PRICING AMERICAN OPTIONS USING FOURIER ANALYSIS

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by

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Certificate

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

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Contents

Abstract

The analytic expression for an American option price under the Black-Scholes model requires the early exercise boundary as one of its inputs, and this is not known a priori. An implicit integral equation can be found for this free boundary, but it has no known closed-form solution, and its numerical solution is highly non-trivial. This has given rise to a number of analytical solution methods and numerical techniques designed to handle the early exercise feature.

The aim of this thesis is to explore Fourier-type solution methods for pricing American options. The price is defined as a free boundary value problem, whose solution satisfies the Black-Scholes PDE with certain final and boundary conditions. This problem is solved using the incomplete Fourier transform method of McKean (1965). The method is generalised to American options with monotonic and convex payoffs in a systematic way, and is further extended by applying it to solve the PIDE for the American call option under Merton's (1976) jump-diffusion model. In this case numerical integration solutions require an intense level of computation. The thesis considers the Fourier-Hermite series expansion method as an alternative approach. This is extended to allow for jump-diffusion with log-normally distributed jump sizes. The main contributions of the thesis are:

- *Evaluation of American Options under Geometric Brownian Motion - Chapters 2 and 3.* The details of McKean's (1965) incomplete Fourier transform are provided for a monotonic payoff function, and several forms for the price and free boundary are reproduced in the case of an American call. A numerical scheme for implementing the equations is given, along with a comparison of several existing numerical solution methods. The applicability of the transform technique to more general payoff types is demonstrated using an American strangle position with interdependent component options. A coupled integral equation system for the two free boundaries is found and solved using numerical integration. The resulting free boundaries are consistently deeper in-the-money than those for the corresponding independent American call and put.
- *Pricing American Options under Jump-Diffusion - Chapter 4.* The incomplete Fourier transform method is applied to the jump-diffusion model of Merton (1976). The PIDE for an American call is solved, and the results are simplified to replicate the integral equations of Gukhal (2001) for the price and free

boundary. An implicit expression for the limit of the free boundary at expiry is derived, and an iterative algorithm is presented for solving the integral expressions numerically. The results are demonstrated to be consistent with existing knowledge of American options under jump-diffusion, and display behaviour that is consist with market-observed volatility smiles.

- *Fourier-Hermite Series Expansions for Options under Jump-Diffusion - Chapter 5.* The Fourier-Hermite series expansion method is extended to the jump-diffusion model of Merton (1976) in the case where the jump sizes are log-normally distributed. With the aid of a suitably calibrated scaling parameter, the method is used to evaluate American call options. The pricing accuracy of this approach is shown to be comparable to both the iterative numerical integration method, and the method of lines technique by Meyer (1998). The series expansion method displays a high computation speed in exchange for some loss of accuracy in the free boundary approximation.

CHAPTER 1

Introduction

1.1. Literature Review and Motivation

The Black-Scholes option pricing formula has been a fundamental tool for option traders in financial markets around the world for many years. Following the derivations by Merton (1973) and Black and Scholes (1973) of the analytic pricing formulae for European call and put options on stocks, the Black-Scholes methodology has been extensively analysed, explored and expanded, with new extensions and applications of the theory arising constantly. One key area of derivatives pricing that remains the focus of a great deal of financial literature is that of American options.

An American option contract is in many ways identical to it's European counterpart. Both provide the holder with the right to buy (call option) or sell (put option) some underlying asset in the future for a pre-determined settlement price, known as the strike price. The key difference lies in the times at which the holder may exercise this right to purchase or sell. For a European option, the holder may only exercise at a pre-determined expiry date. In the case of an American option, however, the holder may exercise the contract at any time from the moment the contract is written, up until and including the expiry date. While this difference in exercise terms is fundamentally simplistic, it results in a pricing problem that is far more complicated to solve mathematically.

American options are commonplace instruments within modern financial markets. They are written on many underlying assets, including stocks, futures and foreign exchange rates. While the instruments themselves are readily found in the financial marketplace, a consistent pricing framework like the Black-Scholes formula is not in use. A key reason for this is that finding the price of an American option under the Black-Scholes framework is equivalent to solving an optimal stopping problem. While an analytic expression for the American option price exists, it is not easy to implement, especially when compared with the European case.

The analytic expression for an American option price requires the optimal early exercise boundary as one of its inputs, and this is not known a priori. While an integral expression can be obtained for this boundary, it results in an implicit integral equation that has no known closed-form solution, and whose numerical solution is highly non-trivial. Such complications do not arise when pricing European options, where special functions are employed to evaluate the relevant pricing formulae. Since extending the Black-Scholes-Merton analysis for European options to the American case does not appear simple on the surface, this has given rise to an extensive range of research into methods for solving the American option pricing problem. In a survey of this work, Myneni (1992) identified several fundamental methods that can be used to price American options. These can broadly be summarised into four distinct classes: approximate solutions; the compound options technique; discrete numerical methods; and the free boundary problem approach.

During the late 1970s, through to the mid 1980s when the Black-Scholes framework was first being extended to American options, computing power was not very extensive. This motivated a considerable amount of work into finding approximation formulae for the prices of American options. Roll (1977) proposed a method whereby the price of an American call could be approximated via a weighted linear sum of three European call options, chosen to replicate the behaviour of the American call when there was only one dividend payment expected during the life of the option. The method was refined by Geske (1979*a*), Whaley (1981) and Geske (1981), who finally demonstrated that the replicating portfolio used was not unique, and hence there was no unique solution for the American call price when found via this approach. Identifying an approximation for the early exercise boundary, Johnson (1983) produced an estimate for the American put price, using the European put as a lower bound. Blomeyer (1986) extended Johnson's approach to the case of an American put option written on an asset with one ex-dividend date prior to expiry, requiring at most 5 European put prices to generate the American put approximation.

The greatest appeal of approximation methods is that they can be implemented with great ease and evaluated very rapidly. MacMillan (1986) derived one such approximation for the American put, based on neglecting small-valued terms in the Black-Scholes partial differential equation (PDE). The same approach was used by Barone-Adesi and Whaley

(1987) to derive a quadratic approximation for American calls and puts written on commodities and futures. Such methods offered a high degree of computational efficiency, but they achieved this at significant cost to the pricing accuracy.

Another way to reduce the complexity of the American option is to instead consider a Bermudan-type option, where there are only a finite number of early exercise dates. Parkinson (1977) found a binomial approximation for the American put using this approach. A more complex method presented itself within the work of Geske (1979*b*), who studied the pricing of compound options under the Black-Scholes framework. When one assumes that the American option can only be exercised at a series of discrete time points, the price can be expressed as a compound option problem. The Geske-Johnson technique, named after the work of Geske and Johnson (1984), offers analytic solutions for American calls and puts under the assumption of discrete early exercise dates. Their solution covers both the presence and absence of dividends, and unlike the approximation approaches, these satisfy the Black-Scholes PDE exactly, along with all the relevant boundary and initial conditions. It was also demonstrated that as the number of early exercise dates was increased, the compound option solution converged to the continuous American option price.

The compound option methodology has subsequently found many American option applications. Whaley (1986) developed an approximation method based on the compound option approach, while Selby and Hodges (1987) and Schroder (1989) presented ways to reduce the required computation time for the method. Generalisation to two-factor models has also been successful, with Ho, Stapleton and Subrahmanyam (1997) extending the Geske-Johnson technique to the case where the risk-free rate evolves stochastically.

Since the early exercise feature added great complexity when finding exact solutions to the Black-Scholes PDE, a common alternative was to use numerical solution techniques, as these could be easily adapted to handle early exercise by using dynamic programming within a time-stepping algorithm. Approximating the continuous stochastic process for the stock price with a binomial tree, Cox, Ross and Rubinstein (1977) presented one of the simplest generalisations of the Black-Scholes framework that could easily handle American options. Binomial trees represent a numerical solution that involves discretising both the time and state variables, another example being finite difference methods. Brennan and Schwartz (1977) were the first to use finite differences to solve the Black-Scholes PDE for the price of an American put. Finite difference techniques are particularly powerful in that they can solve the Black-Scholes PDE for a vast range of American payoff functions without the approximations introduced by the binomial method. Since the method is convergent and unconditionally stable (in the case of implicit and Crank-Nicolson schemes), it has been the subject of many further American option studies, including Wu and Kwok (1997) and Barraquand and Pudet (1996).

The method of lines can be used to solve PDEs by first discretising the time domain to create a system of ordinary differential equations, which are then solved discretely at each time step. Meyer and van der Hoek (1997) applied this to American calls and puts, and proved that the method was convergent. Chiarella, El-Hassan and Kucera (1999) used an alternative method based on path integrals. Expanding the American put price in a Fourier-Hermite series, they derived a backward recursion algorithm for the price and early exercise boundary. The series expansion method was unique in that it required only time discretisation; the price estimate was provided as a continuous function of the underlying asset value. Unlike trees and finite differences, the method of lines and series expansions both demand that the early exercise boundary be estimated to a fine degree of precision during computation. This can make these methods more appealing in applications where a detailed account of the free boundary is sought.

The fourth methodology for American option pricing was adapted from the field of physics. There exists a number of physical problems known as "free boundary problems". Kolodoner (1956) provided an extensive discussion on several such problems, which typically arise from change of phase models, whose dynamics are governed by the well-known PDE called the "heat equation". It is well understood that this PDE is equivalent to the Black-Scholes PDE for option prices, after some suitable transformations.

The connection between free boundary problem theory and American option pricing was first made by McKean (1965) in an appendix to an economic discussion of warrants by Samuelson (1965). The warrants being analysed were the equivalent of American call options. To find the value of this American call, McKean modelled the underlying stock as a stochastic process, and derived the PDE for the American call's price, along with the correct final and boundary conditions. Under this formulation, the free boundary value problem for American options is related to the classical Stefan problem in the area of heat diffusion (see for example Rubinstein (1971) and Crank (1984)). The free boundary problem was solved using an incomplete Fourier transform, which produced an integral expression for the American call price, along with a corresponding integral equation for the free boundary. This approach allowed McKean to produce a solution that was, for all intents and purposes, equivalent to a Black-Scholes formula for American options. The only drawback was that the integral equations could not be solved analytically.

There has since been a large amount of research into American option pricing as a free boundary problem. Some alternative examples were considered by van Moerbeke (1974), (1976), and Karatzas (1988) used martingale methods to reproduce McKean's results. The most significant analysis of McKean's findings was provided by Kim (1990), who updated the Fourier transform results to account for risk-neutral pricing theory. Kim also proved that McKean's integral equations were consistent with the Geske-Johnson solution when the number of finite exercise dates is allowed to become infinitely large. This was the first indication that the compound option and free boundary approaches were mathematically equivalent. The results also yielded a decomposition of the American put price into its European value plus a positive early exercise premium term, an economic interpretation that was further supported by Jacka (1991), and Carr, Jarrow and Myneni (1992). At the same time, Jamshidian (1992) derived expressions for American option prices in the case of equities with a continuous dividend yield, again supporting Kim's findings. Mallier and Alobaidi (2000) also attempted to solve the PDE using partial Laplace transforms, but were unable to find a simple inversion for their result.

With the correct integral equations for the American option price and free boundary well established, it became clear that the greatest computational burden in numerically solving these equations would arise from the free boundary component. Once the free boundary was known, the task of calculating the option price was straightforward. In light of this fact, there has been a great deal of work devoted to finding efficient ways to calculate the early exercise boundary. Underwood and Wang (2000) implemented the integral equation for the boundray using a fixed-point iterative method, but found the method slow to converge. AitSahlia and Lai (1999) transformed the American option problem to a single canonical optimal stopping problem for Brownian motion. Having estimated the free boundary points at a small number of time points, they obtained a continuous approximation for the free boundary using a cubic spline, and then proceeded to numerically price the American option. Little, Pant and Hou (2000) produced a one-dimensional integral representation for the American put free boundary by noting that all values of the underlying asset from zero up to the critical stock price will satisfy a certain equality. They implemented this new representation numerically, and found that it yielded encouraging results.

Allegretto, Barone-Adesi and Elliott (1995) approximated the price of an American put by using a combination of relaxation techniques and numerical methods to quickly estimate the free boundary. Barron and Jensen (1990) used utility arguments to establish an obstacle stopping problem for the American call. The problem was solved using a stochastic optimal control model. Elliott, Myneni and Viswanathan (1990) looked at both finite and perpetual American options using a martingale approach, along with some initial extensions to American portfolios. Buchen, Kelly and Rodolfo (2000) explored an alternative approach, estimating the free boundary using cubic splines, and then applying an iterative scheme to McKean's integral equation for the free boundary. The spline provided an accurate estimate for the free boundary's derivative, and the method was proof that McKean's equations were more tractable than Carr et al. (1992) had claimed.

Huang, Subrahmanyam and Yu (1996) proposed a recursive method for finding the early exercise boundary of American options. They used the Geske-Johnson technique to find the free boundary at a small number of time points. Richardson extrapolation was used to estimate the entire boundary, which was then taken as an input for the analytic American option pricing formula. The method worked well, and this prompted Ju (1998) to propose an alternative approximation for the free boundary along the same lines, but using a piece-wise exponential function. The method was highly accurate, and Ju was able to draw the fundamental conclusion that the American option price was not overly sensitive to the value of the free boundary. Following on from this, AitSahlia and Lai (2001) proposed that the free boundary could be approximated using linear splines with very few knots, again with extremely accurate results. There is considerable value in these findings, as the bulk of the computation time in solving for American option prices is dedicated to solving for the free boundary. When this can be simplified without any loss in pricing accuracy, it becomes possible to compute accurate American option prices at much greater speeds.

Given that the free boundary behaves most erratically near the expiry date, and that McKean's integral equations for the boundary are singular at that time point, small-time expansions have been developed to approximate the free boundary close to expiry. Barles, Burdeau, Romano and Samsoen (1995) provided one such approximation for the free boundary of the American put, while Wilmott, Dewynne and Howison (1993) have conducted analysis for the free boundary of American call options written on dividend paying assets. Kuske and Keller (1998) also derived expressions for the American put early exercise boundary near expiry by solving the relevant integral equations asymptotically. One of the most extensive studies into the American put free boundary is that of Chen and Chadam (2000), who noted that several different approximations for the free boundary near expiry exist, and they sought to determine which ones, if any, were correct. They derived four different approximations, and showed that each was applicable for different time frames. Such approximations offer an alternative to numerical solutions for the free boundary near expiry, in which the free boundary's rapid change in slope is not always well handled.

With such a vast range of solution methods now established for American option problems in the basic Black-Scholes framework, there has been considerable work extending the various methods to more complex models and problems. Kim and Yu (1993) generalised Kim's results to several alternative diffusion processes, including absorbing Gaussian diffusion, residual volatility diffusion, and constant elasticity of variance diffusion. As detailed by Zhang (1997*a*), there exists a large range of option contracts where the payoff function involves more than one underlying asset. The exercise regions for American options on multiple assets have been analysed by Broadie and Detemple (1997) and Villeneuve (1999). American options have been priced in the case of stochastic volatility by Ritchken and Trevor (1999) and Cakici and Topyan (2000) using discrete GARCH models, and as stated previously, Ho et al. (1997) considered American options on equities when the risk-free rate is stochastic. The pricing of American bond options has been conducted by Chesney, Elliott and Gibson (1993) and Yu (1993), while Chiarella and El-Hassan (1999) have priced American bond options using the method of lines. Pham (1997) and Gukhal (2001) also extended the Black-Scholes American option results to allow for jump-diffusion in the asset dynamics.

The primary motivation for this thesis is the incomplete Fourier transform method of McKean (1965) as it applies to the American option pricing problem, and the accompanying simplifications and analysis of Kim (1990). The added complexity involved when pricing American options has given rise to a wide range of solution techniques. At present, no single technique has gained acceptance as a standard method for solving American option problems. We propose that the incomplete Fourier transform method is the most ideal choice for this purpose. It offers a simple generalisation of the European price solution within the Black-Scholes framework, and is readily extended to a broad class of payoff types and more complex stochastic dynamics. The method's flexibility is demonstrated by applying it to an American strangle portfolio, and showing how the transform can be used to solve the partial-integro differential equation (PIDE) in the case of American options under jump-diffusion. As an extension to this research, we generalise the Fourier-Hermite series expansion method of Chiarella et al. (1999) to allow for jump-diffusion in the underlying, and demonstrate its usefulness as an alternative Fourier-style analysis technique for this more complex problem.

1.2. Structure of the Thesis

There are three main topics of research within the thesis. The first two focus on the use of the incomplete Fourier transform method to solve free boundary problems for American options. The first of these is presented in Chapters 2 and 3, which focus on the price and free boundary of American options under the geometric Brownian motion model of Black and Scholes (1973). Chapter 4 uses the incomplete Fourier transform method to evaluate American options under the jump-diffusion model of Merton (1976). The third part, covered in Chapter 5, extends on this by applying the Fourier-Hermite series expansion method to price American options under jump-diffusion. Chapter 6 provides a summary of results and findings, along with potential related paths for future research.

1.2.1. Evaluation of American Options under Geometric Brownian Motion. American calls and puts are common derivatives in modern financial markets. Written on a range of underlying assets, including stocks, futures, and foreign exchange rates, there has long been a real-world demand for an extension of the Merton (1973) and Black and Scholes (1973) results for European call and put options to the American case. This has since been achieved by numerous studies, such as Karatzas (1988), Kim (1990), Jacka

(1991), Carr et al. (1992), and Jamshidian (1992), each using a varied approach, many of which require knowledge of the associated European option solution as a prerequisite. One exception to this is the often underestimated solution method of McKean (1965) which, at its most basic level, is a natural extension of the Black-Scholes solution to the case where early exercise is allowed. That McKean was able to derive the correct result without prior knowledge of the European solution is a testament to the method's broad scope. In Chapter 2 we provide a detailed review of the incomplete Fourier transform technique, and reconcile this with the work of Kim (1990) and Carr et al. (1992).

Using mathematical results from Kolodoner (1956), McKean derived the integral equation for both the price and early exercise boundary of an American call option as the solution to a free boundary problem using an incomplete Fourier transform. Since the equivalent complete transform can be used to solve the problem in the European case, this method was a natural extension of the Black-Scholes European call option solution methodology. Combined with Kim's simplifications and analysis, an analytic formula for the American option price was found, suggesting that it may be possible to solve the problem without resorting to numerical PDE solution techniques such as finite differences (Brennan and Schwartz 1977), or the binomial approximation (Parkinson 1977).

The Fourier transform method is made even more attractive in that it requires no time-discretisation, unlike the compound option method of Geske and Johnson (1984), which yields an equivalent result once the time step sizes are reduced to zero in the limiting sense. Fourier transforms also allow the problem to be solved in relatively general terms. Whereas the form of the payoff needs to be almost specific for most of the other cited methods, the incomplete Fourier transform is capable of generating some form of solution to the free boundary problem for a range of general payoffs. We do not endeavour to prove the existence or uniqueness of the solution in the case of general payoffs, but note that Jacka (1991) has done so for the American put case.

The focus in Chapter 2 is to demonstrate how to apply the incomplete Fourier transform technique for a monotonic payoff function, and then recover several forms for the price and free boundary in the case of an American call option. A numerical scheme for implementing the equations is also provided, accompanied by a selection of results.

The discussion serves to provide a practical explanation of the incomplete Fourier transform, whilst reconciling in full the results of McKean (1965), Kim (1990), and Carr et al. (1992).

To demonstrate the applicability of the transform technique to more general payoff types, in Chapter 3 we apply the method to an American strangle position, research that has since been accepted for publication (Chiarella and Ziogas 2003). Elliott et al. (1990) considered a related problem in the form of an American straddle, deriving the coupled integral equation system for the straddle's free boundaries. There is, however, no clear indication of how the system could be solved, nor what impact the interdependent free boundaries have on the price of the portfolio. The American portfolios under consideration are defined such that if exercised early, the entire payoff is optimally realised. In this way they are fundamentally different to a portfolio formed using independent American calls and puts. Chapter 3 demonstrates how the incomplete Fourier transform is applied to this problem. The coupled integral equation system for the two free boundaries is solved using numerical integration, and the results are compared with a standard strangle formed by combining independent American calls and puts. The topic is of considerable market significance, given that options are typically traded in positions (such as straddles and butterflies) as opposed to individual contracts.

1.2.2. Pricing American Options under Jump-Diffusion. Numerous studies into the returns of stocks and foreign exchange rates (e.g. Jarrow and Rosenfeld (1984), Ball and Torous (1985), Jorion (1988), Ahn and Thompson (1992), and Bates (1996)) indicate that real-world financial data contains leptokurtic features that are better described by jump-diffusion processes, rather than the pure-diffusion process of Black and Scholes (1973). Merton (1976) proposed a model for European options where the underlying asset followed jump-diffusion dynamics. The jumps arrive according to a Poisson process with random jump sizes. The model was later extended to the American option case by Pham (1997) using probability arguments, and Gukhal (2001) who extends the Geske-Johnson compound option approach of Kim (1990) to cater for jumps. In Chapter 4 we demonstrate how the incomplete Fourier transform of McKean (1965) can be used to derive the integral expression in the case of an American call option under jump-diffusion.

There have been several attempts at calculating the price and early exercise boundary for American options in the jump-diffusion setting. The binomial tree method of Amin

(1993) demonstrated that the addition of jumps was able to recreate the so-called "volatility smile" affect observed in market option prices. Zhang (1997*b*) and Carr and Hirsa (2003) used finite difference methods to solve the problem, while Wu and Dai (2001) used a multi-nomial tree approach. Meyer (1998) developed a recursive algorithm for use with the method of lines and d'Halluin, Forsyth and Vetzal (2003) were able to price American puts under jump-diffusion using a fixed-point iteration method.

We use the jump-diffusion model of Merton (1976) to demonstrate an extension of McKean's Fourier transform approach to price American options with more complex price dynamics. Chapter 4 solves the PIDE for an American call using the incomplete Fourier transform technique, and generalises the simplifications of Kim (1990) to recover the integral expressions of Pham (1997) and Gukhal (2001) for both the price and free boundary of the option. An expression for the limit of the free boundary at expiry is found, again based on Kim's analysis for the pure-diffusion case. An iterative algorithm is presented for solving the integral expressions numerically, and the results are demonstrated to be consistent with existing knowledge of American options under jump-diffusion, such as those of Amin (1993). The chapter thereby provides an extension of the methods outlined in Chapters 2 and 3 to the jump-diffusion model.

1.2.3. Fourier-Hermite Series Expansions for Options under Jump-Diffusion. The numerical implementation of the integral expressions for American calls under jump-diffusion in Chapter 4 suggests a simple way to analyse the properties of American option prices and free boundaries in this setting. A shortcoming of this approach is that a substantial amount of computation time is involved, which increases exponentially with the level of time discretisation used. This prompts us to consider another Fourier-type of solution method that can produce a space-continuous approximation for the American call price, along with an estimate of the early exercise boundary for a less cumbersome computational load.

Tree methods, such as Amin (1993) and Wu and Dai (2001) are at best a discrete approximation of the underlying process for asset returns, and this discretisation can introduce bias in prices if not correctly handled. Using finite differences, the problem can be solved as a variational inequality (Zhang 1997*b*), or by direct application of a scheme like Crank-Nicolson to the PIDE (Carr and Hirsa 2003). In this case the option price is found on a discrete space-time grid, and interpolation must be conducted for any spot price that falls between space grid points. Meyer (1998) extended the method of lines solution of Meyer and van der Hoek (1997) to price American calls and puts under jump-diffusion. The method is convergent and fast to compute, and yields an accurate free boundary estimate as part of the solution. This still requires discretisation in both dimensions, however, and is best suited to discrete jump sizes.

Applying ideas from the evaluation of path integrals, Chiarella et al. (1999) demonstrated how Fourier-Hermite series expansions could be used to price both European and American options under pure-diffusion dynamics. The method is fast to compute, simple to implement, and finds an estimate of the early exercise boundary as part of the solution for American options. Furthermore, the option price is estimated by a weighted series of basis functions which are continuous in the underlying asset, avoiding the need for two-dimensional discretisation. In Chapter 5 we extend this Fourier-based numerical method to cater for the jump-diffusion model of Merton (1976) in the case where the jump sizes are log-normally distributed. The method requires no discrete approximation for the jump size density, and with the aid of a suitably calibrated scaling parameter, can be readily applied to evaluate American call options. The results and performance of the series expansion approach are compared with the iterative numerical integration of Chapter 4, along with the method of lines technique of Meyer (1998), and conclusions are drawn on the conditions under which the various methods are most effective.

CHAPTER 2

Pricing American Options under Geometric Brownian Motion

2.1. Introduction

The evaluation of American options under the models of Black and Scholes (1973) and Merton (1973) has been investigated using a vast array of analytic techniques and numerical methods. Unlike European options, American options can be exercised at any time during the term of the derivative contract. As a result there exists some value of the underlying asset at which it is optimal to exercise the option, and this varies with time to maturity. There are numerous ways the problem can be explored within the Black-Scholes framework. McKean (1965) (who seems to have been the first to consider the problem) treats the American call as a free boundary value problem for the Black-Scholes partial differential equation (PDE). Using an incomplete Fourier transform, he obtains an integral expression for the American call price that involves the free boundary. Evaluating this expression at the early exercise boundary produces an integral equation for the free boundary. Although applied to the American call case, in principle this approach should be applicable to any general payoff function.

Parkinson (1977) considers the American put problem by taking series expansions of the solution in transform-space. For numerical implementation, he uses a binomial approximation of the continuous log-normal density for the stock price process. By assuming that the option can only be exercised at a discrete number of time points, Geske and Johnson (1984) treat the American put as a Bermudan option. They solve the problem using compound option theory, and offer the method as a discrete approximation for continuous American put prices. Kim (1990) takes the limit of the Geske-Johnson solution and shows that it yields an integral expression for the American put price, along with an integral equation for its free boundary, however this representation differs from that of McKean (1965). A similar result is found for the American call in the case where the underlying asset pays a continuous dividend yield. In both the call and put examples, the compound option method relies on explicit knowledge of the option payoff in order to proceed. Kim (1990) also shows that the limit of the Geske-Johnson method is equivalent to McKean's solution, as he converts McKean's integral equations into his own representation. Furthermore, Kim (1990) gives a clear economic interpretation to his solution for the American call and put, something that is extremely difficult to deduce from McKean's expression.

Jacka (1991) shows that the American put is equivalent to an optimal stopping problem. He confirms Kim's results, and shows that the solution is unique. Carr et al. (1992) contribute to the American put analysis by obtaining an alternative representation. They decompose the American put into its intrinsic value and time value components, and provide lower and upper bounds for the American put solution. Establishing a suitable hedging argument, Karatzas (1988) finds an expectation for the fair price of American contingent claims under the Black-Scholes model. In this way he finds the Snell envelope representation for the American put. Jaillet, Lamberton and Lapeyre (1990) use variational inequalities to evaluate American options. Little et al. (2000) derive an alternative representation for the free boundary of an American put. They take advantage of information about the stopping region to find an integral equation that is faster to evaluate numerically.

This chapter seeks to consolidate and give some perspective on some of these various contributions. In particular, we shall focus this survey on the results of McKean (1965), Kim (1990) and Carr et al. (1992). Taking the free boundary value problem approach, we use McKean's Fourier transform method to solve the Black-Scholes PDE for an American contingent claim with a general monotonic payoff function. In the process, we derive all the relevant properties of the incomplete Fourier transform. We demonstrate how to go from McKean's representation to that of Kim, and vice versa. The Carr-Jarrow-Myneni representation is also reproduced, and we consider the economic interpretations of the various representations. We draw out the fact that certain methods can cater for payoffs of a fairly general form whilst others (e.g. Kim (1990)) are tied rather strongly to the particular payoff being considered.

A brief comparison of numerical methods for calculating the American option price is provided, along with free boundary estimates where applicable. The numerical solutions presented include numerical integration of Kim's integral equations, the method of lines (Meyer and van der Hoek 1997), Fourier-Hermite series expansions (Chiarella et al. 1999), finite differences (Brennan and Schwartz 1977), and binomial trees (Cox et al. 1977). Comprehensive studies comparing various numerical methods for American option pricing have also been conducted by AitSahlia and Carr (1997), and Broadie and Detemple (1996).

The remainder of this chapter is structured as follows. Section 2.2 details the American option pricing problem under consideration. Section 2.3 proceeds to solve this problem using the incomplete Fourier transform method. We invert the transform of the solution in Section 2.4, and consider the specific example of an American call option in Section 2.5, detailing the various ways in which the integral equations can be represented. Section 2.6 uses the compound option solution method as applied to an American call, and Section 2.7 contains a comparison of the price and, where appropriate, early exercise boundary, as found using a range of numerical techniques. Conclusions follow in Section 2.8, with the various mathematical proofs provided in appendices.

2.2. Problem Statement - Monotonic Payoff

Let $C_a(S,t)$ be the price of an American option written on an underlying asset with price S at time t, and time to expiry $(T-t)$. The underlying pays a continuously compounded dividend at the rate q. Let the payoff function for the option be given by $c(S)$. We assume that $c(S)$ is a non-negative, monotonic increasing function of S (strictly so for all S such that $c(S)>0$), and that $c(S)\to 0$ as $S\to 0$. The early exercise boundary for this American option is denoted by $a(t)$. Figure 2.1 demonstrates the payoff and continuation region for $C_a(S,t)$ in the case where C_a is an American call option, with $c(S)=\max(S-K,0)$.

Under the assumption that the price, S, of the underlying asset is driven by the geometric Brownian motion

$$dS=\mu S dt+\sigma S dW, \tag{2.2.1}$$

with drift μ, volatility σ and Wiener process increments dW, it is known (for example using the standard hedging argument) that C_a satisfies the Black-Scholes PDE

$$\frac{\partial C_a}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C_a}{\partial S^2}+(r-q)S\frac{\partial C_a}{\partial S}-rC_a=0, \quad 0\le t\le T, \tag{2.2.2}$$

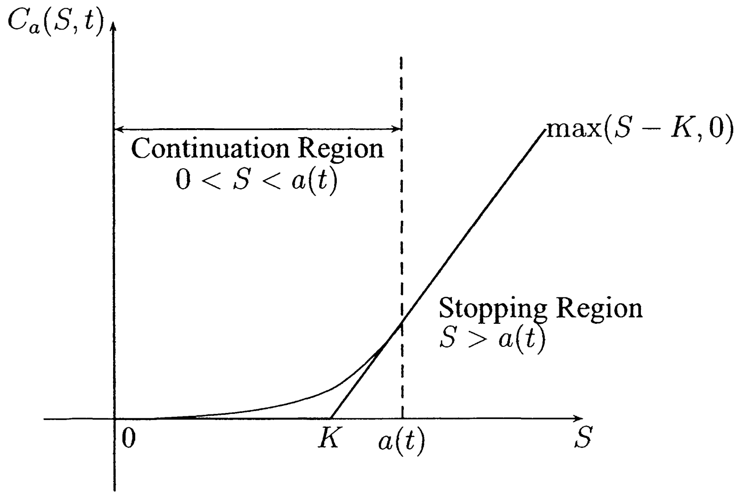

FIGURE 2.1. Continuation region in S-space for an American call option

in the region $0 < S < a(t)$, where r is the risk-free rate, and q is the dividend rate of S (continuously compounded), subject to the following final time and boundary conditions:

$$C_a(S,T) = c(S), \quad 0 < S < \infty, \tag{2.2.3}$$

$$C_a(0,t) = 0, \quad t \geq 0, \tag{2.2.4}$$

$$C_a(a(t),t) = c(a(t)), \quad t \geq 0, \tag{2.2.5}$$

$$\lim_{S \to a(t)} \frac{\partial C_a}{\partial S} = \left.\frac{dc(S)}{dS}\right|_{S=a(t)} = c'(a(t)), \quad t \geq 0. \tag{2.2.6}$$

Condition (2.2.3) is the payoff function for the option at expiry, while conditions (2.2.5)-(2.2.6) are collectively known as the "smooth-pasting" conditions. These ensure that the price, $C_a(S,\tau)$, and its first derivative with respect to S are both continuous. This is necessary to maintain the Black-Scholes assumption of an arbitrage-free market.

It is convenient to first transform the PDE (2.2.2) to a forward-in-time equation, with constant coefficients. Setting $S = e^x$ and $t = T - \tau$, we define the transformed function V_b by

$$C_a(S,t) = C_a(e^x, T-\tau) \equiv V_b(x,\tau). \tag{2.2.7}$$

The transformed PDE for V_b is then

$$\frac{\partial V_b}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V_b}{\partial x^2} + k\frac{\partial V_b}{\partial x} - rV_b, \quad 0 \leq \tau \leq T, \tag{2.2.8}$$

in the region $-\infty < x < \ln b(\tau)$ where $k = r - q - \frac{1}{2}\sigma^2$. The transformed free boundary is denoted by $b(\tau) \equiv a(T-\tau)$, and the payoff is now $v(x) \equiv c(e^x)$. The transformed initial and boundary conditions are

$$V_b(x,0) = v(x), \; -\infty < x < \infty, \tag{2.2.9}$$

$$\lim_{x\to-\infty} V_b(x,\tau) = 0, \quad \tau \geq 0, \tag{2.2.10}$$

$$V_b(\ln b(\tau),\tau) = v(\ln b(\tau)), \quad \tau \geq 0, \tag{2.2.11}$$

$$\lim_{x\to\ln b(\tau)} \frac{\partial V_b}{\partial x} = \left.\frac{dv(x)}{dx}\right|_{x=\ln b(\tau)} = v'(\ln b(\tau)), \quad \tau \geq 0. \tag{2.2.12}$$

In what follows, we will use the notation $b \equiv b(\tau)$ for simplicity, unless there is a particular reason to highlight the maturity dependence of b.

In order to be able to apply integral transform methods to solve this PDE for $V_b(x,\tau)$, the x-domain shall be extended to $-\infty < x < \infty$ by expressing the PDE as

$$H(\ln b - x)\left(\frac{\partial V_b}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 V_b}{\partial x^2} - k\frac{\partial V_b}{\partial x} + rV_b\right) = 0, \tag{2.2.13}$$

where $H(x)$ is the Heaviside step function, defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0, \\ 0, & x < 0. \end{cases} \tag{2.2.14}$$

The reason for the appearance of the factor of $\frac{1}{2}$ at the point of discontinuity is explained below. The initial and boundary conditions remain unchanged.

2.3. Applying the Fourier Transform

We propose to solve the free boundary value problem defined by equations (2.2.8)-(2.2.12) by using the Fourier transform technique to reduce the PDE to an ordinary differential equation (ODE), whose solution is readily obtainable. This is the same method used by McKean (1965). Given that the payoff function for the option is well-known to have a "binding" influence on the price and sensitivities of the associated option contract, we can safely assume the function V_b and its first two derivatives with respect to x can be treated as zero when x tends to $-\infty$. This assumption is subsequently justified by virtue of the fact that the general payoff function under consideration will have both a delta and gamma of zero for some large negative value of x, given the price behaviour specified in (2.2.10). Further justification arises in that the solution obtained satisfies the PDE and associated boundary conditions, and that the solution is unique[1].

Since the x-domain is now $-\infty < x < \infty$, the Fourier transform can be applied to the PDE. The Fourier transform of V_b, $\mathcal{F}\{V_b(x,\tau)\}$, is defined as

$$\mathcal{F}\{V_b(x,\tau)\} = \int_{-\infty}^{\infty} e^{i\eta x} V_b(x,\tau)dx.$$

Thus applying the Fourier transform to the PDE (2.2.13) we obtain

$$\begin{aligned}\mathcal{F}\{H(\ln b - x)\frac{\partial V_b}{\partial \tau}\} &= \frac{1}{2}\sigma^2\mathcal{F}\{H(\ln b - x)\frac{\partial^2 V_b}{\partial x^2}\} \\ &\quad +k\mathcal{F}\{H(\ln b - x)\frac{\partial V_b}{\partial x}\} - r\mathcal{F}\{H(\ln b - x)V_b\}.\end{aligned}$$

By definition

$$\begin{aligned}\mathcal{F}\{H(\ln b - x)V_b(x,\tau)\} &= \int_{-\infty}^{\infty} e^{i\eta x} H(\ln b - x)V_b(x,\tau)dx \\ &= \int_{-\infty}^{\ln b} e^{i\eta x} V_b(x,\tau)dx \\ &\equiv \mathcal{F}^b\{V_b(x,\tau)\} \equiv \hat{V}_b(\eta,\tau), \end{aligned} \tag{2.3.1}$$

where for convenience we introduce the notation $\hat{V}_b(\eta,\tau)$ to also denote the transform. We note that, $\mathcal{F}^b$ is an incomplete Fourier transform, since it is equivalent to a standard Fourier transform applied to $V_b(x,\tau)$ in the x-domain of $-\infty < x < \ln b$. The inversion formula for this incomplete Fourier transform is given in Proposition 2.3.1.

PROPOSITION 2.3.1. *The inverse of the Fourier transform of the function* $f(x,\tau) = H(a-x)g(x,\tau)$*, with* $a \equiv a(\tau)$*, is given by*

$$g(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(x,\tau)e^{i\eta x}dx\right]e^{-i\eta x}d\eta, \quad -\infty < x < a. \tag{2.3.2}$$

Proof: Refer to Appendix 2.2.

□

[1]This is a standard procedure in the solution of PDEs by integral transform methods; see for example Debnath (1995).

Equation (2.3.2) of Proposition 2.3.1 provides the basis for the inversion of the incomplete Fourier transform $\mathcal{F}^b$. Three specific properties of $\mathcal{F}^b$, given in Proposition 2.3.2, will allow us to convert equation (2.2.13) into an ODE for $\mathcal{F}^b$.

PROPOSITION 2.3.2. *Given the definition of $\mathcal{F}^b$ in equation (2.3.1), the following identities exist for $\mathcal{F}^b$:*

$$\mathcal{F}^b\left\{\frac{\partial V_b}{\partial x}\right\} = v(\ln b)e^{i\eta\ln b} - i\eta\hat{V}_b; \tag{2.3.3}$$

$$\mathcal{F}^b\left\{\frac{\partial^2 V_b}{\partial x^2}\right\} = e^{i\eta\ln b}(v'(\ln b) - i\eta v(\ln b)) - \eta^2\hat{V}_b; \tag{2.3.4}$$

$$\mathcal{F}^b\left\{\frac{\partial V_b}{\partial \tau}\right\} = \frac{\partial \hat{V}_b}{\partial \tau} - \frac{b'}{b}e^{i\eta\ln b}v(\ln b). \tag{2.3.5}$$

Proof: Refer to Appendix A2.3.1.

□

Note that in deriving the above results, we make use of the so-called "smooth-pasting" conditions given in equations (2.2.11)-(2.2.12). Applying the results of Proposition 2.3.2 to equation (2.2.13) we have:

PROPOSITION 2.3.3. *The incomplete Fourier transform of the PDE (2.2.13) with respect to x satisfies the ODE*

$$\frac{d\hat{V}_b}{d\tau} + \left(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r\right)\hat{V}_b = F(\eta,\tau), \tag{2.3.6}$$

where

$$F(\eta,\tau) = e^{i\eta\ln b}\left[\frac{\sigma^2 v'(\ln b)}{2} + \left(\frac{b'}{b} - \frac{\sigma^2 i\eta}{2} + k\right)v(\ln b)\right]. \tag{2.3.7}$$

The initial condition

$$\mathcal{F}\{V_b(x,0)\} \equiv \hat{V}_b(\eta,0)$$

may be calculated from equation (2.2.9).

Proof: Refer to Appendix A2.3.2.

□

Instead of solving a PDE for the function $V_b(x,\tau)$, we are now faced with the simpler task of solving the ODE (2.3.6) for the function $\hat{V}_b(\eta,\tau)$. This can then be inverted via the Fourier inversion theorem (see Appendix 2.2) to recover the desired function $V_b(x,\tau)$. Before concluding this section, we obtain the solution to (2.3.6).

PROPOSITION 2.3.4. *The solution $\hat{V}_b(\eta,\tau)$ to the ODE (2.3.6) in Proposition 2.3.3 is given by*

$$\hat{V}_b(\eta,\tau) = \hat{V}_b(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)\tau} + \int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)(\tau-s)}F(\eta,s)ds. \tag{2.3.8}$$

Proof: Recalling that b is a function of τ, the ODE (2.3.6) is of the form

$$\frac{d\hat{V}_b}{d\tau} + \alpha(\eta)\hat{V}_b = F(\eta,\tau),$$

where

$$\alpha(\eta) \equiv \frac{1}{2}\sigma^2\eta^2 + ki\eta + r.$$

Using the integrating factor $e^{\alpha(\eta)\tau}$, the solution to the ODE may be expressed as

$$\hat{V}_b(\eta,\tau)e^{\alpha(\eta)\tau} - \hat{V}_b(\eta,0) = \int_0^\tau F(\eta,s)e^{\alpha(\eta)s}ds,$$

which is readily reduced to equation (2.3.8).

□

2.4. Inverting the Fourier Transform

Having found $\hat{V}_b(\eta,\tau)$, the next step is to recover $V_b(x,\tau)$, the American option price in the x-τ plane. Taking the inverse (complete) Fourier transform of (2.3.8) gives

$$\begin{aligned} H(\ln b - x)V_b(x,\tau) &= \mathcal{F}^{-1}\{\hat{V}_b(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)\tau}\} \\ &\quad +\mathcal{F}^{-1}\left\{\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)(\tau-s)}F(\eta,s)ds\right\}. \end{aligned}$$

Applying the definition of the Heaviside function, this last equation may be expressed as

$$V_b(x,\tau) \equiv V_b^{(1)}(x,\tau) + V_b^{(2)}(x,\tau), \quad -\infty < x < \ln b(\tau). \tag{2.4.1}$$

We now determine explicit expressions for $V_b^{(1)}(x,\tau)$ and $V_b^{(2)}(x,\tau)$.

PROPOSITION 2.4.1. *The function* $V_b^{(1)}(x,\tau)$ *of the representation in equation (2.4.1) is given by*

$$V_b^{(1)}(x,\tau) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\ln b(0^+)} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} v(u)du. \tag{2.4.2}$$

Proof: Refer to Appendix A2.4.1.

□

The proof of Proposition 2.4.1 follows from a simple application of the convolution theorem for Fourier transforms.

PROPOSITION 2.4.2. *The function* $V_b^{(2)}(x,\tau)$ *of the representation in equation (2.4.1) is given by*

$$V_b^{(2)}(x,\tau) = \int_0^{\tau} \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}} [e^{-h(x,s)} Q(x,s)] ds, \tag{2.4.3}$$

where

$$h(x,s) = \frac{(x - \ln b(s) + k(\tau-s))^2}{2\sigma^2(\tau-s)}, \tag{2.4.4}$$

and

$$Q(x,s) = \frac{\sigma^2 v'(\ln b(s))}{2} + \left(\frac{b'(s)}{b(s)} + \frac{1}{2}\left[k - \frac{(x-\ln b(s))}{(\tau-s)}\right]\right) v(\ln b(s)) \tag{2.4.5}$$

for $-\infty < x < \ln b(\tau)$.

Proof: Refer to Appendix A2.4.2.

□

To arrive at Proposition 2.4.2 we apply the inverse transform directly, which subsequently involves evaluating integrals of the exponential of a quadratic function.

Hence with the values of $V_b^{(1)}(x,\tau)$ and $V_b^{(2)}(x,\tau)$ given by Propositions 2.4.1 and 2.4.2, we may use equation (2.4.1) to write the value of the American option in the x-τ plane as

$$V_b(x,\tau) = V_b^{(1)}(x,\tau) + V_b^{(2)}(x,\tau), \tag{2.4.6}$$

for $0 \leq \tau \leq T$ and $-\infty < x < \ln b(\tau)$. Equation (2.4.6) expresses the value of the American option in terms of the early exercise boundary, $b(\tau)$. At present this remains unknown, but we are able to obtain an integral equation that determines the free boundary by requiring the expression for $V_b(x,\tau)$ to satisfy the early exercise boundary condition

(2.2.11). Recalling our definition for the Heaviside function, the free boundary is thus found to satisfy the integral equation

$$\frac{v(\ln b(\tau))}{2} = V_b(\ln b(\tau), \tau), \qquad (2.4.7)$$

where $V_b(x, \tau)$ is given by equation (2.4.6) in conjunction with (2.4.2)-(2.4.5). The factor of $\frac{1}{2}$ appears in the left hand side of (2.4.7) due to properties of the Fourier transform. Recall that the complete Fourier transform was applied to a discontinuous function of the form $H(\ln b - x)f(x, \tau)$. As proved in Dettman (1965), the inverted Fourier transform of a discontinuous function will converge to the midpoint of the discontinuity, as illustrated in Figure 2.2 for the American call option example. Thus in equation (2.4.7), when V_b is evaluated at $\ln b(\tau)$, the factor of $\frac{1}{2}$ must be introduced into the left hand side. This is accounted for by our Heaviside function definition in equation (2.2.14).

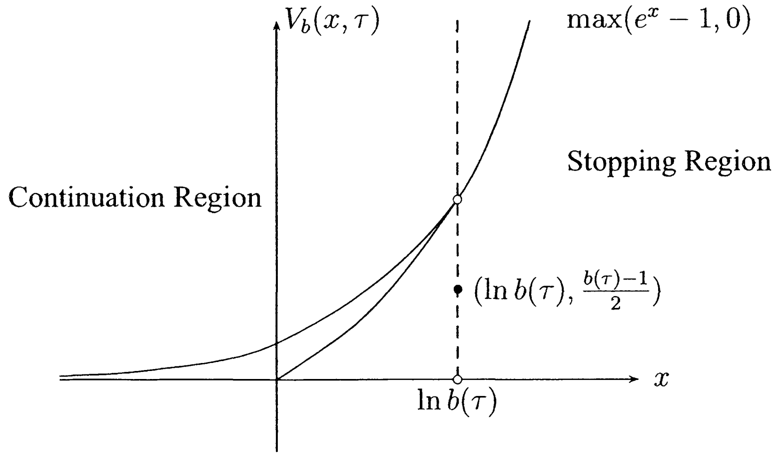

FIGURE 2.2. Behaviour of $V_b(x, \tau)$ at $x = \ln b(\tau)$ in the case of an American call option.

It should also be noted that by using the Fourier transform method, we have been able to derive equations (2.4.6)-(2.4.7) without specifying the exact form of the payoff $v(x)$, beyond a few basic properties. Such generality cannot be easily attained when using Kim's (1990) compound option approach, and demonstrates one of the significant advantages obtained from using integral transform solution techniques. Thus to price an American option with monotonic payoff $v(x)$, one must first solve the integral equation

(2.4.7) using numerical methods to find $b(\tau)$ since an analytical solution seems impossible. Once this is found, it is a simple matter to evaluate $V_b(x,\tau)$ from equation (2.4.6) via numerical integration.

2.5. Alternative Representations of the American Call Value

The Fourier transform approach is capable of handling a broad class of payoff types in a general form, as evidenced by the general price for an American option with monotonic payoff given by equation (2.4.6). While alternative methods, such as Kim (1990) and Carr et al. (1992), are more restrictive in that they require the payoff function to be known explicitly, the results thus obtained are far easier to interpret in an economic sense. Thus to demonstrate that McKean's method can readily yield these alternative representations, we shall consider the example of an American call option. This also allows us to derive the limit of the early exercise boundary at expiry, a task that is far more tractable using Kim's integral equations.

2.5.1. McKean's Representation. Equation (2.4.6) provides us with an integral expression for the price of the American call option in the $x - \tau$ plane. While it is convenient to define the price in terms of time until maturity, the log-transformation has little economic interpretation. Furthermore, we are unable to conduct any further analysis while the form of the payoff function $c(S)$ remains unspecified[2]. For the sake of definiteness, we shall consider an American call option with strike price K, for which $c(S) = \max(S-K,0)$, and hence $v(x) = \max(e^x - K, 0)$. By substituting this expression for $v(x)$ into (2.4.2) and (2.4.3), simplifying and transforming back to the original variable, S, the integral expression for the price of an American call may be written as (see Appendix A2.4.3)

$$\hat{C}_b(S,\tau) = C_E(S,\tau) - Se^{q\tau}N(d_1(S,\tau;b(0^+))) - Ke^{-r\tau}N(d_2(S,\tau;b(0^+))) \qquad (2.5.1)$$
$$+\int_0^\tau \frac{e^{-r(\tau-\xi)-\hat{h}(S,\tau)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\left[\frac{\sigma^2 b(\xi)}{2}+\left(\frac{b'(\xi)}{b(\xi)}+\frac{1}{2}\left[k-\frac{(\ln\frac{S}{b(\xi)})}{\tau-\xi}\right]\right)(b(\xi)-K)\right]d\xi,$$

[2]It is possible to carry forward the analysis by considering an affine payoff, where after the first non-zero value the payoff function is piecewise linear. The complication of this approach is that it introduces time-dependent structural breaks into the early exercise boundary $b(\tau)$ at times $\tau = t_1^*, t_2^*, \ldots$. These t^* values are determined by maintaining continuity in the free boundary at each t^*, in the same manner as described by Broadie and Detemple (1995) for capped American call options. We instead consider the simplest case of an American call option to keep the results presented both clear and concise.

where

$$\begin{aligned}\hat{C}_b(S,\tau) &= C_a(S,t),\\ C_E(S,\tau) &= Se^{-q\tau}N(d_1(S,\tau;K)) - Ke^{-r\tau}N(d_2(S,\tau;K)),\\ \hat{h}(S,\xi) &= \frac{(\ln\frac{S}{b(\xi)} + k(\tau-\xi))^2}{2\sigma^2(\tau-\xi)},\end{aligned}$$

with

$$\begin{aligned}d_1(x,\tau;\beta) &= \frac{(\ln(x/\beta) + (k+\sigma^2)\tau)}{\sigma\sqrt{\tau}},\\ d_2(x,\tau;\beta) &= d_1(x,\tau;\beta) - \sigma\sqrt{\tau},\\ b(0^+) &= \lim_{\tau\to 0^+} b(\tau),\end{aligned}$$

and

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{\alpha^2}{2}}\,d\alpha.$$

Since C_E is in fact the Black-Scholes price for a European call option written on S, equation (2.5.1) represents a decomposition of the American call price into its European value, given by C_E, and the premium paid for early exercise, determined by the remaining terms. This is the solution form presented by McKean (1965), and we henceforth refer to this as McKean's representation for the price of an American call option. While this form is a valid mathematical representation, it seems impossible to develop from it any economic meaning for the early exercise premium. The presence of the derivative of the free boundary, $b'(\tau)$, in the integral equation is also undesirable for the purposes of solving equation (2.5.1) numerically, since it creates numerical difficulties due to the infinite slope of $b(\tau)$ at maturity.

2.5.2. Kim's Representation. An alternative representation of the American call option price can be found by an approach due to Kim (1990). Kim arrived at a simplified form of equation (2.5.1) by taking the limit of the compound option approach to American option pricing (see Section 2.6). By manipulating (2.5.1) and applying integration by parts, Kim showed how the $b'(\tau)$ term could be removed. These manipulations are based on integration by parts, and it is important to note that unless the payoff is given explicitly, further simplification becomes impossible. In this way the simplifications of Kim (1990) are closely tied to the particular payoff function being considered. When the payoff is given explicitly, an important consequence of this re-expression of the American option price is that the early exercise premium becomes more readily interpreted. The result obtained by this approach may be stated as in Proposition 2.5.1.

PROPOSITION 2.5.1. *Using integration by parts, the American call price* $\hat{C}_b(S,\tau)$ *in equation (2.5.1) can be expressed as*

$$\begin{aligned}\hat{C}_b(S,\tau) &= C_E(S,\tau) + \int_0^\tau qSe^{-q(\tau-\xi)}N(d_1(S,\tau-\xi;b(\xi)))d\xi \qquad (2.5.2)\\ &\quad - \int_0^\tau rKe^{-r(\tau-\xi)}N(d_2(S,\tau-\xi;b(\xi)))d\xi,\end{aligned}$$

where $0 < S < b(\tau)$. *Furthermore, the free boundary* $b(\tau)$ *is given by*

$$\begin{aligned}b(\tau) - K &= C_E(b(\tau),\tau) + \int_0^\tau qb(\tau)e^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi\\ &\quad - \int_0^\tau rKe^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi. \qquad (2.5.3)\end{aligned}$$

Proof: Refer to Appendix A2.5.1.

□

Note that the factor of $\frac{1}{2}$ is no longer required when using Kim's integral equation for the American call price. While Kim does not discuss this detail in the original paper, we provide a more complete explanation in Appendix A2.5.1. Furthermore, by following the steps outlined in Appendix A2.5.1 in the reverse order, it is possible to return to McKean's representation for the American call price given by equation (2.5.1). The manipulations are not as intuitively obvious when going from (2.5.2) to (2.5.1), but they are certainly achievable nonetheless.

With Kim's representation it is now possible to give an economic interpretation to the early exercise premium. This premium is comprised of two integral components on the right-hand side of (2.5.2). Should the holder of the call exercise early, borrowing an amount K to purchase the underlying S, then the portfolio held will be of the form $(S-K)$. Thus the early exercise premium is the expected dividend earnings received by holding S, less the expected interest to be repaid on the loan of K. This represents the expected value of the cash flows that the holder of the American call can realise via the early exercise feature.

2.5.3. The Carr-Jarrow-Myneni Representation. There exists a third representation for the American call, first derived by Carr et al. (1992), that focuses on the time value of the American option. This is found by decomposing the value of a European call option into its intrinsic value and delayed exercise value.

PROPOSITION 2.5.2. *By first decomposing $C_E(S,\tau)$ in equation (2.5.2), the American call price $\hat{C}_b(S,\tau)$ can be expressed as*

$$\begin{aligned} \hat{C}_b(S,\tau) &= \max(S-K,0) + \frac{S\sigma^2}{2}\int_0^\tau e^{-q(\tau-\xi)}N'(d_1(S,\tau-\xi;K))d\xi \qquad (2.5.4) \\ &+ \int_0^\tau qSe^{-q(\tau-\xi)}[N(d_1(S,\tau-\xi;b(\xi))) - N(d_1(S,\tau-\xi;K))]d\xi \\ &- \int_0^\tau rKe^{-r(\tau-\xi)}[N(d_2(S,\tau-\xi;b(\xi))) - N(d_2(S,\tau-\xi;K))]d\xi, \end{aligned}$$

where $0 < S < b(\tau)$, *and*

$$N'(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}.$$

Proof: Refer to Appendix A2.5.2.

□

The intrinsic value component of the American call price is given by the present value of the payoff, $\max(S-K,0)$, equivalent to the immediate exercise value of the call. The additional terms represent the added value gained by delaying the exercise, which can also be interpreted as the time value of the American call option.

To better understand the economic meaning behind the integral terms in (2.5.4), consider a portfolio of $\hat{C}_b(S,\tau) - \max(S-K,0)$ held in the continuation region for the American call. The payoff component is achieved by investing K dollars in government bonds and shorting one unit of the underlying asset, S, only when the call is in-the-money. Note that this portfolio will have a net value of zero at expiry, or upon the underlying price entering the stopping region. Given this portfolio, the integral terms in (2.5.4) are the expected present value of the cash flows incurred during the life of this portfolio. The first integral term is the sum of movements in the underlying asset's price about the strike. The last two terms measure the dividends earned, and interest rate expense incurred, while the call is in-the-money but remaining unexercised.

2.5.4. The Free Boundary at Expiry. Before concluding this section, we shall present one additional result regarding the free boundary of the American call option, as found by Kim (1990). Using Kim's representation of the integral equation for the free boundary, it is relatively straight-forward to find the limit of $b(\tau)$ at expiry (i.e. as $\tau \longrightarrow 0^+$). This result is important when trying to solve equation (2.5.3) numerically for $b(\tau)$.

PROPOSITION 2.5.3. *Taking the limit as τ tends to 0^+ in equation (2.5.3), the value of the free boundary, $b(\tau)$, at expiry is given by*

$$b(0^+) = \max\left(K, \frac{r}{q}K\right). \tag{2.5.5}$$

Proof: Refer to Appendix 2.6.

□

Thus the value of $b(0^+)$ depends entirely on the relative parameter values of the risk-free rate, r, and the continuously compounded dividend yield, q. Note that when q is reduced to zero, the value for $b(0^+)$ becomes infinite, which coincides with the well-known result that it is never optimal to exercise an American call option early in the absence of dividends.

When $\tau = 0$, the decision whether or not to exercise the call depends entirely on the value of the underlying, S, when compared with the strike, K. As such, the early exercise boundary at expiry is simply given by $b(0) = K$. It is therefore important to note that a consequence of equation (2.5.5) is that $b(\tau)$ can be discontinuous at $b(0)$, and this occurs specifically when $r > q$. There has been some confusion regarding this detail in the literature, where many have defined $b(0)$ to be the result in equation (2.5.5), rather than $b(0^+) \equiv \lim_{\tau \to 0^+} b(\tau)$. Throughout the thesis we shall adopt this more explicit notation for the limit of $b(\tau)$ to avoid confusion.

2.6. American Call as a Compound Option

Kim (1990) was one of the first to confirm equations (2.5.2)-(2.5.3) for the American call using McKean's method, however his primary derivation was based on the compound option approach of Geske and Johnson (1984). Here we replicate this alternative approach, both to contrast the methodology with the incomplete Fourier transform, and to reiterate the equivalence of the results obtained by the two solution techniques.

For the American call, $\hat{C}_b(S, \tau)$, assume that we can only exercise at a finite number of time points $\tau_k, k = n, n-1, ..., 1, 0$ with $\tau_k - \tau_{k-1} = \Delta\tau$ for all k, and expiry occurs at τ_0 (i.e. $\tau = 0$). Let $p(S_{k-1}, \tau_{k-1}|S_k, \tau_k)$ be the transition density for S_{k-1} at time to maturity τ_{k-1}, given that price S_k was observed at time to maturity τ_k.

Let $U(S_k, k\Delta\tau; b_{k-1})$ denote the value of the unexercised call at time to maturity $k\Delta\tau$, where $b_{k-1} \equiv b((k-1)\Delta\tau)$. Since the holder of the call will not be able to exercise early until time $\Delta\tau$ in the discrete case, we find that at time $\Delta\tau$ prior to expiry, the unexercised call has value

$$\begin{aligned} U(S_1, \Delta\tau; K) = C_E(S_1, \Delta\tau) &= S_1 e^{-q\Delta\tau} N(d_1(S_1, \Delta\tau; K)) \\ &\quad - K e^{-r\Delta\tau} N(d_2(S_1, \Delta\tau; K)), \quad (2.6.1) \end{aligned}$$

which is simply a European call option with $\Delta\tau$ remaining until maturity. The early exercise boundary, b_1, is given by

$$b_1 - K = b_1 e^{-q\Delta\tau} N(d_1(b_1, \Delta\tau; K)) - K e^{-r\Delta\tau} N(d_2(b_1, \Delta\tau; K)). \quad (2.6.2)$$

With the starting case of $U(S_1, \Delta t; K)$ completed, we now develop an induction proof to find $U(S_n, n\Delta\tau; b_{n-1})$.

PROPOSITION 2.6.1. *The price of the unexercised call, U, at time to expiry $2\Delta\tau$ is*

$$\begin{aligned} U(S_2, 2\Delta\tau; b_1) &= C_E(S_2, 2\Delta\tau) + o(\Delta\tau) \quad (2.6.3) \\ &\quad + \int_{b_1}^{\infty} e^{-r\Delta\tau} \left[(1 - e^{-q\Delta\tau})S_1 - (1 - e^{-r\Delta\tau})K\right] p(S_1, \Delta\tau|S_2, 2\Delta\tau) dS_1. \end{aligned}$$

The early exercise price, b_2, at time to expiry $2\Delta\tau$ is defined implicitly by

$$b_2 - K = U(b_2, 2\Delta\tau; b_1).$$

Proof: Refer to Appendix A2.7.1.

□

Having derived an expression for U when $n = 2$, we proceed to find the value of U for a general non-zero integer value of n.

PROPOSITION 2.6.2. *The price of the unexercised call, U, at a general time until maturity $n\Delta\tau$ is*

$$
\begin{aligned}
U(S_n, n\Delta\tau, b_{n-1}) &= \sum_{k=1}^{n-1} e^{-(n-k)r\Delta\tau} \int_{b_k}^{\infty} \left[(1-e^{-q\Delta\tau})S_k - (1-e^{-r\Delta\tau})K\right] \\
&\qquad \times p(S_k, k\Delta\tau | S_n, n\Delta\tau) dS_k \\
&\quad + C_E(S_n, n\Delta\tau) + o(n\Delta\tau). \qquad (2.6.4)
\end{aligned}
$$

This equation is satisfied for $n = 2$, as shown in Proposition 2.6.1, and for $n = m$ and $n = m + 1$, where m is a non-negative integer.

Proof: Refer to Appendix A2.7.2.

□

Equation (2.6.4) provides us with the value of an unexercised American call option with discrete, equally spaced early exercise dates. The integral term calculates the expected present value of the cash flows incurred when holding the portfolio $S - K$ in the stopping region. Up until this point, the solution method is equivalent to that of Geske and Johnson (1984). Kim's contribution was to take the limit of U as $\Delta\tau \rightarrow 0$, thereby returning to the continuous American call case.

PROPOSITION 2.6.3. *The price of the unexercised American call, $\hat{C}_b$, at a general time until maturity $n\Delta\tau$ is*

$$
\begin{aligned}
\hat{C}_b(S_n, n\Delta\tau) &= C_E(S_n, n\Delta\tau) + o(n\Delta\tau) \qquad (2.6.5) \\
&\quad + \sum_{k=1}^{n-1} e^{-(n-k)\Delta\tau} \left(\int_{b_k}^{\infty} [qS_k - rK] p(S_k, k\Delta\tau | S_n, n\Delta\tau) dS_k \right) \Delta\tau.
\end{aligned}
$$

Taking the limit of (2.6.5) as $\Delta\tau \rightarrow 0$, this becomes equation (2.5.2) of Proposition 2.5.1.

Proof: Refer to Appendix A2.7.3.

□

There are several important observations one can make regarding this method. Firstly, it is important to note that the compound option approach requires that the payoff function be known explicitly before the analysis can be carried out. In particular, demonstrating that some of the terms are of order $\Delta\tau$ requires that the payoff be given in an explicit form. The initial steps of the Fourier transform method are not restricted by the need for an explicit payoff function, though we note that some information on the limits and derivative of the payoff are still required for meaningful analysis.

The second detail to note is that the compound option method requires that we first consider a discrete time situation, and then apply limit analysis to find the continuous case. When using Fourier transforms we are able to remain in continuous time at no significant increase in the mathematical complexity of the solution. In this sense the Fourier transform approach is a more natural extension of the PDE solution methods applied to European options. Using the compound option approach for American options introduces an additional level of theoretical complexity that can otherwise be avoided.

2.7. Numerical Examples

Equation (2.5.2) is an explicit expression for the price of an American call option, but it requires the free boundary, $b(\tau)$, to be known before it can be used. While $b(\tau)$ can be found by solving equation (2.5.3), there exists no known closed-form solution for the free boundary. This implies that one must use numerical methods in order to estimate the price and free boundary of the American call option. In this section we apply five existing numerical methods for pricing American options. We consider a 3-month call ($T - t = 0.25$), with strike $K = 100$ and volatility $\sigma = 20\%$. The first call under consideration has risk-free rate $r = 8\%$ and dividend yield $q = 12\%$. For the second call, we take $r = 12\%$ and $q = 8\%$. This allows us to demonstrate the results for the individual cases of $r < q$ and $r > q$.

The first method we use is the binomial tree procedure of Cox et al. (1977). We calculate the risk-neutral transition probabilities as detailed in de Jager (1995) (p.251-252), and structure the tree such that the nodes at expiry are centred about the strike. We use a tree with 10,000 layers ($\Delta t = 2.5 \times 10^{-5}$). Next we consider the finite difference solution for the PDE (2.2.2), subject to the boundary and final conditions (2.2.3)-(2.2.6). The method was first suggested in option pricing by Brennan and Schwartz (1977). The Crank-Nicolson scheme is used, with 4,000 space nodes between $S = 0$ and $S = 200$, and 400 time steps ($\Delta t = 6.25 \times 10^{-4}$). Note that both of these methods do not explicitly compute the early exercise boundary as part of the solution, instead using dynamic programming to check the early exercise condition at each time step.

The other three methods we consider all provide an estimate of the early exercise boundary, as they require this to be calculated in the process of finding the option price. Using techniques frequently applied to Volterra integral equations, we numerically integrate equation (2.5.3) to obtain the free boundary, as suggested by Kim (1990). The resulting free boundary estimate is then used to perform a simple numerical integration to solve (2.5.2) for the call price. We use a simplified version of the implementation outlined in Section 3.6 for the American strangle, and take 100 time steps ($\Delta t = 2.5 \times 10^{-3}$). Note that the method is applied twice, the second time using 200 time steps, and the free boundary estimates are then combined using Richardson extrapolation to ensure that the free boundary estimate is smooth and monotonic. A finer time grid was used for the first 2 time steps to help increase the accuracy near expiry.

The fourth technique is the method of lines, as given by Meyer and van der Hoek (1997). We used cubic splines for any interpolation, and 1,600 time steps ($\Delta t = 1.5625 \times 10^{-4}$). 40,000 space nodes were used in the region $0 < S < 200$. The large number of space nodes is required to help improve the smoothness of the free boundary estimate. We do not apply a finer grid near expiry in this case, and uniform grids are applied throughout. The last technique used is the Fourier-Hermite series expansion of Chiarella et al. (1999). 80 basis functions were used, along with 40 time steps.

In Tables 2.1 and 2.2 we present the price profiles generated by each of the five numerical methods for spot values of $S = 80, 90, 100, 110$ and 120. Table 2.1 considers the case where $r < q$, and we find that all five methods produce prices that are almost always consistent to 2 decimal places, and in many cases consistent to 3 decimal places. The most notable discrepancy appears in the Hermite series results when the call is in-the-money. Table 2.2 presents prices generated by the same methods, but in the case where $r > q$. It is interesting to note that the first 4 methods are all consistent to 3 decimal places for this example, but the Hermite series shows some signs of having more difficulty. At worst the Hermite method is consistent with the others to 1 decimal place, and is most accurate when the call is out-of-the-money.

To complete this comparison, we present plots of the free boundary estimates obtained for these American call options in the case where the strike has been rescaled to $K = 1$. Since the binomial tree and Crank-Nicolson methods do not attempt to estimate the early exercise boundary as part of their solution, we only consider the free boundaries given

S	Binomial	Crank-Nicolson	Integration	Method of Lines	Hermite
80	0.029	0.029	0.029	0.029	0.029
90	0.580	0.580	0.580	0.580	0.580
100	3.525	3.525	3.525	3.524	3.525
110	10.357	10.356	10.357	10.356	10.352
120	20.000	20.000	20.000	20.000	20.000

TABLE 2.1. Comparison of numerical solution methods for American call option prices with parameter values $K = 100$, $r = 0.08$, $q = 0.12$, $\sigma = 0.20$ and $T - t = 0.25$.

S	Binomial	Crank-Nicolson	Integration	Method of Lines	Hermite
80	0.052	0.052	0.052	0.052	0.052
90	0.841	0.841	0.841	0.841	0.840
100	4.396	4.396	4.396	4.396	4.392
110	11.546	11.546	11.546	11.546	11.535
120	20.691	20.691	20.691	20.691	20.676

TABLE 2.2. Comparison of numerical solution methods for American call option prices with parameter values $K = 100$, $r = 0.12$, $q = 0.08$, $\sigma = 0.20$ and $T - t = 0.25$.

by numerical integration, the method of lines and the Fourier-Hermite series expansions. Figure 2.3 presents the free boundary estimates when $r < q$. We find that numerical integration and method of lines provide very similar approximations for $b(\tau)$ in this case. The Hermite solution, however, appears to underestimate the free boundary profile. This is most likely because the polynomial basis functions are poor at estimating the call price when it is close in shape to the piecewise-linear payoff function, introducing some error in the free boundary near $\tau = 0$ that persists for larger values of τ. Note that the Hermite estimate is still smooth, and reaches some fixed level below the other estimates.

Figure 2.4 shows the early exercise boundary estimates when $r > q$. Again we observe that the numerical integration and method of lines approximations are extremely close. The Hermite series, however, is not performing very well by comparison. Note that at expiry, the method must start with $b(0) = K = 1$, and cannot make good use of the knowledge that $b(0^+) = Kr/q = 1.5$ in this case. As in the $r < q$ example, this error is attributable to the Hermite polynomial approximation being unable to provide a good fit for the American call price profile close to expiry. While this helps explain why the Hermite method was the least accurate when pricing the call for $r > q$, it is of value to note that the price differences are still very small, despite the more pronounced error

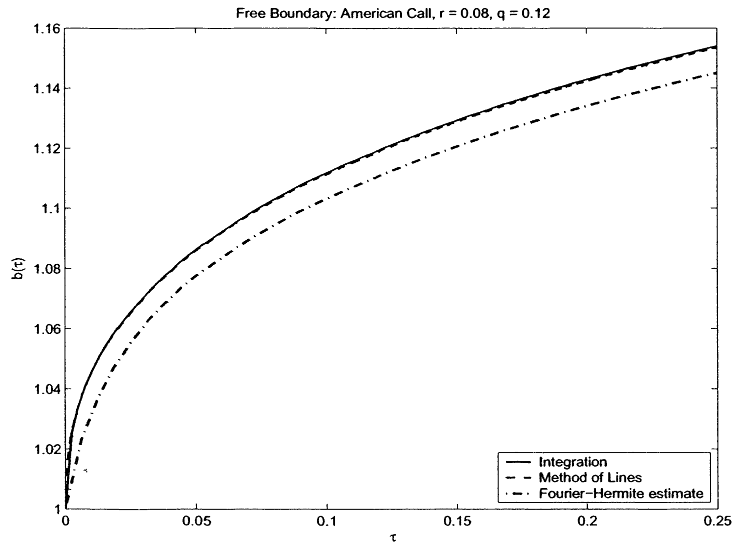

FIGURE 2.3. Comparing free boundary estimates for an American call option with $K = 1, \sigma = 20\%; r = 8\%, q = 12\%$, and $T - t = 0.25$.

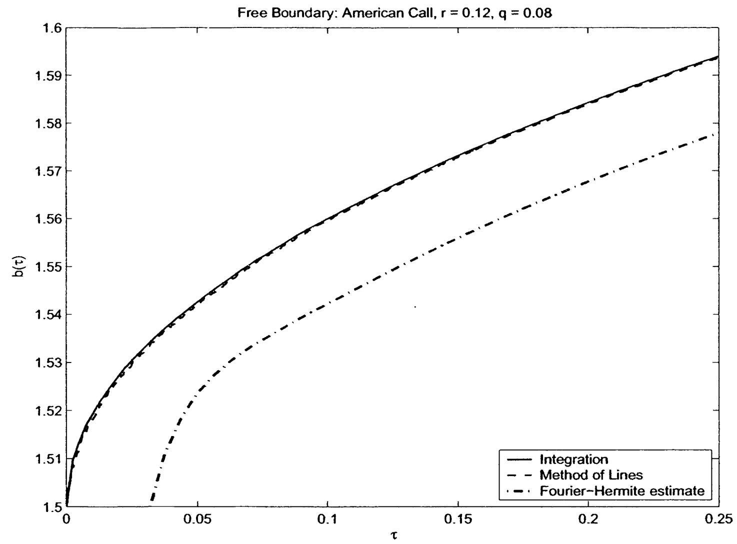

FIGURE 2.4. Comparing free boundary estimates for an American call option with $K = 1, \sigma = 20\%; r = 12\%, q = 8\%$, and $T - t = 0.25$.

that can be seen in the free boundary estimate. It is apparent, however, that the errors in Hermite prices which occur for the in-the-money call are most likely caused by the method providing a suboptimal free boundary estimate.

To compare the relative efficiency of these numerical methods, we provide an overview of the computation time required by each. The code for all five methods was implemented using LAHEY™FORTRAN 95 running on a PC with a Pentium 4 2.40 GHz processer, 512MB of RAM, and running the Windows XP Professional operating system. Table 2.3 lists the time required in seconds. The fastest method by far is the Crank-Nicolson scheme, needing only 0.781 seconds to find the price profile in the space-time grid. As mentioned previously, however, the method does not offer a comprehensive estimate of the early exercise boundary as part of the solution. The Fourier-Hermite series is the second fastest method, needing 0.875 seconds to solve the problem. This includes an estimate of the free boundary, but both the boundary and the in-the-money prices produced show some small degree of inaccuracy. It is interesting to note that such problems are not evident in the original results of Chiarella et al. (1999), where they report values only for the American put with no dividends. It could be that the method performs less well upon the introduction of a continuous dividend yield (which is necessary for the American call problem). Numerical integration is the next fastest, with a runtime of 2.719 seconds. The majority of this time is dedicated to estimating the free boundary. The option price is found at 400 values of S as part of the computation.

Method	Computation Time
Binomial[a]	4.875 sec
Crank-Nicolson[b]	0.781 sec
Integration[c]	2.719 sec
Method of Lines[b]	254.609 sec
Hermite[c]	0.875 sec

TABLE 2.3. Typical computation time for each of the numerical methods. All code was implemented using LAHEY™FORTRAN 95 running on a PC with a Pentium 4 2.40 GHz processer, 512MB of RAM, and running the Windows XP Professional operating system.
[a]This is the time required to find the price for a single value of S.
[b]These methods find the option price at all points within the grid as part of the solution.
[c]These computation times involved finding the option price at 400 different values of S as part of the calculations.

The binomial method needs only 4.875 seconds to find the option price, but this again does not include an accurate free boundary estimate, and the method only provides the price for a single value of S. Further prices require constructing a new tree in each case. Finally, the method of lines was the slowest of the five techniques, needing 254.609 seconds to compute in full. Like the Crank-Nicolson scheme, the method of lines finds option prices at all grid points, but it also provides a very accurate free boundary estimate. Note that the long computation time is clearly caused by the dense space-grid we have applied in order to keep the free boundary estimate monotonic. Thus there is evidence that the integration method, although being quite simple, can offer the best tradeoff in terms of accuracy and time efficiency.

2.8. Conclusion

In this chapter we have presented a survey of the methods for deriving the various integral representations of American option prices, with particular focus on the American call. We revisited McKean's (1965) incomplete Fourier transform method, and demonstrated how his results reconcile with the early exercise premium representation of Kim (1990), and the intrinsic/time value decomposition of Carr et al. (1992). We reviewed the compound option solution technique used by Kim for the American call option, and indicated that the method relies upon explicit knowledge of the payoff function to produce the final integral expression for the American option price. McKean's transform approach, on the other hand, is able to produce an integral expression based only on knowing that the function is monotonic. In this respect the incomplete Fourier transform demonstrates a higher degree of flexibility when considering a broader class of payoff functions within a single framework.

Given that there exists no closed-form solution for the American call option, we compared five existing numerical techniques. We found that binomial trees, the Crank-Nicolson finite difference scheme, direct numerical integration and the method of lines were all able to produce prices of comparable accuracy. The Fourier-Hermite series expansion method was relatively close to these other four methods, but showed some minor pricing inconsistencies. For numerical integration, the method of lines and the Hermite series expansion, we were also able to compare the free boundary estimates produced. While the first two methods were again highly consistent, the Hermite series showed some signs of error, and in particular was ill-suited to the case where the risk-free rate exceeded the continuous dividend yield of the underlying.

In terms of computational efficiency, finite differences proved the fastest method, although no free boundary estimate was generated as part of the solution. The method of lines proved to be the slowest, but this was caused by a highly demanding space-discretisation, designed to maximise the quality of the free boundary estimate. Numerical integration of Kim's integral equation for the early exercise boundary appeared to provide the best compromise between numerical accuracy and computational efficiency. The method seems highly attractive for simple problems, such as the American call under consideration, with most of the computation time being dedicated to finding the early exercise boundary.

This survey implies several directions for further research. Given that the incomplete Fourier transform is well-suited to general monotonic payoff functions, it should be possible to extend the methodology to consider American options with convex or concave payoffs. In Chapter 3 we consider one such example, in the form of an American strangle position. The method can also be applied to evaluate American options with more complex price dynamics, such as jump-diffusion models, which we demonstrate in Chapter 4. Two-dimensional extensions could also be considered, including American options under stochastic volatility, and American options on multiple underlying assets, such as an American basket option. When the asset dynamics are more complicated, direct numerical integration may become less efficient than in the simple case considered in this chapter. Under these circumstances, alternative methods such as Fourier-Hermite series expansions and the method of lines may provide a more optimal accuracy-efficiency tradeoff. In particular, the Hermite series is a computationally efficient method that surrenders some accuracy for gains in runtime. We show in Chapter 5 that in the case of jump-diffusion dynamics, the Fourier-Hermite series expansion offers an attractive level of accuracy coupled with inexpensive computational costs.

Appendix 2.1. Fundamental Results

Here we present a collection of fundamental results that are frequently required throughout the thesis. These results relate to the inversion of Fourier transforms and the evaluation of commonly recurring integrals.

A2.1.1. Convolution Theorem for Fourier Transforms. The Fourier transform of the convolution integral is given by

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f((x-u),\tau_1)g(u,\tau_2)du\right\} = \hat{F}(\eta,\tau_1)\hat{G}(\eta,\tau_2), \tag{A2.1.1}$$

where $\hat{F}$ and $\hat{G}$ are the Fourier transforms, with respect to x, of $f(x,\tau_1)$ and $g(x,\tau_2)$ respectively.

A2.1.2. Integrals of the Exponential-Quadratic Function. Let $\hat{p}$ and $\hat{q}$ be any complex functions not involving the integration variable η, with $Re(\hat{p}) \geq 0$. Furthermore, let n be any non-negative integer. Then the integral of the exponential-quadratic function with respect to η is given by

$$\int_{-\infty}^{\infty} e^{-\hat{p}\eta^2 - \hat{q}\eta}\eta^n d\eta = (-1)^n \sqrt{\frac{\pi}{\hat{p}}} \frac{\partial^n}{\partial \hat{q}^n} e^{\frac{\hat{q}^2}{4\hat{p}}}. \tag{A2.1.2}$$

In addition, we shall consider a more general form of this integral where the limits are finite with $n = 0$. Let $\hat{a}$, α_1 and α_2 be any real functions not involving η, along with $\hat{p}$ and $\hat{q}$ as before. Then the finite integral of the more general exponential-quadratic function with respect to η is given by

$$\int_{\alpha_1}^{\alpha_2} e^{\hat{a}\eta} e^{-\frac{(\hat{q}-\eta)^2}{\hat{p}}} d\eta = \sqrt{\hat{p}\pi} \exp\left\{\frac{(4\hat{q} + \hat{a}\hat{p})\hat{a}}{4}\right\} \{N[f(\alpha_2)] - N[f(\alpha_1)]\}, \tag{A2.1.3}$$

where

$$f(u) = \sqrt{\frac{2}{\hat{p}}} \left(\frac{2u - (2\hat{q} + \hat{a}\hat{p})}{2}\right).$$

Another useful exponential integral result arises when the exponent involves a sum of perfect squares. Define $\hat{p}$, $\hat{q}$, α_1 and α_2 to be real functions not involving η, with $\alpha_1, \alpha_2 > 0$. Then we can readily show that

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{[\eta + \hat{p}]^2}{\alpha_1} - \frac{[\eta + \hat{q}]^2}{\alpha_2}\right\} d\eta = \sqrt{\frac{\pi\alpha_1\alpha_2}{\alpha_1 + \alpha_2}} \exp\left\{-\frac{(\hat{p} - \hat{q})^2}{\alpha_1 + \alpha_2}\right\}. \tag{A2.1.4}$$

Appendix 2.2. The Incomplete Fourier Transform

Our aim is to prove that if $f(x, \tau) = H(a - x)g(x, \tau)$, $a \equiv a(\tau)$, and $H(x)$ is the Heaviside function, then application of the standard Fourier inversion theorem

$$f(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi, \tau) e^{i\eta\xi} d\xi\right] e^{-i\eta x} d\eta, \quad -\infty < x < \infty,$$

yields

$$g(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{a} g(\xi, \tau) e^{i\eta\xi} d\xi\right] e^{-i\eta x} d\eta, \quad -\infty < x < a,$$

which may be regarded as an inversion theorem for the incomplete Fourier transform.

Firstly,

$$\begin{aligned} RHS &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} H(a-\xi)g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta. \end{aligned}$$

Next consider

$$LHS = H(a-x)g(x,\tau) = \begin{cases} g(x,\tau), & -\infty < x < a \\ \frac{g(x,\tau)}{2}, & x = a \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$H(x-a)g(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta, \quad -\infty < x < \infty$$

or alternatively,

$$g(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta, \quad -\infty < x < a$$

and

$$\frac{g(x,\tau)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta, \quad x = a.$$

Refer to Section 2.4 for an explanation regarding the factor of $\frac{1}{2}$ on the left hand side.

Appendix 2.3. Properties of the Incomplete Fourier Transform

A2.3.1. Proof of Proposition 2.3.2. Firstly consider

$$\mathcal{F}^b\left\{\frac{\partial V_b}{\partial x}\right\} = V_b(\ln b,\tau)e^{i\eta\ln b} - i\eta\hat{V}_b(\eta,\tau).$$

By use of the boundary condition (2.2.11),

$$\mathcal{F}^b\left\{\frac{\partial V_b}{\partial x}\right\} = v(\ln b)e^{i\eta\ln b} - i\eta\hat{V}_b.$$

Next consider

$$\begin{aligned} \mathcal{F}^b\left\{\frac{\partial^2 V_b}{\partial x^2}\right\} &= \left.\frac{\partial V_b(x,\tau)}{\partial x}\right|_{x=\ln b}\cdot e^{i\eta\ln b} - i\eta\mathcal{F}^b\left\{\frac{\partial V_b}{\partial x}\right\} \\ &= v'(\ln b)e^{i\eta\ln b} - i\eta[v(\ln b)e^{i\eta\ln b} - i\eta\hat{V}_b], \end{aligned}$$

where the last equality follows by use of the boundary condition (2.2.12), and the transform result (2.3.3). This simplifies to

$$\mathcal{F}^b\left\{\frac{\partial^2 V_b}{\partial x^2}\right\} = e^{i\eta \ln b}(v'(\ln b) - i\eta v(\ln b)) - \eta^2 \hat{V}_b.$$

Finally consider

$$\begin{aligned}\mathcal{F}^b\left\{\frac{\partial V_b}{\partial \tau}\right\} &= \frac{\partial}{\partial \tau}\left[\int_{-\infty}^{\ln b} e^{i\eta x} V_b(x,\tau)dx\right] - \frac{b'}{b}e^{i\eta \ln b}V_b(\ln b,\tau)\\ &= \frac{\partial}{\partial \tau}\left[\mathcal{F}^b\{V_b\}\right] - \frac{b'}{b}e^{i\eta \ln b}V_b(\ln b,\tau),\end{aligned}$$

where $b' \equiv db(\tau)/d\tau$. Applying the boundary condition (2.2.11) we have

$$\mathcal{F}^b\left\{\frac{\partial V_b}{\partial \tau}\right\} = \frac{\partial \hat{V}_b}{\partial \tau} - \frac{b'}{b}e^{i\eta \ln b}v(\ln b).$$

A2.3.2. Proof of Proposition 2.3.3. Taking the incomplete Fourier transform of equation (2.2.8) with respect to x and using (2.3.3) - (2.3.5), we obtain

$$\begin{aligned}&\frac{\partial \hat{V}_b}{\partial \tau} + \left(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r\right)\hat{V}_b\\ &= e^{i\eta \ln b}\left[\frac{b'}{b}v(\ln b) + \frac{1}{2}\sigma^2(v'(\ln b) - i\eta v(\ln b)) + kv(\ln b)\right].\end{aligned}$$

It is a simple matter to rewrite this in terms of $F(\eta,\tau)$ to produce equations (2.3.6)-(2.3.7), and the initial condition is obtained by definition.

Appendix 2.4. Derivation of the American Call Integral Expression

A2.4.1. Proof of Proposition 2.4.1. Recall the definition of $V_b^{(1)}(x,\tau)$, namely

$$V_b^{(1)}(x,\tau) = \mathcal{F}^{-1}\left\{\hat{V}_b(x,0)e^{-(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)\tau}\right\}.$$

We shall evaluate this inverse Fourier transform using the standard Fourier convolution result given in equation (A2.1.1). To apply this convolution we first let

$$F(\eta,\tau_1) = e^{-(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)\tau}.$$

Hence

$$f(x,\tau_1) = \frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2\tau - i\eta(x+k\tau)}d\eta = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{\frac{-(x+k\tau)^2}{2\sigma^2\tau}}$$

by use of equation (A2.1.2) with $\hat{p} = \frac{1}{2}\sigma^2\tau$, $\hat{q} = i(x+k\tau)$ and $n = 0$.

Next we let $G(\eta,\tau_2)=\hat{V}_b(\eta,0)$. Hence we have

$$\begin{aligned} g(x,\tau_2) &= H(\ln b(0^+)-x)V_b(x,0) \\ &= H(\ln b(0^+)-x)v(x). \end{aligned}$$

Thus

$$\begin{aligned} V_b^{(1)}(x,\tau) &= \int_{-\infty}^{\infty}\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}}H(\ln b(0^+)-u)v(u)du \\ &= \int_{-\infty}^{\ln b(0^+)}\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}}v(u)du. \end{aligned}$$

A2.4.2. Proof of Proposition 2.4.2. We recall first that

$$V_b^{(2)}(x,\tau)=\mathcal{F}^{-1}\left\{\int_0^\tau F(\eta,s)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)(\tau-s)}ds\right\},$$

so that

$$V_b^{(2)}(x,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\eta x}\left\{\int_0^\tau F(\eta,s)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)(\tau-s)}ds\right\}d\eta,$$

where from equation (2.3.7),

$$F(\eta,s)=e^{i\eta\ln b(s)}\left[\frac{\sigma^2v'(\ln b(s))}{2}+\left(\frac{b'(s)}{b(s)}-\frac{\sigma^2 i\eta}{2}+k\right)v(\ln b(s))\right].$$

We can rewrite the function $F(\eta,s)$ as

$$F(\eta,s)=e^{i\eta\ln b(s)}\{f_1(s)-\eta f_2(s)\},$$

where we set

$$f_1(s)=\frac{\sigma^2v'(\ln b(s))}{2}+\left(\frac{b'(s)}{b(s)}+k\right)v(\ln b(s)),$$

and

$$f_2(s)=\frac{\sigma^2 i}{2}v(\ln b(s)).$$

Thus the inverse transformation of $V_b^{(2)}(x,\tau)$ becomes

$$V_b^{(2)}(x,\tau)=\frac{1}{2\pi}\int_0^\tau e^{-r(\tau-s)}\left[\int_{-\infty}^{\infty}e^{-\hat{p}\eta^2-\hat{q}\eta}\{f_1(s)-\eta f_2(s)\}d\eta\right]ds,$$

where $\hat{p} = \sigma^2(\tau - s)/2$, and $\hat{q} = i(x + k(\tau - s) - \ln b)$. Using the result in equation (A2.1.2) with $n = 0, 1$ we have

$$\begin{aligned} V_b^{(2)}(x,\tau) &= \frac{1}{2\pi}\int_0^\tau e^{-r(\tau-s)}\left[f_1(s)\sqrt{\frac{\pi}{\hat{p}}}e^{\frac{\hat{q}^2}{4\hat{p}}} + f_2(s)\sqrt{\frac{\pi}{\hat{p}}}e^{\frac{\hat{q}^2}{4\hat{p}}}\frac{\hat{q}}{2\hat{p}}\right] ds \\ &= \int_0^\tau \frac{e^{-r(\tau-s)+\frac{\hat{q}^2}{4\hat{p}}}}{2\sqrt{\pi\hat{p}}} \\ &\quad \times \left[\frac{\sigma^2 v'(\ln b(s))}{2} + \left(\frac{b'(s)}{b(s)} + k\right) v(\ln b(s)) + \frac{\sigma^2 i v(\ln b(s))\hat{q}}{2\sigma^2(\tau - s)}\right] ds. \end{aligned}$$

Substituting for $\hat{p}$ and $\hat{q}$, we find that

$$\begin{aligned} V_b^{(2)}(x,\tau) &= \int_0^\tau \frac{e^{-r(\tau-s)-h(x,s)}}{\sigma\sqrt{2\pi(\tau-s)}} \qquad \text{(A2.4.1)} \\ &\quad \times \left[\frac{\sigma^2 v'(\ln b(s))}{2} + \left(\frac{b'(s)}{b(s)} + \frac{1}{2}\left[k - \frac{(x - \ln b(s))}{(\tau - s)}\right]\right) v(\ln b(s))\right] ds, \end{aligned}$$

where we set

$$h(x,s) \equiv \frac{(x - \ln b(s) + k(\tau - s))^2}{2\sigma^2(\tau - s)}.$$

With a simple change of notation, equation (A2.4.1) may be written as it appears in equations (2.4.3)-(2.4.5).

A2.4.3. McKean's Representation for an American Call. If we set $c(S) = \max(S - K, 0)$, this implies that $v(x) = \max(e^x - K, 0)$, and equation (2.4.2) becomes

$$\begin{aligned} V_b^{(1)}(x,\tau) &= \int_{\ln K}^{\ln b(0^+)} \frac{e^u e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} du - K\int_{\ln K}^{\ln b(0^+)} \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} du \\ &\equiv I_1 - KI_2. \end{aligned}$$

To simplify $V_b^{(1)}(x,\tau)$ further, we shall re-express it in terms of the cumulative standard normal distribution, $N(y)$. For the first term, I_1, we can evaluate this using equation (A2.1.3) with $\alpha_1 = \ln K$, $\alpha_2 = \ln b(0^+)$, $\hat{a} = 1$, $\hat{q} = x + k\tau$ and $\hat{p} = 2\sigma^2\tau$. Recalling that $k = r - q - \frac{1}{2}\sigma^2$, and defining $d_1(x,\tau;\beta) \equiv (\ln(x/\beta) + (k + \sigma^2)\tau)/\sigma\sqrt{\tau}$, I_1 then becomes

$$I_1 = e^x e^{-q\tau}[N(d_1(e^x,\tau;K)) - N(d_1(e^x,\tau;b(0^+)))].$$

For the second term, I_2, by defining $d_2(x,\tau;\beta) \equiv (\ln(x/\beta) + k\tau)/\sigma\sqrt{\tau}$, the integral becomes

$$I_2 = e^{-r\tau}[N(d_2(e^x,\tau;K)) - N(d_2(e^x,\tau;b(0^+)))].$$

Thus it is concluded that

$$\begin{aligned} V_b^{(1)}(x,\tau) &= [e^x e^{-q\tau}N(d_1(e^x,\tau;K)) - Ke^{-r\tau}N(d_2(e^x,\tau;K))] \\ &\quad -[e^x e^{-q\tau}N(d_1(e^x,\tau;b(0^+))) - Ke^{-r\tau}N(d_2(e^x,\tau;b(0^+)))] \end{aligned}$$

It is worth noting that in the case where $r \le q$, $b(0) = K$, as proven by Kim (1990), and this in turn implies that $V_b^{(1)}(x,\tau) = 0$.

Having evaluated $V_b^{(1)}(x,\tau)$, it is a simple matter to evaluate $V_b^{(2)}(x,\tau)$ when $v(x) = \max(e^x - K, 0)$, and reverting back to the original underlying asset variable via $S = e^x$ we obtain (2.5.1).

Appendix 2.5. Alternative Representations of the American Call Price

A2.5.1. Proof of Proposition 2.5.1. We begin by expressing equation (2.5.1) as

$$\hat{C}_b(S,\tau) = C_E(S,\tau) - Se^{-q\tau}N(d_1(S,\tau;b(0^+))) + Ke^{-r\tau}N(d_2(S,\tau;b(0^+))) + R(S,\tau),$$

where

$$\begin{aligned} R(S,\tau) &= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} e^{-\hat{h}(S,\xi)} \\ &\quad \times \left[\frac{\sigma^2 b(\xi)}{2} + \left(\frac{b'(\xi)}{b(\xi)} + \frac{1}{2}\left[k - \frac{\ln\frac{S}{b(\xi)}}{\tau-\xi}\right]\right)(b(\xi) - K)\right] d\xi. \end{aligned}$$

Following Kim (1990), we aim to remove the $b'(\xi)$ term from the integral $R(S,\tau)$. We begin by expressing $\hat{h}(S,\xi)$ as

$$\begin{aligned} \hat{h}(S,\xi) &= \frac{[\ln S - \ln b(\xi) + (r - q - \frac{1}{2}\sigma^2)(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)} \\ &= \frac{1}{2(\tau-\xi)}\left(\frac{\ln S + (r-q-\frac{1}{2}\sigma^2)\tau}{\sigma} - \frac{\ln b(\xi) + (r-q-\frac{1}{2}\sigma^2)\xi}{\sigma}\right)^2 \\ &= \frac{[x - P(\xi)]^2}{2(\tau-\xi)}, \end{aligned}$$

where $x \equiv [\ln S + (r - q - \frac{1}{2}\sigma^2)\tau]/\sigma$ and $P(\xi) \equiv [\ln b(\xi) + (r - q - \frac{1}{2}\sigma^2)\xi]/\sigma$. Note also that $P'(\xi) = \left(\frac{b'(\xi)}{b(\xi)} + (r - q - \frac{1}{2}\sigma^2)\right)/\sigma$. Thus $R(S,\tau)$ may be rewritten as

$$\begin{aligned} R(S,\tau) &= \int_0^\tau \frac{e^{-r(\tau-\xi)-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}}\left[\frac{\sigma b(\xi)}{2} + \frac{1}{\sigma}\left(\frac{b'(\xi)}{b(\xi)} + (r-q-\frac{1}{2}\sigma^2) - (r-q-\frac{1}{2}\sigma^2)\right.\right. \\ &\quad \left.\left. +\frac{1}{2}\left[(r-q-\frac{1}{2}\sigma^2) - \frac{\ln\frac{S}{b(\xi)}}{\tau-\xi}\right]\right)(b(\xi)-K)\right]d\xi. \end{aligned}$$

It follows that

$$\begin{aligned} R(S,\tau) &= \int_0^\tau e^{-r(\tau-\xi)}\frac{e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}}\left[\frac{\sigma b(\xi)}{2} + \left(P'(\xi) - \frac{1}{\sigma}(r-q-\frac{1}{2}\sigma^2)\right.\right. \\ &\quad \left.\left. -\frac{1}{2\sigma}\left[\frac{\ln S - \ln b(\xi) - (r-q-\frac{1}{2}\sigma^2)(\tau-\xi)}{\tau-\xi}\right]\right)(b(\xi)-K)\right]d\xi, \end{aligned}$$

which implies the linear decomposition

$$\begin{aligned} R(S,\tau) &= \int_0^\tau e^{-r(\tau-\xi)}\frac{e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}}\left[\frac{\sigma b(\xi)}{2} + \left(P'(\xi) - \frac{x-P(\xi)}{2(\tau-\xi)}\right)(b(\xi)-K)\right]d\xi \\ &\equiv R_1(S,\tau) - KR_2(S,\tau), \end{aligned}$$

where

$$R_1(S,\tau) = \int_0^\tau e^{-r(\tau-\xi)}b(\xi)\frac{e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}}\left[\frac{\sigma}{2} + P'(\xi) - \frac{x-P(\xi)}{2(\tau-\xi)}\right]d\xi,$$

and

$$R_2(S,\tau) = \int_0^\tau e^{-r(\tau-\xi)}\frac{e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}}\left[P'(\xi) - \frac{x-P(\xi)}{2(\tau-\xi)}\right]d\xi.$$

Beginning with $R_1(S,\tau)$ we have

$$\begin{aligned}
R_1(S,\tau) &= \int_0^\tau e^{-r(\tau-\xi)}\frac{b(\xi)}{\sqrt{\tau-\xi}}\left[\frac{\sigma(\tau-\xi)+2P'(\xi)(\tau-\xi)-x+P(\xi)}{2(\tau-\xi)}\right] \\
&\quad \times\frac{1}{\sqrt{2\pi}}e^{-\frac{[x-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}}e^{(x-P(\xi))\sigma+\frac{\sigma^2}{2}(\tau-\xi)}d\xi \\
&= -\int_0^\tau e^{-q(\tau-\xi)}S\frac{1}{\sqrt{2\pi}}e^{-\frac{[x-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} \\
&\quad \times\left[\frac{\frac{1}{2}\frac{1}{\sqrt{\tau-\xi}}(x-P(\xi)+\sigma(\tau-\xi))-(P'(\xi)+\sigma)\sqrt{(\tau-\xi)}}{(\sqrt{\tau-\xi})^2}\right]d\xi \\
&= -\int_0^\tau e^{-q(\tau-\xi)}S\frac{1}{\sqrt{2\pi}}e^{-\frac{[x-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}}\frac{\partial}{\partial\xi}\left[\frac{x-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right]d\xi \\
&= -\int_0^\tau e^{-q(\tau-\xi)}S\frac{\partial}{\partial\xi}N\left(\frac{x-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)d\xi.
\end{aligned}$$

Repeating this process for $R_2(S,\tau)$, we produce

$$\begin{aligned}
R_2(S,\tau) &= -\int_0^\tau e^{-r(\tau-\xi)}\frac{1}{\sqrt{2\pi}}e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}\left[\frac{-\sqrt{\tau-\xi}P'(\xi)+(x-P(\xi))\frac{1}{2}\frac{1}{\sqrt{\tau-\xi}}}{(\sqrt{\tau-\xi})^2}\right]d\xi \\
&= -\int_0^\tau e^{-r(\tau-\xi)}\frac{1}{\sqrt{2\pi}}e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}\frac{\partial}{\partial\xi}\left[\frac{x-P(\xi)}{\sqrt{\tau-\xi}}\right]d\xi \\
&= -\int_0^\tau e^{-r(\tau-\xi)}\frac{\partial}{\partial\xi}N\left(\frac{x-P(\xi)}{\sqrt{\tau-\xi}}\right)d\xi.
\end{aligned}$$

Substituting for R_1 and R_2 in $R(S,\tau)$ gives

$$\begin{aligned}
R(S,\tau) &= -\int_0^\tau e^{-q(\tau-\xi)}S\frac{\partial}{\partial\xi}N\left(\frac{x-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)d\xi \\
&\quad +K\int_0^\tau e^{-r(\tau-\xi)}\frac{\partial}{\partial\xi}N\left(\frac{x-P(\xi)}{2\sqrt{\tau-\xi}}\right)d\xi \\
&= -S\Bigg\{\left[e^{-q(\tau-\xi)}N\left(\frac{x-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)\right]_0^\tau \\
&\qquad -\int_0^\tau qe^{-q(\tau-\xi)}N\left(\frac{x-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)d\xi\Bigg\} \\
&\quad +K\left\{\left[e^{-r(\tau-\xi)}N\left(\frac{x-P(\xi)}{\sqrt{\tau-\xi}}\right)\right]_0^\tau-\int_0^\tau re^{-r(\tau-\xi)}N\left(\frac{x-P(\xi)}{\sqrt{\tau-\xi}}\right)d\xi\right\},
\end{aligned}$$

where the manipulations follow from an application of integration by parts. In order to further evaluate $R(S,\tau)$, we require the limit result that

$$\lim_{\xi\to\tau}\frac{\ln\frac{S}{b(\xi)}}{\sqrt{\tau-\xi}}=\begin{cases}-\infty, & S<b(\tau),\\ 0, & S=b(\tau),\end{cases}$$

since the equation for $\hat{C}_b(S,\tau)$ must be satisfied for $0<S\leq b(\tau)$ for a live American call. If we introduce a special point-indicator function defined by

$$1_{S=b(\tau)}\equiv\begin{cases}\frac{1}{2}, & S=b(\tau),\\ 0, & \text{otherwise},\end{cases}$$

then further evaluation of $R(S,\tau)$ produces

$$\begin{aligned}R(S,\tau) &= -S\left\{1_{S=b(\tau)}-e^{-q\tau}N\left(\frac{x-P(0)+\sigma\tau}{\sqrt{\tau}}\right)\right.\\ &\quad\left.-\int_0^\tau qe^{-q(\tau-\xi)}N\left(\frac{x-P(\xi)+(\tau-\xi)}{\sqrt{\tau-\xi}}\right)d\xi\right\}\\ &\quad+K\left\{1_{S=b(\tau)}-e^{-r\tau}N\left(\frac{x-P(0)}{\sqrt{\tau}}\right)-\int_0^\tau re^{-r(\tau-\xi)}N\left(\frac{x-P(\xi)}{\sqrt{\tau-\xi}}\right)d\xi\right\}\\ &= Se^{-q\tau}N(d_1(S,\tau;b(0^+)))-Ke^{-r\tau}N(d_2(S,\tau;b(0^+)))-1_{S=b(\tau)}(S-K)\\ &\quad+\int_0^\tau\left[qSe^{-q(\tau-\xi)}N(d_1(S,\tau-\xi;b(\xi)))\right.\\ &\quad\left.-rKe^{-r(\tau-\xi)}N(d_2(S,\tau-\xi;b(\xi)))\right]d\xi.\end{aligned}$$

If we then substitute $R(S,\tau)$ into the expression for $\hat{C}_b(S,\tau)$, the most formal representation for the American call price is

$$\begin{aligned}H(b(\tau)-S)\hat{C}_b(S,\tau) &= C_E(S,\tau)-1_{S=b(\tau)}(S-K)\\ &\quad+\int_0^\tau qSe^{-q(\tau-\xi)}N(d_1(S,\tau-\xi;b(\xi)))d\xi\\ &\quad-\int_0^\tau rKe^{-r(\tau-\xi)}N(d_2(S,\tau-\xi;b(\xi)))d\xi,\end{aligned}$$

where $0<S\leq b(\tau)$. When S is strictly less than $b(\tau)$, this can be written more simply as equation (2.5.2) in Proposition 2.4.2. Furthermore, if we evaluate $\hat{C}_b$ at $S=b(\tau)$, we

find that

$$\begin{aligned}\frac{1}{2}(b(\tau)-K) &= C_E(b(\tau),\tau)-\frac{1}{2}(b(\tau)-K)\\ &\quad+\int_0^{\tau} qb(\tau)e^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi\\ &\quad-\int_0^{\tau} rKe^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi,\end{aligned}$$

which simplifies to

$$\begin{aligned}b(\tau)-K &= C_E(b(\tau),\tau)+\int_0^{\tau} qb(\tau)e^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi\\ &\quad-\int_0^{\tau} rKe^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi,\end{aligned}$$

which is equation (2.5.3), and explains why the factor of $\frac{1}{2}$ is no longer present when evaluating Kim's representation at the free boundary.

A2.5.2. Proof of Proposition 2.5.2. Here we present an alternative method of deriving the American call option representation given in equation (2.5.4), based on the appendix of Carr et al. (1992). In particular, our derivation of this result demonstrates how one can reproduce the Carr-Jarrow-Myneni representation directly from Kim's (1990) form. Taking the European call price, $C_E(S,\tau)$, we can write

$$C_E(S,\tau)=SH(S-K)-SH(S-K)+Se^{-q\tau}N(d_1(S,\tau;K))-Ke^{-r\tau}N(d_2(S,\tau;K)).$$

Given the limit result that

$$\lim_{\tau\to 0} d_1(S,\tau;K)=\lim_{\tau\to 0} d_2(S,\tau;K)=\begin{cases}\infty, & S>K\\ 0, & S=K\\ -\infty, & S<K\end{cases}$$

we can express $C_E(S,\tau)$ as

$$\begin{aligned}
C_E(S,\tau) &= SH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) + [Se^{-qs}N(d_1(S,\tau;K))]_0^\tau \\
&= SH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) \\
&\quad +S\int_0^\tau \left[N'(d_1(S,s;K))\frac{\partial}{\partial s}[d_1(S,s;K)]e^{-qs} - qN(d_1(S,s;K))e^{-qs}\right] ds \\
&= SH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) - qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds \\
&\quad +S\int_0^\tau e^{-qs}N'(d_1(S,s;K))\frac{\partial}{\partial s}[d_2(S,s;K)+\sigma\sqrt{s}]ds \\
&= SH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) - qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds \\
&\quad +S\int_0^\tau e^{-qs}N'(d_1(S,s;K))\left[\frac{\partial}{\partial s}[d_2(S,s;K)] + \frac{\sigma}{2\sqrt{s}}\right] ds \\
&= SH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) - qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds \\
&\quad +S\int_0^\tau e^{-qs}N'(d_1(S,s;K))\frac{\partial}{\partial s}[d_2(S,s;K)]ds \\
&\quad +S\int_0^\tau e^{-qs}N'(d_1(S,s;K))\frac{\sigma}{2\sqrt{s}}ds.
\end{aligned}$$

Noting that $N'(d_1(S,s;K)) = Ke^{-(r-q)s}N'(d_2(S,s;K))/S$, we have

$$\begin{aligned}
C_E(S,\tau) &= (S-K)H(S-K) + KH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) \\
&\quad -qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds + S\int_0^\tau e^{-qs}N'(d_1(S,s;K))\frac{\sigma}{2\sqrt{s}}ds \\
&\quad +K\int_0^\tau e^{-rs}N'(d_2(S,s;K))\frac{\partial}{\partial s}[d_2(S,s;K)]ds \\
&= \max(S-K,0) + \frac{S\sigma^2}{2}\int_0^\tau \frac{e^{-qs}}{\sigma\sqrt{s}}N'(d_1(S,s;K))ds \\
&\quad -qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds \\
&\quad -K\Big\{e^{-r\tau}N(d_2(S,\tau;K)) - H(S-K) \\
&\qquad -\int_0^\tau e^{-rs}N'(d_2(S,s;K))\frac{\partial}{\partial s}[d_2(S,s;K)]ds\Big\} \\
&= \max(S-K,0) + \frac{S\sigma^2}{2}\int_0^\tau \frac{e^{-qs}}{\sigma\sqrt{s}}N'(d_1(S,s;K))ds \\
&\quad -qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds \\
&\quad -K\Big\{\left[e^{-rs}N(d_2(S,s;K))\right]_0^\tau \\
&\qquad -\int_0^\tau e^{-rs}N'(d_2(S,s;K))\frac{\partial}{\partial s}[d_2(S,s;K)]ds\Big\},
\end{aligned}$$

where the last line follows by use of the previous limit result for d_2. After changing the integration variable to $s = \tau - \xi$, we can represent the European call price as

$$\begin{aligned}
C_E(S,\tau) &= \max(S-K,0) + \frac{S\sigma^2}{2}\int_0^\tau e^{-q(\tau-\xi)}N'(d_1(S,\tau-\xi;K))d\xi \\
&\quad -qS\int_0^\tau e^{-q(\tau-\xi)}N(d_1(S,\tau-\xi;K))d\xi \\
&\quad +rK\int_0^\tau e^{-r(\tau-\xi)}N(d_2(S,\tau-\xi;K))d\xi,
\end{aligned}$$

and substituting this into (2.5.2) will yield equation (2.5.4) of Proposition 2.5.2, following a simple rearrangement of terms.

Appendix 2.6. Value of the American Call Free Boundary at Expiry

In deriving equation (2.5.5), it is necessary to analyse the limit of equation (2.5.3) as τ tends to 0^+. Using the method outlined by Kim (1990), we begin by considering

$$\begin{aligned} b(\tau) - K &= b(\tau)e^{-q\tau}N(d_1(b(\tau),\tau;K)) - Ke^{-r\tau}N(d_2(b(\tau),\tau;K)) \\ &\quad + \int_0^\tau qb(\tau)e^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi \\ &\quad - \int_0^\tau rKe^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi. \end{aligned}$$

This equation can be factorised to produce

$$\begin{aligned} &b(\tau)\left\{1 - e^{-q\tau}[N(d_1(b(\tau),\tau;K)) - \int_0^\tau qe^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi\right\} \\ &= K\left\{1 - e^{-r\tau}N(d_2(b(\tau),\tau;K)) - \int_0^\tau re^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))]d\xi\right\}, \end{aligned}$$

which then yields the following implicit equation for $b(\tau)$:

$$\begin{aligned} \frac{b(\tau)}{K} &= \left(1 - e^{-r\tau}N(d_2(b(\tau),\tau;K)) - \int_0^\tau re^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi\right) \qquad \text{(A2.6.1)} \\ &\times\left(1 - e^{-q\tau}N(d_1(b(\tau),\tau;K)) - \int_0^\tau qe^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi\right)^{-1}. \end{aligned}$$

Before proceeding further, it should be noted that $b(\tau) \geq K$. To find the value of $b(0^+)$, we take the limit of equation (A2.6.1) as τ tends to 0^+. In order to evaluate this limit, we need to find two limits involving d_1 and d_2. The first to consider is

$$\lim_{\tau\to 0^+} d_2(b(\tau),\tau;K) = \lim_{\tau\to 0^+} \frac{\ln\frac{b(\tau)}{K}}{\sigma\sqrt{\tau}} = \begin{cases} 0, & b(0^+) = K \\ \infty, & b(0^+) > K. \end{cases} \qquad \text{(A2.6.2)}$$

Similarly the following limit for d_1 can be shown to be

$$\lim_{\tau\to 0^+} d_1(b(\tau),\tau;K) = \begin{cases} 0, & b(0^+) = K \\ \infty, & b(0^+) > K. \end{cases} \qquad \text{(A2.6.3)}$$

Note also that $N(0) = 0.5$ and $N(\infty) = 1$. Given that the limits (A2.6.2) and (A2.6.3) depend on the value of $b(0^+)$ relative to K, there are two cases to consider when finding the limit of equation (A2.6.1). Consider the first case where $b(0^+) = K$. Taking the limit of equation (A2.6.1) as τ tends to 0^+, and using the results from equations (A2.6.2)-(A2.6.3), we obtain

$$\lim_{\tau\to 0^+} \frac{b(\tau)}{K} = 1, \tag{A2.6.4}$$

and thus $b(0^+) = K$ is one possible solution for $b(0^+)$.

Now consider the second case, where $b(0^+) > K$. The limit as τ tends to zero of equation (A2.6.1) is now of the form $\frac{0}{0}$, and therefore L'Hopital's rule can be applied. Firstly, let

$$\lim_{\tau\to 0^+} \frac{b(\tau)}{K} = \lim_{\tau\to 0^+} \frac{\hat{N}(\tau)}{\hat{D}(\tau)},$$

where

$$\begin{aligned}\hat{N}(\tau) &\equiv 1 - e^{-r\tau} N(d_2(b(\tau),\tau;K)) \\ &\quad - \int_0^\tau re^{-r(\tau-\xi)} N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi,\end{aligned}$$

and

$$\begin{aligned}\hat{D}(\tau) &\equiv 1 - e^{-q\tau} N(d_1(b(\tau),\tau;K)) \\ &\quad - \int_0^\tau qe^{-q(\tau-\xi)} N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi.\end{aligned}$$

To apply L'Hopital's rule, we must differentiate both $\hat{N}(\tau)$ and $\hat{D}(\tau)$ with respect to τ, and take their limits as τ tends to 0^+. For $\hat{N}(\tau)$ we have

$$\begin{aligned}\hat{N}'(\tau) &= re^{-r\tau} N(d_2(b(\tau),\tau;K)) - e^{-r\tau} N'(d_2(b(\tau),\tau;K))\frac{\partial}{\partial\tau}[d_2(b(\tau),\tau;K)] \\ &\quad -rN(d_2(b(\tau),0;b(\tau))) \\ &\quad -r\int_0^\tau \{-re^{-r(\tau-\xi)} N(d_2(b(\tau),\tau-\xi;b(\xi))) \\ &\qquad +e^{-r(\tau-\xi)} N'(d_2(b(\tau),\tau-\xi;b(\xi)))\frac{\partial}{\partial\tau}[d_2(b(\tau),\tau-\xi;b(\xi))]\}d\xi,\end{aligned}$$

Note that as $x \to \infty$, $N'(x) \to 0$ at a faster rate than any other terms observed in $\hat{N}'(\tau)$ (see Kim, 1990). We also note that

$$\lim_{\xi\to\tau} d_2(b(\tau),\tau-\xi;b(\xi)) = 0.$$

Combining all these limit results, it is concluded that

$$\lim_{\tau\to 0^+} \hat{N}'(\tau) = -\frac{r}{2}. \tag{A2.6.5}$$

Similarly for $\hat{D}'(\tau)$ it can be shown that

$$\lim_{\tau\to 0^+} \hat{D}(\tau) = -\frac{q}{2}. \tag{A2.6.6}$$

Thus it is concluded that

$$\lim_{\tau\to 0^+} \frac{b(\tau)}{K} = \frac{r}{q}. \tag{A2.6.7}$$

Recalling that this result only holds when $b(0^+) > K$, it follows that we must have $r > q$. Finally, combining the results from equations (A2.6.4) and (A2.6.7) gives

$$\lim_{\tau\to 0^+} b(\tau) = K \max\left(1, \frac{r}{q}\right)$$

which is equation (2.5.5) of Proposition 2.5.3.

Appendix 2.7. Induction Proof for the American Call Option Price

The details of this appendix are drawn from the proofs presented in Kim (1990).

A2.7.1. Proof of Proposition 2.6.1. Given that

$$\hat{C}_b(S_1, \Delta\tau) = \begin{cases} U(S_1, \Delta\tau; K), S_1 < b_1, \\ S_1 - K, \qquad S_1 \geq b_1, \end{cases}$$

the value of $U(S_2, 2\Delta\tau; b_1)$ is

$$\begin{aligned} U(S_2, 2\Delta\tau; b_1) &= \int_0^{b_1} e^{-r\Delta\tau} U(S_1, \Delta\tau; K) p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1 \\ &\quad + \int_{b_1}^{\infty} e^{-r\Delta\tau} (S_1 - K) p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1 \\ &= \int_0^{\infty} e^{-r\Delta\tau} U(S_1, \Delta\tau; K) p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1 \\ &\quad - \int_{b_1}^{\infty} e^{-r\Delta\tau} U(S_1, \Delta\tau; K) p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1 \\ &\quad + \int_{b_1}^{\infty} e^{-r\Delta\tau} (S_1, -K) p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1 \\ &= C_E(S_2, 2\Delta\tau) + I(b_1), \end{aligned}$$

where

$$I(b_1) \equiv \int_{b_1}^{\infty} e^{-r\Delta\tau} [S_1 - K - C_E(S_1, \Delta\tau)] p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1.$$

Consider $I(b_1)$, which can be manipulated to produce

$$\begin{aligned}
I(b_1) &= \int_{b_1}^{\infty} e^{-r\Delta\tau}(S_1 - K)p(S_1,\Delta\tau|S_2,2\Delta\tau)dS_1 \\
&\quad - \int_{b_1}^{\infty} e^{-r\Delta\tau}\left[\int_{K}^{\infty} e^{-r\Delta\tau}(S_0 - K)p(S_0,0|S_1,\Delta\tau)dS_0\right] \\
&\qquad \times p(S_1,\Delta\tau|S_2,2\Delta\tau)dS_1 \\
&= \int_{b_1}^{\infty} e^{-r\Delta\tau}(S_1 - K)p(S_1,\Delta\tau|S_2,2\Delta\tau)dS_1 \\
&\quad - \int_{b_1}^{\infty}\left\{e^{-r\Delta\tau}\left[\int_{0}^{\infty} e^{-r\Delta\tau}(S_0 - K)p(S_0,0|S_1,\Delta\tau)dS_0\right.\right. \\
&\quad \left.\left. - \int_{0}^{K} e^{-r\Delta\tau}(S_0 - K)p(S_0,0|S_1,\Delta\tau)dS_0\right] p(S_1,\Delta\tau|S_2,2\Delta\tau)\right\} dS_1 \\
&= \int_{b_1}^{\infty} e^{-r\Delta\tau}(S_1 - K)p(S_1,\Delta\tau|S_2,2\Delta\tau)dS_1 \\
&\quad - \int_{b_1}^{\infty}\left\{e^{-r\Delta\tau}\left[e^{-r\Delta\tau}(S_1 e^{(r-q)\Delta\tau} - K)\right.\right. \\
&\quad \left.\left. - \int_{0}^{K} e^{-r\Delta\tau}(S_0 - K)p(S_0,0|S_1,\Delta\tau)dS_0\right] p(S_1,\Delta\tau|S_2,2\Delta\tau)\right\} dS_1 \\
&= \int_{b_1}^{\infty} e^{-r\Delta\tau}\left[(S_1 - K) - S_1 e^{-q\Delta\tau} + K e^{-r\Delta\tau}\right] p(S_1,\Delta\tau|S_2,2\Delta\tau)dS_1 \\
&\quad + \int_{b_1}^{\infty} e^{-r\Delta\tau}\left[\int_{0}^{K} e^{-r\Delta\tau}(S_0 - K)p(S_0,0|S_1,\Delta\tau)dS_0)\right] \\
&\qquad \times p(S_1,\Delta\tau|S_2,2\Delta\tau)dS_1 \\
&= \int_{b_1}^{\infty} e^{-r\Delta\tau}\left[(1 - e^{-q\Delta\tau})S_1 - (1 - e^{-r\Delta\tau})K\right] p(S_1,\Delta\tau|S_2,2\Delta\tau)dS_1 \\
&\quad + \int_{b_1}^{\infty} e^{-r\Delta\tau} p(S_1,\Delta\tau|S_2,2\Delta\tau) \\
&\qquad \times \left[\int_{0}^{K} e^{-r\Delta\tau}(S_0 - K)p(S_0,0|S_1,\Delta\tau)dS_0)\right] dS_1.
\end{aligned}$$

Let L_1 be defined as

$$\begin{aligned}
L_1 &\equiv \int_{b_1}^{\infty} e^{-r\Delta\tau} p(S_1,\Delta\tau|S_2,2\Delta\tau)\left[\int_{0}^{K} e^{-r\Delta\tau}(K - S_0)p(S_0,0|S_1,\Delta\tau)dS_0)\right] dS_1 \\
&= \int_{b_1}^{\infty} e^{-r\Delta\tau} p(S_1,\Delta\tau|S_2,2\Delta\tau)P_E(S_1,\Delta\tau)dS_1,
\end{aligned}$$

where $P_E(S_1, \Delta\tau) = Ke^{-r\Delta\tau}N(-d_2(S_1, \Delta\tau; K)) - S_1 e^{-q\Delta\tau}N(-d_1(S_1, \Delta\tau; K))$, which is the price of a European put written on S with strike K. Since $P_E(S_1, \Delta\tau)$ is a decreasing function of S_1, an upper bound for L_1 is

$$L_1 < \int_{b_1}^{\infty} e^{-r\Delta\tau} p(S_2, 2\Delta\tau | S_1, \Delta\tau) P_E(b_1, \Delta\tau) dS_1,$$

which evaluates to

$$L_1 < e^{-r\Delta\tau} N(d_2(S_2, \Delta\tau; b_1)) P_E(b_1, \Delta\tau).$$

Note that as $(\tau_2 - \tau_1) \to 0$, $b_2 \to b_1$. Since $S_2 < b_2$ for an unexercised call we find that

$$\lim_{\Delta\tau \to 0} N(d_2(S_2, \Delta\tau; b_1)) = 0.$$

For $P_E(b_1, \Delta\tau)$, since $b_1 \geq K$, we have

$$\lim_{\Delta\tau \to 0} \frac{\ln \frac{K}{b_1}}{\sqrt{\Delta\tau}} = \begin{cases} -\infty, & b_1 > K, \\ 0, & b_1 = K. \end{cases}$$

In either case $\lim_{\Delta\tau \to 0} P_E(b_1, \Delta\tau) = 0$, and hence the term L_1 is of $o(\Delta\tau)$.

The price of the unexercised call at time to maturity $2\Delta\tau$ is therefore

$$\begin{aligned} U(S_2, 2\Delta\tau; b_1) &= C_E(S_2, 2\Delta\tau) + o(\Delta\tau) \\ &+ \int_{b_1}^{\infty} e^{-r\Delta\tau} \left[(1 - e^{-q\Delta\tau})S_1 - (1 - e^{-r\Delta\tau})K\right] p(S_1, \Delta\tau | S_2, \Delta\tau) dS_1, \end{aligned}$$

as given in equation (2.6.3).

A2.7.2. Proof of Proposition 2.6.2. To find the price of the unexercised call at a general time step $n\Delta\tau$, Kim (1990) uses an indication proof. Assume that the unexercised call price at time $m\Delta\tau$ is given by

$$\begin{aligned} U(S_m, m\Delta\tau, b_{m-1}) &= \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} \left[(1 - e^{-q\Delta\tau})S_k - (1 - e^{-r\Delta\tau})K\right] \\ &\qquad \times p(S_k, k\Delta\tau | S_m, m\Delta\tau) dS_k \\ &\quad + C_E(S_m, m\Delta\tau) + o(m\Delta\tau), \end{aligned}$$

as stated in equation (2.6.4). It is simple to show that this holds for $m = 2$ (see Appendix A2.7.1). We must now prove that this relationship holds for $m + 1$.

Given that

$$b_m - K = U(b_m, m\Delta\tau; b_{m-1}),$$

the value of the unexercised call at $(m+1)\Delta\tau$ is

$$\begin{aligned}
&U(S_{m+1}, (m+1)\Delta\tau; b_m) \\
&\quad = \int_0^{b_m} e^{-r\Delta\tau} U(S_m, m\Delta\tau; b_{m-1}) p(S_m, m\Delta\tau | S_{m+1}(m+1)\Delta\tau) dS_m \\
&\qquad + \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K) p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_m \\
&\quad = \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K - U(S_m, m\Delta\tau; b_{m-1})) \\
&\qquad\qquad \times p(S_m, m\Delta\tau) | S_{m+1}, (m+1)\Delta\tau) dS_m \\
&\qquad + \int_0^{\infty} e^{-r\Delta\tau} U(S_m, m\Delta\tau, b_{m-1}) \\
&\qquad\qquad \times p(S_m, m\Delta\tau | S_{m+1}(m+1)\Delta\tau) dS_m \\
&\quad \equiv U_1 + U_2.
\end{aligned}$$

Consider firstly the term U_2, which can be simplified as

$$\begin{aligned}
U_2 &= \int_0^{\infty} e^{-r\Delta\tau} C_E(S_m, m\Delta\tau) p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_m + o(m\Delta\tau) \\
&\quad + \int_0^{\infty} e^{-r\Delta\tau} \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} \left[(1 - e^{-q\Delta\tau}) S_k - (1 - e^{-r\Delta\tau}) K\right] \\
&\quad \times p(S_k, k\Delta\tau | S_m, m\Delta\tau) dS_k \\
&\qquad \times p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_m \\
&= C_E(S_{m+1}, (m+1)\Delta\tau) + o(m\Delta\tau) \\
&\quad + \sum_{k=1}^{m-1} \int_{b_k}^{\infty} e^{-(m-k+1)r\Delta\tau} \left[(1 - e^{-q\Delta\tau}) S_k - (1 - e^{-r\Delta\tau}) K\right] \\
&\quad \times \int_0^{\infty} p(S_k, k\Delta\tau | S_m, m\Delta\tau) p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_m dS_k \\
&= C_E(S_{m+1}, (m+1)\Delta\tau) + o(m\Delta\tau) \\
&\quad + \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau} \int_{b_k}^{\infty} \left[(1 - e^{-q\Delta\tau}) S_k - (1 - e^{-r\Delta\tau}) K\right] \\
&\qquad\qquad \times p(S_k, k\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_k
\end{aligned}$$

Next consider the term U_1. Extensive manipulations yield

$$\begin{aligned}
U_1 &= \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad - \int_{b_m}^{\infty} e^{-r\Delta\tau}C_E(S_m, m\Delta\tau)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad + \int_{b_m}^{\infty} e^{-r\Delta\tau}\sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau}\left[\int_{b_k}^{\infty}\left(e^{-q\Delta\tau}S_k - e^{-r\Delta\tau}K\right)\right. \\
&\quad \left.\times p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k\right]p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad - \int_{b_m}^{\infty} e^{-r\Delta\tau}\sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau}\int_{b_k}^{\infty}(S_k - K)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k \\
&\quad \times p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m + o(m\Delta\tau) \\
&= \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad - \int_{b_m}^{\infty} e^{-r\Delta\tau}C_E(S_m, m\Delta\tau)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad + \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau}\int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{b_k}^{\infty}\left(e^{-q\Delta\tau}S_k - e^{-r\Delta\tau}K\right)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k dS_m \\
&\quad - \sum_{k=1}^{m-2} e^{-(m-k+1)r\Delta\tau}\int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{b_k}^{\infty}(S_k - K)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k dS_m \\
&\quad - e^{2r\Delta\tau}\int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{b_{m-1}}^{\infty}(S_{m-1} - K)p(S_{m-1}, (m-1)\Delta\tau|S_m, m\Delta\tau)dS_{m-1}dS_m \\
&\quad + o(m\Delta\tau).
\end{aligned}$$

If we set

$$\begin{aligned}
L_m^{(1)} &\equiv \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau}\int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{b_k}^{\infty}\left(e^{-q\Delta\tau}S_k - e^{-r\Delta\tau}K\right)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k dS_m,
\end{aligned}$$

then further manipulations yield

$$
\begin{aligned}
U_1 &= \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad - \int_{b_m}^{\infty} e^{-r\Delta\tau} \int_{K}^{\infty} e^{-rm\Delta\tau}(S_0 - K)p(S_0, 0|S_m, m\Delta\tau)dS_0 \\
&\quad \times p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad + L_m^{(1)} - \sum_{k=1}^{m-2} e^{-(m-k+1)r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{b_k}^{\infty} (S_k - K)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k dS_m \\
&\quad - e^{-2r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{0}^{\infty} (S_{m-1} - K)p(S_{m-1}, (m-1)\Delta\tau|S_m, m\Delta\tau)dS_{m-1}dS_m \\
&\quad + e^{-2r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{0}^{b_{m-1}} (S_{m-1} - K)p(S_{m-1}, (m-1)\Delta\tau|S_m, m\Delta\tau)dS_{m-1}dS_m \\
&\quad\quad + o(m\Delta\tau) \\
&= \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad - \sum_{k=0}^{m-2} e^{-(m-k+1)r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{b_k}^{\infty} (S_k - K)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k dS_m \\
&\quad - e^{-2r\Delta\tau} \int_{b_m}^{\infty} (S_m e^{(r-q)\Delta\tau} - K)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad + L_m^{(1)} + L_m^{(2)} + o(m\Delta\tau),
\end{aligned}
$$

where

$$
\begin{aligned}
L_m^{(2)} &\equiv e^{-2r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{0}^{b_{m-1}} (S_{m-1} - K)p(S_{m-1}, (m-1)\Delta\tau|S_m, m\Delta\tau)dS_{m-1}dS_m,
\end{aligned}
$$

and we use the notation $b_0 \equiv b(0) = K$. Thus if we set

$$L_m^{(3)} \equiv -\sum_{k=1}^{m-1} e^{-(m-k+2)r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau)$$
$$\times \int_{b_{k-1}}^{\infty} (S_{k-1} - K) p(S_{k-1}, (k-1)\Delta\tau | S_m, m\Delta\tau) dS_{k-1} dS_m,$$

U_1 becomes

$$U_1 = \int_{b_m}^{\infty} e^{-r\Delta\tau}[(1-e^{-q\Delta\tau})S_m - (1-e^{-r\Delta\tau})K] p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_m$$
$$+ L_m^{(1)} + L_m^{(2)} + L_m^{(3)} + o(m\Delta\tau),$$

and defining $L_m \equiv L_m^{(1)} + L_m^{(2)} + L_m^{(3)}$, $U(S_{m+1}, (m+1)\Delta\tau; b_m)$ reduces to

$$U(S_{m+1}, (m+1)\Delta\tau; b_m) = C_E(S_{m+1}, (m+1)\Delta\tau) + o(m\Delta\tau) + L_m$$
$$+ \sum_{k=1}^{m} e^{-(m-k+1)r\Delta\tau} \int_{b_k}^{\infty} [(1-e^{-q\Delta\tau})S_k - (1-e^{-r\Delta\tau})K]$$
$$\times p(S_k, k\Delta\tau | S_{m+1}, (m+1)\Delta\tau) S_k.$$

All that remains is to prove that L_m is of $o(\Delta\tau)$. We begin by noting that when $S_m = b_m$, $U(S_m, m\Delta\tau; b_{m-1})$ becomes

$$b_m - K = \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} [(1-e^{-q\Delta\tau})S_k - (1-e^{-r\Delta\tau})K]$$
$$\times p(S_k, k\Delta\tau | b_m, m\Delta\tau) dS_k$$
$$+ C_E(b_m, m\Delta\tau) + o(m\Delta\tau)$$
$$= \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} (S_k - K) p(S_k, k\Delta\tau | b_m, m\Delta\tau) dS_k$$
$$- \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} (e^{-q\Delta\tau} S_k - e^{-r\Delta\tau} K) p(S_k, k\Delta\tau | b_m, m\Delta\tau) dS_k$$
$$+ e^{-rm\Delta\tau} \int_{K}^{\infty} (S_0 - K) p(S_0, 0 | b_m, m\Delta\tau) dS_0 + o(m\Delta\tau),$$

and thus

$$\begin{aligned} b_m - K &= \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau} \int_{b_{k-1}}^{\infty} (S_{k-1} - K) p(S_{k-1}, (k-1)\Delta\tau | b_m, m\Delta\tau) dS_{k-1} \\ &\quad - \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} (e^{-q\Delta\tau} S_k - e^{-r\Delta\tau} K) p(S_k, k\Delta\tau | b_m, m\Delta\tau) dS_k \\ &\quad + e^{-r\Delta\tau} \int_{b_{m-1}}^{\infty} (S_{m-1} - K) p(S_{m-1}, (m-1)\Delta\tau | b_m, m\Delta\tau) dS_{m-1} \\ &\quad + o(m\Delta\tau), \end{aligned}$$

where again $b_0 \equiv b(0) = K$. If we take $S_m > b_m$, we have

$$\begin{aligned} S_m - K &> \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau} \int_{b_{k-1}}^{\infty} (S_{k-1} - K) p(S_{k-1}, (k-1)\Delta\tau | S_m, m\Delta\tau) dS_{k-1} \\ &\quad - \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} (e^{-q\Delta\tau} S_k - e^{-r\Delta\tau} K) p(S_k, k\Delta\tau | S_m, m\Delta\tau) dS_k \\ &\quad + e^{-r\Delta\tau} \int_{0}^{\infty} (S_{m-1} - K) p(S_{m-1}(m-1)\Delta\tau | S_m, m\Delta\tau) dS_{m-1} \\ &\quad - e^{-r\Delta\tau} \int_{0}^{b_{m-1}} (S_{m-1} - K) p(S_{m-1}(m-1)\Delta\tau | S_m, m\Delta\tau) dS_{m-1}. \end{aligned}$$

Thus we arrive at the inequality

$$\begin{aligned} & S_m - K - e^{-r\Delta\tau} \int_{0}^{\infty} (S_{m-1} - K) p(S_{m-1}, (m-1)\Delta\tau | S_m, m\Delta\tau) dS_{m-1} \\ & > \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau} \int_{b_{k-1}}^{\infty} (S_{k-1} - K) p(S_{k-1}, (k-1)\Delta\tau | S_m, m\Delta\tau) dS_{k-1} \\ & \quad - \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} (e^{-q\Delta\tau} S_k - e^{-r\Delta\tau} K) p(S_k, k\Delta\tau | S_m, m\Delta\tau) dS_k \\ & \quad - e^{-r\Delta\tau} \int_{0}^{b_{m-1}} (S_{m-1} - K) p(S_{m-1}, (m-1)\Delta\tau | S_m, m\Delta\tau) dS_{m-1}. \end{aligned}$$

Rewriting L_m gives

$$\begin{aligned}
L_m &= e^{-r\Delta\tau}\int_{b_m}^{\infty}\left\{\sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau}\int_{b_k}^{\infty}(e^{-q\Delta\tau}S_k - e^{-r\Delta\tau}K)p(S_k,k\Delta\tau|S_m,m\Delta\tau)dS_k\right.\\
&\quad + \int_0^{b_{m-1}} e^{-r\Delta\tau}(S_{m-1}-K)p(S_{m-1}(m-1)\Delta\tau|S_m,m\Delta\tau)dS_{m-1}\\
&\quad \left.-\sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau}\int_{b_{k-1}}^{\infty}(S_{k-1}-K)p(S_{k-1},(k-1)\Delta\tau|S_m,m\Delta\tau)dS_{k-1}\right\}\\
&\quad \times p(S_m,m\Delta\tau|S_{m+1},(m+1)\Delta\tau)dS_m\\
-L_m &< e^{-r\Delta\tau}\int_{b_m}^{\infty}\left\{S_m - K - e^{-r\Delta\tau}\int_0^{\infty}(S_{m-1}-K)\right.\\
&\qquad \left.\times p(S_{m-1},(m-1)\Delta\tau|S_m,m\Delta\tau)dS_{m-1}\right\}\\
&\qquad \times p(S_m,m\Delta\tau|S_{m+1},(m+1)\Delta\tau)dS_m\\
-L_m &< e^{-r\Delta\tau}\int_{b_m}^{\infty}\left\{S_m - K - e^{-r\Delta\tau}(S_m e^{(r-q)\Delta\tau} - K)\right.\\
&\qquad \left.\times p(S_m,m\Delta\tau|S_{m+1},(m+1)\Delta\tau)dS_m\right\}\\
-L_m &< e^{-r\Delta\tau}\int_{b_m}^{\infty}\{(1-e^{-q\Delta\tau})S_m - (1-e^{-r\Delta\tau})K\}\\
&\qquad \times p(S_m,m\Delta\tau|S_{m+1},(m+1)\Delta\tau)dS_m\\
-L_m &< (1-e^{-q\Delta\tau})S_{m+1}e^{-q\Delta\tau}N(d_1(S_{m+1},\Delta\tau;b_m))\\
&\quad -(1-e^{-r\Delta\tau})Ke^{-r\Delta\tau}N(d_2(S_{m+1},\Delta\tau;b_m)).
\end{aligned}$$

We now consider the limit of this bound for $|L_m|$ as $\Delta\tau \to 0$. Firstly we note that

$$\lim_{\Delta\tau\to 0}(1-e^{-q\Delta\tau}) = \lim_{\Delta\tau\to 0}(1-e^{-r\Delta\tau}) = 0.$$

As $\Delta\tau \to 0$, $b_m \to b_{m+1}$, and since $S_{m+1} < b_{m+1}$ for an unexercised American call, we have

$$\lim_{\Delta\tau\to 0} N(d_1(S_{m+1},\Delta\tau;b_m)) = \lim_{\Delta\tau\to 0} N(d_2(S_{m+1},\Delta\tau;b_m)) = 0,$$

and thus $|L_m| \to 0$ as $\Delta\tau \to 0$. Hence L_m is of $o(\Delta\tau)$ and $U(S_{m+1},(m+1)\Delta\tau;b_m)$ is given by

$$\begin{aligned}U(S_{m+1},(m+1)\Delta\tau;b_m) &= C_E(S_{m+1},(m+1)\Delta\tau)+o((m+1)\Delta\tau)\\&+\sum_{k=1}^{m}e^{-(m-k+1)r\Delta\tau}\int_{b_k}^{\infty}[(1-e^{-q\Delta\tau})S_k-(1-e^{-r\Delta\tau})K]\\&\times p(S_k,k\Delta\tau|S_{m+1},(m+1)\Delta\tau)dS_k,\end{aligned}$$

and equation (2.6.4) is satisfied for $n=m+1$, completing the induction proof.

A2.7.3. Proof of Proposition 2.6.3. An obvious use of equivalent notation in (2.6.4) produces

$$\begin{aligned}\hat{C}_b(S_n,n\Delta\tau) &= C_E(S_n,n\Delta\tau)+o(n\Delta\tau)\\&+\sum_{k=1}^{n-1}e^{-(n-k)r\Delta\tau}\int_{b_k}^{\infty}[(1-e^{-q\Delta\tau})S_k-(1-e^{-r\Delta\tau})K]\\&\times p(S_k,k\Delta\tau|S_n,n\Delta\tau)dS_k,\end{aligned}$$

which is the unexercised American call option price with n discrete early exercise dates, occurring after every time step $\Delta\tau$. Using Taylor series we have

$$(1-e^{-\alpha\Delta\tau})=\alpha\Delta\tau+o(\Delta\tau),$$

where α is a constant, and thus the price becomes

$$\begin{aligned}\hat{C}_b(S_n,n\Delta\tau) &= C_E(S_n,n\Delta\tau)+o(n\Delta\tau)\\&+\sum_{k=1}^{n-1}e^{-(n-k)r\Delta\tau}\int_{b_k}^{\infty}[qS_k-rK]p(S_k,k\Delta\tau|S_n,n\Delta\tau)dS_k\Delta\tau.\end{aligned}$$

Finally, to find the value of the continuous American call, we set $n\Delta\tau=\tau, S_n=S$, and take the limit as $\Delta\tau\to 0$ to produce

$$\begin{aligned}\hat{C}_b(S,\tau) &= C_E(S,\tau)+\int_0^{\tau}e^{-r(\tau-\xi)}\int_{b(\xi)}^{\infty}(qS_\xi-rK)p(S_\xi,\xi|S,\tau)dS_\xi d\xi\\&= C_E(S,\tau)+\int_0^{\tau}qSe^{-q(\tau-\xi)}N(d_1(S,\tau-\xi;b(\xi)))d\xi\\&-\int_0^{\tau}rKe^{-r(\tau-\xi)}N(d_2(S,\tau-\xi;b(\xi)))d\xi,\end{aligned}$$

which is equation (2.5.2) of Proposition 2.5.1.

CHAPTER 3

Evaluation of American Option Portfolios

3.1. Introduction

In Chapter 2 we explored how McKean (1965) and Kim (1990) successfully extended the Black-Scholes European option pricing methodology to American calls and puts, and in particular showed that the incomplete Fourier transform method is able to readily cater for general monotonic payoff functions. This chapter extends the transform methodology to a special type of American strangle position, where the early exercise of one side of the position will knock-out the remaining side. Through this example, a general American option pricing framework for convex payoffs is provided. A numerical comparison is conducted between the two different American strangle definitions, demonstrating that McKean's method leads to useful representations for the price and free boundaries of American option portfolios.

Despite the large amount of research conducted into the American option pricing problem, as discussed at length in Chapters 1 and 2, there is still no universal framework with which one can derive the integral equations and free boundaries for a generic payoff function, either monotonic, convex or concave. Elliott et al. (1990) considered the American straddle pricing problem, deriving the coupled integral equation system for the straddle's free boundaries, however it is not clear how their approach could be extended to general convex/concave payoff functions. While they used probability theory in their analysis, in this chapter we revisit McKean's incomplete Fourier transform method for American call options and derive the coupled integral equation system via this approach. We extend the findings of Elliott et al. (1990) by applying McKean's method to a special kind of American strangle. If exercised early, the entire payoff is optimally realised, making this fundamentally different to an American strangle formed using individual calls and puts. Thus the strangle under consideration is the analogue of the straddle considered by Elliott et al. (1990). Alobaidi and Mallier (2002) also considered the American straddle problem, using incomplete Laplace transforms to derive integral equations for both its price and free boundary. The primary drawback in their method is that while a solution is readily found in transform-space, inverting the solution appears analytically impossible, making the final result very difficult to implement numerically. We demonstrate that the incomplete Fourier transform is a preferable solution technique, leading to solutions that are straight-forward to invert, and thereby more suitable for numerical implementation.

The American strangle is an example of a more general American option position with a convex payoff function and indeed the methodology developed in this chapter may be applied to evaluate such positions. Although other methods can be applied to a broad class of market-relevant American portfolios, we have chosen to use the Fourier transform method because it provides a systematic approach for solving such problems. The integral equations for capped American call options presented by Broadie and Detemple (1995) can also be derived using McKean's analysis, and provides the basis for evaluating American bull/bear spreads and butterflies. The application of McKean's method in this chapter is based on the exposition in Chapter 2. We also transform the results into the equivalent of Kim's integral equation form, and then proceed to implement these equations numerically to find firstly the strangle's early exercise boundaries, and finally the strangle's price.

We use this American strangle contract as our illustrative example for several reasons. The American strangle is a natural generalisation of the results of Elliott et al. (1990). It is typical for option traders to deal in positions rather than single options, implying that there exists a market for option portfolios comprised of American options. By applying our analysis to this alternative form of American strangle, we are given the opportunity to explore the differences between the two American strangle definitions, both analytically and numerically. Such analysis has not been presented in the existing literature to our knowledge.

The alternative strangle definition contains several implicit advantages over a "traditional" American strangle constructed using a long position in both an American call and an American put. The new strangle is self-closing, since exercising one side of the position will knock-out the other. These implicit knock-out barriers will make this strangle cheaper than a "traditional" one, and may have market applications within pure-volatility strategies. It is important to note that the proposed strangle loses the flexibility to be decomposed into its component options. An integral equation exists for the delta of the strangle, and this can be solved in the same manner as the pricing integral equation. Thus the strangle can be hedged in the same manner as any American call or put.

This chapter is structured as follows. Section 3.2 outlines McKean's free boundary problem that arises from this special American strangle option pricing problem. Section 3.3 applies the incomplete Fourier transform to solve the PDE in terms of a transform variable. The transform is inverted in Section 3.4 to provide a McKean-type integral expression for the American strangle price, and a corresponding integral equation system for the strangle's two early exercise boundaries. Section 3.5 provides the transform from McKean-style equations to Kim-style equations. Section 3.6 outlines the numerical solution method for both the free boundaries and strangle price. A selection of numerical results are provided in Section 3.7, with concluding remarks presented in Section 3.8. Appendices follow the final section, with detailed proofs for the various propositions.

3.2. Problem Statement - American Strangle

Let $A_{a_1,a_2}(S,t)$ be the price of an American strangle position written on an underlying asset with price S at time t, and time to expiry $(T-t)$. This position is formed using a long put with strike K_1, and a long call with strike K_2, and we specify that $K_1 < K_2$. Let the early exercise boundary on the put side be denoted by $a_1(t)$, and the early exercise boundary on the call side be denoted by $a_2(t)$. Figure 3.1 demonstrates the payoff and continuation region for $A_{a_1,a_2}(S,t)$.

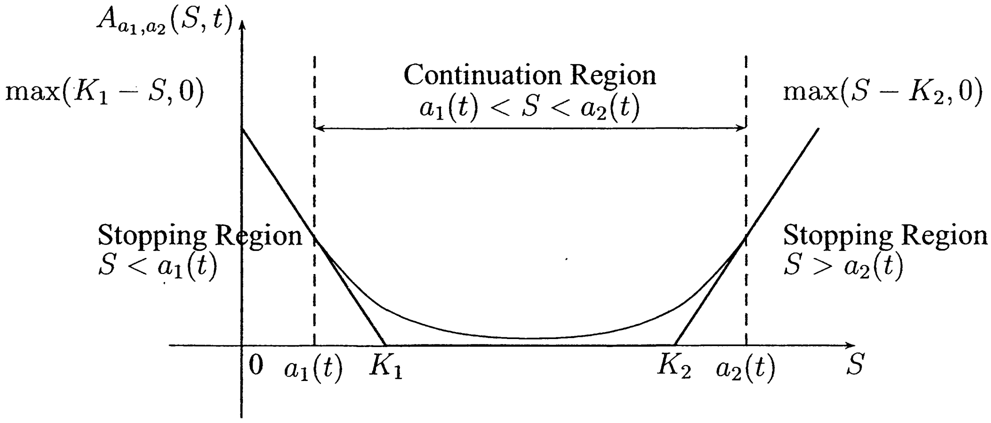

FIGURE 3.1. Continuation region for the American strangle in S-space.

Under the assumption that the price of the underlying asset is driven by the geometric Brownian motion in equation (2.2.1), it is known that A satisfies the Black-Scholes PDE

$$\frac{\partial A}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 A}{\partial S^2} + (r-q)S\frac{\partial A}{\partial S} - rA = 0, \quad 0 \le t \le T, \tag{3.2.1}$$

in the region $a_1(t) < S < a_2(t)$, where r is the risk-free rate, and q is the dividend rate of S (continuously compounded), subject to the following final time and boundary conditions:

$$A_{a_1,a_2}(S,T) = \max(K_1 - S, 0) + \max(S - K_2, 0), \tag{3.2.2}$$

$$0 < S < \infty,$$

$$A_{a_1,a_2}(a_1(t),t) = K_1 - a_1(t), \quad t \ge 0, \tag{3.2.3}$$

$$A_{a_1,a_2}(a_2(t),t) = a_2(t) - K, \quad t \ge 0, \tag{3.2.4}$$

$$\lim_{S\to a_1(t)} \frac{\partial A}{\partial S} = -1, \quad \lim_{S\to a_2(t)} \frac{\partial A}{\partial S} = 1, \quad t \ge 0. \tag{3.2.5}$$

Condition (3.2.2) is the payoff function for the strangle at expiry, while conditions (3.2.3)-(3.2.5) are collectively known as the "smooth-pasting" conditions. These ensure that the price and first derivative with respect to S of $A_{a_1,a_2}(S,t)$ are both continuous. This is necessary to maintain the Black-Scholes assumption of an arbitrage-free market.

Firstly, we shall transform the PDE (3.2.1) to a forward-in-time equation. Setting $S = e^x$ and $t = T - \tau$, we define the transformed function V by

$$A_{a_1,a_2}(S,t) = V_{c_1,c_2}(x,\tau). \tag{3.2.6}$$

The transformed PDE for V is then

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + k\frac{\partial V}{\partial x} - rV, \quad 0 \le \tau \le T, \tag{3.2.7}$$

in the region $\ln c_1(\tau) < x < \ln c_2(\tau)$ where $k = r - q - \frac{1}{2}\sigma^2$, and the transformed free boundaries are given by $c_1(\tau) = a_1(t)$ and $c_2(\tau) = a_2(t)$. The transformed initial and boundary conditions are

$$V_{c_1,c_2}(x,0) = \max(K_1 - e^x, 0) + \max(e^x - K_2, 0), \quad (3.2.8)$$
$$-\infty < x < \infty,$$
$$V_{c_1,c_2}(\ln c_1(\tau), \tau) = K_1 - c_1(\tau), \quad \tau \geq 0, \quad (3.2.9)$$
$$V_{c_1,c_2}(\ln c_2(\tau), \tau) = c_2(\tau) - K_2, \quad \tau \geq 0, \quad (3.2.10)$$
$$\lim_{x \to \ln c_1(\tau)} \frac{\partial V}{\partial x} = -c_1(\tau), \quad (3.2.11)$$
$$\lim_{x \to \ln c_2(\tau)} \frac{\partial V}{\partial x} = c_2(\tau). \quad (3.2.12)$$

In what follows, we will use the notation $c_1 \equiv c_1(\tau)$ and $c_2 \equiv c_2(\tau)$ for simplicity.

By a natural extension of the approach detailed in Chapter 2, the x-domain shall be extended to $-\infty < x < \infty$ by expressing the PDE as

$$H(\ln c_2 - x)H(x - \ln c_1)\left(\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - k\frac{\partial V}{\partial x} + rV\right) = 0,$$

where $H(x)$, the Heaviside step function, has been defined in equation (2.2.14). The initial and boundary conditions remain unchanged. We are now ready to begin solving this PDE for $V_{c_1,c_2}(x,\tau)$.

3.3. Applying the Fourier Transform

As in Chapter 2 we shall solve the problem defined by equations (3.2.7)-(3.2.12) using the Fourier transform technique. One particular difference in the strangle case is that the solution is bounded by the interval (c_1, c_2), and we know with certainty that the function V and its first two derivatives with respect to x are finite at the limits of the space-domain for the transform.

Since the x-domain is now $-\infty < x < \infty$, the Fourier transform of the PDE can be found. The Fourier transform of V, $\mathcal{F}\{V_{c_1,c_2}(x,\tau)\}$, is defined as

$$\mathcal{F}\{V_{c_1,c_2}(x,\tau)\} = \int_{-\infty}^{\infty} e^{i\eta x} V_{c_1,c_2}(x,\tau)dx.$$

Thus, the transformed PDE appears as

$$\begin{aligned}\mathcal{F}\{H(\ln c_2 - x)H(x-\ln c_1)\frac{\partial V}{\partial \tau}\} &= \frac{1}{2}\sigma^2\mathcal{F}\{H(\ln c_2 - x)H(x-\ln c_1)\frac{\partial^2 V}{\partial x^2}\}\\ &\quad +k\mathcal{F}\{H(\ln c_2 - x)H(x-\ln c_1)\frac{\partial V}{\partial x}\}\\ &\quad -r\mathcal{F}\{H(\ln c_2 - x)H(x-\ln c_1)V\}.\end{aligned}$$

By definition

$$\begin{aligned}&\mathcal{F}\{H(\ln c_2 - x)H(x-\ln c_1)V_{c_1,c_2}(x,\tau)\}\\ &\quad = \int_{-\infty}^{\infty} e^{i\eta x}H(\ln c_2 - x)H(x-\ln c_1)V_{c_1,c_2}(x,\tau)dx\\ &\quad = \int_{\ln c_1}^{\ln c_2} e^{i\eta x}V_{c_1,c_2}(x,\tau)dx\\ &\quad \equiv \mathcal{F}^c\{V_{c_1,c_2}(x,\tau)\} \equiv \hat{V}_{c_1,c_2}(\eta,\tau),\end{aligned} \tag{3.3.1}$$

where for convenience we introduce the notation $\hat{V}_{c_1,c_2}(\eta,\tau)$ to also denote the transform. We note that $\mathcal{F}^c$ is an incomplete Fourier transform, since it is equivalent to a standard Fourier transform applied to $V_{c_1,c_2}(x,\tau)$ in the domain $\ln c_1 < x < \ln c_2$. In Appendix 3.1 we show how the incomplete Fourier transform may be derived as a consequence of the standard Fourier transform, and there derive the corresponding inversion theorem. To apply the incomplete Fourier transform to the PDE (3.2.7), we need to consider three specific properties of $\mathcal{F}^c$.

PROPOSITION 3.3.1. *Given the definition of $\mathcal{F}^c$ in equation (3.3.1), the following identities exist for $\mathcal{F}^c$:*

$$\mathcal{F}^c\left\{\frac{\partial V}{\partial x}\right\} = (c_2-K_2)e^{i\eta\ln c_2} - (K_1-c_1)e^{i\eta\ln c_1} - i\eta\hat{V}; \tag{3.3.2}$$

$$\begin{aligned}\mathcal{F}^c\left\{\frac{\partial^2 V}{\partial x^2}\right\} &= e^{i\eta\ln c_2}(c_2 - i\eta(c_2-K_2))\\ &\quad -e^{i\eta\ln c_1}(-c_1 - i\eta(K_1-c_1)) - \eta^2\hat{V};\end{aligned} \tag{3.3.3}$$

$$\mathcal{F}^c\left\{\frac{\partial V}{\partial \tau}\right\} = \frac{\partial \hat{V}}{\partial \tau} - \frac{c_2'}{c_2}e^{i\eta\ln c_2}(c_2-K_2) + \frac{c_1'}{c_1}e^{i\eta\ln c_1}(K_1-c_1). \tag{3.3.4}$$

Proof: Refer to Appendix A3.2.1. Note that in deriving the above results, we make use of the so-called "smooth-pasting" conditions given in equations (3.2.9)-(3.2.12).

□

The PDE can now be transformed, as required.

PROPOSITION 3.3.2. *The incomplete Fourier transform of the PDE (3.2.7) with respect to x satisfies the ODE*

$$\frac{d\hat{V}}{d\tau} + \left(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r\right)\hat{V} = F(\eta, \tau), \tag{3.3.5}$$

where

$$\begin{aligned} F(\eta,\tau) &= e^{i\eta \ln c_2}\left[\frac{\sigma^2 c_2}{2} + (\frac{c_2'}{c_2} - \frac{\sigma^2 i\eta}{2} + k)(c_2 - K_2)\right] \\ &\quad - e^{i\eta \ln c_1}\left[-\frac{\sigma^2 c_1}{2} + (\frac{c_1'}{c_1} - \frac{\sigma^2 i\eta}{2} + k)(K_1 - c_1)\right], \end{aligned} \tag{3.3.6}$$

with initial condition

$$\mathcal{F}\{V_{c_1,c_2}(x,0)\} \equiv \hat{V}_{c_1,c_2}(\eta, 0).$$

being calculated from equation (3.2.8).

Proof: Refer to Appendix A3.2.2.

□

Instead of solving a PDE for the function $V_{c_1,c_2}(x,\tau)$, we are now faced with the simpler task of solving the ODE (3.3.5) for the function $\hat{V}_{c_1,c_2}(\eta,\tau)$. This can then be inverted via the Fourier inversion theorem (see Appendix 3.1) to recover the desired function $V_{c_1,c_2}(x,\tau)$. Before concluding this section, we obtain the solution to (3.3.5).

PROPOSITION 3.3.3. *The solution for $\hat{V}_{c_1,c_2}(\eta,\tau)$ is given by*

$$\hat{V}_{c_1,c_2}(\eta,\tau) = \hat{V}_{c_1,c_2}(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)\tau} + \int_0^\tau e^{-(\frac{\sigma^2\eta^2}{2} + ki\eta + r)(\tau - s)}F(\eta,s)ds. \tag{3.3.7}$$

Proof: Recall that c_1 and c_2 are functions of τ, the ODE (3.3.5) is of the form

$$\frac{d\hat{V}}{d\tau} + \alpha(\eta)\hat{V} = F(\eta,\tau),$$

whose solution is given in Proposition 2.3.4.

□

3.4. Inverting the Fourier Transform

Having now found $\hat{V}_{c_1,c_2}(\eta,\tau)$, it is necessary to recover $V_{c_1,c_2}(x,\tau)$, the American strangle price in the x-τ plane. Taking the inverse (complete) Fourier transform of (3.3.7) gives

$$\begin{aligned} V_{c_1,c_2}(x,\tau) &= \mathcal{F}^{-1}\{\hat{V}(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)\tau}\} \\ &\quad +\mathcal{F}^{-1}\left\{\int_0^\tau e^{-(\frac{\sigma^2\eta^2}{2}+ki\eta+r)(\tau-s)}F(\eta,s)ds\right\} \\ &\equiv V^{(1)}_{c_1,c_2}(x,\tau)+V^{(2)}_{c_1,c_2}(x,\tau); \quad \ln c_1(\tau)<x<\ln c_2(\tau). \end{aligned}$$

We must now determine $V^{(1)}_{c_1,c_2}(x,\tau)$ and $V^{(2)}_{c_1,c_2}(x,\tau)$.

PROPOSITION 3.4.1. *The function $V^{(1)}_{c_1,c_2}(x,\tau)$ is given by*

$$\begin{aligned} V^{(1)}_{c_1,c_2}(x,\tau) &= [K_1e^{-r\tau}N(-d_2(e^x,\tau;K_1))-e^xe^{-q\tau}N(-d_1(e^x,\tau;K_1))] \\ &\quad +[e^xe^{-q\tau}N(d_1(e^x,\tau;K_2))-K_2e^{-r\tau}N(d_2(e^x,\tau;K_2))] \\ &\quad -[K_1e^{-r\tau}N(-d_2(e^x,\tau;c_1(0^+)))-e^xe^{-q\tau}N(-d_1(e^x,\tau;c_1(0^+)))] \\ &\quad -[e^xe^{-q\tau}N(d_1(e^x,\tau;c_2(0^+)))-K_2e^{-r\tau}N(d_2(e^x,\tau;c_2(0^+)))] \end{aligned} \tag{3.4.1}$$

where

$$\begin{aligned} d_1(S,\tau;\beta) &= \frac{\ln(S/\beta)+\left(r-q+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \\ d_2(S,\tau;\beta) &= d_1(S,\tau;\beta)-\sigma\sqrt{\tau}, \\ c_i(0^+) &= \lim_{\tau\to0^+}c_i(\tau), \quad i=1,2, \end{aligned}$$

and

$$N(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-\frac{\alpha^2}{2}}d\alpha.$$

Proof: Refer to Appendix A3.3.1.

□

PROPOSITION 3.4.2. *The function $V^{(2)}_{c_1,c_2}(x,\tau)$ is given by*

$$V^{(2)}_{c_1,c_2}(x,\tau)=\int_0^\tau\frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}}[e^{-h_2(x,s)}Q_2(x,s)+e^{-h_1(x,s)}Q_1(x,s)]ds, \tag{3.4.2}$$

where

$$h_j(x,s) = \frac{(x - \ln c_j(s) + k(\tau - s))^2}{2\sigma^2(\tau - s)}, \tag{3.4.3}$$

and

$$Q_j(x,s) = \frac{\sigma^2 c_j(s)}{2} + \left(\frac{c_j'(s)}{c_j(s)} + \frac{1}{2}\left[k - \frac{(x - \ln c_j(s))}{(\tau - s)}\right]\right)(c_j(s) - K_j) \tag{3.4.4}$$

for $j = 1, 2$ *and* $\ln c_1(\tau) < x < \ln c_2(\tau)$.

Proof: Refer to Appendix A3.3.2.

□

Hence, the value of the American strangle in the x-τ plane is given by

$$V_{c_1,c_2}(x,\tau) = V^{(1)}_{c_1,c_2}(x,\tau) + V^{(2)}_{c_1,c_2}(x,\tau), \tag{3.4.5}$$
$$0 \le \tau \le T; \quad \ln c_1(\tau) < x < \ln c_2(\tau).$$

Equation (3.4.5) expresses the value of the American strangle position in terms of the early exercise boundaries $c_1(\tau)$ and $c_2(\tau)$. At this point these remain unknown, but we are able to obtain an integral equation system that determines them by requiring the expression for $V_{c_1,c_2}(x,\tau)$ to satisfy the early exercise boundary conditions (3.2.9) and (3.2.10). Recalling our definition for the Heaviside function, the integral equation system

$$\frac{c_2(\tau) - K_2}{2} = V_{c_1,c_2}(\ln c_2(\tau),\tau), \tag{3.4.6}$$
$$\frac{K_1 - c_1(\tau)}{2} = V_{c_1,c_2}(\ln c_1(\tau),\tau), \tag{3.4.7}$$

is obtained, where $V_{c_1,c_2}(x,\tau)$ is given by equation (3.4.5) in conjunction with (3.4.1)-(3.4.4). The factor of $\frac{1}{2}$ appears in the left hand side of (3.4.6) and (3.4.7) due to properties of the Fourier transform. Recall that the complete Fourier transform was applied to discontinuous functions of the form $H(\ln c_1 - x)H(x - \ln c_2)f(x,\tau)$. As proved in Dettman (1965) (p.360), the inverted Fourier transform of a discontinuous function will converge to the midpoint of the discontinuity. Thus in equations (3.4.6)-(3.4.7), when V is evaluated at either $\ln c_1(\tau)$ or $\ln c_2(\tau)$, the factor of $\frac{1}{2}$ must be introduced into the left hand side, as was seen in Chapter 2. This is accounted for by our Heaviside function definition in equation (2.2.14).

The system of integral equations (3.4.6)-(3.4.7) must be solved simultaneously using numerical methods to find $c_1(\tau)$ and $c_2(\tau)$, since analytical solutions seem impossible. Once these are found, it is a simple matter to evaluate $V_{c_1,c_2}(x,\tau)$ via numerical integration.

3.5. The Kim-Type Representation

To make the task of numerical implementation less complicated, we will transform equations (3.4.1)-(3.4.5) into the form presented by Kim (1990). This has the effect of removing the $c_1'(\tau)$ and $c_2'(\tau)$ terms from the integral through the use of integration by parts. The first step is to re-write the pricing equation in terms of the original underlying asset S.

PROPOSITION 3.5.1. *The solution to the free boundary value problem (3.2.1)-(3.2.5) in terms of S and τ is given by*

$$A_{c_1,c_2}(S,\tau) = A^{(1)}_{c_1,c_2}(S,\tau) + A^{(2)}_{c_1,c_2}(S,\tau), \tag{3.5.1}$$

where

$$\begin{aligned} A^{(1)}_{c_1,c_2}(S,\tau) &= [K_1 e^{-r\tau} N(-d_2(A,\tau;K_1)) - Se^{-q\tau} N(-d_1(S,\tau;K_1))] \\ &\quad +[Se^{-q\tau}N(d_1(S,\tau;K_2)) - K_2 e^{-r\tau}N(d_2(x,\tau;K_2))] \\ &\quad -[K_1 e^{-r\tau}N(-d_2(S,\tau;c_1(0^+))) - Se^{-q\tau}N(-d_1(S,\tau;c_1(0^+)))] \\ &\quad -[Se^{-q\tau}N(d_1(S,\tau;c_2(0^+))) - K_2 e^{-r\tau}N(d_2(S,\tau;c_2(0^+)))], \end{aligned} \tag{3.5.2}$$

with d_1, d_2 as in Proposition 3.4.1, and

$$A^{(2)}_{c_1,c_2}(S,\tau) = \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}}[e^{-\hat{h}_2(S,\xi)}\hat{Q}_2(S,\xi) + e^{-\hat{h}_1(S,\xi)}\hat{Q}_1(S,\xi)]d\xi, \tag{3.5.3}$$

where

$$\hat{h}_j(S,\xi) = \frac{(\ln(S/c_j(\xi)) + k(\tau-\xi))^2}{2\sigma^2(\tau-\xi)}, \tag{3.5.4}$$

and

$$\hat{Q}_j(S,\xi) = \frac{\sigma^2 c_j(\xi)}{2} + \left(\frac{c_j'(\xi)}{c_j(\xi)} + \frac{1}{2}\left[k - \frac{\ln(S/c_j(\xi))}{(\tau-\xi)}\right]\right)(c_j(\xi) - K_j) \tag{3.5.5}$$

for $j = 1,2$ and $c_1(\tau) < S < c_2(\tau)$.

Proof: Recall that $x = \ln S$, and substitute this into equations (3.4.1)-(3.4.5).

□

PROPOSITION 3.5.2. *Equations (3.5.1)-(3.5.5) can be expressed as follows:*

$$A_{c_1,c_2}(S,\tau) = AP(S,\tau) + AC(S,\tau), \tag{3.5.6}$$

where

$$\begin{aligned} AP(S,\tau) &= K_1 e^{-r\tau} N(-d_2(S,\tau;K_1)) - Se^{-q\tau} N(-d_1(S,\tau;K_1)) \\ &+ \int_0^{\tau} [K_1 r e^{-r(\tau-\xi)} N(-d_2(S,\tau-\xi;c_1(\xi))) \\ &\qquad -Sqe^{-q(\tau-\xi)} N(-d_1(S,\tau-\xi;c_1(\xi)))]d\xi, \end{aligned} \tag{3.5.7}$$

and

$$\begin{aligned} AC(S,\tau) &= Se^{-q\tau} N(d_1(S,\tau;K_2)) - K_2 e^{-r\tau} N(d_2(S,\tau;K_2)) \\ &+ \int_0^{\tau} [Sqe^{-q(\tau-\xi)} N(d_1(S,\tau-\xi;c_2(\xi))) \\ &\qquad -K_2 r e^{-r(\tau-\xi)} N(d_2(S,\tau-\xi;c_2(\xi)))]d\xi. \end{aligned} \tag{3.5.8}$$

Proof: The above is derived using integration by parts[1], as demonstrated by Kim (1990), and outlined in Appendix A2.5.1. Note that equations (3.5.7)-(3.5.8) do not involve the derivatives c_1' and c_2'.

□

3.5.1. The Perpetual American Strangle. Before concluding this section, we shall consider the American strangle in the case where the position has no fixed exercise date. Kim (1990) demonstrates how to derive the price and early exercise boundary of perpetual American calls and puts using his integral representation for the options. Here we extend his results to the perpetual American strangle.

[1]We note that it is possible to arrive directly at this form of the solution by solving an inhomogeneous version of (3.2.1) in the domain $0 < S < \infty$, where the inhomogeneous term, representing the cash flows associated with the strangle in the stopping regions, is given by

$$H(a_1(t) - S)(rK_1 - qS) + H(S - a_2(t))(qS - rK_2).$$

Such an approach was first explained by Jamshidian (1992), and requires a clear understanding of the cash flows that take place when the American portfolio has been exercised early. The Fourier transform analysis may be more mathematically cumbersome than that of Jamshidian (1992), but its strength lies in forgoing any such cash flow analysis. The method is also the only intuitive way to derive McKean's representation for the integral equations, something that cannot be readily understood using the Jamshidian technique.

PROPOSITION 3.5.3. *The price of a perpetual American strangle, $A_{c_1,c_2}(S,\infty)$ is given by*

$$A_{c_1,c_2}(S,\infty) = \left(\frac{S}{c_1(\infty)}\right)^{\beta_-}\left[\frac{(k+\sigma^2)c_1(\infty)-kK_1}{2\sqrt{k^2+2\sigma^2 r}}+\frac{K_1-c_1(\infty)}{2}\right] + \left(\frac{S}{c_2(\infty)}\right)^{\beta_+}\left[\frac{(k+\sigma^2)c_2(\infty)-kK_2}{2\sqrt{k^2+2\sigma^2 r}}+\frac{c_2-K_2(\infty)}{2}\right], \quad (3.5.9)$$

where the early exercise boundaries are given by

$$c_1(\infty) = \frac{K_1(\beta_-)}{(\beta_-)-1}-\frac{2}{(\beta_-)-1}\left((\beta_-)+\frac{k}{\sigma^2}\right)\left(\frac{c_1(\infty)}{c_2(\infty)}\right)^{\beta_+} \times\left[\frac{(k+\sigma^2)c_2(\infty)-kK_2}{2\sqrt{k^2+2\sigma^2 r}}+\frac{c_2(\infty)-K_2}{2}\right], \quad (3.5.10)$$

and

$$c_2(\infty) = \frac{K_2(\beta_+)}{(\beta_+)-1}+\frac{2}{(\beta_+)-1}\left((\beta_+)+\frac{k}{\sigma^2}\right)\left(\frac{c_1(\infty)}{c_2(\infty)}\right)^{\beta_-} \times\left[\frac{(k+\sigma^2)c_1(\infty)-kK_1}{2\sqrt{k^2+2\sigma^2 r}}+\frac{K_1-c_1(\infty)}{2}\right], \quad (3.5.11)$$

with $\beta_\pm = (-k\pm\sqrt{2r\sigma^2+k^2})/\sigma^2$ and $k = r-q-\sigma^2/2$.

Proof: Refer to Appendix 3.4.

□

The interdependence of the two early exercise boundaries prevents us from finding a closed-form solution for $A_{c_1,c_2}(S,\infty)$, $c_1(\infty)$ and $c_2(\infty)$, as opposed to the individual call and put cases considered by Kim (1990). Nevertheless, the system of equations (3.5.10)-(3.5.11) for the free boundaries is readily solved using numerical techniques.

3.6. Numerical Implementation

The integral equation system for the free boundaries $c_1(\tau)$ and $c_2(\tau)$ is now

$$K_1 - c_1(\tau) = A_{c_1,c_2}(c_1(\tau),\tau), \quad (3.6.1)$$

$$c_2(\tau) - K_2 = A_{c_1,c_2}(c_2(\tau),\tau). \quad (3.6.2)$$

It is of value to note that equation (3.5.6) is simply the sum of the integral pricing expressions for an American put and an American call. The added complexity in pricing an

American strangle therefore arises from the early exercise boundaries. Each free boundary is dependent upon the other free boundary in the system (3.6.2)-(3.6.1), and therefore these boundaries are not equal to those found when valuing an American call and put option separately. Thus it is important to understand the nature of the early exercise boundaries for American option portfolios in order to obtain the correct free boundary values.

To solve this system, we propose using a numerical scheme similar to that usually applied to Volterra integral equations. Firstly, discretise the time variable τ into n equally spaced intervals of length h. Thus $\tau_i = ih$ for $i = 0, 1, 2, \ldots, n$, and $h = T/n$. Following the methods of Kim (1990), it can be readily shown that the initial values are given by (see Appendix 3.5)

$$c_1(0^+) = \min\left(K_1, K_1\frac{r}{q}\right), \quad \text{and} \quad c_2(0^+) = \max\left(K_2, K_2\frac{r}{q}\right). \tag{3.6.3}$$

Thus by starting at $i = 1$, there are only two unknown values in the system (3.6.2)-(3.6.1) for each i, namely $c_1(\tau_i)$ and $c_2(\tau_i)$. We use Simpson's rule to evaluate the integral terms.

For each i beginning with $i = 1$, the bisection method is applied to the following nonlinear equation to find $c_1(\tau_i)$:

$$c_1(ih) = K_1 - \hat{AP}(c_1(ih), ih; c_1) - \hat{AC}(c_1(ih), ih; c_2), \tag{3.6.4}$$

where

$$\begin{aligned} \hat{AP}(c_1(ih), ih; c_1) \quad = \quad & K_1 e^{-rih} N(-d_2(c_1(ih), ih; K_1)) \\ & -c_1(ih) e^{-qih} N(-d_1(c_1(ih), ih; K_1)) \\ & +h \sum_{j=0}^{i} w_j [K_1 r e^{-rh(i-j)} N(-d_2(c_1(ih), h(i-j); c_1(jh))) \\ & -c_1(ih) q e^{-qh(i-j)} N(-d_1(c_1(ih), h(i-j); c_1(jh)))], \end{aligned} \tag{3.6.5}$$

and

$$\begin{aligned}\hat{AC}(c_1(ih), ih; c_2) &= c_1(ih)e^{-qih}N(d_1(c_1(ih), ih; K_2)) \\ &\quad -K_2e^{-rih}N(-d_2(c_1(ih), ih; K_2)) \\ &\quad +h\sum_{j=0}^{i} w_j[c_1(ih)qe^{-qh(i-j)}N(d_1(c_1(ih), h(i-j); c_2(jh)))] \\ &\qquad -K_2re^{-rh(i-j)}N(d_2(c_1(ih), h(i-j); c_2(jh)))].\end{aligned} \tag{3.6.6}$$

The summation weights, w_j, are those dictated by the numerical integration scheme, in this case Simpson's rule and the extended Simpson's rule (used on the end furthest from expiry whenever i is odd). The bisection method was chosen over more complex techniques, such as Newton's method, because the monotonic nature of the free boundaries enables us to efficiently reduce our search region for the unknown root as i increases. The bisection method also saves us from having to evaluate the derivatives of the free boundary integral equations.

Similarly, we apply the bisection method to the following to find $c_2(\tau_i)$:

$$c_2(ih) = K_2 + \hat{AP}(c_2(ih), ih; c_1) + \hat{AC}(c_2(ih), ih; c_2), \tag{3.6.7}$$

where

$$\begin{aligned}\hat{AP}(c_2(ih), ih; c_1) &= K_1e^{-rih}N(-d_2(c_2(ih), ih; K_1)) \\ &\quad -c_2(ih)e^{-qih}N(-d_1(c_2(ih), ih; K_1)) \\ &\quad +h\sum_{j=0}^{i} w_j[K_1re^{-rh(i-j)}N(-d_2(c_2(ih), h(i-j); c_1(jh))) \\ &\qquad -c_2(ih)qe^{-qh(i-j)}N(-d_1(c_2(ih), h(i-j); c_1(jh)))],\end{aligned} \tag{3.6.8}$$

and

$$\begin{aligned}\hat{AC}(c_2(ih), ih; c_2) &= c_2(ih)e^{-qih}N(d_1(c_2(ih), ih; K_2)) \\ &\quad -K_2e^{-rih}N(-d_2(c_2(ih), ih; K_2)) \\ &\quad +h\sum_{j=0}^{i} w_j[c_2(ih)qe^{-qh(i-j)}N(d_1(c_2(ih), h(i-j); c_2(jh)))] \\ &\qquad -K_2re^{-rh(i-j)}N(d_2(c_2(ih), h(i-j); c_2(jh)))].\end{aligned} \tag{3.6.9}$$

It is important to note that at $j = i$, both d_1 and d_2 are singular. This can be handled by using the limits

$$\lim_{\xi \to \tau} N(d_1(c_1(\tau), \tau - \xi; c_2(\xi)) = \lim_{\xi \to \tau} N(-d_1(c_2(\tau), \tau - \xi; c_1(\xi)) = 0, \qquad (3.6.10)$$

and

$$\lim_{\xi \to \tau} N(d_1(c_2(\tau), \tau - \xi; c_2(\xi)) = \lim_{\xi \to \tau} N(-d_1(c_1(\tau), \tau - \xi; c_1(\xi)) = 0.5. \qquad (3.6.11)$$

These limits are the same for d_2. It is also important to see that while, for example, (3.6.4)-(3.6.6) depends upon $c_2(\tau_i)$ it does not explicitly require $c_2(\tau_i)$, due to the need to use the limits (3.6.10)-(3.6.11) at $j = i$. Hence the simultaneous integral equation system (3.6.2)-(3.6.1) can be solved by finding $c_1(\tau_i)$ using all known values of $c_1(\tau_j)$ and $c_2(\tau_j)$, $j = 0, 1, 2, \ldots, i-1$. That is, the interdependence at the ith time point is removed due to the need to consider the limits of d_1 and d_2 when evaluating (3.6.4)-(3.6.6) and (3.6.7)-(3.6.9).

The above numerical scheme is firstly carried out using using a time-step size of h, and then repeated using $h/2$. In each case, since it is necessary to alternate between two different numerical integration schemes (for odd and even values of i) it turns out that the free boundaries have non-monotonic gradients. This is rectified by combining the two estimates using Richardson extrapolation. Pricing the American strangle is then achieved via numerical integration using Simpson's rule, combined with the estimates of $c_1(\tau_i)$ and $c_2(\tau_i)$.

The algorithm *American Strangle Price* presented in Appendix 3.6 demonstrates how Richardson extrapolation was implemented when solving the integral equation system for the free boundaries. Note that the algorithm allows for a finer grid to be used close to the expiry date of the strangle, since this is the region where the free boundaries experience the most rapid change. The final American strangle price is readily obtained using numerical integration once the free boundaries have been estimated.

3.7. Results

Firstly we examined the accuracy of the numerical scheme being implemented. The method was applied to a 1-year American strangle position, using $n = 100, 200, 400$ and 800. This has been compared against an optimised Crank-Nicolson scheme using 4 time steps per day. The r and q parameters were chosen to be non-zero and unequal. The results are summarised in Table 3.1 for 5 spot prices, chosen to represent at-the-money, in-the-money and out-of-the-money prices for the strangle within the continuation region.

S	CN 60,000	CN 120,000	Kim 100	Kim 200	Kim 400	Kim 800
0.75	0.275648	0.275648	0.275647	0.275647	0.275647	0.275647
1.00	0.100322	0.100319	0.100332	0.100332	0.100332	0.100332
1.25	0.038560	0.038560	0.038563	0.038562	0.038561	0.038561
1.50	0.092316	0.092314	0.092344	0.092341	0.092341	0.092340
1.75	0.255619	0.255619	0.255631	0.255632	0.255633	0.255633

TABLE 3.1. American Strangle price found numerically. The Crank-Nicolson finite difference scheme involved 4 time steps per day, using 60,000 and 120,000 space-nodes. The numerical scheme for solving Kim's integral equations used $n = 100, 200, 400$ and 800 respectively, as indicated in the table. The parameter values were $r = 5\%$, $q = 10\%$, $T - t = 1$; $K_1 = 1$; $K_2 = 1.5$; $\sigma = 20\%$.

From Table 3.1 it can be seen that for the strangle at-the-money on the call side, the Crank-Nicolson scheme has converged to 4 decimal places, while it has converged to 5 decimal places at the other spot values. Thus we take the Crank-Nicolson results as being the true solution to an accuracy of around 4 decimal places.

For all the values of n used, the American strangle prices found using Kim's integral equation system match those found using Crank-Nicolson to 4 decimal places. We conclude from these results that the numerical method employed in solving the integral equations has an accuracy of 4 decimal places. It can also be seen that the numerical scheme for Kim's integral equations has converged to 5 decimal places for n as low as 100. We therefore select $n = 200$ for the purposes of generating all further results. The algorithm was implemented using LAHEY™FORTRAN 95 on a PC with a Pentium 4 2.40 GHz processer, 512MB of RAM, and running the Windows XP Professional operating system. With $n = 200$, the code takes approximately 66 seconds to solve the integral equation system for the American strangle's free boundaries.

By considering the number of calculations required by the code for a given n, we can provide some insight into how the value of n affects the code's run-time. Assume that when using the bisection method to solve the integral equation system, the mean number of iterations required for each application is m. In practice we find that m is approximately 50. When finding the ith pair of free boundary values for $i = 1, 2, ..., n$, we must evaluate the integrand at $i + 1$ points. Therefore to estimate each free boundary using the bisection method, we are required to make $m[(n + 1)(n + 2)/2]$ integrand evaluations. Thus it can be concluded that the number of operations required by our code is of order mn^2.

To demonstrate the early exercise boundaries and price properties of the American strangle, we implemented the method using $n = 200$ time nodes. To improve the accuracy of the method where the free boundaries change rapidly, a finer grid was used between the first three nodes (specifically, 40 nodes between $i = 0$ and $i = 2$). The method was also applied in the same manner to the American call and put contracts which define the components of the strangle's payoff function. By comparing the results for the strangle against those of the independent call and put, we can demonstrate how the American strangle's free boundaries and price are affected by the interdependence between $c_1(\tau)$ and $c_2(\tau)$.

Firstly, consider an American strangle with one year until maturity. Let the put-side strike be 1 and the call-side strike be 1.1, with the volatility of the underlying at 20%. In Figures 3.2-3.5 we present the call- and put-side boundaries for the American strangle, with $r > q$ (Figures 3.2 and 3.5), $r < q$ (Figures 3.3 and 3.4), and finally $r = q$ (Figures 3.4 and 3.5). In all cases, we include the free boundary for the corresponding American call or put. The same results are repeated again in Figures 3.2, 3.3, 3.6 and 3.7, but with the call-side strike having been reduced to 1.001, moving the strangle position closer to a straddle.

There are several distinct features that can be ascertained from these free boundary plots. The first is that the relative values of r and q directly affect whether or not the American strangle free boundaries will show significant divergence from the corresponding American call and put boundaries. In particular, when $r > q$, only the put-side boundaries diverge, and when $r < q$ only the call-side boundaries diverge[2]. When $r = q$, there is divergence in both boundaries, but it is smaller than in the other two cases.

Since the early exercise of the in-the-money side of the strangle will knock-out the other side, it is expected that the strangle will have to be deeper in-the-money to warrant early exercise than one formed using independent American calls and puts. In all cases,

[2]Note that in Figure 3.2 the strangle free boundaries are almost equal to the corresponding call free boundaries, thus making only two of the four boundaries visible in the plot. For the put side, shown in Figure 3.3, all four free boundaries in the plot are almost equal, and only a single boundary can be seen.

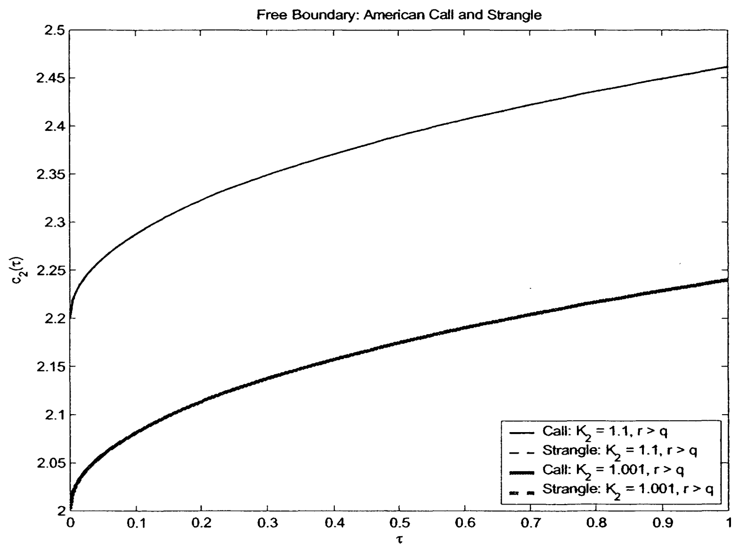

FIGURE 3.2. $K_1 = 1, \sigma = 20\%; r = 10\%, q = 5\%$

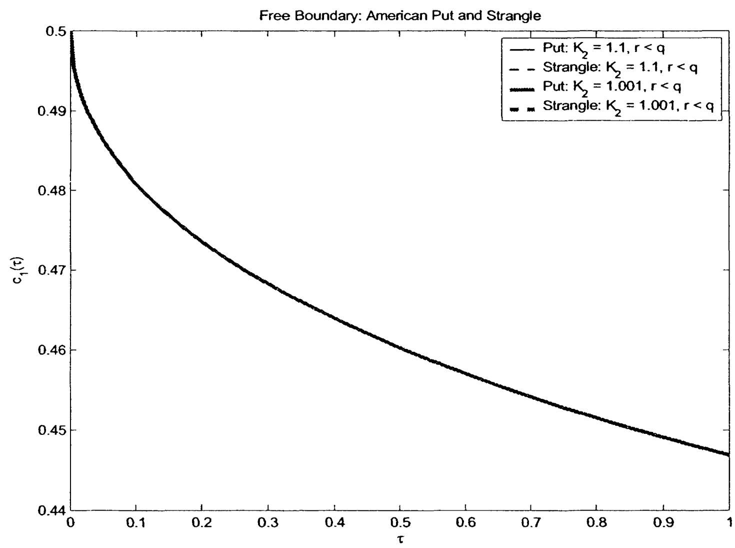

FIGURE 3.3. $K_1 = 1, \sigma = 20\%, r = 5\%, q = 10\%$

the call-side free boundary for the strangle is always greater than or equal to that of the corresponding American call free boundary, while the put-side is always less than or equal to that of the corresponding American put free boundary. This is in keeping with the economic intuition behind the American strangle position.

In all three cases of r and q values, moving the call-side's strike closer to the put-side's strike increases any divergence between the American strangle free boundaries and those of the corresponding American call and put. This is again as one would expect, since the

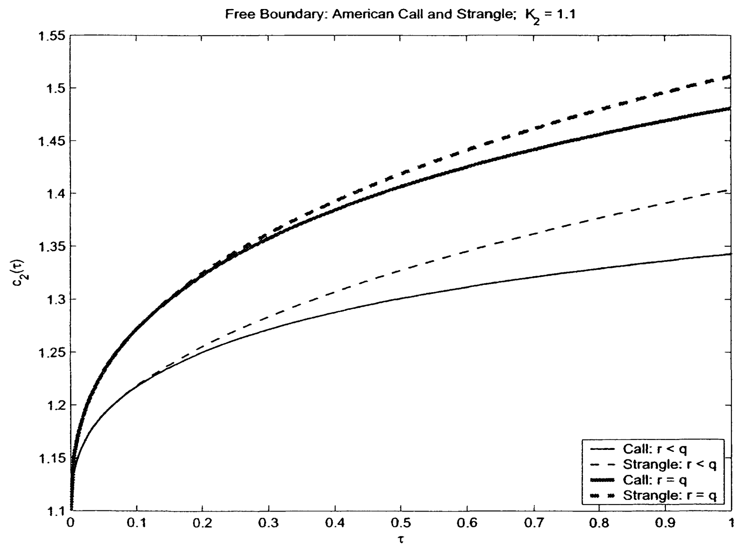

FIGURE 3.4. $K_1 = 1, K_2 = 1.1, \sigma = 20\%, q = 10\%; r = 5\%$ when $r < q$

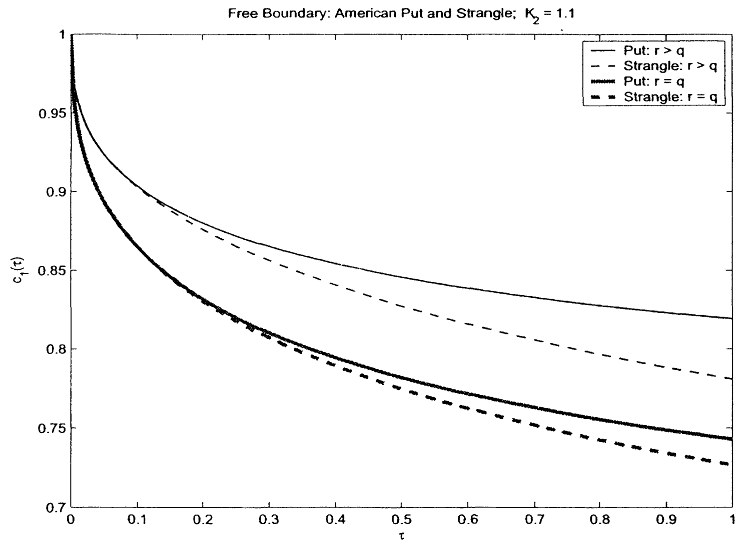

FIGURE 3.5. $K_1 = 1, K_2 = 1.1, \sigma = 20\%, r = 10\%; q = 5\%$ when $r > q$

closer the strangle is to being a straddle, the more intrinsic value the out-of-the-money strangle component will contribute to the early exercise decision. It can also be seen that as the time to maturity increases, the divergence between the strangle free boundaries and the corresponding call and put boundaries increases. When the strangle has a very short time to maturity, say 2 weeks or less, then the divergence between the two free boundaries becomes minimal.

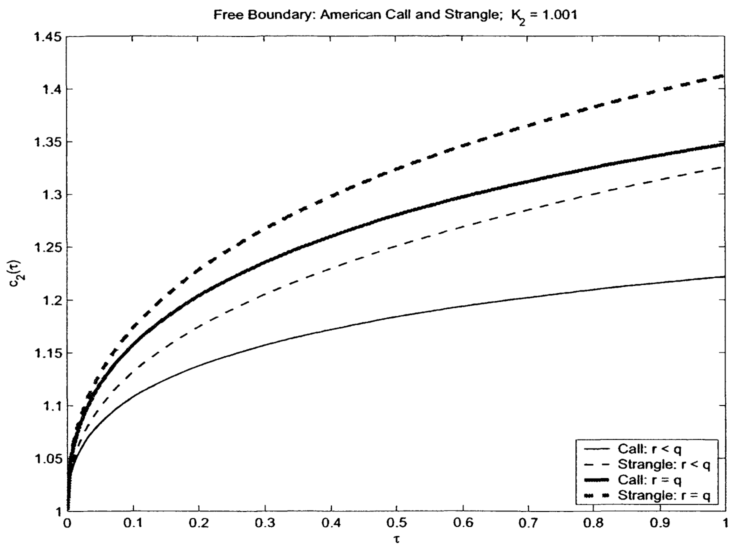

FIGURE 3.6. $K_1 = 1, K_2 = 1.001$, $\sigma = 20\%$, $q = 10\%$; $r = 5\%$ when $r < q$

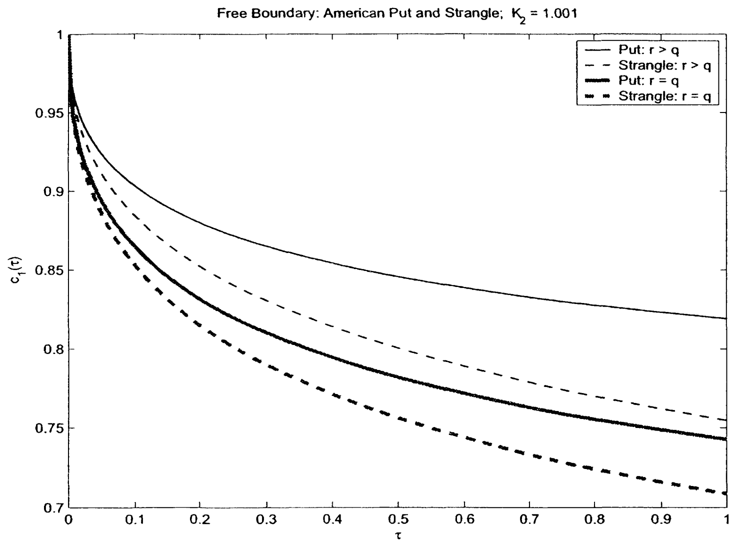

FIGURE 3.7. $K_1 = 1, K_2 = 1.001$, $\sigma = 20\%$, $r = 10\%$; $q = 5\%$ when $r > q$

Figure 3.8 demonstrates how the early exercise boundary of the American call and the call-side free boundary of the strangle vary with changes in the volatility of S. We focus on the case where $r < q$, since this is when the call-side differences are most pronounced. As the volatility increases, the divergence between the corresponding free boundaries becomes larger. A similar result can be seen for the put side of the strangle, and it's corresponding American put option, as displayed in Figure 3.9, although the increased

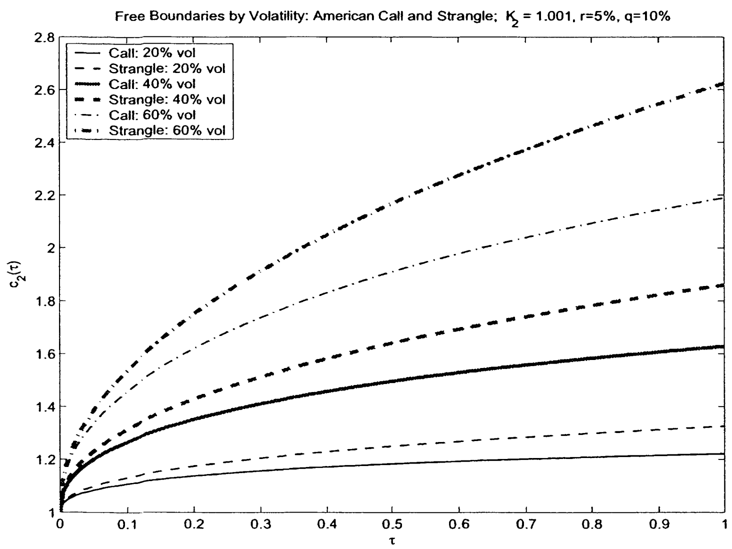

FIGURE 3.8. Changes in the call-side free boundaries for different values of σ.

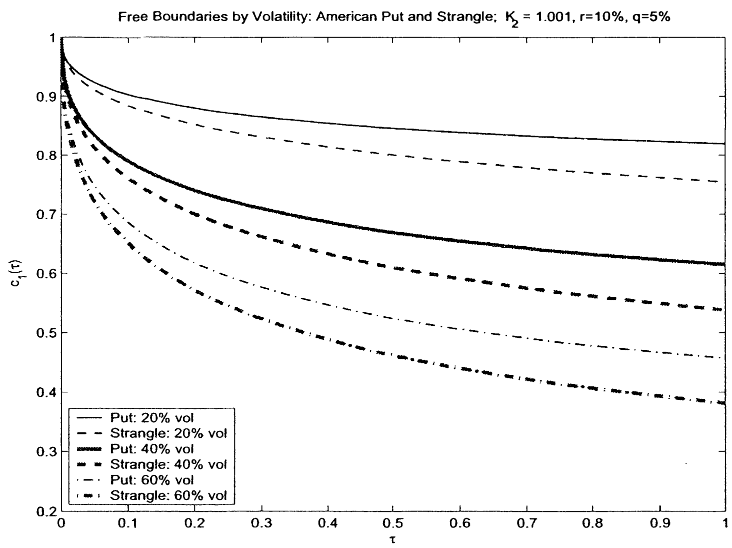

FIGURE 3.9. Changes in the put-side free boundaries for different values of σ.

divergence is far less obvious. We used $r > q$, again so that we could focus on the case where the put-side differences were most extreme. Smaller values of r and q were also explored, and this produced similar behaviour in the location of the free boundaries as presented for changes in the volatility. These results have been omitted for the sake of brevity.

To demonstrate the long-term impact of the interdependence on the early exercise boundaries of the strangle, we numerically evaluated the system of simultaneous equations (3.5.10)-(3.5.11) for $c_1(\infty)$ and $c_2(\infty)$, and compared these with the corresponding perpetual American put and call early exercise boundaries. In solving this system we used the same basic root-finding technique as for the fixed maturity American strangle. The system was solved using an iterative method, with the perpetual American put and call free boundaries (as given by Kim (1990)) used as the initial approximations for $c_1(\infty)$ and $c_2(\infty)$ respectively. Table 3.2 summarises the results for a range of r, q, and K_2 values, with $K_1 = 1$ and $\sigma = 20\%$. We can see that as for the American strangle with finite maturity, the interdependence requires that the strangle under consideration should be exercised deeper in-the-money than the corresponding strangle formed with independent American calls and puts. Once again, the relative values of r and q influence which side (call or put) displays the most substantial difference, and the closer K_2 is to a fixed value of K_1, the larger the observed difference on the put side becomes. Note that in all cases we can see that there is a clear difference between the strangle and the independent call and put, indicating that as time to maturity increases, the free boundaries for the strangle will always be deeper in-the-money than they are for the corresponding American call and put.

		Put $c_1(\infty)$	Strangle $c_1(\infty)$	Call $c_2(\infty)$	Strangle $c_2(\infty)$
$r > q$	$K_2 = 1.100$	0.7566	0.5950	2.9077	2.9223
	$K_2 = 1.001$	0.7566	0.5885	2.6460	2.6648
$r < q$	$K_2 = 1.100$	0.3783	0.3761	1.4539	1.8166
	$K_2 = 1.001$	0.3783	0.3756	1.3230	1.7005

TABLE 3.2. Comparing the early exercise boundaries for perpetual American strangles with corresponding perpetual American calls and puts. When $r > q$, $r = 10\%$, $q = 5\%$, and vice versa when $r < q$. $K_1 = 1$ and $\sigma = 20\%$.

While it is clear that the early exercise boundaries for the finite strangle are not always equivalent to those of the component American call and put in the examples provided, the difference never exceeds 0.1, which in relative terms is no more than 10% of the put-side's strike price. Past research into American options, such as Ju (1998) and AitSahlia and Lai (2001), has found that the price of American call and put options is not greatly affected by the free boundary estimate used. While a 10% difference in the free boundary has obvious early exercise timing repercussions, it remains to be seen whether the price of the strangle using these free boundaries is far removed from that of a strangle priced simply using the sum of an American call and an American put. To explore the effect of these free boundary differences on the strangle's price, we compare the price of the American strangle against the "traditional" American call plus American put approach. The prices were found using Simpson's rule with 100 nodes (implying no need for interpolation when using our $c_1(\tau)$ and $c_2(\tau)$ estimates), and were compiled for a range of volatilities (Tables 3.3-3.4) and call-side strikes (Tables 3.5-3.6). In all cases, the prices were found for a range of underlying asset values, S, between 0 and 300,000. Thus, these results are indicative of a position in a contract involving several thousand American strangle contracts. The tables present only the prices for which the difference between the strangle and the call-put sum was greatest. The time to maturity is always set at 1 year.

σ (%)	Max Price Difference	S (000's)	Relative Difference
20	1,201.13	80	5.84%
40	1,156.51	60	2.85%
60	1,078.79	40	1.80%
80	1,137.30	30	1.62%

TABLE 3.3. Maximum price differences between the American Strangle and the same position formed using an American Call and an American Put for a range of σ values. $r = 10\%$, $q = 5\%$, $T - t = 1$; K_1 =100,000; K_2 =100,100; prices of underlying range from $S = 0$ to S =300,000 in steps of 10,000.

σ (%)	Max Price Difference	S (000's)	Relative Difference
20	1,478.68	130	4.93%
40	1,996.79	170	2.83%
60	2,776.88	240	1.98%
80	3,458.06	300	1.71%

TABLE 3.4. Maximum price differences between the American Strangle and the same position formed using an American Call and an American Put for a range of σ values. $r = 5\%$, $q = 10\%$, $T - t = 1$; K_1 =100,000; K_2 =100,100; prices of underlying range from $S = 0$ to S =300,000 in steps of 10,000.

It should be noted that in all cases the American strangle price is always less than or equal to the sum of the corresponding American call and put prices. This is as expected, since the American strangle is equivalent to combining a long knock-out American call

K_2 (000's)	Max Price Difference	S (000's)	Relative Difference
100.01	1,207.49	80	5.87%
101.00	1,138.72	80	5.56%
110.00	630.92	80	3.14%
150.00	11.23	80	0.06%

TABLE 3.5. Maximum price differences between the American Strangle and the same position formed using an American Call and an American Put for a range of K_2 values. $r = 10\%, q = 5\%, T - t = 1$; K_1 =100,000; $\sigma = 20\%$; prices of underlying range from $S = 0$ to S =300,000 in steps of 10,000.

K_2 (000's)	Max Price Difference	S (000's)	Relative Difference
100.01	1,482.61	130	4.93%
101.00	1,433.60	130	4.92%
110.00	748.99	140	2.50%
150.00	15.16	180	0.05%

TABLE 3.6. Maximum price differences between the American Strangle and the same position formed using an American Call and an American Put for a range of K_2 values. $r = 5\%, q = 10\%, T - t = 1$; K_1 =100,000; $\sigma = 20\%$; prices of underlying range from $S = 0$ to S =300,000 in steps of 10,000.

and a long knock-out American put, where the knock-out barriers are $c_1(\tau)$ and $c_2(\tau)$ for the call and put respectively. The decrease in the strangle's price reflects the presence of these implicit knock-out barriers, and hence the inability to separate the call and put sides in this new strangle position.

From Table 3.3, we see that when $r > q$, the largest difference appears on the put-side, as one would expect. The difference remains around 1,000 for all the volatilities, but as the volatility increases, the relative difference decreases, and is at most between 5-6%. The greatest differences occur when the put is deep in-the-money, and this maximum occurs deeper in-the-money as the volatility increases.

When $r < q$, the maximum difference occurs on the call-side. Table 3.4 shows that this can exceed 3,000 for a large enough volatility, but as is the case in Table 3.3, the smaller the volatility, the greater the relative price difference is. This difference never exceeds 5%, and the greatest differences arise when the strangle is deep in-the-money on the call side. Thus the largest relative price deviations will occur for low volatilities. While this result appears counter-intuitive, it is important to note that "realistic" volatilities (e.g. 20%) produce the greatest relative price differences.

Table 3.5 considers the maximum price differences for a range of call-side strikes, with $r > q$. As in Table 3.3, the greatest differences occur deep in-the-money on the put-side, and become smaller as $K_2 - K_1$ increases. A similar result is shown in Table 3.6, where $r < q$, and the difference is now greatest deep in-the-money on the call side. Once the call-side strike reaches 150,000, the relative price difference is at most less than 0.1%, while the largest relative differences, when the strangle is effectively a straddle, are still no more than 6%. Overall, it appears that a 10% difference in one of the early exercise boundaries will produce at most a 6% difference in the price, when comparing this American strangle with a position formed by going long in both an American call and an American put. From a market perspective, it appears that the reduction in the strangle's premium by foregoing the flexibility to separate the strangle's components is relatively small. There appears little premium advantage in creating an American strangle with early exercise triggered knock-out features for the out-of-the-money side of the position. This suggests that American strangles could generally be of little value to investors and traders from a premium perspective. To what extent this alternate definition would impact on transaction and investment costs to the holder remains unknown, since the greatest differences arise in the timing of early exercise, and the volume of transactions required to close-out the strangle position.

3.8. Conclusion

In this chapter we have presented a generalisation of McKean's free boundary value problem for pricing American options. We have considered the example of an American strangle position, where exercising one side of the position early will knock-out the remaining side. McKean's integral expression for this strangle's price were derived, along with the integral equation system for its two free boundaries. The integral equations were re-expressed in a more economically intuitive form using Kim's simplifications. It was shown that analytically the free boundaries for the American strangle are not equal to those found when valuing independent American calls and puts.

Kim's form of the integral equation system was solved using a scheme typically applied to nonlinear Volterra integral equations. It was found that numerically the early exercise boundary of this strangle only differed significantly from the boundaries of corresponding American calls and puts for certain values of the risk-free rate and continuous dividend yield parameters. The differences became larger as the distance between the strangle's strikes was reduced, and as the time to expiry increased. Comparing the prices of this new strangle to those of a strangle formed using a long American call and a long American put, we showed that for several call-side strikes and volatilities, our strangle was cheaper than the "traditional" one by no more than 6%, and that these differences were most apparent when the strangle was deep in-the-money. Economically, this pricing difference can be viewed as the reduction in value caused by introducing the knock-out effect into the new strangle, and foregoing the freedom to separate the call and put sides.

The early exercise boundaries for our strangle required that the position be deeper in-the-money than a "traditional" strangle, to compensate the intrinsic value forgone on the out-of-the-money side. If one does not calculate these free boundaries correctly, there is the potential to exercise the American strangle presented in this chapter too early. Despite these early exercise differences, the prices of the two strangles were usually very close, and an important contribution of this chapter has been to quantify this difference. An investor interested in an American strangle position may be indifferent when choosing between this proposal and a "traditional" American strangle, since only a small premium is required for the added flexibility of the latter. Whether or not the reduced transaction costs from the self-closing strangle would benefit the investor is a matter we leave to future study.

The methodology developed here is applicable to American positions with quite general convex or concave payoffs. One avenue for future research would be to consider other complex payoff types, such as an American butterfly (i.e. concave payoff), or an American bear/bull spread (i.e. monotonic payoff). These positions can be constructed with similar early exercise conditions to our American strangle, and can be evaluated using our generalisation of McKean's framework. The numerical method presented should be rigourously tested against existing techniques, such as binomial trees and finite differences. Better numerical techniques for solving the integral equation system also need to be investigated, such as the method demonstrated by Kallast and Kivinukk (2003). In addition, one could explore the potential to numerically solve the McKean-type integral equation system in its original form.

Appendix 3.1. The Two-Sided Incomplete Fourier Transform

In Appendix 2.2 we presented the Fourier integral theorem for the one-sided incomplete Fourier transform. For the American strangle problem we must extend this result to the case where the solution is bounded between two free boundaries. Generalising the discussion in Appendix 2.2, we can readily show that if $f(x,\tau) = H(b-x)H(a-x)g(x,\tau)$, where $H(x)$ is the Heaviside step function, $a \equiv a(\tau)$, $b \equiv b(\tau)$, and $a < b$ for all $0 \le \tau \le T$, then the Fourier integral theorem

$$f(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,\tau)e^{i\eta x}dx\right]e^{-i\eta x}d\eta, \quad -\infty < x < \infty,$$

will yield

$$g(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{a}^{b} g(x,\tau)e^{i\eta x}dx\right]e^{-i\eta x}d\eta, \quad a < x < b,$$

which we refer to as the two-sided incomplete Fourier transform of $g(x,\tau)$. Note that

$$\frac{g(x,\tau)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{a}^{b} g(x,\tau)e^{i\eta x}dx\right]e^{-i\eta x}d\eta, \quad x = a, b,$$

and refer to Section 3.4 for an explanation regarding the factor of $\frac{1}{2}$ on the left hand side.

Appendix 3.2. Properties of the Two-Sided Incomplete Fourier Transform

A3.2.1. Proof of Proposition 3.3.1. Firstly consider

$$\mathcal{F}^c\left\{\frac{\partial V}{\partial x}\right\} = V_{c_1,c_2}(\ln c_2,\tau)e^{i\eta\ln c_2} - V_{c_1,c_2}(\ln c_1,\tau)e^{i\eta\ln c_1} - i\eta\hat{V}_{c_1,c_2}(\eta,\tau).$$

Finally by use of boundary conditions (3.2.9) and (3.2.10),

$$\mathcal{F}^c\left\{\frac{\partial V}{\partial x}\right\} = (c_2 - K_2)e^{i\eta\ln c_2} - (K_1 - c_1)e^{i\eta\ln c_1} - i\eta\hat{V}.$$

Next consider

$$\begin{aligned}\mathcal{F}^c\left\{\frac{\partial^2 V}{\partial x^2}\right\} &= e^{i\eta\ln c_2}\cdot\left.\frac{\partial V_{c_1,c_2}(x,\tau)}{\partial x}\right|_{x=\ln c_2} - e^{i\eta\ln c_1}\cdot\left.\frac{\partial V_{c_1,c_2}(x,\tau)}{\partial x}\right|_{x=\ln c_1} \\ &\quad -i\eta\mathcal{F}^c\left\{\frac{\partial V}{\partial x}\right\} \\ &= c_2e^{i\eta\ln c_2} + c_1e^{i\eta\ln c_1} - i\eta[(c_2 - K_2)e^{i\eta\ln(c_2)} - (K_1 - c_1)e^{i\eta\ln(c_1)} - i\eta\hat{V}],\end{aligned}$$

where the last equality follows by use of the boundary conditions (3.2.11) and (3.2.12), and the transform result (3.3.2). The last equation simplifies to

$$\mathcal{F}^c\left\{\frac{\partial^2 V}{\partial x^2}\right\} = e^{i\eta \ln c_2}(c_2 - i\eta(c_2 - K_2)) - e^{i\eta \ln c_1}(-c_1 - i\eta(K_1 - c_1)) - \eta^2 \hat{V}.$$

Finally consider

$$\begin{aligned}\mathcal{F}^c\left\{\frac{\partial V}{\partial \tau}\right\} &= \frac{\partial}{\partial \tau}\left[\int_{\ln c_1}^{\ln c_2} e^{i\eta x} V_{c_1,c_2}(x,\tau)dx\right] \\ &\quad -\frac{c_2'}{c_2} e^{i\eta \ln c_2} V_{c_1,c_2}(\ln c_2, \tau) + \frac{c_1'}{c_1} e^{i\eta \ln c_1} V_{c_1,c_2}(\ln c_1, \tau) \\ &= \frac{\partial}{\partial \tau}\left[\mathcal{F}^c\{V\}\right] - \frac{c_2'}{c_2} e^{i\eta \ln c_2} V_{c_1,c_2}(\ln c_2, \tau) + \frac{c_1'}{c_1} e^{i\eta \ln c_1} V_{c_1,c_2}(\ln c_1, \tau),\end{aligned}$$

where $c_j' \equiv dc_j(\tau)/d\tau, \quad j = 1, 2$. Applying the boundary conditions (3.2.9) and (3.2.10) we have

$$\mathcal{F}^c\left\{\frac{\partial V}{\partial \tau}\right\} = \frac{\partial \hat{V}}{\partial \tau} - \frac{c_2'}{c_2} e^{i\eta \ln c_2}(c_2 - K_2) + \frac{c_1'}{c_1} e^{i\eta \ln c_1}(K_1 - c_1).$$

A3.2.2. Proof of Proposition 3.3.2. Taking the incomplete Fourier transform of equation (3.2.7) with respect to x and using (3.3.2) - (3.3.4), we obtain

$$\begin{aligned}&\frac{\partial \hat{V}}{\partial \tau} + \left(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r\right)\hat{V} \\ &= e^{i\eta \ln c_2}\left[\frac{c_2'}{c_2}(c_2 - K_2) + \frac{1}{2}\sigma^2(c_2 - i\eta(c_2 - K_2)) + k(c_2 - K_2)\right] \\ &\quad - e^{i\eta \ln c_1}\left[\frac{c_1'}{c_1}(K_1 - c_1) + \frac{1}{2}\sigma^2(-c_1 - i\eta(K_1 - c_1)) + k(K_1 - c_1)\right].\end{aligned}$$

It is a simple matter to rewrite this in terms of $F(\eta, \tau)$ to produce equations (3.3.5)-(3.3.6), and the initial condition is obtained by definition.

Appendix 3.3. Derivation of the American Strangle Integral Equations

A3.3.1. Proof of Proposition 3.4.1. We shall evaluate the inverse Fourier transform, $V^{(1)}_{c_1,c_2}(x,\tau)$, using the standard Fourier convolution result given by equation (A2.1.1). Let

$$F(\eta,\tau_1) = e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)\tau}.$$

Hence

$$f(x,\tau_1) = \frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2\tau-i\eta(x+k\tau)}d\eta = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{\frac{-(x+k\tau)^2}{2\sigma^2\tau}}$$

by use of equation (A2.1.2) with $\hat{p}=\frac{1}{2}\sigma^2\tau$, $\hat{q}=i(x+k\tau)$ and $n=0$. Next let $G(\eta,\tau_2) = \hat{V}_{c_1,c_2}(\eta,0)$. Hence we have

$$\begin{aligned} g(x,\tau_2) &= H(\ln c_2(0^+)-x)H(x-\ln c_1(0^+))V_{c_1,c_2}(x,0)\\ &= H(\ln c_2(0^+)-x)H(x-\ln c_1(0^+))\\ &\quad\times[H(\ln K_1-x)(K_1-e^x)+H(x-\ln K_2)(e^x-K_2)]. \end{aligned}$$

Thus

$$\begin{aligned} V^{(1)}_{c_1,c_2}(x,\tau) &= \int_{-\infty}^{\infty}\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}}H(\ln c_2(0^+)-u)H(u-\ln c_1(0^+))\\ &\quad\times[H(\ln K_1-u)(K_1-e^u)+H(u-\ln K_2)(e^u-K_2)]du. \end{aligned}$$

It is known that $c_1(0^+)\leq K_1$ and $c_2(0^+)\geq K_2$. Hence

$$H(\ln K_1-u)H(\ln c_2(0^+)-u)H(u-\ln c_1(0^+)) = H(\ln K_1-u)H(u-\ln c_1(0^+)),$$

and

$$H(u-\ln K_2)H(\ln c_2(0^+)-u)H(u-\ln c_1(0^+)) = H(u-\ln K_2)H(\ln c_2(0^+)-u).$$

Therefore

$$\begin{aligned} V^{(1)}_{c_1,c_2}(x,\tau) &= \int_{\ln c_1(0^+)}^{\ln K_1}\frac{K_1e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}}du - \int_{\ln c_1(0^+)}^{\ln K_1}\frac{e^ue^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}}du\\ &\quad+\int_{\ln K_2}^{\ln c_2(0^+)}\frac{e^ue^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}}du - \int_{\ln K_2}^{\ln c_2(0^+)}\frac{K_2e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}}du\\ &\equiv I_1-I_2+I_3-I_4. \end{aligned}$$

To simplify $V^{(1)}_{c_1,c_2}(x,\tau)$ further, we shall express it in terms of the cumulative standard normal distribution, $N(y)$. For the first term, I_1, by defining $d_2(S,\tau;\beta) \equiv (\ln(S/\beta) + k\tau)/\sigma\sqrt{\tau}$, the integral then becomes

$$I_1 = K_1 e^{-r\tau}[N(-d_2(e^x,\tau;K_1)) - N(-d_2(e^x,\tau;c_1(0^+)))].$$

We can evaluate the second term, I_2, using equation (A2.1.3) with $\alpha_1 = \ln c_1(0^+)$, $\alpha_2 = \ln K_1$, $\hat{a} = 1$, $\hat{p} = 2\sigma^2\tau$ and $\hat{q} = x + k\tau$. Recalling that $k = r - q - \frac{1}{2}\sigma^2$, and defining $d_1(S,\tau;\beta) \equiv (\ln(S/\beta) + (k+\sigma^2)\tau)/\sigma\sqrt{\tau}$, I_2 then becomes

$$I_2 = -e^x e^{-q\tau}[N(-d_1(e^x,\tau;K_1)) - N(-d_1(e^x,\tau;c_1(0^+)))].$$

Similarly it can be shown that

$$I_3 = e^x e^{-q\tau}[N(d_1(e^x,\tau;K_2)) - N(d_1(e^x,\tau;c_2(0^+)))],$$

and

$$I_4 = K_2 e^{-r\tau}[N(d_2(e^x,\tau;K_2)) - N(d_2(e^x,\tau;c_2(0^+)))].$$

Thus it is concluded that

$$\begin{aligned} V^{(1)}_{c_1,c_2}(x,\tau) = {} & [K_1 e^{-r\tau}N(-d_2(e^x,\tau;K_1)) - e^x e^{-q\tau}N(-d_1(e^x,\tau;K_1))] \\ & +[e^x e^{-q\tau}N(d_1(e^x,\tau;K_2)) - K_2 e^{-r\tau}N(d_2(e^x,\tau;K_2))] \\ & -[K_1 e^{-r\tau}N(-d_2(e^x,\tau;c_1(0^+))) - e^x e^{-q\tau}N(-d_1(e^x,\tau;c_1(0^+)))] \\ & -[e^x e^{-q\tau}N(d_1(e^x,\tau;c_2(0^+))) - K_2 e^{-r\tau}N(d_2(e^x,\tau;c_2(0^+)))]. \end{aligned}$$

A3.3.2. Proof of Proposition 3.4.2. We begin by noting that

$$\begin{aligned} V^{(2)}_{c_1,c_2}(x,\tau) = {} & \mathcal{F}^{-1}\left\{\int_0^\tau F_2(\eta,s)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)(\tau-s)}ds\right\} \\ & -\mathcal{F}^{-1}\left\{\int_0^\tau F_1(\eta,s)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)(\tau-s)}ds\right\}, \end{aligned}$$

where

$$F_2(\eta,s) = e^{i\eta\ln c_2(s)}\left[\frac{\sigma^2 c_2(s)}{2} + \left(\frac{c_2'(s)}{c_2(s)} - \frac{\sigma^2 i\eta}{2} + k\right)(c_2(s) - K_2)\right],$$

and

$$F_1(\eta, s) = e^{i\eta \ln c_1(s)} \left[-\frac{\sigma^2 c_1(s)}{2} + \left(\frac{c_1'(s)}{c_1(s)} - \frac{\sigma^2 i\eta}{2} + k \right) (K_1 - c_1(s)) \right]$$

Following the approach outlined in Appendix 2.4, $V^{(2)}_{c_1,c_2}(x,\tau)$ evaluates to

$$\begin{aligned} V^{(2)}_{c_1,c_2}(x,\tau) &= \int_0^\tau \left[\frac{\sigma^2 c_2(s)}{2} + \left(\frac{c_2'(s)}{c_2(s)} + \frac{1}{2}\left[k - \frac{(x - \ln c_2(s))}{(\tau - s)} \right] \right) (c_2(s) - K_2) \right] \\ &\quad \times \frac{e^{-g_2(x,s)}}{\sigma\sqrt{2\pi(\tau - s)}} ds \\ &\quad - \int_0^\tau \left[-\frac{\sigma^2 c_1(s)}{2} + \left(\frac{c_1'(s)}{c_1(s)} + \frac{1}{2}\left[k - \frac{(x - \ln c_1(s))}{(\tau - s)} \right] \right) (K_1 - c_1(s)) \right] \\ &\quad \times \frac{e^{-g_1(x,s)}}{\sigma\sqrt{2\pi(\tau - s)}} ds, \end{aligned} \tag{A3.3.1}$$

where we set

$$g_2(x,s) = \frac{(x - \ln c_2(s) + k(\tau - s))^2}{2\sigma^2(\tau - s)} + r(\tau - s), \tag{A3.3.2}$$

and

$$g_1(x,s) = \frac{(x - \ln c_1(s) + k(\tau - s))^2}{2\sigma^2(\tau - s)} + r(\tau - s). \tag{A3.3.3}$$

With a simple change of notation, equation (A3.3.1) may be written as it is appears in equations (3.4.2)-(3.4.4).

Appendix 3.4. Derivation for the Perpetual American Strangle

To derive the results for the perpetual American strangle, given in Proposition 3.5.3, we begin with equation (3.5.6). Following Kim (1990)[3], we change the integration variable according to $u = \tau - \xi$, to produce

$$\begin{aligned} A_{c_1,c_2}(S,\tau) &= K_1 e^{-r\tau} N(-d_2(S,\tau;K_1)) - S e^{-q\tau} N(-d_1(S,\tau;K_1)) \\ &\quad + \int_0^\tau [K_1 r e^{-ru} N(-d_2(S,u;c_1(\tau - u))) \\ &\quad - S q e^{-qu} N(-d_1(S,u;c_1(\tau - u)))] du \\ &\quad + S e^{-q\tau} N(d_1(S,\tau;K_2)) - K_2 e^{-r\tau} N(d_2(S,\tau;K_2)) \\ &\quad + \int_0^\tau [S q e^{-qu} N(d_1(S,u;c_2(\tau - u))) \\ &\quad - K_2 r e^{-ru} N(d_2(S,u;c_2(\tau - u)))] du. \end{aligned} \tag{A3.4.1}$$

[3]Note that one could also proceed by solving the time independent Black-Scholes equation in the domain $c_1 < S < c_2$. An example of this approach can be found in Appendix 4.6, where it is applied to the perpetual American call option under jump-diffusion.

Let $\lim_{\tau\to\infty} A_{c_1,c_2}(S,\tau) \equiv A_{c_1,c_2}(S,\infty)$, where $A_{c_1,c_2}(S,\infty)$ is independent of τ. Furthermore we assume, as per Kim (1990), that as $\tau \longrightarrow \infty$ both $c_1(\tau)$ and $c_2(\tau)$ tend toward some constant bound. Thus we assume that $\lim_{\tau\to\infty} c_1(\tau) = c_1(\infty)$ and $\lim_{\tau\to\infty} c_2(\tau) = c_2(\infty)$, where $c_1(\infty)$ and $c_2(\infty)$ are both constant. Taking the limit of equation (A3.4.1) as $\tau \longrightarrow \infty$ results in

$$\begin{aligned} A_{c_1,c_2}(S,\tau) &= \int_0^\infty [K_1 r e^{-ru} N(-d_2(S,u;c_1(\infty))) \\ &\quad - Sqe^{-qu} N(-d_1(S,u;c_1(\infty)))]du \\ &\quad + \int_0^\infty [Sqe^{-qu} N(d_1(S,u;c_2(\infty))) \\ &\quad - K_2 r e^{-ru} N(d_2(S,u;c_2(\infty)))]du. \end{aligned} \tag{A3.4.2}$$

To produce equation (3.5.9) of Proposition 3.5.3, all that remains is to evaluate the integral terms in (A3.4.2). Following the details in Kim (1990) (p.570)[4] the required integral results are

$$\begin{aligned} V_1(S,K,c) &= \int_0^\infty [Sqe^{-qu} N(d_1(S,u;c(\infty))) - Kre^{-ru} N(d_2(S,u;c(\infty)))]du \\ &= (S-K)H(S-c) \\ &\quad + \left(\frac{S}{c}\right)^{\beta(S,c)} \left[\frac{(k+\sigma^2)c - kK}{2\sqrt{k^2+2\sigma^2 r}} - \rho(S,c)\frac{c-K}{2}\right], \end{aligned} \tag{A3.4.3}$$

and

$$\begin{aligned} V_2(S,K,c) &= \int_0^\infty [Kre^{-ru} N(-d_2(S,u;c(\infty))) - Sqe^{-qu} N(-d_1(S,u;c(\infty)))]du \\ &= K - S + V_1(S,K,c), \end{aligned} \tag{A3.4.4}$$

---

[4]It is useful to note that the last equation on p.570 of Kim (1990) contains an error: "2α" should be "α". Note also that equation (9) of the main text should read "$K/(\beta-1)$". The integrals on page 570 simplify to

$$\int_0^\infty e^{-a^2x^2-b^2/x^2}dx = \frac{\sqrt{\pi}}{2|a|}e^{-2|a||b|},$$

as given by Abramowitz and Stegun (1970) (p.304, equation (7.4.33)), and

$$\int_0^\infty \frac{1}{x^2}e^{-a^2x^2-b^2/x^2}dx = \frac{\sqrt{\pi}}{2|b|}e^{-2|a||b|}.$$

This last result was found using Mathematica 5.0. We thank Peter Buchen for drawing our attention to these results.

where we set $k \equiv r - q - \sigma^2/2$, $\beta(S,c) \equiv (-k - \rho(S,c)\sqrt{2r\sigma^2 + k^2})/\sigma^2$, $H(x)$ is the Heaviside step function defined by equation (2.2.14), and

$$\rho(S,c) \equiv \begin{cases} 1, & S > c, \\ -1, & S < c. \end{cases} \tag{A3.4.5}$$

Note that equation (A3.4.4) is obtained by use of the well-known result $N(x) = 1 - N(-x)$.

Next we shall derive integral equations for the boundaries $c_1(\infty)$ and $c_2(\infty)$. Beginning with $c_1(\infty)$, we evaluate (3.5.9) at $S = c_1(\infty)$ to give

$$\begin{aligned} K_1 - c_1(\infty) &= \int_0^\infty K_1 r e^{-ru} N\left(-\frac{k}{\sigma}\sqrt{u}\right) du \\ &\quad - \int_0^\infty c_1(\infty) q e^{-qu} N\left(-\frac{k+\sigma^2}{\sigma}\sqrt{u}\right) du \\ &\quad + \int_0^\infty c_1(\infty) q e^{-qu} N(d_1(c_1(\infty), u; c_2(\infty))) du \\ &\quad - \int_0^\infty K_2 r e^{-ru} N(d_2(c_1(\infty), u; c_2(\infty))) du. \end{aligned}$$

Applying integration by parts to the first two integral terms, it is simple to show that

$$\begin{aligned} K_1 - c_1(\infty) &= \frac{K_1}{2} - K_1 \int_0^\infty e^{-ru} \frac{k}{2\sigma\sqrt{2\pi u}} e^{-\left(\frac{k^2}{2\sigma^2}\right)u} du \\ &\quad - \frac{c_1(\infty)}{2} + c_1(\infty) \int_0^\infty e^{-qu} \frac{k+\sigma^2}{2\sigma\sqrt{2\pi u}} e^{-\left(\frac{(k+\sigma^2)^2}{2\sigma^2}\right)u} du \\ &\quad + \int_0^\infty c_1(\infty) q e^{-qu} N(d_1(c_1(\infty), u; c_2(\infty))) du \\ &\quad - \int_0^\infty K_2 r e^{-ru} N(d_2(c_1(\infty), u; c_2(\infty))) du. \end{aligned}$$

To simplify further manipulations, let

$$\begin{aligned} I_1 &\equiv \int_0^\infty c_1(\infty) q e^{-qu} N(d_1(c_1(\infty), u; c_2(\infty))) du \\ &\quad - \int_0^\infty K_2 r e^{-ru} N(d_2(c_1(\infty), u; c_2(\infty))) du. \end{aligned}$$

Using the result that $\int_0^\infty (e^{-au}/\sqrt{2\pi u})du = 1/\sqrt{2a}$, we have

$$\begin{aligned}\frac{K_1 - c_1(\infty)}{2} &= -\frac{K_1 k}{2}\frac{1}{\sqrt{2r\sigma^2+k^2}} + \frac{c_1(\infty)(k+\sigma^2)}{2}\frac{1}{\sqrt{2q\sigma^2+(k+\sigma^2)^2}} + I_1 \\ &= \frac{1}{2}\left[c_1(\infty)(k+\sigma^2) - K_1 k\right]\frac{1}{\sqrt{2r\sigma^2+k^2}} + I_1\end{aligned}$$

Making $c_1(\infty)$ the subject, we have

$$c_1(\infty) = \frac{K_1(\beta_-)}{(\beta_-)-1} - \frac{2I_1}{(\beta_-)-1}\left((\beta_-) + \frac{k}{\sigma^2}\right),$$

where $\beta_\pm = (-k \pm \sqrt{2r\sigma^2+k^2})/\sigma^2$, and finally, substituting for I_1 and evaluating by use of equation (A3.4.3) yields the result in equation (3.5.10). Similarly, by evaluating (3.5.9) at $S = c_2(\infty)$ we find that

$$c_2(\infty) = \frac{K_2(\beta_+)}{(\beta_+)-1} + \frac{2I_2}{(\beta_+)-1}\left((\beta_+) + \frac{k}{\sigma^2}\right),$$

where

$$\begin{aligned}I_2 &\equiv \int_0^\infty K_1 r e^{-ru} N(-d_2(c_2(\infty), u; c_1(\infty)))du \\ &\quad - \int_0^\infty c_2(\infty) q e^{-qu} N(-d_1(c_2(\infty), u; c_1(\infty)))du,\end{aligned}$$

which yields equation (3.5.11) after applying equation (A3.4.4).

Appendix 3.5. Value of the American Strangle Free Boundaries at Expiry

In deriving equation (3.6.3), it is necessary to analyse the limit of equations (3.6.2) and (3.6.1) as τ tends to 0^+. Using the method outlined by Kim (1990), we begin by considering equation (3.6.1):

$$\begin{aligned}K_1 - c_1(\tau) &= c_1(\tau)e^{-q\tau}[N(d_1(c_1(\tau), \tau; K_2)) - N(-d_1(c_1(\tau), \tau; K_1))] \\ &\quad - e^{-r\tau}[K_2 N(d_2(c_1(\tau), \tau; K_2)) - K_1 N(-d_2(c_1(\tau), \tau; K_1))] \\ &\quad + \int_0^\tau q c_1(\xi) e^{-q(\tau-\xi)}[N(d_1(c_1(\tau), \tau-\xi; c_2(\xi))) \\ &\qquad\qquad - N(-d_1(c_1(\tau), \tau-\xi; c_1(\xi)))]d\xi \\ &\quad - \int_0^\tau r e^{-r(\tau-\xi)}[K_2 N(d_2(c_1(\tau), \tau-\xi; c_2(\xi))) \\ &\qquad\qquad - K_1 N(-d_2(c_1(\tau), \tau-\xi; c_1(\xi)))]d\xi.\end{aligned}$$

This equation can be factorised to produce

$$
\begin{aligned}
&c_1(\tau)\Bigg\{1+e^{-q\tau}[N(d_1(c_1(\tau),\tau;K_2))-N(-d_1(c_1(\tau),\tau;K_1))]\\
&\quad+\int_0^{\tau} qe^{-q(\tau-\xi)}[N(d_1(c_1(\tau),\tau-\xi;c_2(\xi)))-N(-d_1(c_1(\tau),\tau-\xi;c_1(\xi)))]d\xi\Bigg\}\\
&=K_1+e^{-r\tau}[K_2N(d_2(c_1(\tau),\tau;K_2))-K_1N(-d_2(c_1(\tau),\tau;K_1))]\\
&\quad+\int_0^{\tau} re^{-r(\tau-\xi)}[K_2N(d_2(c_1(\tau),\tau-\xi;c_2(\xi)))-K_1N(-d_2(c_1(\tau),\tau-\xi;c_1(\xi)))]d\xi,
\end{aligned}
$$

which then yields the following implicit equation for $c_1(\tau)$:

$$
\begin{aligned}
c_1(\tau) \;=\; &\Bigg(K_1+e^{-r\tau}[K_2N(d_2(c_1(\tau),\tau;K_2))-K_1N(-d_2(c_1(\tau),\tau;K_1))]\\
&+\int_0^{\tau} re^{-r(\tau-\xi)}[K_2N(d_2(c_1(\tau),\tau-\xi;c_2(\xi)))\\
&\qquad -K_1N(-d_2(c_1(\tau),\tau-\xi;c_1(\xi)))]d\xi\Bigg)\\
&\times\Bigg(1+e^{-q\tau}[N(d_1(c_1(\tau),\tau;K_2))-N(-d_1(c_1(\tau),\tau;K_1))]\\
&+\int_0^{\tau} qe^{-q(\tau-\xi)}[N(d_1(c_1(\tau),\tau-\xi;c_2(\xi)))\\
&\qquad -N(-d_1(c_1(\tau),\tau-\xi;c_1(\xi)))]d\xi\Bigg)^{-1}. \qquad \text{(A3.5.1)}
\end{aligned}
$$

Before proceeding further, it should be noted that $K_1 < K_2$, $c_1(\tau) \leq K_1$, and $c_2(\tau) \geq K_2$. To find the value of $c_1(0^+)$, we take the limit of equation (A3.5.1) as τ tends to 0^+. In order to evaluate this limit, we need to find four limits involving d_1 and d_2. The first to consider is

$$\lim_{\tau\to0^+} d_2(c_1(\tau),\tau;K_2)=\lim_{\tau\to0^+}\frac{\ln\frac{c_1(\tau)}{K_2}}{\sigma\sqrt{\tau}}=-\infty,\ \text{since } c_1(\tau)<K_2. \qquad \text{(A3.5.2)}$$

Secondly, we have

$$\lim_{\tau\to0^+} d_2(c_1(\tau),\tau;K_1)=\lim_{\tau\to0^+}\frac{\ln\frac{c_1(\tau)}{K_1}}{\sigma\sqrt{\tau}}=\begin{cases}0, & c_1(0^+)=K_1\\ -\infty, & c_1(0^+)<K_1.\end{cases} \qquad \text{(A3.5.3)}$$

Similarly the following limits can be shown to be

$$\lim_{\tau\to0^+} d_1(c_1(\tau),\tau;K_2)=-\infty,\ \text{since } c_1(\tau)<K_2, \qquad \text{(A3.5.4)}$$

and

$$\lim_{\tau \to 0^+} d_1(c_1(\tau), \tau; K_1) = \begin{cases} 0, & c_1(0^+) = K_1 \\ -\infty, & c_1(0^+) < K_1. \end{cases} \tag{A3.5.5}$$

Note also that $N(-\infty) = 0$, $N(0) = 0.5$ and $N(\infty) = 1$. Given that the limits (A3.5.3) and (A3.5.5) depend on the value of $c_1(0^+)$ relative to K_1, there are two cases to consider when finding the limit of equation (A3.5.1). Consider the first case where $c_1(0^+) = K_1$. Taking the limit of equation (A3.5.1) as τ tends to zero, and using the results from equations (A3.5.2) - (A3.5.5), we obtain

$$\lim_{\tau \to 0^+} c_1(\tau) = K_1. \tag{A3.5.6}$$

Now consider the second case, where $c_1(0^+) < K_1$. The limit as τ tends to 0^+ of equation (A3.5.1) is now of the form $\frac{0}{0}$, and L'Hopital's rule can be applied. Firstly, let

$$\lim_{\tau \to 0^+} c_1(\tau) = \lim_{\tau \to 0^+} \frac{\hat{N}_1(\tau)}{\hat{D}_1(\tau)}$$

where

$$\begin{aligned} \hat{N}_1(\tau) \equiv\; & K_1 + e^{-r\tau}[K_2 N(d_2(c_1(\tau), \tau; K_2)) - K_1 N(-d_2(c_1(\tau), \tau; K_1))] \\ & + \int_0^\tau r e^{-r(\tau-\xi)}[K_2 N(d_2(c_1(\tau), \tau - \xi; c_2(\xi))) \\ & \qquad - K_1 N(-d_2(c_1(\tau), \tau - \xi; c_1(\xi)))]d\xi, \end{aligned}$$

and

$$\begin{aligned} \hat{D}_1(\tau) \equiv\; & 1 + e^{-q\tau}[N(d_1(c_1(\tau), \tau; K_2)) - N(-d_1(c_1(\tau), \tau; K_1))] \\ & + \int_0^\tau q e^{-q(\tau-\xi)}[N(d_1(c_1(\tau), \tau - \xi; c_2(\xi))) \\ & \qquad - N(-d_1(c_1(\tau), \tau - \xi; c_1(\xi)))]d\xi. \end{aligned}$$

To apply L'Hopital's rule, we must differentiate both $\hat{N}_1(\tau)$ and $\hat{D}_1(\tau)$, and take their limit as τ tends to 0^+. For $\hat{N}_1(\tau)$ we have

$$\begin{aligned}
\hat{N}_1'(\tau) &= -re^{-r\tau}[K_2N(d_2(c_1(\tau),\tau;K_2)) - K_1N(-d_2(c_1(\tau),\tau;K_1))] \\
&\quad +e^{-r\tau}\left[K_2N'(d_2(c_1(\tau),\tau;K_2))\frac{\partial d_2(c_1(\tau),\tau;K_2)}{\partial\tau}\right. \\
&\quad \left. +K_1N'(-d_2(c_1(\tau),\tau;K_1))]\frac{\partial d_2(c_1(\tau),\tau;K_1)}{\partial\tau}\right] \\
&\quad +r[K_2N(d_2(c_1(\tau),0;c_2(\tau))) - K_1N(-d_2(c_1(\tau),0;c_1(\tau)))] \\
&\quad +r\int_0^\tau \{-re^{-r(\tau-\xi)}[K_2N(d_2(c_1(\tau),\tau-\xi;c_2(\xi))) \\
&\quad -K_1N(-d_2(c_1(\tau),\tau-\xi;c_1(\xi)))] \\
&\quad +e^{-r(\tau-\xi)}[K_2N'(d_2(c_1(\tau),\tau-\xi;c_2(\xi)))\theta_1 \\
&\quad +K_1N'(-d_2(c_1(\tau),\tau-\xi;c_1(\xi)))\theta_2]\}d\xi,
\end{aligned}$$

where

$$\theta_1 \equiv \frac{\partial d_2(c_1(\tau),\tau-\xi;c_2(\xi))}{\partial\tau} \quad \text{and} \quad \theta_2 \equiv \frac{\partial d_2(c_1(\tau),\tau-\xi;c_1(\xi))}{\partial\tau}.$$

Note that $N'(x) = e^{-\frac{x^2}{2}}/\sqrt{2\pi}$, and that as $x \to \infty$, $N'(x) \to 0$ at a faster rate than any other terms observed in $\hat{N}_1'(\tau)$ (see Kim (1990)). We also note that

$$\lim_{\xi\to\tau} d_2(c_1(\tau),\tau-\xi;c_1(\tau)) = 0$$

and

$$\lim_{\xi\to\tau} d_2(c_1(\tau),\tau-\xi;c_2(\tau)) = -\infty.$$

Combining all these limit results, it is concluded that

$$\lim_{\tau\to0^+} \hat{N}_1'(\tau) = \frac{r}{2}K_1. \qquad \text{(A3.5.7)}$$

Similarly for $\hat{D}_1'(\tau)$ it can be shown that

$$\lim_{\tau\to0^+} \hat{D}_1'(\tau) = \frac{q}{2}. \qquad \text{(A3.5.8)}$$

Thus it is concluded that

$$\lim_{\tau\to0^+} c_1(\tau) = \frac{r}{q}K_1. \qquad \text{(A3.5.9)}$$

Recalling that this result only holds when $c_1(0^+) < K_1$, it follows that we must have $r < q$. Combining the results from equations (A3.5.6) and (A3.5.9) we find that

$$\lim_{\tau \to 0^+} c_1(\tau) = \min\left(K_1, \frac{r}{q}K_1\right),$$

which is the first part of equation (3.6.3). Similarly the process can be repeated for equation (3.6.1), yielding

$$\lim_{\tau \to 0^+} c_2(\tau) = \max\left(K_2, \frac{r}{q}K_2\right),$$

which is the second part of equation (3.6.3).

Appendix 3.6. Algorithm for Evaluating the American Strangle

Here we present the algorithm *American Strangle Price* which outlines the main steps involved in using Richardson extrapolation for pricing the American strangle portfolio.

Algorithm *American Strangle Price*

Input: S, r, q, σ, K_1, K_2, T (time to expiry), n (number of time intervals), n_{sml} (number of starting time intervals for finer grid), n_s (number of time intervals within the fine-grid region).

Output: AS (American strangle price), c_1, c_2 (early exercise boundaries).

$h = T/n$; $h_s = h * n_{sml}/n_s$
$c_{1,0} = K_1 * \min(r/q, 1)$; $c_{2,0} = K_2 * \max(r/q, 1)$
$a_{1,0} = c_{1,0}$; $a_{2,0} = c_{2,0}$; $a^{(2)}_{1,0} = c_{1,0}$; $a^{(2)}_{2,0} = c_{2,0}$
$b_{1,0} = c_{1,0}$; $b_{2,0} = c_{2,0}$; $b^{(2)}_{1,0} = c_{1,0}$; $b^{(2)}_{2,0} = c_{2,0}$
for $i = 1$ **to** n_s
    do solve the integral equation system for $b_{1,i}$ and $b_{2,i}$ using time step-size h_s
for $i = 1$ **to** $n_s * 2$
    do solve the integral equation system for $b^{(2)}_{1,i}$ and $b^{(2)}_{2,i}$ using time step-size $h_s/2$
for $i = 1$ **to** $n_{sml} * 2$
    do $j = i * n_s/(2 * n_{sml})$
        $a^{(2)}_{1,i} = (2^4 * b^{(2)}_{1,2*j} - b_{1,j})/(2^4 - 1)$
        $a^{(2)}_{2,i} = (2^4 * b^{(2)}_{2,2*j} - b_{2,j})/(2^4 - 1)$
for $i = 1$ **to** n_{sml}
    do $a_{1,i} = a^{(2)}_{1,i*2}$
        $a_{2,i} = a^{(2)}_{2,i*2}$

for $i = 1 + n_{sml}$ **to** n

do solve the integral equation system for $a_{1,i}$ and $a_{2,i}$ using time step-size h

for $i = 1 + n_{sml}$ **to** $n * 2$

do solve the integral equation system for $a_{1,i}^{(2)}$ and $a_{2,i}^{(2)}$ using time step-size $h/2$

for $i = 1 + n_{sml}$ **to** n

do $c_{1,i} = (2^4 * a_{1,2*i}^{(2)} - a_{1,i})/(2^4 - 1)$

$c_{2,i} = (2^4 * a_{2,2*i}^{(2)} - a_{2,i})/(2^4 - 1)$

calculate AS using c_1, c_2 and S

CHAPTER 4

Pricing American Options under Jump-Diffusion

4.1. Introduction

In Chapters 2 and 3 we assumed that the returns for the underlying asset were best modelled using the pure-diffusion process proposed by Black and Scholes (1973) and Merton (1973). Following these seminal works on option pricing, there have been a large number of studies into the applicability of this model to real-world financial data. A significant amount of evidence has accumulated which indicates that stocks and foreign exchange rates are better modelled by jump-diffusion processes, rather than pure-diffusion processes. Some of these studies include Jarrow and Rosenfeld (1984), Ball and Torous (1985), Jorion (1988), Ahn and Thompson (1992), and Bates (1996). This evidence implies that there is empirical justification for considering American option pricing under jump-diffusion models. While Pham (1997) and Gukhal (2001) have extended Merton's (1976) jump-diffusion model for European option prices to the American option case, each has done so using different techniques, and neither author considers how to implement their pricing equations numerically. The purpose of this chapter is to firstly show how the incomplete Fourier transform method of McKean (1965), as presented in Chapter 2, may be extended to the case of American call options written on assets with jump-diffusion price dynamics. Using the simplifications of Kim (1990) we are able to reconcile this approach with the results of Gukhal (2001), and provide a method of analysis that extends very naturally from Merton's framework. We also present an iterative numerical method that provides a practical way to solve the resulting integral equation system for the American call's price, along with its early exercise boundary. Using this procedure we also find further support for the observations of Amin (1993) in relation to the impact of jumps on the price and free boundary of American call options.

Merton (1976) was the first to extend the Black-Scholes model to consider European options on assets following jump-diffusion processes, showing that the Black-Scholes

PDE becomes a partial integro-differential equation (PIDE). He derived[1] the jump-diffusion equivalent of the Black-Scholes formula for European calls where the arrival times of the jumps followed a Poisson distribution, and the distribution for the jump sizes was of a general form. He assumed that the risk associated with the jump term could be diversified away by the holder of the call option. As a particular example, Merton considered the case where the jumps were log-normally distributed, resulting in a natural extension of the Black-Scholes results.

Pham (1997) was one of the first to expand the Merton jump-diffusion model to American options. Using probability arguments, Pham derived the integral equation for the price of an American put under jump-diffusion, along with an integral equation for the put's free boundary. Unlike Merton, he does not assume that the jump risk can be diversified away. Performing analysis on these integral equations, Pham demonstrated that the value of the American put under jump-diffusion is greater than that of an American put under pure-diffusion, and that the increased risk from the jump term makes the option holder more sensitive to the decision of early exercise. Mullinaci (1996) also considered the American put option under jump-diffusion. Using a discrete time model, Mullinaci finds the Snell envelope for the American put option, resulting in a numerical technique for pricing the American put.

Another exploration of American options under jump-diffusion was presented by Gukhal (2001). By extending the Geske-Johnson limiting technique of Kim (1990) to Merton's jump-diffusion model, Gukhal derived the integral equations for the prices of both American calls and puts, along with the integral equations for their free boundaries. His results were for a general jump-size distribution, and he also provided more specific equations in the case of binomial and log-normally distributed jump sizes. In particular, Gukhal offered a very intuitive decomposition for the prices of American options under jump-diffusion. The first two components, namely the European value and early exercise premium, were already familiar from Kim's pure-diffusion results. The third component introduced by the presence of jumps was identified as an adjustment cost made by the holder of the option when the underlying asset jumps from the stopping region back into

[1] Merton does not indicate how he obtained the solution he gives. In an appendix he verifies that the solution given satisfies the PIDE, but of course this procedure requires one to know the form of the solution.

the continuation region. This cost is incurred because only jumps out of the continuation region will be self-financing.

While the results of Pham (1997) and Gukhal (2001) provide extensions of various American option pricing techniques to the jump-diffusion case, to our knowledge there is no work in the current literature that extends McKean's incomplete Fourier transform technique to the jump-diffusion case. Transform methods are a highly validated and accepted technique for solving PDEs, and thereby extending the Fourier transform to the jump-diffusion case fills an important gap in the existing literature. Chiarella (2003) demonstrates how the Fourier transform method, which is able to readily solve the Black-Scholes PDE for European option prices, can be used to solve Merton's partial integro-differential equation for European option prices under jump-diffusion. In Chapter 3 we generalised the American straddle presented by Elliott et al. (1990), deriving the integral expression for an American strangle portfolio using Fourier transforms, and solved the resulting linked integral equation system for the strangle's free boundaries. In this chapter we use the methods presented in Chapter 2 to extend McKean's method to solve for the price of an American call option under Merton's jump-diffusion framework. As we stated in chapters 2 and 3, the main advantage of the Fourier transform method is that it is broadly applicable to a wide variety of option pricing problems, and thus it provides a natural means of extending the Black-Scholes analysis to non-European payoffs.

Several authors have explored a range of numerical methods for pricing American options under jump-diffusion. Amin (1993) used an extension of the binomial tree method to demonstrate a number of interesting properties for American option prices and free boundaries in the presence of jumps. Zhang (1997*b*) used finite difference methods to solve the problem as a variational inequality, while Wu and Dai (2001) used a multinomial tree approach. The method of lines was used by Meyer (1998) to price American puts under jump-diffusion. Carr and Hirsa (2003) solved the partial-integro differential equation numerically via a Crank-Nicolson finite difference scheme. Using a fixed-point iteration method, d'Halluin et al. (2003) were able to price American put options under jump-diffusion.

It is interesting to note, however, that there has been little work in the existing literature on the implementation of the integral equations for the price and free boundary of American options under jump-diffusion. While some authors such as Pham (1997) and

Gukhal (2001) derive these integral equations, they do not offer any thoughts on how they can be solved numerically. Here we extend on the approach used for the American strangle in Chapter 3 by applying a modified version of the method to solve the linked integral equation system that arises for the American call and its free boundary in the case of jump-diffusion. While the focus of this chapter is not on finding optimal numerical methods for American option prices with jumps, we are able to demonstrate that the Fourier transform technique leads to integral equation forms that are tractable for numerical implementation. We shall return to the issue of accuracy and efficiency of numerical solution methods for this problem in Chapter 5.

The remainder of this chapter is structured as follows. Section 4.2 outlines the free boundary problem that arises from pricing an American call option under Merton's jump-diffusion model. Section 4.3 applies McKean's incomplete Fourier transform to solve the PIDE in terms of a transform variable. The transform is inverted in Section 4.4, providing a McKean-style integral equation for the American call price, and a corresponding integral equation for the call's early exercise boundary. A feature of the solution is that the integral equation for the call value and the integral equation for the free boundary are interdependent, so that the convenient two-pass procedure of the non-jump case is no longer applicable. Section 4.5 analyses the integral equations in the case where the jump sizes follow a log-normal distribution, as suggested by Merton (1976). This section also includes a discussion of the transformation from McKean's representation to Kim's representation, and relates our findings to those of Gukhal (2001). Section 4.6 outlines the numerical solution method for solving the linked integral equation system for both the free boundary and price of the American call. A selection of numerical results for the American call option and its early exercise boundary are also provided. Concluding remarks are presented in Section 4.7. Most of the lengthy mathematical derivations are given in appendices.

4.2. Problem Statement - Merton's Model

Let $C(S,t)$ be the price of an American option written on the underlying asset S at time t, with time to expiry $(T-t)$, and strike price K. We assume that S pays a continuous dividend yield of rate q. Let $a(t)$ denote the early exercise boundary at time t, and assume

S follows the jump-diffusion process

$$dS = (\mu - \lambda k)Sdt + \sigma SdW + (Y-1)Sd\bar{q}, \tag{4.2.1}$$

where μ is the instantaneous return per unit time, σ is the instantaneous volatility per unit time, W is a standard Wiener process and $\bar{q}$ is a Poisson process whose increments satisfy

$$d\bar{q} = \begin{cases} 1, & \text{with probability } \lambda dt, \\ 0, & \text{with probability } (1 - \lambda dt). \end{cases}$$

Let the jump size, Y, be a random variable whose probability measure we denote by Q_Y, and W, Y and $\bar{q}$ are all independent. We use $G(Y)$ to denote the corresponding probability density function. Thus the expected jump size, k, is given by

$$k = \mathbb{E}^{Q_Y}[Y-1] = \int_0^\infty (Y-1)G(Y)dY.$$

Following Merton's (1976) argument and assuming that the jump risk is fully diversifiable[2], it is known that C satisfies the partial integro-differential equation (PIDE)

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - q - \lambda k)S\frac{\partial C}{\partial S} - rC + \lambda \int_0^\infty [C(SY,t) - C(S,t)]G(Y)dY = 0, \tag{4.2.2}$$

in the region $0 \le t \le T$ and $0 < S < a(t)$, where r is the risk-free rate.

In the case of an American call option, the PIDE (4.2.2) is subject to the final time and boundary conditions

$$\begin{aligned} C(S,T) &= \max(S-K,0), \quad 0 < S < \infty & (4.2.3)\\ C(0,t) &= 0, \quad t \ge 0, & (4.2.4)\\ C(a(t),t) &= a(t) - K, \quad t \ge 0, & (4.2.5)\\ \lim_{S \to a(t)} \frac{\partial C}{\partial S} &= 1, \quad t \ge 0. & (4.2.6) \end{aligned}$$

Condition (4.2.3) is the payoff function for the call at expiry, and condition (4.2.4) ensures that the option is worthless if S falls to zero. The value-matching condition (4.2.5) forces the value of the call option to be equal to its payoff on the early exercise boundary, and the smooth-pasting condition (4.2.6) sets the call's delta to be continuous at the free boundary

[2]We make this assumption for convenience. The derivation that follows would carry through if we were to assume a constant market price of jump risk.

to guarantee arbitrage-free prices. As shown in Appendix 4.6, optimality conditions for the call price impose these boundary conditions naturally, at least in the perpetual case. For the finite call under consideration, we note that the standard arbitrage arguments that justify condition (4.2.6) are not readily applied under Merton's jump-diffusion model, since this depends upon the price process for S being continuous. The corresponding boundary conditions were proven by Pham (1997) for the American put case, and we shall assume here that this result for the put will extend naturally to the American call problem, as per Gukhal (2001). Figure 4.1 demonstrates the payoff, price profile and early exercise boundary for the American call under consideration.

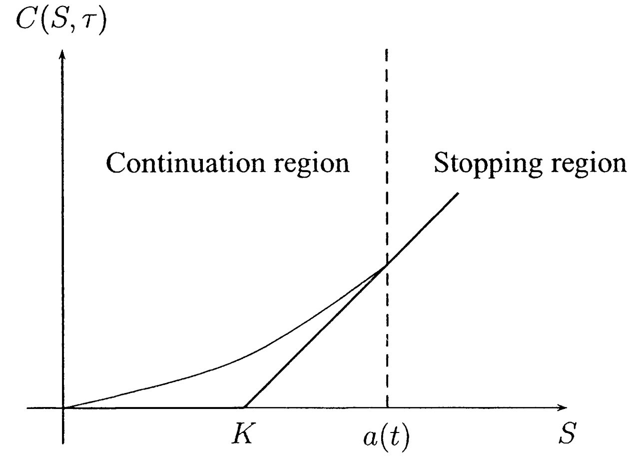

FIGURE 4.1. Continuation region for the American call option.

Our first step is to transform the PIDE to a forward-in-time equation, with constant coefficients and a "standardised" strike of 1. Let $S = Ke^x$, $t = T - \tau$, and

$$C(S,t) = KV(x,\tau).$$

The transformed PIDE for V is then[3]

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \phi\frac{\partial V}{\partial x} - rV + \lambda\int_0^\infty [V(x+\ln Y,\tau) - V(x,\tau)]G(Y)dY,$$

[3]It should be noted that

$$\begin{aligned} C(SY,t) &= KV(\ln(\frac{SY}{K}),\tau) \\ &= KV(x+\ln Y,\tau). \end{aligned}$$

where $\phi \equiv r - q - \lambda k - \frac{\sigma^2}{2}$. The transformed PIDE can be simplified to

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + \phi \frac{\partial V}{\partial x} - (r+\lambda)V + \lambda \int_0^\infty V(x + \ln Y, \tau) G(Y) dY, \tag{4.2.7}$$

in the region $0 \le \tau \le T$, $-\infty \le x \le \ln b(\tau)$, where $b(\tau) = a(t)/K$ is the free boundary re-scaled by the strike price. It is this latter quantity that will be a particular focus of our subsequent analysis. The initial and boundary conditions assume the form

$$\begin{aligned} V(x,0) &= \max(e^x - 1, 0), \quad -\infty < x < \infty, && (4.2.8) \\ \lim_{x \to -\infty} V(x,\tau) &= 0, \quad \tau \ge 0, && (4.2.9) \\ V(\ln b(\tau), \tau) &= b(\tau) - 1, \quad \tau \ge 0, && (4.2.10) \\ \lim_{x \to \ln b(\tau)} \frac{\partial V}{\partial x} &= b(\tau), \quad \tau \ge 0. && (4.2.11) \end{aligned}$$

For simplicity, we shall denote $b(\tau)$ by $b \equiv b(\tau)$ when it is clear at which time this function is being evaluated.

As for the pure-diffusion case detailed in Chapter 2, the x-domain shall be extended to $-\infty < x < \infty$ to facilitate the application of the Fourier transform method. We achieve this by expressing the PIDE (4.2.7) as

$$\mathrm{H}(\ln b - x) \left(\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - \phi \frac{\partial V}{\partial x} + (r+\lambda)V - \lambda \int_0^\infty V(x + \ln Y, \tau) G(Y) dY \right) = 0 \tag{4.2.12}$$

where $H(x)$, the Heaviside step function, is defined in equation (2.2.14). The initial and boundary conditions remain unchanged.

4.3. Applying the Fourier Transform

In Chapter 2 we used the incomplete Fourier transform to reduce the PDE (2.2.13) to an ODE. Here we shall apply this method to solve the problem defined by equations (4.2.7)-(4.2.11), reducing the PIDE (4.2.12) to an integro-differential equation. Note that the function V and its first two derivatives with respect to x behave as outlined for the pure-diffusion problem in Chapter 2. This knowledge is required to eliminate limit terms that arise in integration by parts (see Appendix 4.1).

Since the x-domain is now $-\infty < x < \infty$, the Fourier transform of the PIDE can be found. Define the Fourier transform of V, $\mathcal{F}\{V(x,\tau)\}$, as

$$\mathcal{F}\{V(x,\tau)\} = \int_{-\infty}^{\infty} e^{i\eta x} V(x,\tau)dx.$$

Applying this Fourier transform to the PIDE (4.2.12),

$$\begin{aligned}\mathcal{F}\left\{H(\ln b - x)\frac{\partial V}{\partial \tau}\right\} &= \frac{\sigma^2}{2}\mathcal{F}\left\{H(\ln b - x)\frac{\partial^2 V}{\partial x^2}\right\} + \phi\mathcal{F}\left\{H(\ln b - x)\frac{\partial V}{\partial C}\right\} \\ &\quad -(r+\lambda)\mathcal{F}\left\{H(\ln b - x)V\right\} \qquad (4.3.1) \\ &\quad +\lambda\mathcal{F}\left\{H(\ln b - x)\int_0^{\infty} V(x+\ln Y,\tau)G(Y)dY\right\}.\end{aligned}$$

By the definition of the Fourier transform, we have

$$\mathcal{F}\left\{H(\ln b - x)V(x,\tau)\right\} = \int_{-\infty}^{\ln b} e^{i\eta x}V(x,\tau)dx \equiv \mathcal{F}^b\{V(x,\tau)\} \equiv \hat{V}^b(\eta,\tau). \quad (4.3.2)$$

It should be noted that $\mathcal{F}^b$ is an incomplete Fourier transform, since it is a standard Fourier transform applied to $V(x,\tau)$ in the domain $-\infty < x < b(\tau)$. We now apply this transform to carry out the transform operations in (4.3.1).

PROPOSITION 4.3.1. *Using the initial and boundary conditions (4.2.8)-(4.2.11), the incomplete Fourier transform of the PIDE (4.2.12) with respect to x satisfies the integro-differential equation*

$$\frac{\partial \hat{V}}{\partial \tau} + \left[\frac{\sigma^2\eta^2}{2} + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right]\hat{V} = F(\eta,\tau) \quad (4.3.3)$$

where

$$\begin{aligned}F(\eta,\tau) &= e^{i\eta \ln x}\left[\frac{\sigma^2 b}{2} + \left(\frac{b'}{b} - \frac{\sigma^2 i\eta}{2} + \phi\right)(b-1)\right] + \lambda\Phi(\eta,\tau), \quad (4.3.4) \\ A(\eta) &= \int_0^{\infty} e^{-i\eta\ln Y}G(Y)dY, \quad (4.3.5) \\ \Phi(\eta,\tau) &= \int_0^{\infty} e^{-i\eta\ln Y}G(Y)\left[\int_{\ln b}^{\ln bY} e^{i\eta z}V(z,\tau)dz\right]dY, \quad (4.3.6)\end{aligned}$$

and $b' \equiv db(\tau)/d\tau$. Furthermore, the solution to the integro-differential equation (4.3.3) is given by

$$\begin{aligned}\hat{V}(\eta,\tau) &= \hat{V}(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-s)} \\ &\quad + \int_0^{\tau} e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-s)}F(\eta,s)ds. \qquad (4.3.7)\end{aligned}$$

Proof: Refer to Appendix 4.1.

□

4.4. Inverting the Fourier Transform

Now that $\hat{V}(\eta,\tau)$ has been found, we may invert it to recover $V(x,\tau)$, the American call price in the x-τ plane. By taking the inverse Fourier transform of (4.3.7), we have

$$\begin{aligned}V(x,\tau) &= (\mathcal{F}^b)^{-1}\left\{\hat{V}(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))\tau}\right\} \\ &\quad +(\mathcal{F}^b)^{-1}\left\{\int_0^{\tau} e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-s)}F(\eta,s)ds\right\} \\ &\equiv V_1(x,\tau)+V_2(x,\tau) \\ &\equiv \frac{1}{K}\left[C_1(S,\tau)+C_2(S,\tau)\right] = \frac{1}{K}C(S,\tau) \qquad (4.4.1)\end{aligned}$$

where $-\infty < x < \ln b(\tau)$, and the forms of the functions $C_1(S,\tau)$ and $C_2(S,\tau)$ are given by Propositions 4.4.1, 4.4.2 and 4.4.3 below.

PROPOSITION 4.4.1. *The function $C_1(S,\tau)$ in equation (4.4.1) is given by*

$$\begin{aligned}C_1(S,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\tau}^{(n)}\{C_E[SX_ne^{-\lambda k\tau},K,1,r,q,\tau,\sigma^2] \\ &\qquad -C_E[SX_ne^{-\lambda k\tau},K,b(0^+),r,q,\tau,\sigma^2]\} \qquad (4.4.2)\end{aligned}$$

where

$$
\begin{aligned}
C_E[S,K,\beta,r,q,\tau,\sigma^2] &= Se^{-q\tau}N[d_1(S,K\beta,r,q,\tau,\sigma^2)] \\
&\qquad -Ke^{-r\tau}N[d_2(S,K\beta,r,q,\tau,\sigma^2)], \\
d_1(S,\kappa,r,q,\tau,\sigma^2) &= \frac{\ln\frac{S}{\kappa}+\left(r-q+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \\
d_2(S,\kappa,r,q,\tau,\sigma^2) &= d_1(S,\kappa,r,q,\tau,\sigma^2)-\sigma\sqrt{\tau}, \\
b(0^+) &= \lim_{\tau\to0^+} b(\tau), \\
N[\alpha] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-\frac{\eta^2}{2}}d\eta, \\
X_n &\equiv Y_1Y_2...Y_n; \quad X_0\equiv 1,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}_\tau^{(n)}\{f(X_n)\} &\equiv \int_0^\infty\int_0^\infty ...\int_0^\infty G(Y_1)G(Y_2)...G(Y_n)f(X_n)dY_1dY_2...dY_n \\
&= \int_0^\infty G(X_n)f(X_n)dX_n.
\end{aligned}
$$

Proof: Refer to Appendix A4.2.1.

□

Next we consider the more complicated function $V_2(x,\tau)$. The first step is to break the function down into two linear components that arise from the form of function F in equation (4.3.4):

$$
\begin{aligned}
V_2(x,\tau) &= (\mathcal{F}^b)^{-1}\left\{\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-s)}F(\eta,s)ds\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\eta x}\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-s)}F(\eta,s)dsd\eta \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\eta x}\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-s)} \\
&\qquad \times e^{i\eta\ln b(s)}\left[\frac{\sigma^2 b(s)}{2}+\left(\frac{b'(s)}{b(s)}-\frac{\sigma^2 i\eta}{2}+\phi\right)(b(s)-1)\right]dsd\eta \\
&\quad +\frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\eta x}\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta))(\tau-s)}\lambda\Phi(\eta,s)dsd\eta \\
&\equiv V_2^{(1)}(x,\tau)+V_2^{(2)}(x,\tau) \\
&\equiv \frac{1}{K}\left[C_2^{(1)}(S,\tau)+C_2^{(2)}(S,\tau)\right]=\frac{1}{K}C_2(S,\tau). \qquad (4.4.3)
\end{aligned}
$$

We start by considering the function $C_2^{(1)}(S,\tau)$.

PROPOSITION 4.4.2. *The term $C_2^{(1)}(S,\tau)$ in equation (4.4.3) is given by*

$$\begin{aligned} C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_\tau^{(n)}\{C_E[SX_n e^{-\lambda k\tau}, K, b(0^+), r, q, \tau, \sigma^2]\} \qquad (4.4.4) \\ &+\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\bigg\{\int_0^\tau (\tau-\xi)^{n-1}\left[SX_n e^{-\lambda k(\tau-\xi)}e^{-(q+\lambda)(\tau-\xi)}\right. \\ &\times[(\lambda[k+1]+q)(\tau-\xi)-n]N[d_1(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)] \\ &-Ke^{-(r+\lambda)(\tau-\xi)}[(\tau-\xi)(r+\lambda)-n] \\ &\left.\times N[d_2(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)]\right]d\xi\bigg\}, \end{aligned}$$

where the function C_E and operator $\mathbb{E}_\tau^{(n)}$ have been defined in Proposition 4.4.1.

Proof: Refer to Appendix A4.2.2.

□

Before proceeding further, it is worth noting that if we now combine $C_1(S,\tau)$ with $C_2^{(1)}(S,\tau)$, some of the terms will cancel, leaving us with

$$\begin{aligned} C_1(S,\tau)+C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_\tau^{(n)}\left\{C_E\left[SX_n e^{-\lambda k\tau}, K, 1, r, q, \tau, \sigma^2\right]\right\} \\ &+\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\bigg\{\int_0^\tau (\tau-\xi)^{n-1} SX_n e^{-\lambda k(\tau-\xi)}e^{-(q+\lambda)(\tau-\xi)} \qquad (4.4.5) \\ &\times[(\lambda[k+1]+q)(\tau-\xi)-n] \\ &\times N[d_1(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)]d\xi\bigg\} \\ &-\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\bigg\{\int_0^\tau (\tau-\xi)^{n-1} Ke^{-(r+\lambda)(\tau-\xi)} \\ &\times[(\tau-\xi)(r+\lambda)-n] \\ &\times N\left[d_2(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)\right]d\xi\bigg\}. \end{aligned}$$

The last remaining term to be evaluated is $C_2^{(2)}(S,\tau)$, which is the extra term introduced into the expression for the American option price by the presence of jumps in the stochastic process for S.

PROPOSITION 4.4.3. *The term $C_2^{(2)}(S,\tau)$ is given by*

$$
\begin{aligned}
C_2^{(2)}(S,\tau) &= -\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\left\{\int_0^\tau (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)}\right. \\
&\qquad \times\left[\int_0^1 G(Y)\int_{Kb(\xi)}^{Kb(\xi)/Y} C(\omega Y,\xi)J(\omega,\xi,SX_n,\tau)d\omega dY\right. \qquad (4.4.6)\\
&\qquad \left.\left.-\int_1^\infty G(Y)\int_{Kb(\xi)/Y}^{Kb(\xi)}(\omega Y-K)J(\omega,\xi,SX_n,\tau)d\omega dY\right]d\xi\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
J(\omega,\xi,SX_n,\tau) &\equiv \frac{1}{\omega\sigma\sqrt{2\pi(\tau-\xi)}} \qquad (4.4.7)\\
&\qquad \times\exp\left\{\frac{-\left[\left(r-q-\lambda k-\frac{\sigma^2}{2}\right)(\tau-\xi)+\ln\frac{SX_n}{\omega}\right]^2}{2\sigma^2(\tau-\xi)}\right\},
\end{aligned}
$$

and the operator $\mathbb{E}_\tau^{(n)}$ has been defined in Proposition 4.4.1.

Proof: Refer to Appendix A4.2.3.

□

Now that we have derived the functions $C_1(S,\tau)$ and $C_2(S,\tau)$, we can provide an integral equation for the price of the American call, $C(S,\tau)$.

PROPOSITION 4.4.4. *The integral equation for the price of the American call option, $C(S,\tau)$, is*

$$
\begin{aligned}
C(S,\tau) \;=\; & \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\tau}^{(n)}\{C_E[SX_n e^{-\lambda k\tau}, K, 1, r, q, \tau, \sigma^2]\} \qquad (4.4.8)\\
& + \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)}\Big\{ \int_0^{\tau} (\tau-\xi)^{n-1} SX_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} \\
& \qquad \times [(\lambda[k+1]+q)(\tau-\xi)-n] \\
& \qquad \times N\left[d_1\left(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2\right)\right] d\xi \Big\} \\
& - \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)}\Big\{ \int_0^{\tau} (\tau-\xi)^{n-1} K e^{-(r+\lambda)(\tau-\xi)} [(\tau-\xi)(r+\lambda)-n] \\
& \qquad \times N\left[d_2\left(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2\right)\right] d\xi \Big\} \\
& - \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)}\Big\{ \int_0^{\tau} (\tau-\xi)^{n} e^{-(r+\lambda)(\tau-\xi)} \\
& \qquad \times \Big[\int_0^1 G(Y) \int_{Kb(\xi)}^{Kb(\xi)/Y} C(\omega Y,\xi) J(\omega,\xi,SX_n,\tau)\, d\omega\, dY \\
& \qquad - \int_1^{\infty} G(Y) \int_{Kb(\xi)/Y}^{Kb(\xi)} (\omega Y - K) J(\omega,\xi,SX_n,\tau)\, d\omega\, dY \Big] d\xi \Big\},
\end{aligned}
$$

where the function C_E and operator $\mathbb{E}_{\tau}^{(n)}$ have been defined in Proposition 4.4.1, and the function J is defined in Proposition 4.4.3.

Proof: Equation (4.4.8) follows from substituting equations (4.4.2), (4.4.4) and (4.4.6) into equation (4.4.1).

□

A key feature of equation (4.4.8) is that it is an integral equation rather than the integral expression obtained for the American call price in the no-jump case, because of the appearance of the option price in the integrals in the final summation term on the right-hand side. As we have pointed out, the presence of this term is due to the jump process. It should also be noted that, as in the no-jump option pricing case, in order to implement (4.4.8) we need to know the free boundary b. An integral equation for this will be derived below. Finally, we can perform some algebraic manipulations to equation (4.4.8) to obtain the American call integral equations in the form presented by Gukhal (2001).

PROPOSITION 4.4.5. *Algebraic manipulation of equation (4.4.8) allows the American call price, $C(S,\tau)$ of Proposition 4.4.4, to be expressed as*

$$
\begin{aligned}
C(S,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_{\tau}^{(n)}\{C_E[SX_ne^{-\lambda k\tau},K,1,r,q,\tau,\sigma^2]\} \qquad (4.4.9)\\
&+\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\tau}^{(n)}\Bigg\{\int_0^{\tau}(\tau-\xi)^n e^{-\lambda(\tau-\xi)}qSX_ne^{-\lambda k(\tau-\xi)}e^{-q(\tau-\xi)}\\
&\qquad\times N\left[d_1\left(SX_ne^{-\lambda k(\tau-\xi)},Kb(\xi),r,q,\tau-\xi,\sigma^2\right)\right]d\xi\Bigg\}\\
&-\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\tau}^{(n)}\Bigg\{\int_0^{\tau}(\tau-\xi)^n e^{-\lambda(\tau-\xi)}rKe^{-r(\tau-\xi)}\\
&\qquad\times N\left[d_2\left(SX_ne^{-\lambda k(\tau-\xi)},Kb(\xi),r,q,\tau-\xi,\sigma^2\right)\right]d\xi\Bigg\}\\
&-\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_{\tau}^{(n)}\Bigg\{\int_0^{\tau}(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)}\\
&\times\left[\int_0^1 G(Y)\int_{Kb(\xi)}^{Kb(\xi)/Y}[C(\omega Y,\xi)-(\omega Y-K)]J(\omega,\xi,SX_n,\tau)d\omega dY\right]d\xi\Bigg\}.
\end{aligned}
$$

Proof: Refer to Appendix 4.3.

□

The four additive components of the call value in equation (4.4.9) each have a clear economic interpretation, as outlined by Gukhal (2001). The first term represents the European component of the American call option's value, while the remaining three terms combine to form the total early exercise premium. The middle terms are natural extensions of the early exercise premium that arises in the pure-diffusion case. More specifically, the term containing qS calculates the dividend received when holding the underlying, and the term involving rK captures the interest payable on a loan of size K. Thus these terms capture the potential income to the option holder should the option be exercised to buy the underlying, borrowing K to do so.

The fourth term arises entirely due to the introduction of jumps in the price process for S. Note that if no jumps are present (λ=0) then this term will be zero, and equation (4.4.9) simplifies to the American call price under pure-diffusion. This term captures the rebalancing costs incurred by the option holder whenever the price of the underlying jumps down from the stopping region into the continuation region. Figure 4.2 illustrates this effect in detail. If the holder of the option has observed that the underlying price is at $S_- > Kb(\tau)$, then the call will be optimally exercised. If an instant after exercising a jump of size Y occurs such that $S_+ = YS_- < Kb(\tau)$, then the portfolio $S - K$ held by the investor will now be worth less than the unexercised American call. This difference is the cost being captured by the fourth term in (4.4.9).

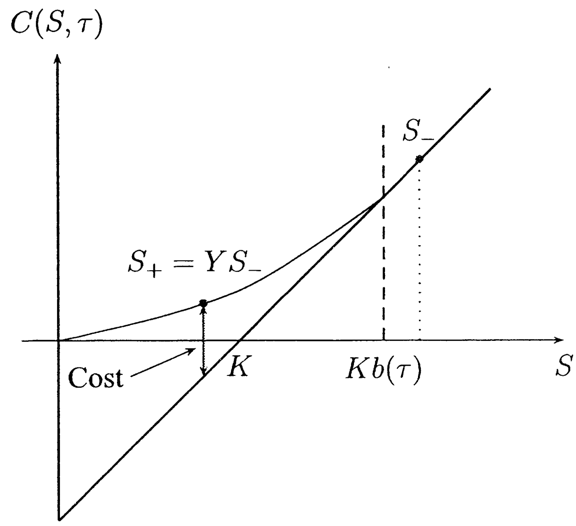

FIGURE 4.2. Cost incurred by the investor from downward jumps in S.

In equation (4.4.9), the value of the American call option is expressed as a function of the original underlying variable S, and the new time variable τ, which is a measure of time to maturity. As we have already noted, equation (4.4.9) also depends upon the unknown early exercise boundary, now defined as $b(\tau) = a(t)/K$. By requiring the expression for $C(S,\tau)$ to satisfy the boundary condition (4.2.5), we can derive a similar integral equation for the value of $b(\tau)$. This integral equation is given by

$$\begin{aligned}
K(b(\tau)-1) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_\tau^{(n)}\{C_E[Kb(\tau)X_ne^{-\lambda k\tau},K,1,r,q,\tau,\sigma^2]\} \qquad (4.4.10)\\
&+\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\left\{\int_0^\tau(\tau-\xi)^ne^{-\lambda(\tau-\xi)}qKb(\tau)X_ne^{-\lambda k(\tau-\xi)}e^{-q(\tau-\xi)}\right.\\
&\qquad\left.\times N\left[d_1\left(b(\tau)X_ne^{-\lambda k(\tau-\xi)},b(\xi),r,q,\tau-\xi,\sigma^2\right)\right]d\xi\right\}\\
&-\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\left\{\int_0^\tau(\tau-\xi)^ne^{-\lambda(\tau-\xi)}rKe^{-r(\tau-\xi)}\right.\\
&\qquad\left.\times N\left[d_2\left(b(\tau)X_ne^{-\lambda k(\tau-\xi)},b(\xi),r,q,\tau-\xi,\sigma^2\right)\right]d\xi\right\}\\
&-\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\left\{\int_0^\tau(\tau-\xi)^ne^{-(r+\lambda)(\tau-\xi)}\right.\\
&\qquad\times\left[\int_0^1 G(Y)\int_{Kb(\xi)}^{Kb(\xi)/Y}[C(\omega Y,\xi)-(\omega Y-K)]\right.\\
&\qquad\left.\left.\times\ J(\omega,\xi,Kb(\tau)X_n,\tau)d\omega dY\right]d\xi\right\}.
\end{aligned}$$

It is particularly crucial to note that the integral equation (4.4.10) depends upon the unknown call value $C(S,\tau)$, and this dependence arises from integral terms that have been introduced by the presence of jumps in the dynamics for S.

The general form of the integral equation system consisting of (4.4.9) and (4.4.10) can be written as

$$C(S,\tau) = \Omega_C(S,\tau)+\int_0^\tau\Psi_C[C(S,\xi),b(\xi),\xi,\tau,S]d\xi, \qquad (4.4.11)$$

$$b(\tau) = \Omega_b(b(\tau),\tau)+\int_0^\tau\Psi_b[C(Kb(\tau),\xi),b(\xi),\xi,\tau,Kb(\tau)]d\xi, \qquad (4.4.12)$$

where the definitions of the functions Ψ_C, Ψ_b, Ω_C and Ω_b are implied by the right hand sides of equations (4.4.9) and (4.4.10) respectively. The interdependence of (4.4.11) and (4.4.12) is obvious, and it is this interdependence that makes numerical implementation much more involved than for the corresponding no-jump problem. Thus in order to implement these integral equations for the free boundary and call price, we need to develop numerical techniques to solve the linked integral equation system (4.4.9)-(4.4.10).

4.5. American Call with Log-Normal Jumps

The primary difficulty in solving for the American call price under jump-diffusion dynamics is caused by the dependence of equation (4.4.9), and subsequently equation (4.4.10), on the unknown American call price, $C(S,\tau)$. To overcome the complication brought about by this interdependence, in Section 4.6 we propose an iterative numerical scheme similar to the one employed in Chapter 3 for solving the American strangle problem. Before we begin, however, we must specify the density, $G(Y)$, for the jump sizes. We shall consider a log-normal distribution for the jump sizes, Y, in accordance with a model suggested by Merton (1976). The probability density function for Y is given by

$$G(Y) = \frac{1}{Y\delta\sqrt{2\pi}}e^{-\frac{(\ln Y-(\gamma-\delta^2/2))^2}{2\delta^2}}, \tag{4.5.1}$$

where we set $\gamma \equiv \ln(1+k)$, and δ^2 is the variance of $\ln Y$. Furthermore $\mathbb{E}^{Q_Y}[Y] = e^{\gamma}$, a fact which is relevant to the numerical experiments reported later. Gukhal (2001) assumes that $\gamma = -\delta^2/2$ when deriving his equation (5.1) for the American call option price, but here we forego this assumption and provide a more general form of Gukhal's results.

PROPOSITION 4.5.1. *In the case where $G(Y)$ is given by equation (4.5.1), the integral equation for $C(S,\tau)$ in (4.4.9) becomes*

$$\begin{aligned} C(S,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}C_E[S,K,1,r_n(\tau),q,\tau,v_n^2(\tau)] \\ &+\sum_{n=0}^{\infty}\frac{[\lambda']^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-\lambda'(\tau-\xi)}qSe^{-q(\tau-\xi)} \\ &\qquad\times N\left[d_1(S,Kb(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))\right]d\xi \\ &-\sum_{n=0}^{\infty}\frac{[\lambda']^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-\lambda'(\tau-\xi)}rKe^{-r_n(\tau-\xi)(\tau-\xi)} \\ &\qquad\times N\left[d_2\left(S,Kb(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi)\right)\right]d\xi \\ &-\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\ &\quad\times\int_0^1 G(Y)\int_{Kb(\xi)}^{Kb(\xi)/Y}\frac{[C(\omega Y,\xi)-(\omega Y-K)]}{\omega v_n(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\ &\qquad\times\exp\left\{-\frac{1}{2}[d_2(S,\omega,r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))]^2\right\}d\omega dY d\xi, \end{aligned} \tag{4.5.2}$$

where $\lambda' = \lambda(1+k)$, $r_n(\tau) = r - \lambda k + n\gamma/\tau$ *and* $v_n^2(\tau) = \sigma^2 + n\delta^2/\tau$.

Proof: Refer to Appendix 4.4.

□

While equation (4.5.2) has incorporated the distribution for Y, the last term involving the triple-integral needs to be further simplified before attempting to implement it numerically.

PROPOSITION 4.5.2. *By simplifying the cost term, $C(S,\tau)$ in Proposition 4.5.1 can be expressed as*

$$
\begin{aligned}
C(S,\tau) \;=\; & \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} C_E[S,K,1,r_n(\tau),q,\tau,v_n^2(\tau)] \qquad (4.5.3)\\
& + \sum_{n=0}^{\infty} \frac{[\lambda']^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-\lambda'(\tau-\xi)} qSe^{-q(\tau-\xi)} \\
& \qquad \times N\left[d_1(S,Kb(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))\right] d\xi \\
& - \sum_{n=0}^{\infty} \frac{[\lambda']^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-\lambda'(\tau-\xi)} rKe^{-r_n(\tau-\xi)(\tau-\xi)} \\
& \qquad \times N\left[d_2\left(S,Kb(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi)\right)\right] d\xi \\
& - \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \int_0^1 \frac{[C(ZKb(\xi),\xi) - (ZKb(\xi)-K)]}{Zv_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
& \qquad \times \exp\left\{ \frac{-[d_2(S,ZKb(\xi),r_{n+1}(\tau-\xi),q,\tau-\xi,v_{n+1}^2(\tau-\xi))]^2}{2} \right\} \\
& \qquad \times N[D_n(S,Kb(\xi),\tau-\xi,ZKb(\xi))] dZ d\xi,
\end{aligned}
$$

where

$$
\begin{aligned}
& D_n(S,Kb(\xi),\tau-\xi,\beta) \\
& = \frac{\delta^2 \ln\frac{S}{\beta} + \left[\left(\ln\frac{\beta}{Kb(\xi)}\right) v_{n+1}^2(\tau-\xi) + \delta^2[r_n(\tau-\xi) - q] - \gamma v_n^2(\tau-\xi)\right](\tau-\xi)}{v_n(\tau-\xi)v_{n+1}(\tau-\xi)\delta(\tau-\xi)}.
\end{aligned}
$$

Proof: Refer to Appendix 4.5.

□

We draw the reader's attention to the fact that in the form (4.5.3) the last term now only involves a double integral which will result in a considerable saving in computational effort.

4.5.1. The Perpetual American Call with Jumps. As demonstrated by Kim (1990) in the pure-diffusion case, the perpetual American call provides an upper bound for the early exercise boundary of the American call with finite maturity. Here we extend this concept to the jump-diffusion case, providing an expression for the perpetual American call option under jump-diffusion. Let $C(S,\infty)$ denote the value of the perpetual American call, with early exercise boundary $Kb(\infty)$, which we assume to be constant. Dewynne (2004) demonstrates how to solve the time-invariant Black-Scholes PIDE (which is simply an integro-differential equation) for the price and early exercise boundary of a perpetual American call under jump-diffusion, and the results are given in Proposition 4.5.3.

PROPOSITION 4.5.3. *The value of the perpetual American call, $C(S,\infty)$, is given by*

$$C(S,\infty) = K(b(\infty)-1)\left(\frac{S}{Kb(\infty)}\right)^{\alpha_+}, \tag{4.5.4}$$

where the constant early exercise boundary, $b(\infty)$, is given by

$$b(\infty) = \frac{\alpha_+}{\alpha_+ - 1}. \tag{4.5.5}$$

The parameter α_+ is the positive root of the quadratic equation

$$\frac{\sigma^2}{2}\alpha(\alpha-1) + (r-q)\alpha - r + \lambda[e^{\alpha\gamma+\frac{\delta^2}{2}\alpha(\alpha-1)} - e^{\gamma}] = 0, \tag{4.5.6}$$

under the condition that

$$0 \le r + \lambda(e^{\gamma}-1). \tag{4.5.7}$$

This root is bounded by

$$1 \le \alpha_+ \le \frac{-\rho + \sqrt{\rho^2 + 2\sigma^2 r}}{\sigma^2}, \tag{4.5.8}$$

where $\rho \equiv r - q - \sigma^2/2$. Note also that

$$\left.\frac{\partial C(S,\infty)}{\partial S}\right|_{S=Kb(\infty)} = 1. \tag{4.5.9}$$

Proof: See Appendix 4.6.

□

Equations (4.5.4)-(4.5.6) provide an analytic expression for the perpetual American call price and free boundary under jump-diffusion. In order to use equations (4.5.4)-(4.5.5), we must first solve (4.5.6) numerically for α_+. This can be easily achieved using the bisection method, where we take advantage of the bounds for α_+ given in equation (4.5.8). In particular, note that when $q = 0$, $\alpha = 1$ is always a solution to (4.5.6). In this case the early exercise boundary is infinite, and it is never optimal to exercise the perpetual American call in the absence of dividends.

For the perpetual American call price to exist, condition (4.5.7) must be satisfied. The condition can be interpreted as requiring that the net effect of risk-free appreciation and jumps on the value of S be non-negative. This will always be satisfied when the jumps are expected to be upward ($\gamma > 0$) or neutral ($\gamma = 0$). When downward jumps are expected ($\gamma < 0$) the existence of the solution depends upon the relative values of r, λ and γ. Furthermore, it is interesting to note that the solution satisfies the smooth pasting condition, as stated in equation (4.5.9). This condition is not specified as part of the problem (see Appendix 4.6) but is satisfied when we determine $b(\infty)$ in an optimal manner. This result proves that the perpetual American call under jump-diffusion with log-normal jump sizes will satisfy the smooth pasting condition.

4.5.2. Properties of the Free Boundary at Expiry. Understanding the value of the free boundary at $\tau = 0^+$ is very important in the pure-diffusion case, yet to our knowledge such analysis has not been extended to the jump-diffusion problem within the literature cited earlier. In Appendix 4.7 we derive this limit which is presented in Proposition 4.5.4.

PROPOSITION 4.5.4. *The limit of the early exercise boundary $b(\tau)$ as $\tau \longrightarrow 0^+$ is given by*

$$b(0^+) = \max\left(1, \frac{r + \lambda N[(-\ln b(0^+) - (\gamma - \frac{\delta^2}{2}))/\delta]}{q + \lambda' N[(-\ln b(0^+) - (\gamma + \frac{\delta^2}{2}))/\delta]}\right). \tag{4.5.10}$$

Proof: Refer to Appendix 4.7.

□

It is worthwhile to observe that when $\lambda = 0$ equation (4.5.10) simplifies to the limit derived by Kim (1990) for the pure-diffusion American call free boundary. Note that (4.5.10) is an implicit expression for $b(0^+)$, but it can be solved quickly and accurately using standard root-finding techniques. Furthermore, as $q \longrightarrow 0$ the solution to the implicit part of equation (4.5.10) increases without bound. Thus when $q = 0$, $b(0^+) = \infty$, and we observe the well-known property that it is never optimal to exercise an American call option early in the absence of dividends.

Before concluding this section, we shall take a closer look at the properties of equation (4.5.10), specifically with a view to better understanding the solution to

$$b(0^+) = f(b(0^+)), \tag{4.5.11}$$

where

$$f(b(0^+)) = \frac{r + \lambda N[(-\ln b(0^+) - (\gamma - \frac{\delta^2}{2}))/\delta]}{q + \lambda' N[(-\ln b(0^+) - (\gamma + \frac{\delta^2}{2}))/\delta]}.$$

Once (4.5.11) is solved, then the $\max[\ ]$ operator can be applied. Since the value of the underlying is always non-negative, we must consider the domain $b(0^+) \geq 0$ when finding the solution to (4.5.11). It is not possible to provide a simple, explicit summary of the behaviour of (4.5.11) for various values of $b(0^+)$, because the cumulative normal density functions depend upon $b(0^+)$, and the function $f(b(0^+))$ involves the parameters r, q, λ, γ and δ, all of which have a significant impact on the value of $f(b(0^+))$. Nevertheless, we can offer some insight into the nature of (4.5.11).

Firstly, we see that is is simple to evaluate $f(b(0^+))$ at the limits of the domain. Specifically, we can show that

$$f(0) = \frac{r + \lambda}{q + \lambda e^{\gamma}} \geq 0, \tag{4.5.12}$$

and

$$\lim_{b(0^+) \to \infty} f(b(0^+)) \equiv f(\infty) = \frac{r}{q}. \tag{4.5.13}$$

Thus for $f(b(0^+))$ to be finite at each extremity of the domain, it is sufficient that we have $q > 0$. In this case, it is clear that $f(b(0^+))$ is continuous, and (4.5.11) will have at least one solution. We shall demonstrate by example that $f(b(0^+))$ is not monotonic, nor is it strictly bounded by the end values (4.5.12)-(4.5.13). This makes it difficult to prove that for $q > 0$ equation (4.5.11) has at most one solution. Since $b(0^+)$ appears only inside cumulative normal functions within $f(b(0^+))$, we can safely claim that the behaviour of $f(b(0^+))$ with respect to $b(0^+)$ will be bounded by the behaviour of $N(\ln x)$. In particular, we recall that $0 \leq N(\ln x) \leq 1$, and that $N(\ln x)$ is well-known to be a smooth, continuous function of x, where $x \geq 0$. From this we postulate that the function $f(b(0^+))$ will not display any oscillating features within the domain under consideration, nor will it display frequent changes of slope.

To provide evidence in support of our claims regarding equation (4.5.11), we now present some numerical examples. Firstly, we demonstrate the limits (4.5.12)-(4.5.13) for

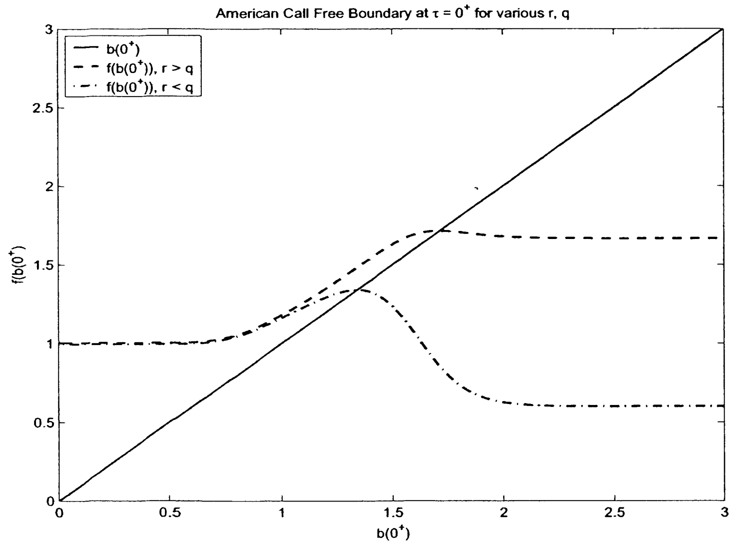

FIGURE 4.3. Behaviour of equation (4.5.11) when $\lambda = 5$, $\gamma = 0$ and $\delta = 0.2$. When $r > q$ we set $r = 0.05$, $q = 0.03$, and $r = 0.03$, $q = 0.05$ when $r < q$.

varying values of r and q. Setting $\lambda = 5$, $\gamma = 0$ and $\delta = 0.2$, we plot the functions $y = b(0^+)$ and $y = f(b(0^+))$ for various values of r and q, as shown in Figure 4.3. When $r = 0.05$ and $q = 0.03$, we can see that $f(0) < f(\infty)$. On the other hand, when $r = 0.03$ and $q = 0.05$, we now have $f(0) > f(\infty)$. In both cases it is clear that $f(b(0^+))$ is not bounded by these endpoint values. Thus we can see that the relative values of r and q directly influence the values of $f(0)$ and $f(\infty)$

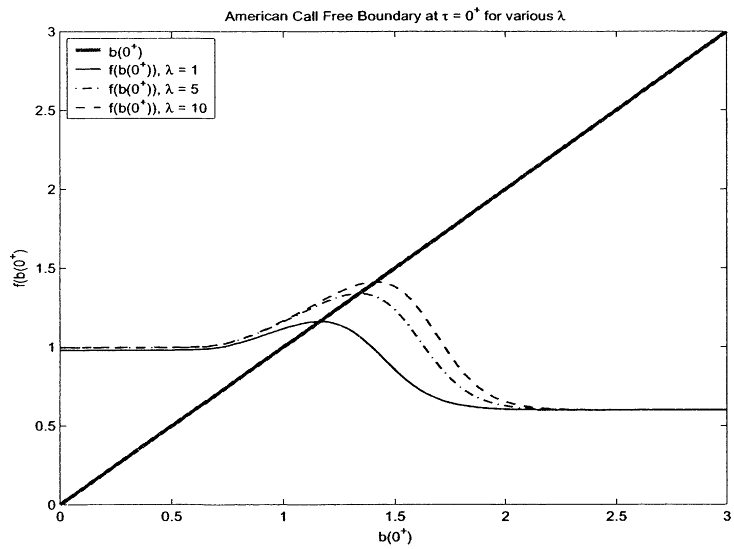

FIGURE 4.4. Behaviour of equation (4.5.11) when $r = 0.03$, $q = 0.05$, $\gamma = 0$ and $\delta = 0.2$, for various values of λ.

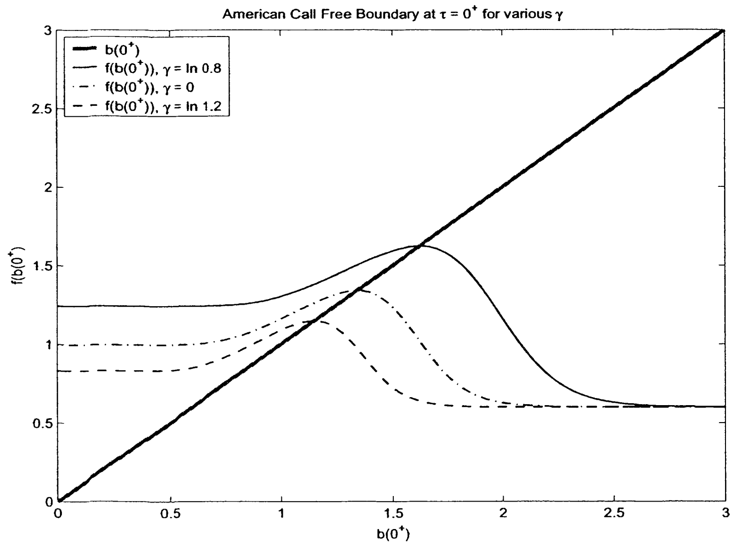

FIGURE 4.5. Behaviour of equation (4.5.11) when $r = 0.03$, $q = 0.05$, $\lambda = 5$ and $\delta = 0.2$, for various values of γ.

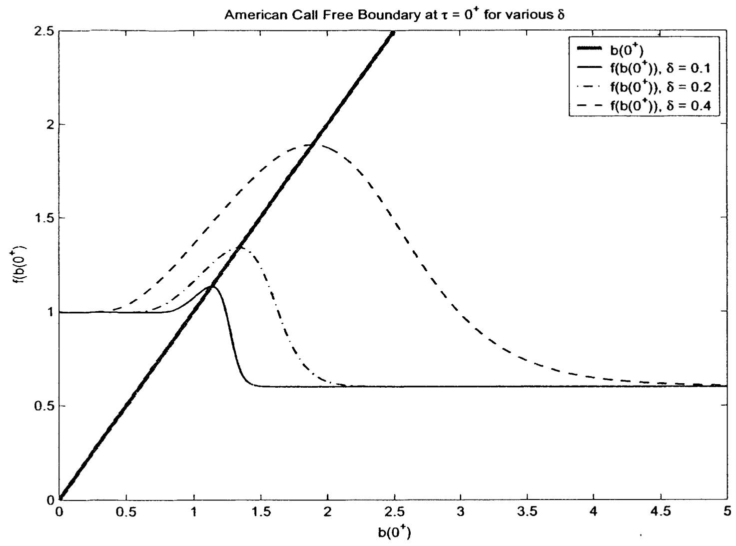

FIGURE 4.6. Behaviour of equation (4.5.11) when $r = 0.03$, $q = 0.05$, $\lambda = 5$ and $\gamma = 0$, for various values of δ

Since it is difficult to appreciate the impact of the jump-parameters on $f(b(0^+))$ using comparative statics, we again provide numerical examples to highlight the properties of $f(b(0^+))$. In all cases we set $r = 0.03$ and $q = 0.05$, with default jump-parameter values as used in generating Figure 4.3. In Figure 4.4 we see how $f(b(0^+))$ is affected by changes in λ. Aside from the obvious impact this has on $f(0)$, we can see that as λ increases, the peak of $f(b(0^+))$ also increases. Next we vary γ to produce Figure 4.5.

In addition to varying the value of $f(0)$, changes in γ affect the size and location of the "hump" in $f(b(0^+))$. As γ is increased, the "hump" feature reduces in size and shifts towards the origin. Finally we observe the impact of varying δ values in Figure 4.6. We see that as δ increases, the width of the "hump" feature in $f(b(0^+))$ increases. Thus the jump-parameters primarily influence the shape and location of the non-linear features of $f(b(0^+))$, with λ and γ also affecting the value of $f(0)$. The r and q parameters only affect the endpoint values of $f(b(0^+))$. It should be noted that in all the cases presented thus far, there is clearly only one solution to equation (4.5.11), given by the intercept of $y = b(0^+)$ and $y = f(b(0^+))$.

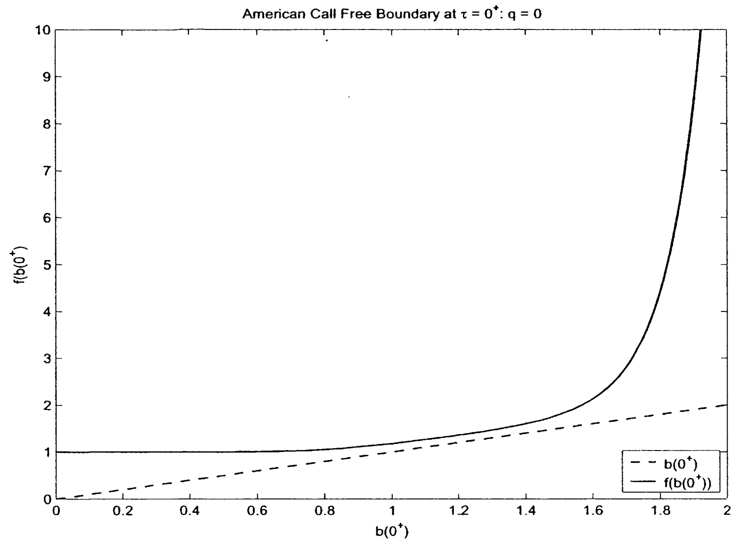

FIGURE 4.7. Behaviour of equation (4.5.11) when $q = 0$. Other parameter values are $r = 0.03$, $\lambda = 10$, $\gamma = 0$ and $\delta = 0.2$.

The last scenario to consider is when $q = 0$. We consider only the case where $\lambda > 0$, since when $\lambda = 0$, equation (4.5.11) reduces to the pure-diffusion result derived by Kim (1990). In this case, $f(\infty)$ is no longer finite, instead increasing without bound as $b(0^+) \to \infty$. Figure 4.7 demonstrates the behaviour of $f(b(0^+))$ with $q = 0$ for a selection of additional parameter values. It is clear from the plot that there is no solution for $b(0^+) = f(b(0^+))$. Furthermore, the only way that equation (4.5.11) will be satisfied when $q = 0$ is by taking the limit as $b(0^+) \to \infty$, in which case both sides of (4.5.11) will have the same infinite limit. Thus we infer that when $q = 0$, the free boundary at $\tau = 0^+$ increases without bound, and it is never optimal to exercise an American call early in the absence of dividends.

4.6. Numerical Implementation and Results

We now provide an iterative numerical scheme with which to evaluate the linked integral equation system formed by (4.5.3) as stated, and (4.5.3) evaluated at $S = Kb(\tau)$. We firstly discretise the time variable, τ, into N equally spaced intervals of length h. Thus $\tau = ih$ for $i = 0, 1, 2, ..., N$, and $h = T/N$. Denote the call price profile at time step i by $C(S, ih) = C_i(S)$, and similarly the free boundary at time step i by $b(ih) = b_i$. Using the same numerical technique that is applied to Volterra integral equations, we can solve the system for increasing values of i, until eventually the entire free boundary and price profile are calculated. In calculating the infinite summations, we continued adding terms until a pre-determined level of accuracy was reached, using a tolerance level of 10^{-16}. For the parameter values under consideration, we found that such a level of convergence was reached using 20 terms. In order to start the algorithm we require the initial values $C_0(S)$ and b_0, where $b_0 \equiv b(0^+)$. $C_0(S)$ is simply the payoff function for the call, namely $C_0(S) = \max(S - K, 0)$, and b_0 is given by equation (4.5.10).

Given that (4.5.3) depends upon $C(S, \tau)$, an initial approximation will be needed for $C_i(S)$ at each time step. A suitable approximation to $C_i(S)$ is given by $C_{i-1}(S)$, which is simply the American call price at the previous time step. Note that $C_0(S)$ is simply the payoff for the call option. The price at the $(i - 1)th$ time step is calculated for a suitably large number of evenly-spaced S values, and linear interpolation is applied to the profile as required. The algorithm *American Call Price* in Appendix 4.8 outlines how the iterative procedure is carried out for each i. Note that as the value of i increases, the computational burden will also increase at a "faster than linear" rate, since the integration at step i depends on all values of b_j and $C_j(S)$ for $j = 0, 1, 2, ...i - 1$. The time-integrals are evaluated using suitable applications of Simpson's rule (see Section 3.6), while the integral with respect to Z is evaluated using an appropriate Gauss-quadrature scheme. We use the bisection method to solve equation (4.5.3) for b_i when $S = Kb_i$, and the sums over the expected number of jumps are computed for ascending values of n until each integral term converges to a pre-specified level of accuracy (typically ten decimal places or more).

To demonstrate this algorithm we present some numerical results that enable us to observe the impact of infrequent jumps on the price of an American call option. Here we consider an American call with a strike of 1 and 6 months until expiry. $N = 50$ time steps were used, with a finer grid of 10 points used in the first 2 intervals to improve the quality of the free boundary approximation close to expiry. Price profiles at each time step were generated using 400 evenly spaced points between $S = 0$ and $S = 4K$. The fine space-grid was selected to help minimise the errors caused by using linear interpolation of the price profile in the numerical integration. The Gauss-quadrature scheme used 40 nodes[4]. To improve the accuracy of the results, the method was applied a second time using twice as many time-steps, and the resulting free boundaries were combined using Richardson extrapolation. For more details regarding the fundamentals of the numerical method, including the implementation of Simpson's rule and Richardson extrapolation, see Section 3.6.

The free boundary profile was generated in the case where $r = 8\%$ and $q = 12\%$. We take a jump-size volatility of $\delta^2 = 0.05$ and use $\lambda = 5$ for the expected number of jumps per year. The free boundaries were found for the pure-diffusion case (i.e. $\lambda = 0$), and then for various values of e^{γ} (we recall that $\mathbb{E}^{Q_Y}[Y] = e^{\gamma}$), specifically 0.95, 1, and 1.05 in order to gauge the effect of average up-jumps and average down-jumps. Table 4.1 summarises the values of σ used to ensure that the global volatility was the same for each combination of γ and λ values. The results are displayed in Figure 4.8. The most obvious feature of these results is the dramatic effect the presence of jumps has on the profile for the free boundary. Close to expiry, the free boundary with jumps is significantly larger than in the pure-diffusion case. This follows from the increased probability of large price movements near expiry, made possible by the presence of jumps within the return dynamics. Thus the holder of the call is less likely to exercise near expiry under the jump-diffusion model to best minimise the potential costs from downward jumps.

As time to expiry increases, we see that the pure-diffusion boundary increases more rapidly, since the jump component becomes less dominant within the underlying dynamics for large time intervals. While jumps are more likely, they become less influential overall, since there are sufficient opportunities for the jumps to be reversed, either by jumps in the opposite direction, or through the diffusion term. Therefore when far from maturity the holder of the call is more likely to exercise early under jump-diffusion than in the pure-diffusion case. These findings coincide with those of Amin (1993), who also notes

[4]Details regarding Gauss-quadrature can be found in Abramowitz and Stegun (1970).

that for a sufficiently large time to expiry, the probability density for the underlying converges under both models, such that there is no clear distinction between pure-diffusion and jump-diffusion.

Using the results in Proposition 4.5.3, we are able to find the free boundary and price of the perpetual American call for the parameter values listed in Table 4.1, and the results are provided within this table. We can see that Amin's results regarding long-maturities are certainly verified in the case where $\gamma = 0$, as the at-the-money perpetual American call price and free boundary are very similar in the pure-diffusion and jump-diffusion cases. This is not so when $\gamma > 0$, as here the at-the-money call price and free boundary are both larger than the pure-diffusion result. Amin's conclusions were made for the case where $\gamma = 0$, and it is interesting to note that the result regarding the convergence of the distributions does not apply for other values of γ. Note that there is no solution for the perpetual call when $\gamma < 0$, since we have $r + \lambda[e^{\gamma} - 1] = -0.17 < 0$, violating condition (4.5.7). We also note that Amin does not provide any formal evidence relating to the limit of the free boundary at expiry, although his numerical results are consistent with the limiting value given by equation (4.5.10).

One further observation we can make from Figure 4.8 is the impact of the value of γ on the free boundary. As γ increases, the value of the early exercise boundary decreases. This is attributable to the potential for the option holder to incur a rebalancing cost when the price jumps from the stopping region back down into the continuation region. Recall that $\gamma > 0$ implies upward jumps on average, thus making the expected cost of downward jumps quite small. When $\gamma < 0$, we expect downward jumps on average, and the holder will therefore require that S be even larger before exercising the call early. It should be noted that the free boundary estimates when jumps are present are not entirely "smooth", in that the slope of the curves are not strictly decreasing as time to maturity increases. This is attributed to mild numerical inaccuracies within the algorithm, which could be overcome by using more sophisticated and robust techniques, particularly when interpolating the call prices during integration. Such inaccuracies are not reflected in the corresponding price profiles however, as the American call price is not particularly sensitive to minor changes in the early exercise boundary.

In Figure 4.9 we present the American call price profiles at time $\tau = 0.5$ prior to expiry, in the case where $r = 5\%$ and $q = 3\%$. Here we set $\lambda = 1$ and $\delta = 0.15$. Table 4.2

σ^2	λ	e^γ	δ^2	$C(K,\infty)$	$b(\infty)$
0.3064	0	-	-	0.35055	2.69606
0.0625	5.00	0.95	0.05	DNE	DNE
0.0500	5.00	1.00	0.05	0.34865	2.68041
0.0112	5.00	1.05	0.05	0.49512	4.32812

TABLE 4.1. Parameter values used to generate the free boundaries in Figure 4.8. The global volatility was fixed at $s = 55.35\%$, determined by $s^2 = \sigma^2 + \lambda[e^{2\gamma+\delta^2} - 2e^\gamma + 1]$. The price for the at-the-money perpetual American call, $C(K,\infty)$, is included, along with the corresponding free boundary, $b(\infty)$. "DNE" indicates that the perpetual American call solution does not exist for these parameters.

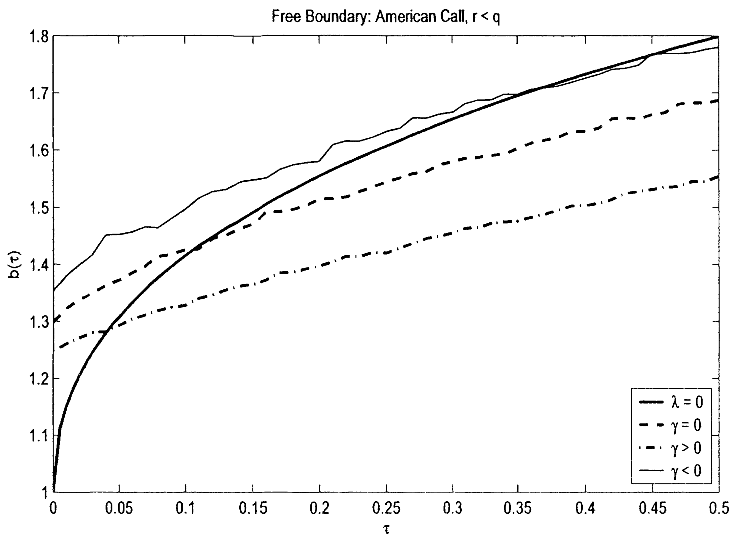

FIGURE 4.8. Early exercise boundaries for the American call option, for a range of γ values, compared with the pure-diffusion case of $\lambda = 0$. Other parameter values are $K = 1$, $T = 0.5$, $r = 8\%$, $q = 12\%$, $\lambda = 5$ and $\delta^2 = 0.05$. See Table 4.1 for further details.

summarises the parameter values of σ used, and we varied γ in the same manner as was done for generating the previous free boundaries. It can be seen that regardless of the value of γ being considered, the 6-month at-the-money American call with jumps is consistently worth less than the corresponding American call under pure-diffusion. Furthermore, as S moves away from the strike, the value of the call with jumps increases relative to the pure-diffusion case, until the jump-diffusion prices are eventually the greater of the two. The value of γ has some impact on the rate of this change as one varies the value of S, and obviously the fact that both cases must eventually be equal to the payoff for large values of S minimises the impact of this behviour when the call is deep in-the-money.

σ	λ	e^{γ}	δ
0.2000	0	-	-
0.1302	1.00	0.95	0.15
0.1313	1.00	1.00	0.15
0.1114	1.00	1.05	0.15

TABLE 4.2. Parameter values used to generate the price profiles in Figure 4.9, and the corresponding relative differences in Figure 4.10. The global volatility was fixed at $s = 20\%$, determined by
$s^2 = \sigma^2 + \lambda[e^{2\gamma+\delta^2} - 2e^{\gamma} + 1]$.

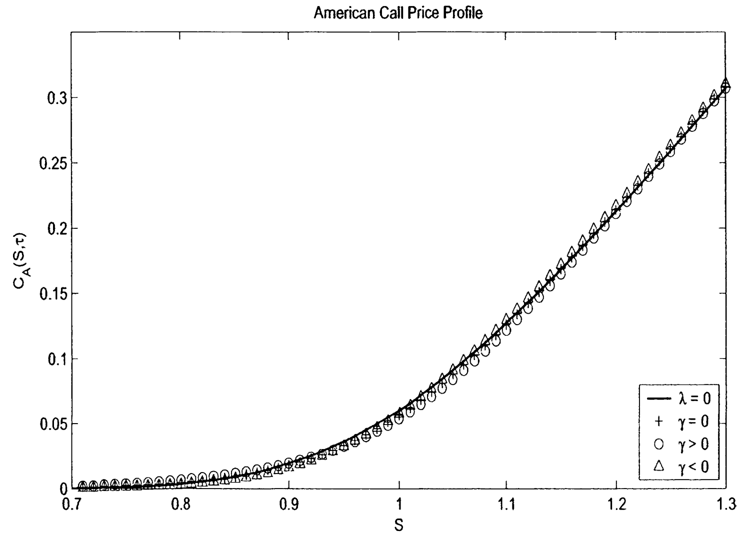

FIGURE 4.9. Price profile of an American call option for various values of γ, and compared with the pure-diffusion case of $\lambda = 0$. Other parameter values are $K = 1, T = 0.5, r = 5\%, q = 3\%, \lambda = 1$ and $\delta = 0.15$. See Table 4.2 for further details.

The relative changes in the call prices caused by jumps can be discerned more clearly in Figure 4.10, where we now plot the relative price difference between the pure-diffusion and jump-diffusion American call prices for various values of γ, using the same parameter values as in Figure 4.9. We observe that the jump-diffusion model results in a 5-10% decrease in the value of the at-the-money American call for these parameter values. When the call is deep out-of-the-money, the jump-diffusion model gives higher prices than the diffusion case, and a similar result occurs for the price when deep in-the-money, although this behaviour is capped by the presence and relative value of the early exercise boundary. This implies that the jump-diffusion model is able to reflect the basic volatility smile structure observed in market option prices. We have elected not to demonstrate this result

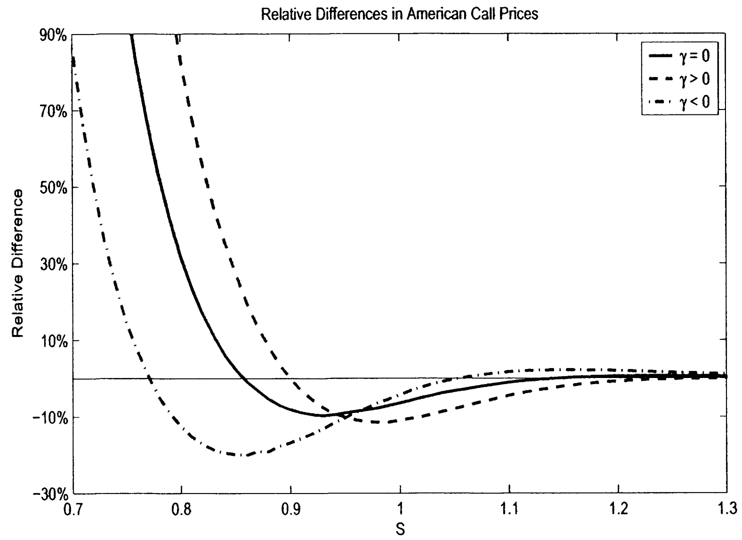

FIGURE 4.10. Relative price differences between the pure-diffusion American call and the corresponding contract under jump-diffusion, for various values of γ. Other parameter values are $K = 1, T = 0.5, r = 5\%$, $q = 3\%$, $\lambda = 1$ and $\delta = 0.15$. See Table 4.2 for further details.

using Black-Scholes implied volatilities, as this procedure only makes theoretical sense in the case of European options. Nevertheless, it is clear from the relative price differences that the jump-diffusion dynamics have the potential to capture volatility smile behaviour.

4.7. Conclusion

This chapter has presented an extension of McKean's (1965) free boundary value problem for the American call option to the case where the underlying asset follows a jump-diffusion process, as originally proposed by Merton (1976). Using the incomplete Fourier transform approach, we solved the PIDE to obtain a coupled integral equation system for the price and free boundary of the American call, where jumps occur according to a Poisson process, with a general distribution for the jump sizes. We showed how these may be manipulated into the integral equations for the American call derived by Gukhal (2001). This new approach both recovers Gukhal's results found via the compound option method, and has the advantage of naturally extending the more broadly applicable Fourier transform technique to the jump-diffusion model. This approach has the benefit that the various expectation operations remain clearly defined, with explicit distinction between the diffusion of the price process and the random jump sizes.

We have also derived a simplification of the triple integral expression within the integral equations in the case where the jump-sizes are lognormally distributed. This reduces the computational burden when one proceeds to numerical implementation. In addition the limit of the free boundary at expiry has been derived, as this is a necessary input into the numerical procedure. We also provided results regarding the price and free boundary for perpetual American calls under jump-diffusion.

A means of numerically implementing the coupled integral equation system for the American call was provided, based on the numerical approach in the American strangle analysis of Chapter 3. An iterative method was proposed to deal with the interdependence between the call price and free boundary. While the method is mathematically simple, it provides a means with which to numerically analyse the behaviour of the integral equation system, something not addressed by existing literature. The numerical results presented from this method demonstrate that even a small frequency of expected jumps has a dramatic impact on the price profile of the call option, particularly at-the-money. In addition, the mean expected jump size has a significant impact on the free boundary for the American call. This is due in part to the added cost to the option holder incurred whenever the underlying's price jumps downward from the stopping region into the continuation region, a feature identified by Gukhal (2001) and reinforced in our findings.

The numerical results presented have replicated the findings of Amin (1993) in relation to the impact of jumps on the behaviour of the early exercise boundary of the American call. We provided further evidence that early exercise of the call is more likely under jump-diffusion near expiry, while away from expiry early exercise is less likely. Secondly, we demonstrated that the addition of jumps increases the value of out-of-the-money American calls, while at-the-money calls become less valuable. This behaviour is in accordance with the well-known volatility smile phenomenon observed in option prices within financial markets. The in-the-money value can also be greater with jumps, but this depends largely upon the option parameters, time to maturity and relative values of the free boundary under each model for asset returns.

As mentioned previously, the method presented here is readily applicable to a range of jump size distributions and payoff functions, and one avenue for future research would be to explore these alternatives. Merton's model for the jump process assumes that jump risk is fully diversifiable. This assumption could be relaxed within the Fourier transform framework, but only certain kinds of jump risk could be catered for. In addition the numerical algorithm presented is primarily a first-pass solution for the integral equation system. Further analysis needs to be conducted to improve the speed, accuracy and efficiency of the method provided by using more advanced numerical analysis techniques. In particular we shall consider an alternative numerical method in Chapter 5, and compare this with the numerical integration approach.

Appendix 4.1. Properties of the Incomplete Fourier Transform

According to Appendix A2.3.1, from the pure-diffusion case (i.e. the model with no jumps) we know that

$$\mathcal{F}^b\left\{\frac{\partial V}{\partial x}\right\} = (b-1)e^{i\eta\ln b} - i\eta\hat{V}, \tag{A4.1.1}$$

$$\mathcal{F}^b\left\{\frac{\partial^2 V}{\partial x^2}\right\} = e^{i\eta\ln b}(b - i\eta(b-1)) - \eta^2\hat{V}, \tag{A4.1.2}$$

and $$\mathcal{F}^b\left\{\frac{\partial V}{\partial \tau}\right\} = \frac{\partial \hat{V}}{\partial \tau} - \frac{b'}{b}e^{i\eta\ln b}(b-1), \tag{A4.1.3}$$

where $b' \equiv db(\tau)/d\tau$. This leaves one term to be evaluated, namely

$$\mathcal{F}\left\{H(\ln b - x)\int_0^\infty V(x+\ln Y,\tau)G(Y)dY\right\}$$
$$= \int_{-\infty}^{\ln b} e^{i\eta x}\int_0^\infty V(x+\ln Y,\tau)G(Y)dY dx. \tag{A4.1.4}$$

Using the change of variable $z = x + \ln Y$, equation (A4.1.4) becomes

$$\begin{aligned}
&\mathcal{F}\left\{H(\ln b - x)\int_0^\infty V(x+\ln Y,\tau)G(Y)d(Y)\right\} \\
&= \int_0^\infty \int_{-\infty}^{\ln b + \ln Y} e^{i\eta(z-\ln Y)}V(z,\tau)G(Y)dz dY \\
&= \int_0^\infty \left[\int_{-\infty}^{\ln b} e^{i\eta(z-\ln Y)}V(z,\tau)G(Y)dz \right. \\
&\quad \left. + \int_{\ln b}^{\ln Y + \ln b} e^{i\eta(z-\ln Y)}V(z,\tau)G(Y)dz\right] dY \\
&= \int_0^\infty e^{-i\eta\ln Y}G(Y)dY \int_{-\infty}^{\ln b} V(z,\tau)e^{i\eta z}dz \\
&\quad + \int_0^\infty e^{-i\eta\ln Y}G(Y)\left[\int_{\ln b}^{\ln Y+\ln b} e^{i\eta z}V(z,\tau)dz\right] dY \\
&= A(\eta)\hat{V}(\eta,\tau) + \Phi(\eta,\tau),
\end{aligned}$$

where

$$A(\eta) \equiv \int_0^\infty e^{i\eta \ln Y} G(Y) dY,$$

and

$$\Phi(\eta,\tau) \equiv \int_0^\infty e^{-i\eta \ln Y} G(Y) \left[\int_{\ln b}^{\ln Y + \ln b} e^{i\eta \ln z} V(z,\tau) dz \right] dY.$$

Hence, our PIDE is transformed into the integro-differential equation

$$\begin{aligned} \frac{\partial \hat{V}}{\partial \tau} - \frac{b'}{b} e^{i\eta \ln b}(b-1) &= \frac{\sigma^2}{2}\left(e^{i\eta \ln b}(b - i\eta(b-1)) - \eta^2 \hat{V} \right) \\ &\quad + \phi\left((b-1)e^{i\eta \ln b} - i\eta \hat{V} \right) - (r+\lambda)\hat{V} \\ &\quad + \lambda[A(\eta)\hat{V}(\eta,\tau) + \Phi(\eta,\tau)], \end{aligned}$$

which is readily simplified to

$$\frac{\partial \hat{V}}{\partial \tau} + \left[\frac{\sigma^2 \eta^2}{2} + \phi i\eta + (r+\lambda) - \lambda A(\eta) \right] \hat{V} = F(\eta,\tau),$$

where

$$F(\eta,\tau) = e^{i\eta \ln b}\left[\frac{\sigma^2 b}{2} + \left(\frac{b'}{b} - \frac{\sigma^2 i\eta}{2} + \phi \right)(b-1) \right] + \lambda \Phi(\eta,\tau).$$

The solution to this integro-differential equation is given by

$$\begin{aligned} \hat{V}(\eta,\tau) &= \hat{V}(\eta,0) e^{-(\frac{1}{2}\sigma^2\eta^2 + \phi i\eta + (r+\lambda) - \lambda A(\eta))\tau} \\ &\quad + \int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2 + \phi i\eta + (r+\lambda) - \lambda A(\eta))(\tau - s)} F(\eta,s) ds, \end{aligned}$$

where $\mathcal{F}^b\{V(x,0)\} \equiv \hat{V}(\eta,0)$.

Appendix 4.2. Derivation of the American Call Integral Equations

A4.2.1. Proof of Proposition 4.4.1. Consider the function $V_1(x,\tau)$, given by

$$V_1(x,\tau) = (\mathcal{F}^b)^{-1}\left\{ \hat{V}(\eta,0) e^{-\left(\frac{1}{2}\sigma^2\eta^2 + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right)\tau} \right\}.$$

To evaluate this inversion, recall the convolution result for Fourier transforms given by equation (A2.1.1). If we let

$$\hat{F}(\eta,\tau_1) = e^{-\left(\frac{1}{2}\sigma^2\eta^2 + \phi i\eta + (r+\lambda) - \lambda A(\eta)\right)\tau},$$

then $f(x,\tau_1)$ is given by

$$\begin{aligned} f(x,\tau_1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)\tau} e^{-i\eta x} d\eta \\ &= \frac{e^{-(\lambda+r)\tau}}{2\pi}\int_{-\infty}^{\infty} e^{-\left[\frac{1}{2}\sigma^2\eta^2-\lambda A(\eta)\right]\tau-i\eta[\phi\tau+x]} d\eta. \end{aligned}$$

Furthermore, let

$$\hat{G}(\eta,\tau_2) = \hat{V}(\eta,0).$$

Hence $g(x,\tau_2)$ will simply be the payoff function in the continuation region, given by

$$g(x,\tau_2) = H(\ln b(0^+) - x)\max(e^x - 1, 0) = H(\ln b(0^+) - x)H(x)(e^x - 1).$$

Thus $V_1(x,\tau)$ becomes

$$\begin{aligned} V_1(x,\tau) &= \int_{-\infty}^{\infty}\Big[H(\ln b(0^+) - u)H(u)(e^u - 1) \\ &\qquad \times \frac{e^{-(\lambda+r)\tau}}{2\pi}\int_{-\infty}^{\infty} e^{-\left[\frac{1}{2}\sigma^2\eta^2-\lambda A(\eta)\right]\tau-i\eta[\phi\tau+x-u]} d\eta\Big] du. \end{aligned}$$

The expression for $V_1(x,\tau)$ can now be further simplified to

$$\begin{aligned} V_1(x,\tau) &= \frac{e^{-(\lambda+r)\tau}}{2\pi}\int_{-\infty}^{\ln b(0^+)}\left[H(u)(e^u - 1)\int_{-\infty}^{\infty} e^{-\left[\frac{1}{2}\sigma^2\eta^2-\lambda A(\eta)\right]\tau-i\eta[\phi\tau+x-u]} d\eta\right] du \\ &= \frac{e^{-(\lambda+r)\tau}}{2\pi}\int_{-\infty}^{\infty}\left[H(u)(e^u - 1)\int_{-\infty}^{\infty} e^{-\left[\frac{1}{2}\sigma^2\eta^2-\lambda A(\eta)\right]\tau-i\eta[\phi\tau+x-u]} d\eta\right] du \\ &\quad - \frac{e^{-(\lambda+r)\tau}}{2\pi}\int_{\ln b(0^+)}^{\infty}\left[H(u)(e^u - 1)\int_{-\infty}^{\infty} e^{-\left[\frac{1}{2}\sigma^2\eta^2-\lambda A(\eta)\right]\tau-i\eta[\phi\tau+x-u]} d\eta\right] du. \end{aligned}$$

Letting $C_1(S,\tau) = KV_1(x,\tau)$, the problem can be re-expressed in terms of the original space variable S as

$$\begin{aligned} C_1(S,\tau) &= e^{-(\lambda+r)\tau}\int_{-\infty}^{\infty} KH(u)(e^u - 1)\hat{K}(u,S,\tau)du \\ &\quad - e^{-(\lambda+r)\tau}\int_{\ln b(0^+)}^{\infty} KH(u)(e^u - 1)\hat{K}(u,S,\tau)du, \end{aligned}$$

where

$$\hat{K}(u,S,\tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left[\frac{1}{2}\sigma^2\eta^2-\lambda A(\eta)\right]\tau-i\eta[\phi\tau+\ln\frac{S}{K}-u]} d\eta.$$

We shall now consider further the function $\hat{K}(u,S,\tau)$. Using a Taylor series expansion, the expression for $\hat{K}(u,S,\tau)$ becomes

$$\begin{aligned}\hat{K}(u,S,\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2\tau - i\eta[\phi\tau+\ln\frac{S}{K}-u]}\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}A(\eta)^n d\eta \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2\tau - i\eta[\phi\tau+\ln\frac{S}{K}-u]}A(\eta)^n d\eta.\end{aligned}$$

Note that by definition

$$\begin{aligned}A(\eta)^n &= \left\{\int_0^{\infty} e^{-i\eta\ln Y}G(Y)dY\right\}^n \\ &= \int_0^{\infty} e^{-i\eta\ln Y_1}G(Y_1)dY_1 ... \int_0^{\infty} e^{-i\eta\ln Y_n}G(Y_n)dY_n.\end{aligned}$$

Using this definition in the expression for $\hat{K}(u,S,\tau)$, we have

$$\begin{aligned}\hat{K}(u,S,\tau) &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2\tau - i\eta[\phi\tau+\ln\frac{S}{K}-u]} \\ &\quad\times\left\{\int_0^{\infty} e^{-i\eta\ln Y_1}G(Y_1)dY_1 ... \int_0^{\infty} e^{-i\eta\ln Y_n}G(Y_n)dY_n\right\}d\eta \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}\int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} G(Y_1)G(Y_2)...G(Y_n) \\ &\quad\times\left\{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2\tau - i\eta[\phi\tau+\ln\frac{S}{K}-u] - i\eta\ln(Y_1Y_2...Y_n)}d\eta\right\}dY_1dY_2...dY_n \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}\int_0^{\infty}...\int_0^{\infty} G(Y_1)...G(Y_n)I(SX_n,\tau)dY_1...dY_n\end{aligned}$$

where

$$I(SX_n,\tau) \equiv \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2\tau - i[\phi\tau+\ln\frac{SX_n}{K}-u]\eta}d\eta,$$

and $X_n \equiv Y_1Y_2...Y_n$, with $X_0 \equiv 1$. $I(SX_n,\tau)$ can be evaluated using equation (A2.1.2) with $\hat{p} = \frac{1}{2}\sigma^2\tau$ and $\hat{q} = i[\phi\tau + \ln(SX_n/K) - u]$. Thus we have

$$I(SX_n,\tau) = \sqrt{\frac{2\pi}{\sigma^2\tau}}\exp\left\{\frac{-[u-\ln\frac{SX_n}{K}-\phi\tau]^2}{2\sigma^2\tau}\right\}.$$

Therefore the expression for $\hat{K}(u,S,\tau)$ becomes

$$\hat{K}(u,S,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \exp\left\{ \frac{-[u - \ln\frac{SX_n}{K} - \phi\tau]^2}{2\sigma^2\tau} \right\} \right\},$$

where we note that

$$\mathbb{E}_\tau^{(n)}\{(\cdot)\} = \int_0^\infty \int_0^\infty ... \int_0^\infty (\cdot) G(Y_1)G(Y_2)...G(Y_n) dY_1 dY_2 ... dY_n,$$

and $\mathbb{E}_\tau^{(0)}\{(\cdot)\} \equiv (\cdot)$.

To further simplify the expression for $C_1(S,\tau)$, let

$$C_1^{(1)}(S,\tau) \equiv e^{-(\lambda+r)\tau} \int_{-\infty}^{\infty} KH(u)(e^u - 1)\hat{K}(u,S,\tau)du$$

and

$$C_1^{(2)}(S,\tau) \equiv e^{-(\lambda+r)\tau} \int_{\ln b(0)}^{\infty} KH(u)(e^u - 1)\hat{K}(u,S,\tau)du.$$

Firstly, for $C_1^{(1)}(S,\tau)$, we have

$$\begin{aligned}
C_1^{(1)}(S,\tau) &= \frac{e^{-(\lambda+r)\tau}K}{\sigma\sqrt{2\pi\tau}} \int_0^\infty (e^u - 1) \\
&\quad \times \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \exp\left\{ \frac{-[u - \ln\frac{SX_n}{K} - \phi\tau]^2}{2\sigma^2\tau} \right\} \right\} du \\
&= \sum_{n=0}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!} \\
&\quad \times \mathbb{E}_\tau^{(n)} \left\{ \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_0^\infty (Ke^u - K) \exp\left[\frac{-[u - \ln\frac{SX_n}{K} - \phi\tau]^2}{2\sigma^2\tau} \right] du \right\}.
\end{aligned}$$

Using the change of variable $e^z = Ke^u$, we obtain

$$\begin{aligned}
C_1^{(1)}(S,\tau) &= \sum_{n=0}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!} \\
&\quad \times \mathbb{E}_\tau^{(n)} \left\{ \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^\infty (e^z - K) \exp\left[\frac{-[z - \ln SX_n - \phi\tau]^2}{2\sigma^2\tau} \right] dz \right\} \\
&= \sum_{n=0}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ C_E[SX_n e^{-\lambda k\tau}, K, 1, r, q, \tau, \sigma^2] \right\},
\end{aligned}$$

where

$$C_E[S,K,\beta,r,q,\tau,\sigma^2] = Se^{-r\tau}N[d_1(S,K\beta,r,q,\tau,\sigma^2)] \\ -Ke^{-r\tau}N[d_2(S,K\beta,r,q,\tau,\sigma^2)]$$

with

$$d_1(S,\kappa,r,q,\tau,\sigma^2) = \frac{\ln\frac{S}{\kappa} + (r-q+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},$$

and

$$d_2(S,\kappa,r,q,\tau,\sigma^2) = d_1(S,\kappa,r,q,\tau,\sigma^2) - \sigma\sqrt{\tau}.$$

The details for this conclusion can be found in Chiarella (2003).

Next, for $C_1^{(2)}(S,\tau)$ we have

$$\begin{aligned} C_1^{(2)}(S,\tau) &= \frac{e^{-(\lambda+r)\tau}K}{\sigma\sqrt{2\pi\tau}}\int_{\ln b(0^+)}^{\infty} H(u)(e^u-1) \\ &\quad \times\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}\mathbb{E}_\tau^{(n)}\left\{\exp\left[\frac{-[u-\ln\frac{SX_n}{K}-\phi\tau]^2}{2\sigma^2\tau}\right]du\right\} \\ &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \\ &\quad \times\mathbb{E}_\tau^{(n)}\left\{\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{\ln b(0^+)}^{\infty}(Ke^u-K)\exp\left[\frac{-[u-\ln\frac{SX_n}{K}-\phi\tau]^2}{2\sigma^2\tau}\right]du\right\}. \end{aligned}$$

Note that since $a(T)\geq K$, we know that $\ln b(0^+)\geq 0$. Thus using the change of variable $e^z = Ke^u$, we obtain

$$\begin{aligned} C_1^{(2)}(S,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \\ &\quad \times\mathbb{E}_\tau^{(n)}\left\{\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{\ln[Kb(0^+)]}^{\infty}(e^z-K)\exp\left[\frac{-[z-\ln SX_n-\phi\tau]^2}{2\sigma^2\tau}\right]dz\right\}. \end{aligned}$$

Hence it is readily shown that

$$C_1^{(2)}(S,\tau) = \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_\tau^{(n)}\left\{C_E[SX_ne^{-\lambda k\tau},K,b(0^+),r,q,\tau,\sigma^2]\right\},$$

and thus the final expression for $C_1(S,\tau)$ is given by

$$
\begin{aligned}
C_1(S,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_\tau^{(n)}\left\{C_E[SX_ne^{-\lambda k\tau},K,1,r,q,\tau,\sigma^2]\right\}\\
&\quad -\sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_\tau^{(n)}\left\{C_E[SX_ne^{-\lambda k\tau},K,b(0^+),r,q,\tau,\sigma^2]\right\}.
\end{aligned}
$$

A4.2.2. Proof of Proposition 4.4.2. We begin this proof by examining the function $V_2^{(1)}(x,\tau)$.

$$
\begin{aligned}
V_2^{(1)}(x,\tau) &= \frac{1}{2\pi}\int_0^{\tau} e^{-(r+\lambda)(\tau-s)}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2\eta^2}{2}(\tau-s)-i\eta[\phi(\tau-s)+x-\ln b(s)]}e^{\lambda A(\eta)(\tau-s)}\\
&\quad\times\left[\frac{\sigma^2 b(s)}{2}+\left(\frac{b'(s)}{b(s)}-\frac{\sigma^2 i\eta}{2}+\phi\right)(b(s)-1)\right]dsd\eta.
\end{aligned}
$$

We let

$$
\begin{aligned}
f_1(s) &= \frac{\sigma^2 b(s)}{2}+\left(\frac{b'(s)}{b(s)}+\phi\right)(b(s)-1),\\
f_2(s) &= \frac{\sigma^2 i}{2}(b(s)-1),
\end{aligned}
$$

$\hat{p}=\sigma^2(\tau-s)/2$, and $\hat{q}(X_n)=i[x+\ln(X_n)+\phi(\tau-s)-\ln b(s)]$, where X_n is as defined in Appendix A4.2.1. Applying a Taylor series expansion to $e^{\lambda A(\eta)(\tau-s)}$, we can rewrite

$V_2^{(1)}(x,\tau)$ as

$$\begin{aligned}
V_2^{(1)}(x,\tau) &= \frac{1}{2\pi}\int_0^\tau e^{-(r+\lambda)(\tau-s)}\left[\int_{-\infty}^{\infty} e^{-\hat{p}\eta^2-i\eta[\phi(\tau-s)+x-\ln b(s)]}\sum_{n=0}^{\infty}\frac{\lambda(\tau-s)^n A(\eta)^n}{n!}\right.\\
&\qquad \left.\times\{f_1(s)-\eta f_2(s)\}d\eta\right]ds\\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi}\frac{1}{n!}\int_0^\tau e^{-(r+\lambda)(\tau-s)}(\lambda[\tau-s])^n\\
&\qquad\times\left[\int_{-\infty}^{\infty} e^{-\hat{p}\eta^2-i\eta[\phi(\tau-s)+x-\ln b(s)]}\left\{\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} e^{-\eta\ln X_n}\right.\right.\\
&\qquad\left.\left.\times G(Y_1)G(Y_2)...G(Y_n)dY_1dY_2...dY_n\right\}\{f_1(s)-\eta f_2(s)\}d\eta\right]ds\\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi}\frac{1}{n!}\int_0^{\infty}\cdots\int_0^{\infty}\int_0^\tau e^{-(r+\lambda)(\tau-s)}(\lambda[\tau-s])^n\\
&\qquad\times\left[\int_{-\infty}^{\infty} e^{-\hat{p}\eta^2-\hat{q}(X_n)\eta}\{f_1(s)-\eta f_2(s)\}d\eta\right]ds\\
&\qquad\times G(Y_1)G(Y_2)...G(Y_n)dY_1dY_2...dY_n\\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\left\{\int_0^\tau e^{-(r+\lambda)(\tau-s)}(\tau-s)^n\right.\\
&\qquad\left.\times\left[\int_{-\infty}^{\infty} e^{-\hat{p}\eta^2-\hat{q}(X_n)\eta}\{f_1(s)-\eta f_2(s)\}d\eta\right]ds\right\},
\end{aligned}$$

where $\mathbb{E}_\tau^{(n)}$ is as stated in Appendix A4.2.1.

Following the details in Appendix A2.4.2 we can readily show that

$$\begin{aligned}
V_2^{(1)}(x,\tau) &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\left\{\int_0^\tau \frac{e^{-g_n(x,s)}(\tau-s)}{\sigma\sqrt{2\pi(\tau-s)}}\left[\frac{\sigma^2 b(s)}{2}\right.\right.\\
&\qquad\left.\left.+\left(\frac{b'(s)}{b(s)}+\frac{1}{2}\left[\phi-\frac{x+\ln(X_n)-\ln b(s)}{\tau-s}\right]\right)\right](b(s)-1)ds\right\},
\end{aligned}$$

where

$$g_n(x,s)\equiv\frac{(x+\ln(X_n)+\phi(\tau-s)-\ln b(s))^2}{2\sigma^2(\tau-s)}+(r+\lambda)(\tau-s).$$

Next we return to the original space variable, S, by setting $C_2^{(1)}(S,\tau) = KV_2^{(1)}(x,\tau)$, with $S = Ke^x$. This results in

$$\begin{aligned} C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \Bigg\{ \int_0^\tau \frac{e^{-h_n(S,\xi)}(\tau-\xi)^n}{\sigma\sqrt{2\pi(\tau-\xi)}} \\ &\times \left[\frac{K\sigma^2 b(\xi)}{2} + \left(\frac{b'(\xi)}{b(\xi)} + \frac{1}{2}\left[\phi - \frac{\ln\frac{S}{K} + \ln(X_n) - \ln b(\xi)}{\tau-\xi} \right] \right) \right] \\ &\times (Kb(\xi) - K) d\xi \Bigg\}, \end{aligned}$$

where

$$h_n(S,\xi) \equiv \frac{\left[\ln\frac{SX_n}{Kb(\xi)} + \phi(\tau-\xi)\right]^2}{2\sigma^2(\tau-\xi)} + (r+\lambda)(\tau-\xi)$$

Therefore $C_2^{(1)}(S,\tau)$ is given by

$$\begin{aligned} C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \Bigg\{ \int_0^\tau \frac{e^{-h_n(S,\xi)}(\tau-\xi)^n}{\sigma\sqrt{2\pi(\tau-\xi)}} \\ &\times \left[\frac{\sigma^2 K b(\xi)}{2} + \left(\frac{b'(\xi)}{b(\xi)} + \frac{1}{2}\left[\phi - \frac{\ln\frac{SX_n}{Kb(\xi)}}{\tau-\xi} \right] \right) \right] \\ &\times (Kb(\xi) - K) d\xi \Bigg\}. \end{aligned}$$

We now aim to simplify the expression for $C_2^{(1)}(S,\tau)$ using the methods of Kim (1990). For simplicity of notation, we define $G(\xi) \equiv Kb(\xi)$. The first step is to rewrite $h_n(S,\xi)$ as

$$\begin{aligned} h_n(S,\xi) &= \frac{\left[\ln(SX_n) - \ln G(\xi) + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)(\tau-\xi)\right]^2}{2\sigma^2(\tau-\xi)} + (r+\lambda)(\tau-\xi) \\ &= \frac{[y_n - Q(\xi)]^2}{2(\tau-\xi)} + (r+\lambda)(\tau-\xi), \end{aligned}$$

where

$$y_n \equiv \frac{\ln SX_n + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\tau}{\sigma},$$

and

$$Q(\xi) \equiv \frac{\ln G(\xi) + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\xi}{\sigma}.$$

It is important to note that the derivative of $Q(\xi)$ with respect to ξ is given by

$$Q'(\xi) = \frac{1}{\sigma}\left(\frac{G'(\xi)}{G(\xi)} + \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\right).$$

Using these results, $C_2^{(1)}(S,\tau)$ becomes

$$\begin{aligned}
C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Bigg\{\int_0^\tau \frac{(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \\
&\quad\times\Bigg[\frac{\sigma G(\xi)}{2} + \frac{1}{\sigma}\Bigg(\frac{G'(\xi)}{G(\xi)} + \left(r-q-\lambda k-\frac{\sigma^2}{2}\right) - \left(r-q-\lambda k-\frac{\sigma^2}{2}\right) \\
&\quad+\frac{1}{2}\left[\left(r-q-\lambda k-\frac{\sigma^2}{2}\right) - \frac{\ln\frac{SX_n}{G(\xi)}}{\tau-\xi}\right]\Bigg)(G(\xi)-K)\Bigg]d\xi\Bigg\} \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Bigg\{\int_0^\tau e^{-(r+\lambda)(\tau-\xi)}(\tau-\xi)^n\frac{e^{-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \\
&\quad\times\Bigg[\frac{\sigma G(\xi)}{2} + \Bigg(Q'(\xi) - \frac{1}{\sigma}\left(r-q-\lambda k-\frac{\sigma^2}{2}\right) \\
&\quad-\frac{1}{2\sigma}\left[\frac{\ln SX_n - \ln G(\xi) - \left(r-q-\lambda k-\frac{\sigma^2}{2}\right)(\tau-\xi)}{\tau-\xi}\right]\Bigg) \\
&\quad\times(G(\xi)-K)\Bigg]d\xi \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Bigg\{\int_0^\tau e^{-(r+\lambda)(\tau-\xi)}(\tau-\xi)^n\frac{e^{-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \\
&\quad\times\left[\frac{\sigma G(\xi)}{2} + \left(Q'(\xi) - \frac{y_n-Q(\xi)}{2(\tau-\xi)}\right)(G(\xi)-K)\right]d\xi\Bigg\}.
\end{aligned}$$

Thus we arrive at a new expression for $C_2^{(1)}(S,\tau)$, given by

$$\begin{aligned}
C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Bigg\{\int_0^\tau G(\xi)e^{-(r+\lambda)(\tau-\xi)}(\tau-\xi)^n\frac{e^{-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \\
&\quad\times\left[\frac{\sigma}{2} + Q'(\xi) - \frac{y_n-Q(\xi)}{2(\tau-\xi)}\right]d\xi \\
&\quad-K\int_0^\tau e^{-(r+\lambda)(\tau-\xi)}(\tau-\xi)^n\frac{e^{-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \\
&\quad\times\left[Q'(\xi) - \frac{y_n-Q(\xi)}{2(\tau-\xi)}\right]d\xi\Bigg\}. \qquad \text{(A4.2.1)}
\end{aligned}$$

In order to simplify equation (A4.2.1), we must derive two results. For the first result, we have

$$
\begin{aligned}
&(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} G(\xi) \frac{e^{-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \left[\frac{\sigma}{2}+Q'(\xi)-\frac{[y_n-Q(\xi)]}{2(\tau-\xi)}\right] \\
&= (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \frac{G(\xi)}{\sqrt{\tau-\xi}} \left[\frac{\sigma(\tau-\xi)+2Q'(\xi)(\tau-\xi)-y_n+Q(\xi)}{2(\tau-\xi)}\right] \\
&\quad \times \frac{1}{\sqrt{2\pi}} e^{-\frac{[y_n-Q(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} e^{\frac{\sigma^2}{2}(\tau-\xi)+\sigma(y_n-Q(\xi))} \\
&= -(\tau-\xi)^n S X_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} \frac{1}{\sqrt{2\pi}} e^{-\frac{[y_n-Q(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} \\
&\quad \times \left[\frac{\frac{1}{2}\frac{1}{\sqrt{\tau-\xi}}(y_n-Q(\xi)+\sigma(\tau-\xi))-(Q'(\xi)+\sigma)\sqrt{\tau-\xi}}{(\sqrt{\tau-\xi})^2}\right] \\
&= -(\tau-\xi)^n S X_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} \frac{\partial}{\partial \xi} N\left(\frac{y_n-Q(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right). \qquad \text{(A4.2.2)}
\end{aligned}
$$

For the second result, consider

$$
\begin{aligned}
&(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \frac{e^{-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \left[Q'(\xi)-\frac{[y_n-Q(\xi)]}{2(\tau-\xi)}\right] \\
&= -(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \frac{1}{\sqrt{2\pi}} e^{-\frac{[y_n-Q(\xi)]^2}{2(\tau-\xi)}} \left[\frac{-Q'(\xi)\sqrt{\tau-\xi}+\frac{1}{2}(y_n-Q(\xi))\frac{1}{\sqrt{\tau-\xi}}}{(\sqrt{\tau-\xi})^2}\right] \\
&= -(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \frac{\partial}{\partial \xi} N\left(\frac{y_n-Q(\xi)}{\sqrt{\tau-\xi}}\right). \qquad \text{(A4.2.3)}
\end{aligned}
$$

Using equations (A4.2.2) and (A4.2.3) in equation (A4.2.1), $C_2^{(1)}(S,\tau)$ becomes

$$
\begin{aligned}
C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \Big\{ -\int_0^\tau (\tau-\xi)^n S X_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} \\
&\quad \times \frac{\partial}{\partial \xi} N\left(\frac{y_n-Q(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi \\
&\quad +K \int_0^\tau (\tau-\xi)^n e^{-(\lambda+r)(\tau-\xi)} \frac{\partial}{\partial \xi} N\left(\frac{y_n-Q(\xi)}{\sqrt{\tau-\xi}}\right) d\xi. \qquad \text{(A4.2.4)}
\end{aligned}
$$

By applying integration by parts, we can evaluate equation (A4.2.4) as

$$\begin{aligned}
C_2^{(1)}(S,\tau) &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Bigg\{SX_n e^{-\lambda k\tau}e^{-(q+\lambda)\tau}\tau^n N\left(\frac{y_n - Q(0^+)+\sigma\tau}{\sqrt{\tau}}\right) \\
&\quad + \int_0^\tau SX_n e^{-\lambda k(\tau-\xi)}e^{-(q+\lambda)(\tau-\xi)}(\tau-\xi)^{n-1}[(\lambda[k+1]+q)(\tau-\xi)-n] \\
&\qquad \times N\left(\frac{y_n - Q(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)d\xi \\
&\quad - Ke^{-(r+\lambda)\tau}\tau^n N\left(\frac{y_n - Q(0^+)}{\sqrt{\tau}}\right) \\
&\quad - \int_0^\tau K(\tau-\xi)^{n-1}e^{-(r+\lambda)(\tau-\xi)}\left[(\tau-\xi)(r+\lambda)-n\right] \\
&\qquad \times N\left(\frac{y_n - Q(\xi)}{\sqrt{\tau-\xi}}\right)d\xi\Bigg\} \\
&= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_\tau^{(n)}\left\{C_E\left[SX_n e^{-\lambda k\tau},K,b(0^+),r,q,\tau,\sigma^2\right]\right\} \\
&\quad + \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Bigg\{\int_0^\tau(\tau-\xi)^{n-1}\Big[SX_n e^{-\lambda k(\tau-\xi)}e^{-(q+\lambda)(\tau-\xi)} \\
&\qquad \times[(\lambda[k+1]+q)(\tau-\xi)-n] \\
&\qquad \times N\left[d_1\left(SX_n e^{-\lambda k(\tau-\xi)},Kb(\xi),r,q,\tau-\xi,\sigma^2\right)\right] \\
&\quad - Ke^{-(r+\lambda)(\tau-\xi)}[(\tau-\xi)(r+\lambda)-n] \\
&\qquad \times N\left[d_2\left(SX_n e^{-\lambda k(\tau-\xi)},Kb(\xi),r,q,\tau-\xi,\sigma^2\right)\right]\Big]d\xi\Bigg\},
\end{aligned}$$

which is the final result stated in Proposition 4.4.2.

A4.2.3. Proof of Proposition 4.4.3. The term $V_2^{(2)}(x,\tau)$ is given by

$$V_2^{(2)}(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^\tau e^{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)(\tau-s)}e^{-i\eta x}\lambda\Phi(\eta,s)dsd\eta,$$

where we recall that

$$\Phi(\eta,s) = \int_0^\infty e^{-i\eta\ln Y}G(Y)\left[\int_{\ln b(s)}^{\ln Y+\ln b(s)}e^{i\eta z}V(z,s)dz\right]dY.$$

We begin by changing the order of integration within $V_2^{(2)}(x,\tau)$:

$$\begin{aligned} V_2^{(2)}(x,\tau) &= \frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\int_0^{\tau} e^{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)(\tau-\xi)-i\eta x} \\ &\quad \times \int_0^{\infty} e^{-i\eta\ln Y}G(Y)\int_{\ln b(\xi)}^{\ln Y+\ln b(\xi)} e^{i\eta z}V(z,\xi)dzdYd\xi d\eta \\ &= \lambda\int_0^{\tau}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{1}{2}\sigma^2\eta^2+\phi i\eta+(r+\lambda)-\lambda A(\eta)\right)(\tau-\xi)-i\eta x} \\ &\quad \times \int_0^{\infty} e^{-i\eta\ln Y}G(Y)\int_{\ln b(\xi)}^{\ln Yb(\xi)} e^{i\eta z}V(z,\xi)dzdYd\xi d\eta \\ &= \lambda\int_0^{\tau} e^{-(r+\lambda)(\tau-\xi)}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda A(\eta)(\tau-\xi)}e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(\phi(\tau-\xi)+x)} \\ &\quad \times \int_0^{\infty} e^{-i\eta\ln Y}G(Y)\int_{\ln b(\xi)}^{\ln Yb(\xi)} e^{i\eta z}V(z,\xi)dzdYd\eta d\xi. \end{aligned}$$

Applying a Taylor series expansion to $e^{\lambda A(\eta)(\tau-\xi)}$, we can rewrite $V_2^{(2)}(x,\tau)$ as

$$\begin{aligned} V_2^{(2)}(x,\tau) &= \sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2\eta^2}{2}(\tau-\xi)-i\eta(\phi(\tau-\xi)+x)} \\ &\quad \times \int_0^{\infty}\cdots\int_0^{\infty} e^{-i\eta\ln Y_1\ldots Y_n}G(Y_1)\ldots G(Y_n)dY_1\ldots dY_n \\ &\quad \times \int_0^{\infty} e^{-i\eta\ln Y}G(Y)\int_{\ln b(\xi)}^{\ln Yb(\xi)} e^{i\eta z}V(z,\xi)dzdYd\eta d\xi \\ &= \sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!}\mathbb{E}_{\tau}^{(n)}\left\{\int_0^{\tau}(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)}\int_0^{\infty}G(Y)\right. \\ &\quad \times\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(k_1(\tau-\xi)+x+\ln X_nY)} \\ &\quad \left.\times\int_{\ln b(\xi)}^{\ln Yb(\xi)} e^{i\eta z}V(z,\xi)dzd\eta dYd\xi\right\}. \end{aligned}$$

Consider the two innermost integrals, $I(x,\tau,Y,\xi)$, defined as

$$\begin{aligned} I(x,\tau,Y,\xi) &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(\phi(\tau-\xi)+x+\ln Y\ln X_n)} \\ &\quad \times\int_{\ln b(\xi)-\ln Y}^{\ln b(\xi)} e^{i\eta(x+\ln Y)}V(x+\ln Y,\xi)dxd\eta, \end{aligned}$$

where the integral with respect to x has been derived using the change of variable $z = x + \ln Y$. To evaluate I, we shall express it as the inverse Fourier transform

$$I(x,\tau,Y,\xi) = -\mathcal{F}^{-1}\left\{ e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(\phi(\tau-\xi)+\ln X_n)} \times \int_{\ln b(\xi)}^{\ln b(\xi)-\ln Y} e^{i\eta x} V(x+\ln Y,\xi)dx \right\}. \quad \text{(A4.2.5)}$$

Since we know that $0 < Y < \infty$, we must now consider two separate cases to evaluate equation (A4.2.5). The first case to consider is when $0 < Y < 1$, which means we can rewrite (A4.2.5) as

$$I(x,\tau,Y,\xi) = -\mathcal{F}^{-1}\left\{ e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(\phi(\tau-\xi)+\ln X_n)} \times \int_{-\infty}^{\infty} H(\ln b(\xi) - \ln Y - x)H(x - \ln b(\xi))e^{i\eta x}V(x+\ln Y,\xi)dx \right\}.$$

To evaluate this inversion, we again refer to the convolution result for Fourier transforms given by equation (A2.1.1). Let

$$\hat{F}(\eta,\xi) = e^{-\frac{1}{2}\sigma^2\eta^2(\tau-\xi)-i\eta(\phi(\tau-\xi)+\ln X_n)},$$

so that $f(x,\xi)$ is given by

$$f(x,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\hat{p}\eta^2-\hat{q}\eta}d\eta,$$

with $\hat{p} = \sigma^2(\tau-\xi)/2$ and $\hat{q} = i(\phi(\tau-\xi)+\ln X_N + x)$. Using equation (A2.1.2), $f(x,\xi)$ evaluates to

$$f(x,\xi) = \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}}\exp\left\{-\frac{[\phi(\tau-\xi)+\ln X_n + x]^2}{2\sigma^2(\tau-\xi)}\right\}.$$

For the second part of the convolution, let

$$\hat{G}(\eta,\xi) = \int_{-\infty}^{\infty} H(\ln b(\xi) - \ln Y - x)H(x - \ln b(\xi))e^{i\eta x}V(x+\ln Y,\xi)dx.$$

Therefore $g(x,\xi)$ is simply

$$g(x,\xi) = H(\ln b(\xi) - \ln Y - x)H(x - \ln b(\xi))V(x+\ln Y,\xi).$$

Combining f and g, the inverse Fourier transform, I, becomes

$$I(x,\tau,Y,\xi) = -\int_{\ln b(\xi)}^{\ln b(\xi)-\ln Y} \frac{V(u+\ln Y,\xi)}{\sigma\sqrt{2\pi(\tau-\xi)}} \times \exp\left\{-\frac{[\phi(\tau-\xi)+(x-u)\ln X_n]^2}{2\sigma^2(\tau-\xi)}\right\} du. \tag{A4.2.6}$$

In the second case we have $1 < Y < \infty$, which means we can now rewrite equation (A4.2.5) as

$$\begin{aligned} I(x,\tau,Y,\xi) &= \mathcal{F}^{-1}\left\{ e^{-\frac{\sigma^2\eta^2}{2}(\tau-\xi)-i\eta(\phi(\tau-\xi)+\ln X_n)} \right. \\ &\quad \left. \times \int_{-\infty}^{\infty} H(\ln b(\xi)-x)H(x-\ln b(\xi)+\ln Y)e^{i\eta x}V(x+\ln Y,\xi)dx \right\}. \end{aligned}$$

Following the same method as used in the first case, we find that

$$I(x,\tau,Y,\xi) = \int_{(\ln b(\xi)-\ln Y)}^{\ln b(\xi)} \frac{V(u+\ln Y,\xi)}{\sigma\sqrt{2\pi(\tau-\xi)}} \times \exp\left\{-\frac{[\phi(\tau-\xi)+(x-u)+\ln X_n]^2}{2\sigma^2(\tau-\xi)}\right\} du. \tag{A4.2.7}$$

Since results (A4.2.6) and (A4.2.7) depend entirely upon the relevant value of Y, we can integrate piecewise over the Y-domain, and thereby express $V_2^{(2)}(x,\tau)$ as

$$\begin{aligned} V_2^{(2)}(x,\tau) &= \lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\left\{\int_0^\tau (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \int_0^\infty G(Y)I(x,\tau,Y,\xi)dYd\xi\right\} \\ &= \lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\left\{\int_0^\tau (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \right. \\ &\quad \times \left[\int_0^1 G(Y)\int_{\ln b(\xi)}^{\ln b(\xi)-\ln Y} -\frac{V(u+\ln Y,\xi)}{\sigma\sqrt{2\pi(\tau-\xi)}} \right. \\ &\quad \times \exp\left\{-\frac{[\phi(\tau-\xi)+(x-u)+\ln X_n]^2}{2\sigma^2(\tau-\xi)}\right\} dudY \\ &\quad + \int_1^\infty G(Y)\int_{\ln b(\xi)-\ln Y}^{\ln b(\xi)} \frac{V(u+\ln Y,\xi)}{\sigma\sqrt{2\pi(\tau-\xi)}} \\ &\quad \left.\left. \times \exp\left\{-\frac{[\phi(\tau-\xi)+(x-u)+\ln X_n]^2}{2\sigma^2(\tau-\xi)}\right\} dudY\right] d\xi\right\}. \end{aligned}$$

Setting $C_2^{(2)}(S,\tau) = KV_2^{(2)}(x,\tau)$, we have

$$\begin{aligned} C_2^{(2)}(S,\tau) &= -\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \bigg\{ \int_0^1 (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\ &\times \bigg[\int_0^1 G(Y) \int_{\ln b(\xi)}^{\ln[b(\xi)/Y]} \frac{C(Ke^uY,\xi)}{\sigma\sqrt{2\pi(\tau-\xi)}} \\ &\qquad \times \exp\left\{ -\frac{[\phi(\tau-\xi)+(x-u)+\ln X_n]^2}{2\sigma^2(\tau-\xi)} \right\} dudY \\ &\quad - \int_1^{\infty} G(Y) \int_{\ln[b(\xi)/Y]}^{\ln b(\xi)} \frac{C(Ke^uY,\xi)}{\sigma\sqrt{2\pi(\tau-\xi)}} \\ &\qquad \times \exp\left\{ -\frac{[\phi(\tau-\xi)+(x-u)+\ln X_n]^2}{2\sigma^2(\tau-\xi)} \right\} dudY \bigg] d\xi \bigg\}. \end{aligned}$$

Finally, we shall introduce some additional notation and a change of variable to simplify the expression for $C_2^{(2)}(S,\tau)$. Letting $\omega = Ke^u$,we have

$$\begin{aligned} C_2^{(2)}(S,\tau) &= -\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \bigg\{ \int_0^{\tau} (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\ &\times \bigg[\int_0^1 G(Y) \int_{Kb(\xi)}^{Kb(\xi)/Y} \frac{C(\omega Y,\xi)}{\sigma\sqrt{2\pi(\tau-\xi)}} \\ &\qquad \times \exp\left\{ -\frac{[\phi(\tau-\xi)+\ln\frac{SX_n}{\omega}]^2}{2\sigma^2(\tau-\xi)} \right\} \frac{1}{\omega} d\omega dY \\ &\quad - \int_1^{\infty} G(Y) \int_{Kb(\xi)/Y}^{Kb(\xi)} \frac{C(\omega Y,\xi)}{\sigma\sqrt{2\pi(\tau-\xi)}} \\ &\qquad \times \exp\left\{ -\frac{[\phi(\tau-\xi)+\ln\frac{SX_n}{\omega}]^2}{2\sigma^2(\tau-\xi)} \right\} \frac{1}{\omega} d\omega dY \bigg] d\xi \bigg\}. \end{aligned}$$

Next, we consider carefully the domains for the integrals with respect to ω. For the first integral, the domain for ωY is

$$YKb(\xi) < \omega Y < Kb(\xi).$$

Thus ωY lies in the continuation region, and the value of $C(\omega Y,\xi)$ is unknown. For the second integral, the domain for ωY is

$$Kb(\xi) < \omega Y < YKb(\xi).$$

Thus ωY lies in the stopping region, and therefore the value of $C(\omega Y, \xi)$ is known to be

$$C(\omega Y, \xi) = \omega Y - K, \quad \text{where} \quad \omega > \frac{K}{Y}.$$

Thus $C_2^{(2)}(S, \tau)$ can be written more simply as

$$\begin{aligned} C_2^{(2)}(S,\tau) &= -\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \Bigg\{ \int_0^\tau (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\ &\quad \times \Bigg[\int_0^1 G(Y) \int_{Kb(\xi)}^{Kb(\xi)/Y} C(\omega Y, \xi) J(\omega, \xi, SX_n, \tau) d\omega dY \\ &\quad - \int_1^\infty G(Y) \int_{Kb(\xi)/Y}^{Kb(\xi)} (\omega Y - K) J(\omega, \xi, SX_n, \tau) d\omega dY \Bigg] d\xi \Bigg\}, \end{aligned}$$

where

$$J(\omega, \xi, SX_n, \tau) \equiv \frac{1}{\omega\sigma\sqrt{2\pi(\tau-\xi)}} \exp\left\{ -\frac{[\phi(\tau-\xi) + \ln \frac{SX_n}{\omega}]^2}{2\sigma^2(\tau-\xi)} \right\}$$

and

$$\phi = r - q - \lambda k - \frac{\sigma^2}{2}.$$

Appendix 4.3. Deriving Proposition 4.4.5

Rewrite the result in Proposition 4.4.4 as

$$\begin{aligned} C(S,\tau) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ C_E[SX_n e^{-\lambda k \tau}, K, 1, r, q, \tau, \sigma^2] \right\} \\ &\quad + (A_1^{(1)} + A_1^{(2)} - A_1^{(3)}) - (A_2^{(1)} + A_2^{(2)} - A_2^{(3)}) - (J_1 - J_2) \end{aligned}$$

where

$$\begin{aligned} A_1^{(1)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ q \int_0^\tau (\tau-\xi)^n f(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\}, \\ A_1^{(2)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \lambda[k+1] \int_0^\tau (\tau-\xi)^n f(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\}, \\ A_1^{(3)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ n \int_0^\tau (\tau-\xi)^{n-1} f(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\}, \end{aligned}$$

$$A_2^{(1)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ r \int_0^\tau (\tau-\xi)^n K g(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\},$$

$$A_2^{(2)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \lambda \int_0^\tau (\tau-\xi)^n K g(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\},$$

$$A_2^{(3)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ n \int_0^\tau (\tau-\xi)^{n-1} K g(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2) d\xi \right\},$$

$$J_1 = \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \times \left[\int_0^1 G(Y) \int_{Kb(\xi)}^{Kb(\xi)/Y} C(\omega Y, \xi) J(\omega, \xi, SX_n, \tau) d\omega dY \right] d\xi \right\},$$

and

$$J_2 = \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \times \left[\int_1^\infty G(Y) \int_{Kb(\xi)/Y}^{Kb(\xi)} (\omega Y - K) J(\omega, \xi, SX_n, \tau) d\omega dY \right] d\xi \right\},$$

with

$$f(SX_n e^{-\lambda(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2) \equiv SX_n e^{-\lambda k(\tau-\xi)} e^{-(q+\lambda)(\tau-\xi)} N[d_1(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)],$$

and

$$g(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2) \equiv e^{-(r+\lambda)(\tau-\xi)} N[d_2(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)].$$

Rearrange $C(S,\tau)$ to obtain the form

$$C(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ [SX_n e^{-\lambda k\tau}, K, 1, r, q, \tau, \sigma^2] \right\} + [A_1^{(1)} - A_2^{(1)}] - J_1 + \{A_1^{(2)} - A_1^{(3)} - (A_2^{(2)} - A_2^{(3)}) + J_2\}.$$

We shall now simplify the term $\{A_1^{(2)} - A_1^{(3)} - (A_2^{(2)} - A_2^{(3)}) + J_2\}$.

A4.3.1. Simplifying J_2. Consider the integral

$$I(Y,\xi) = \int_{Kb(\xi)/Y}^{Kb(\xi)} (\omega Y - K) J(\omega, \xi, SX_n, \tau) d\omega,$$

where

$$J(\omega, \xi, SX_n, \tau) = \frac{1}{\omega\sigma\sqrt{2\pi(\tau-\xi)}} \exp\left\{ \frac{-[(r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi) + \ln\frac{SX_n}{\omega}]^2}{2\sigma^2(\tau-\xi)} \right\}.$$

Making the change of variable $\omega = e^u$, $I(Y,\xi)$ becomes

$$\begin{aligned} I(Y,\xi) &= \frac{Y}{\sigma\sqrt{2\pi(\tau-\xi)}} \\ &\quad \times \int_{\ln[Kb(\xi)/Y]}^{\ln Kb(\xi)} e^u \exp\left\{ \frac{-[(r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi) + \ln SX_n - u]^2}{2\sigma^2(\tau-\xi)} \right\} du \\ &\quad - \frac{K}{\sigma\sqrt{2\pi(\tau-\xi)}} \\ &\quad \times \int_{\ln[Kb(\xi)/Y]}^{\ln Kb(\xi)} \exp\left\{ \frac{-[(r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi) + \ln SX_n - u]^2}{2\sigma^2(\tau-\xi)} \right\} du. \end{aligned}$$

These integrals can be simplified using the result in equation (A2.1.3), with $\alpha_1 = \ln\frac{Kb(\xi)}{Y}$, $\alpha_2 = \ln Kb(\xi)$, $\hat{p} = 2\sigma^2(\tau-\xi)$, $\hat{q} = (r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi) + \ln SX_n$, and $\hat{a} = 1, 0$ for the first and second integrals respectively. Hence $I(Y,\xi)$ can be simplified to

$$\begin{aligned} I(Y,\xi) &= \frac{Y}{\sigma\sqrt{2\pi(\tau-\xi)}} \sqrt{\hat{p}\pi} \exp\left\{ \frac{(4\hat{q}+\hat{p})}{4} \right\} \\ &\qquad \times \left\{ N[\hat{f}(\ln Kb(\xi))] - N\left[\hat{f}\left(\ln\frac{Kb(\xi)}{Y}\right)\right] \right\} \\ &\quad - \frac{K}{\sigma\sqrt{2\pi(\tau-\xi)}} \sqrt{\hat{p}\pi} \left\{ N[\hat{g}(\ln Kb(\xi))] - N\left[\hat{g}\left(\ln\frac{Kb(\xi)}{Y}\right)\right] \right\}, \end{aligned}$$

where

$$\hat{f}(u) = \sqrt{\frac{2}{\hat{p}}} \left(\frac{2u - (2\hat{q}+\hat{p})}{2} \right),$$

and

$$\hat{g}(u) = \sqrt{\frac{2}{\hat{p}}}(u - \hat{q}).$$

Applying the definitions of $\hat{p}$ and $\hat{q}$, we find that

$$\frac{\sqrt{\hat{p}\pi}}{\sigma\sqrt{2\pi(\tau-\xi)}} = \frac{\sqrt{2\sigma^2(\tau-\xi)}}{\sigma\sqrt{2(\tau-\xi)}} = 1, \tag{A4.3.1}$$

and

$$\exp\left\{\frac{4\hat{q}+\hat{p}}{4}\right\} = SX_n e^{-\lambda k(\tau-\xi)} e^{r(\tau-\xi)} e^{-q(\tau-\xi)}. \tag{A4.3.2}$$

Furthermore, the functions $\hat{f}(u)$ and $\hat{g}(u)$ simplify to produce

$$\hat{f}(u) = -d_1(SX_n e^{-\lambda k(\tau-\xi)}, e^u, r, q, \tau-\xi, \sigma^2), \tag{A4.3.3}$$

and

$$\hat{g}(u) = -d_2(SX_n e^{-\lambda k(\tau-\xi)}, e^u, r, q, \tau-\xi, \sigma^2). \tag{A4.3.4}$$

Substituting equations (A4.3.1)-(A4.3.4) into $I(Y,\xi)$, we have

$$\begin{aligned} I(Y,\xi) &= YSX_n e^{-\lambda k(\tau-\xi)} e^{r(\tau-\xi)} e^{-q(\tau-\xi)} \\ &\quad \times\Bigg\{ N\left[-d_1\left(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2\right)\right] \\ &\quad -N\left[-d_1\left(SX_n e^{-\lambda k(\tau-\xi)}, \frac{Kb(\xi)}{Y}, r, q, \tau-\xi, \sigma^2\right)\right]\Bigg\} \\ &\quad -K\Bigg\{ N\left[-d_2\left(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2\right)\right] \\ &\quad -N\left[-d_2\left(SX_n e^{-\lambda k(\tau-\xi)}, \frac{Kb(\xi)}{Y}, r, q, \tau-\xi, \sigma^2\right)\right]\Bigg\}. \end{aligned}$$

Using the relationship $N(-x) = 1 - N(x)$, the expression for J_2 becomes

$$\begin{aligned} J_2 &= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \Bigg\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \\ &\quad \times \int_1^\infty G(Y)\Big[J_2^{(1)}(Y,\xi,X_n) - J_2^{(2)}(Y,\xi,X_n) \\ &\quad -(J_2^{(3)}(Y,\xi,X_n) - J_2^{(4)}(Y,\xi,X_n))\Big] dY d\xi \Bigg\}, \end{aligned}$$

where

$$J_2^{(1)}(Y,\xi,X_n) = YSX_n e^{-\lambda k(\tau-\xi)} e^{-q(\tau-\xi)} N\left[d_1\left(SX_n e^{-\lambda k(\tau-\xi)}, \frac{Kb(\xi)}{Y}, r, q, \tau-\xi, \sigma^2\right)\right],$$

$$J_2^{(2)}(Y,\xi,X_n) = YSX_n e^{-\lambda k(\tau-\xi)} e^{-q(\tau-\xi)} N\left[d_1\left(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2\right)\right],$$

$$J_2^{(3)}(Y,\xi,X_n) = Ke^{-r(\tau-\xi)} N\left[d_2\left(SX_n e^{-\lambda k(\tau-\xi)}, \frac{Kb(\xi)}{Y}, r, q, \tau-\xi, \sigma^2\right)\right]$$

and

$$J_2^{(4)}(Y,\xi,X_n) = Ke^{-r(\tau-\xi)} N\left[d_2\left(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2\right)\right].$$

A4.3.2. Simplifying $A_1^{(2)}$ and $A_2^{(2)}$. First recall that

$$k \equiv \int_0^\infty (Y-1)G(Y)dY = \int_0^\infty YG(Y)dY - 1.$$

Substituting this into $A_1^{(2)}$, we have

$$\begin{aligned}
A_1^{(2)} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \lambda \left(\int_0^\infty YG(Y)dY \right) \right. \\
&\qquad \left. \times \int_0^\tau (\tau-\xi)^n f(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2), d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^\infty G(Y) Y S X_n e^{-\lambda k(\tau-\xi)} e^{-q(\tau-\xi)} \right. \\
&\qquad \left. \times N[d_1(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)] dY d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^1 G(Y) J_2^{(2)}(Y,\xi,X_n) dY d\xi \right\} \\
&\qquad + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_1^\infty G(Y) J_2^{(2)}(Y,\xi,X_n) dY d\xi \right\}.
\end{aligned}$$

For $A_2^{(2)}$, we note that by the properties of $G(Y)$,

$$\int_0^\infty A_2^{(2)} G(Y)dY = A_2^{(2)},$$

since $A_2^{(2)}$ does not involve Y. Hence $A_2^{(2)}$ can be rewritten as

$$\begin{aligned}
A_2^{(2)} &= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n \right. \\
&\qquad \left. \times \int_0^\infty KG(Y) g(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2) dY d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \int_0^\infty (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^\infty G(Y) K e^{-r(\tau-\xi)} \right. \\
&\qquad \left. \times N[d_2(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)] dY d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_0^1 G(Y) J_2^{(4)}(Y,\xi,X_n) dY d\xi \right\} \\
&\qquad + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_\tau^{(n)} \left\{ \int_0^\tau (\tau-\xi)^n e^{-\lambda(\tau-\xi)} \int_1^\infty G(Y) J_2^{(4)}(Y,\xi,X_n) dY d\xi \right\}.
\end{aligned}$$

A4.3.3. Simplifying $A_1^{(3)}$ and $A_2^{(3)}$. Since the first term in the summations for $A_1^{(3)}$ and $A_2^{(3)}$ are zero[5], we can rewrite them as sums beginning at $n = 1$. First, $A_1^{(3)}$ becomes

$$\begin{aligned}
A_1^{(3)} &= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)} \left\{ n \int_0^{\tau} (\tau - \xi)^{n-1} f(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau - \xi, \sigma^2) d\xi \right\} \\
&= \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \mathbb{E}_{\tau}^{(n-1)} \left\{ \int_0^{\infty} G(Y) \right. \\
&\qquad \times \left. \int_0^{\tau} (\tau - \xi)^{n-1} f(YSX_{n-1} e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau - \xi, \sigma^2) d\xi dY \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)} \left\{ \int_0^{\tau} (\tau - \xi)^n e^{-\lambda(\tau-\xi)} \int_0^{\infty} G(Y) SX_n Y e^{-\lambda k(\tau-\xi)} e^{-q(\tau-\xi)} \right. \\
&\qquad \left. \times N\left[d_1 \left(SX_n e^{-\lambda k(\tau-\xi)}, \frac{Kb(\xi)}{Y}, r, q, \tau - \xi, \sigma^2 \right) \right] dY d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)} \left\{ \int_0^{\tau} (\tau - \xi)^n e^{-\lambda(\tau-\xi)} \int_0^{1} G(Y) J_2^{(1)}(Y, \xi, X_n) dY d\xi \right\} \\
&\quad + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)} \left\{ \int_0^{\tau} (\tau - \xi)^n e^{-\lambda(\tau-\xi)} \int_1^{\infty} G(Y) J_2^{(1)}(Y, \xi, X_n) dY d\xi \right\}.
\end{aligned}$$

Similarly for $A_2^{(3)}$,

$$\begin{aligned}
A_2^{(3)} &= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)} \left\{ n \int_0^{\tau} (\tau - \xi)^{n-1} g(SX_n e^{-\lambda(\tau-\xi)}, Kb(\xi), r, q, \tau - \xi, \sigma^2) d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)} \left\{ \int_0^{\tau} (\tau - \xi)^n e^{-\lambda(\tau-\xi)} \int_0^{\infty} G(Y) K e^{-r(\tau-\xi)} \right. \\
&\qquad \left. \times N\left[d_2 \left(SX_n e^{-\lambda k(\tau-\xi)}, \frac{Kb(\xi)}{Y}, r, q, \tau - \xi, \sigma^2 \right) \right] dY d\xi \right\} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)} \left\{ \int_0^{\tau} (\tau - \xi)^n e^{-\lambda(\tau-\xi)} \int_0^{1} G(Y) J_2^{(3)}(Y, \xi, X_n) dY d\xi \right\} \\
&\quad + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)} \left\{ \int_0^{\tau} (\tau - \xi)^n e^{-\lambda(\tau-\xi)} \int_1^{\infty} G(Y) J_2^{(3)}(Y, \xi, X_n) dY d\xi \right\}.
\end{aligned}$$

A4.3.4. Obtaining Gukhal's Expression. Combining the results from Sections A4.3.1-A4.3.3, the integral equation for the American call price becomes

$$C(S, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}_{\tau}^{(n)} \left\{ C_E[SX_n e^{-\lambda k\tau}, K, 1, r, q, \tau, \sigma^2] \right\} + [A_1^{(1)} - A_2^{(1)}] - J_1 + Q,$$

[5]The first term in each sum is multiplied by n, and the sums begin with $n = 0$.

where

$$
\begin{aligned}
Q &= \lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Big\{\int_0^\tau(\tau-\xi)^n e^{-\lambda(\tau-\xi)} \\
&\qquad\times\int_0^1 G(Y)\Big[J_2^{(2)}(Y,\xi,X_n)-J_2^{(1)}(Y,\xi,X_n) \\
&\qquad\qquad -(J_2^{(4)}(Y,\xi,X_n)-J_2^{(3)}(Y,\xi,X_n))\Big]dYd\xi\Big\}.
\end{aligned}
$$

To demonstrate that our findings are identical to those of Gukhal (2001), we shall re-express the term $[J_2^{(2)}-J_2^{(1)}-(J_2^{(4)}-J_2^{(3)})]$ as an integral over the kernel $J(\omega,\xi,SX_n,\tau)$. Comparing the expression for Q with the simplification of J_2 in Section A4.3.1, we can readily conclude that

$$
\begin{aligned}
Q &= \lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Big\{\int_0^\tau(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\
&\qquad\times\left[\int_0^1 G(Y)\int_{Kb(\xi)}^{Kb(\xi)/Y}(\omega Y-K)J(\omega,\xi,SX_n,\tau)d\omega dY\right]d\xi\Big\}.
\end{aligned}
$$

Finally, substituting for $A_1^{(1)}$, $A_2^{(1)}$, J and Q, the integral equation for $C(S,\tau)$ becomes

$$
\begin{aligned}
C(S,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_\tau^{(n)}\left\{C_E[SX_ne^{-\lambda k\tau},K,1,r,q,\tau,\sigma^2]\right\} \\
&+\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Big\{\int_0^\tau(\tau-\xi)^n e^{-\lambda(\tau-\xi)}qSX_ne^{-\lambda k(\tau-\xi)}e^{-q(\tau-\xi)} \\
&\qquad\times N[d_1(SX_ne^{-\lambda k(\tau-\xi)},Kb(\xi),r,q,\tau-\xi,\sigma^2)]d\xi\Big\} \\
&-\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Big\{\int_0^\tau(\tau-\xi)^n e^{-\lambda(\tau-\xi)}rKe^{-r(\tau-\xi)} \\
&\qquad\times N[d_2(SX_ne^{-\lambda k(\tau-\xi)},Kb(\xi),r,q,\tau-\xi,\sigma^2)]d\xi\Big\} \\
&-\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\Big\{\int_0^\tau(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\
&\times\left[\int_0^1 G(Y)\int_{Kb(\xi)}^{Kb(\xi)/Y}[C(\omega Y,\xi)-(\omega Y-K)]J(\omega,\xi,SX_n,\tau)d\omega dY\right]d\xi\Big\},
\end{aligned}
$$

which is the result given in Proposition 4.4.5.

Appendix 4.4. American Call Evaluation for Log-Normal Jumps

Consider the case where $G(Y)$ is given by

$$G(Y)=\frac{1}{Y\delta\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\ln Y-(\gamma-\frac{\delta^2}{2})}{\delta}\right)^2\right\},$$

which subsequently implies that

$$\mathbb{E}_\tau^{(n)}\{f(X_n)\}=\int_0^\infty f(X_n)\frac{1}{X_n\delta\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\ln X_n-n(\gamma-\frac{\delta^2}{2})}{\delta\sqrt{n}}\right)^2\right\}dX_n.$$

We shall use this to evaluate all of the $\mathbb{E}_\tau^{(n)}$ operators in Proposition 4.4.5.

A4.4.1. European Component. Using the results from Merton (1976), the European component becomes

$$\sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\mathbb{E}_\tau^{(n)}\left\{C_E[SX_ne^{-\lambda k\tau},K,1,r,q,\tau,\sigma^2]\right\}$$
$$=\sum_{n=0}^{\infty}\frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}C_E[S,K,1,r_n(\tau),q,\tau,v_n^2(\tau)],$$

where $\lambda'=\lambda(1+k)$, $r_n(\tau)=r-\lambda k+n\gamma/\tau$, and $v_n^2(\tau)=\sigma^2+n\delta^2/\tau$, with C_E as defined in Proposition 4.4.1.

A4.4.2. Early Exercise Premium - First Term. Consider the first part of the early exercise premium, given by

$$\begin{aligned}C_A^{(1)}(S,\tau) &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}_\tau^{(n)}\left\{\int_0^\tau(\tau-\xi)^ne^{-\lambda(\tau-\xi)}qSX_ne^{-\lambda k(\tau-\xi)}e^{-q(\tau-\xi)}\right.\\ &\qquad\left.\times N[d_1(SX_ne^{-\lambda k(\tau-\xi)},Kb(\xi),r,q,\tau-\xi,\sigma)]\right\}d\xi\\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_0^\tau(\tau-\xi)^ne^{-\lambda(\tau-\xi)}qSe^{-\lambda k(\tau-\xi)}e^{-q(\tau-\xi)}\\ &\qquad\times\mathbb{E}_\tau^{(n)}\{X_nN[d_1(SX_ne^{-\lambda k(\tau-\xi)},Kb(\xi),r,q,\tau-\xi,\sigma^2)]\}d\xi.\end{aligned}$$

Referring to Chiarella (2003), we can show that

$$\mathbb{E}_\tau^{(n)}\{X_nN[d_1(SX_ne^{-\lambda k(\tau-\xi)},Kb(\xi),r,q,\tau-\xi,\sigma^2)]\}$$
$$=e^{n\gamma}N[d_1(S,Kb(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))].$$

Noting that $e^{n\gamma} = (k+1)^n$, $C_A^{(1)}$ becomes

$$C_A^{(1)}(S,\tau) = \sum_{n=0}^{\infty} \frac{[\lambda']^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-\lambda'(\tau-\xi)} q S e^{-q(\tau-\xi)} \times N[d_1(S, Kb(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi))]d\xi.$$

A4.4.3. Early Exercise Premium - Second Term. Next we consider the second part of the early exercise premium, which is given as

$$\begin{aligned} C_A^{(2)}(S,\tau) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\tau}^{(n)} \Bigg\{ \int_0^{\tau} (\tau-\xi)^n e^{-\lambda(\tau-\xi)} r K e^{-r(\tau-\xi)} \\ &\qquad \times N[d_2(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)]d\xi \Bigg\} \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-\lambda(\tau-\xi)} r K e^{-r(\tau-\xi)} \\ &\qquad \times \mathbb{E}_{\tau}^{(n)}\{N[d_2(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)]\}d\xi. \end{aligned}$$

Once again referring to Chiarella (2003), we find that

$$\begin{aligned} &\mathbb{E}_{\tau}^{(n)}\{N[d_2(SX_n e^{-\lambda k(\tau-\xi)}, Kb(\xi), r, q, \tau-\xi, \sigma^2)]\} \\ &\quad = N[d_2(S, Kb(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi))]. \end{aligned}$$

Thus $C_A^{(2)}$ evaluates to

$$\begin{aligned} C_A^{(2)}(S,\tau) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-\lambda(\tau-\xi)} r K e^{-r_n(\tau-\xi)(\tau-\xi)} e^{-\lambda k(\tau-\xi)+n\gamma} \\ &\qquad \times N[d_2(S, Kb(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi))]d\xi \\ &= \sum_{n=0}^{\infty} \frac{[\lambda']^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-\lambda'(\tau-\xi)} r K e^{-r_n(\tau-\xi)(\tau-\xi)} \\ &\qquad \times N[d_2(S, Kb(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi))]d\xi. \end{aligned}$$

A4.4.4. Cost Term from Downward Jumps. The final term to consider is the cost incurred when S jumps from the stopping region into the continuation region. We rewrite this term as

$$C_A^{(3)}(S,\tau) = \lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)}\Bigg[\int_0^1 G(Y) \times\int_{Kb(\xi)}^{Kb(\xi)/Y}[C(\omega Y,\xi)-(\omega Y-K)]\mathbb{E}_{\tau}^{(n)}\{J(\omega,\xi,SX_n,\tau)\}d\omega dY\Bigg]d\xi,$$

and begin by evaluating the $\mathbb{E}_{\tau}^{(n)}$ operator. From the definition of J, we have

$$\mathbb{E}_{\tau}^{(n)}\{J(\omega,\xi,SX_n,\tau)\} = \int_0^{\infty}\frac{1}{X_n\delta\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{[\ln X_n-n(\gamma-\frac{\delta^2}{2})]^2}{\delta^2 n}\right\} \times\frac{1}{\omega\sigma\sqrt{2\pi(\tau-\xi)}} \times\exp\left\{\frac{-[(r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi)+\ln\frac{SX_n}{\omega}]^2}{2\sigma^2(\tau-\xi)}\right\}dX_n.$$

Making the change of variable $x_n = \ln X_n$, the expectation becomes

$$\mathbb{E}_{\tau}^{(n)}\{J(\omega,\xi,SX_n,\tau)\} = \frac{1}{\omega\sigma\sqrt{2\pi(\tau-\xi)}}\frac{1}{\delta\sqrt{2\pi}} \times\int_{-\infty}^{\infty}\exp\left\{-\frac{[x_n+\hat{p}]^2}{\alpha_1}-\frac{[x_n+\hat{q}]^2}{\alpha_2}\right\}dx_n,$$

where $\alpha_1 = 2\delta^2 n$, $\alpha_2 = 2\sigma^2(\tau-\xi)$, $\hat{p} = -n\left(\gamma-\frac{\sigma^2}{2}\right)$, and $\hat{q} = \ln\frac{S}{\omega}+\left(r-q-\lambda k-\frac{\sigma^2}{2}\right)(\tau-\xi)$. Using the integration result in equation (A2.1.4), this integral can be evaluated to give

$$\begin{aligned}
&\mathbb{E}_{\tau}^{(n)}\{J(\omega,\xi,SX_n,\tau)\}\\
&= \frac{1}{\omega\sigma\sqrt{2\pi(\tau-\xi)}}\frac{1}{\delta\sqrt{2\pi n}}\sqrt{\frac{\pi\alpha_1\alpha_2}{\alpha_1+\alpha_2}}\exp\left\{-\frac{(\hat{p}-\hat{q})^2}{\alpha_1+\alpha_2}\right\}\\
&= \frac{1}{\omega\sqrt{2\pi}\sqrt{\delta^2 n+\sigma^2(\tau-\xi)}}\\
&\qquad\times\exp\left\{-\frac{[\ln\frac{S}{\omega}+(r-q-\lambda k-\frac{\sigma^2}{2})(\tau-\xi)+n\gamma-\frac{n\delta^2}{2}]^2}{2(\sigma^2(\tau-\xi)+\delta^2 n)}\right\}\\
&= \frac{1}{\omega\sqrt{2\pi(\tau-\xi)v_n^2(\tau-\xi)}}\\
&\qquad\times\exp\left\{-\frac{[\ln\frac{S}{\omega}+(r_n(\tau-\xi)-q-\frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\}.
\end{aligned}$$

Thus the expression for $C_A^{(3)}$ becomes

$$\begin{aligned}
C_A^{(3)}(S,\tau) &= \lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\
&\quad\times\int_0^1 \frac{1}{Y\delta\sqrt{2\pi}}\exp\left\{-\frac{[\ln Y-(\gamma-\frac{\delta^2}{2})]^2}{2\delta^2}\right\} \\
&\quad\times\int_{Kb(\xi)}^{Kb(\xi)/Y}\frac{[C(\omega Y,\xi)-(\omega Y-K)]}{\omega v_n(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
&\quad\times\exp\left\{-\frac{\ln\frac{S}{\omega}+(r_n(\tau-\xi)-q-\frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\}d\omega dY d\xi.
\end{aligned}$$

A4.4.5. Final Result - Proposition 4.5.1. Combining Sections A4.4.1-A4.4.4, we find that the integral equation for $C(S,\tau)$ in the case of log-normal jumps is given by

$$\begin{aligned}
C(S,\tau) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}C_E[S,K,1,r_n(\tau),q,\tau,V_n^2(\tau)] \\
&\quad+\sum_{n=0}^{\infty}\frac{[\lambda']^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-\lambda'(\tau-\xi)}qSe^{-q(\tau-\xi)} \\
&\qquad\times N[d_1(S,Kb(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))]d\xi \\
&\quad-\sum_{n=0}^{\infty}\frac{[\lambda']^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-\lambda'(\tau-\xi)}rKe^{-r_n(\tau-\xi)(\tau-\xi)} \\
&\qquad\times N[d_2(S,Kb(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))]d\xi \\
&\quad-\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \\
&\quad\times\int_0^1 \frac{1}{Y\delta\sqrt{2\pi}}\exp\left\{-\frac{[\ln Y-(\gamma-\frac{\delta^2}{2})]^2}{2\delta^2}\right\} \\
&\quad\times\int_{Kb(\xi)}^{Kb(\xi)/Y}\frac{[C(\omega Y,\xi)-(\omega Y-K)]}{\omega v_n(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
&\quad\times\exp\left\{-\frac{[\ln\frac{S}{\omega}+(r_n(\tau-\xi)-q-\frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\}d\omega dY d\xi,
\end{aligned}$$

which is the result in Proposition 4.5.1.

Appendix 4.5. Simplifying the Cost Term

Recall the term $C_A^{(3)}(S,\tau)$ from Section A4.4.4, which can be expressed as

$$C_A^{(3)}(S,\tau) = \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} I(S,\tau,\xi) d\xi,$$

where

$$\begin{aligned} I(S,\tau,\xi) \quad &\equiv \quad \int_0^1 \frac{1}{Y\delta\sqrt{2\pi}} \exp\left\{-\frac{[\ln Y - (\gamma - \frac{\delta^2}{2})]^2}{2\delta^2}\right\} \\ &\times \int_{Kb(\xi)}^{Kb(\xi)/Y} \frac{[C(\omega Y,\xi) - (\omega Y - K)]}{\omega v_n(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\ &\times \exp\left\{-\frac{[\ln \frac{S}{\omega} + (r_n(\tau-\xi) - q - \frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\} d\omega dY. \end{aligned}$$

To simplify $I(S,\tau,\xi)$ we need to make a change of variable that reduces the dependence of C on the integration variables. If we let $Z = \omega Y$ in the innermost integral, $I(S,\tau,\xi)$ becomes

$$\begin{aligned} I(S,\tau,\xi) \quad &= \quad \int_0^1 \int_{YKb(\xi)}^{Kb(\xi)} \frac{[C(Z,\xi) - (Z - K)]}{YZv_n(\tau-\xi)\delta 2\pi\sqrt{\tau-\xi}} \exp\left\{-\frac{[\ln Y - (\gamma - \frac{\delta^2}{2})]^2}{2\delta^2}\right\} \\ &\times \exp\left\{-\frac{[\ln \frac{SY}{Z} + (r_n(\tau-\xi) - q - \frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\} dZdY. \end{aligned}$$

Changing the order of integration gives

$$\begin{aligned} I(S,\tau,\xi) \quad &= \quad \int_0^{Kb(\xi)} \int_0^{\frac{Z}{Kb(\xi)}} \frac{[C(Z,\xi) - (Z - K)]}{YZv_n(\tau-\xi)\delta 2\pi\sqrt{\tau-\xi}} \exp\left\{-\frac{[\ln Y - (\gamma - \frac{\delta^2}{2})]^2}{2\delta^2}\right\} \\ &\times \exp\left\{-\frac{[\ln \frac{SY}{Z} + (r_n(\tau-\xi) - q - \frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_n^2(\tau-\xi)(\tau-\xi)}\right\} dYdZ. \end{aligned}$$

Letting $y = \ln Y$, we have

$$\begin{aligned} I(S,\tau,\xi) \quad &= \quad \int_0^{Kb(\xi)} \frac{[C(Z,\xi) - (Z - K)]}{Zv_n(\tau-\xi)\delta 2\pi\sqrt{\tau-\xi}} \\ &\times \int_{-\infty}^{\ln \frac{Z}{Kb(\xi)}} \exp\left\{-\frac{[y+\hat{p}]^2}{\alpha_1} - \frac{[y+\hat{q}]^2}{\alpha_2}\right\} dydZ, \end{aligned}$$

where $\alpha_1 = 2\delta^2$, $\alpha_2 = 2v_n^2(\tau-\xi)(\tau-\xi)$, $\hat{p} = -(\gamma - \frac{\delta^2}{2})$, and $\hat{q} = -\ln\frac{S}{Z} + (r_n(\tau-\xi) - q - \frac{v_n^2(\tau-\xi)}{2})(\tau-\xi)$. It is readily shown that

$$\begin{aligned}
I(S,\tau,\xi) &= \int_0^{Kb(\xi)} \frac{[C(Z,\xi)-(Z-K)]}{Zv_n(\tau-\xi)\delta 2\pi\sqrt{\tau-\xi}} \exp\left\{\frac{-(\hat{p}-\hat{q})^2}{\alpha_1+\alpha_2}\right\} \\
&\quad \times \int_{-\infty}^{\ln[Z/Kb(\xi)]} \exp\left\{-\left(\frac{y\sqrt{\alpha_1+\alpha_2}+\frac{\hat{p}\alpha_2+\hat{q}\alpha_1}{\sqrt{\alpha_1+\alpha_2}}}{\sqrt{\alpha_1\alpha_2}}\right)^2\right\} dydZ.
\end{aligned}$$

Make the further change of variable $x = \sqrt{2}\left(y\sqrt{\alpha_1+\alpha_2}+\frac{\hat{p}\alpha_2+\hat{q}\alpha_1}{\sqrt{\alpha_1+\alpha_2}}\right)/\sqrt{\alpha_1\alpha_2}$, and note that when $y = \ln\frac{Z}{Kb(\xi)}$,

$$\begin{aligned}
x &= \frac{\delta^2\ln\frac{S}{Z} + \left[(\ln\frac{Z}{Kb(\xi)})v_{n+1}^2(\tau-\xi) + \delta^2[r_n(\tau-\xi)-q] - \gamma v_n^2(\tau-\xi)\right](\tau-\xi)}{v_n(\tau-\xi)v_{n+1}(\tau-\xi)\delta(\tau-\xi)} \\
&\equiv D_n(S,Kb(\xi),\tau-\xi,Z).
\end{aligned}$$

Thus we have

$$\begin{aligned}
I(S,\tau,\xi) &= \int_0^{Kb(\xi)} \frac{[C(Z,\xi)-(Z-K)]}{Zv_n(\tau-\xi)\delta 2\pi\sqrt{\tau-\xi}} \\
&\quad \times \exp\left\{-\frac{[\ln\frac{S}{Z} + (r_{n+1}(\tau-\xi) - q - \frac{v_{n+1}^2(\tau-\xi)}{2})(\tau-\xi)]^2}{2v_{n+1}^2(\tau-\xi)(\tau-\xi)}\right\} \\
&\quad \times \int_{-\infty}^{D_n(S,Kb(\xi),\tau-\xi,Z)} e^{-\frac{x^2}{2}}\sqrt{\frac{\alpha_1\alpha_2}{2(\alpha_1+\alpha_2)}}dxdZ \\
&= \int_0^{Kb(\xi)} \frac{[C(Z,\xi)-(Z-K)]}{Zv_n(\tau-\xi)\delta 2\pi\sqrt{\tau-\xi}} \frac{2\delta v_n(\tau-\xi)\sqrt{\tau-\xi}}{2\sqrt{\delta^2+v_n^2(\tau-\xi)(\tau-\xi)}} \\
&\quad \times\sqrt{2\pi}\exp\left\{-\frac{1}{2}[d_2(S,Z,r_{n+1}(\tau-\xi),q,\tau-\xi,v_{n+1}^2(\tau-\xi))]^2\right\} \\
&\quad \times N[D_n[S,Kb(\xi),\tau-\xi,Z]dZ \\
&= \int_0^{Kb(\xi)} \frac{[C(Z,\xi)-(Z-K)]}{Zv_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
&\quad \times\exp\left\{-\frac{1}{2}[d_2(S,Z,r_{n+1}(\tau-\xi),q,\tau-\xi,v_{n+1}^2(\tau-\xi))]^2\right\} \\
&\quad \times N[D_n(S,Kb(\xi),\tau-\xi,Z)]dZ.
\end{aligned}$$

Substituting for $I(S,\tau,\xi)$ into the expression for $C_A^{(3)}(S,\tau)$ and rescaling the integral with respect to Z to have an upper limit of 1 gives the result in Proposition 4.5.2.

Appendix 4.6. Deriving the Perpetual American Call under Jump-Diffusion

Since the perpetual American call does not depend on τ, its price , $C(S,\infty)$, satisfies the time-invariant integro-differential equation

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2C}{dS^2} + (r-q-\lambda k)S\frac{dC}{dS} - rC + \lambda \int_0^\infty [C(SY,\infty) - C(S,\infty)]G(Y)dY, \quad \text{(A4.6.1)}$$

subject to the boundary conditions

$$C(0,\infty) = 0, \quad \text{(A4.6.2)}$$

and

$$C(Kb(\infty),\infty) = K(b(\infty) - 1). \quad \text{(A4.6.3)}$$

To solve the problem defined by (A4.6.1)-(A4.6.3), we assume a solution of the form $C(S,\infty) = S^\alpha$. Substituting this into equation (A4.6.1), we find that

$$\left[\frac{1}{2}\alpha(\alpha-1) + (r-q)\alpha - r + \lambda \int_0^\infty (Y^\alpha - Y)G(Y)dY\right] S^\alpha = 0.$$

Thus non-trivial solutions for $C(S,\infty)$ will occur when

$$\frac{1}{2}\alpha(\alpha-1) + (r-q)\alpha - r + \lambda \int_0^\infty (Y^\alpha - Y)G(Y)dY = 0.$$

We assume that the jump sizes, Y, are log-normally distributed, with the density given by equation (4.5.1). In this case we must evaluate the integral

$$\begin{aligned} \mathbb{E}^{Q_Y}[Y^\alpha] &= \int_0^\infty Y^\alpha G(Y)dY \\ &= \int_0^\infty \frac{Y^\alpha}{Y\delta\sqrt{2\pi}} \exp\left\{\frac{-[\ln Y - (\gamma - \frac{\delta^2}{2})]^2}{2\delta^2}\right\} dY. \end{aligned}$$

Changing the integration variable to $x = \ln Y$, this becomes

$$\begin{aligned} \mathbb{E}^{Q_Y}[Y^\alpha] &= \int_{-\infty}^\infty \frac{e^{\alpha x}}{\delta\sqrt{2\pi}} \exp\left\{\frac{-[x - (\gamma - \frac{\delta^2}{2})]^2}{2\delta^2}\right\} dx \\ &= \frac{1}{\delta\sqrt{2\pi}} \exp\left\{\frac{-(\gamma - \frac{\delta^2}{2})^2}{2\delta^2}\right\} \int_{-\infty}^\infty e^{-\hat{p}x^2 - \hat{q}x} dx, \end{aligned}$$

where $\hat{p} = 1/(2\delta^2)$ and $\hat{q} = -[\delta^2\alpha + (\gamma - \delta^2/2)]/\delta^2$. Using equation (A2.1.2) from Appendix A2.1.2, this evaluates to

$$\begin{aligned}
\mathbb{E}^{Q_Y}[Y^\alpha] &= \frac{1}{\delta\sqrt{2\pi}} \exp\left\{\frac{-(\gamma - \frac{\delta^2}{2})^2}{2\delta^2}\right\} \sqrt{\frac{\pi}{\hat{p}}} e^{\frac{\hat{q}^2}{4\hat{p}}} \\
&= \exp\left\{\frac{-(\gamma - \frac{\delta^2}{2})^2}{2\delta^2}\right\} \exp\left\{\frac{[\delta^2\alpha + (\gamma - \frac{\delta^2}{2})]^2}{2\delta^2}\right\} \\
&= \exp\left\{\gamma\alpha + \frac{\delta^2}{2}\alpha(\alpha - 1)\right\}.
\end{aligned}$$

Since $\mathbb{E}^{Q_Y}[Y] = e^\gamma$, α is the solution to the quadratic equation

$$f(\alpha) = \frac{1}{2}\alpha(\alpha - 1) + (r - q)\alpha - r + \lambda[e^{\gamma\alpha + \frac{\delta^2}{2}\alpha(\alpha-1)} - e^\gamma] = 0. \qquad \text{(A4.6.4)}$$

Equation (A4.6.4) has two roots, which we denote by α_1 and α_2. To satisfy condition (A4.6.2), we require that one of these roots is positive, and the other negative. That is, we must have $\alpha_1 \equiv \alpha_- < 0$ and $\alpha_2 \equiv \alpha_+ > 0$. Since $f(\alpha)$ is quadratic in form[6], and $d^2f/d\alpha^2 > 0$, the condition of having roots with opposite signs will be satisfied whenever $f(0) < 0$, which implies that

$$\lambda[1 - e^\gamma] \le r \qquad \text{(A4.6.5)}$$

for a solution to exist. In this case the solution to the intergo-differential equation (A4.6.1) is

$$C(S, \infty) = A_1 S^{\alpha_-} + A_2 S^{\alpha_+},$$

where A_1 and A_2 are constants. From condition (A4.6.2) it follows that $A_1 = 0$, and by use of condition (A4.6.3) we find that

$$A_2 = \frac{K(b(\infty) - 1)}{[Kb(\infty)]^{\alpha_+}}.$$

Hence the value of the perpetual American call is given by

$$C(S, \infty) = K[b(\infty) - 1]\left(\frac{S}{Kb(\infty)}\right)^{\alpha_+}. \qquad \text{(A4.6.6)}$$

[6]If a quadratic function $f(x)$ satisfies $d^2f/dx^2 > 0$ for all x, then its roots can only be of opposite sign if the function has a negative value at the origin i.e. $f(0) < 0$. Note that while equation (A4.6.4) is not a simple quadratic, the exponential-quadratic term also displays quadratic-type behaviour, and thus $f(\alpha)$ is "quadratic" in the sense that it has at most two roots, and $d^2f/d\alpha^2 > 0$.

To find the free boundary, $b(\infty)$, we note that the optimal value of $b(\infty)$ must maximise the value of the call. Differentiating (A4.6.6) with respect to $b(\infty)$ and setting this to zero, we have

$$\frac{\partial C(S,\infty)}{\partial b(\infty)} = K\left(\frac{S}{Kb(\infty)}\right)^{\alpha_+}\left[1 - \frac{b(\infty)-1}{b(\infty)}\alpha_+\right] = 0,$$

which simplifies to give

$$b(\infty) = \frac{\alpha_+}{\alpha_+ - 1}. \tag{A4.6.7}$$

It is of interest to find the value of the perpetual American call's delta at the free boundary. This is given by

$$\begin{aligned} \left.\frac{\partial C(S,\infty)}{\partial S}\right|_{S=Kb(\infty)} &= \alpha_+\frac{b(\infty)-1}{b(\infty)} \\ &= \alpha_+\frac{\alpha_+ - 1}{\alpha_+}\left[\frac{\alpha_+}{\alpha_+ - 1} - 1\right] \\ &= 1. \end{aligned}$$

Hence the delta for a perpetual American call under jump-diffusion is equal to 1 at the free boundary, and the "smooth pasting" condition is satisfied.

Finally, we shall consider the task of solving equation (A4.6.4) for α_+. Since numerical methods are required, it is of value to determine bounds for α_+. Firstly, consider the quadratic in α given by

$$\frac{1}{2}\alpha(\alpha-1) + (r-q)\alpha - r = 0. \tag{A4.6.8}$$

This has solution

$$\alpha = \frac{-\rho \pm \sqrt{\rho^2 + 2\sigma^2 r}}{\sigma^2},$$

where $\rho = r - q - \sigma^2/2$. Next consider the exponential-quadratic

$$\lambda[e^{\gamma\alpha + \frac{\delta^2}{2}\alpha(\alpha-1)} - e^{\gamma}] = 0, \tag{A4.6.9}$$

whose solutions are $\alpha = 1, -2\gamma/\delta^2$. Since the roots of equation (A4.6.4) are bounded by the roots of (A4.6.8) and (A4.6.9)[7], we conclude that

$$1 \le \alpha_+ \le \frac{-\rho \pm \sqrt{\rho^2 + 2\sigma^2 r}}{\sigma^2}.$$

[7]This result is readily verified graphically. Note that (A4.6.4) = (A4.6.8) + (A4.6.9).

We also note that when $q = 0$, $\alpha_+ = 1$, and the free boundary is no longer finite. Thus it is never optimal to exercise a perpetual American call in the absence of dividends, just as Kim (1990) demonstrates for the pure-diffusion case.

Appendix 4.7. Value of the Free Boundary at Expiry

Evaluating equation (4.5.3) from Proposition 4.5.2 at $S = a(t) = Kb(\tau)$, we have

$$\begin{aligned}
K(b(\tau)-1) = & \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} C_E[Kb(\tau), K, 1, r_n(\tau), q, \tau, v_n^2(\tau)] \\
& + \sum_{n=0}^{\infty} \frac{[\lambda']^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-\lambda'(\tau-\xi)} qKb(\tau) e^{-q(\tau-\xi)} \\
& \times N[d_1(b(\tau), b(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi))] d\xi \\
& - \sum_{n=0}^{\infty} \frac{[\lambda']^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-\lambda'(\tau-\xi)} rKe^{-r_n(\tau-\xi)(\tau-\xi)} \\
& \times N[d_2(b(\tau), b(\xi), r_n(\tau-\xi), q, \tau-\xi, v_n^2(\tau-\xi))] d\xi \\
& - \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^{\tau} (\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)} \int_0^{Kb(\xi)} \frac{[C(Z,\xi)-(Z-K)]}{Zv_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
& \times \exp\left\{-\frac{1}{2}\left[d_2(Kb(\tau), Z, r_{n+1}(\tau-\xi), q, \tau-\xi, v_{n+1}^2(\tau-\xi))\right]^2\right\} \\
& \times N[D_N(Kb(\tau), Kb(\xi), \tau-\xi, Z)] dZ d\xi. \qquad \text{(A4.7.1)}
\end{aligned}$$

Define a particular linear decomposition of $C(Z,\xi)$ as

$$C(Z,\xi) \equiv C_1(Z,\xi) - KC_2(Z,\xi),$$

such that $C_1(Z,0) = ZH(Z-K)$, and $C_2(Z,0) = H(Z-K)$, where H is the Heaviside function. The steps that follow do not depend upon this decomposition, and it is possible to arrive at the same result without it. The linear decomposition simply creates a more "symmetrical" form for the limit calculations.

Noting the expressions for C_E, d_1 and d_2, and dividing throughout by K in equation (A4.7.1), we can rewrite the integral equation for $b(\tau)$ as

$$\begin{aligned}
b(\tau)-1 \;=\; & \sum_{n=0}^{\infty}\frac{e^{-\lambda'\tau}(\lambda\tau)^n}{n!}\Big(b(\tau)e^{-q\tau}N[d_1(b(\tau),1,r_n(\tau),q,\tau,v_n^2(\tau))] \\
& -e^{-r_n(\tau)\tau}N[d_2(b(\tau),1,r_n(\tau),q,\tau,v_n^2(\tau))]\Big) \\
& +\sum_{n=0}^{\infty}\frac{[\lambda']^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-\lambda'(\tau-\xi)}qb(\tau)e^{-q(\tau-\xi)} \\
& \times N[d_1(b(\tau),b(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))]d\xi \\
& -\sum_{n=0}^{\infty}\frac{[\lambda']^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-\lambda'(\tau-\xi)}re^{-r_n(\tau-\xi)} \\
& \times N[d_2(b(\tau),b(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))]d\xi \\
& -\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)}\int_0^{Kb(\xi)}\frac{C_1(Z,\xi)-Z}{KZv_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
& \times\exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r_{n+1}(\tau-\xi),q,\tau-\xi,Z)]^2\right\} \\
& \times N[D_n(Kb(\tau),Kb(\xi),\tau-\xi,Z)]dZd\xi \\
& +\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_0^{\tau}(\tau-\xi)^n e^{-(r+\lambda)(\tau-\xi)}\int_0^{Kb(\xi)}\frac{C_2(Z,\xi)-1}{Zv_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
& \times\exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r_{n+1}(\tau-\xi),q,\tau-\xi,v_{n+1}^2(\tau-\xi))]^2\right\} \\
& \times N[D_n(Kb(\tau),Kb(\xi),\tau-\xi,Z)]dZd\xi.
\end{aligned}$$

Following Kim (1990), we factorise this expression according to

$$b(\tau)-1=\sum_{n=0}^{\infty}\left[b(\tau)\left(f_n^{(1)}(b(\tau),\tau)-g_n^{(1)}(b(\tau),\tau)\right)-\left(f_n^{(2)}(b(\tau),\tau)-g_n^{(2)}(b(\tau),\tau)\right)\right],$$

where

$$\begin{aligned}
f_n^{(1)}(b(\tau),\tau) \;\equiv\; & \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}e^{-q\tau}N[d_1(b(\tau),1,r_n(\tau),q,\tau,v_n^2(\tau))] \\
& +\int_0^{\tau}\frac{[\lambda'(\tau-\xi)]^n e^{-\lambda'(\tau-\xi)}}{n!}qe^{-q(\tau-\xi)} \\
& \times N[d_1(b(\tau),b(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))]d\xi,
\end{aligned}$$

$$
\begin{aligned}
f_n^{(2)}(b(\tau),\tau) &\equiv \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}e^{-r_n(\tau)\tau}N[d_2(b(\tau),1,r_n(\tau),q,\tau,v_n^2(\tau))] \\
&+\int_0^\tau \frac{[\lambda'(\tau-\xi)]^n e^{-\lambda'(\tau-\xi)}}{n!} r e^{-r_n(\tau-\xi)(\tau-\xi)} \\
&\quad \times N[d_2(b(\tau),b(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))]d\xi, \\
g_n^{(1)}(b(\tau),\tau) &\equiv \frac{\lambda}{b(\tau)}\int_0^\tau \frac{[\lambda(\tau-\xi)]^n e^{-\lambda(\tau-\xi)}}{n!}e^{-r(\tau-\xi)} \\
&\quad \times \int_0^{Kb(\xi)} \frac{C_1(Z,\xi)-Z}{KZv_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
&\quad \times \exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r_{n+1}(\tau-\xi),q,\tau-\xi,v_{n+1}^2(\tau-\xi))]^2\right\} \\
&\quad \times N[D_n(Kb(\tau),Kb(\xi),\tau-\xi,Z)]dZd\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
g_n^{(2)}(b(\tau),\tau) &\equiv \lambda\int_0^\tau \frac{[\lambda(\tau-\xi)]^n e^{-\lambda(\tau-\xi)}}{n!}e^{-r(\tau-\xi)} \\
&\quad \times \int_0^{Kb(\xi)} \frac{C_2(Z,\xi)-1}{Zv_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
&\quad \times \exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r_{n+1}(\tau-\xi),q,\tau-\xi,v_{n+1}^2(\tau-\xi))]^2\right\} \\
&\quad \times N[D_n(Kb(\tau),Kb(\xi),\tau-\xi,Z)]dZd\xi.
\end{aligned}
$$

Rearranging the factorised expression we have

$$
b(\tau)-b(\tau)\sum_{n=0}^{\infty}\left[f_n^{(1)}(b(\tau),\tau)-g_n^{(1)}(b(\tau),\tau)\right]=1-\sum_{n=0}^{\infty}\left[f_n^{(2)}(b(\tau),\tau)-g_n^{(2)}(b(\tau),\tau)\right],
$$

and therefore the equation we wish to evaluate in the limit as $\tau \to 0^+$ is

$$
b(\tau)=\frac{1-\sum_{n=0}^{\infty}\left[f_n^{(2)}(b(\tau),\tau)-g_n^{(2)}(b(\tau),\tau)\right]}{1-\sum_{n=0}^{\infty}\left[f_n^{(1)}(b(\tau),\tau)-g_n^{(1)}(b(\tau),\tau)\right]}. \tag{A4.7.2}
$$

Since we know that $b(\tau) \geq 1$ for all $\tau \geq 0$, we shall consider separately the cases $b(0^+) = 1$ and $b(0^+) > 1$.

Firstly, when $b(0^+) = 1$, consider the following limits:

$$
\lim_{\tau\to 0^+}\sum_{n=0}^{\infty} f_n^{(1)}(b(\tau),\tau) = N[0] = \frac{1}{2};
$$

$$\lim_{\tau\to 0^+}\sum_{n=0}^{\infty} f_n^{(2)}(b(\tau),\tau) = N[0] = \frac{1}{2};$$

$$\lim_{\tau\to 0^+}\sum_{n=0}^{\infty} g_n^{(i)}(b(\tau),\tau) = 0, (i = 1,2).$$

The last limit follows from the fact that the integrand is well behaved for small values of τ (as we know that the option price is finite near expiry for a given value of S). Thus we have

$$\lim_{\tau\to 0^+} b(\tau) = \frac{1-\frac{1}{2}}{1-\frac{1}{2}} = 1,$$

and $b(0^+) = 1$ is one possible solution for the free boundary at expiry.

Secondly, when $b(0^+) > 1$, we need to consider:

$$\lim_{\tau\to 0^+}\sum_{n=0}^{\infty} f_n^{(1)}(b(\tau),\tau) = N[\infty] = 1;$$

$$\lim_{\tau\to 0^+}\sum_{n=0}^{\infty} f_n^{(2)}(b(\tau),\tau) = N[\infty] = 1;$$

$$\lim_{\tau\to 0^+}\sum_{n=0}^{\infty} g_n^{(i)}(b(\tau),\tau) = 0, (i = 1,2).$$

Thus

$$\lim_{\tau\to 0^+} b(\tau) = \frac{1-1}{1-1} = \frac{0}{0},$$

an indeterminate form which can be resolved using L'Hopitals's rule, giving

$$\lim_{\tau\to 0} b(\tau) = \lim_{\tau\to 0}\frac{-\frac{\partial}{\partial\tau}\left(\sum_{n=0}^{\infty}[f_n^{(2)}(b(\tau),\tau) - g_n^{(2)}(b(\tau),\tau)]\right)}{-\frac{\partial}{\partial\tau}\left(\sum_{n=0}^{\infty}[f_n^{(1)}(b(\tau),\tau) - g_n^{(1)}(b(\tau),\tau)]\right)}$$

$$= \lim_{\tau\to 0}\frac{\frac{\partial}{\partial\tau}f_0^{(2)}(b(\tau),\tau) - \frac{\partial}{\partial\tau}g_0^{(2)}(b(\tau),\tau) + \sum_{n=1}^{\infty}\left[\frac{\partial}{\partial\tau}f_n^{(2)}(b(\tau),\tau) - \frac{\partial}{\partial\tau}g_n^{(2)}(b(\tau),\tau)\right]}{\frac{\partial}{\partial\tau}f_0^{(1)}(b(\tau),\tau) - \frac{\partial}{\partial\tau}g_0^{(1)}(b(\tau),\tau) + \sum_{n=1}^{\infty}\left[\frac{\partial}{\partial\tau}f_n^{(1)}(b(\tau),\tau) - \frac{\partial}{\partial\tau}g_n^{(1)}(b(\tau),\tau)\right]}. \tag{A4.7.3}$$

We now consider the 8 linear terms within (A4.7.3) individually, finding limits for each. For the first term in the denominator[8],

$$\begin{aligned}
&\frac{\partial}{\partial\tau} f_0^{(1)}(b(\tau),\tau)\\
&\quad = e^{-(\lambda'+q)\tau} N'[d_1(b(\tau),1,r-\lambda k,q,\tau,\sigma^2)]\\
&\qquad \times \frac{\partial}{\partial\tau}[d_1(b(\tau),1,r-\lambda k,q,\tau,\sigma^2)]\\
&\quad -(\lambda'+q)e^{-(\lambda'+q)\tau} N[d_1(b(\tau),1,r-\lambda k,q,\tau,\sigma^2)] + qN[0]\\
&\quad + \int_0^\tau \frac{\partial}{\partial\tau}\left[q e^{-(\lambda'+q)\tau} e^{-q(\tau-\xi)} N[d_1(b(\tau),b(\xi),r-\lambda k,q,\tau-\xi,\sigma^2)]\right] d\xi.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\lim_{\tau\to 0^+} \frac{\partial}{\partial\tau} f_0^{(1)}(b(\tau),\tau) &= -(\lambda'+q)N[\infty] + \frac{q}{2}\\
&= -\left(\lambda' + \frac{q}{2}\right).
\end{aligned}$$

Next for $n \geq 1$ consider

$$\begin{aligned}
\frac{\partial}{\partial\tau} f_n^{(1)}(b(\tau),\tau) &= \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}\left[e^{-q\tau} N'[d_1(b(\tau),1,r_n(\tau),q,\tau,v_n^2(\tau))]\right.\\
&\quad \times \frac{\partial}{\partial\tau}[d_1(b(\tau),1,r_n(\tau),q,\tau,v_n^2(\tau))]\\
&\quad \left. - q e^{-q\tau} N[d_1(b(\tau),1,r_n(\tau),q,\tau,v_n^2(\tau))]\right]\\
&\quad + \frac{e^{-q\tau}}{n!} N[d_1(b(\tau),1,r_n(\tau),\tau,v_n^2(\tau))]\\
&\quad \times [e^{-\lambda'\tau} n(\lambda'\tau)^{n-1}\lambda' + (\lambda'\tau)^n e^{-\lambda'\tau}(-\lambda')]\\
&\quad + \int_0^\tau \frac{\partial}{\partial\tau}\left\{\frac{[\lambda'(\tau-\xi)]^n e^{-\lambda'(\tau-\xi)}}{n!} q e^{-q(\tau-\xi)}\right.\\
&\quad \left. \times N[d_1(b(\tau),b(\xi),r_n(\tau-\xi),q,\tau-\xi,v_n^2(\tau-\xi))]\right\} d\xi.
\end{aligned}$$

We can safely infer that the integral term will tend to zero as $\tau \to 0^+$ for all applicable n values, since all terms under the integral sign are bounded. Thus when $n = 1$ we have

$$\begin{aligned}
\lim_{\tau\to 0^+} \frac{\partial}{\partial\tau} f_1^{(1)}(b(\tau),\tau) &= N[d_1(b(0^+),1,r_1(0),q,0,v_1^2(0))]\lambda'\\
&= \lambda' N\left[\frac{\ln b(0^+) + \gamma + \frac{\delta^2}{2}}{\delta}\right],
\end{aligned}$$

[8]We recall that $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

and when $n > 1$ we have

$$\lim_{\tau \to 0^+} \frac{\partial}{\partial \tau} f_n^{(1)}(b(\tau), \tau) = 0.$$

Next consider the term

$$\begin{aligned}
&\frac{\partial}{\partial \tau} f_0^{(2)}(b(\tau), \tau) \\
&\quad = e^{-(\lambda' + r - \lambda k)\tau} N'[d_2(b(\tau), 1, r - \lambda k, q, \tau, \sigma^2)] \\
&\qquad \times \frac{\partial}{\partial \tau}[d_2(b(\tau), 1, r - \lambda k, q, \tau, \sigma^2)] \\
&\quad -(\lambda' + r - \lambda k) e^{-(\lambda' + r - \lambda k)\tau} N[d_2(b(\tau), 1, r - \lambda k, q, \tau, \sigma^2)] + rN[0] \\
&\quad + \int_0^\tau \frac{\partial}{\partial \tau} \left[r e^{-\lambda'(\tau - \xi)} e^{-(r - \lambda k)(\tau - \xi)} N[d_2(b(\tau), b(\xi), r - \lambda k, q, \tau - \xi, \sigma^2)] \right] d\xi.
\end{aligned}$$

Hence we find that

$$\begin{aligned}
\lim_{\tau \to 0^+} \frac{\partial}{\partial \tau} f_0^{(2)}(b(\tau), \tau) &= -(\lambda' + r - \lambda k) N[\infty] + \frac{r}{2} \\
&= -\left(\lambda + \frac{r}{2}\right).
\end{aligned}$$

In the case where $n > 0$ we have the term

$$\begin{aligned}
\frac{\partial}{\partial \tau} f_n^{(2)}(b(\tau), \tau) &= \frac{e^{-\lambda' \tau} (\lambda' \tau)^n}{n!} \left[e^{-r_n(\tau)\tau} N'[d_2(b(\tau), 1, r_n(\tau), q, \tau, v_n^2(\tau))] \right. \\
&\qquad \frac{\partial}{\partial \tau} \left[d_2(b(\tau), 1, r_n(\tau), q, \tau, v_n^2(\tau)) \right] \\
&\qquad\qquad \left. -(r - \lambda k) e^{-r_n(\tau)\tau} N[d_2(b(t), 1, r_n(\tau), q, \tau, v_n^2(\tau))] \right] \\
&\quad + \frac{e^{-r_n(\tau)\tau}}{n!} N \left[d_2(b(\tau), I, r_n(\tau), q, \tau, v_n^2(\tau)) \right] \\
&\qquad \times \left[e^{-\lambda' \tau} n (\lambda' \tau)^{n-1} \lambda' + (\lambda' \tau)^n e^{-\lambda' \tau} (-\lambda') \right] \\
&\quad + \int_0^\tau \frac{\partial}{\partial \tau} \left\{ \frac{[\lambda'(\tau - \xi)]^n e^{-\lambda'(\tau - \xi)}}{n!} r e^{-r_n(\tau - \xi)(\tau - \xi)} \right. \\
&\qquad \left. \times N[d_2(b(\tau), b(\xi), r_n(\tau - \xi), q, \tau - \xi, v_n^2(\tau - \xi))] \right\} d\xi.
\end{aligned}$$

Again we begin with $n = 1$, finding that

$$\begin{aligned}
\lim_{\tau \to 0^+} \frac{\partial}{\partial \tau} f_1^{(2)}(b(\tau), \tau) &= e^{-\gamma} N[d_2, b(0^+), 1, r_1(0), q, 0, v_1^2(0)] \lambda' \\
&= \lambda N \left[\frac{\ln b(0^+) + \gamma - \frac{\delta^2}{2}}{\delta} \right],
\end{aligned}$$

and when $n > 1$ we have

$$\lim_{\tau\to 0+}\frac{\partial}{\partial\tau}f_n^{(2)}(b(\tau),\tau) = 0.$$

In order to evaluate the derivative $\frac{\partial}{\partial\tau}g_0^{(1)}(b(\tau),\tau)$, we will need to know

$$\begin{aligned}
&\lim_{\xi\to\tau} D_0(Kb(\tau), Kb(\xi), \tau-\xi, Z)\\
&= \lim_{\xi\to\tau}\frac{\delta^2\ln\frac{Kb(\tau)}{Z} + [\ln\frac{Z}{Kb(\xi)}v_1^2(\tau-\xi) + \delta^2[r-\lambda k - q] - \gamma\sigma^2](\tau-\xi)}{\sigma v_1(\tau-\xi)\delta(\tau-\xi)}\\
&= \lim_{\xi\to\tau}\frac{\delta^2\ln\frac{b(\tau)}{b(\xi)} + [\ln\frac{Z}{Kb(\xi)}\sigma^2 + \delta^2[r-\lambda k - q] - \gamma\sigma^2](\tau-\xi)}{\sigma\delta\sqrt{\sigma^2(\tau-\xi)+\delta^2}\sqrt{\tau-\xi}}\\
&= \lim_{\xi\to\tau}\frac{\delta^2\ln\frac{b(\tau)}{b(\xi)}}{\sigma\delta\sqrt{[\sigma^2(\tau-\xi)+\delta^2](\tau-\xi)}}\\
&= 0.
\end{aligned}$$

Using this result we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\tau}g_0^{(1)}(b(\tau),\tau)\\
&= \frac{\lambda}{b(\tau)}\int_0^{Kb(\tau)}\frac{C_1(Z,\tau)-Z}{KZ\delta\sqrt{2\pi}}\\
&\quad\times\exp\left\{-\frac{1}{2}[d_2(Kb(\tau),z,r_1(0),q,0,v_1^2(0))]\right\}N[0]dZ\\
&+\frac{\lambda}{b(\tau)}\int_0^{\tau}\frac{\partial}{\partial\tau}\{e^{-\lambda(\tau-\xi)}e^{-r(\tau-\xi)}\int_0^{Kb(\xi)}\frac{C_1(Z,\xi)-Z}{KZv_1(\tau-\xi)\sqrt{2\pi(\tau-\xi)}}\\
&\quad\times\exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r_1(\tau-\xi),q_1,\tau-\xi,v_1^2(\tau-\xi))]^2\right\}\\
&\quad\times N[D_0(Kb(\tau),Kb(\xi),\tau-\xi,Z)]dZ\}d\xi\\
&-\frac{\lambda}{(b(\tau))^2}b'(\tau)\int_0^{\tau}e^{-\lambda(\tau-\xi)}e^{-r(\tau-\xi)}\int_0^{Kb(\xi)}\frac{C_1(Z,\xi)-Z}{KZv_1(\tau-\xi)\sqrt{2\pi(\tau-\xi)}}\\
&\quad\times\exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r_1(\tau-\xi),q,\tau-\xi,v_1^2(\tau-\xi))]^2\right\}\\
&\quad\times N[D_0(Kb(\tau),Kb(\xi),\tau-\xi,Z)]dZd\xi.
\end{aligned}$$

Since all integrals with respect to ξ will be finite as $\tau \longrightarrow 0^+$, and that the second integral tends to zero faster than $b'(\tau)$ increases[9], we find that

$$\begin{aligned}\lim_{\tau\to 0^+} g_0^{(1)}(b(\tau),\tau) &= \frac{\lambda}{2b(0^+)}\int_0^{Kb(0^+)} \frac{C_1(Z,0)-Z}{KZ\delta\sqrt{2\pi}} \exp\left\{\frac{-[\ln\frac{Kb(0^+)}{Z}]+\gamma-\frac{\delta^2}{2}}{2\delta^2}\right\} dZ \\ &= \frac{\lambda}{2b(0^+)}\int_0^{Kb(0^+)} \frac{ZH(Z-K)-Z}{KZ\delta\sqrt{2\pi}} \exp\left\{\frac{-[\ln\frac{Kb(0^+)}{Z}]+\gamma-\frac{\delta^2}{2}}{2\delta^2}\right\} dZ \\ &= \frac{\lambda}{2b(0^+)}\int_0^{K} \frac{-Z}{KZ\delta\sqrt{2\pi}} \exp\left\{\frac{-[\ln\frac{Kb(0^+)}{Z}]+\gamma-\frac{\delta^2}{2}}{2\delta^2}\right\} dZ.\end{aligned}$$

Setting $Z = Ke^x$, and using the fact that $b(0^+) > 1$, we have

$$\begin{aligned}\lim_{\tau\to 0^+} g_0^{(1)}(b(\tau),\tau) &= \frac{-\lambda}{2b(0^+)}\int_{-\infty}^{0} \frac{e^x}{\delta\sqrt{2\pi}} \exp\left\{\frac{-[x-\ln b(0^+)-\gamma+\frac{\delta^2}{2}]^2}{2\delta^2}\right\} dx \\ &= \frac{-\lambda}{2b(0^+)\delta\sqrt{2\pi}} \lim_{\alpha_1\to-\infty}\int_{\alpha_1}^{\alpha_2} e^{\hat{a}x} e^{-\frac{(\hat{q}-x)^2}{\hat{p}}} dx,\end{aligned}$$

where $\alpha_2 = 0$, $\hat{a} = 1$, $\hat{p} = 2\delta^2$ and $\hat{q} = \ln b(0^+) + \gamma - \frac{\delta^2}{2}$. Using the result in equation (A2.1.3), we can evaluate the integral term to give

$$\lim_{\tau\to 0^+} g_0^{(1)}(b(\tau),\tau) = \frac{-\lambda}{2b(0^+)\delta\sqrt{2\pi}} \lim_{\alpha_1\to-\infty} \sqrt{\hat{p}\pi}\exp\left\{\frac{(4\hat{q}+\hat{p})}{4}\right\}\{N[\hat{f}(\alpha_2)]-N[\hat{f}(\alpha_1)]\},$$

where

$$\hat{f}(u) = \frac{u-\ln b(0^+)-\gamma-\frac{\delta^2}{2}}{\delta}.$$

By the properties of the cumulative normal density function, this limit finally evaluates to

$$\lim_{\tau\to 0^+} g_0^{(1)}(b(\tau),\tau) = \frac{-\lambda e^{\gamma}}{2} N\left[\frac{-\ln b(0^+)-\gamma-\frac{\delta^2}{2}}{\delta}\right].$$

[9]We do not provide a proof for this particular limit, but note that the term $b'(\tau)$ does not arise if we forego the linear decomposition $C(Z,\xi) = C_1(Z,\xi) - KC_2(Z,\xi)$. In this case the analysis proceeds without any occurrence of $b'(\tau)$, and the same final result is obtained for $b(0^+)$.

The sixth term for consideration is

$$\begin{aligned}
&\frac{\partial}{\partial \tau} g_n^{(1)}(b(\tau),\tau) \\
&= \frac{\lambda}{b(\tau)} \int_0^\tau \frac{\partial}{\partial \tau} \left\{ \frac{[\lambda(\tau-\xi)]^n e^{-\lambda(\tau-\xi)}}{n!} e^{-r(\tau-\xi)} \right. \\
&\quad \times \int_0^{Kb(\xi)} \frac{C_1(Z,\xi)-Z}{KZv_n(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
&\qquad \times \exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r_{n+1}(\tau-\xi),q,\tau-\xi,v_{n+1}^2(\tau-\xi))]^2\right\} \\
&\qquad \left. \times N[D_n(Kb(\tau),Kb(\xi),\tau-\xi,Z)]dZ \right\} d\xi \\
&\quad -\frac{\lambda b'(\tau)}{(b(\tau))^2} \int_0^\tau \frac{[\lambda(\tau-\xi)]^n e^{-\lambda(\tau-\xi)}}{n!} \int_0^{Kb(\xi)} \frac{C_1(Z,\xi)-Z}{KZv_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \\
&\quad \times \exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r_{n+1}(\tau-\xi),q,\tau-\xi,v_{n+1}^2(\tau-\xi))]^2\right\} \\
&\quad \times N[D_n(Kb(\tau),Kb(\xi),\tau-\xi,Z)]dZ\} \, d\xi.
\end{aligned}$$

Thus we can conclude that for $n \geq 1$

$$\lim_{\tau \to 0^+} \frac{\partial}{\partial \tau} g_n^{(1)}(b(\tau),\tau) = 0.$$

For the next term we have

$$\begin{aligned}
\frac{\partial}{\partial \tau} g_0^{(2)}(b(\tau),\tau) &= \lambda \int_0^{Kb(\tau)} \frac{C_2(Z,\xi)-1}{Z\delta\sqrt{2\pi}} \\
&\quad \times \exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r_1(0),q,0,v_1^2(0))]^2\right\} N[0]dZ \\
&\quad +\lambda \int_0^\tau \frac{\partial}{\partial \tau} \left\{ e^{-\lambda(\tau-\xi)} e^{-r(\tau-\xi)} \int_0^{Kb(\xi)} \frac{C_2(Z,\xi)-1}{Zv_1(\tau-\xi)\sqrt{2\pi(\tau-\xi)}} \right. \\
&\quad \times \exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r,(\tau-\xi),q,\tau-\xi,v_1^2(\tau-\xi))]^2\right\} \\
&\quad \left. \times N[D_0(Kb(\tau),Kb(\xi),\tau-\xi,Z)]dZ \right\} d\xi.
\end{aligned}$$

Taking the limit gives

$$\begin{aligned}
\lim_{\tau\to 0^+} g_0^{(2)}(b(\tau),\tau) &= \frac{\lambda}{2}\int_0^{Kb(0^+)} \frac{C_2(Z,0)-1}{Z\delta\sqrt{2\pi}} \exp\left\{\frac{-[\ln\frac{Kb(0^+)}{Z}+\gamma-\frac{\delta^2}{2}]^2}{2\delta^2}\right\} dZ \\
&= \frac{\lambda}{2}\int_0^{Kb(0^+)} \frac{H(Z-K)-1}{Z\delta\sqrt{2\pi}} \exp\left\{\frac{-[\ln\frac{Kb(0^+)}{Z}+\gamma-\frac{\delta^2}{2}]^2}{2\delta^2}\right\} dZ \\
&= -\frac{\lambda}{2}\int_0^{K} \frac{1}{Z\delta\sqrt{2\pi}} \exp\left\{\frac{-[\ln\frac{Z}{K}-\ln b(0^+)-\gamma+\frac{\delta^2}{2}]^2}{2\delta^2}\right\} dZ.
\end{aligned}$$

Let $\alpha = (\ln\frac{Z}{K} - \ln b(0^+) - \gamma + \frac{\delta^2}{2})/\delta$, so that we now have

$$\begin{aligned}
\lim_{\tau\to 0^+} g_0^{(2)}(b(\tau),\tau) &= -\frac{\lambda}{2\sqrt{2\pi}}\int_{-\infty}^{-[\ln b(0^+)-\gamma+\frac{\delta^2}{2}]/\delta} e^{-\frac{\alpha^2}{2}}d\alpha \\
&= -\frac{\lambda}{2}N\left[\frac{-\ln b(0^+)-\gamma+\frac{\delta^2}{2}}{\delta}\right].
\end{aligned}$$

For the eighth and final term, we find that

$$\begin{aligned}
&\frac{\partial}{\partial\tau} g_n^{(2)}(b(\tau),\tau) \\
&:= \lambda\int_0^{\tau}\frac{\partial}{\partial\tau}\left\{\frac{[\lambda(\tau-\xi)]^n e^{-\lambda(\tau-\xi)}}{n!}e^{-r(\tau-\xi)}\int_0^{Kb(\xi)}\frac{C_2(Z,\xi)-1}{Zv_{n+1}(\tau-\xi)\sqrt{2\pi(\tau-\xi)}}\right. \\
&\times\exp\left\{-\frac{1}{2}[d_2(Kb(\tau),Z,r_{n+1}(\tau-\xi),q,\tau-\xi v_{n+1}^2(\tau-\xi))]\right\} \\
&\quad\left.\times N[D_0(Kb(\tau),Kb(\xi),\tau-\xi,Z)]dZ\right\}d\xi.
\end{aligned}$$

We can conclude that for this term

$$\lim_{\tau\to 0^+} g_n^{(2)}(b(\tau),\tau) = 0,$$

where $n \geq 1$.

Combining these eight limits we find that

$$\begin{aligned}
\lim_{\tau\to 0^+} b(\tau) &= \frac{-(\lambda+\frac{r}{2})+\lambda N\left[(-\ln b(0^+)+\gamma-\frac{\delta^2}{2})/\delta\right]+\frac{\lambda}{2}N\left[(-\ln b(0^+)-\gamma+\frac{\delta^2}{2})/\delta\right]}{-(\lambda'+\frac{q}{2})+\lambda' N\left[(\ln b(0^+)+\gamma+\frac{\delta^2}{2})/\delta\right]+\frac{\lambda'}{2}N\left[(-\ln b(0^+)-\gamma-\frac{\delta^2}{2})/\delta\right]} \\
&= \frac{r+\lambda N\left[(-\ln b(0^+)-(\gamma-\frac{\delta^2}{2}))/\delta\right]}{q+\lambda' N\left[(-\ln b(0^+)-(\gamma+\frac{\delta^2}{2}))/\delta\right]},
\end{aligned}$$

and since $b(0^+) \geq 1$, we conclude that

$$b(0^+) = \max\left(1, \frac{r + \lambda N[(-\ln b(0^+) - (\gamma - \frac{\delta^2}{2}))/\delta]}{q + \lambda' N[(-\ln b(0^+) - (\gamma + \frac{\delta^2}{2}))/\delta]}\right),$$

which is the result given in equation (4.5.10) of Proposition 4.5.4.

Appendix 4.8. Algorithm for Evaluating the American Call Option under Jump-Diffusion

Here we present the algorithm *American Call Price* which outlines the iterative scheme for evaluating the price and free boundary of an American call option under jump-diffusion.

Algorithm *American Call Price*

Input: S, r, q, σ, K, T (time to expiry), λ, γ, δ, N (number of time intervals).

Output: C (American call price), b (early exercise boundary).

solve equation (4.5.10) for b_0
$C_0(S) = \max(S - K, 0)$
$TOL = 10^{-10}$
for $i = 1$ **to** N
    do calculate initial estimate $C_i^{(0)}(S) = C_{i-1}(S)$
        set $b_i^{(0)} = b_{i-1}$; $j = 0$
        repeat
            $j = j + 1$;
            solve equation (4.5.3) with $S = b_i^{(j)}$ for $b_i^{(j)}$
            calculate $C_i^{(j)}(S)$ using equation (4.5.3)
        until $|b_i^{(j)} - b_i^{(j-1)}| < TOL$ **and** $||C_i^{(j)}(S) - C_i^{(j-1)}(S)|| < TOL$
        $b_i = b_i^{(j)}$
        $C_i(S) = C_i^{(j)}(S)$

CHAPTER 5

Fourier-Hermite Series Expansion for American Calls under Jump-Diffusion

5.1. Introduction

When the underlying asset is assumed to follow a jump-diffusion process, the task of valuing American options becomes far more complex. In Chapter 4 we showed how the incomplete Fourier transform method can be extended to allow for the presence of jumps, thereby producing an integral equation system for the price and early exercise boundary of the American call. In the case where the jump sizes are log-normally distributed, as proposed by Merton (1976), we used an iterative method based on numerical integration to solve the free boundary problem. At present there are relatively few numerical solution alternatives for the American option under jump-diffusion, and only some of these display sufficiently high levels of computational efficiency for practical purposes. The aim of this chapter is to suggest an alternative approach by extending the Fourier-Hermite series expansion method of Chiarella et al. (1999) to the jump-diffusion case, and demonstrate that this numerical technique can offer a highly efficient alternative to existing methods in the task of pricing American call options.

While iterative numerical integration can be used to solve the integral equations of the American call pricing problem, the method is computationally cumbersome. Several alternative methods have been explored, with a view to finding one that offers more efficiency for the same level of accuracy. Amin (1993) extends the binomial tree model to demonstrate the impact jump-diffusion has on the free boundary and option price when compared with a pure-diffusion model. This idea is further extended by Wu and Dai (2001) in the form of a multi-nomial tree. By considering the American option problem as a variational inequality, Zhang (1997*b*) is able to implement a finite difference method. Carr and Hirsa (2003) also use finite differences, applying the Crank-Nicolson scheme to the partial-integro differential equation for the American put. Mullinaci (1996) uses a discrete time solution for the underlying stochastic differential equation, leading to explicit formulae for the Snell envelope. In the case of American puts, d'Halluin et al. (2003) apply a fixed-point iteration method.

In the pure-diffusion case, Meyer and van der Hoek (1997) use the method of lines to find both the price and free boundary for American call and put options. They demonstrate that the method is highly efficient, and produces accurate results that converge to the true solution as the level of discretisation is increased. Meyer (1998) subsequently extends this idea to Merton's jump-diffusion model, in the case where the density for the jump size is discrete. For a small number of potential jump sizes, Meyer demonstrates that the method of lines can be applied iteratively to find both the price and free boundary for American calls and puts. Again, the method is proven to be convergent, and it displays a substantial level of accuracy.

Chiarella et al. (1999) demonstrate how Fourier-Hermite series expansions can be used to price both European and American options under pure-diffusion dynamics. In Chapter 2 we presented results for the pure-diffusion American call using Hermite series, and demonstrated that the method is extremely fast to compute, yielding accurate prices at the expense of accuracy in the free boundary estimate. An additional benefit is that unlike any of the approaches cited previously, Fourier-Hermite series require only that the time dimension be discretised, since our estimate of the price will be given in terms of continuous basis functions of the underlying asset price. Furthermore, the option price sensitivities, such as delta and gamma, can be readily calculated from the polynomial price estimate using direct differentiation.

In this chapter we explore another alternative numerical method for the evaluation of American call options under Merton's jump-diffusion model. We propose to extend the path-integral approach of Chiarella et al. (1999) to the jump-diffusion case by considering an American call option where the density for the jump sizes is log-normal. This corresponds to the problem considered in Chapter 4, for which Merton (1976) provides closed-form prices in the case of European calls and puts. It is shown that the Fourier-Hermite method is well-suited to this problem, since the log-normal density is naturally related to the orthogonality-weighting function for Hermite polynomials.

The discussion in this chapter shall be as follows. Section 5.2 establishes the pricing problem in the case of a European call option with log-normally distributed jump sizes. Section 5.3 details how Fourier-Hermite series can be used to approximate the solution for a European call. The method is expanded to the American call case in Section 5.4, with a discussion of numerical implementation issues given in Section 5.5. Some numerical results are presented in Section 5.6, with price and free boundary comparisons between the series expansion, numerical integration and method of lines solutions. Conclusions are provided in Section 5.7, with details for all necessary mathematical proofs provided in appendices.

5.2. Problem Statement - Log-Normal Jumps

Let $C(S_t, t)$ denote the price of an option contract written on the underlying asset S_t at present time t. $C(S_t, t)$ has strike price K, and matures at time $T > t$. We assume that S_t follows a jump-diffusion process, whose risk-neutral dynamics are given by equation (4.2.1), with $S \equiv S_t$ and $\mu \equiv r - q$. For the purpose of this chapter we shall assume that $G(Y)$ is a log-normal density, specifically the density given by equation (4.5.1).

Given the stochastic differential equation (SDE) for S_t, we can solve the corresponding Kolmogorov backward equation to find the transition density for S_t. Let $p(S_T, T|S_t, t)$ denote the probability of observing the price, S_T, at future time T, given that we observe the price, S_t, at the current time t, where S_t follows the risk-neutral dynamics in (4.2.1). The transition density is therefore (see e.g. Chiarella (2003))

$$\begin{aligned} p(S_T, T|S_t, t) &= \frac{e^{-\lambda(T-t)}}{S_T\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n}{n! v_n \sqrt{T-t}} \\ &\quad \times \exp\left\{ \frac{-[\ln(S_T/S_t) - (r_n - q - v_n^2/2)(T-t)]^2}{2v_n^2(T-t)} \right\} \end{aligned} \tag{5.2.1}$$

where $r_n = r - \lambda k + n\gamma/(T-t)$ and $v_n^2 = \sigma^2 + n\delta^2/(T-t)$. Thus $p(S_T, T|S_t, t)$ is a Poisson-weighted sum of log-normal density functions, where each density in the sum is considered on the condition that n jumps have been observed in the time interval $(T - t)$.

As in Merton (1976), we assume that the jump-risk can be fully diversified by the option holder. Applying the Feynman-Kac formula, the price of $C(S_t, t)$ is given by

$$C(S_t, t) = e^{-r(T-t)}\mathbb{E}_t[g(S_T)] = e^{-r(T-t)} \int_0^{\infty} g(S_T) p(S_T, T|S_t, t) dS_T, \tag{5.2.2}$$

where $g(S_T) \equiv C(S_T, T)$ is the payoff function for $C(S_t, t)$.

In order to apply the Fourier-Hermite expansion technique, we will need to transform equation (5.2.2) to one where the domain of integration spans the interval $(-\infty, \infty)$. This is achieved by the change of variable $\xi_T = \ln(S_T/K)/\theta$, where θ is a "volatility scaling" constant[1] whose value depends upon the relative values of σ, λ, γ and δ. Furthermore, let $Kf(\xi_t, t) = C(S_t, t)$. Under this transformation, equation (5.2.2) becomes

$$f(\xi_t, t) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{g(Ke^{\theta\xi_T})}{K} \Pi(\xi_T, T|\xi_t, t) d\xi_T, \qquad (5.2.3)$$

where

$$\Pi(\xi_T, T|\xi_t, t) = \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n e^{-\lambda(T-t)}\theta}{n! v_n \sqrt{2\pi(T-t)}} \times \exp\left\{ \frac{-[\xi_T - v_n \theta^{-1}\sqrt{2(T-t)}\mu_n(\xi_t, T-t)]^2}{2v_n^2(T-t)\theta^{-2}} \right\}, \qquad (5.2.4)$$

with

$$\mu_n(\xi_t, T-t) = \frac{\theta}{v_n\sqrt{2(T-t)}} \left[\xi_t + \left(r_n - q - \frac{v_n^2}{2} \right) \frac{(T-t)}{\theta} \right]. \qquad (5.2.5)$$

Using the Chapman-Kolmogorov equation, it is possible to form a backward recursion for the transformed price, $f(\xi_t, t)$. Firstly, discretise the time domain into J sub-intervals, each of length Δt. Introducing the notation $f^j(\xi_j) \equiv f(\xi_{j\Delta t}, j\Delta t)$, with $f^J(\xi_J) = g(Ke^{\theta\xi_T})/K$, we can apply the same methods as used in Chiarella et al. (1999) to express $f^{j-1}(\xi_{j-1})$ as

$$f^{j-1}(\xi_{j-1}) = e^{-r\Delta t} \int_{-\infty}^{\infty} f^j(\xi_j) \Pi(\xi_j, t_j|\xi_{j-1}, t_{j-1}) d\xi_j, \quad (j = J, J-1, \cdots, 1). \quad (5.2.6)$$

Note that $f^0(\xi_0)$ represents the transformed option price at the current time t.

To evaluate the integral term in equation (5.2.6) we will estimate $f^j(\xi_j)$ using a Fourier-Hermite series expansion. Chiarella et al. (1999) recommend the use of Hermite polynomials because their weighting function is closely related to the functional form of $\Pi(\xi_j, t_j|\xi_{j-1}, t_{j-1})$. Furthermore, series expansions have the advantage that they result in a price estimate which is a continuous function of the underlying, eliminating the need to extrapolate prices for various values of ξ_t.

5.3. Evaluation of European Call Options

We begin our application of the Fourier-Hermite series expansion method by firstly considering the case of a European call option. This example will allow us to provide a

[1]In the pure-diffusion case, Chiarella et al. (1999) set $\theta \equiv \sigma$.

clear explanation of how the Hermite series method works before considering the added complexity that results from having an early exercise feature. In addition, there are several key results that arise from the European case which are required for the American option, making the European problem an efficient starting point for the American call.

In the case of the European call, the payoff function $g(S_T)$ becomes

$$g(S_T) = \max(S_T - K, 0),$$

and therefore

$$f^J(\xi_J) = \max(e^{\theta\xi_J} - 1, 0).$$

Substituting for $\Pi(\xi_j, t_j|\xi_{j-1}, t_{j-1})$ in equation (5.2.6), we have

$$\begin{aligned} f^{j-1}(\xi_{j-1}) &= e^{-r\Delta t}\int_{-\infty}^{\infty} f^j(\xi_j)\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n e^{-\lambda\Delta t}\theta}{n! v_n\sqrt{2\pi\Delta t}} \\ &\quad \times \exp\left\{\frac{-[\xi_j - v_n\theta^{-1}\sqrt{2\Delta t}\mu_n(\xi_{j-1},\Delta t)]^2}{2v_n^2\Delta t\theta^{-2}}\right\} d\xi_j, \end{aligned}$$

where we note that $r_n = r - \lambda k + n\gamma/\Delta t$, and set $\hat{v}_n^2 \equiv (\sigma^2 + n\delta^2/\Delta t)/\theta^2 = v_n^2/\theta^2$. Changing the variable of integration from ξ_j to $\hat{v}_n\sqrt{2\Delta t}\xi_j$ gives

$$f^{j-1}(\xi_{j-1}) = \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_{-\infty}^{\infty} f^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2}d\xi_j. \quad (5.3.1)$$

Next we expand $f^j(\xi_j)$ in a Fourier-Hermite series according to

$$f^j(\xi_j) = \sum_{m=0}^{\infty}\alpha_m^j H_m(\xi_j), \quad (5.3.2)$$

where the α_m^j coefficients are given by[2]

$$\alpha_m^j = \frac{1}{2^m m!}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_j^2} f^j(\xi_j)H_m(\xi_j)d\xi_j. \quad (5.3.3)$$

For practical purposes, we must truncate the summation in (5.3.2) at some finite number of basis functions, N. Our goal now is to determine the coefficients α_m^j.

[2]Refer to Abramowitz and Stegun (1970) for standard results regarding Hermite polynomials.

PROPOSITION 5.3.1. *The coefficients α_m^j can be generated recursively using the relationship*

$$\alpha_m^{j-1} = e^{-(r+\lambda)\Delta t} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \sum_{i=0}^{N} A_{m,i}^{(n)} \alpha_i^j, \quad (j = J-1, J-2, \cdots, 2, 1), \tag{5.3.4}$$

where the $A_{m,i}^{(n)}$ terms are given by

$$A_{m,i}^{(n)} = \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_i(b_n + z w_n) dz, \tag{5.3.5}$$

with

$$b_n = \left(r_n - q - \frac{v_n^2}{2} \right) \frac{\Delta t}{\theta},$$

and

$$w_n = \sqrt{1 + 2\Delta t \hat{v}_n^2}.$$

Proof: Refer to Appendix A5.1.1.

□

In order to implement the recursion (5.3.4) for the coefficients of the Hermite expansions, we must first evaluate (5.3.5). By using the recurrence relations for Hermite polynomials (Abramowitz and Stegun 1970) we can also generate recursions for the $A_{m,i}^{(n)}$ terms.

PROPOSITION 5.3.2. *The terms $A_{m,i}^{(n)}$, defined by equation (5.3.5), can be found using the recurrence relation*

$$A_{m,i}^{(n)} = \frac{i}{m} A_{m-1,i-1}^{(n)}, \quad (m, i = 1, 2, \cdots, N), \tag{5.3.6}$$

where

$$A_{0,i}^{(n)} = 2b_n A_{0,i-1}^{(n)} + 2(i-1)(w_n^2 - 1) A_{0,i-2}^{(n)}, \quad (i = 2, 3, \cdots, N), \tag{5.3.7}$$

$$A_{0,0}^{(n)} = 1, \quad A_{0,1}^{(n)} = 2b_n,$$

and

$$A_{m,i}^{(n)} = 0, \textit{ for } m > i.$$

Proof: Refer to Appendix A5.1.2.

□

Combining the results of Propositions 5.3.1 and 5.3.2, we now have all that is required to determine the α_m^j coefficients, with the exception of those at time step $(J-1)$. As recommended by Chiarella et al. (1999) we avoid expanding the piecewise linear payoff function $f^J(\xi_J)$ in a Fourier-Hermite series, and instead evaluate the initial α_m^{J-1} coefficients directly.

PROPOSITION 5.3.3. *The coefficients at the first time step prior to expiry, α_m^{J-1}, are given by the recurrence relation*

$$\alpha_m^{J-1} = \frac{\theta}{2m}\left[\alpha_{m-1}^{J-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^{m-1}(m-1)!} \times \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^{m-1}\sqrt{\pi}}H_{m-2}\left(-\frac{b_n}{w_n}\right)e^{-(b_n/w_n)^2}\right], \quad (m=2,3,\cdots,N), \tag{5.3.8}$$

with

$$\alpha_0^{J-1} = \frac{e^{-(r+\lambda)\Delta t}}{2} \times \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\left\{e^{\theta b_n}e^{\frac{\theta^2 w_n^2}{4}}erfc\left(-\frac{b_n}{w_n}-\frac{\theta w_n}{2}\right) - erfc\left(-\frac{b_n}{w_n}\right)\right\}, \tag{5.3.9}$$

and

$$\alpha_1^{J-1} = \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{\theta}{2}\left\{e^{\theta b_n}e^{\frac{\theta^2 w_n^2}{4}}erfc\left(-\frac{b_n}{w_n}-\frac{\theta w_n}{2}\right)\right\}, \tag{5.3.10}$$

where erfc is the complementary error function, given by

$$erfc(y) = \frac{2}{\sqrt{\pi}}\int_y^{\infty}e^{-\alpha^2}d\alpha. \tag{5.3.11}$$

Proof: Refer to Appendix A5.1.3.

□

At this point we now have all that is required to find the European call price using Hermite series expansions, with the exception of the value of the scaling parameter θ. The issue of selecting appropriate θ values is discussed at length in Section 5.5, but we note here that Merton (1976) provides a closed-form solution for the European call price under the dynamics given by (4.2.1) with (4.5.1). It is thereby possible for us to choose θ such that the Hermite-series technique accurately reproduces the closed-form solution.

5.4. Evaluation of American Call Options

With the European call solution using Fourier-Hermite series established, we now address the task of pricing an American call option. Given the same underlying dynamics from (4.2.1) in conjunction with (4.5.1), the American call price is given by

$$C_A(S_t, t) = \max_{t \le \tau \le T} \{\mathbb{E}_t[e^{-r(\tau - t)} \max(S_\tau - K, 0)]\}. \tag{5.4.1}$$

The expectation is taken over the range of possible stopping times, τ. The optimal stopping time, τ^*, is the smallest time for which it is optimal to exercise early, and is defined according to

$$\tau^* = \inf\{s \in [t, T] : F(S_s, s) = S_s - K\}.$$

Applying the same time discretisation as was used for the European call, we can evaluate the American call price using the backward recursion

$$C_A(S_t, t) = \max\{\max(S_\tau - K, 0), e^{-r\Delta t}\mathbb{E}_t[C_A(S_{t+\Delta t}, t + \Delta t)]\}, \quad (0 \le t \le T).$$

This is equivalent to finding the discounted expected call value at time step t, given the value at time $t + \Delta t$, and then applying the external max[ ] operator to the price profile for all relevant values of S to determine at which underlying asset values early exercise has become optimal. This is the same method commonly applied when pricing American options using binomial trees and finite difference methods.

Using the same change of variable for the underlying from Section 5.2, and defining $KF^j(\xi_j) \equiv C_A(S_{j\Delta t}, j\Delta t)$, the value of the American call becomes

$$F^{j-1}(\xi_{j-1}) = \max\{\max(e^{\theta\xi_{j-1}} - 1, 0), e^{-r\Delta t}\mathbb{E}_{t_{j-1}}[F^j(\xi_j)]\}, \quad (j = J, J-1, \cdots, 1).$$

As demonstrated by Chiarella et al. (1999), we can account for the early exercise feature within the Fourier-Hermite series expansion method by way of a three-step procedure, implemented for $j = J, J-1, \cdots, 1$:

Step 1: Determine $V^{j-1}(\xi_{j-1})$, which is given by

$$\begin{aligned} V^{j-1}(\xi_{j-1}) &= e^{-r\Delta t}\mathbb{E}_{t_{j-1}}[F^j(\xi_j)] \\ &= e^{-r\Delta t}\int_{-\infty}^{\infty} \Pi(\xi_j, t_j | \xi_{j-1}, t_{j-1}) F^j(\xi_j) d\xi_j. \end{aligned} \tag{5.4.2}$$

This is the value at t_{j-1} of the American call option unexercised.

Step 2: Solve for the early exercise value of the state variable at time t_{j-1}, denoted by ξ^*_{j-1}. This is the value of ξ which solves

$$V^{j-1}(\xi) = e^{\theta\xi} - 1. \tag{5.4.3}$$

Step 3: The value of the American call at time t_{j-1} is determined by

$$F^{j-1}(\xi_{j-1}) = \begin{cases} V^{j-1}(\xi_{j-1}) & \text{for } -\infty < \xi_{j-1} < \xi^*_{j-1}, \\ e^{\theta\xi_{j-1}} - 1 & \text{for } \xi^*_{j-1} < \xi_{j-1} < \infty. \end{cases} \tag{5.4.4}$$

The most complicated component in this three-step procedure is the calculation of $V^{j-1}(\xi_{j-1})$ in step 1. This calculation is achieved by first expanding $V^{j-1}(\xi_{j-1})$ in a Fourier-Hermite series according to

$$V^{j-1}(\xi_{j-1}) = \sum_{m=0}^{N} \alpha_m^{j-1} H_m(\xi_{j-1}), \tag{5.4.5}$$

whose coefficients are given by Proposition 5.4.1.

PROPOSITION 5.4.1. *The coefficients α_m^{j-1} are generated recursively using the relationship*

$$\alpha_m^{j-1} = \gamma_m^{j-1} + e^{-(r+\lambda)\Delta t} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \sum_{i=0}^{N} A_{m,i}^{j,n} \alpha_i^j, \quad (j = J-1, J-2, \cdots, 2, 1), \tag{5.4.6}$$

where the $A_{m,i}^{j,n}$ terms are given by

$$\begin{aligned} A_{m,i}^{j,n} &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k^{(n)}} e^{-z^2} H_m(z) H_i(b_n + w_n z) dz, \\ &\quad (m, i = 0, 1, 2, \cdots, N). \end{aligned} \tag{5.4.7}$$

The γ_m^{j-1} terms are found recursively using

$$\begin{aligned} \gamma_m^{j-1} &= \frac{\theta \gamma_{m-1}^{j-1}}{2m} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m!} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{e^{-(z_j^{(n)})^2}}{w_n^m \sqrt{\pi}} \\ &\quad \times \left\{ H_{m-1}(z_j^{(n)}) [e^{\theta(b_n + w_n z_j^{(n)})} - 1] + w_n \theta H_{m-2}(z_j^{(n)}) \right\}, \quad (m = 2, \cdots, N), \end{aligned} \tag{5.4.8}$$

where

$$\gamma_0^{j-1} = \frac{e^{-(r+\lambda)\Delta t}}{2} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \left\{ e^{b_n \theta} e^{\frac{w_n^2 \theta^2}{4}} erfc\left(z_j^{(n)} - \frac{w_n \theta}{2} \right) - erfc(z_j^{(n)}) \right\},$$

$$\gamma_1^{j-1} = \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!w_n}\left\{\frac{1}{\sqrt{\pi}}e^{-(z_j^{(n)})^2}e^{\theta(w_m z_j^{(n)}+b_n)} + \frac{\theta w_n}{2}e^{b_n\theta}e^{\frac{w_n^2\theta^2}{4}}\mathit{erfc}\left(z_j^{(n)} - \frac{w_n\theta}{2}\right) - \frac{1}{\sqrt{\pi}}e^{-(z_j^{(n)})^2}\right\},$$

with

$$z_j^{(n)} = \frac{\xi_j^* - b_n}{w_n},$$

and b_n, w_n as defined in Proposition 5.3.1.

Proof: Refer to Appendix A5.2.1.

□

At this point we are again required to evaluate an integral equation, in this case (5.4.7), in order to implement the recurrence for α_m^{j-1}. By use of the recurrence relations for Hermite polynomials, we can develop a recurrence to find the $A_{m,i}^{j,n}$ terms for the American call.

PROPOSITION 5.4.2. *The terms $A_{m,i}^{j,n}$, defined by equation (5.4.7), can be generated by use of the recurrence relation*

$$A_{m,i}^{j,n} = \frac{i}{m}A_{m-1,i-1}^{j,n} - \frac{1}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}H_{m-1}(z_j^{(n)})H_i(b_n + w_n z_j^{(n)})e^{-(z_j^{(n)})^2}, \quad (m, i = 1, 2, \cdots, N), \tag{5.4.9}$$

where

$$A_{m,0}^{j,n} = -\frac{1}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}e^{-(z_j^{(n)})^2}H_{m-1}(z_j^{(n)}), \quad (m = 1, 2, \cdots, N), \tag{5.4.10}$$

$$A_{0,i}^{j,n} = 2(w_n^2 - 1)(i-1)A_{0,i-2}^{j,n} + 2b_n A_{0,i-1}^{j,n} - \frac{w_n}{\sqrt{\pi}}H_{i-1}(b_n + w_n z_j^{(n)})e^{-(z_j^{(n)})^2}, \quad (i = 2, 3, \cdots, N), \tag{5.4.11}$$

with

$$A_{0,0}^{j,n} = \frac{1}{2}\mathit{erfc}(-z_j^{(n)}),$$

and

$$A_{0,1}^{j,n} = b_n \mathit{erfc}(-z_j^{(n)}) - \frac{w_n}{\sqrt{\pi}}e^{-(z_j^{(n)})^2}.$$

Proof: Refer to Appendix A5.2.2.

□

All that remains is to initiate the algorithm with respect to time. As was shown for the European call, this requires us to calculate α_m^{J-1}. Since the American call has the same payoff as the European call, and the early exercise condition is simply given by the value of the underlying asset relative to the strike price[3], the α_m^{J-1} coefficients for the American call are the same as those for the corresponding European option. Thus for the first time step, the α_m^{J-1} are given by equations (5.3.8)-(5.3.10) from Proposition 5.3.3.

5.5. Numerical Implementation - American Call

In order to numerically implement the three-step backwards recursion for the American call, we must address two further issues. The first is the matter of solving for the optimal exercise boundary, ξ_j^*, at each time step. This is achieved by applying a root-finding method to equation (5.4.3) in step 2. Here we use the same iterative method supplied by Chiarella et al. (1999) for the pure-diffusion case. Specifically, ξ_{j-1}^* is given by

$$\xi_{j-1}^{i+1} = \frac{1}{\theta}\ln(1 + V^{j-1}(\xi_{j-1}^i)) \text{ for } i = 0, 1, 2, \cdots \tag{5.5.1}$$

which is iterated until $|\xi_{j-1}^{i+1} - \xi_{j-1}^i| < \varepsilon$ for some arbitrarily small ε, where $\xi_{j-1}^0 = \xi_j^*$. We also assume that $\xi_J^* = 0$, since it is known that for an American call, $\xi_J^* \geq 0$. This method typically displays fast convergence, but in the cases where it does not, it can be replaced with an appropriate alternative, such as the bisection method.

The second unresolved issue at this point is the form of the scaling parameter θ. In the pure-diffusion case (i.e. when $\lambda = 0$), Chiarella et al. (1999) set $\theta = \sigma$. This has the effect of transforming the problem to one with a unit coefficient for the diffusion term. While the authors present no details on the purpose of this transformation, practical experiments demonstrate that the results of the Hermite series expansion method are far more accurate when this volatility scaling transformation is applied.

In the jump-diffusion case, it is not as simple to perform an equivalent volatility scaling to the jump-diffusion SDE (4.2.1). The theoretical equivalent to the pure-diffusion case would be to define θ as

$$\theta^2 = \sigma^2 + \lambda(e^{2\gamma+\delta^2} - 2e^\gamma + 1),$$

[3]Strictly speaking, the free boundary at expiry time, J, is equivalent to the strike price, K. We also know the limit of the free boundary as time to expiry tends to 0^+, as demonstrated in Section 4.5. This limit has no impact on the value of the payoff function at expiry, thus making the value of $z_J^{(n)}$ irrelevant for the purpose of calculating α_m^{J-1}.

however in practice this does not consistently produce sufficiently accurate prices. In particular, when the jump component is significantly volatile, such a definition appears to consistently underestimate θ. Furthermore, there is evidence that when the diffusion volatility is significantly large in relation to the volatility contributed by the jump term, then $\theta = \sigma$ can often prove sufficient, and the more complex definition leads to an over-estimation of θ.

While there is no closed-form solution for the American call price under the dynamics in (4.2.1), there is a formula for the corresponding European call, derived by Merton (1976). By comparing the Fourier-Hermite series solution for the European call to the exact solution, we are able to numerically explore the values of θ that maximise the accuracy of the method. Such analysis demonstrates that θ is clearly a function of the four price-process parameters which add to the global price volatility, such that

$$\theta \equiv \theta(\sigma, \lambda, \gamma, \delta).$$

Determining the exact functional form of θ, however, is not as straightforward, due to the most natural starting point proving ineffective, and the complex four-dimensional form required.

Without a specific function for θ, we instead propose a simple optimisation method based on European options. Given Merton's closed-form solution for the European call, we first select a value of θ such that the Hermite series solution is sufficiently accurate in a neighbourhood around the strike. This accuracy can be assessed using an arbitrary error measure (such as the root mean square error) for a range of spot prices centred at K. When generating our results, we estimate θ by trial and error to around 2-3 significant figures. We do not aim to develop an efficient optimisation technique for selecting θ in this chapter, but rather to demonstrate that a sufficiently optimal value of θ exists, and that this can be confirmed by use of the pricing formula for the European call.

The algorithm *Fourier-Hermite American Call Price* provided in Appendix 5.3 details the main steps involved when implementing the Fourier-Hermite series expansion for an American call. In particular the algorithm summarises the order in which the coefficients must be calculated, and the exact initial values required. Note that we do not provide explicit details for the process of estimating θ, but it is clear that this must be completed before any of the coefficients can be computed.

5.6. Results

We now demonstrate the accuracy and efficiency of the Fourier-Hermite series expansion method by generating prices for the American call under a range of parameter values. As a basis for comparison, we also calculate the call prices using two alternative methods. The first method is direct numerical integration of the integral equations for the price and free boundary of the American call. A derivation of these equations, using McKean's incomplete Fourier transform method, was provided in Chapter 4, along with a corresponding numerical integration scheme. In using this method, we initially discretise the time-domain into 50 steps. The process is then repeated using 100 time steps, and the two results are combined into a final solution using Richardson extrapolation. We also apply a fine grid for the initial 4 time steps, consisting of 40 sub-steps, to help improve the free boundary estimate near expiry.

Since the existing literature offers no specific numerical method as the "true" solution for the problem at hand, call prices are also generated by a second method for comparison purposes. Given that Meyer (1998) proves the method of lines is convergent for American calls and puts with discrete jumps, we shall use it as an additional benchmark for the Fourier-Hermite method. We implement the method of lines for the American call as outlined by Meyer, with a few minor modifications. For all necessary interpolations we use cubic splines rather than the cubic Lagrangian suggested by Meyer. 50 time steps are used to maintain consistency with the numerical integration results. We apply 10,000 space steps in the region $0 \leq S \leq 4K$. The large number of space steps was necessary to ensure that the resulting free boundary was sufficiently smooth. Since the method demands that the distribution for the jump sizes be discrete, an approximation was used for the log-normal density, $G(Y)$, consisting of 200 evenly-spaced values in the region $-10 \leq \ln Y \leq 10$.

When implementing the Fourier-Hermite series, we again set the number of time steps to be 50, and use $N = 40$ basis functions for the series expansion of the price. We consider a 6-month American call option with a strike of 100 for a range of parameter values. The infinite sums were computed term-by-term until convergence was obtained in the sum. This typically required 20 terms or less for a tolerance level of 10^{-16}. In all cases we first find the exact price of the corresponding European call option, and then apply the Fourier-Hermite method to the European case for several values of θ, until the relative errors in the prices at $S = 80, 90, 100, 110$, and 120 are sufficiently small (usually less than 1%, and always less than 0.1% at the strike). The required values of θ were found to vary as σ, γ, δ and λ were varied, but remained unaffected by changes in r and q. In all cases we found that we required $\theta > \sigma$. The final value was determined using simple trial and error, but could be readily computed via a suitable optimisation algorithm.

The code for all three methods was implemented using LAHEY™FORTRAN 95 running on a PC with a Pentium 4 2.40 GHz processer, 512MB of RAM, and running the Windows XP Professional operating system. The typical computation time for each of the numerical methods is reported in Table 5.1. Numerical integration is by far the slowest method, taking over 29 minutes to compute, and this value increases exponentially as the number of time steps increases. The method of lines provides a significant saving, with only 93.578 seconds required to solve the problem. The main contributions to this runtime are the large number of space steps required to achieve a monotonic early exercise boundary, and the large number of discrete jump sizes used to approximate the log-normal distribution in equation (4.5.1). When compared to the method of lines, the Fourier-Hermite series is exceptionally fast, requiring only 1.359 seconds to calculate the call price and free boundary. This does not include the time spent determining the optimal value of θ, but since the method requires even less computation for the European call, a good optimisation method should add very little to this runtime, which we anticipate to be no more than 10-15 seconds in total. This fast computation is attributable to the method's heavy reliance on recurrence relations, both for the Hermite polynomial evaluations and the various coefficient calculations.

Method	Computation Time
McKean (Integration)	29 min 16.578 sec
Method of Lines	1 min 33.578 sec
Fourier-Hermite	1.359 sec

TABLE 5.1. Typical computation time for each of the numerical methods. All code was implemented using LAHEY™FORTRAN 95 running on a PC with a Pentium 4 2.40 GHz processer, 512MB of RAM, and running the Windows XP Professional operating system.

A range of American call prices are presented in tables 5.2-5.5. In all of these tables we report the price of the American call at spot values of $S = 80, 90, 100, 110$ and 120. The relative difference between the numerical integration and Fourier-Hermite series methods are also included, as they are devoid of any discretisation error that may be introduced when approximation $G(Y)$ for the method of lines solution. Tables 5.2-5.4 focus on the prices as the mean jump size, e^{γ}, is changed for various values of r and q, and with $\sigma = 0.40$. Table 5.5 considers two additional cases with smaller diffusion coefficients of $\sigma = 0.20$.

Table 5.2 presents the 6-month American call price for $e^{\gamma} = 1$, representing jumps centered around the current underlying asset price. In Table 5.3 we have $e^{\gamma} = 1.05$, indicating upwards jumps on average, whilst Table 5.4 has $e^{\gamma} = 0.95$, implying that downward jumps are expected. The value of δ was adjusted in each case to ensure that the volatility of $\ln Y$ was fixed at 20% and the Poisson intensity is set at $\lambda = 1$ throughout. In all cases the relative difference between the numerical integration and Fourier-Hermite methods is less than 1%. This appears insensitive to the relative values of r and q. In most cases the three methods are found to be equivalent to the first 2-3 significant figures.

$\mathbb{E}[Y] = e^{\gamma} = 1.00$	S	McKean (Integration)	Method of Lines	Fourier-Hermite	Relative Difference
$r = 0.05, q = 0.03$					
	80	4.05	4.09	4.07	0.5097%
	90	7.67	7.69	7.70	0.3875%
	100	12.68	12.67	12.72	0.2633%
	110	18.94	18.91	18.97	0.1670%
	120	26.22	26.19	26.25	0.1048%
$r = 0.03, q = 0.05$					
	80	3.66	3.70	3.69	0.7724%
	90	7.04	7.06	7.08	0.5470%
	100	11.80	11.80	11.86	0.4790%
	110	17.84	17.82	17.90	0.3611%
	120	24.96	24.93	25.01	0.2127%

TABLE 5.2. Comparing the Fourier-Hermite American call price with results obtained from numerical integration and the method of lines, in the case where $\gamma = 0$. Other parameter values are $\sigma = 0.40$, $K = 100$, $T - t = 0.50$, $\lambda = 1$, $\delta = 0.1980$ and $\theta = 0.60$ for the Fourier-Hermite scaling parameter. The relative difference is calculated as $|C_{McKean} - C_{Fourier-Hermite}|/C_{McKean}$.

Given that the diffusion coefficient of $\sigma = 0.40$ is quite large, Table 5.5 considers two cases where this has been reduced to 0.20. The first example in Table 5.5 reduces σ while maintaining $\lambda = 1$. The Fourier-Hermite series continues to yield prices of suitable magnitude, with the largest relative difference being around 2.1%. In the second part of

$\mathbb{E}[Y] = e^{\gamma} = 1.05$	S	McKean (Integration)	Method of Lines	Fourier-Hermite	Relative Difference
$r = 0.05, q = 0.03$					
	80	4.12	4.19	4.14	0.5325%
	90	7.71	7.77	7.75	0.4136%
	100	12.68	12.72	12.71	0.2865%
	110	18.89	18.91	18.93	0.1863%
	120	26.14	26.15	26.17	0.1197%
$r = 0.03, q = 0.05$					
	80	3.74	3.81	3.76	0.7495%
	90	7.10	7.16	7.14	0.5555%
	100	11.82	11.86	11.88	0.5037%
	110	17.82	17.84	17.88	0.3800%
	120	24.91	24.92	24.96	0.2225%

TABLE 5.3. Comparing the Fourier-Hermite American call price with results obtained from numerical integration and the method of lines, in the case where $\gamma = 0.0488$. Other parameter values are $\sigma = 0.40$, $K = 100$, $T - t = 0.50$, $\lambda = 1$, $\delta = 0.1888$ and $\theta = 0.60$ for the Fourier-Hermite scaling parameter. The relative difference is calculated as $|C_{McKean} - C_{Fourier-Hermite}|/C_{McKean}$.

$\mathbb{E}[Y] = e^{\gamma} = 0.95$	S	McKean (Integration)	Method of Lines	Fourier-Hermite	Relative Difference
$r = 0.05, q = 0.03$					
	80	4.07	4.07	4.10	0.7333%
	90	7.76	7.73	7.80	0.5239%
	100	12.83	12.77	12.88	0.3359%
	110	19.14	19.06	19.18	0.2066%
	120	26.46	26.37	26.49	0.1255%
$r = 0.03, q = 0.05$					
	80	3.67	3.67	3.70	0.7495%
	90	7.11	7.08	7.16	0.5555%
	100	11.92	11.86	12.00	0.5037%
	110	18.00	17.93	18.09	0.3800%
	120	25.15	25.07	25.22	0.2225%

TABLE 5.4. Comparing the Fourier-Hermite American call price with results obtained from numerical integration and the method of lines, in the case where $\gamma = -0.0513$. Other parameter values are $\sigma = 0.40$, $K = 100$, $T - t = 0.50$, $\lambda = 1$, $\delta = 0.2082$ and $\theta = 0.67$ for the Fourier-Hermite scaling parameter. The relative difference is calculated as $|C_{McKean} - C_{Fourier-Hermite}|/C_{McKean}$.

Table 5.5, we increase the Poisson intensity to $\lambda = 5$, and observe the impact of more

$\mathbb{E}[Y] = e^{\gamma} = 1.00$	S	McKean (Integration)	Method of Lines	Fourier-Hermite	Relative Difference
$\lambda = 1, \theta = 0.50$					
	80	1.10	1.20	1.10	0.5630%
	90	3.03	3.13	3.09	2.0669%
	100	6.95	6.98	7.07	1.8161%
	110	13.11	13.09	13.23	0.9130%
	120	21.06	21.01	21.17	0.5091%
$\lambda = 5, \theta = 0.675$					
	80	4.29	4.54	4.29	0.0306%
	90	7.69	7.91	7.72	0.3908%
	100	12.45	12.57	12.52	0.6008%
	110	18.50	18.52	18.58	0.4195%
	120	25.64	25.59	25.68	0.1514%

TABLE 5.5. Comparing the Fourier-Hermite American call price with results obtained from numerical integration and the method of lines, in the case where $\gamma = 0$, with the smaller diffusion volatility of $\sigma = 0.20$. Other parameter values are $r = 0.03$, $q = 0.05$, $K = 100$, $T - t = 0.50$, and $\delta = 0.1980$. The relative difference is calculated as $|C_{McKean} - C_{Fourier-Hermite}|/C_{McKean}$.

frequent jumps on the results. It is interesting to note that this leads to relative differences that are again consistently less than 1%.

To complete the analysis, we provide some free boundary profiles for the three methods under consideration. In figures 5.1-5.2, we present the early exercise boundary for two different 6-month American call options with strike price $K = 1.00$. For Figure 5.1 we have set $r = 3\%$, $q = 5\%$, $\lambda = 1$, $\gamma = 0$ and $\delta = 0.1988$. Given that the numerical integration and method of lines results are extremely close together, we shall assume that these best represent the true free boundary. The Fourier-Hermite result deviates from the other methods in two critical ways. Firstly, the free boundary near expiry, $\tau = 0$, is quite poor. The Fourier-Hermite estimate is significantly less than the true solution. The second discrepancy arises near the current time, $\tau = 0.50$. While the Fourier-Hermite result is now quite close to the true solution, it appears to have converged to a function that contains a systematic error relative to the desired result. Figure 5.2 repeats the results of Figure 5.1, but this time we take $\sigma = 0.20$, $\gamma = 0.0488$ and $\delta = 0.1888$. Once again, it is clear that the numerical integration and method of lines results are extremely close, while the Fourier-Hermite solution deviates greatly near expiry, and contains a systematic error in the result near the current time.

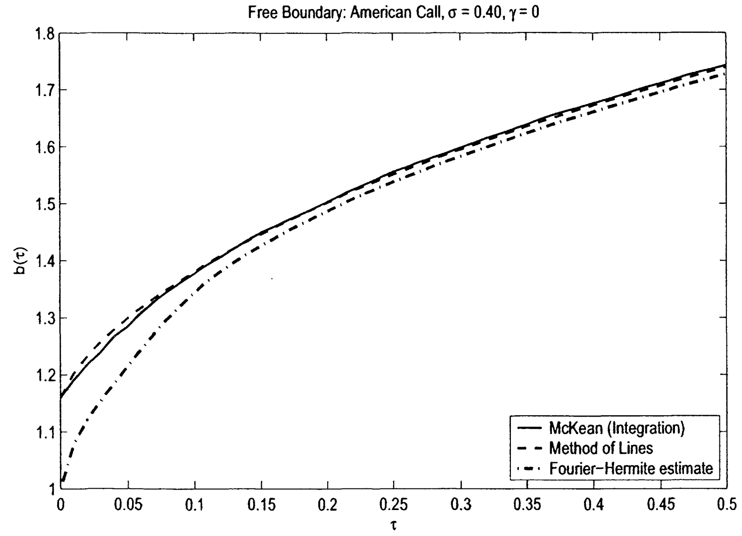

FIGURE 5.1. Comparing the early exercise boundary approximation for the American call using numerical integration, method of lines, and Fourier-Hermite series, where the diffusion volatility is $\sigma = 0.40$ and $\gamma = 0$. Other parameters are $K = 1$, $r = 0.03$, $q = 0.05$, $T - t = 0.50$, $\lambda = 1$, $\delta = 0.1988$ and $\theta = 0.60$ for the Fourier-Hermite method.

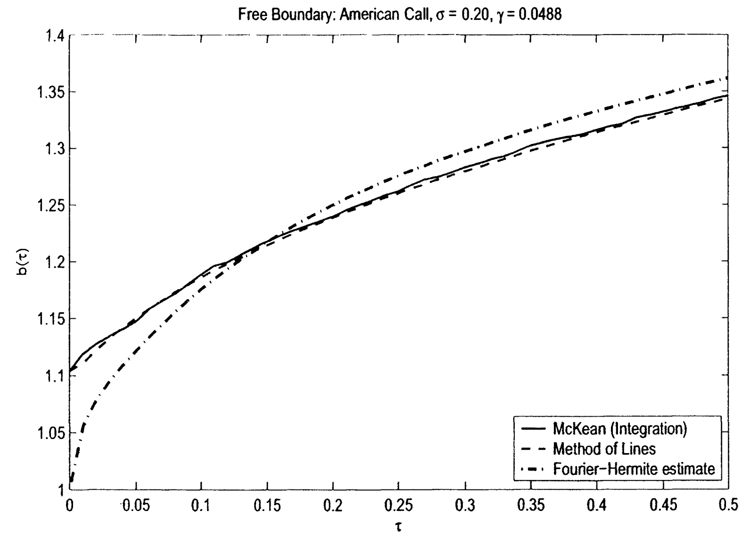

FIGURE 5.2. Comparing the early exercise boundary approximation for the American call using numerical integration, method of lines, and Fourier-Hermite series, where the diffusion volatility is $\sigma = 0.20$ and $\gamma = 0.0488$. Other parameters are $K = 1$, $r = 0.03$, $q = 0.05$, $T - t = 0.50$, $\lambda = 1$, $\delta = 0.1888$ and $\theta = 0.485$ for the Fourier-Hermite method.

Given the nature of the Fourier-Hermite solution, it is possible to offer some justification for the observed free boundary estimates, as well as their anticipated impact on the American call price. Near expiry, it is clear that the result is tending to some definite function that is essentially the true solution plus some systematic error. The difference between the Fourier-Hermite solution and the exact free boundary is most likely due to the fact that the series approximation for the American call price is centred about the strike. Since the observed free boundaries in figures 5.1-5.2 are quite far from the strike for any significant amount of time prior to expiry, it is not unsurprising to find that the series expansion contains some small margin of error when approximating the free boundary for τ values greater than 0.15.

Near expiry, however, the differences are far more dramatic. This is because the option price, for small values of τ, is very close in shape to the piecewise-linear payoff function for the call. In this time-region the option price will not be well approximated by a Fourier-Hermite series, since we are fitting an N-degree polynomial to a function that is almost piecewise-linear. It is interesting to note, however, that despite the poor approximation near expiry, the prices for the 6-month call options produced by the Fourier-Hermite method are still accurate. In particular, the minor error in the free boundary for $\tau > 0.2$ seems to have had no significant impact on the prices produced by the method. This is in keeping with the well known result that the prices of American options are highly insensitive to small changes in the free boundary[4]. Hence one major shortcoming of polynomial series expansions is that they cannot easily handle piecewise-linear functions. In particular, to ensure that the method remains stable for all time steps after the first, we must use $b(0) = 1$ at the start of the time-stepping procedure, and cannot take advantage of our knowledge of the limit $b(0^+)$ from Proposition 4.5.4. Thus there appears no robust way to extract a more accurate free boundary approximation for small values of τ. Should one require a precise estimate of the free boundary near expiry, this could be quickly achieved using an alternative method, such as the method of lines, applied to the interval $0 \leq \tau \leq 0.15$. Another possible method would be to form a small-time expansion for the free boundary near expiry, and use this to approximate $b(\tau)$ when τ is near zero. Given, however, that numerous difficulties have occurred when applying this idea to the pure-diffusion problem, there is currently little chance that such analysis could be carried out for the jump-diffusion case.

[4]See for example AitSahlia and Lai (2001), and Chiarella and Ziogas (2003).

5.7. Conclusion

In this chapter we have presented a generalisation of the Fourier-Hermite series expansion method of Chiarella et al. (1999) for the pricing of European and American call options. This extension applies the Fourier-Hermite series method to the jump-diffusion model of Merton (1976), where the jump sizes are log-normally distributed. We derived the recurrence relations for both the European and American call option under jump-diffusion, and presented the special time-stepping algorithm to account for early exercise in the American case. When implementing the method for the jump-diffusion model, an unspecified scaling parameter is required to be known. Using Merton's closed-form solution for the European call price, we provide a means for estimating this scaling parameter's value for a given global volatility level.

The series expansion method was used to generate a range of American call prices, and the results compared with those generated using the numerical integration method from Chapter 4, as well as the method of lines approach of Meyer (1998). We find that all three methods produce relatively consistent prices, and in particular that the Fourier-Hermite prices are always within 1% of the numerical integration results, with only two reported exceptions. The results indicate that for a sufficiently large global volatility, the Fourier-Hermite method yields excellent levels of accuracy when compared with the standards displayed in the existing literature on the subject. Furthermore, the Fourier-Hermite method proved to be extremely efficient, requiring significantly less computation time than either of the alternatives presented.

The most notable short-coming for the Fourier-Hermite approach was in estimating the early exercise boundary. The method was incapable of reproducing the correct free boundary near expiry, and was only able to achieve a solution involving a systematic error near the current time. The expiry issue we attribute to the poor performance of polynomial approximations when estimating functions that are close to piecewise linear in form, such as the value of an American call or put near expiry . For the current-time discrepancy, we suggest that the centralisation of the series expansions around the strike were a likely cause. This cannot be easily remedied without foregoing price accuracy in the critical region around the strike. It has been of interest to note that even with these small inaccuracies in the free boundary estimate, the resulting prices have been accurate. This demonstrates that the series expansion technique has a potential trade-off in the form of increased computation speed at the cost of accuracy in estimating the early exercise boundary, most predominantly near expiry. This does not diminish the value of the method as an efficient means of pricing American options under jump-diffusion processes where the jump sizes follow a specified continuous distribution. Further computation time is saved in that there is never any need to interpolate option prices for various values of the spot, since the price estimate is a continuous function of the underlying asset. It is also trivial to estimate the delta and gamma for the American call once the Fourier-Hermite series approximation has been found.

There are several avenues that these results suggest for future research. Given that the free boundary estimate near expiry is suboptimal, some alternative estimate would be of significant value. A small-time expansion of the free boundary near expiry remains unaddressed for American calls under jump-diffusion. In presenting multiple benchmark prices for the American call option, we acknowledge that there is still no clear consensus as to what the exact price is for the American call under consideration. This continues to cast some doubt regarding the accuracy of any numerical method being considered. While the presented method has the advantage of being well-suited to the case where jump sizes follow a log-normal distribution, it is not yet known how the method would perform for jump sizes with discrete distributions. Finally, we have not offered an explicit optimisation routine for selecting the scaling parameter prior to finding the American call price. Determining and verifying an explicit optimisation routine, or an explicit form for the scaling parameter in terms of the global volatility of the jump-diffusion process, would further increase the robustness of the method.

Appendix 5.1. Hermite Coefficients for the European Call

A5.1.1. Proof of Proposition 5.3.1. From equation (5.3.3), α_m^{j-1} is given by

$$\alpha_m^{j-1} = \frac{1}{2^m m!} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} f^{j-1}(\xi_{j-1}) H_m(\xi_{j-1}) d\xi_{j-1}.$$

Substituting for $f^{j-1}(\xi_{j-1})$ from (5.3.1), we obtain

$$\begin{aligned}\alpha_m^{j-1} &= \frac{1}{2^m m!}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} H_m(\xi_{j-1}) \\ &\quad\times \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n(\Delta t)^n}{n!} \int_{-\infty}^{\infty} f^j(\hat{v}_n\sqrt{2\Delta t}\xi_j) e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2} d\xi_j d\xi_{j-1} \\ &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n(\Delta t)^n}{n!} \int_{-\infty}^{\infty} f^j(\hat{v}_n\sqrt{2\Delta t}\xi_j) I_m^{(n)}(\xi_j) d\xi_j,\end{aligned}$$

where

$$I_m^{(n)}(\xi_j) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2-\xi_{j-1}^2} H_m(\xi_{j-1}) d\xi_{j-1}. \tag{A5.1.1}$$

To evaluate $I_m^{(n)}(\xi_j)$ we complete the square in the exponent. Recalling the definition of μ_n from (5.2.5), it is simple to show that

$$[x-\mu_n(\xi,\Delta t)]^2 + \xi^2 = \left[\frac{w_n\xi}{\hat{v}_n\sqrt{2\Delta t}} - \frac{(x\hat{v}_n\sqrt{2\Delta t}-b_n)}{w_n\hat{v}_n\sqrt{2\Delta t}}\right]^2 + \left[\frac{\hat{v}_n\sqrt{2\Delta t}x - b_n}{w_n}\right]^2, \tag{A5.1.2}$$

where we set $b_n \equiv (r_n - q - \frac{v_n^2}{2})\Delta t/\theta$ and $w_n \equiv \sqrt{1+2\Delta t\hat{v}_n^2}$. Thus $I_m^{(n)}(\xi_j)$ can be expressed as

$$\begin{aligned}I_m^{(n)}(\xi_j) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\left\{-\left[\frac{w_n\xi_{j-1}}{\hat{v}_n\sqrt{2\Delta t}} - \frac{(\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n)}{w_n\hat{v}_n\sqrt{2\Delta t}}\right]^2\right\} \\ &\quad\times \exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j - b_n}{w_n}\right]^2\right\} H_m(\xi_{j-1}) d\xi_{j-1}.\end{aligned}$$

If we make the change of variable $y = w_n\xi_{j-1}/(\hat{v}_n\sqrt{2\Delta t})$, $I_m^{(n)}(\xi_j)$ becomes

$$\begin{aligned}I_m^{(n)}(\xi_j) &= \frac{1}{\sqrt{\pi}}\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j - b_n}{w_n}\right]^2\right\}\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n} \\ &\quad\times \int_{-\infty}^{\infty} \exp\left\{-\left[y - \frac{(\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n)}{w_n\hat{v}_n\sqrt{2\Delta t}}\right]^2\right\} H_m\left(\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n}y\right) dy.\end{aligned} \tag{A5.1.3}$$

To evaluate this integral, we refer to a result from Erdélyi, Magnus, Oberhettinger and Tricomi (1953) (p.195, equation (30)), which states that

$$\frac{1}{\sqrt{2\pi u}}\int_{-\infty}^{\infty} H_m(z)\exp\left\{-\frac{(z-v)^2}{2u}\right\}dz = (1-2u)^{\frac{m}{2}} H_m\left(\frac{v}{\sqrt{1-2u}}\right).$$

Letting $y = z/\sqrt{2u}$, this becomes

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H_m(\sqrt{2u}y)\exp\left\{-\left[y-\frac{v}{\sqrt{2u}}\right]^2\right\}dy = (1-2u)^{\frac{m}{2}} H_m\left(\frac{v}{\sqrt{1-2u}}\right).$$

Thus if we equate $u = \hat{v}_n^2\Delta t/w_n^2$ and $v = (\xi_k\hat{v}_n\sqrt{2\Delta t} - b_n)/w_n^2$, we can now evaluate $I_m^{(n)}(\xi_j)$ as

$$\begin{aligned} I_m^{(n)}(\xi_j) &= \exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j - b_n}{w_n}\right]^2\right\}\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n}\left(1-\frac{\hat{v}_n^2 2\Delta t}{w_n^2}\right)^{\frac{m}{2}} \\ &\quad \times H_m\left(\frac{(\xi_j\hat{v}_n\sqrt{2\Delta t} - b_n)}{w_n^2}\sqrt{\frac{w_n^2}{w_n^2 - 2\hat{v}_n^2\Delta t}}\right) \\ &= \frac{\hat{v}_n\sqrt{2\Delta t}}{w_n^{m+1}} H_m\left(\frac{\xi_j\hat{v}_n\sqrt{2\Delta t} - b_n}{w_n}\right)\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j - b_n}{w_n}\right]^2\right\}. \end{aligned} \tag{A5.1.4}$$

Using equation (A5.1.4), the expression for α_m^{j-1} becomes

$$\begin{aligned} \alpha_m^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_{-\infty}^{\infty} f^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n^{m+1}} \\ &\quad \times H_m\left(\frac{\xi_j\hat{v}_n\sqrt{2\Delta t} - b_n}{w_n}\right)\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j - b_n}{w_n}\right]^2\right\}d\xi_j. \end{aligned}$$

If we define $z = (\xi_j\hat{v}_n\sqrt{2\Delta t} - b_n)/w_n$, we now have

$$\alpha_m^{j-1} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n! w_n^m}\int_{-\infty}^{\infty} f^j(b_n + zw_n)H_m(z)e^{-z^2}dz.$$

Expanding $f^j(b_n + zw_n)$ in a Fourier-hermite series as defined in (5.3.2) we obtain

$$\begin{aligned}\alpha_m^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n(\Delta t)^n}{n! w_n^m} \sum_{i=0}^{\infty} \alpha_i^k \int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_i(b_n + zw_n) dz \\ &= e^{-(r+\lambda)\Delta t} \sum_{n=0}^{\infty} \frac{\lambda^n(\Delta t)^n}{n!} \sum_{i=0}^{\infty} \alpha_i^k A_{m,i}^{(n)},\end{aligned}$$

where

$$A_{m,i}^{(n)} \equiv \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_i(b_n + zw_n) dz.$$

Truncating the number of basis functions at order N, we obtain equations (5.3.4) - (5.3.5) of Proposition 5.3.1. Note that while we must truncate the order of the Hermite-series expansion, the same is not true for the summation over the number of observed jumps, n. This must be computed for increasing values of n until convergence is obtained, according to some pre-specified accuracy level.

A5.1.2. Proof of Proposition 5.3.2. To develop a recurrence relation for $A_{m,i}^{(n)}$, we note from Abramowitz and Stegun (1970) that the recurrence relation for Hermite polynomials is

$$H_m(z) = 2zH_{m-1}(z) - 2(m-1)H_{m-2}(z),$$

and furthermore, the derivative of a Hermite polynomial can be expressed recursively as

$$H_m'(z) = 2mH_{m-1}(z).$$

Applying the recurrence relation to equation (5.3.5), we have

$$\begin{aligned}A_{m,i}^{(n)} &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_i(b_n + zw_n) dz \\ &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2ze^{-z^2} H_{m-1}(z) H_i(b_n + zw_n) dz \\ &\quad - \frac{2(m-1)}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_{m-2}(z) H_i(b_n + zw_n) dz.\end{aligned}$$

Using integration by parts on the first integral, we have

$$
\begin{aligned}
A_{m,i}^{(n)} &= \frac{1}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}\left\{ \left[-e^{-z^2}H_{m-1}(z)H_i(b_n+zw_n)\right]_{-\infty}^{\infty} \right. \\
&\qquad + \int_{-\infty}^{\infty} e^{-z^2}[2iw_n H_i(b_n+zw_n)H_{m-1}(z) \\
&\qquad\qquad \left. +2(m-1)H_{m-2}(z)H_i(b_n+zw_n)]dz \right\} \\
&\quad -\frac{2(m-1)}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{m-2}(z)H_i(b_n+zw_n)dz \\
&= \frac{2iw_n}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{i-1}(b_n+zw_n)H_{m-1}(z)dz \\
&= \frac{i}{m}\frac{1}{2^{m-1}(m-1)!w_n^{m-1}}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{i-1}(b_n+zw_n)H_{m-1}(z)dz \\
&= \frac{i}{m}A_{m-1,i-1}^{(n)}, \quad (m,i=1,2,\cdots,N),
\end{aligned}
$$

which is equation (5.3.6) of Proposition 5.3.2.

To implement the recurrence for $A_{m,i}^{(n)}$, we require expressions for $A_{m,0}^{(n)}$, $A_{0,i}^{(n)}$ and $A_{0,0}^{(n)}$. Firstly, $A_{m,0}^{(n)}$ is given by

$$
\begin{aligned}
A_{m,0}^{(n)} &= \frac{1}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_m(z)dz \\
&= 0 \text{ for } m \neq 0,
\end{aligned}
$$

where the last equality follows from the orthogonality result for Hermite polynomials. This subsequently implies that $A_{m,i}^{(n)} = 0$ for all $m > i$. Through use of the Hermite polynomial recurrence relation, $A_{0,i}^{(n)}$ is given by

$$
\begin{aligned}
A_{0,i}^{(n)} &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_i(b_n+zw_n)dz \\
&= \frac{w_n}{\sqrt{\pi}}\int_{-\infty}^{\infty} 2ze^{-z^2}H_{i-1}(b_n+zw_n)dz + \frac{2b_n}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{i-1}(b_n+zw_n)dz \\
&\quad \frac{-2(i-1)}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{i-2}(b_n+zw_n)dz.
\end{aligned}
$$

Applying integration by parts to the first integral term, $A_{0,i}^{(n)}$ becomes

$$\begin{aligned} A_{0,i}^{(n)} &= \frac{w_n}{\sqrt{\pi}}\left\{\left[-e^{-z^2}H_{i-1}(b_n+zw_n)\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}e^{-z^2}2(i-1)H_{i-2}(b_n+zw_n)w_n dz\right\} \\ &\quad +2b_nA_{0,i-1}^{(n)}-2(i-1)A_{0,i-2}^{(n)} \\ &= 2b_nA_{0,i-1}^{(n)}+2(i-1)(w_n^2-1)A_{0,i-2}^{(n)}, \quad (i=2,3,\cdots,N), \end{aligned}$$

which is a recurrence relation for $A_{0,i}^{(n)}$ as given in equation (5.3.7). It is straightforward to show that

$$A_{0,0}^{(n)}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-z^2}dz=1,$$

and to use the recurrence for $A_{0,i}^{(n)}$, we also require $A_{0,1}^{(n)}$, which can be evaluated as

$$\begin{aligned} A_{0,1}^{(n)} &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-z^2}H_1(b_n+zw_n)dz \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-z^2}2(b_n+zw_n)dz \\ &= 2b_n. \end{aligned}$$

A5.1.3. Proof of Proposition 5.3.3. To generate α_m^{J-1}, recall that at time step $j=J$

$$f^J(\xi_J)=\max(e^{\theta\xi_J}-1,0),$$

and the transition to f^{J-1} is given by

$$f^{J-1}(\xi_{J-1})=\frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_0^{\infty}(e^{v_n\sqrt{2\Delta t}\xi_J}-1)e^{-[\xi_J-\mu_n(\xi_{J-1},\Delta t)]^2}d\xi_J.$$

Expanding the solution at time step $j=J-1$ in a Fourier-Hermite series according to

$$f^{J-1}(\xi_{J-1})=\sum_{m=0}^{\infty}\alpha_m^{J-1}H_m(\xi_{J-1}),$$

the expression for α_m^{J-1} becomes

$$\alpha_m^{J-1}=\frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_0^{\infty}(e^{v_n\sqrt{2\Delta t}\xi_J}-1)I_m^{(n)}(\xi_J)d\xi_J,$$

where $I_m^{(n)}(\xi_J)$ is given by equation (A5.1.4). Substituting $I_m^{(n)}(\xi_J)$ into α_m^{J-1} we have

$$\begin{aligned}\alpha_m^{J-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n^{m+1}} \\ &\quad\times\int_0^{\infty}(e^{v_n\sqrt{2\Delta t}\xi_J}-1)H_m\left(\frac{\hat{v}_n\sqrt{2\Delta t}\xi_J-b_n}{w_n}\right) \\ &\quad\times\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_J-b_n}{w_n}\right]^2\right\}d\xi_J.\end{aligned}$$

Making the change of variable $z=(\hat{v}_n\sqrt{2\Delta t}\xi_J-b_n)/w_n$, α_m^{J-1} becomes

$$\begin{aligned}\alpha_m^{J-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty}(e^{\theta w_n z}e^{\theta b_n}-1)e^{-z^2}H_m(z)dz \\ &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m}\left\{e^{\theta b_n}\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right)-\Omega_m^{(n)}\left(-\frac{b_n}{w_n}\right)\right\},\end{aligned}$$

where

$$\Omega_m^{(n)}\left(-\frac{b_n}{w_n}\right)\equiv\frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty}e^{-z^2}H_m(z)dz, \tag{A5.1.5}$$

and

$$\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right)\equiv\frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty}e^{-z^2}e^{\theta w_n z}H_m(z)dz. \tag{A5.1.6}$$

Firstly consider the integral $\Omega_m^{(n)}$. Using the three-term recurrence relation for $H_m(z)$, we have

$$\Omega_m^{(n)}\left(-\frac{b_n}{w_n}\right)=\frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty}2ze^{-z^2}H_{m-1}(z)dz-2(m-1)\Omega_{m-2}^{(n)}\left(-\frac{b_n}{w_n}\right).$$

Applying integration by parts, we find that

$$\Omega_m^{(n)}\left(-\frac{b_n}{w_n}\right)=\frac{1}{\sqrt{\pi}}H_{m-1}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2}. \tag{A5.1.7}$$

Note that when $m=0$ we have

$$\Omega_0^{(n)}\left(-\frac{b_n}{w_n}\right)=\frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty}e^{-z^2}dz=\frac{1}{2}\text{erfc}\left(-\frac{b_n}{w_n}\right),$$

and when $m=1$,

$$\Omega_1^{(n)}\left(-\frac{b_n}{w_n}\right)=\frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty}e^{-z^2}2zdz=\frac{1}{\sqrt{\pi}}e^{-\left(\frac{b_n}{w_n}\right)^2}.$$

Next we consider $\Psi_m^{(n)}$. Again using the three-term recurrence we find that

$$\begin{aligned}\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right) &= \frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty} 2ze^{-z^2}e^{\theta w_n z}H_{m-1}(z)dz - 2(m-1)\Psi_{m-2}^{(n)}\left(-\frac{b_n}{w_n}\right)\\ &= \Phi_m^{(n)}\left(-\frac{b_n}{w_n}\right) - 2(m-1)\Psi_{m-2}^{(n)}\left(-\frac{b_n}{w_n}\right)\end{aligned}$$

where

$$\Phi_m^{(n)}\left(-\frac{b_n}{w_n}\right) \equiv \frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty} 2ze^{-z^2}e^{\theta w_n z}H_{m-1}(z)dz.$$

Through the use of integration by parts, $\Phi_m^{(n)}$ becomes

$$\begin{aligned}\Phi_m^{(n)}\left(-\frac{b_n}{w_n}\right) &= \frac{1}{\sqrt{\pi}}\left\{\left[-e^{-z^2}H_{m-1}(z)e^{\theta w_n z}\right]_{-\frac{b_n}{w_n}}^{\infty}\right.\\ &\quad\left.+\int_{-\frac{b_n}{w_n}}^{\infty} e^{-z^2}[e^{\theta w_n z}2(m-1)H_{m-2}(z) + H_{m-1}(z)w_n e^{w_n z}\theta]dz\right\}\\ &= \frac{1}{\sqrt{\pi}}H_{m-1}\left(-\frac{b_n}{w_n}\right)e^{-b_n\theta}e^{-\left(\frac{b_n}{w_n}\right)^2} + \theta w_n\Psi_{m-1}^{(n)}\left(-\frac{b_n}{w_n}\right)\\ &\quad+2(m-1)\Psi_{m-2}^{(n)}\left(-\frac{b_n}{w_n}\right).\end{aligned}$$

Thus the recurrence for $\Psi_m^{(n)}$ is given by

$$\begin{aligned}\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right) &= \frac{1}{\sqrt{\pi}}H_{m-1}\left(-\frac{b_n}{w_n}\right)e^{-b_n\theta}e^{-\left(\frac{b_n}{w_n}\right)^2}\\ &\quad+w_n\theta\Psi_{m-1}^{(n)}\left(-\frac{b_n}{w_n}\right), (m=1,2,\cdots,N).\end{aligned}\tag{A5.1.8}$$

For $m = 0$ we have

$$\begin{aligned}\Psi_0^{(n)}\left(-\frac{b_n}{w_n}\right) &= \frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty} e^{-z^2}e^{\theta w_n z}dz\\ &= \frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty} e^{-(z-\frac{\theta w_n}{2})^2}e^{\frac{\theta^2 w_n^2}{4}}dz\\ &= \frac{1}{2}e^{\frac{\theta^2 w_n^2}{4}}\text{erfc}\left(-\frac{b_n}{w_n}-\frac{\theta w_n}{2}\right),\end{aligned}$$

and when $m = 1$,

$$\begin{aligned}\Psi_1^{(n)}\left(-\frac{b_n}{w_n}\right) &= \frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty} 2ze^{-z^2}e^{\theta w_n z}dz \\ &= \frac{1}{\sqrt{\pi}}\left[-e^{-z^2}e^{\theta w_n z}\right]_{-\frac{b_n}{w_n}}^{\infty} + \int_{-\frac{b_n}{w_n}}^{\infty}\theta e^{-z^2}w_n e^{w_n z}dz \\ &= \frac{1}{\sqrt{\pi}}\exp\left\{-\left(\frac{b_n}{w_n}\right)^2 - \theta b_n\right\} + \frac{\theta w_n}{2}e^{\frac{\theta^2 w_n^2}{4}}\operatorname{erfc}\left(-\frac{b_n}{w_n} - \frac{\theta w_n}{2}\right).\end{aligned}$$

Hence the coefficients α_m^{J-1} are given by

$$\alpha_m^{J-1} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m!}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m}\left\{e^{\theta b_n}\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right) - \frac{1}{\sqrt{\pi}}H_{m-1}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2}\right\}.$$

We can now use equation (A5.1.8) to derive a recurrence for α_m^{J-1}, independent of $\Psi_m^{(n)}$. Firstly, rearrange the expression for α_m^{J-1} to give

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{\theta b_n}}{w_n^m}\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right) &= e^{(r+\lambda)\Delta t}2^m m!\alpha_m^{J-1} \\ &+ \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m\sqrt{\pi}}H_{m-1}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2}.\end{aligned} \tag{A5.1.9}$$

In addition, from equation (A5.1.8) we can readily show that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{\theta b_n}}{w_n^m}\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right) &= \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m\sqrt{\pi}}H_{m-1}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2} \\ &+ \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{\theta e^{\theta b_n}}{w_n^{m-1}}\Psi_{m-1}^{(n)}\left(-\frac{b_n}{w_n}\right)\end{aligned} \tag{A5.1.10}$$

Substituting (A5.1.9) into (A5.1.10) we have

$$\begin{aligned}e^{(r+\lambda)\Delta t}2^m m!\alpha_m^{J-1} &= \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{\theta b_n}}{w_n^{m-1}}\Psi_{m-1}^{(n)}\left(-\frac{b_n}{w_n}\right) \\ &= \theta e^{(r+\lambda)\Delta t}2^{m-1}(m-1)!\alpha_{m-1}^{J-1} \\ &+ \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{\theta}{w_n^{m-1}\sqrt{\pi}}H_{m-2}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2},\end{aligned}$$

and hence the recurrence relation for α_m^{J-1} is

$$\alpha_m^{J-1} = \frac{\theta}{2m}\left[\alpha_{m-1}^{J-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^{m-1}(m-1)!}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^{m-1}\sqrt{\pi}}H_{m-2}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2}\right],$$
$$(m = 2,3,\cdots,N),$$

as stated in (5.3.8) of Proposition 5.3.3. To initiate this recurrence, we note that for $m = 0$,

$$\begin{aligned}\alpha_0^{J-1} &= e^{-(r+\lambda)\Delta t}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\left\{e^{\theta b_n}\Psi_0^{(n)}\left(-\frac{b_n}{w_n}\right) - \Omega_0^{(n)}\left(-\frac{b_n}{w_n}\right)\right\}\\ &= \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\left\{e^{\theta b_n}e^{\frac{\theta^2 w_n^2}{4}}\operatorname{erfc}\left(-\frac{b_n}{w_n}-\frac{\theta w_n}{2}\right) - \operatorname{erfc}\left(-\frac{b_n}{w_n}\right)\right\},\end{aligned}$$

and when $m = 1$,

$$\begin{aligned}\alpha_1^{J-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n}\left\{e^{\theta b_n}\Psi_1^{(n)}\left(-\frac{b_n}{w_n}\right) - \Omega_1^{(n)}\left(-\frac{b_n}{w_n}\right)\right\}\\ &= \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{\theta}{2}\left\{e^{\theta b_n}e^{\frac{\theta^2 w_n^2}{4}}\operatorname{erfc}\left(-\frac{b_n}{w_n}-\frac{\theta w_n}{2}\right)\right\}.\end{aligned}$$

Appendix 5.2. Hermite Coefficients for the American Call

A5.2.1. Proof of Proposition 5.4.1. Substituting the transition density (5.2.4) into equation (5.4.2), the expression for $V^{j-1}(\xi_{j-1})$ becomes

$$V^{j-1}(\xi_{j-1}) = \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_{-\infty}^{\infty}e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2}F^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)d\xi_j.$$

Using the value of $F^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)$ from equation (5.4.4), we have

$$\begin{aligned}V^{j-1}(\xi_{j-1}) &= \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\\ &\quad\times\left\{\int_{-\infty}^{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2}V^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)d\xi_j\right.\\ &\quad\left.+\int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}^{\infty}e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2}(e^{\theta\xi_j\hat{v}_n\sqrt{2\Delta t}}-1)d\xi_j\right\}\\ &= \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_{-\infty}^{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2}V^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)d\xi_j\\ &\quad+h^{j-1}(\xi_{j-1}),\end{aligned}$$

where

$$h^{j-1}(\xi_{j-1}) \equiv \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n(\Delta t)^n}{n!} \int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}^{\infty} e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2}(e^{v_n\sqrt{2\Delta t}\xi_j}-1)d\xi_j.$$

Next, expand the functions V^j, V^{j-1} and h^{j-1} in Fourier-Hermite series, such that

$$V^{j-1}(\xi_{j-1}) = \sum_{m=0}^{\infty} \alpha_m^{j-1} H_m(\xi_{j-1}),$$

and

$$h^{j-1}(\xi_j) = \sum_{m=0}^{\infty} \gamma_m^{j-1} H_m(\xi_{j-1}).$$

From the orthogonality conditions for Hermite polynomials, the coefficients for these expansions are given by

$$\alpha_m^{j-1} = \frac{1}{2^m m!}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} V^{j-1}(\xi_{j-1}) H_m(\xi_{j-1}) d\xi_{j-1}$$

for V^{j-1}, and

$$\gamma_m^{j-1} = \frac{1}{2^m m!}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} h^{j-1}(\xi_{j-1}) H_m(\xi_{j-1}) d\xi_{j-1},$$

for h^{j-1}.

Now we must develop recurrence relations for α and γ. Starting with the γ coefficients, substitute the expression for $h^{j-1}(\xi_{j-1})$ into the γ_m^{j-1} equation to obtain

$$\begin{aligned}
\gamma_m^{j-1} &= \frac{1}{2^m m!\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} H_m(\xi_{j-1}) \left\{ \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n(\Delta t)^n}{n!} \right. \\
&\qquad \left. \times \int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta}}}^{\infty} e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2}(e^{v_n\sqrt{2\Delta}\xi_j}-1)d\xi_j \right\} d\xi_{j-1} \\
&= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\pi} \sum_{n=0}^{\infty} \frac{\lambda^n(\Delta t)^n}{n!} \left\{ \int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}^{\infty} (e^{v_n\sqrt{2\Delta t}\xi_j}-1) \right. \\
&\qquad \left. \times \int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} H_m(\xi_{j-1}) e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2} d\xi_{j-1} d\xi_j \right\}.
\end{aligned}$$

Using equation (A5.1.2), γ_m^{j-1} becomes

$$\begin{aligned}\gamma_m^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\pi}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!} \\ &\times\left\{\int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}^{\infty}(e^{v_n\sqrt{2\Delta t}\xi_j}-1)\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j-b_n}{w_n}\right]^2\right\}\right. \\ &\times\left.\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{w_n\xi_{j-1}}{\hat{v}_n\sqrt{2\Delta t}}-\frac{(\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n)}{w_n\hat{v}_n\sqrt{2\Delta t}}\right]^2\right\}H_m(\xi_{j-1})d\xi_{j-1}d\xi_j\right\}.\end{aligned}$$

A change of integration variable to $y = w_n\xi_{j-1}/\hat{v}_n\sqrt{2\Delta t}$ yields

$$\begin{aligned}\gamma_m^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\pi}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!} \\ &\times\left\{\int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}^{\infty}(e^{v_n\sqrt{2\Delta t}\xi_j}-1)\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j-b_n}{w_n}\right]^2\right\}\right. \\ &\times\int_{-\infty}^{\infty}\exp\left\{-\left[y-\frac{(\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n)}{w_n\hat{v}_n\sqrt{2\Delta t}}\right]^2\right\} \\ &\times\left.H_m\left(\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n}y\right)\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n}dyd\xi_j\right\} \\ &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}^{\infty}(e^{v_n\sqrt{2\Delta t}\xi_j}-1)I_m^{(n)}(\xi_j)d\xi_j\end{aligned}$$

where $I_m^{(n)}(\xi_j)$ is given by equation (A5.1.3). Since $I_m^{(n)}(\xi_j)$ can be evaluated to produce (A5.1.4), γ_m^{j-1} becomes

$$\begin{aligned}\gamma_{m-1}^{j} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\left\{\int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}^{\infty}(e^{v_n\sqrt{2\Delta t}\xi_j}-1)\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n^{m+1}}\right. \\ &\times\left.H_m\left(\frac{\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n}{w_n}\right)\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j-b_n}{w_n}\right]^2\right\}d\xi_j\right\}.\end{aligned}$$

If we now let $z = (\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n)/w_n$, and define $z_j^{(n)} \equiv (\xi_j^* - b_n)/w_n$, the integral for γ becomes

$$\gamma_{m-1}^{j} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!w_n^m}\int_{z_j^{(n)}}^{\infty}(e^{(w_nz+b_n)\theta}-1)H_m(z)e^{-z^2}dz.$$

To find a recurrence relation for γ_m^{j-1}, note that

$$\begin{aligned}\gamma_m^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n! w_n^m}\left\{\frac{e^{\theta b_n}}{\sqrt{\pi}}\int_{z_j^{(n)}}^{\infty} e^{-z^2}e^{\theta w_n z}H_m(z)dz\right.\\ &\left. -\frac{1}{\sqrt{\pi}}\int_{z_j^{(n)}}^{\infty} e^{-z^2}H_m(z)dz\right\}\\ &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n! w_n^m}\left\{e^{\theta b_n}\Psi_m^{(n)}(z_j^{(n)}) - \Omega_m^{(n)}(z_j^{(n)})\right\},\end{aligned}$$

where $\Omega_m^{(n)}$ and $\Psi_m^{(n)}$ are defined by equations (A5.1.5) and (A5.1.6) respectively. Using (A5.1.7), we can easily show that

$$\Omega_m^{(n)}(z_j^{(n)}) = \frac{1}{\sqrt{\pi}}H_{m-1}(z_j^{(n)})e^{-(z_j^{(n)})^2},$$

and similarly, equation (A5.1.8) implies that the recurrence for $\Psi_m^{(n)}(z_j^{(n)})$ is

$$\Psi_m^{(n)}(z_j^{(n)}) = \frac{1}{\sqrt{\pi}}H_{m-1}(z_j^{(n)})e^{\theta w_n z_j^{(n)}-(z_j^{(n)})^2} + \theta w_n \Psi_{m-1}^{(n)}(z_j^{(n)}).$$

Thus the expression for γ_m^{j-1} becomes

$$\gamma_m^{j-1} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m!}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n! w_n^m}\left\{e^{\theta b_n}\Psi_m^{(n)}(z_j^{(n)}) - \frac{1}{\sqrt{\pi}}H_{m-1}(z_j^{(n)})e^{-(z_j^{(n)})^2}\right\},$$

which can be rearranged to produce

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{\theta b_n}}{w_n^m}\Psi_m^{(n)}(z_j^{(n)}) &= e^{(r+\lambda)\Delta t}2^m m!\gamma_m^{j-1}\\ &+\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n! w_n^m}\frac{1}{\sqrt{\pi}}H_{m-1}(z_j^{(n)})e^{-(z_j^{(n)})^2}.\end{aligned}$$

From the recurrence for $\Psi_m^{(n)}$ we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{\theta b_n}}{w_n^m}\Psi_m^{(n)}(z_j^{(n)}) &= \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m\sqrt{\pi}}H_{m-1}(z_j^{(n)})e^{\theta b_n+\theta w_n z_j^{(n)}-(z_j^{(n)})^2}\\ &+\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{\theta b_n}}{w_n^{m-1}}\theta\Psi_{m-1}^{(n)}(z_j^{(n)}),\end{aligned}$$

and by substitution we find that

$$
\begin{aligned}
e^{(r+\lambda)\Delta t}2^m m!\gamma_m^{j-1} &= \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m\sqrt{\pi}}H_{m-1}(z_j^{(n)})e^{-(z_j^{(n)})^2}\left[e^{\theta(b_n+w_n z_j^{(n)})}-1\right] \\
&\quad +\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{\theta b_n}}{w_n^{m-1}}\theta\Psi_{m-1}^{(n)}(z_j^{(n)}) \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m\sqrt{\pi}}H_{m-1}(z_j^{(n)})e^{-(z_j^{(n)})^2}\left[e^{\theta(b_n+w_n z_j^{(n)})}-1\right] \\
&\quad +\theta e^{(r+\lambda)\Delta t}2^{m-1}(m-1)!\gamma_{m-1}^{j-1} \\
&\quad +\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!w_n^{m-1}}\frac{\theta}{\sqrt{\pi}}H_{m-2}(z_j^{(n)})e^{-(z_j^{(n)})^2}.
\end{aligned}
$$

Hence the recurrence relation for γ_m^{j-1} is

$$
\begin{aligned}
\gamma_m^{j-1} &= \frac{\theta\gamma_{m-1}^{j-1}}{2m}+\frac{e^{-(r+\lambda)\Delta t}}{2^m m!}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{-(z_j^{(n)})^2}}{w_n^m\sqrt{\pi}} \\
&\quad \times\left\{H_{m-1}(z_j^{(n)})[e^{\theta(b_n+w_n z_j^{(n)})}-1]+w_n\theta H_{m-2}(z_j^{(n)})\right\},
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_0^{j-1} &= e^{-(r+\lambda)\Delta t}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\left\{e^{\theta b_n}\Psi_0^{(n)}(z_j^{(n)})-\Omega_0^{(n)}(z_j^{(n)})\right\} \\
&= \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\left\{e^{b_n\theta}e^{\frac{w_n^2\theta^2}{4}}\text{erfc}\left(z_j^{(n)}-\frac{w_n\theta}{2}\right)-\text{erfc}(z_j^{(n)})\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_1^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!w_n}\left\{e^{\theta b_n}\Psi_1^{(n)}(z_j^{(n)})-\Omega_1^{(n)}(z_j^{(n)})\right\} \\
&= \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!w_n}\left\{\frac{1}{\sqrt{\pi}}e^{-(z_j^{(n)})^2}e^{\theta(w_m z_j^{(n)}+b_n)}\right. \\
&\quad \left.+\frac{\theta w_n}{2}e^{b_n\theta}e^{\frac{w_n^2\theta^2}{4}}\text{erfc}\left(z_j^{(n)}-\frac{w_n\theta}{2}\right)-\frac{1}{\sqrt{\pi}}e^{-(z_j^{(n)})^2}\right\}.
\end{aligned}
$$

Next we consider the α coefficients. Substituting the expression for $V^{j-1}(\xi_{j-1})$ into the equation for α_m^{j-1}, we have

$$\begin{aligned}
\alpha_m^{j-1} &= \frac{1}{2^m m!\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} h^{j-1}(\xi_{j-1})H_m(\xi_{j-1})d\xi_{j-1} \\
&\quad + \frac{1}{2^m m!\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} H_m(\xi_{j-1})\frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}}\sum_{n-0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!} \\
&\quad \times \int_{-\infty}^{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}} e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2} V^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)d\xi_j d\xi_{j-1} \\
&= \gamma_m^{j-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_{-\infty}^{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}} V^j(\hat{v}_n\sqrt{2\Delta t}\xi_j) \\
&\quad \times \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2} H_m(\xi_{j-1})d\xi_{j-1}d\xi_j \\
&= \gamma_m^{j-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_{-\infty}^{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}} V^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)I_m^{(n)}(\xi_j)d\xi_j,
\end{aligned}$$

where $I_m^{(n)}(\xi_j)$ is given by equation (A5.1.1), and evaluated to produce (A5.1.4). With the change of variable $z = (\hat{v}_n\sqrt{2\Delta t}\xi_j - b_n)/w_n$, the expression for α_m^{j-1} becomes

$$\alpha_m^{j-1} = \gamma_m^{j-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n! w_n^m}\int_{-\infty}^{z_j^{(n)}} e^{-z^2}H_m(z)V^j(b_n+w_n z)dz.$$

Substituting the Fourier-Hermite expansion for $V^j(b_n+w_n z)$ into the expression for α_m^{j-1}, and truncating the series at term N, we obtain

$$\begin{aligned}
\alpha_m^{j-1} &= \gamma_m^{j-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n! w_n^m}\int_{-\infty}^{z_j^{(n)}} e^{-z^2}H_m(z)\sum_{i=0}^{N}\alpha_i^j H_i(b_n+w_n z)dz \\
&= \gamma_m^{j-1} + e^{-(r+\lambda)\Delta t}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\sum_{i=0}^{N}A_{m,i}^{j,n}\alpha_i^j, \quad (j=J,J-1,\cdots,1),
\end{aligned}$$

where

$$A_{m,i}^{j,n} \equiv \frac{1}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{z_j^{(n)}} e^{-z^2}H_m(z)H_i(b_n+w_n z)dz, \quad (m,i=0,1,2,\cdots,N).$$

A5.2.2. Proof of Proposition 5.4.2. If we apply the three-term Hermite polynomial recurrence relation from Appendix A5.1.2 to equation (5.4.7) we find that

$$A_{m,i}^{j,n} = \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} 2ze^{-z^2} H_{m-1}(z) H_i(b_n + w_n z) dz - \frac{1}{2mw_n^2} A_{m-2,i}^{j,n}.$$

By use of integration by parts, this becomes

$$\begin{aligned} A_{m,i}^{j,n} &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \Bigg\{ \left[-e^{-z^2} H_{m-1}(z) H_i(b_n + w_n z) \right]_{-\infty}^{z_j^{(n)}} \\ &\quad + \int_{-\infty}^{z_k^{(n)}} e^{-z^2} [2(m-1) H_{m-2}(z) H_i(b_n + w_n z) \\ &\qquad + 2iw_n H_{i-1}(b_n + w_n z) H_{m-1}(z)] dz \Bigg\} - \frac{1}{2mw_n^2} A_{m-2,i}^{j,n} \end{aligned}$$

Thus the recurrence for $A_{m,i}^{j,n}$ is

$$A_{m,i}^{j,n} = \frac{i}{m} A_{m-1,i-1}^{j,n} - \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} H_{m-1}(z_j^{(n)}) H_i(b_n + w_n z_j^{(n)}) e^{-(z_j^{(n)})^2}.$$

To use this recurrence, we require $A_{m,0}^{j,n}$, $A_{0,i}^{j,n}$ and $A_{0,0}^{j,n}$. Beginning with $A_{m,0}^{j,n}$,

$$\begin{aligned} A_{m,0}^{j,n} &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} e^{-z^2} H_m(z) dz \\ &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} 2ze^{-z^2} H_{m-1}(z) dz - \frac{1}{2mw_n^2} A_{m-2,0}^{j,n}. \end{aligned}$$

By an application of integration by parts, this simplifies to

$$\begin{aligned} A_{m,0}^{j,n} &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \left\{ \left[-e^{-z^2} H_{m-1}(z) \right]_{-\infty}^{z_j^{(n)}} + \int_{-\infty}^{z_j^{(n)}} e^{-z^2} 2(m-1) H_{m-2}(z) dz \right\} \\ &\quad - \frac{1}{2mw_n^2} A_{m-2,0}^{j,n} \\ &= -\frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} e^{-(z_j^{(n)})^2} H_{m-1}(z_j^{(n)}), \quad (m = 1, 2, 3, \cdots). \end{aligned}$$

Next we consider $A_{0,i}^{j,n}$, which is given by

$$\begin{aligned} A_{0,i}^{j,n} &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} e^{-z^2} H_i(b_n + w_n z) dz \\ &= \frac{w_n}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} 2ze^{-z^2} H_{i-1}(b_n + w_n z) dz + 2b_n A_{0,i-1}^{j,n} - 2(i-1) A_{0,i-2}^{j,n}. \end{aligned}$$

With an application of integration by parts, this becomes

$$\begin{aligned} A_{0,i}^{j,n} &= \frac{w_n}{\sqrt{\pi}}\left\{\left[-e^{-z^2}H_{i-1}(b_n+w_nz)\right]_{-\infty}^{z_j^{(n)}}+\int_{-\infty}^{z_j^{(n)}}e^{-z^2}2(i-1)w_nH_{i-2}(b_n+w_nz)dz\right\} \\ &\quad -2(i-1)A_{0,i-2}^{j,n}+2b_nA_{0,i-1}^{j,n}, \end{aligned}$$

and hence the recurrence for $A_{0,i}^{j,n}$ is

$$A_{0,i}^{j,n}=2(w_n^2-1)(i-1)A_{0,i-2}^{j,n}+2b_nA_{0,i-1}^{j,n}-\frac{w_n}{\sqrt{\pi}}H_{i-1}(b_n+w_nz_j^{(n)})e^{-(z_j^{(n)})^2}.$$

Finally, to implement the recurrence for $A_{m,i}^{j,n}$, we must obtain the initial values $A_{0,0}^{j,n}$ and $A_{0,1}^{j,n}$, which are given by

$$A_{0,0}^{j,n}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{z_j^{(n)}}e^{-z^2}dz=\frac{1}{\sqrt{\pi}}\int_{-z_j^{(n)}}^{\infty}e^{-z^2}dz=\frac{1}{2}\text{erfc}(-z_j^{(n)}),$$

and

$$\begin{aligned} A_{0,1}^{j,n} &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{z_j^{(n)}}e^{-z^2}2(b_n+w_nz)dz \\ &= b_n\text{erfc}(-z_j^{(n)})-\frac{w_n}{\sqrt{\pi}}e^{-(z_j^{(n)})^2}. \end{aligned}$$

Appendix 5.3. Fourier-Hermite Algorithm for the American Call Option under Jump-Diffusion

Here we present the algorithm *Fourier-Hermite American Call Price* which outlines the iterative scheme for evaluating the price and free boundary of an American call option under jump-diffusion using Fourier-Hermite series expansions.

Algorithm *Fourier-Hermite American Call Price*

Input: S, r, q, σ, K, T (time to expiry), λ, γ, δ, J (number of time intervals), N (number of basis functions).

Output: C (American call price), a (early exercise boundary).

1. estimate θ using the European call under jump-diffusion
2. calculate α_0^{J-1}, α_1^{J-1}
3. **for** $m=2$ **to** N
4. **do** calculate α_m^{J-1}
5. solve for ξ_{J-1}^* using equation (5.5.1)

for $j = J - 1$ **downto** 1
    do calculate $A^j_{0,0}$, $A^j_{0,1}$, $A^j_{1,0}$
        for $m, i = 2$ **to** N
            do calculate $A^j_{m,0}$, $A^j_{0,i}$
        calculate γ_0^{k-1}, γ_1^{k-1}
        for $m = 2$ **to** N
            do calculate γ_m^{j-1}
        for $m = 0$ **to** N
            do calculate α_m^{j-1}
        solve for ξ^*_{j-1} using equation (5.5.1)
for $j = 0$ **to** J
        $a_j = Ke^{\theta \xi^*_j}$
if $S < a(0)$
    then $C(S, 0) = K \sum_{m=0}^{N} \alpha_m^0 H_m(\ln(S/K)/\theta)$
    else $C(S, 0) = S - K$

CHAPTER 6

Conclusion

6.1. Summary of Findings

The option pricing framework pioneered by Black and Scholes (1973) and Merton (1973) provides the basis for modern derivative security pricing. Although frequently traded within financial markets, American options remain the subject of much contemporary research. It is the holder's right to exercise the option prior to the expiry date that makes pricing American options far more complex than their European counterparts. An American option price is a function of its unknown early exercise boundary, which must be found as part of the solution. There are many ways to structure this problem, including the free boundary formulation of McKean (1965) and the compound option approach of Kim (1990). The problem can then be solved using a variety of numerical techniques.

In this thesis we have explored how Fourier-type solution methods can be extended to evaluate American option pricing problems with complex payoff structures and more general dynamics for the price of the underlying asset. The first part of the thesis explored McKean's incomplete Fourier transform method, and how it can be generalised to solve problems beyond the basic American call. For pure-diffusion processes we demonstrated that the transform method provides a systematic approach for deriving the integral expression for American option prices, along with the corresponding integral equations for their free boundaries. The American strangle portfolio is then considered as an example of a general convex payoff function. A further extension of the transform approach is provided for the jump-diffusion model of Merton (1976). We have shown that the incomplete Fourier transform approach can be extended to the jump-diffusion problem with a general jump-size distribution, and provided an iterative numerical integration method to evaluate the American call in the case of log-normal jump sizes. Finally, we extended the Fourier-Hermite series expansion method of Chiarella et al. (1999) to Merton's jump-diffusion with log-normal jumps for American call options.

6.1.1. Evaluation of American Portfolios. When evaluating American options as free boundary value problems, it is possible to obtain several forms for the pricing integral expressions and early exercise boundary integral equations. In Chapter 2 we presented a survey deriving these various integral representations of American option prices, focusing on the American call example. We revisited McKean's (1965) incomplete Fourier transform method, and demonstrated how his results reconcile with the early exercise premium representation of Kim (1990), and the intrinsic/time value decomposition of Carr et al. (1992). In particular, we found that the Fourier transform method had the distinct advantage of being able to proceed with very general knowledge of the payoff function, whereas the compound option solution technique of Kim (1990) required explicit knowledge of the payoff.

With no known closed-form solution for the American call option, it is necessary to solve for the price and early exercise boundary using numerical methods. We compared five existing numerical techniques in Chapter 2, and found that binomial trees, the Crank-Nicolson finite difference scheme, direct numerical integration and the method of lines were all able to produce prices of comparable accuracy. The Fourier-Hermite series expansion method was relatively close to these other four methods, but showed some relatively minor pricing inconsistencies for in-the-money calls. Numerical integration, the method of lines and the Hermite series expansion also required that the free boundary be estimated as part of the solution. The first two methods yielded highly consistent free boundary estimates, whilst the Hermite series showed some signs of error, and in particular was ill-suited to the case were the risk-free rate exceeded the continuous dividend yield of the underlying.

In terms of computational efficiency, finite differences proved the fastest method, although no free boundary estimate was generated as part of the solution. This was followed closely by the Hermite series method, which was able to solve the problem very quickly at the cost of some accuracy in the free boundary estimate. The method of lines proved to be the slowest, as a very fine space-discretisation was needed to keep the free boundary estimate monotonic. Numerical integration of Kim's integral equation for the early exercise boundary appeared to provide the best compromise between numerical accuracy and computational efficiency, and was able to overcome oscillations in the free boundary by use of Richardson extrapolation.

In Chapter 3 we used an American strangle portfolio to demonstrate how McKean's incomplete Fourier transform method can be extended to more complex payoff structures. The American strangle was defined such that exercising one side of the position early would knock-out the remaining side. A McKean-type of integral expression for this strangle's price were derived, along with the integral equation system for its two free boundaries. The integral equations were re-expressed in a more economically intuitive form using Kim's simplifications. We also established the important result that the free boundaries for the American strangle are not equal to those found when valuing independent American calls and puts.

Solving the resulting system of nonlinear Volterra-style integral equations, it was found that the early exercise boundary of the strangle only differed significantly from the boundaries of corresponding American calls and puts for certain values of the risk-free rate and continuous dividend yield parameters. The differences became larger as the distance between the strangle's strikes was reduced, and also as the time to expiry increased. This latter point was highlighted by considering the perpetual American strangle. In terms of pricing, the strangle under consideration was cheaper than the "traditional" one by no more than 6% for the parameters considered, and these differences were most apparent when the strangle was deep in-the-money. Economically, this pricing difference is interpreted as the reduction in value caused by introducing the knock-out effect into the new strangle, and foregoing the freedom to separate the call and put sides.

The early exercise boundaries for our strangle required that the position be deeper in-the-money than a "traditional" strangle, to compensate the intrinsic value forgone on the out-of-the-money side. The prices of the two strangles were usually very close despite the free boundary differences, and an important contribution of this thesis has been to quantify this difference. An investor interested in an American strangle position may be indifferent when choosing between this proposal and a "traditional" American strangle based on initial costs, as only a small increase in premium is required to obtain the added flexibility of the latter.

6.1.2. Evaluating American Call Options under Jump-Diffusion using Fourier Transforms. To further demonstrate the broad applicability of McKean's (1965) incomplete Fourier transform method, Chapter 4 generalises his free boundary value problem for the American call option to the case where the underlying asset follows a jump-diffusion process, as originally proposed by Merton (1976). We solved the PIDE using this approach to obtain a coupled integral equation system for the price and free boundary of the American call, where the jumps arrive according to a Poisson process, and the jump sizes follow some general distribution. We showed how these may be manipulated into the integral equations for the American call derived by Gukhal (2001). This new approach both recovers Gukhal's results found via the compound option method, and demonstrates that the transform technique can be readily used to solve the American call pricing problem under jump-diffusion dynamics.

We derived a simplification of the triple integral expression in the integral equations in the case where the jump-sizes are log-normally distributed. This reduces the computational burden when one proceeds to numerical implementation. A vital result has been the derivation of the limit of the free boundary at expiry for the jump-diffusion model with log-normal jump sizes. This is a new result that has not previously been presented, as far as we are aware, even though existing numerical results (such as Amin (1993)) clearly imply that the free boundary just prior to expiry does indeed change when jumps are introduced into the underlying dynamics. Numerical experiments indicated that this limit displays all the correct properties required, and it conforms with existing published free boundary results, although no formal proofs for the behaviour of the free boundary at expiry have yet been obtained. We also presented results concerning the perpetual American call under jump-diffusion.

A means of numerically implementing the coupled integral equation system for the American call was provided, based on the numerical approach in the American strangle analysis of Chapter 3. An iterative method was proposed to deal with the interdependence between the call price and free boundary. The method is mathematically simple, and intended to provide an initial means for numerically analysing the behaviour of the integral equation system, something not addressed by existing literature. The numerical results presented from this method demonstrated that even a small frequency of expected jumps has a dramatic impact on the price profile of the call option, particularly at-the-money. The mean expected jump size was also shown to have a significant impact on the free boundary for the American call, which we attributed in part to the added cost incurred by the option holder whenever the underlying's price jumps downward from the stopping region into the continuation region. This feature was previously identified analytically by Gukhal (2001), and has been reinforced in our numerical findings.

These integration results have replicated the conclusions of Amin (1993) regarding the impact of jumps on the behaviour of the early exercise boundary for the American call. We provided further evidence that early exercise of the call is more likely under jump-diffusion near expiry, while away from expiry early exercise is less likely. Secondly, we demonstrated that the addition of jumps increased the value of out-of-the-money American calls, while at-the-money calls became less valuable. This behaviour is in accordance with the well-known volatility smile phenomenon observed in option prices within financial markets. The in-the-money value can also be greater with jumps, but the impact of jumps on the price in this case is always restricted by the proximity of the free boundary.

6.1.3. Evaluating American Call Options under Jump-Diffusion using Fourier-Hermite Series Expansions. The numerical integration solution method outlined in Chapter 4 provided an initial approximation for the American call price and free boundary under jump-diffusion, but the method is highly computationally-intensive in its present form. In Chapter 5 we presented a generalisation of the Fourier-Hermite series expansion method of Chiarella et al. (1999) as an alternative to the time-consuming numerical integration approach. This extension considered the jump-diffusion model of Merton (1976) in the case where the jump sizes are log-normally distributed. We derived the recurrence relations for both the European and American call option under jump-diffusion, and presented the special time-stepping algorithm to account for early exercise in the American case. An unspecified scaling parameter is required to be known when implementing this method, and we demonstrated how one can estimate its value using Merton's closed-form solution for the European call price.

We generated a range of American call prices using the series expansion method, and the results were compared with those generated using the numerical integration method from Chapter 4, as well as the method of lines approach by Meyer (1998). We found that all three methods produced relatively consistent prices, and in particular that the Fourier-Hermite prices were always within 1% of the numerical integration results, with only two reported exceptions. When the global volatility of the price process was sufficiently large, the Fourier-Hermite method yielded excellent levels of accuracy when compared with the standards displayed in the existing literature. The Fourier-Hermite method also proved to be extremely efficient, requiring significantly less computation time than either of the alternatives presented. This demonstrated that for pricing purposes, the Fourier-Hermite method is a highly competitive alternative when the pricing dynamics are more complicated than the basic Black-Scholes diffusion process.

The most notable short-coming for the Fourier-Hermite approach was in estimating the early exercise boundary. The method could not reproduce the correct free boundary near expiry, and was only able to achieve a solution containing a systematic error near the current time. The expiry issue we attributed to difficulties encountered when estimating functions that are close to piecewise linear in form using polynomials, such as the value of an American call or put near expiry. For the current-time discrepancy, we suggested that the centralisation of the series expansions around the strike could be a contributing factor, and this cannot be easily overcome without foregoing price accuracy in the critical region around the strike. It is important to note, however, that even with small inaccuracies in the free boundary estimate, the resulting prices have remained accurate. This demonstrated that the series expansion technique has a potential trade-off in the form of increased computational speed at the cost of accuracy in estimating the early exercise boundary, most predominantly near expiry. Thus the method is still an extremely valuable alternative for efficiently pricing American options under jump-diffusion processes where the jump sizes follow a specified continuous distribution. Further computation time is saved by virtue of the continuous estimating of the option price in terms of Hermite polynomials. This also makes estimating the delta and gamma for the American call simple (and also more accurate than with finite difference and tree methods) once the Fourier-Hermite coefficients have been found.

6.2. Directions for Future Research

The results presented in this thesis imply a number of avenues for future research. Given that the incomplete Fourier transform has been shown to readily generalise to handle American options with more complex payoff functions and pricing dynamics, there are many other models that could be explored. Two-dimensional extensions could also be considered, including American options under stochastic volatility, and American options on multiple underlying assets, such as an American basket option. The transform-based methodology developed in this thesis is applicable to American positions with quite general convex or concave payoffs. It would be of interest to consider other complex payoff types, such as an American butterfly (concave payoff), or an American bear/bull spread (monotonic payoffs). These are collectively equivalent to the capped American call problem of Broadie and Detemple (1995), and can be constructed with similar early exercise conditions to the American strangle, thus facilitating its evaluation using our generalisation of McKean's framework.

Whether or not the reduced transaction costs from the self-closing American strangle would benefit the investor is a matter we leave to future study. The numerical method presented for the strangle should be rigourously tested against existing techniques, such as binomial trees and finite differences. Better numerical techniques for solving the integral equation system also need to be investigated. In addition, one could explore the potential to numerically solve McKean's integral equation in its original form, extending the methods of Buchen et al. (2000) to more general payoff types.

In the case of jump-diffusion, the transform method presented in this thesis is readily applicable to a range of jump size distributions and payoff functions, and one area for future research would be to explore these alternatives. We assumed, as in Merton's model, that any jump risk associated with the underlying is fully diversifiable. This assumption could be relaxed within the Fourier transform framework, but only for certain cases, such as constant and time-dependent jump risk. For the limit of the free boundary at expiry, there is clearly a need to provide explicit proofs for its behaviour when the dividend yield is zero, and to ensure the explicit expression has a unique solution for non-zero dividend yields. The numerical algorithm presented for the jump-diffusion integral equation system is clearly a "first-pass" solution for the American call price and free boundary. Further analysis needs to be conducted to improve the speed, accuracy and efficiency of the method by using more advanced numerical analysis techniques.

With regards to the Fourier-Hermite series expansion, given that the free boundary estimate near expiry is suboptimal, some alternative estimate would be of significant value. A small-time expansion of the free boundary near expiry remains unaddressed for American calls under jump-diffusion. This could prove extremely challenging in light of the fact that the free boundary just before expiry is not well understood at present for the jump-diffusion model, and even under pure-diffusion, there are many suggestions on what the small-time expansions should be (see for example Chen and Chadam (2000)). There is still no clear consensus as to what the exact price is for the American call under jump-diffusion with log-normal jump sizes, and this continues to cast some doubt regarding the accuracy of any numerical method being considered. While the Fourier-Hermite series method has the advantage of being well-suited to the case where jump sizes follow a log-normal distribution, it is not yet known how the method would perform for jump sizes with discrete distributions. Finally, we have not offered an explicit optimisation routine for selecting the scaling parameter prior to finding the American call price. Determining and verifying an explicit optimisation routine, or an explicit form for the scaling parameter in terms of the global volatility of the jump-diffusion process, would further increase the robustness of the method. All these additional extensions and further details we leave to future research.

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