

# BOUNDARY CROSSING PROBLEMS IN INSURANCE

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# Abstract

In the actuarial sense, a risk process models a surplus of an insurance company. The company is allowed to invest money with a constant interest rate. Some generalizations of the constant interest rate models are also considered. Ruin is defined to have occurred when the risk process reaches some certain level, which is less than the initial capital. In particular the level is assumed to be zero.

Papers such as Harrison [17], Schmidli [37] and Embrechts & Schmidli [11] consider similar models with constant interest rate and obtain explicit solutions as well as diffusion approximations for the probability of ruin in infinite time. Our main approach is to use Martingale techniques in order to obtain exact solutions for probabilities of ruin in the finite time horizon which are further compared with numerical simulations. Furthermore, we analyse models with more general interest rate and propose a series of methods which can be used in order to determine the finite time ruin probabilities.