

On the One-Shot Zero-Error Classical Capacity of Classical-Quantum Channels Assisted by Quantum Non-signalling Correlations

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Abstract

Duan and Winter studied the one-shot zero-error classical capacity of a quantum channel assisted by quantum non-signalling correlations, and formulated this problem as a semidefinite program depending only on the Kraus operator space of the channel. For the class of classical-quantum channels, they showed that the *asymptotic* zero-error classical capacity assisted by quantum non-signalling correlations, minimized over all classical-quantum channels with a confusability graph G , is exactly $\log \vartheta(G)$, where $\vartheta(G)$ is the celebrated Lovász theta function. In this paper, we show that the same result holds in the *one-shot* setting for a class of *circulant graphs* defined by *equal-sized cyclotomic cosets*, which include the cycle graphs of odd length, the Paley graphs of prime vertices, and the cubit residue graphs of prime vertices. Examples of other graphs are also discussed. This endows the Lovász θ function with a more straightforward operational meaning.

I. INTRODUCTION

Shannon discussed the communication problem in the setting of zero errors and connected this problem to the graph theory [1]. Let $N : V \rightarrow W$ be a channel with discrete alphabets V and W . Two distinct messages can be confused if their channel outputs are equal with a nonzero probability. The maximum messages that can be sent through the channel N without confusion is equivalent to the maximum number of independent vertices $\alpha(G)$ of its *confusability graph* G . The confusability graph G of N has a vertex set V , the channel input alphabet, and two vertices are connected if their channel outputs are likely to be confused. Using the channel N twice in parallel corresponds to a confusability graph $G \boxtimes G$, where \boxtimes is the graph *strong product*. (For two graphs G_1, G_2 with vertex sets V_1, V_2 , and edge sets E_1, E_2 , respectively, their strong product $G_1 \boxtimes G_2$ has a vertex set $V_1 \times V_2$, and two vertices (v_1, v_2) and $(w_1, w_2) \in V_1 \times V_2$ are connected if $v_1 w_1 \in E_1$ and $v_2 w_2 \in E_2$; or $v_1 w_1 \in E_1$ and $v_2 = w_2$; or $v_1 = w_1$ and $v_2 w_2 \in E_2$, where we use vw to denote an edge connecting vertices v with w .) The Shannon capacity of a graph G is defined as

$$\Theta(G) = \sup_n \sqrt[n]{\alpha(G^{\boxtimes n})} = \lim_{n \rightarrow \infty} \sqrt[n]{\alpha(G^{\boxtimes n})}.$$

The quantity $\Theta(G)$ is difficult to determine, even for simple graphs, such as cycle graphs \mathcal{C}_n of odd length. In [2], Lovász proposed an upper bound $\vartheta(G)$ (to be defined in Sec. II) on $\Theta(G)$, and it is tight in some cases. For example, $\Theta(\mathcal{C}_5) = \vartheta(\mathcal{C}_5)$. Although $\Theta(\mathcal{C}_n)$ for odd $n \geq 7$ are still unknown, it seems close to $\vartheta(\mathcal{C}_n)$. However, Haemers showed that it is possible that there is a gap between $\vartheta(G)$ and $\Theta(G)$ for some graphs [3], [4]. It is desired to find operational meanings for $\vartheta(G)$, apart from an upper bound for $\Theta(G)$.

Recently the problem of zero-error communication has been studied in quantum information theory [5], [6]. Some unexpected phenomena were observed in the quantum case. For example, very noisy channels can be super-activated [7], [8], [9], [10]. It is also likely that entanglement can increase the zero-error capacity of classical channels [11], [12]. Again, entanglement-assisted zero-error capacity is upper-bounded by the Lovász ϑ function [13]. For a classical channel, it is suspected that its entanglement-assisted zero-error capacity is exactly the Lovász ϑ function [6].

Non-signalling correlations have been studied in relativistic causality of quantum operations [14], [15], [16], [17], [18]. In [19], Cubitt *et al.* considered non-signalling correlations in the zero-error classical communications. Duan and Winter further introduced quantum non-signalling correlations (QNSCs) in the zero-error communication problem [20]. QNSCs are completely positive and trace-preserving linear maps shared between two parties so that they cannot send any information to each other by using these linear maps. Resources, such as shared randomness, entanglement, and classical non-signalling correlations, can be considered as special types of QNSCs. The one-shot zero-error classical capacity of a quantum channel \mathcal{N} assisted by a QNSC Π is equivalent to the largest integer M so that a noiseless classical channel that can send M messages can be simulated by the composition of \mathcal{N} and Π . Duan and Winter formulated this problem as a semidefinite program (SDP) [21]. For the class of classical-quantum (C-Q) channels, the *one-shot* zero-error classical capacity assisted by QNSCs is the integral part of the solution $\Upsilon(\mathcal{N})$ to an SDP (see Eq. (2)) [20]. Moreover, they proved that the *asymptotic* zero-error classical capacity assisted

by QNSCs, minimized over all C-Q channels with a confusability graph G , is exactly $\log \vartheta(G)$. This provides an operational meaning of the Lovász ϑ function. (The definition of a confusability graph can be generalized quantum channels. For C-Q channels, see Sec. III.)

In this article we focus on the same problem in the *one-shot* setting. We consider the type of C-Q channel $\mathcal{N} : |k\rangle\langle k| \mapsto |u_k\rangle\langle u_k|$, where $\{|u_k\rangle\}$ is an *orthonormal representation* of a graph G in some Hilbert space \mathcal{B} . We will provide a class of *circulant graphs*, defined by *equal-sized cyclotomic cosets*, and their orthonormal representations so that the one-shot QNSC-assisted zero-error classical capacity of a C-Q channel \mathcal{N} induced from these orthonormal representations is the integral part of

$$\Upsilon(\mathcal{N}) = \vartheta(G).$$

Moreover, since ϑ is multiplicative, the asymptotic QNSC-assisted zero-error classical capacity of \mathcal{N} is

$$C_{0,\text{NS}}(\mathcal{N}) = \lim_{m \rightarrow \infty} \frac{1}{m} \log \Upsilon(\mathcal{N}^{\otimes m}) = \log \vartheta(G).$$

This provides a more straightforward operational meaning for the Lovász ϑ function. In particular, our results apply to the cycles \mathcal{C}_n of odd length. There are some work trying to connect the Shannon capacity $\Theta(\mathcal{C}_n)$ and independence number $\alpha(\mathcal{C}_n^{\boxtimes m})$ to $\vartheta(\mathcal{C}_n)$ [22], [23], [24], [25]. Now we know that with the assistance of quantum non-signalling correlations, $\Upsilon(\mathcal{N}) = \vartheta(\mathcal{C}_n)$. This may explain why it is difficult to build equality between $\Theta(\mathcal{C}_n)$ and $\log \vartheta(\mathcal{C}_n)$.

This paper is organized as follows. We first give definitions of graphs, orthonormal representations, and the Lovász ϑ function in the next section. QNSC-assisted zero-error communication is introduced in Sec. III. In Sec. IV, we provide an orthonormal representation for any circulant graph. Then we explicitly construct feasible solutions to the SDP for the one-shot QNSC-assisted zero-error classical capacity of a C-Q channel, whose confusability graph is a circulant graph defined by equal-sized cyclotomic cosets. This type of circulant graphs are characterized in Sec. V, and they include three families of graphs: the cycle graphs \mathcal{C}_n of odd length, the Paley graphs \mathcal{QR}_p , where p is a prime congruent to 1 modulo 4, and the cubic residue graphs \mathcal{CR}_p , where p is a prime congruent to 1 modulo 3. Finally we conclude with a discussion on other graphs with $\Upsilon(\mathcal{N}) = \vartheta(G)$ in Sec. VI.

II. LOVÁSZ ϑ FUNCTION AND GRAPHS

In this article the vertex set V of a graph G under consideration is the ring of integers modulo n for $n = |V|$. That is, $V = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n = \{0, 1, \dots, n-1\}$. Let E be the edge set of G and let vw denote an edge connecting vertices v with w . Let $[M]_{i,j}$ denote the (i, j) entry of a matrix M . The adjacency matrix A_G of G has entries

$$[A_G]_{i,j} = \begin{cases} 1, & \text{if } ij \in E; \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues and eigenvectors of a graph G are the eigenvalues and eigenvectors of its adjacency matrix A_G . An *automorphism* on a graph G is a permutation on its vertex set V that preserves the adjacency. Consequently, the adjacency matrix A_G is invariant under the conjugation of an automorphism. A graph is called *asymmetric* if it has no nonidentity automorphism. If for any two edges of G , there exists an automorphism mapping one edge to the other, then G is *edge-transitive*.

In order to estimate $\Theta(G)$, Lovász proposed an upper bound $\vartheta(G)$ on the Shannon capacity of a graph G [2], which is the minimum *value* of an *orthonormal representation* of the graph.

We use a more general definition of an orthonormal representation as follows.

Definition 1. Suppose $\{P_k\} \in \mathbb{C}^{d \times d}$ is a set of projectors so that

$$\text{Tr } P_i P_j = 0$$

if $ij \notin E$. Then $\{P_k\}$ is an orthonormal representation of G . The value of $\{P_k\}$ is defined as

$$\eta(\{P_k\}) = \min_{\substack{\sigma \geq 0 \\ \text{Tr } \sigma = 1}} \max_k \frac{1}{\text{Tr } P_k \sigma}.$$

(This definition of η is different from that in [20].) The trace-one, positive semidefinite operator σ that yields the minimum value is called the *handle* of the representation. Then $\vartheta(G)$ is defined as

$$\vartheta(G) = \min_{\{P_k\}} \eta(\{P_k\}).$$

We also say that $\vartheta(G)$ is the *Lovász number* of G . An *optimal* orthonormal representation (OOR) of G is a representation with value $\vartheta(G)$. If P_k and σ are restricted to rank-one matrices, this is exactly the definition in [2]. Following [20], [26], one can show that the definition is well-defined even allowing P_k and σ to have rank greater than one. In the case of $P_k = |u_k\rangle\langle u_k|$, we also say that $\{|u_k\rangle\}$ is an orthonormal representation of G , without ambiguity.

Finally, in [2, Theorem 3], Lovász showed that $\vartheta(G)$ is the minimum of the largest eigenvalue of any symmetric matrix A such that

$$[A]_{i,j} = 1 \text{ if } i = j \text{ or } ij \notin E. \quad (1)$$

Thus $\vartheta(G)$ can be determined by solving an SDP, and it serves as a practical upper bound on $\Theta(G)$.

III. ZERO-ERROR COMMUNICATION ASSISTED WITH QUANTUM NON-SIGNALLING CORRELATIONS

Quantum non-signalling correlations are completely positive and trace-preserving linear maps $\Pi : \mathcal{L}(\mathcal{A}_i) \otimes \mathcal{L}(\mathcal{B}_i) \rightarrow \mathcal{L}(\mathcal{A}_o) \otimes \mathcal{L}(\mathcal{B}_o)$ shared between two parties Alice and Bob (with Hilbert spaces \mathcal{A} and \mathcal{B} , respectively, and the subscripts i and o stand for input and output, respectively) so that they cannot send classical information to each other by using Π . Let the Choi matrix of Π be

$$\Omega_{\mathcal{A}'_i \mathcal{A}_o \mathcal{B}'_i \mathcal{B}_o} = (\text{id}_{\mathcal{A}'_i} \otimes \text{id}_{\mathcal{B}'_i} \otimes \Pi)(\Phi_{\mathcal{A}_i \mathcal{A}'_i} \otimes \Phi_{\mathcal{B}_i \mathcal{B}'_i}),$$

where $\text{id}_{\mathcal{A}} \in \mathcal{L}(\mathcal{A})$ is the identity operator on the Hilbert space \mathcal{A} , $\Phi_{\mathcal{A}_i \mathcal{A}'_i} = |\Phi_{\mathcal{A}_i \mathcal{A}'_i}\rangle\langle\Phi_{\mathcal{A}_i \mathcal{A}'_i}|$, and $|\Phi_{\mathcal{A}_i \mathcal{A}'_i}\rangle = \sum_k |k_{\mathcal{A}_i}\rangle|k_{\mathcal{A}'_i}\rangle$ is the un-normalized maximally-entangled state. For Π to be a QNSC, Duan and Winter derived the following constraints [20]:

$$\begin{aligned} \Omega_{\mathcal{A}'_i \mathcal{A}_o \mathcal{B}'_i \mathcal{B}_o} &\geq 0, \\ \text{Tr}_{\mathcal{A}_o \mathcal{B}_o} \Omega_{\mathcal{A}'_i \mathcal{A}_o \mathcal{B}'_i \mathcal{B}_o} &= \mathbb{I}_{\mathcal{A}'_i \mathcal{B}'_i}, \\ \text{Tr}_{\mathcal{A}_o \mathcal{A}'_i} \Omega_{\mathcal{A}'_i \mathcal{A}_o \mathcal{B}'_i \mathcal{B}_o} X_{\mathcal{A}'_i}^T &= 0, \forall \text{Tr } X = 0, \\ \text{Tr}_{\mathcal{B}_o \mathcal{B}'_i} \Omega_{\mathcal{A}'_i \mathcal{A}_o \mathcal{B}'_i \mathcal{B}_o} Y_{\mathcal{B}'_i}^T &= 0, \forall \text{Tr } Y = 0, \end{aligned}$$

where \mathbb{I} is the identity matrix of appropriate dimension, X and Y are Hermitian operators, and X^T is the transpose of X .

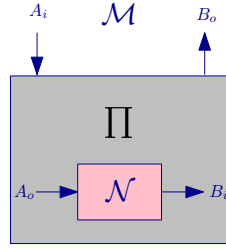


Fig. 1. A general simulation network: implementing a channel \mathcal{M} using another channel \mathcal{N} once, and the QNSC Π between Alice and Bob.

Suppose $\mathcal{N} : |k\rangle\langle k| \rightarrow \rho_k \in \mathcal{L}(\mathcal{B})$ is a C-Q channel that maps a set of classical states $|k\rangle\langle k|$ for $k = 0, \dots, n-1$ into some quantum states $\rho_k \in \mathcal{L}(\mathcal{B})$. Suppose that P_k are the projectors onto the support of ρ_k , respectively. Then $\{P_k\}$ defines a confusability graph G with vertex set \mathbb{Z}_n and two vertices i and j are connected if and only if $\text{Tr } P_i P_j \neq 0$.

Let \mathcal{M} be the composition channel of \mathcal{N} and a QNSC Π as illustrated in Fig. 1. The *one-shot zero-error classical capacity* of \mathcal{N} assisted by Π is the largest integer m so that \mathcal{M} can simulate a noiseless classical channel that can send m messages. In [20], Duan and Winter showed that this one-shot capacity is the integral part of the solution $\Upsilon(\mathcal{N})$ to the following SDP with variables $s_k \in \mathbb{R}$ and $R_k \in \mathcal{L}(\mathcal{B})$:

$$\begin{aligned} \Upsilon(\mathcal{N}) &= \max \sum_k s_k \\ \text{subject to: } &s_k \geq 0, \\ &0 \leq R_k \leq s_k(\mathbb{I} - P_k), \\ &\sum_k (s_k P_k + R_k) = \mathbb{I}. \end{aligned} \quad (2)$$

For an arbitrary graph G , Duan and Winter considered the case of asymptotically many channel uses and showed that

$$\min_{\mathcal{N}} \lim_{m \rightarrow \infty} \frac{1}{m} \log \Upsilon(\mathcal{N}^{\otimes m}) = \log \vartheta(G),$$

where the minimization is over all C-Q channels \mathcal{N} with confusability graph G .

Herein we try to solve the one-shot capacity $\Upsilon(\mathcal{N})$. Apparently $\Upsilon(\mathcal{N}) \geq \alpha(G)$, the independence number of G . This lower bound can be achieved as follows. We choose a maximum independent set \mathcal{I} of size $\alpha(G)$ and set $s_k = 1$ if $k \in \mathcal{I}$ and $s_k = 0$, otherwise. For some $s_{k^*} = 1$, let $R_{k^*} = \mathbb{I} - \sum_{k \in \mathcal{I}} P_k$ and $R_k = 0$ for $k \neq k^*$. Then the constraints of (2) are satisfied and $\Upsilon(\mathcal{N}) \geq \sum_k s_k = \alpha(G)$.

To find an upper bound on $\Upsilon(\mathcal{N})$, we consider the dual problem of (2):

$$\begin{aligned} \hat{\Upsilon}(\mathcal{N}) &= \min \text{Tr } T \\ \text{subject to: } &\text{Tr } P_k T - \text{Tr } (\mathbb{I} - P_k) Q_k \geq 1, \\ &Q_k + T \geq 0, \\ &Q_k \geq 0, \end{aligned} \tag{3}$$

where T is Hermitian. It can be verified that

$$\text{Tr } T - \sum_k \text{Tr } R_k (T + Q_k) \geq \sum_k s_k$$

and the duality gap is zero when $\text{Tr } R_k (T + Q_k) = 0$ for $s_k \neq 0$. By choosing $Q_k = 0$ for all k and $T = \eta(\{P_k\})\sigma$, where σ is the handle of $\{P_k\}$, we have

$$\hat{\Upsilon}(\mathcal{N}) \leq \eta(\{P_k\}).$$

It is suspected that equality may hold for graphs with nontrivial automorphisms. In the rest of this article, we will directly solve the SDP (2) for the C-Q channel $\mathcal{N} : |k\rangle\langle k| \rightarrow |u_k\rangle\langle u_k|$, where $\{|u_k\rangle\}$ is an orthonormal representation for some circulant graph G , defined by equal-sized cyclotomic cosets.

IV. CIRCULANT GRAPHS

In this section we first discuss the definition of a circulant graph and its properties, and then derive an orthonormal representation $\{|u_k\rangle\}$ with $|u_k\rangle = U^k|u_0\rangle$, where U is a unitary operator. Then we show that a circulant graph G , defined by equal-sized cyclotomic cosets modulo n , will induce a C-Q channel \mathcal{N} so that $\Upsilon(\mathcal{N}) = \eta(\{|u_k\rangle\})$. This is done by explicitly constructing s_k and R_k , which lead to a feasible solution to the above SDP with object function $\sum_k s_k = \eta(\{|u_k\rangle\})$.

A. Orthonormal Representation of Circulant Graphs

Let C be a subset of $\mathbb{Z}_n \setminus \{0\}$ so that $-C = C$. A circulant graph $G = X(\mathbb{Z}_n, C)$, defined by the connection set C , has an edge set $\{ij : i - j \in C\}$. Consequently its adjacency matrix A_G has entries $[A_G]_{i,j} = 1$ if and only if $i - j \in C$. (For example, a cycle graph C_n is defined by the connection set $C = \{1, n-1\}$.) Define a unitary matrix

$$U = \text{diag} \left(1, e^{-2\pi i/n}, \dots, e^{-2(n-1)\pi i/n} \right). \tag{4}$$

Let $|\mathbf{1}\rangle = (1 \ 1 \cdots 1)$ be the vector whose entries are all ones. It can be easily verified that the eigenvectors of A_G are $|v_k\rangle = U^{-k}|\mathbf{1}\rangle$ with corresponding eigenvalues

$$\lambda_k = \sum_{j \in C} e^{2\pi i j k / n} \tag{5}$$

for $k = 0, \dots, n-1$. Let λ_{\max} and λ_{\min} be the largest and the smallest eigenvalues of A_G , respectively. It is easy to see that $\lambda_{\max} = \lambda_0 = |C|$. For a circulant graph G that is edge-transitive, its Lovász number is $\vartheta(G) = \frac{-n\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}$ [2]. Note that $\lambda_{\min} < 0$ since $\text{tr}(A_G) = 0$. Below we provide an orthonormal representation for an arbitrary circulant graph.

Theorem 2. Consider a circulant graph $G = X(\mathbb{Z}_n, C)$. Let $\eta = \frac{-n\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}$. Define

$$|u_0\rangle = \frac{1}{\sqrt{\eta}} \left(1, \sqrt{\frac{\lambda_1 - \lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}}, \dots, \sqrt{\frac{\lambda_{n-1} - \lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}} \right)$$

and

$$|u_k\rangle = U^k|u_0\rangle, \quad k = 0, \dots, n-1, \tag{6}$$

where U is the unitary operator defined in (4). Then $\{|u_k\rangle\}$ is an orthonormal representation of the circulant graph G with value η and handle $|c\rangle = (1, 0, \dots, 0)$. Moreover,

$$\langle u_k | u_{k+m} \rangle = \frac{[A_G]_{k+m,k}}{-\lambda_{\min}} + \delta_{m,0}$$

for any k , where $\delta_{m,j}$ is the Kronecker delta function. If G is edge-transitive, then $\{|u_k\rangle\}$ is an OOR with value $\eta = \vartheta(G)$. \square

Proof: It is straightforward to verify that $\{|u_k\rangle\}$ is an orthonormal representation:

$$\begin{aligned}
\langle u_k | u_{k+m} \rangle &= \frac{1}{\vartheta(G)} \sum_{j=0}^{n-1} \frac{\lambda_j - \lambda_{\min}}{\lambda_{\max} - \lambda_{\min}} e^{-2\pi i j m / n} \\
&= -\frac{1}{n\lambda_{\min}} \sum_{j=0}^{n-1} \lambda_j e^{-2\pi i j m / n} + \frac{1}{n} \sum_{j=0}^{n-1} e^{-2\pi i j m / n} \\
&= \frac{1}{-\lambda_{\min}} \sum_{j=0}^{n-1} \sum_{l \in C} \frac{e^{2\pi i j (l-m) / n}}{n} + \delta_{m,0} \\
&= \frac{1}{-\lambda_{\min}} \sum_{l \in C} \sum_{j=0}^{n-1} \frac{e^{2\pi i j (l-m) / n}}{n} + \delta_{m,0} \\
&= \frac{[AG]_{k,k+m}}{-\lambda_{\min}} + \delta_{m,0}.
\end{aligned}$$

It is obvious that

$$\frac{1}{|\langle c | u_k \rangle|^2} = \eta, \quad k = 0, \dots, n-1.$$

■

Remark: If λ_{\min} is of multiplicity m , then m entries of $|u_k\rangle$ are zeros. Also, it is straightforward to see that a graph with an orthonormal representation in the form of Eq. (6) must be circulant.

B. Circulant Graphs defined by Cyclotomic Cosets Modulo n

In the following we will define circulant graphs by cyclotomic cosets modulo n . Cyclotomic cosets usually appear in the application of coding theory for minimal polynomials over finite fields or integer rings [27]. We use a more general concept here.

Let $\mathbb{Z}_n^\times = (\mathbb{Z}/n\mathbb{Z})^\times$ denote the multiplicative group of \mathbb{Z}_n , which consists of the units in \mathbb{Z}_n and its size is determined by the Euler's totient function: $|\mathbb{Z}_n^\times| = \varphi(n)$. Suppose $q \in \mathbb{Z}_n^\times$. The cyclotomic coset modulo n over q which contains $s \in \mathbb{Z}_n$ is

$$C_{(s)} = \{s, sq, sq^2, \dots, sq^{r_s-1}\},$$

where r_s is the smallest positive integer r so that $sq^r \equiv s \pmod{n}$. The subscript s is called the coset representative of $C_{(s)}$. Since q and n are relatively prime, we have $q^{\varphi(n)} \equiv 1 \pmod{n}$ by Fermat-Euler theorem. Thus r_s exists for any s and the cyclotomic cosets are well-defined: $C_{(\alpha)} = C_{(\beta)}$ if and only if $\alpha = \beta q^c \pmod{n}$ for some $c \in \mathbb{Z}$. Any element in a coset can be the coset representative, though it is usually the smallest number in the coset. As a consequence, the integers modulo n are partitioned into disjointed cyclotomic cosets:

$$\mathbb{Z}_n = \bigcup_{j=0}^t C_{(\alpha_j)},$$

where $\{\alpha_0 = 0, \alpha_1, \dots, \alpha_t\}$ is a set of (disjointed) coset representatives. We consider $t > 1$, while the case $t = 1$ is trivial. Since q is relatively prime to n , we always have $C_{(0)} = \{0\}$. It suffices to consider partitions of $\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\}$.

If $-1 \in C_{(1)}$, we can define a circulant graph $X(\mathbb{Z}_n, C_{(\alpha)})$ for any $\alpha \neq 0$. Therefore, the cyclotomic cosets $C_{(0)}, C_{(\alpha_1)}, \dots, C_{(\alpha_t)}$ define a t -class association scheme with $t+1$ symmetric relations $\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_t$ so that $(x, y) \in \mathcal{R}_j$ if $x - y \in C_{(\alpha_j)}$. Each relation \mathcal{R}_j defines a circulant graph $X(\mathbb{Z}_n, C_{(\alpha_j)})$.

Assume further that these cyclotomic cosets are *equal-sized*, except $C_{(0)} = \{0\}$. That is, $|C_{(\alpha)}| = |C_{(1)}|$ for any $\alpha \neq 0$, and $n = t|C_{(1)}| + 1$. A circulant graph defined by these cyclotomic cosets have some interesting properties that are critical to the proof of our main theorem. First, by (5), the eigenvalues of $X(\mathbb{Z}_n, C_{(\alpha_j)})$ are

$$\lambda_k^{(\alpha_j)} = \sum_{l \in C_{(k\alpha_j)}} e^{2\pi i l},$$

which depends only on its cyclotomic coset. Each $\lambda_k^{(\alpha_j)}$ is of multiplicity $|C_{(1)}|$, except for λ_0 , which is of multiplicity 1. It can be seen that these graphs $X(\mathbb{Z}_n, C_{(\alpha_j)})$ are equivalent and it suffices to consider $G = X(\mathbb{Z}_n, C_{(1)})$.

On the other hand, suppose $\beta \in \mathbb{Z}_n^\times \setminus C_{(1)}$. Let $\tau_\beta(C_{(\alpha)}) = C_{(\alpha\beta)}$. It can be checked that τ_β is a permutation on the cyclotomic cosets of order at most t . One can delve into more about the structure of τ_β , but we only need the following

equation in the proof of our main theorem:

$$\mathbb{Z}_n = \bigcup_{j=0}^t C_{(\alpha_j)} = \bigcup_{j=0}^t C_{(\alpha_j \beta)}.$$

(Note that the indices are under modulo n and we will always omit “mod n ” as it is clear from the context.)

Example 1. For $\mathbb{Z}_{17}^\times = \langle 3 \rangle$, $-1 \equiv 3^8$ and $13 \equiv 3^4$. Let $C_{(1)} = \langle 13 \rangle$ and we have

$$\begin{aligned} C_{(0)} &= \{0\}, \\ C_{(1)} &= \{1, 13, 16, 4\}, \\ C_{(2)} &= \{2, 9, 15, 8\}, \\ C_{(3)} &= \{3, 5, 14, 12\}, \\ C_{(6)} &= \{6, 10, 11, 7\}. \end{aligned}$$

The circulant graph $X(\mathbb{Z}_{17}, C_{(1)})$ is shown in Fig. 2.

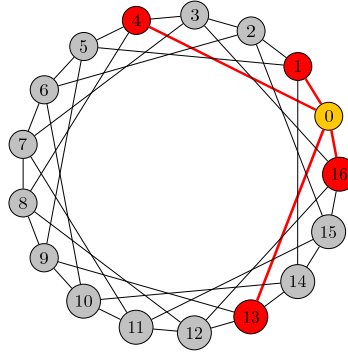


Fig. 2. The circulant graph $X(\mathbb{Z}_{17}, \{1, 13, 16, 4\})$.

□

Now we are ready to derive our main theorem. Characterization of equal-sized cyclotomic cosets is left to the next section.

Theorem 3. Suppose $\mathbb{Z}_n^* = \bigcup_{j=1}^t C_{(\alpha_j)}$, where $\{C_{(\alpha_j)}\}$ are cyclotomic cosets modulo n over q of equal size for some q relatively prime to n and $C_{(1)} = C_{(-1)}$. Let \mathcal{N} be the C-Q channel induced by the orthonormal representation $\{|u_k\rangle\}$ of $G = X(\mathbb{Z}_n, C_{(1)})$ in Theorem 2. Then

$$\Upsilon(\mathcal{N}) = \eta(\{|u_k\rangle\}).$$

Moreover, a set of solution to the SDP (2) is $s_k = \frac{1}{n} \eta(\{|u_k\rangle\})$, $R_k = U^k R_0 U^{-k}$, and

$$R_0 = \frac{1}{n} \left(\mathbb{I} - \sum_{j=0}^{n-1} x_j P_j \right), \quad (7)$$

where U is defined in (4) and $x_j = \frac{\lambda_{j\beta} - \lambda_\beta}{\lambda_0 - \lambda_\beta}$, given $\lambda_\beta = \lambda_{\min}$ for some $\beta \in \mathbb{Z}_n^\times$. (In particular, $x_0 = 1$ and $x_j = 0$ for $j \in C_{(1)}$.)

If G is edge-transitive, we have $\Upsilon(\mathcal{N}) = \vartheta(G)$. Since ϑ is multiplicative, the asymptotic QNSC-assisted zero-error classical capacity of \mathcal{N} is

$$C_{0,\text{NS}}(\mathcal{N}) = \lim_{m \rightarrow \infty} \frac{1}{m} \log \Upsilon(\mathcal{N}^{\otimes m}) = \log \vartheta(G).$$

□

Proof: Apparently, $\Upsilon(\mathcal{N}) = \sum_k s_k = \eta(\{|u_k\})$. Also,

$$\begin{aligned} \sum_k (s_k P_k + R_k) &= \frac{1}{n} \sum_k \left(\eta(\{|u_k\}) P_k - \sum_{j=0}^{n-1} x_j U^k P_j U^{-k} \right) + \mathbb{I} \\ &= \frac{1}{n} \left(\eta(\{|u_k\}) - \sum_{j=0}^{n-1} x_j \right) \sum_k P_k + \mathbb{I} \\ &= \frac{1}{n} \left(- \sum_{j=0}^{n-1} \frac{\lambda_{j\beta}}{\lambda_0 - \lambda_\beta} \right) \sum_k P_k + \mathbb{I} \\ &= \mathbb{I}, \end{aligned}$$

where the last equality is because $\sum_{j=0}^{n-1} \lambda_{j\beta} = \sum_{j=0}^{n-1} \lambda_j = 0$. It remains to verify $0 \leq R_0 \leq s_0(\mathbb{I} - P_0)$.

Let $D = \sum_{j=0}^{n-1} x_j P_j$. From Theorem 2, we have

$$[P_j]_{a,b} = \frac{1}{\eta(\{|u_k\})} \sqrt{\frac{(\lambda_a - \lambda_\beta)(\lambda_b - \lambda_\beta)}{(\lambda_0 - \lambda_\beta)^2}} e^{-2\pi i j(a-b)/n}.$$

Thus for $a \neq b$,

$$\begin{aligned} [D]_{a,b} &= \frac{1}{\eta(\{|u_k\})} \sqrt{\frac{(\lambda_a - \lambda_\beta)(\lambda_b - \lambda_\beta)}{(\lambda_0 - \lambda_\beta)^2}} \sum_{j=0}^{n-1} \frac{\lambda_{j\beta} - \lambda_\beta}{\lambda_0 - \lambda_\beta} e^{-2\pi i j(a-b)/n} \\ &= \frac{1}{-n\lambda_\beta} \sqrt{\frac{(\lambda_a - \lambda_\beta)(\lambda_b - \lambda_\beta)}{(\lambda_0 - \lambda_\beta)^2}} \sum_{j=0}^{n-1} \lambda_{j\beta} e^{-2\pi i j(a-b)/n} \\ &= \frac{1}{-n\lambda_\beta} \sqrt{\frac{(\lambda_a - \lambda_\beta)(\lambda_b - \lambda_\beta)}{(\lambda_0 - \lambda_\beta)^2}} \sum_{j=0}^{n-1} \sum_{k \in C_{(\beta)}} e^{2\pi i j(k-(a-b))/n} \\ &= \begin{cases} \frac{1}{-\lambda_\beta} \sqrt{\frac{(\lambda_a - \lambda_\beta)(\lambda_b - \lambda_\beta)}{(\lambda_0 - \lambda_\beta)^2}}, & \text{if } a - b \in C_{(\beta)}; \\ 0, & \text{if } a - b \notin C_{(\beta)}. \end{cases} \end{aligned}$$

Similarly, we have

$$[D]_{a,a} = \sqrt{\frac{(\lambda_a - \lambda_\beta)(\lambda_a - \lambda_\beta)}{(\lambda_0 - \lambda_\beta)^2}}$$

for $0 \leq a \leq n-1$. Therefore, D is a nonnegative matrix. Observe that $|u_0\rangle$ is a positive eigenvector of D with eigenvalue 1. Thus by Perron-Frobenius theorem, the largest eigenvalue of D is 1. Consequently, $R_0 = \frac{1}{n}(\mathbb{I} - D) \geq 0$. Also, $R_0 \leq s_0(\mathbb{I} - P_0)$ as long as $\eta(\{|u_k\}) \geq 1$. ■

V. CHARACTERIZATION OF EQUAL-SIZED CYCLOTOMIC COSETS

In this section we characterize some properties of the equal-sized cyclotomic cosets. Then we provide three families of graphs that fit Theorem 3: the cycle graphs, the Paley graphs, and the cubic residue graphs.

Observe that the cyclotomic coset of 1 modulo n over q is $C_{(1)} = \langle q \rangle$, which is a cyclic subgroup of the multiplicative group \mathbb{Z}_n^\times . Thus $|C_{(1)}|$ divides $\varphi(n)$. Since $C_{(1)} = C_{(-1)}$, $|C_{(1)}|$ is even, which implies n is odd. Consequently, $|C_{(1)}|$ is a common divisor of $\varphi(n)$ and $n-1$. Let

$$\Gamma_d^n = \{a \in \mathbb{Z}_{n+1}^* : \gcd(a, n) = n/d\}$$

and then $|\Gamma_d^n| = \varphi(d)$. We have $\mathbb{Z}_n = \bigcup_{d:d|n} \Gamma_d^n$. For each $\alpha \neq 0$, $C_{(\alpha)} \subseteq \Gamma_{d_\alpha}^n$ for some $d_\alpha | n$. Therefore we have the following lemma.

Lemma 4. If $\{C_{\alpha_1}, \dots, C_{\alpha_t}\}$ is a set of equal-sized cyclotomic cosets modulo n , then $|C_{(1)}|$ must be a common divisor of $\varphi(d)$ for all $d|n$ and $d > 1$. □

It remains to find conditions so that $C_{(1)} = C_{(-1)}$. In the following we provide several families of graphs.

Remark: Lemma 4 is a necessary condition that equal-sized cyclotomic cosets modulo n exist for a certain n . It is likely also a sufficient condition. However, we did not find composite n so that the nontrivial equal-sized cyclotomic cosets has $C_{(1)} = C_{(-1)}$.

A. Trivial Equal-sized Cyclotomic Cosets

For any odd $n \geq 3$, there exists a trivial connection set $C_{(1)} = \{1, n-1\}$, which is a cyclotomic coset modulo n over $n-1$.

Example 2. For $n = 7$ and $q = 6$, we have

$$\begin{aligned} C_{(0)} &= \{0\}, \\ C_{(1)} = C_{(6)} &= \{1, 6\}, \\ C_{(2)} = C_{(5)} &= \{2, 5\}, \\ C_{(3)} = C_{(4)} &= \{3, 4\}. \end{aligned}$$

Each of the coset, except $C_{(0)}$, defines a circulant graph equivalent to the cycle graph \mathcal{C}_7 .

If $\mathcal{N}_1 : |k\rangle\langle k| \rightarrow \rho_k \in \mathcal{L}(\mathcal{B})$ is a C-Q channel induced from the OOR of \mathcal{C}_7 as in Theorem 2, then ρ_k is a state in a 5-dimensional Hilbert space and we have $\Upsilon(\mathcal{N}_1) = \vartheta(\mathcal{C}_7) = 3.317$. □

As shown in Example 2, $C_{(1)}$ defines the cycle graph \mathcal{C}_n and we have $\mathbb{Z}_n = \bigcup_{j=0}^{\frac{n-1}{2}} C_{(j)}$. Each nontrivial eigenvalue has multiplicity 2, as can be seen from $|C_{(j)}| = 2$ for $j \neq 0$, and $\lambda_{\min} = \lambda_{\frac{n-1}{2}} = \lambda_{\frac{n+1}{2}} = -2 \cos \frac{\pi}{n}$.

Corollary 5. Suppose \mathcal{N} is a C-Q channel induced by the OOR of the cycle graph \mathcal{C}_n as in Theorem 2. Then

$$\Upsilon(\mathcal{N}) = \vartheta(\mathcal{C}_n) = \frac{n \cos \frac{\pi}{n}}{1 + \cos \frac{\pi}{n}}.$$
□

B. Nontrivial Equal-sized Cyclotomic Cosets

When n is a prime power, \mathbb{Z}_n^\times is cyclic. Let $\mathbb{Z}_n^\times = \langle \alpha \rangle$, and α is of order $\varphi(n)$. Consequently, $-1 \equiv \alpha^{\varphi(n)/2}$. Therefore, $-1 \in C_{(1)} = \langle q \rangle$ if $q = \alpha^b$ for some $b \mid (\varphi(n)/2)$, and then $|C_{(1)}| = \frac{\varphi(n)}{b}$. It is clear that \mathbb{Z}_n^\times is equally partitioned by $C_{(1)}$. Furthermore, if $\mathbb{Z}_n^* \setminus \mathbb{Z}_n^\times$ can also be equally partitioned by $C_{(1)}$, then $X(\mathbb{Z}_n, C_{(1)})$ fits Theorem 3.

We first consider the case when n is not a prime.

Theorem 6. Let $n = p^r$ be a prime power. Suppose $\mathbb{Z}_n^\times = \langle \alpha \rangle$ for $\alpha \in \mathbb{Z}_p$. Then the graph $X(\mathbb{Z}_{p^r}, \langle \alpha^{p^{r-1}} \rangle)$ fits Theorem 3. □

Proof: We have $\varphi(n) = p^{r-1}(p-1)$ and then $\alpha^{p^{r-1}(p-1)} \equiv 1 \pmod{p}$. Let $C_{(1)}^r$ be the cyclotomic coset modulo p^r over $\alpha^{p^{r-1}}$ that contains 1. Thus $|C_{(1)}^r| = p-1$, which divides $\varphi(p^a)$ for $a = 1, \dots, r$. Also, $-1 \equiv (\alpha^{p^{r-1}})^{\frac{p-1}{2}} \in C_{(1)}^r$.

Let $pC = \{p\alpha : \alpha \in C\}$ for a set $C \subseteq \mathbb{Z}_{p^r}$. First, we have $\mathbb{Z}_p^* = \Gamma_p^p = C_{(1)}^1$. Also $\mathbb{Z}_{p^2}^* = \Gamma_{p^2}^{p^2} \cup \Gamma_{p^2}^p = \Gamma_{p^2}^{p^2} \cup p\Gamma_{p^2}^p$. Since $\Gamma_{p^a}^{p^a} = \mathbb{Z}_{p^a}^\times$ can be equally partitioned by the cyclotomic coset $C_{(1)}^a$ for any a as in the proof of Theorem 7, $\mathbb{Z}_{p^2}^*$ can be partitioned into cosets of size $p-1$. Observe that

$$\begin{aligned} \mathbb{Z}_{p^r}^* &= \Gamma_{p^r}^{p^r} \cup \Gamma_{p^{r-1}}^{p^r} \cup \dots \cup \Gamma_p^{p^r} \\ &= \Gamma_{p^r}^{p^r} \cup p \left\{ \Gamma_{p^{r-1}}^{p^{r-1}} \cup \dots \cup \Gamma_p^{p^{r-1}} \right\} \\ &= \Gamma_{p^r}^{p^r} \cup \left\{ \bigcup_{j=1}^{r-1} p^j \Gamma_{p^{r-j}}^{p^{r-j}} \right\}, \end{aligned} \tag{8}$$

where $\Gamma_{p^r}^{p^r} = \mathbb{Z}_{p^r}^\times$ can be equally partitioned by the cyclotomic coset $C_{(1)}^r$. Thus by induction, $\mathbb{Z}_{p^r}^*$ can be partitioned into cosets of size $p-1$.

Let $C_{(1)}^r \pmod{p^a} = \{a \pmod{p^a} : a \in C_{(1)}^r\}$. Since $\langle \alpha \pmod{p^r} \rangle = \mathbb{Z}_{p^r}^\times$, $\langle \alpha \pmod{p^a} \rangle = \mathbb{Z}_{p^a}^\times$ for any $a \leq r$. An interesting property is

$$C_{(1)}^r \pmod{p^a} = C_{(1)}^a.$$

Therefore, these cosets are exactly the cyclotomic cosets modulo p^r over $\alpha^{p^{r-1}}$ of equal size. Suppose $\Gamma_{p^a}^{p^a}$ is partitioned into the cyclotomic cosets $\{C_{(\alpha_1)}^a, C_{(\alpha_2)}^a, \dots, C_{(\alpha_{p-a-1})}^a\}$. Then by (8), the cyclotomic cosets of $\mathbb{Z}_{p^r}^*$ are

$$\{p^{r-a} C_{(\alpha_j)}^a\}.$$
■

Example 3. For $\mathbb{Z}_{125} = \langle 2 \rangle$, $-1 \equiv 2^{50}$ and $57 \equiv 2^{25}$. Let $C_{(1)}^3 = \langle 57 \rangle$ and we have

$$\begin{aligned} C_{(1)}^2 &= \{1, 7, 24, 18\}, \\ C_{(2)}^2 &= \{2, 14, 23, 11\}, \\ C_{(3)}^2 &= \{3, 21, 22, 4\}, \\ C_{(6)}^2 &= \{6, 17, 19, 8\}, \\ C_{(9)}^2 &= \{9, 13, 16, 12\} \end{aligned}$$

and

$$C_{(5)}^2 = \{5, 10, 20, 15\} = 5\{1, 2, 4, 3\} = 5\{C_{(1)}^1\}.$$

Consequently, $\mathbb{Z}_{125}^* \setminus \mathbb{Z}_{125}^\times = 5\{C_{(1)}^2 \cup C_{(2)}^2 \cup C_{(3)}^2 \cup C_{(6)}^2 \cup C_{(9)}^2\} \cup 25C_{(1)}^1$. \square

It is simpler for the case that n is a prime.

Theorem 7. Let $p = 2st + 1$ be a prime. Suppose $\mathbb{Z}_p^* = \langle \alpha \rangle$. Then the graph $X(\mathbb{Z}_p, \langle \alpha^t \rangle)$ fits Theorem 3. \square

Proof: In this case $\mathbb{Z}_n^* = \mathbb{Z}_n^\times$ and $\varphi(n) = n - 1$. Since $\alpha^{2st} \equiv 1 \pmod{p}$, the cyclotomic cosets modulo p over α^t are $C_{(1)}, C_{(\alpha)}, \dots, C_{(\alpha^{t-1})}$. Also, $-1 \equiv (\alpha^t)^s \in C_{(1)}$. These cosets are equal-sized and $\mathbb{Z}_p^* = \bigcup_{j=1}^t C_{(\alpha^j)}$. If $|C_{(\beta)}| < |C_{(1)}|$ for some β , then $\beta \alpha^{t|C_{(\beta)}|} \equiv \beta$. Since β is a unit in \mathbb{Z}_p^* , we must have $\alpha^{t|C_{(\beta)}|} \equiv 1$, which is a contradiction to the order of α . Then the result is straightforward. \blacksquare

Example 4. Consider $\mathbb{Z}_{37} = \langle 2 \rangle$. The following graphs satisfy the conditions in Theorem 7: $\mathcal{C}_{37} = X(\mathbb{Z}_{37}, \{1, 36\})$, $X(\mathbb{Z}_{37}, \{1, 6, 36, 31\})$, $X(\mathbb{Z}_{37}, \langle 27 \rangle)$, $\mathcal{CR}_{37} = X(\mathbb{Z}_{37}, \langle 8 \rangle)$, $\mathcal{QR}_{37} = X(\mathbb{Z}_{37}, \langle 4 \rangle)$. \square

C. Paley Graphs

When $t = 2$, the cosets in Theorem 7 lead to exactly the Paley graphs or the quadratic residue graphs \mathcal{QR}_p .

A nonzero integer a is called a quadratic residue modulo n if $a = b^2 \pmod{n}$ for some integer b ; otherwise, a is a quadratic nonresidue modulo n . Note that 0 is neither a quadratic residue, nor a nonresidue. Suppose $p \equiv 1 \pmod{4}$ is a Pythagorean prime. Let Q denote the set of quadratic residues modulo p and N the set of nonresidues. Since $p \equiv 1 \pmod{4}$, $-1 \in Q$. Then $\mathcal{QR}_p = X(\mathbb{Z}_p, Q)$ [28].

Suppose α is a primitive element of \mathbb{Z}_p . Then $Q = \{\alpha^c : c \text{ even}\}$ and $N = \{\alpha^c : c \text{ odd}\}$. It is clear that $|Q| = |N| = (p-1)/2$ and $\mathbb{Z}_p = Q \cup N \cup \{0\}$. By Eq. (5) and the formula for quadratic Gauss sum: $\sqrt{p} = \sum_{j=0}^{p-1} e^{2\pi i j^2/p}$, the eigenvalues of \mathcal{QR}_p are

$$\lambda_j = \begin{cases} (-1 + \sqrt{p})/2, & \text{if } j \in Q; \\ (-1 - \sqrt{p})/2, & \text{if } j \in N; \\ (p-1)/2, & \text{if } j = 0, \end{cases}$$

The Paley graphs are self-complimentary and consequently $\Theta(\mathcal{QR}_p) = \vartheta(\mathcal{QR}_p) = \sqrt{p}$ [2, Theorem 12]. In fact, $\alpha(\mathcal{QR}_p^{\boxtimes 2}) = p$ [29]. Let $b \in N$ and then $\{(a, ab \pmod{p}) : a \in \mathbb{F}_p\}$ is an independent set of size p in $\mathcal{QR}_p^{\boxtimes 2}$. For example, the smallest Paley graph is $\mathcal{QR}_5 = \mathcal{C}_5$, and $\{(0, 0), (1, 2), (2, 4), (3, 1), (4, 3)\}$ is an independent set of size five in $\mathcal{C}_5^{\boxtimes 2}$. This shows that the capacity can be achieved by two uses of a channel corresponding to \mathcal{QR}_p .

Corollary 8. Suppose \mathcal{N} is a C-Q channel induced by the OOR of the Paley graph \mathcal{QR}_p as in Theorem 2. Then

$$\Upsilon(\mathcal{N}) = \vartheta(\mathcal{QR}_p) = \sqrt{p}.$$

Proof:

The proof for Paley graphs is easier than the general proof in Theorem 3 since there are only three cyclotomic cosets and two nontrivial eigenvalues. The SDP (2) can be achieved by

$$R_0 = \frac{1}{p} \left(\mathbb{I} - P_0 - \frac{2}{\sqrt{p}+1} \sum_{j \in N} P_j \right).$$

One can show that

$$p[R_0]_{a,b} = \begin{cases} 1, & \text{if } a = b \in N; \\ \frac{-1+\sqrt{p}}{1+\sqrt{p}}, & \text{if } a = b \in Q; \\ -\left(\frac{2}{1+\sqrt{p}}\right)^2, & \text{if } a, b \in Q \text{ and } a - b \in N; \\ 0, & \text{otherwise.} \end{cases}$$

A key observation here is that

$$\sum_{b:b \neq a} |p[R_0]_{a,b}| = \begin{cases} p[R_0]_{a,a}, & \text{if } a \in Q; \\ 0, & \text{if } a \in N. \end{cases}$$

Then by Gershgorin's disk theorem, the eigenvalues of pR_0 are either 1 or lie in the disks with center $p[R_0]_{a,a}$ and radius $p[R_0]_{a,a}$ for $a \in Q$. Also note that R_0 is Hermitian and it has real eigenvalues. As a consequence, the eigenvalues of R_0 are nonnegative and thus $R_0 \geq 0$.

The null space of R_0 are spanned by $\sum_{j:j \in N} |u_j\rangle$ and $|u_0\rangle$, which implies $(1, 0, \dots, 0)$ is an eigenvector of R_0 with eigenvalue 0. ■

D. Cubic Residue Graphs

When $t = 3$, the cosets in Theorem 7 lead to the cubic residue graphs \mathcal{CR}_p [30]. A nonzero integer a is called a cubic residue modulo p if $a = b^3 \pmod p$ for some integer b . The cyclotomic coset $C_{(1)}$ consists of cubic residues.

$\mathcal{CR}_p = X(\mathbb{Z}_p, C_{(1)})$ has three nontrivial eigenvalues, which can be found by the formula for cubic Gauss sum. These three eigenvalues are the roots of $x^3 - 3px - ap = 0$, where $4p = a^2 + b^2$ for some integers $a \equiv 1 \pmod 3$ and b [31]. Currently the closed form for $\vartheta(\mathcal{CR}_p)$ is still unknown, since it is related to the determination of Gauss sums [32], [33].

These discussions can be extended to $t \geq 3$.

VI. DISCUSSION

We have shown that $\Upsilon(\mathcal{N}) = \eta(\{|u_k\rangle\})$ for \mathcal{N} induced by a class of circulant graphs that are defined by equal-sized cyclotomic cosets. The type of circulant graphs defined by equal-sized cyclotomic cosets bear very a strong symmetry.

It is interesting to see if there are other graphs that have this property. Let us focus on the case $\eta(\{|u_k\rangle\}) = \vartheta(G)$ in the following discussion. Most asymmetric graphs have $\vartheta(G) = \alpha(G)$ and they naturally lead to C-Q channels with $\Upsilon(\mathcal{N}) = \vartheta(G)$. Now we consider graphs with $\vartheta(G) > \alpha(G)$.

We say a graph G' is a *degenerate* graph of G if an orthonormal representation of G is also an orthonormal representation of G' , and hence their Lovász numbers are equal: $\vartheta(G) = \vartheta(G')$. We say a graph \hat{G} is *essential* if it has no proper subgraph $H \subset \hat{G}$ with $\vartheta(H) = \vartheta(\hat{G})$. Suppose $\{P_k\}$ is an orthonormal representation of the essential graph \hat{G} . Then two vertices i and j are connected if and only if $\text{Tr } P_i P_j \neq 0$. Apparently, for any graph G , it has an essential subgraph \hat{G} that lies in the intersections of all degenerate graphs of G .

Example 5. Consider Fig. 3. G_1 is an asymmetric graph and G_2 is a degenerate graph of G_1 , which consists of the cycle \mathcal{C}_5 and two segments. Their essential graph is the union of \mathcal{C}_5 and two isolated points and thus $\vartheta(G_1) = \vartheta(G_2) = \vartheta(\mathcal{C}_5) + 2 = \sqrt{5} + 2$.

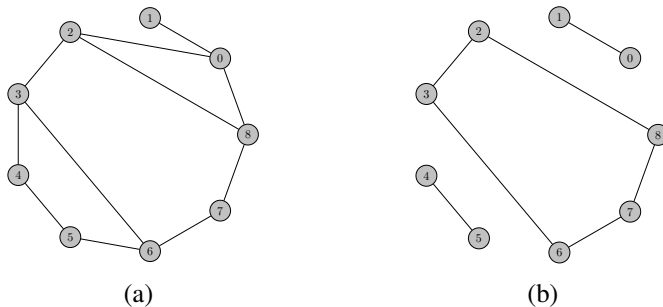


Fig. 3. (a) an asymmetric graph G_1 ; (b) a degenerate graph G_2 of G_1 .

As shown in Example 5, it is possible that an asymmetric graph has a degenerate graph with nonidentity automorphisms and $\vartheta(G) > \alpha(G)$.

Corollary 9. Suppose a graph G leads to a C-Q channel \mathcal{N} with $\Upsilon(\mathcal{N}) = \vartheta(G)$. Then so do the degenerate graphs of G .

Suppose a graph G has an essential graph \hat{G} with $\Upsilon(\hat{\mathcal{N}}) = \vartheta(\hat{G})$. It is also possible that G leads to $\Upsilon(\mathcal{N}) = \vartheta(G)$ as shown in the following example.

Example 6. Fig. 4 is a graph G whose essential graph is \mathcal{C}_5 . An orthonormal representation of \mathcal{C}_5 can be easily extended to an orthonormal representation of G by choosing $|u_5\rangle = |u_0\rangle$.

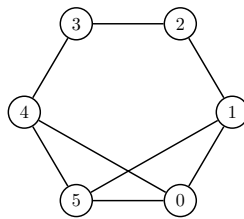


Fig. 4. A graph with essential graph C_5 .

However, generally it is nontrivial to construct an orthonormal representation of a graph from an orthonormal representation of its proper essential graph.

So far we have shown that circulant graphs defined by equal-sized cyclotomic cosets and their degenerate graphs would induce C-Q channels with $\Upsilon(\mathcal{N}) = \vartheta(G)$. Now let consider graphs other than these. First we show how to find an OOR for any graph G , following the proof of [2, Theorem 3]. After solving the SDP for $\vartheta(G)$ [2, Theorem 3], we end up with a symmetric matrix A satisfying (1) with the largest eigenvalue $\vartheta(G)$. Then there exist vectors $|x_1\rangle, \dots, |x_n\rangle \in \mathbb{R}^{d+1}$, where $d = \text{rank}(\lambda I - A)$, such that

$$\lambda \delta_{i,j} - [A]_{i,j} = \langle x_i | x_j \rangle$$

and the first entry of $|x_k\rangle$ is 0 for all k . Let $|c\rangle = (1, 0 \dots, 0) \in \mathbb{R}^{d+1}$ and

$$|u_k\rangle = \frac{1}{\sqrt{\lambda}}(|c\rangle + |x_k\rangle). \quad (9)$$

Then $\{|u_k\rangle\}$ is an OOR of G with value

$$\vartheta(G) = \frac{1}{|\langle c | u_k \rangle|^2}, \quad \forall k.$$

For a C-Q channel induced by $\{|u_k\rangle\}$ to have $\Upsilon(\mathcal{N}) = \vartheta(G)$ in the SDP (2), we must have

$$\langle c | R_k | c \rangle = 0, \quad \forall k. \quad (10)$$

That is, the first row and the first column of R_k are all zeros.

Example 7. Consider the Möbius ladder $M_8 = X(\mathbb{Z}_8, \{1, 4\})$ as shown in Fig. 5, which is circulant but beyond the scope of Theorem 3. Clearly we may choose $s_k = \vartheta(G)/n$ for all k . Let

$$R_0 = \frac{s_0}{\vartheta(G)} \left(\mathbb{I} - P_0 - \sum_{k=1}^7 x_k P_k \right)$$

and R_k can be defined by permuting the indices of P_k in R_0 appropriately. Since vertices 1, 8, and 4 are neighbors of vertex 0, we may choose $x_1 = x_7 = x_4 = 0$. We define a map:

$$\Gamma : i \mapsto n - i.$$

Apparently, Γ is an automorphism of M_8 . Assume $x_2 = x_6$ and $x_3 = x_5$. By solving the linear system from (10), we have $x_2 = x_6 = 0.5$ and $x_3 = x_5 = \frac{\vartheta(G)}{2} - 1$. Surprisingly, $\sum_k x_k = \vartheta(G)$, $0 \leq R_k \leq s_k(\mathbb{I} - P_k)$, and $\sum_k s_k P_k + R_k = \mathbb{I}$. Thus $\Upsilon(\mathcal{N}) = \vartheta(G)$ for $G = M_8$.

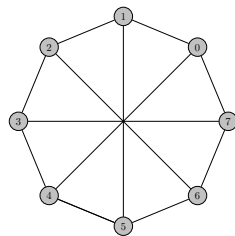


Fig. 5. Möbius ladder $M_8 = X(\mathbb{Z}_8, \{1, 4\})$.

To sum up, we have found many more graphs $\Upsilon(\mathcal{N}) = \vartheta(G)$. If this holds for general graphs, it would imply that $\vartheta(G)$ can be achieved by a single channel use. In fact, the techniques used in Example 7 can be generalized to other graphs. However,

we do not know how to prove $R_k \geq 0$. For example, we have $\Upsilon(\mathcal{N}) = \vartheta(G)$ for the graph $G = Z_7$ in Fig. 6 and its OOR constructed in (9). At the same time, the dual program (3) is satisfied with $T = \vartheta(G)|c\rangle\langle c|$ and $Q_k = 0$. Thus there should be a more unifying theory than Theorem 3 and this is our future research direction. Finally, Theorem 3 says that the one-shot QNSC-assisted zero-error capacity of a C-Q channel \mathcal{N} defined by an orthonormal representation $\{|u_k\rangle\}$ from Theorem 2 of a certain circulant graph is equal to the value of the representation: $\Upsilon(\mathcal{N}) = \eta(\{|u_k\rangle\})$. This suggests an even stronger conjecture: is it true that $\Upsilon(\mathcal{N}) = \eta(\{|u_k\rangle\})$ for an arbitrary graph and its arbitrary orthonormal representation?

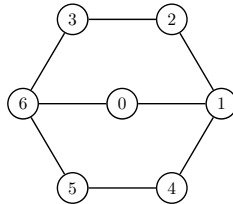


Fig. 6. A graph Z_7 that is not circulant, regular, or edge-transitive.

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