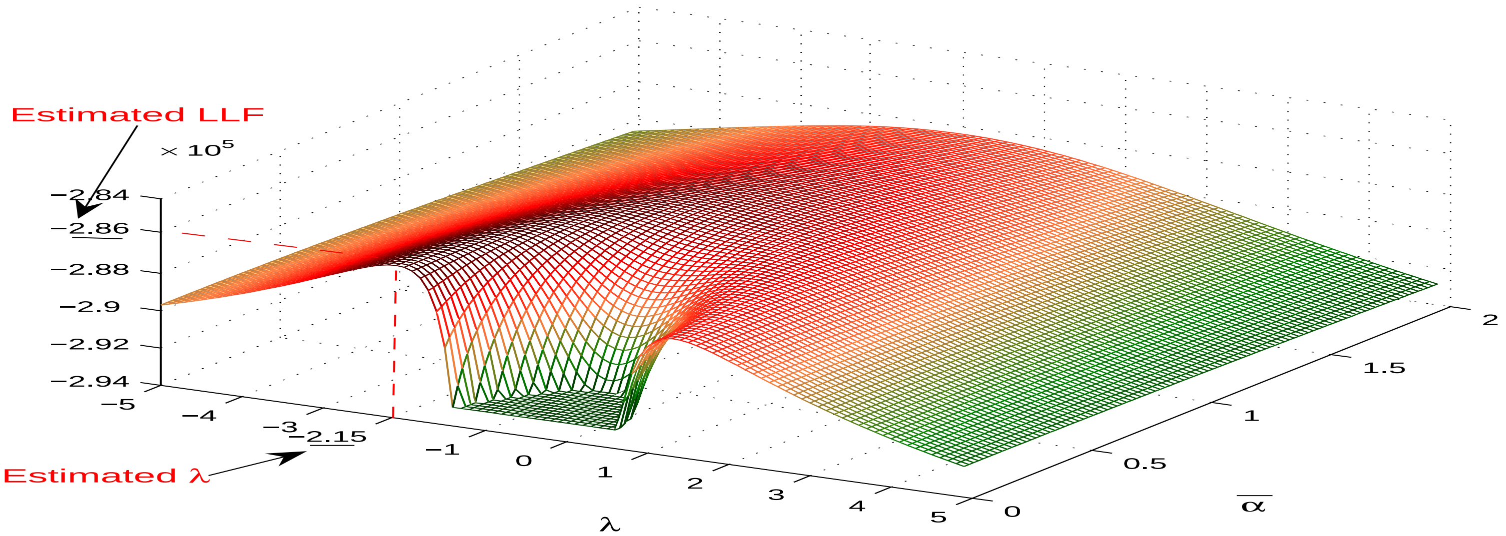
Index construction

Pl. & Rendek (2008):



Results for log-returns of the EWI104s

Pl. & Rendek (2008)



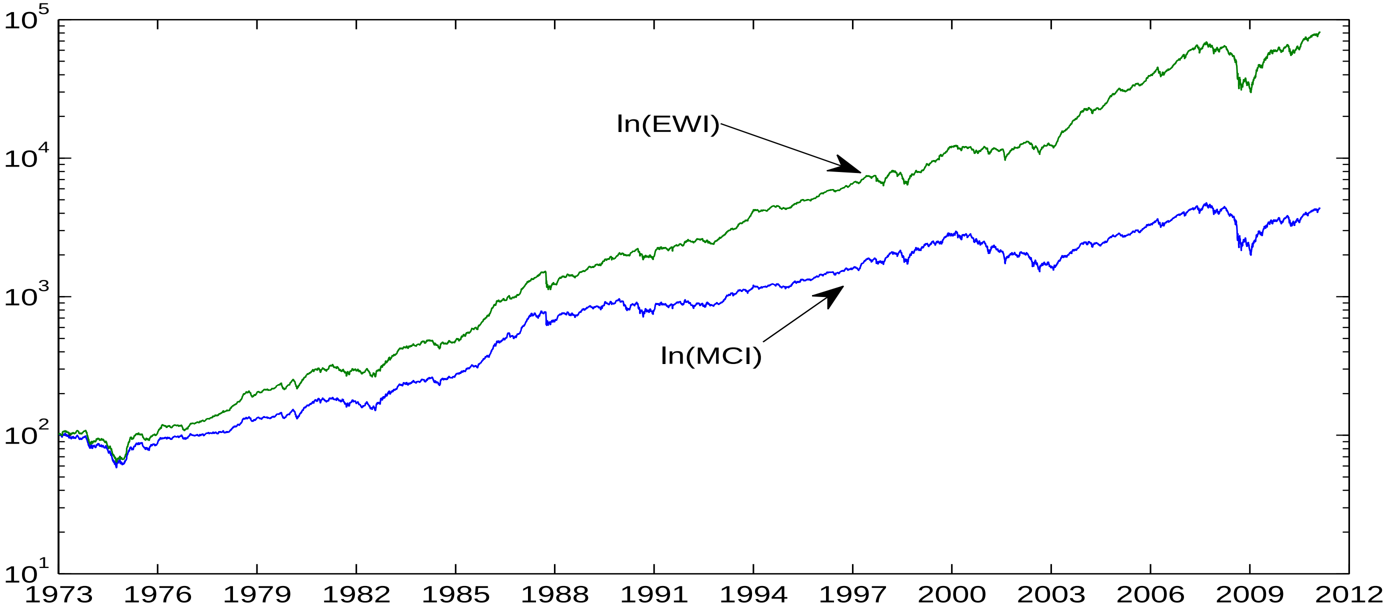
	SGH	Student-t	NIG	Hyperbolic	VG
σ	0.98	0.72	0.97	0.96	0.96
$\bar{\alpha}$	0.00		0.97	0.72	
λ	-2.16				1.49
ν		4.33			
$\ln(\mathcal{L}^*)$	-285796.39	-285796.39	-286448.94	-287152.08	-287499.83
L_n		**0.0000004**	1305.10	2711.38	3406.88

$L_n = 0.0000004 < \chi^2_{0.001,1} \approx 0.000002$

Approximating the Numéraire Portfolio by Naive Diversification

Pl. & Rendek (2012a)

EWI114: Equi-weighted index, 2000 constituents, 40 bp. transaction cost



Sharpe Ratio: 1.29 (EWI), 0.54 (MCI)

Naive Diversification Theorem

In a well-securitized financial market the sequence of benchmarked equi-weighted indices, with fractions given by

$$\pi^j_{\delta_{EWI\ell},t} = \begin{cases} \frac{1}{\ell} & \textit{for} \quad j \in \{1, 2, \ldots, \ell\} \\ 0 & \textit{otherwise}, \end{cases}$$

is a sequence of benchmarked approximate numéraire portfolios.

Statistics for the EWI114 with various transaction cost and reallocation terms

Transaction cost	0	5	40	80	200
Reallocation terms	1				
Final value	139338.64	130111.93	80543.07	46555.04	8988.23
Annualised average return	0.1979	0.1961	0.1834	0.1689	0.1254
Annualised volatility	0.1135	0.1135	0.1135	0.1135	0.1134
Sharpe ratio	**1.4205**	1.4046	1.2930	1.1654	0.7822
Reallocation terms	2				
Final value	124542.04	119369.00	88697.63	63166.73	22808.64
Annualised average return	0.1949	0.1938	0.1859	0.1770	0.1500
Annualised volatility	0.1134	0.1134	0.1134	0.1134	0.1135
Sharpe ratio	1.3955	1.3856	1.3163	1.2369	0.9987
Reallocation terms	4				
Final value	111899.82	108230.16	85698.25	65628.82	29467.48
Annualised average return	0.1921	0.1912	0.1850	0.1780	0.1568
Annualised volatility	0.1135	0.1135	0.1134	0.1134	0.1134
Sharpe ratio	1.3699	1.3622	1.3080	1.2459	1.0591

The Affine Nature of Aggregate Wealth Dynamics

Object: normalized units of wealth

Total wealth: $Y_{\tau_i}^{\Delta}$, $\tau_i = i\Delta$

Wealth unit value: $\sqrt{\Delta}$

Economic activity: until τ_{i+1} "projects" consume $\eta\Delta$ fraction of wealth; $\beta\sqrt{\Delta}$ new units generated (branching process) on average

Mean for increment of aggregate wealth: $\left(\beta - \eta Y_{\tau_i}^{\Delta}\right)\Delta$

Assumption 1: Outcomes of "projects" are independent.

Assumption 2: each "project" generates in the period $[\tau_i, \tau_{i+1})$ wealth with variance $v^2 \Delta^{\frac{3}{2}}$

Number of wealth units: $\frac{Y^{\Delta}_{\tau_i}}{\sqrt{\Delta}}$

Then: the variance of the increment of the aggregate wealth is $\upsilon^2 Y^{\Delta}_{\tau_i} \Delta$

$$\text{for} \quad \Delta \to 0 \qquad Y^{\Delta}_{\tau_{i+1}} - Y^{\Delta}_{\tau_i} = \left(\beta - \eta Y^{\Delta}_{\tau_i}\right) \Delta + \upsilon \sqrt{Y^{\Delta}_{\tau_i}} \Delta W_{\tau_i}$$

$E(\Delta W_{\tau_i}) = 0, \qquad E((\Delta W_{\tau_i})^2) = \Delta$

conjectures drift and diffusion terms

Week convergence to the square root process:
Kleoden & Pl. (1999), Alfonsi (2005), Diop (2003)
parameter reduction arises: $\beta = \eta = \upsilon = 1$

$$dY_{\tau_t} = (1 - Y_{\tau_t})\, d\tau_t + \sqrt{Y_{\tau_t}} dW_{\tau_t}$$

Quadratic Variation:

$$[Y_{\tau_\cdot}]_t = \int_0^t Y_{\tau_s} d\tau_s = \int_0^t Y_{\tau_s} M_s ds$$

Market Activity:

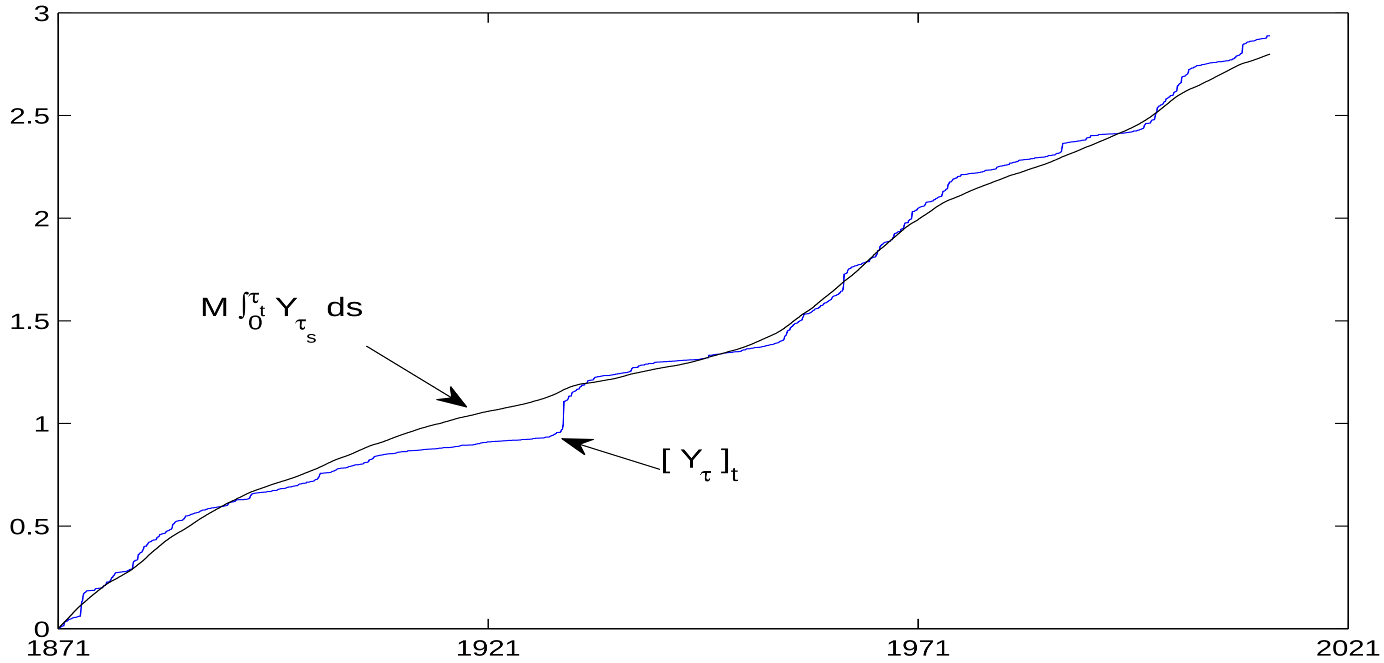
$$M_t = \frac{d\tau_t}{dt}$$

Integrated Normalized Index:

$$M \int_0^t Y_{\tau_s} ds \approx [Y_{\tau_\cdot}]_t$$

Quadratic variation and integrated normalized S&P500

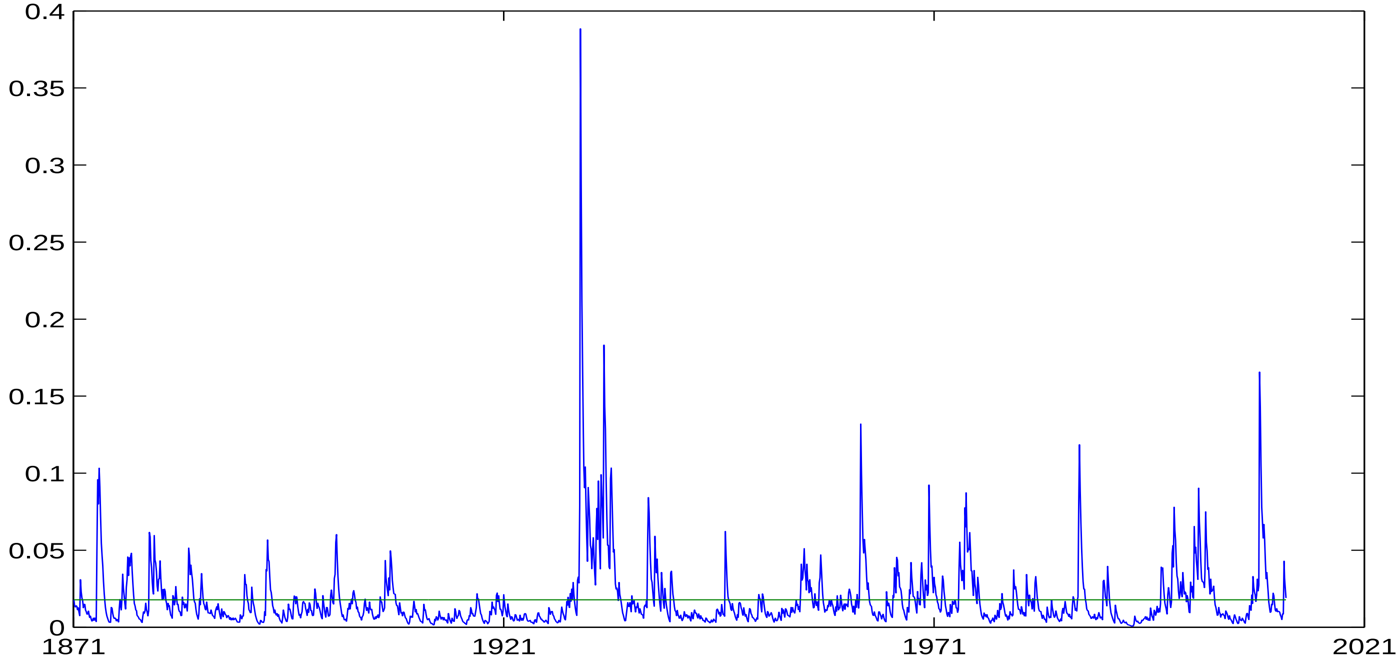
monthly data, calendar time



$M \approx 0.0178$

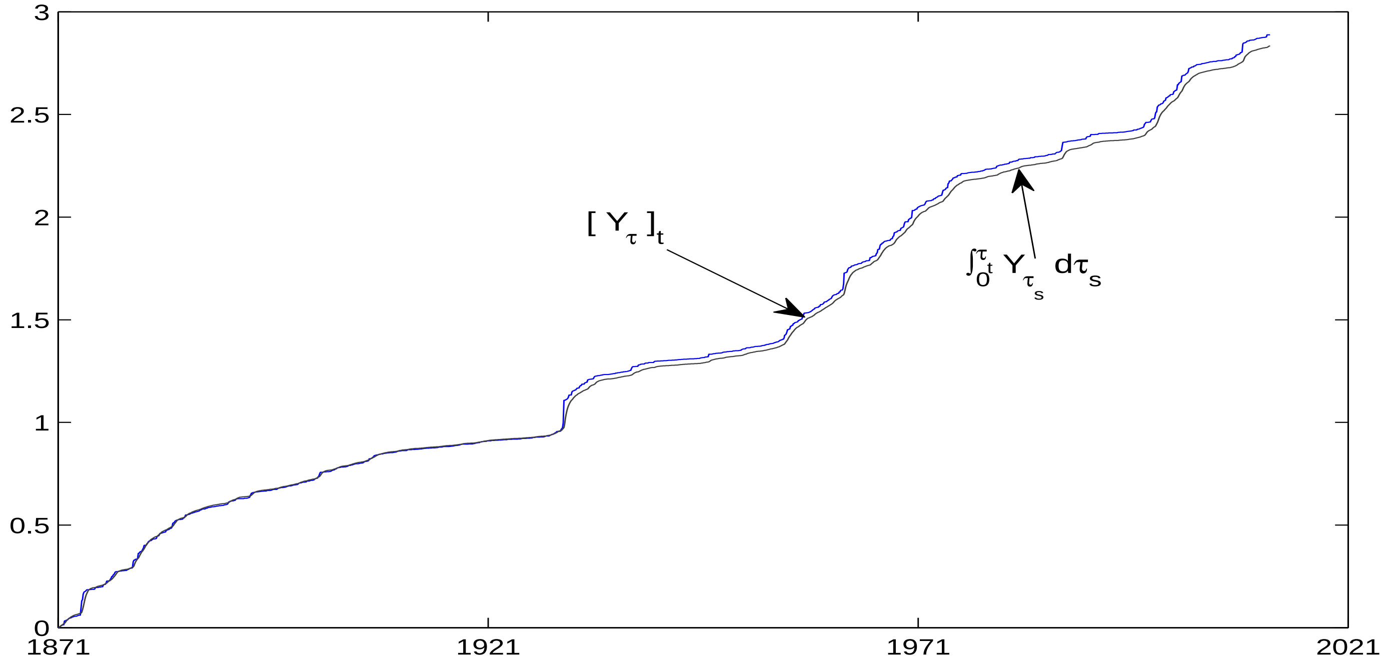
average long term fit

Market Activity: $M_t = \frac{d\tau_t}{dt}$ from model



Quadratic variation and integrated normalized S&P500

monthly data, τ-time



Stylized Empirical Facts

- falsify potential models, Popper (1959)
- TOTMKWD in 26 currency denominations

about 1000 years of daily data

(i) uncorrelated log-returns

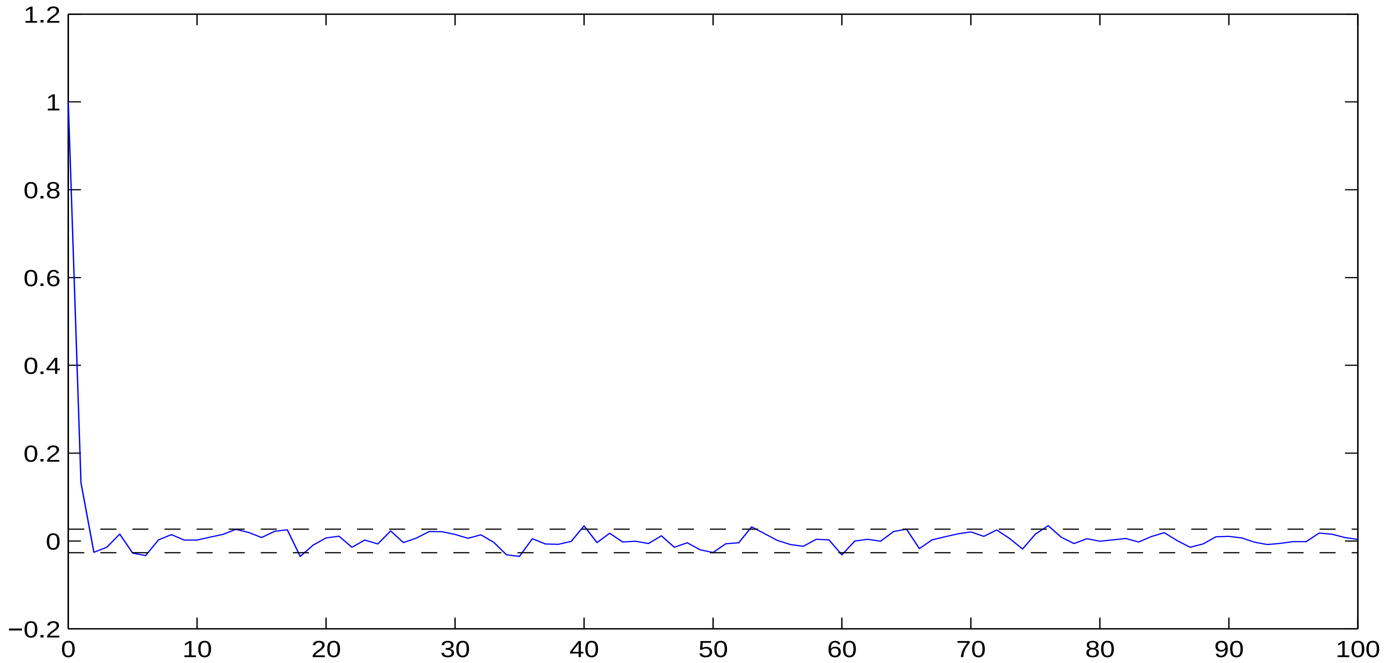


Fig. 2: Average autocorrelation function for log-returns

(ii) correlated absolute log-returns

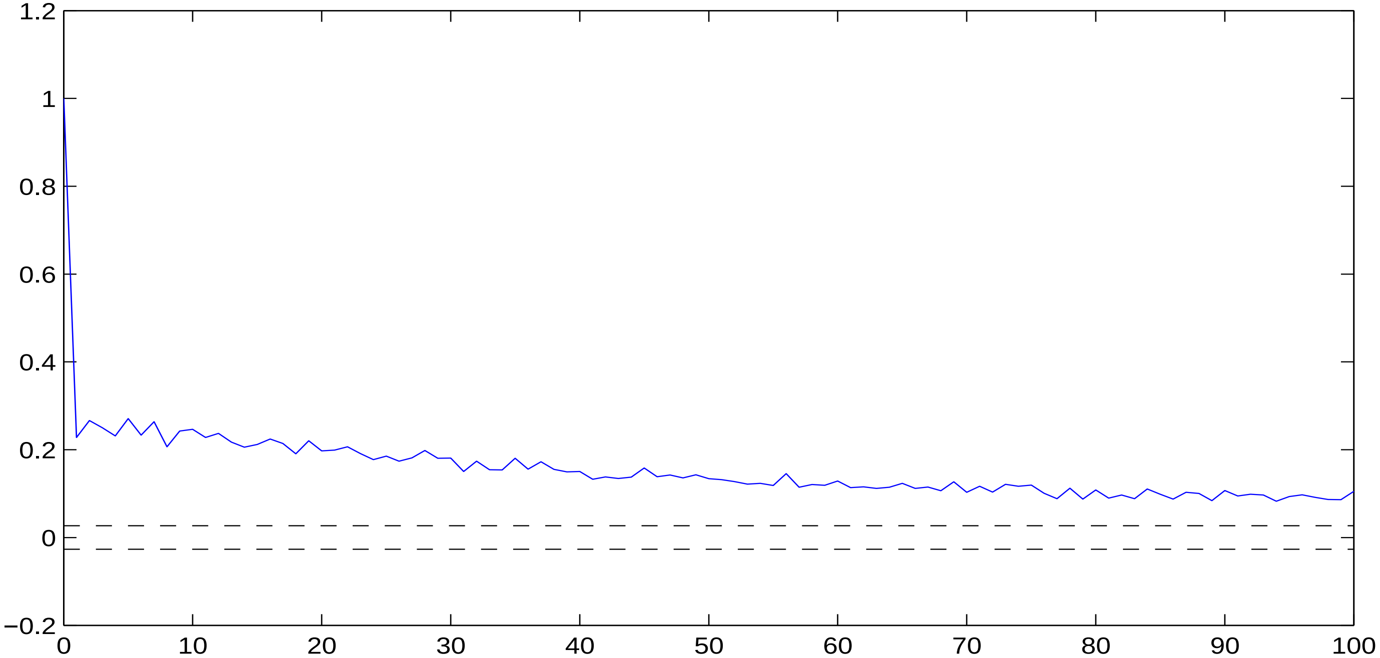


Fig. 3: Average autocorrelation function for absolute log-returns

(iii) Student-t distributed log-returns

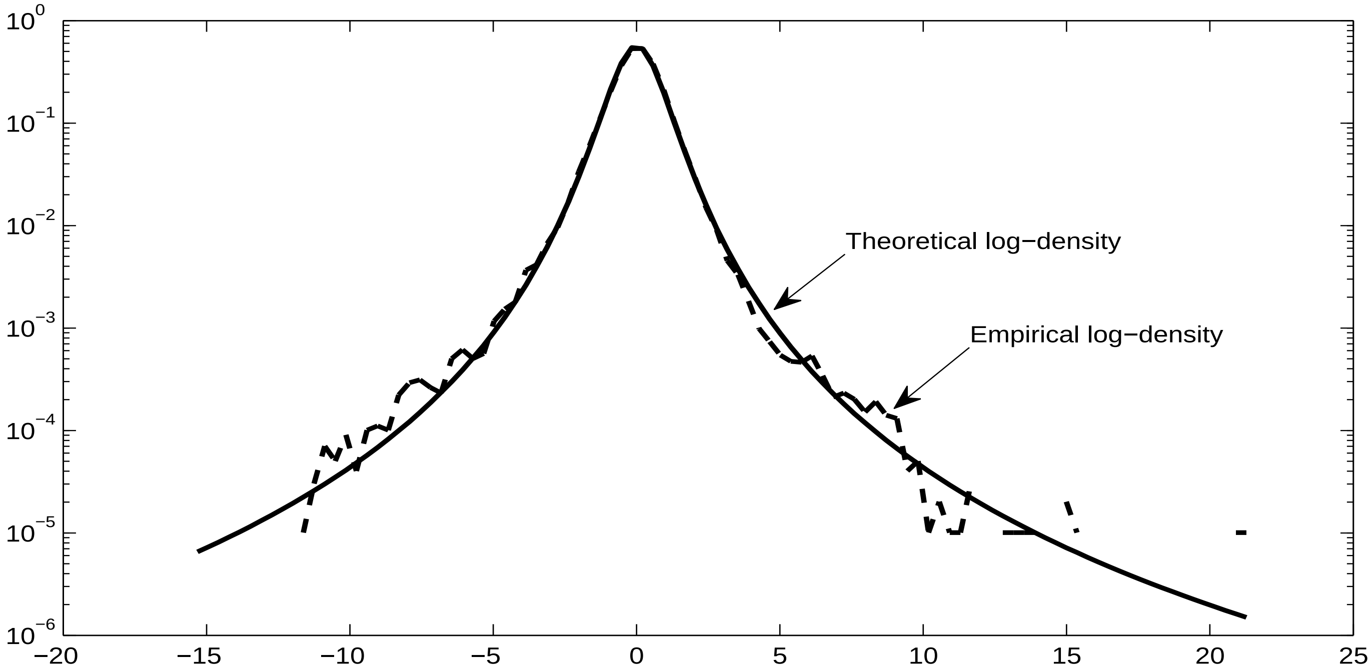


Fig. 4: Logarithm of empirical density of normalized log-returns with Student-t density

(iv) volatility clustering

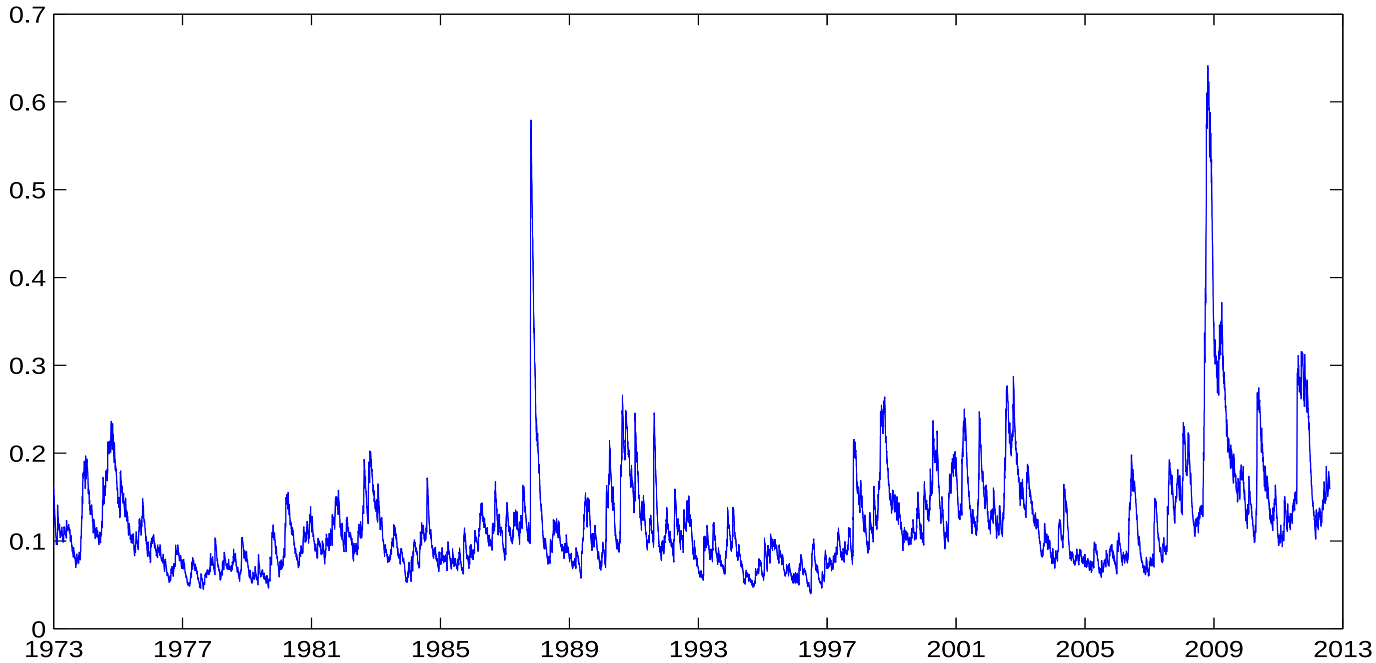


Fig. 5: Estimated volatility

(v) long term exponential growth

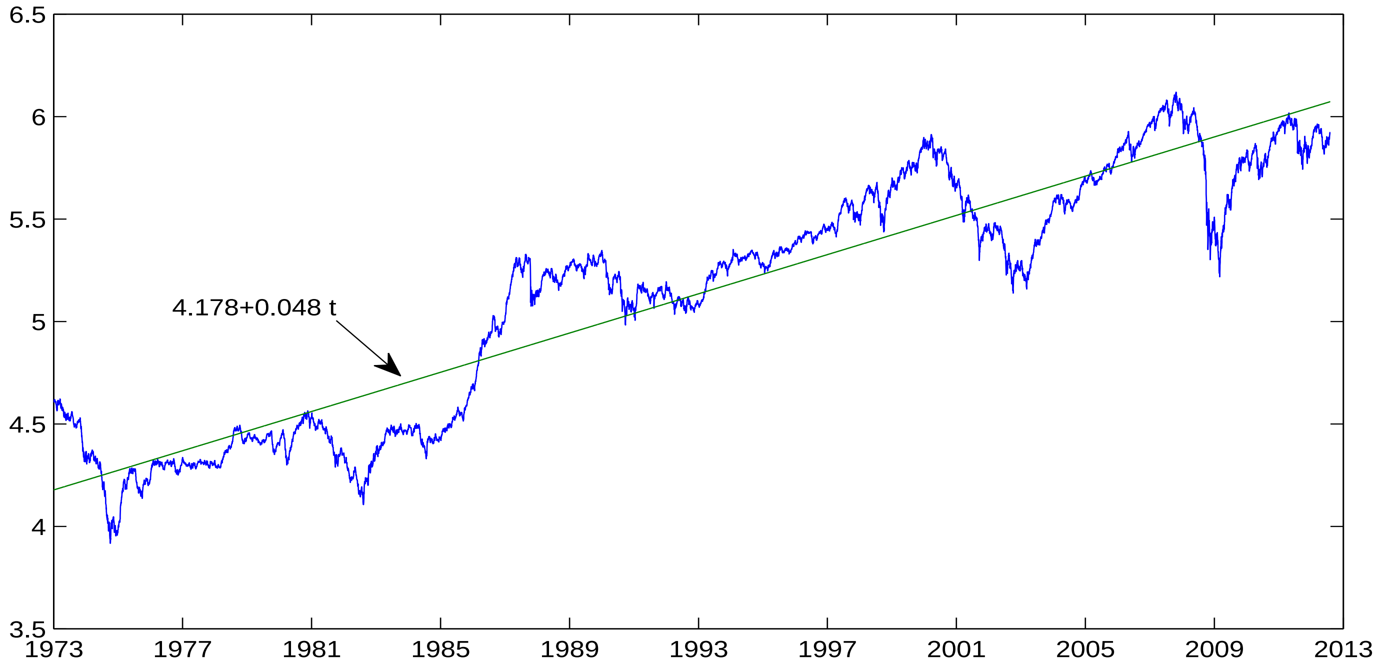


Fig. 6: Logarithm of index with trend line

(vi) leverage effect

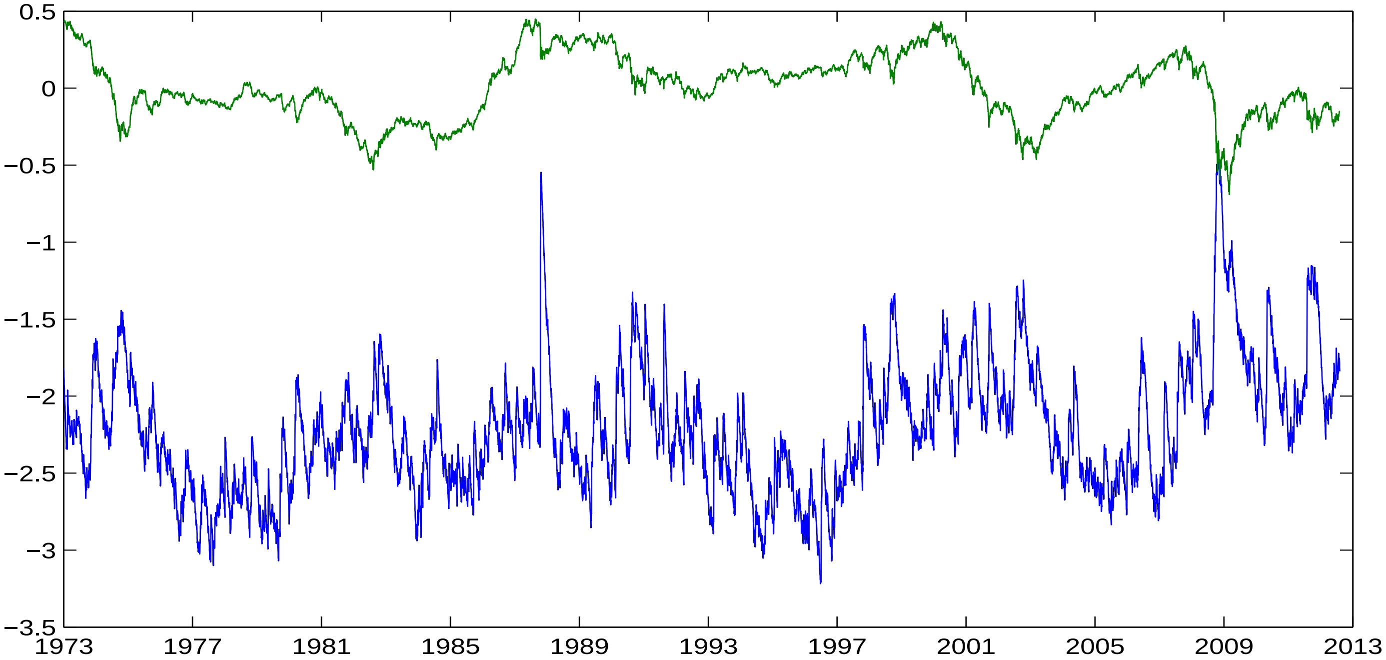


Fig. 7: Logarithms of normalized index and its volatility

(vii) extreme volatility at major downturns

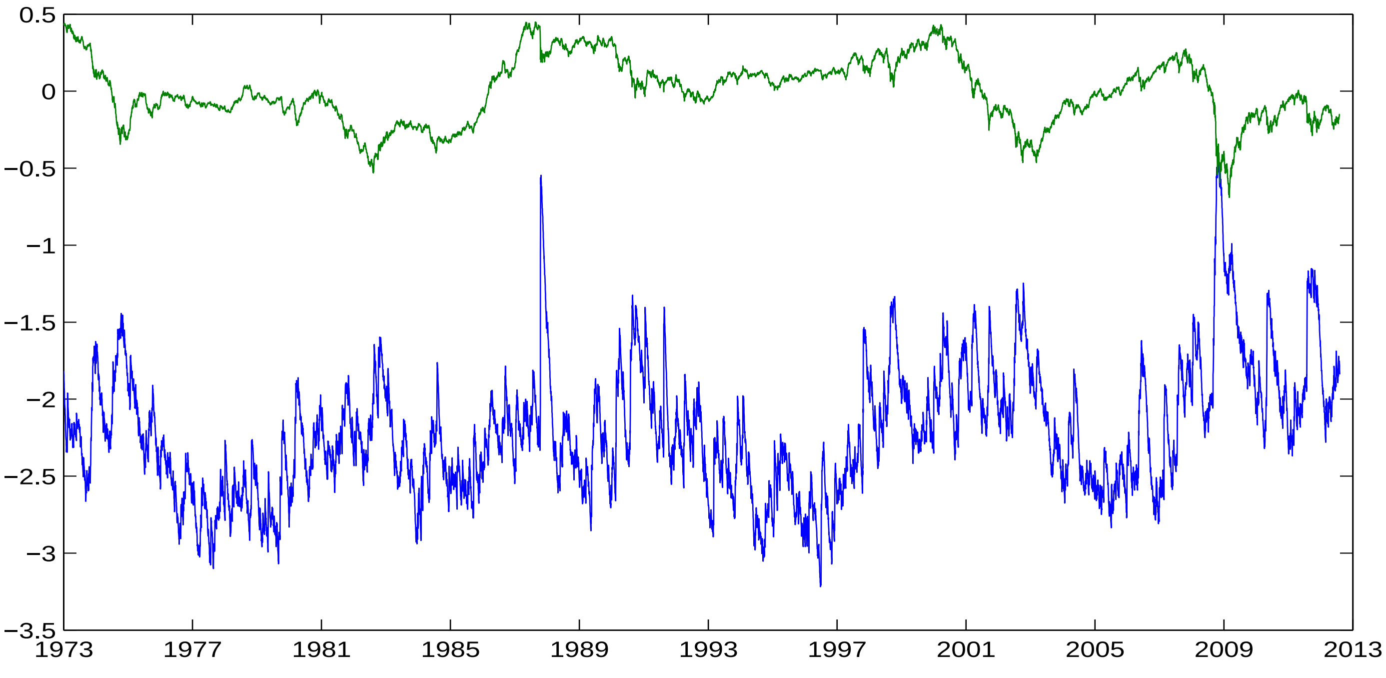


Fig. 7: Logarithms of normalized index and its volatility

$\Rightarrow$ **Discounted Index Model**

$$S_t = A_{\tau_t}(Y_{\tau_t})^q,$$

$$A_{\tau_t} = A \exp\{a\tau_t\}$$

Normalized index: $(Y_{\tau_t})^q = \frac{S_t}{A_{\tau_t}}$

$$dY_\tau = \left(\frac{\delta}{4} - \frac{1}{2} \left(\frac{\Gamma\left(\frac{\delta}{2} + q\right)}{\Gamma\left(\frac{\delta}{2}\right)} \right)^{\frac{1}{q}} Y_\tau \right) d\tau + \sqrt{Y_\tau}\; dW(\tau)$$

Long term mean: $\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (Y)_s^q \, ds = 1 \quad$ P-a.s

Market activity time: $d\tau_t = M_t dt$

Inverse of market activity:

$$d\left(\frac{1}{M_t}\right) = \left(\frac{\nu}{4}\gamma - \epsilon\frac{1}{M_t}\right)dt + \sqrt{\frac{\gamma}{M_t}}dW_t,$$

where

$$dW(\tau_t) = \sqrt{\frac{d\tau_t}{dt}}dW_t = \sqrt{M_t}dW_t$$

◇ only one W_t

◇ two component model

Discounted index SDE:

$$dS_t = S_t \left(\mu_t dt + \sigma_t dW_t\right)$$

Expected rate of return:

$$\mu_t = \left(\frac{a}{M_t} - \frac{q}{2} \left(\frac{\Gamma\left(\frac{\delta}{2} + q\right)}{\Gamma\left(\frac{\delta}{2}\right)} \right)^{\frac{1}{q}} + \left(\frac{\delta}{4} q + \frac{1}{2} q (q-1) \right) \frac{1}{M_t Y_{\tau_t}} \right) M_t$$

Volatility:

$$\sigma_t = q \sqrt{\frac{M_t}{Y_{\tau_t}}}$$

Pl. & Rendek (2012c)

Benchmark Approach

$\hat{B}_t$ – benchmark savings account

$$d\hat{B}_t = \hat{B}_t \left(\left(-\mu_t + \sigma_t^2 \right) dt - \sigma_t dW_t \right)$$

$$\begin{aligned} \sigma_t^2 \leq \mu_t \;\; &\Rightarrow \;\; \hat{B}_t \text{ is an } (\underline{\mathcal{A}}, P)\text{-supermartingale} \\ &\Rightarrow \;\; \text{no strong arbitrage; Pl. (2011)} \end{aligned}$$

Assumptions:

A1. $\delta = 2(q+1)$ **A2**. $\frac{q}{2}\left(\frac{\Gamma(2q+1)}{\Gamma(q+1)}\right)^{\frac{1}{q}} \leq a$

Pricing: Real world conditional expectation of the benchmarked payoff $\Rightarrow$ benchmarked derivative price

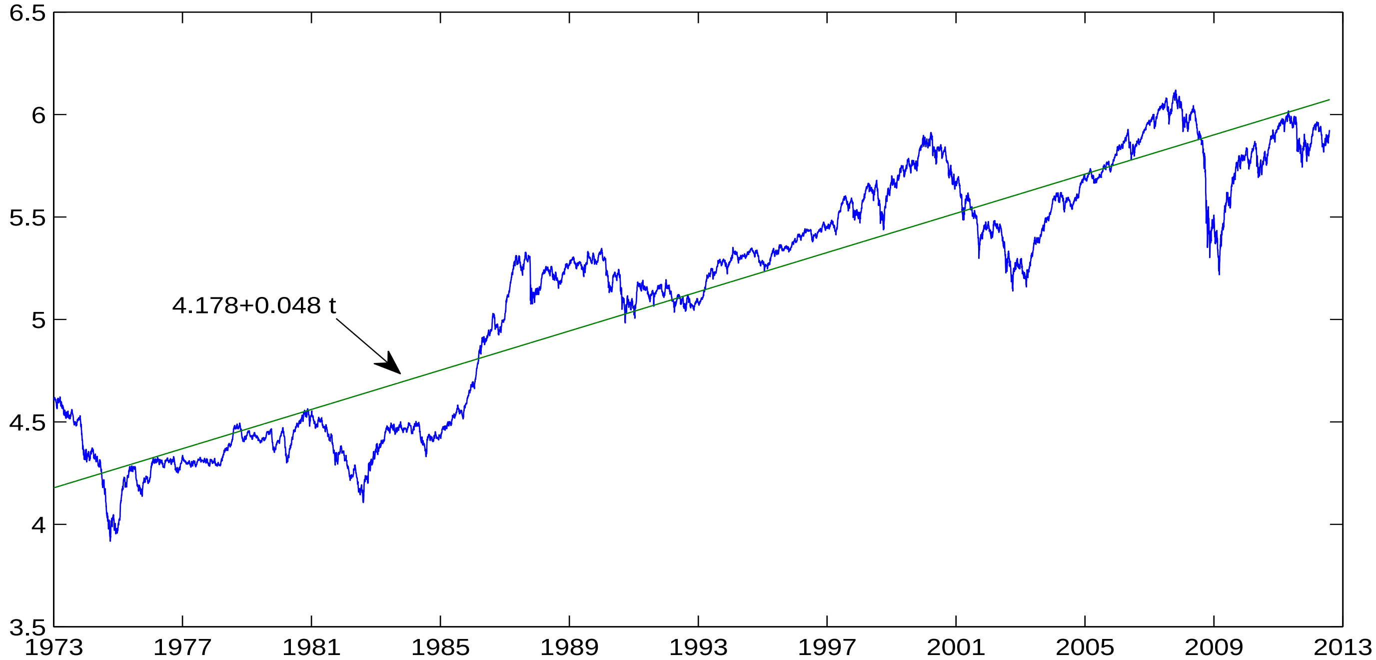
Real world pricing formula:

$$V_t = S_t E\left(\frac{H_T}{S_T} | \mathcal{A}_t\right)$$

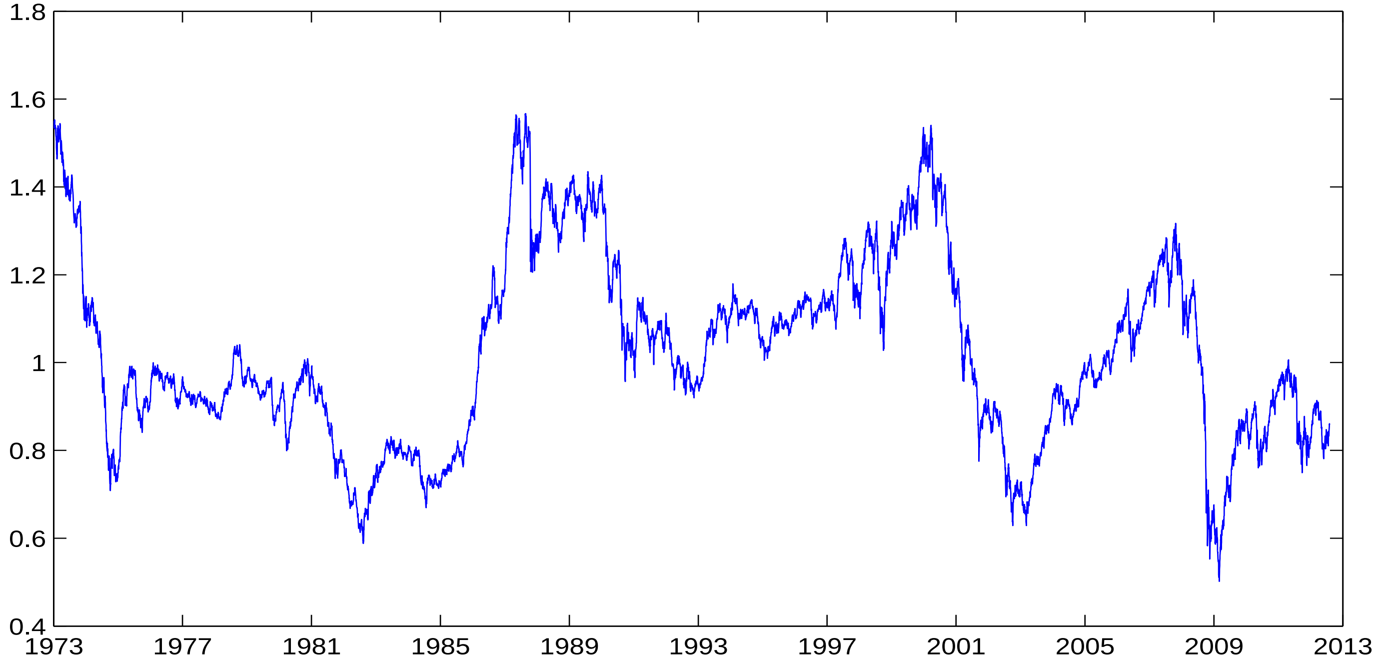
Fitting the model to TOTMKWD

Step 1: Normalization of Index

$A_{\tau_t} \approx A \exp\{\frac{4a\epsilon}{\gamma(\nu-2)} t\} \Rightarrow A = 65.21,\ \frac{4a\epsilon}{\gamma(\nu-2)} \approx 0.048$



Normalized TOTMKWD



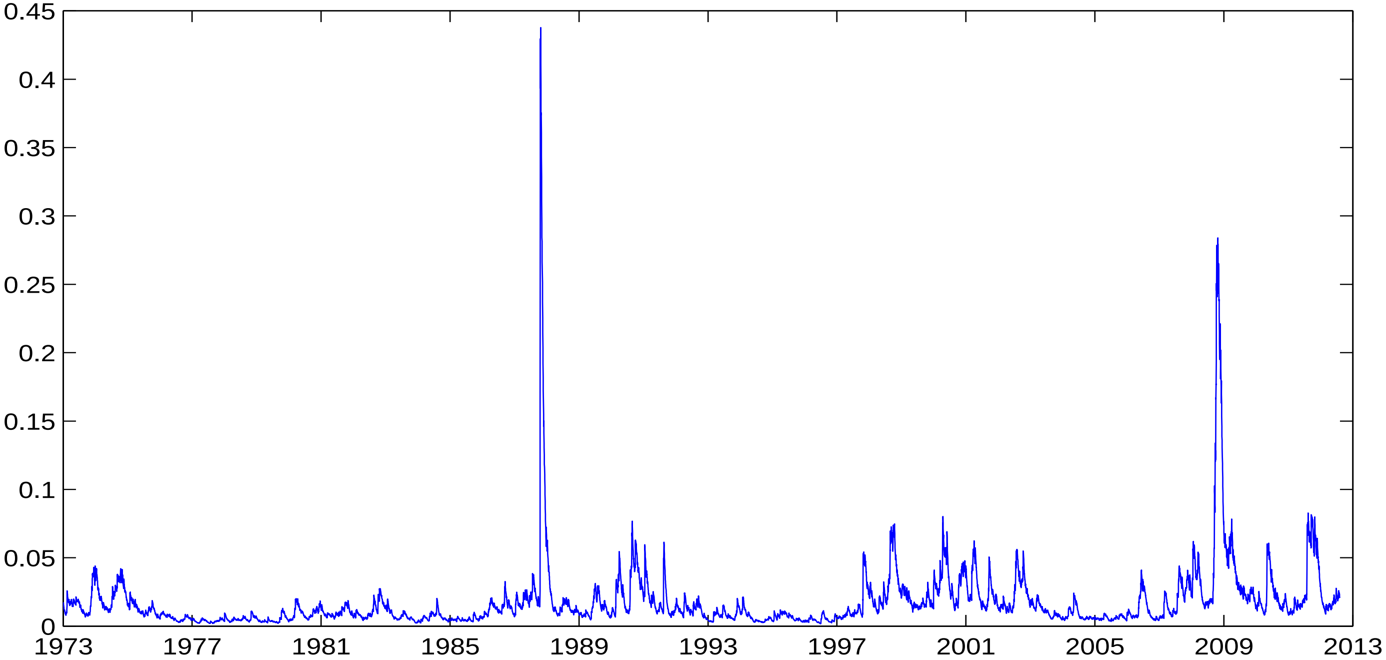
Step 2: Power q: $\delta \approx 4 \Rightarrow q = \frac{\delta}{2} - 1 \approx 1$
Affine nature $\Rightarrow q = 1$
Step 3: Observing Market Activity:

$$\frac{d[\sqrt{Y}]_{\tau_t}}{dt} = \frac{1}{4}\frac{d\tau_t}{dt} = \frac{M_t}{4}$$

$$\hat{Q}_{t_i} \approx \frac{[\sqrt{Y}]_{\tau_{t_{i+1}}} - [\sqrt{Y}]_{\tau_{t_i}}}{t_{i+1} - t_i}$$

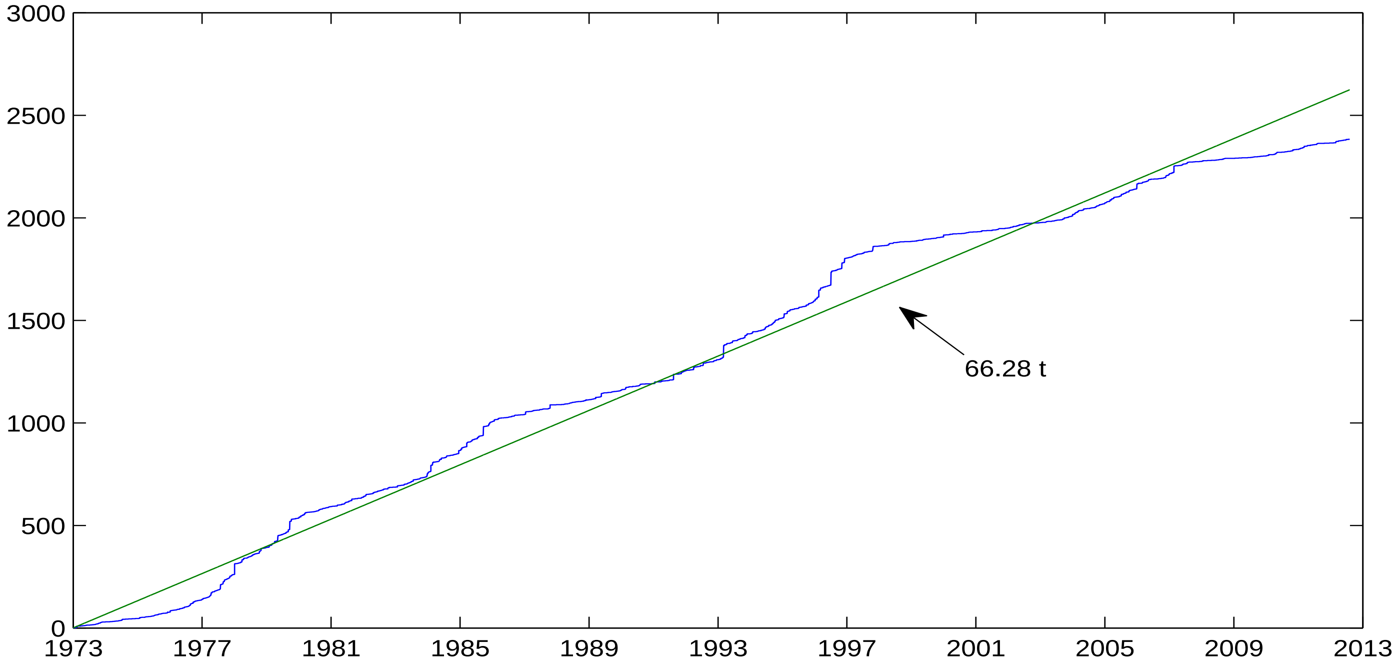
$$\tilde{Q}_{t_{i+1}} = \alpha\sqrt{t_{i+1} - t_i}\hat{Q}_{t_i} + (1 - \alpha\sqrt{t_{i+1} - t_i})\tilde{Q}_{t_i},\ \alpha = 0.92$$

Market activity: $M_t \approx 4\tilde{Q}_t$



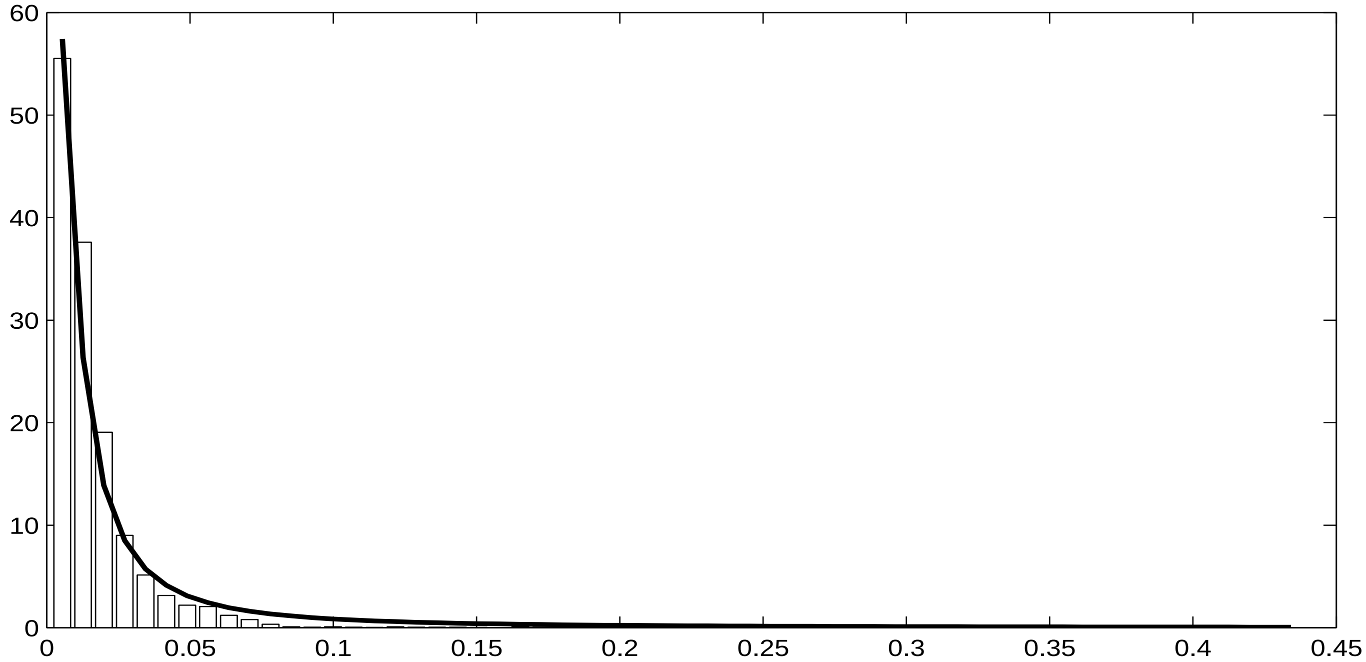
$M_0 = 0.0175$

Step 4: Parameters γ:



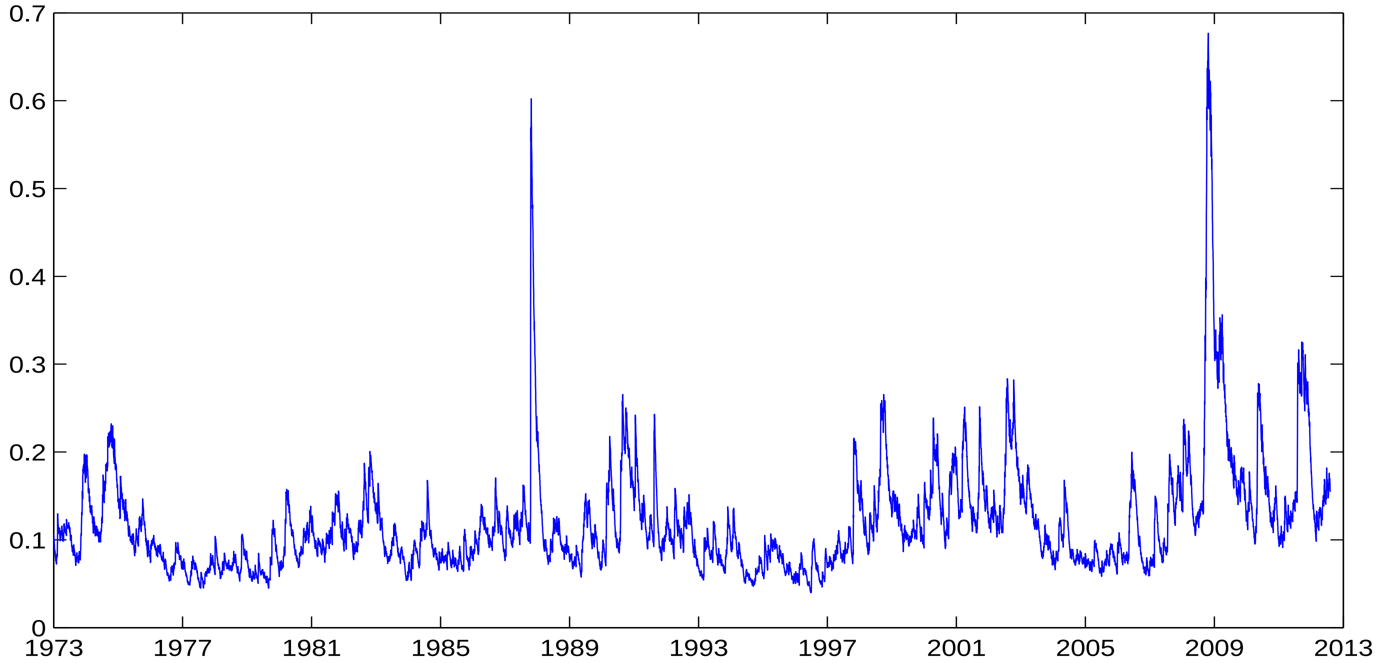
$$\gamma = 265.12$$

Step 5: Parameters ν and ϵ:
Step 6: Long Term Average Net Growth Rate a:



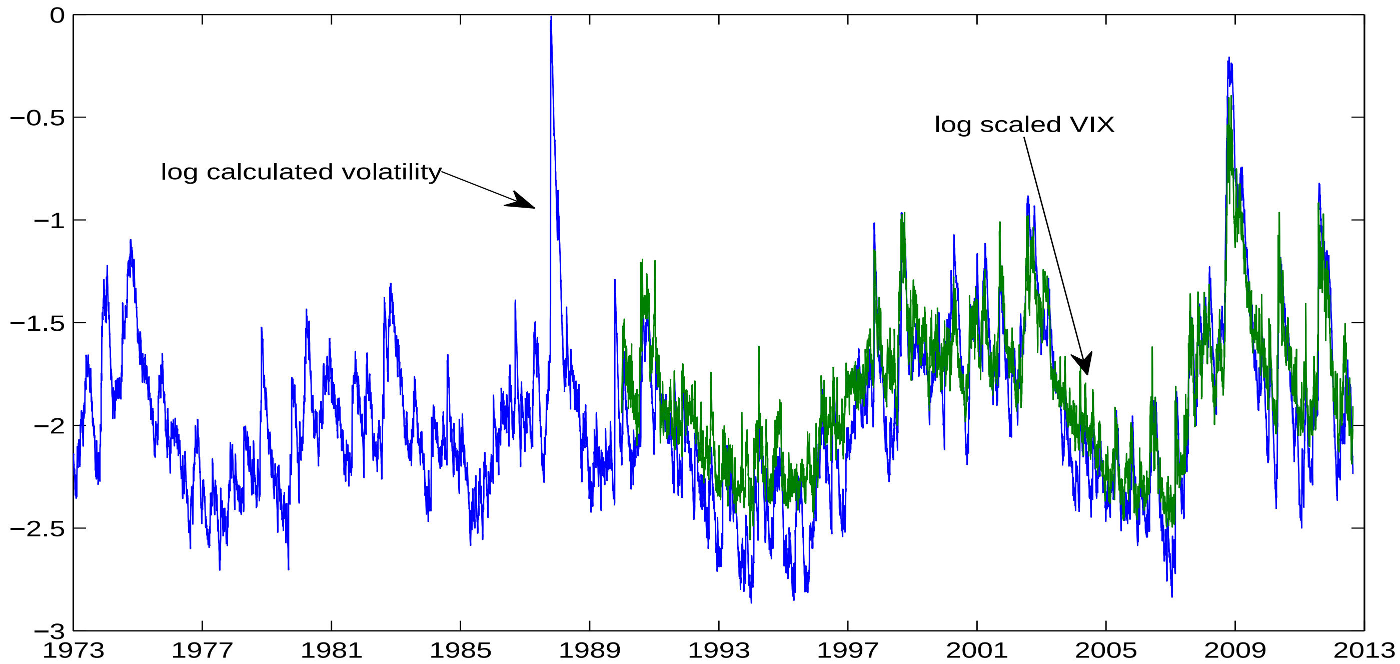
$\nu \approx 4$, $\epsilon \approx 2.18 \Rightarrow a = 2.55 \Rightarrow$ no strong arbitrage

Calculated Volatility



$\sigma_t \approx \sqrt{\frac{4\tilde{Q}_t}{Y_{\tau_t}}}$, average volatility: 11.9%

S&P500 and VIX



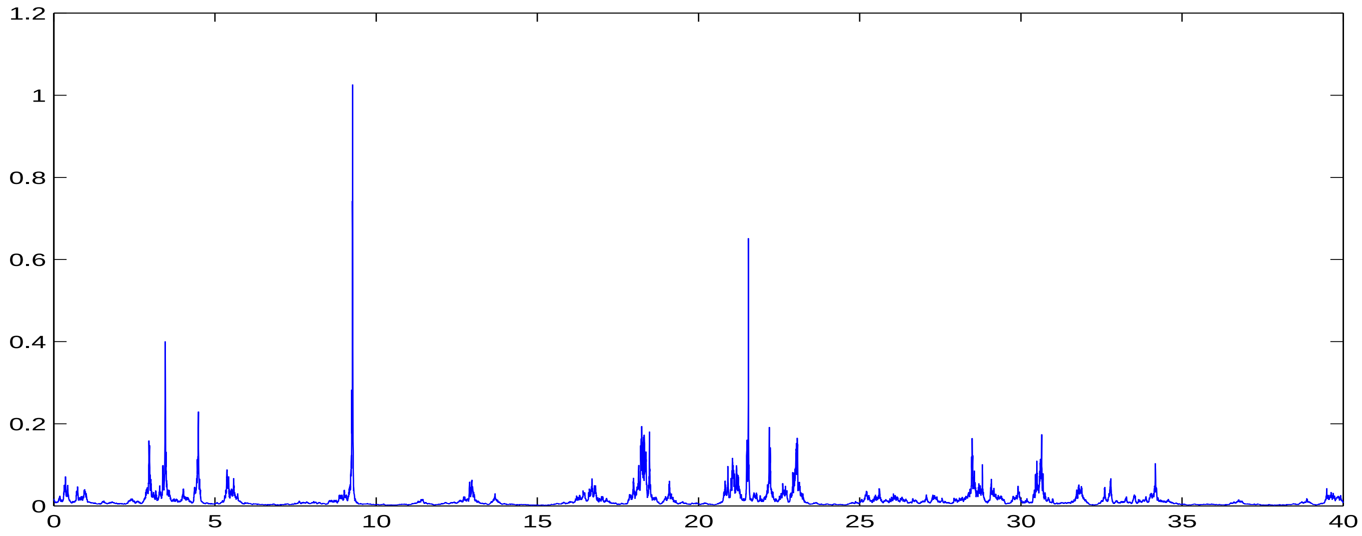
$A = 52.09, \epsilon = 2.15, \gamma = 172.3, a = 1.5$

Model applies to proxies of numéraire portfolio

Simulation Study

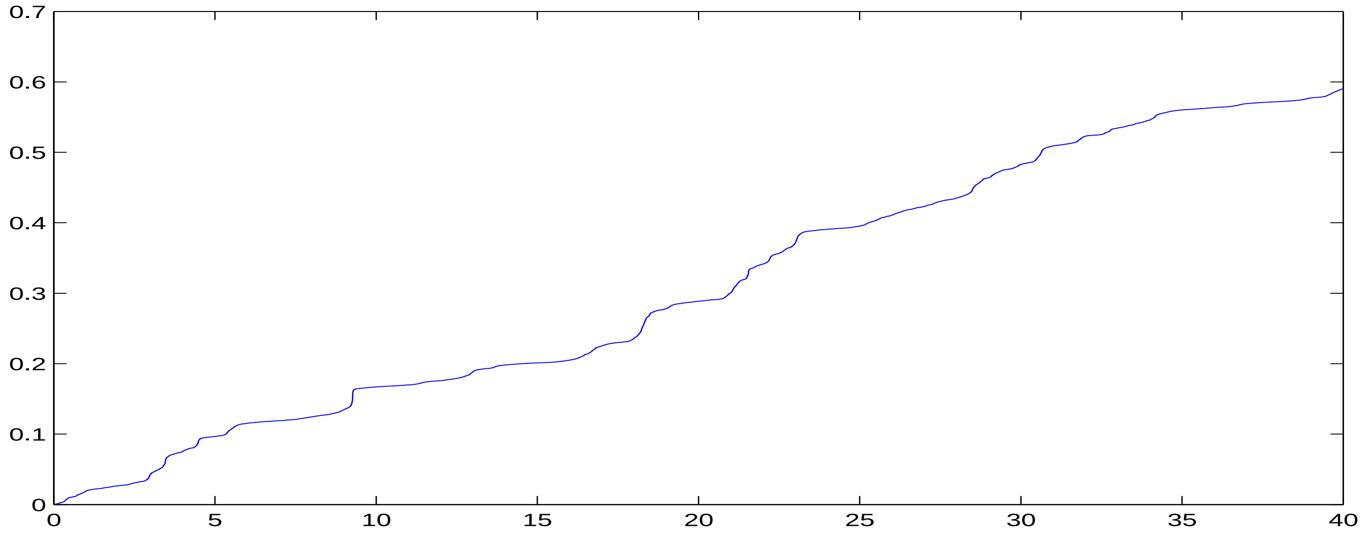
Step 1: Market activity:

$$\frac{1}{M_{t_{i+1}}} = \frac{\gamma(1 - e^{-\epsilon(t_{i+1}-t_i)})}{4\epsilon}\left(\chi^2_{3,i} + \left(\sqrt{\frac{4\epsilon e^{-\epsilon(t_{i+1}-t_i)}}{\gamma(1 - e^{-\epsilon(t_{i+1}-t_i)})}\frac{1}{M_{t_i}}} + Z_i\right)^2\right)$$



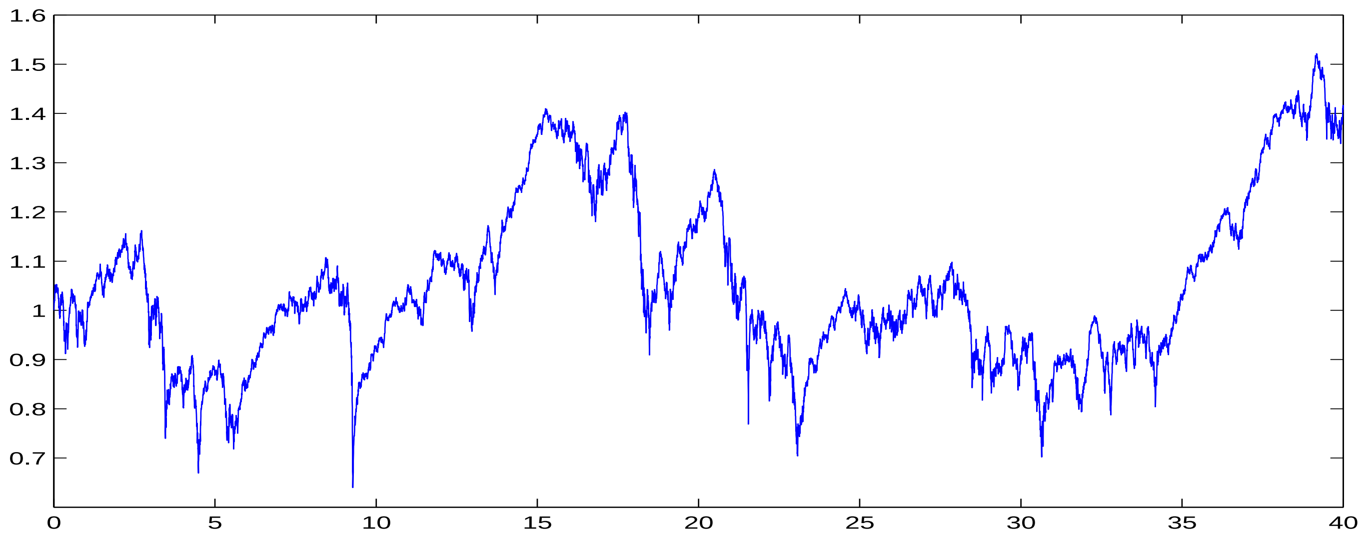
Step 2: τ-time:

$$\tau_{t_{i+1}} - \tau_{t_i} = \int_{t_i}^{t_{i+1}} M_s ds \approx M_{t_i}(t_{i+1} - t_i)$$



Step 3: Normalized index:

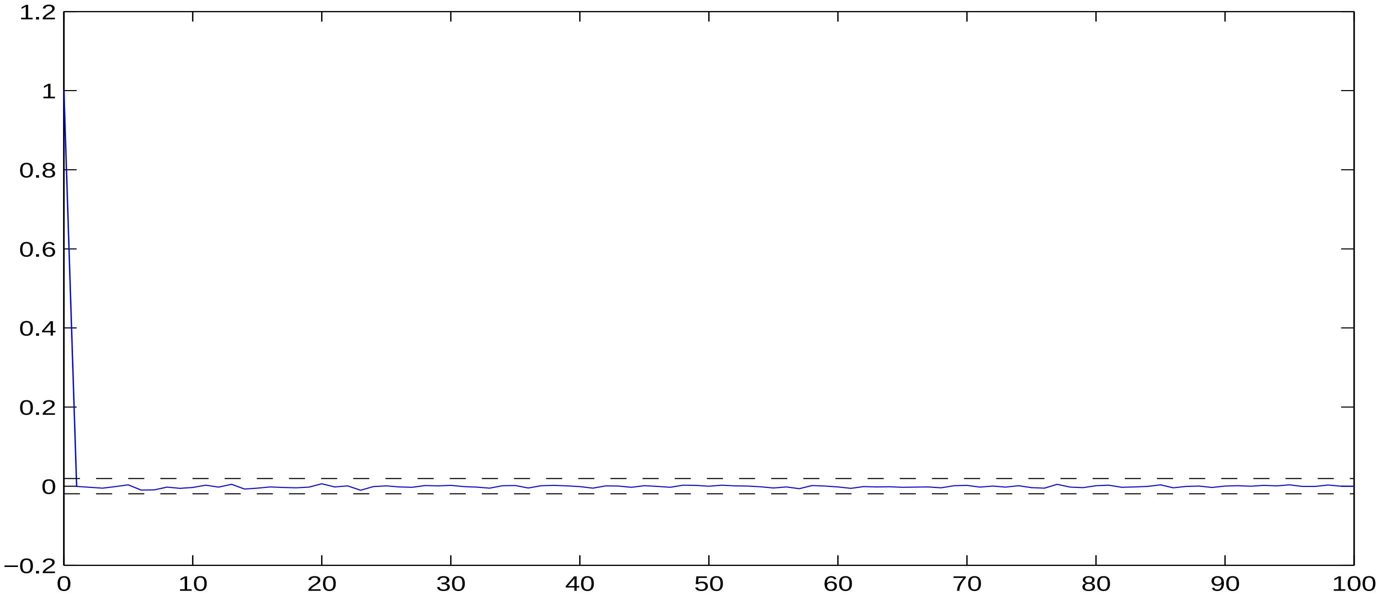
$$Y_{\tau_{t_{i+1}}} = \frac{1 - e^{-(\tau_{t_{i+1}} - \tau_{t_i})}}{4} \left(\chi^2_{3,i} + \left(\sqrt{\frac{4e^{-(\tau_{t_{i+1}} - \tau_{t_i})}}{1 - e^{-(\tau_{t_{i+1}} - \tau_{t_i})}}} Y_{\tau_{t_i}} + Z_i \right)^2 \right)$$



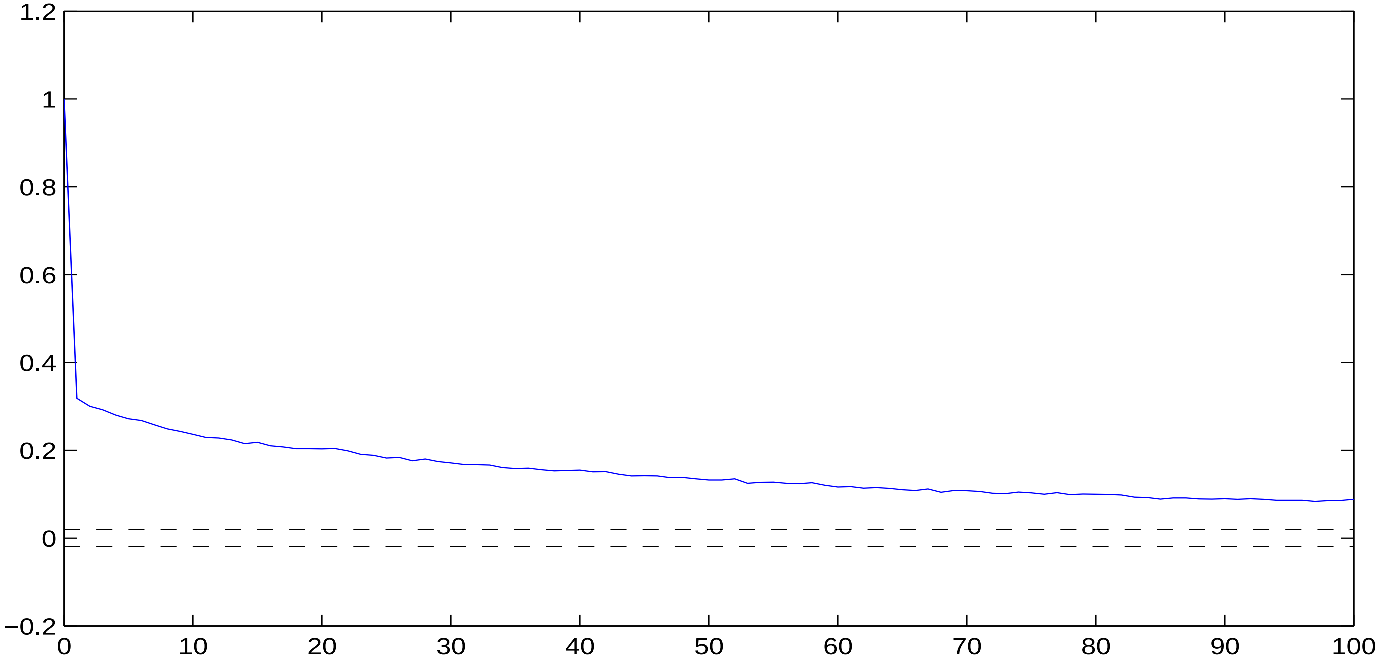
Model recovers stylized empirical facts:

Model is difficult to falsify: Popper (1934)

1. Uncorrelated returns



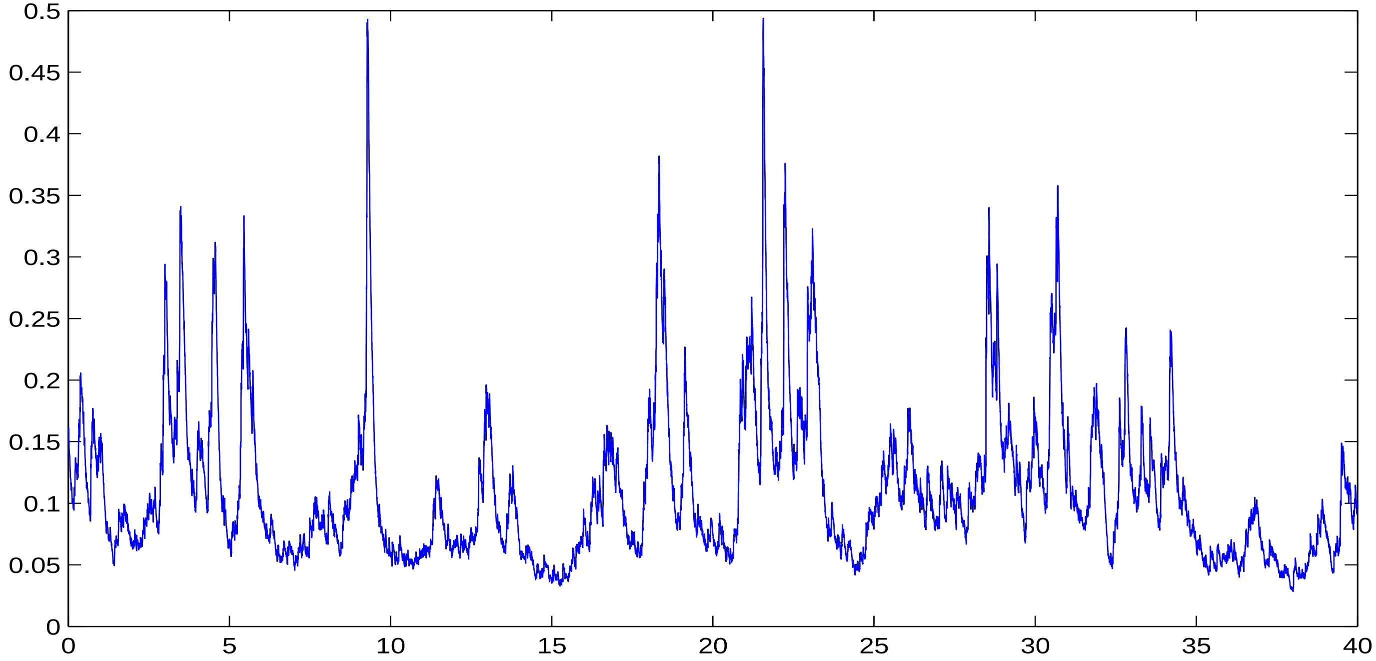
2. Correlated absolute returns



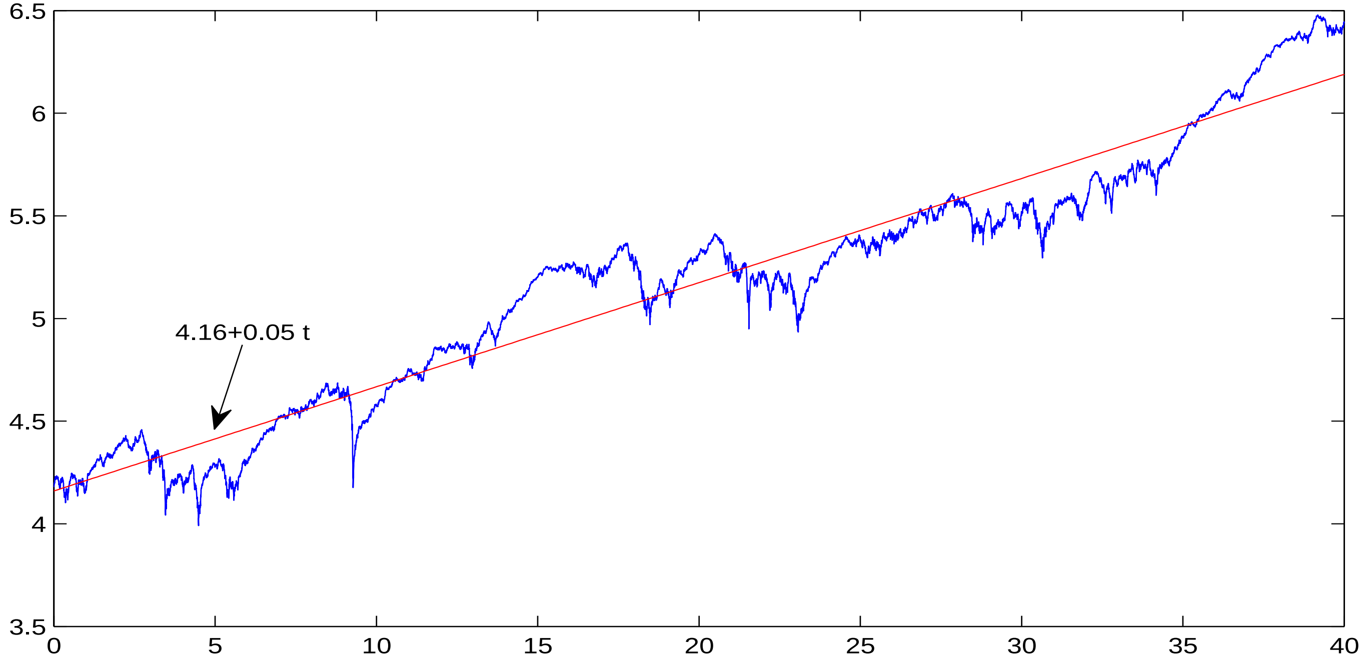
3. Student-t distributed returns

Simulation	Student-t	NIG	Hyperbolic	VG	ν
1	0.008934	37.474149	102.719638	131.240780	4.012850
2	11.485226	11.175028	96.457136	132.916256	3.450916
3	0.000000	100.928524	244.190151	294.719960	3.734148
4	9.002421	35.759464	347.060676	331.014904	2.579009
5	8.767003	11.551178	121.190482	144.084964	3.170449
6	0.401429	60.570898	205.788160	252.591737	3.432435
7	12.239056	4.354888	46.411554	78.273485	3.957696
8	1.693411	23.910523	94.408789	130.623174	3.849691
9	1.232454	47.830407	202.073144	237.168411	3.236322
10	0.000000	43.037206	128.807757	162.582353	3.774957
11	0.433645	47.782681	172.736397	208.847632	3.431803
12	0.000000	56.019354	146.077121	185.624888	3.899403
13	7.137154	48.219756	579.922931	477.383441	2.293363
14	5.873948	16.515390	107.770531	135.508299	3.388307
15	0.000000	54.718046	184.112794	217.304105	3.402049
16	6.982560	3.991610	29.192198	47.105125	4.268740
17	2.966916	22.914863	108.513143	138.044416	3.553629
18	0.000000	52.066364	129.790856	160.373085	3.959605
19	0.006909	39.568695	111.398645	143.914350	3.982892
20	0.000001	56.845664	169.915512	211.260626	3.651091
21	1.674578	17.681088	61.710576	90.679738	4.265834
22	14.010840	3.279722	47.433693	73.789313	3.770825
23	11.198940	12.074044	114.888817	143.800553	3.257146
24	0.455557	27.676102	86.841704	114.452947	4.006528

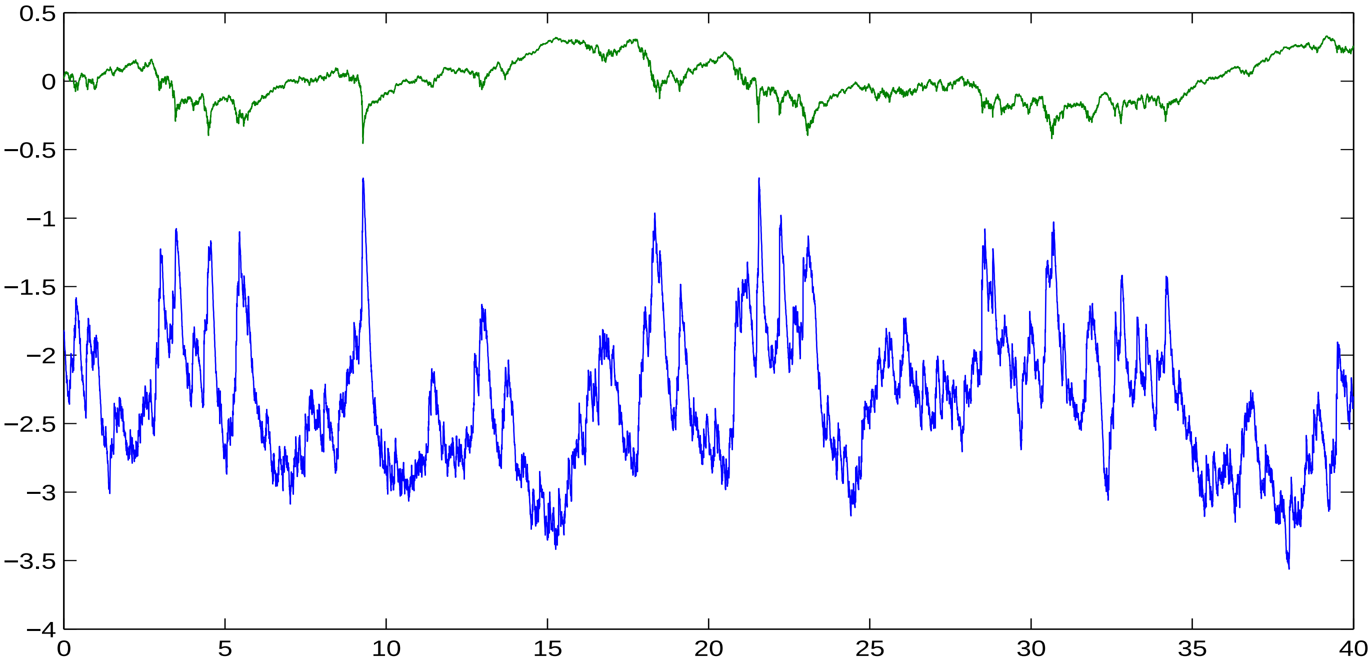
4. Volatility clustering



5. Long term exponential growth



6. **Leverage effect**
7. **Extreme volatility at major market downturns**



Conclusions:

$\diamond$ equity index model: 3 initial parameters, 3 structural parameters and 1 Wiener process (nondiversifiable uncertainty)

$\diamond$ model recovers 7 stylized empirical facts

$\diamond$ long dated derivative pricing under benchmark approach

$\diamond$ leads outside classical theory