BOUNDARY CROSSING PROBLEMS IN INSURANCE

By

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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN MATHEMATICAL SCIENCES
AT
UNIVERSITY OF TECHNOLOGY, SYDNEY
SYDNEY, AUSTRALIA
JUN 2006

UNIVERSITY OF TECHNOLOGY, SYDNEY
CERTIFICATE OF AUTHORSHIP/ORGINALITY

Date: **Jun 2006**

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Title: **Boundary Crossing Problems in Insurance**

Department: **Mathematical Sciences**

Degree: **M.Sc.** Convocation: **Mar** Year: **2005**

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ACKNOWLEDGMENTS

There are many people I would like to acknowledge for their support, both technical and emotional, during my MSc studies. I list but a few here.

I would like to thank Professor Alexander Novikov, my supervisor, for providing opportunity to take up a MSc and for continuing advice and enthusiasm throughout the research.

My thanks go to the staff and students of the Mathematical Science Department for their support and for friendly atmosphere. On the financial note, I wish to express my gratitude to my supervisor Professor Alexander Novikov and also Professor Eckhard Platen. Further, I would like to thank Dale Roberts for his editorial assistance.

Finally to my friends, my family and to Lubos, thank you all for being there.

Table of Contents

List of Tables

List of Figures

Abstract

In the actuarial sense, a risk process models a surplus of an insurance company. The company is allowed to invest money with a constant interest rate. Some generalizations of the constant interest rate models are also considered. Ruin is defined to have occurred when the risk process reaches some certain level, which is less than the initial capital. In particular the level is assumed to be zero.

Papers such as Harrison [17], Schmidli [37] and Embrechts & Schmidli [11] consider similar models with constant interest rate and obtain explicit solutions as well as diffusion approximations for the probability of ruin in infinite time. Our main approach is to use Martingale techniques in order to obtain exact solutions for probabilities of ruin in the finite time horizon which are further compared with numerical simulations. Furthermore, we analyse models with more general interest rate and propose a series of methods which can be used in order to determine the finite time ruin probabilities.

Introduction

0.1 Thesis Outline

The subject of this thesis is risky investment modelling for insurance businesses. In particular, we look at models with deterministic interest rate, both constant and dependent on time. Further, we define and determine some explicit formulas for ruin probabilities in infinite and finite time horizon. In case of complex models, when explicit formulas of ruin probabilities are not given, we propose some approximations for such probabilities.

We restrict our attention to one specific insurance portfolio. Such portfolio is characterised by a number of ingredients of both deterministic and stochastic nature. Classical models, such as the Cramér – Lundberg model introduced in detail in section 2.1.1, take into account the following factors:

- time period,
- starting position – the initial capital,
- premium income, and
- stochastic nature of claims.

Some of the more advanced models in the literature are additionally equipped with deterministic or stochastic interest rate, inflation or dividends.

The classical model mentioned above is the Cramér – Lundberg model given by the following formula

$$X_t = x + ct - \sum_{i=1}^{N_t} \xi_i, \tag{0.1.1}$$

where the time parameter $t \geq 0$, x is the initial capital of the insurance company, c is the premium income which increases with time linearly and $\sum_{i=1}^{N_t} \xi_i$ is the accumulated claims amount up to time t. N_t is a Poisson process with parameter $\lambda > 0$ and ξ_i are random variables governed by the distribution function $F(x)$. In principal, any distribution concentrated on the nonnegative half line, can be used as a claim size distribution $F(x)$. However, we will make a distinction between *well – behaved* distributions and dangerous distributions with a *heavy tail.* Concepts like *well – behaved* or *heavy – tailed* distributions belong to the common vocabulary of actuaries. Roughly speaking, the class of *well – behaved* distributions consists of those distributions F with an exponentially bounded tail. This means that large claims are not impossible, but the probability of their occurrence decreases at least exponentially fast to zero as the threshold x becomes larger and larger. This thesis is mainly concerned with the *well – behaved* claim size distributions, in particular, with claim sizes following exponential distributions.

The most of the attention is devoted to a model with the deterministic interest rate. This model is given by the following Stochastic Differential Equation (SDE)

$$dX_t = \beta_t X_t dt + dL_t, \tag{0.1.2}$$

where $t \geq 0$, β_t is the time dependent deterministic interest rate, in particular $\beta_t = \beta = const$. Further, L_t is a Lévy process responsible for the randomness in the model. When $L_t = ct - \sum_{i=1}^{N_t} \xi_i$, the model 0.1.2 is a generalization of the model 0.1.1 to a model with a deterministic interest rate.

0.2 Research Motivation and Objectives

Ruin theory has always been a vital part of actuarial mathematics. Calculations of and approximation to ruin probabilities have been a constant source of inspiration and technique developed in actuarial mathematics. The actuary has to make decisions, for instance, which premium should be charged or which type of reinsurance to take. These are often determined by the means of minimization of the probability of ruin. To be more specific, consider the risk reserve X_t and define the random variable

$$\tau_l = \inf\{t \geq 0 : X_t \leq l\}.$$

Hence, the ruin is defined as a first crossing time through a level l less then the initial capital x by the process X_t. We stress that τ_l is dependent on all the stochastic elements in the risk reserve process X_t as well as on the deterministic value x.

In the literature we located some explicit results related mainly to the infinite time ruin probabilities such as Harrison [17], Schmidli [37] and Embrechts & Schmidli [11]. Further, we also found research pertaining to finite time probabilities, however the results were obtained by the solution of Partial Integro Differential Equations (PIDE) as in Paulsen & Gjessing [27] or by the Extreme Value Theory (EVT) as in Tang [40] and those solutions cover just some special cases of the problem. We use a martingale method as, in many instances, more suitable and more general approach, in contrast to those methods (chapter 3). Additionally, most of the research is devoted to the constant interest rate models. There is still a gap to fill in the research concerning time dependent interest rate models. Motivated by Roberts and Shortland [32] we analyse a model with an interest rate dependent on time but deterministic.

The main aim of this thesis is to determine finite time ruin probabilities for the model 0.1.2. As one can expect, ruin probabilities will depend heavily on the claim size distribution. If the latter is *well – behaved* the ruin probabilities will turn out to be typically exponentially bounded as the initial capital becomes large. However, when the claim size distribution has a heavy tail, then one single large claim may be responsible for the ultimate ruin of the portfolio. Additionally, it turns out that models powered by *well – behaved* distributions can be successfully approximated by diffusion processes, which are relatively easier to analyze.

The results for ruin probabilities on which we will focus in this thesis can be characterized by the following features and mathematical tools.

- only in the easiest cases we will succeed in getting explicit formulas for the ruin probabilities as in section 3.6.1
- martingale methods are used in order to obtain Laplace transforms for a stopping time given by $\tau_l = \inf\{t \geq 0 : X_t \leq l\}$ as in section 3.4.2 , which may be further inverted numerically,
- the theory of Piecewise Deterministic Markov Processes (PDMP) is employed in order to support the martingale solution as in section 3.5,
- diffusion approximation is used for more complex models as in chapter 4,
- Piecewise Linear Approximation (PLA) to more general boundaries is used for approximation of ruin probabilities in the finite time horizon as in section 4.3,
- simulation methods are utilized for both illustration and computation purposes as in sections 3.6.3 and 4.4.

0.3 Structure of the Thesis

The thesis is organized into the following chapters:

- Chapter I presents some fundamental mathematical definitions and theorems including martingale basics, the theory of stochastic integration and introduction to Lévy processes. These topics are dealt with only briefly, just enough information is provided to make the thesis self – sufficient.

- Chapter II introduces the classical Cramér – Lundberg model as well as generalizes the model to so called Ornstein – Uhlenbeck processes. Further, it focuses on the theory of Piecewise Deterministic Markov Processes (PDMP), since the main model of this thesis as well as some techniques used in this text are based on this theory. It also divides the general model into three groups with respect to the interest rate nature.

- Chapter III is fully devoted to the models with constant interest rate and calculation of the finite time ruin probabilities using martingale techniques and also integro – differential equations in the case of exponential distribution of claim sizes. It also focuses on special cases of the problem and compares the explicit formulas with results of the Monte Carlo simulations.

- Chapter IV generalizes model analyzed in chapter III for the case of time – dependent interest rate but deterministic. It incorporates some different approaches to the problem of ruin such as diffusion approximation and Piecewise Linear Approximation (PLA) to more general boundaries. It also focuses on a special case of this problem which can be solved analytically. Numerical results are also provided.

Chapter 1

Mathematical Preliminaries

This chapter provides introductory material underlining subsequent chapters. It is not intended as a detailed reference to any of the topics covered as it contains only definitions and theorems related to the research of this thesis. The literature used to write this chapter consists of Applebaum [2], Borovkov [3], Cont & Tankov [8], Fristed & Lawrence [12], Liptser & Shiryaev [19, 20] and Revuz & Yor [31].

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The set Ω of elementary events is equipped with a $\sigma -$ algebra $\mathcal{F}$. A probability measure on $(\Omega, \mathcal{F})$ is a positive finite measure $\mathbb{P}$ with total mass 1. A measurable set $A \in \mathcal{F}$, called an event, is therefore a set of events to which a probability $\mathbb{P}(A)$ can be assigned. Hence,

$$\mathbb{P} : \mathcal{F} \rightarrow [0, 1].$$

Moreover, given two measurable spaces $(\Omega, \mathcal{F})$ and $(E, \mathcal{G})$, define a random variable X as follows:

Definition 1.0.1 (Random Variable). A random variable X taking values in E is a measurable function

$$X : \Omega \to E.$$

Hence, it is a function such that $X^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{G}$.

1.1 Concepts of Probability Theory

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = X(\omega)$ a nonnegative random variable.

Definition 1.1.1 (Expectation). Expectation of X denoted by EX is the Lebesque integral

$$EX = \int_{\Omega} X(\omega)P(d\omega) = \int_{\Omega} XdP.$$

Denote by $X^+ = \max(X,0)$ and $X^- = -\min(X,0)$.

Definition 1.1.2 (Integrable Random Variable). The random variable X is said to be integrable if $E|X| = EX^+ + EX^- < \infty$.

CONVERGENCE OF RANDOM VARIABLES

Definition 1.1.3 (Convergence in Probability). We say that the sequence of random variables $X_n, n = 1, 2, \ldots,$ converges in probability to a random variable X if, for any $\varepsilon > 0$, $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$.

Definition 1.1.4 (Almost Sure Convergence). The sequence of random variables $X_n, n = 1, 2, \ldots,$ is called convergent to a random variable with probability 1, or almost surely, if the set $\{\omega : X_n(\omega) \to X(\omega)\}$ has P – measure one.

Note that convergence with probability 1 (P – a.s.) implies convergence in probability.

Definition 1.1.5 (Weak Convergence). The sequence of random variables $X_n, n = 1, 2, \ldots,$ with $E|X_n| < \infty$, is called weakly convergent to a random variable X if

$$\lim_{n \to \infty} E(f(X_n)) = E(f(X)),$$

for any bounded function f.

Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub – σ – algebra.

Definition 1.1.6 (Conditional Expectation). The conditional expectation of X with respect to $\mathcal{G}$, denoted $E(X|\mathcal{G})$ is by definition any $\mathcal{G}$ – measurable function $Y = Y(\omega)$, for which EY is defined, such that for any $\Lambda \in \mathcal{G}$

$$\int_\Lambda X(\omega) P(d\omega) = \int_\Lambda Y(\omega) P(d\omega).$$

BASIC PROPERTIES OF CONDITIONAL EXPECTATION

1. $E(X|\mathcal{G}) \geq 0$, if $X \geq 0$ (P – a.s.);

2. $E(1|\mathcal{G}) = 1$ (P – a.s.);

3. $E(X + Y|\mathcal{G}) = E(X|\mathcal{G}) + E(Y|\mathcal{G})$ (P – a.s.) if $E(X)$ and $E(Y)$ exist;

4. $E(XY|\mathcal{G}) = XE(Y|\mathcal{G})$ if $E(XY)$ exists and X is $\mathcal{G}$ – measurable;

5. If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then (P – a.s.) $E(X|\mathcal{G}_1) = E[E(X|\mathcal{G}_2)|\mathcal{G}_1]$;

6. If $\mathcal{F}^{\mathcal{Y}}$ is the smallest σ – algebra with respect to which the random element $Y(\omega)$ is measurable and if $\mathcal{G}$ and $\mathcal{F}^{\mathcal{Y}}$ are independent, then (P – a.s.) $E(X|\mathcal{G}) = E(X)$.

Definition 1.1.7 (Absolutely continuous distribution). A distribution P of a random variable ξ is said to be absolutely continuous if, for any Borel set B,

$$P(B) = P(\xi \in B) = \int_B f(x)dx,$$

where $f(x) \geq 0$, $\int_{-\infty}^{\infty} f(x)dx = 1$.

1.2 Stochastic Processes and Stopping Times

Define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathcal{F}_t$, understood as a nondecreasing family of σ – algebras from $\mathcal{F}$.

Definition 1.2.1 (Stochastic Process). A stochastic process is a family $(X_t)_{t\in D}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The set D is called the index set of the process. Formally, a stochastic process is a mapping

$$X : D \times \Omega \to \mathbb{R}.$$

When D is a countable subset of the real line (e.g. $D = \{0, 1, 2, \dots\}$) the stochastic process X_t is said to be a discrete – time process. If D is an interval of the real line (e.g. $D \in [0, \infty)$), the stochastic process is said to be a continuous – time process. If D is a set of multidimensional indexes then X_t is called a spatial process or a random field. If not stated otherwise we will assume that $D = \mathbb{R}_+$, since this thesis deals with continuous time stochastic processes. Further, denote by $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ and $\mathcal{F}_{t-} = \sigma\Big(\bigcup_{s<t} \mathcal{F}_s\Big)$.

In general, we do not require the measurability of X, however if D is a subset of $\mathbb{R}$ and X is measurable with respect to the product – σ – algebra $\mathcal{B}(D) \times \mathcal{F}$, then we say the stochastic process $(X_t)_{t\in D}$ is *measurable*.

For each fixed $\omega \in \Omega$, the function $t \to X(t, \omega)$ is called a *sample path* or *trajectory*, however we usually drop the dependence on $\omega \in \Omega$. This thesis deals with processes having sample paths belonging to one of the following two spaces:

- the space of continuous functions $g : \mathbb{R}_+ \to \mathbb{R}$ denoted by $\mathcal{C}(\mathbb{R}_+)$,

- the space of right – continuous functions $g : \mathbb{R}_+ \to \mathbb{R}$ with left – hand limits denoted by $\mathcal{D}(\mathbb{R}_+)$. We say that a stochastic process with sample paths from $\mathcal{D}(\mathbb{R}_+)$ is *cádlág.*

Theorem 1.2.1 (Fubini 1, [20], p.21). *Let $(X_t)_{t\in D}$ be a measurable stochastic process. Then:*

1. *Almost all trajectories of this process are measurable functions of $t \in D$;*
2. *If EX_t exists for all $t \in D$, then $m_t = EX_t$ is a measurable function of $t \in D$;*
3. *If S is a measurable set in $D = [0, \infty)$ and $\int_S E|X_t|dt < \infty$, then*

$$\int_S |X_t|dt < \infty \quad (P - a.s.) \tag{1.2.1}$$

and

$$\int_S EX_t dt = E \int_S X_t dt. \tag{1.2.2}$$

Theorem 1.2.2 (Fubini 2, [12], p.130). *Let $(\Psi, \mathcal{G}, \mu)$ and $(\Theta, \mathcal{H}, \nu)$ be two σ – finite measure spaces, and let φ be an $\bar{\mathbb{R}} = [-\infty, \infty]$ valued measurable function defined on the product measure space $(\Psi, \mathcal{G}, \mu) \times (\Theta, \mathcal{H}, \nu)$. If*

$$\int_{\Psi\times\Theta} \varphi d(\mu \times \nu)$$

exists, then

$$x \to \int_\Theta \varphi(x, y)\nu(dy)$$

is a μ – almost everywhere defined measurable function from $(\Psi, \mathcal{G})$ to $\bar{\mathbb{R}}$, and

$$\int_{\Psi\times\Theta} \varphi d(\mu \times \nu) = \int_\Psi \Big(\int_\Theta \varphi(x, y)\nu(dy)\Big)\mu(dx).$$

Definition 1.2.2 (Adapted Process). We say that a measurable stochastic process $X_t, t \in D$ is adapted to a family of σ – algebras $\mathcal{F}_t$ if, for any $t \in D$, X_t are $\mathcal{F}_t$ – measurable.

Definition 1.2.3 (Increasing Predictable Process). We say that a stochastic process X_t is an increasing predictable process if for each $t \in D$ the random variable X_t is $\mathcal{F}_{t-}$ – measurable.

Theorem 1.2.3. *Let $X_t, t \in D$ be a right continuous integrable increasing process, $\mathcal{F} = \mathcal{F}_{t+},\ t \geq 0$. Then for each $t > 0$ the variables X_t are $\mathcal{F}_{t-}$ – measurable.*

SOME CLASSES OF STOCHASTIC PROCESSES

Definition 1.2.4 (Stationary Process in a Narrow Sense). The stochastic process $X_t, t \in D$ is said to be stationary in narrow sense (strictly stationary) if for any real δ the finite – dimensional distributions do not change with the shift on δ:

$$P(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = P(X_{t_1+\delta} \in A_1, \ldots, X_{t_n+\delta} \in A_n),$$

for $t_1, \ldots, t_n, t_1 + \delta, \ldots, t_n + \delta \in D$.

Definition 1.2.5 (Stationary Process in a Wide Sense). The stochastic process $X_t, t \in D$ is called stationary in a wide sense if $EX_t^2 < \infty$ and $EX_t = EX_{t+\delta}$, $E(X_s X_t) = E(X_{s+\delta} X_{t+\delta})$, i.e. if the first and second moments do not change with the shift.

Definition 1.2.6 (Process with Independent Increments). The process X_t, $t \in D$ is a process with independent increments if, for any $t_n > t_{n-1} > \cdots > t_1 > 0$, the increments $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ yield a system of independent random variables.

Definition 1.2.7 (Homogeneous process). A process with independent increments is called homogeneous if the distribution of the probabilities of the increments $X_t - X_s$ depends only on the differences $t - s$.

Definition 1.2.8 (Martingales). The adapted stochastic process $X_t, t \in D$ is called a martingale with respect to $\mathcal{F}_t$ if $E|X_t| < \infty$ and

$$E(X_t|\mathcal{F}_s) = X_s \quad (P - a.s.), \quad t \geq s. \tag{1.2.3}$$

Denote by $M(\mathcal{F}_t, \mathbb{P})$ a class of martingales with respect to $(\mathcal{F}_t, \mathbb{P})$.

STOPPING TIME AND THE OPTIONAL STOPPING THEOREM

Definition 1.2.9 (Stopping time). A random moment τ is said to be a stopping time or, Markov time with respect to $\mathcal{F}_t$ if the process

$$\mathbb{I}(\tau \leq t) \ is\ \mathcal{F}_t \ - \ adapted,$$

i.e. for all $t \in D$, $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$.

Proposition 1.2.4. *If $m_t \in M(\mathcal{F}_t, \mathbb{P})$ then*

$$E(m_t) = E(m_0).$$

Theorem 1.2.5 (Optional Stopping Theorem, [31], p.67). *If $m_t \in M(\mathcal{F}_t, \mathbb{P})$, $t \geq 0$, then for any stopping time τ*

$$X_t := m_{\min(\tau,t)} \in M(\mathcal{F}_t, \mathbb{P})$$

and hence for any fixed $t \geq 0$

$$E(m_{\min(\tau,t)}) = E(m_0).$$

1.3 Gaussian Processes

All stochastic processes in this section are examples of so – called Gaussian processes. The definition of a Gaussian process is presented below.

Definition 1.3.1 (Gaussian process). A stochastic process X_t, $t \in D$, is called a Gaussian process if any random vector $\mathbf{X} = (X(t_1), \dots, X(t_n))^T$ is Gaussian.

Proposition 1.3.1. *If X_t, $t \in D$, is a Gaussian process then to determine any joint distribution of a Gaussian process*

$$F_{X(t_1),\dots,X(t_n)}(x_1, \dots, x_n) = P\{X(t_1) \le x_1, \dots, X(t_n) \le x_n\}$$

is sufficient to know only two functions:

$$m(t) = E(X(t)), \quad R(t,s) = cov(X(t), X(s)).$$

GAUSSIAN WHITE NOISE AND GAUSSIAN RANDOM WALK

Two well known examples of Gaussian processes are a discrete time Gaussian white noise and a Gaussian random walk.

Definition 1.3.2 (Gaussian White Noise). A stochastic process $X(t)$, $t \in D$, $D = \{0, \Delta, 2\Delta, \dots\}$, $\Delta > 0$ is said to be a Gaussian white noise, if $X(t)$ are independent Gaussian random variables $X(t) \sim N(m, \sigma^2)$ with

$$R(t,s) = \begin{cases} \sigma^2, & t = s; \\ 0, & t \neq s. \end{cases}$$

Definition 1.3.3 (Gaussian Random Walk). Let $X(t)$, $t \in D = \{0, \Delta, 2\Delta, \dots\}$, $\Delta > 0$ be a Gaussian white noise and

$$S_t := X(\Delta) + \dots + X(t) = \sum_{k=1}^{n} X(k\Delta), \quad (t = k\Delta, S_0 = 0)$$

The process S_t is a Gaussian process called a Gaussian random walk with $t = n\Delta$. $S_t \sim N(tm/\Delta, t\sigma^2/\Delta)$.

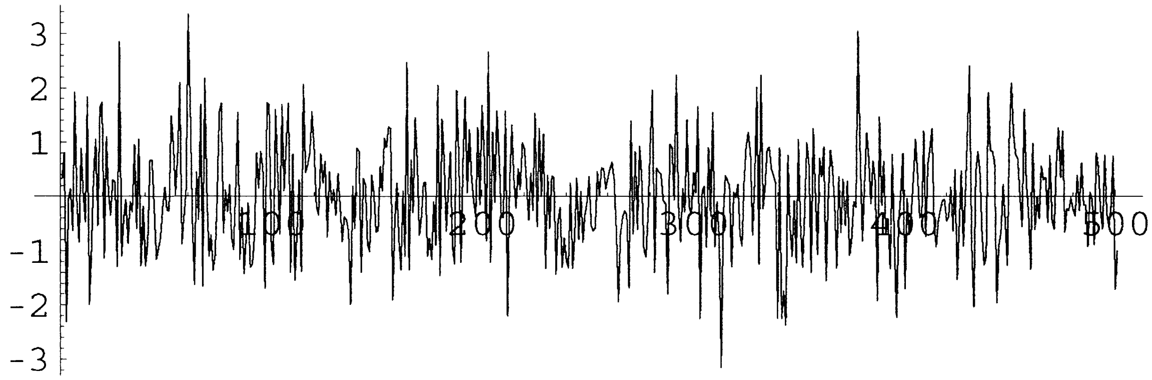

Figure 1.1: Trajectory of a Gaussian White Noise, $\Delta = 1, m = 0$ and $\sigma = 1$.

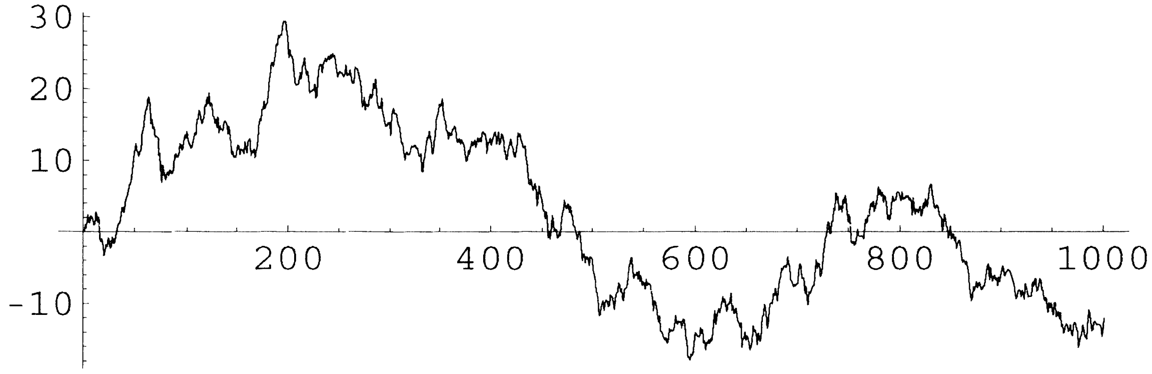

Figure 1.2: Trajectory of a Gaussian Random Walk, $\Delta = 1, m = 0$ and $\sigma = 1$.

Trajectories of the Gaussian white noise and the Gaussian random walk are presented in Figures 1.1 and 1.2 respectively. Both, the Gaussian random walk and the Gaussian white noise are simulated with $\Delta = 1, m = 0$ and $\sigma = 1$. Mathematica codes used to obtain these graphs are included in appendices A.0.1 and A.0.2, respectively.

WIENER PROCESSES

Probably, the most important and widely used Gaussian process is described in this section. It is called a Wiener process or Brownian Motion. Below, the definition of this process is presented as well as its main properties.

Definition 1.3.4 (Wiener Process, Brownian Motion). A stochastic process $W(t)$, $t \in [0, \infty)$ is said to be a Wiener process (or Brownian Motion), if $W(t)$ is a Gaussian process with

$$EW(t) = mt, \quad R(t,s) = \sigma^2 \min(t,s).$$

When $m = 0$ and $\sigma = 1$ the process $W(t)$ is called *Standard Brownian Motion.*

PROPERTIES OF WIENER PROCESS

Proposition 1.3.2 (Independence of Increments). *Denote increments of a Brownian motion $W(t)$ as $\xi(t,h) = W(t+h) - W(t), h > 0$. Then*

$$E(\xi(t,h)) = mh, \quad Var(\xi(t,h)) = \sigma^2 h$$

and

$$cov(\xi(t,h), \xi(s,h)) = \begin{cases} \sigma^2 h, & t = s; \\ 0, & \mid t - s \mid \geq h. \end{cases}$$

Proposition 1.3.3. *Brownian motion has almost surely continuous but almost surely nondifferentiable trajectories.*

Trajectories of a Wiener process in case of $m = 0$ and $\sigma = 1$ (Standard Brownian Motion) can be simulated taking

$$B(t + \Delta) = B(t) + \Delta B(t),$$

where $\Delta B(t) = \sqrt{\Delta} Z$. Moreover, $Z \sim N(0,1)$ and $t \in [0,\infty)$. This simulation is based on the independence of increments property 1.3.2.

Generally, the Wiener process may be simulated using the following representation

$$\Delta W(t) = \sigma \Delta B(t) + m\Delta, \ t \in [0,\infty).$$

Trajectories of Standard Brownian Motion and a Wiener process with $m = 10$ and $\sigma = 5$ are illustrated in Figures 1.3 and 1.4, respectively. Mathematica code used for this simulation is included in the appendix A.0.3.

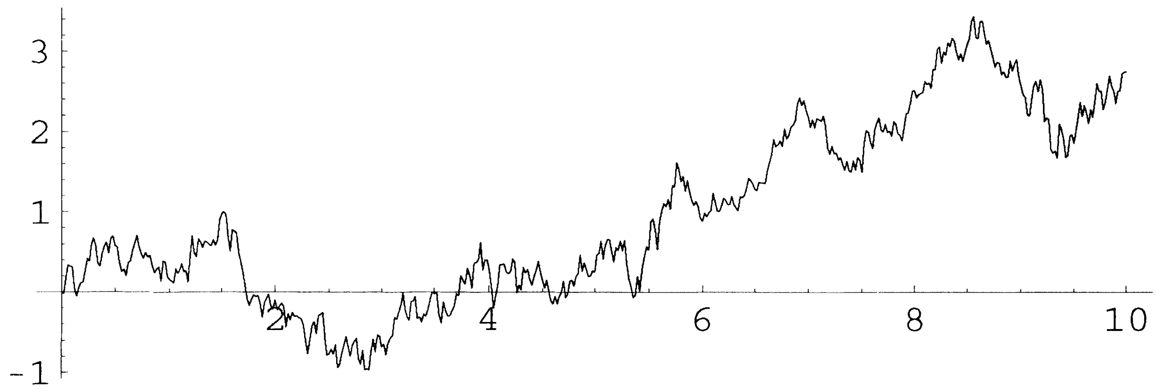

Figure 1.3: Trajectories of Standard Brownian Motion.

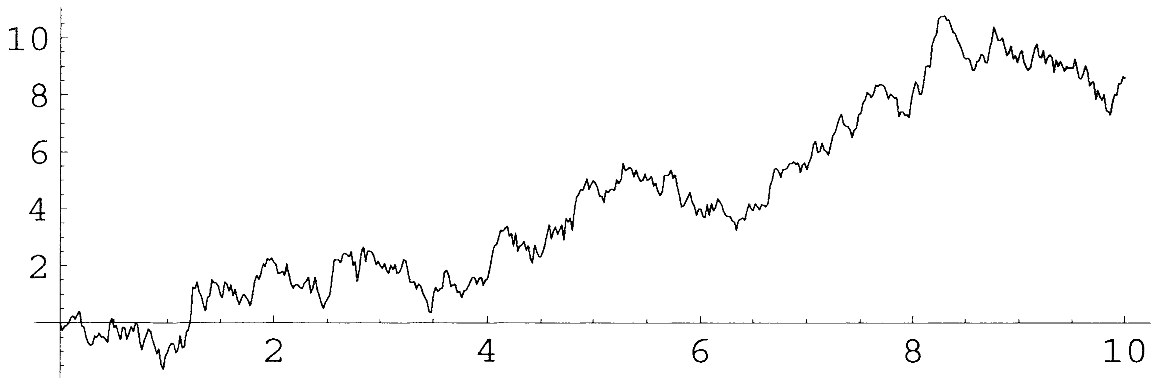

Figure 1.4: Trajectories of a Wiener Process, $m = 0.5$ and $\sigma = 2$.

Eventually, the following theorem states the weak convergence of the sequence of Gaussian random walks $S_{\lfloor nt \rfloor}$ to the Wiener process X_t.

Theorem 1.3.4 (Donsker, [12], p.374). *If* $\sigma^2 = Var(X_1) < \infty$, *then for* $t \geq 0$

$$X_t^{(n)} = \frac{S_{\lfloor nt \rfloor} - E(S_{\lfloor nt \rfloor})}{\sqrt{n}} \to X_t \sim N(0, \sigma^2 t)$$

in distribution. Furthermore, $X^{(n)} \to X$ *where* X *is a Brownian motion with diffusion coefficient* σ.

Theorem 1.3.5 ([12], p.386). *Let* B_t *be a standard Brownian motion and consider*

$$\sigma_a = \inf\{t \geq 0 : B_t \geq a\}$$

where $a > 0$. *Then the density of* σ_a *is*

$$p_{\sigma_a} = \frac{a}{\sqrt{2\pi s^3}} \exp\{-\frac{a^2}{2s}\}, \ s > 0.$$

1.4 The Poisson Process and the Compound Poisson Process

A Poisson process is defined as a counting process. It counts the number of random times T_n, which occur between 0 and t, where $(T_n - T_{n-1})_{n\geq 1}$ is an i.i.d. sequence of exponential variables.

Definition 1.4.1 (Poisson process). Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. A process $(N_t)_{t\geq 0}$ defined by

$$N_t = \sum_{n\geq 1} \mathbb{I}(t \geq T_n)$$

is called the Poisson process with intensity λ.

Proposition 1.4.1. *Let $(N_t)_{t\geq 0}$ be a Poisson process.*

1. *For any $t > 0$, N_t is almost surely finite.*

2. *For any ω, the sample path $t \to N_t(\omega)$ is piecewise constant and increases by jumps of size 1.*

3. *The sample paths $t \to N_t$ are right continuous with left limit (cádlág), i.e. belong to the class $\mathcal{D}(\mathbb{R}_+)$.*

4. *For any $t > 0$, $N_{t-} = N_t$ with probability 1.*

5. *N_t is continuous in probability:*

$$\forall t > 0,\ N_s \to N_t \textit{ in probability, when } s \to t.$$

6. *For any $t > 0$, N_t follows a Poisson distribution with parameter λt:*

$$\forall n \in \mathbb{N}, \ \mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

7. *The characteristic function of N_t is given by*

$$E[e^{iuN_t}] = exp\{\lambda t(e^{iu} - 1)\}, \ \forall u \in \mathbb{R}.$$

Definition 1.4.2 (Compound Poisson Process). A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process X_t defined as

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where jumps sizes Y_i are i.i.d. with distribution f and N_t is a Poisson process with intensity λ, independent from $(Y_i)_{i\geq 1}$.

Proposition 1.4.2 (Characteristic Function of the Compound Poisson Process, [8], proposition 3.4). *Let X_t be a compound Poisson process. Its characteristic function has the following representation:*

$$E[\exp\{iuX_t\}] = \exp\left\{t\lambda \int_{-\infty}^{\infty} (e^{iux} - 1)F(dx)\right\}, \quad \forall u \in \mathbb{R},$$

where λ denotes the jump intensity and F the jump size distribution.

SIMULATION ALGORITHM

ALGORITHM 1.4.1 (Simulation of compound Poisson process). The steps for simulation of the compound Poisson process are as follows:

1. Simulate a random variable N from Poisson distribution with parameter λT. N gives the total number of jumps on the interval $[0, T]$.

2. Simulate N independent r.v., U_i, uniformly distributed on the interval $[0, T]$. These variables correspond to the jump times.

3. Simulate jump sizes: N independent r.v. Y_i with identical distribution f. The trajectory is given by
$$X(t) = \sum_{i=1}^{N} \mathbb{I}(U_i < t) Y_i.$$

The Poisson process itself can be obtained as a compound Poisson process on $\mathbb{R}$ such that $Y_i \equiv 1$.

Trajectories of the Poisson process as well as the Compound Poisson process are obtained with the use of Algorithm 1.4.1 and are illustrated in Figures 1.5 and 1.6, respectively. Mathematica codes used in order to obtain these results are included in appendices A.0.4 and A.0.5, respectively.

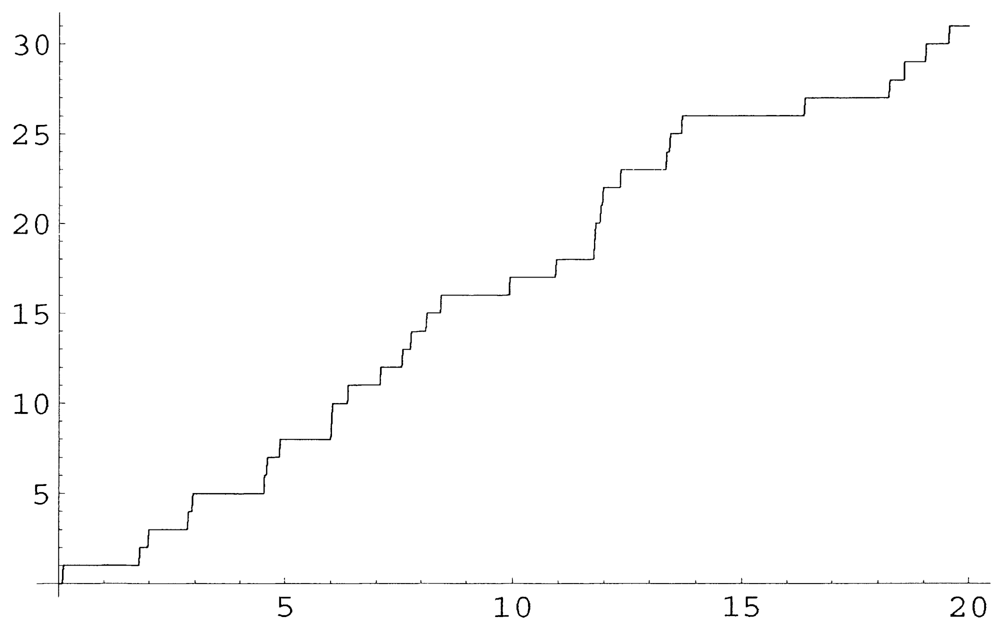

Figure 1.5: Trajectories of a Poisson Process, $\lambda = 1$.

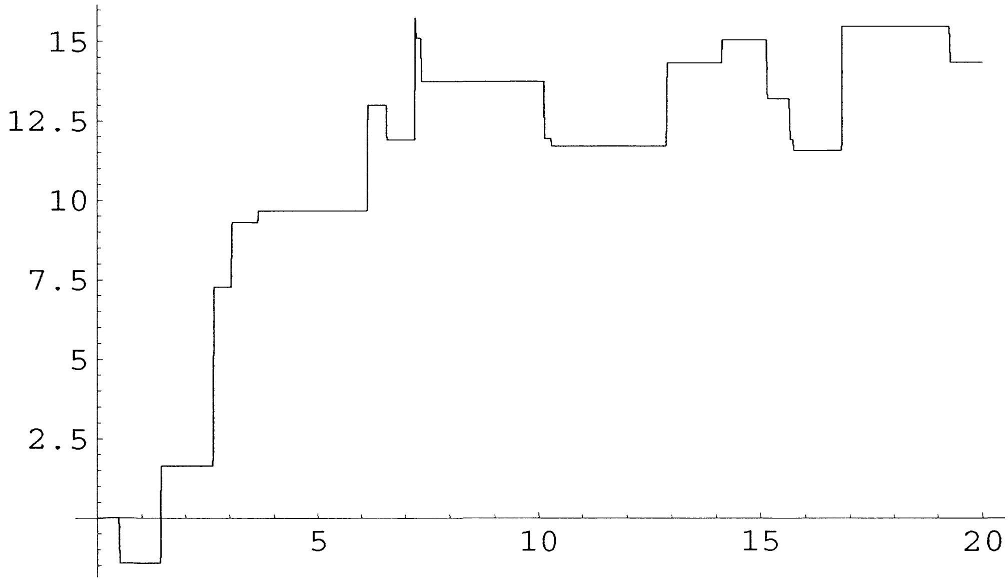

Figure 1.6: Trajectories of a Compound Poisson Process, $\lambda = 1$ and $N(0.5, 2)$ distribution of jumps.

1.5 Stochastic Integrals with respect to Brownian Motion and Itô's Formula

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a probability space equipped with filtration $\mathcal{F}_t \subseteq \mathcal{F}$.

Denote by B_t a standard Brownian motion adapted to $\mathcal{F}_t$. Let f_t be an adapted process. Then, consider a uniform partition $t_k = k\frac{t}{n}$, $k = 0, 1, \ldots, n, \ldots$ and denote

$$Z_m = \sum_{k=1}^{m} f_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}), \quad m = 1, 2, \ldots, n.$$

It may be shown that under the assumption

$$P\{\int_0^t f_s^2 ds < \infty\} = 1$$

there exists $\lim_{n\to\infty} Z_n$ (in probability). This limit is called a **stochastic integral** of f_t with respect to B_t and is denoted as

$$\lim_{n\to\infty} Z_n = \int_0^t f_s dB_s.$$

The following theorem is a special case of the Dambis, Dubins – Schwarz theorem [31], p.170.

Theorem 1.5.1. *Let f_t be an adapted random function such that $P(\int_0^t f_s^2 ds < \infty) = 1$ for any $0 \leq t < \infty$, $X_t = \int_0^t f_s dB_s$ and*

$$\int_0^\infty f_s^2 ds = \infty \ a.s.$$

Set

$$\tau_b = \inf\{t \geq 0 : \int_0^t f_s^2 ds \geq b\}, \ b \geq 0.$$

Then

$$X_{\tau_b} \sim N(0, b),$$

the process $\tilde{B}(b) = X_{\mathcal{T}_b}$ *is a standard Brownian Motion with respect to filtration* $\mathcal{F}_b = \sigma\{X_{\mathcal{T}_s},\ s \leq b\}$ *such that* $\int_0^t f_s^2 ds$ *is a stopping time with respect to* $\mathcal{F}_b$ *and*

$$X_t = \tilde{B}(\int_0^t f_s^2 ds).$$

Definition 1.5.1 (Itô Process). The continuous stochastic process $X_t, 0 \leq t \leq T$, is called an Itô process if there exists two adapted processes a_t and b_t, $0 \leq t \leq T$, such that

$$P\{\int_0^T |a_t| dt < \infty\} = 1,$$
$$P\{\int_0^T b_t^2 dt < \infty\} = 1$$

and with probability 1 for $0 \leq t \leq T$,

$$X_t = X_0 + \int_0^t a(s,\omega)dx + \int_0^t b(s,\omega)dW_s$$

or

$$dX_t = a(t,\omega)dt + b(t,\omega)dW_t.$$

Theorem 1.5.2 (Itô's Lemma, [12], p.669). *Let* X_t *be give by*

$$dX_t = a_t dt + b_t dW_t$$

and let $f(t,x)$ *be twice continuously differentiable. Set* $Y_t = f(t, X_t)$. *Then*

$$dY_t = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(dX_t)^2,$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ *is computed according to the rules*

$$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0$$

and

$$dW_t \cdot dW_t = dt.$$

1.6 Stochastic Integrals with respect to Square Integrable Martingales and their Properties

A right continuous martingale $M_t, t \in D$ is a square integrable martingale when

$$E \sup_{t \geq 0} M_t^2 < \infty.$$

Denote by $\mathcal{M}$ a class of square integrable martingales.

Theorem 1.6.1 ([20], p.161). *For each $X \in \mathcal{M}$ there exists a unique predictable increasing process $A_t \equiv \langle X \rangle_t$, $t \leq T$, such that for all $t, 0 \leq t \leq T$,*

$$X_t^2 = m_t + \langle X \rangle_t \quad (P - a.s.)$$

where $m_t \in M(\mathcal{F}_t, \mathbb{P})$.

The process $\langle X \rangle_t$ is called a predictable quadratic characteristic of the stochastic process X_t.

Denote by Φ_1 a class of $\mathcal{F}_t$ – adapted functions and by Φ_2 a class of predictable functions. Further let $A_t = \langle M \rangle_t$ and $L_A^2(\Phi_i)$ be a class of functions from Φ_i satisfying the condition

$$E \int_0^\infty f^2(s, \omega) dA_s < \infty.$$

Definition 1.6.1 (Simple Function). The function $f \in L_A^2(\Phi_2)$ is called a simple function if there exists a finite decomposition $0 = t_0 < \cdots < t_n < \infty$, such that

$$f(t, \omega) = \sum_{k=0}^{n-1} f(t_k, \omega) \mathbb{I}_{(t_k, t_{k+1}]}(t).$$

The class of simple functions is denoted by $\mathcal{E}$.

Let $X_t \in \mathcal{M}$ and $A_t = \langle X \rangle_t$, $t \geq 0$. We shall define the stochastic integral $I(f)$, denoted by $\int_0^\infty f(s,\omega)dX_s$, over a simple function $f = f(t,\omega)$, as follows

$$I(f) = \sum_{k=0}^{n-1} f(t_k,\omega)(X_{t_{k+1}} - X_{t_k}).$$

PROPERTIES OF THE STOCHASTIC INTEGRAL

This stochastic integral has the following properties (f, f_1, f_2 are simple functions):

1. $I_t(af_1 + bf_2) = aI_t(f_1) + bI_t(f_2) \quad (P - a.s.), \quad a, b = const., \quad t \geq 0;$
2. $\int_0^t f(s,\omega)dX_s = \int_0^u f(s,\omega)dX_s + \int_u^t f(s,\omega)dX_s \quad (P - a.s.);$
3. $I_t(f)$ is a right continuous function over $t \geq 0 \quad (P - a.s);$
4. $E\Big[\int_0^t f(u,\omega)dX_u \Big| \mathcal{F}_s\Big] = \int_0^s f(u,\omega)dX_u \quad (P - a.s.);$
5. $E\Big[\int_0^t f_1(u,\omega)dX_u \int_0^t f_2(u,\omega)dX_u\Big] = E\int_0^t f_1(u,\omega)f_2(u,\omega)dA_u;$

In particular from 4 and 5 it follows:

1. $E\int_0^t f(u,\omega)dX_u = 0;$
2. $E\Big[\int_0^t f(u,\omega)dX_u\Big]^2 = E\int_0^t f^2(u,\omega)dA_u.$

Lemma 1.6.2 ([20], p.186). *Let $X_t \in \mathcal{M}$, and let $A_t = \langle X \rangle_t$, $t \geq 0$, be the predictable increasing process corresponding to the martingale X. Then the space $\mathcal{E}$ of simple functions is dense in $L_A^2(\Phi_2)$.*

The above lemma enable us to define the stochastic integrals $I(f)$ over the martingale $X_t \in \mathcal{M}$ for $f \in L_A^2(\Phi_2)$ as the limit in the mean square of the integrals $I(f_n)$, where the f_n are simple functions approximating f in terms of

$$E\int_0^\infty |f(t,\omega) - f_n(t,\omega)|^2 dA_t \to 0, \quad n \to \infty.$$

Further, we have the following useful properties valid almost surely:

Proposition 1.6.3 ([19], pp. 96 – 97). *Let* $M \in \mathcal{M}$ *and* $f \in L^2_{\langle M \rangle}(\Phi_2)$. *Then for* $X_t = \int_0^t f_s dM_s$

$$\langle X \rangle_t = \int_0^t f_s^2 d\langle M \rangle_s$$

and $X \in \mathcal{M}$.

Additionally, denote by $X_\infty = \lim_{t\to\infty} X_t$ and by $\{X \to\}$ the set on which X_∞ exists and it is a finite random variable.

Theorem 1.6.4 ([19], p. 136). *Let* $X \in \mathcal{M}$, $X_0 = 0$. *Then*

$$\{\langle X \rangle_\infty < \infty\} \subseteq \{X \to\}.$$

1.7 Lévy Processes

The Lévy process is a continuous time process with independent and stationary increments. These processes provide the ingredients for building continuous time stochastic models in insurance and finance.

Definition 1.7.1 (Lévy Process). A *cádlág* stochastic process $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$ such that $X_0 = 0$ is called a Lévy process if it possesses the following properties:

1. Independent increments: for every increasing sequence of times $t_0, \ldots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

2. Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.

It is useful to notice that a Lévy process can be obtained as a limit of a sequence of random walks as in Cont & Tankov [8]. The next theorem states this fact.

Theorem 1.7.1. *Define a sequence of random walks as*

$$X_t^{(n)} = S_{\lfloor nt \rfloor}^{(n)} \to X_t \ \textit{ in distribution}, \ \textit{as } n \to \infty.$$

Then, $X^{(n)} \to X$ in distribution where X is a Lévy process.

Further, to every *cádlág* process $(X_t)_{t\geq 0}$ on $\mathbb{R}$ one can associate a random measure on $[0, \infty[\times\mathbb{R}$ describing the jumps of X: for any measurable set $B \subset [0, \infty[\times\mathbb{R}$

$$N_X(B) = \sharp\{(t, \Delta X_t) \in B\},$$

where $\Delta X_t = X_t - X_{t-}$ are jump sizes of the process X_t. Therefore, for every measurable set $A \subset \mathbb{R}$, $N_X([t_1, t_2] \times A)$ counts the number of jump times of X between t_1 and t_2 such that their jump sizes are in A.

Definition 1.7.2 (Lévy Measure). Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. The measure ν on $\mathbb{R}$ defined by:

$$\nu(A) = E[\sharp\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R})$$

is called the Lévy measure of X : $\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A.

Definition 1.7.3 (Poisson Random Measure). Given a random measure M on a measurable space $(S, \mathcal{A})$ we say that we have a Poisson random measure $N(t, A)$ if each $M(A), A \in \mathcal{A}$ has a Poisson distribution whenever $M(A) < \infty$.

Definition 1.7.4 (Compensated Poisson Random Measure). For each $t \geq 0$ and A bounded below, we define the compensated Poisson random measure by

$$\tilde{N}(t, A) = N(t, A) - t\nu(A).$$

Theorem 1.7.2 (Lévy – Khintchine, [8], theorem 3.1). *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ with characteristic triplet $(\sigma^2, \mu, \nu(dx))$ such that $\int \min(x^2, 1) d\nu(x) < \infty$. Then*

$$E[e^{izX_t}] = e^{t\psi(z)}, \quad z > 0,$$

with

$$\psi(z) = i\mu z - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty} \Big(e^{izx} - 1 - izx\mathbb{I}(|x| \leq 1)\Big)\nu(dx).$$

Theorem 1.7.3 (Lévy – Itô Decomposition, [8], proposition 3.7). *If X is a Lévy process, then there exists $b \in \mathbb{R}$, a Brownian motion B_t and an independent Poisson random measure N such that, for each $t \geq 0$,*

$$X(t) = bt + B_t + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx).$$

In the Lévy – Itô decomposition the process $Y_t = B_t + \int_{|x|<1} x\tilde{N}(t, dx)$ is a martingale, while the process $Z_t = \int_{|x|\geq 1} xN(t, dx)$ is a compound Poisson process.

Proposition 1.7.4 ([8], lemma 15.1). *Let $f : [0, T] \to \mathbb{R}$ be left – continuous and Z_t be a Lévy process. Then*

$$E\Big\{ \exp\Big(i \int_0^T f(t) dZ_t\Big)\Big\} = \exp\Big\{ \int_0^T \psi(f(t)) dt \Big\},$$

where $\psi(u)$ is the characteristic exponent of Z.

Chapter 2

Stochastic Modelling in Insurance

This chapter is devoted to the topic of stochastic modelling in insurance. Its main concern is to introduce an Ornstein – Uhlenbeck (OU) process that is further analysed in chapters 3 and 4. The OU – process is classified with respect to the interest rate into three insurance model groups. Moreover, the definition and analysis of the OU model is placed in the context of the Piecewise Deterministic Markov Processes (PDMP) theory, which is utilized in the analysis of so – called 'Ruin Problem'.

2.1 Risk Processes

By risk process X_t we refer to an insurance model with the risk being associated with the incoming stochastic claims from insurance clients. The first risk process described in this section originates from the classical risk theory and it has significantly contributed to the development of the modern risk models, which take into account the possibility of investment. Specifically, the Cramér – Lundberg process is a special case of the OU process assuming that the interest rate is equal to zero.

2.1.1 Cramér – Lundberg Model

We consider the classical continuous time claim process in insurance risk theory with the following structure:

- claims occur at times $\{T_n; n \in \mathbb{N}\}$ of a Poisson process with rate λ and corresponding counting process $\{N(t); t \geq 0\}$ where $N(t) = \sum_{k=1}^{\infty} \mathbb{I}(t \geq T_k)$;
- the inter – arrival times

$$Y_1 = T_1, \quad Y_k = T_k - T_{k-1}, \quad k = 2, 3, \ldots$$

  are *i.i.d.* exponentially distributed with finite mean $EY_1 = \frac{1}{\lambda}$.
- the claim sizes $\{U_k; k \in \mathbb{N}\}$ are independent, identically distributed random variables, having common distribution function F with $F(x) = 0$ when $x < 0$ and finite mean μ;
- N and U_k are assumed to be independent;
- $\sum_{i=1}^{N(t)} U_i$ represents the accumulated claims up to time t.

In the Cramér-Lundberg model the free reserves process X is defined by

$$X_t = u + ct - \sum_{i=1}^{N_t} U_i,$$

where u is the initial capital and $c > 0$ is the premium income per unit time.

The total claim amount process $\{S(t); t \geq 0\}$ of the underling portfolio is given by

$$S(t) = \begin{cases} \sum_{i=1}^{N(t)} U_i, & N(t) > 0; \\ 0, & N(t) = 0. \end{cases}$$

Further, the total claim amount distribution is represented by

$$G_t(u) = P(S(t) \leq u) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} F^{n*}(u), \quad u \geq 0, \quad t \geq 0,$$

where $F^{n*}(u) = P(\sum_{i=1}^{n} U_i \leq u)$ is the n-fold convolution of F.

Therefore, the Cramér Lundberg model can be also written as

$$X_t = u + ct - S(t).$$

The models we consider will typically have the property that there exists a constant ρ such that

$$\frac{1}{t} \sum_{i=1}^{N_t} U_i \to \rho, \quad t \to \infty, \quad a.s.$$

The interpretation of ρ is as the average amount of claim per unit time.

Definition 2.1.1. The *relative safety loading* η is defined by

$$\eta = \frac{c - \rho}{\rho}$$

and defines the relative amount by which the premium rate c exceeds ρ.

We will assume that $\eta > 0$ and $\rho = \lambda\mu$. Figure 2.1 shows the trajectory of the process X_t for the initial value of X_0 equal to 100, premium rate c equal to 3 and the rate of the Poisson process λ equal to 1.

This thesis does not focus on the analysis of the risk process described in this section, however the results for this model can be obtained from the analysis of the OU process since the Cramér – Lundberg model can be obtained from the OU model by setting the interest rate to zero.

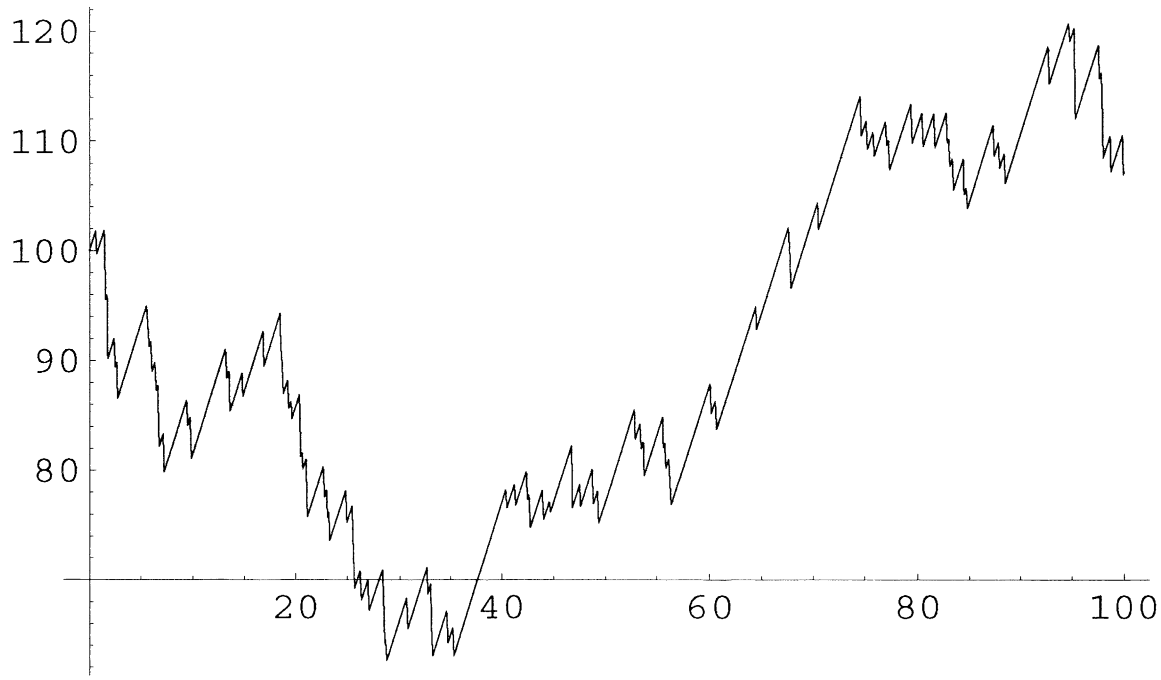

Figure 2.1: Trajectory of a Cramér Lundberg model with $\lambda = 1$, $c = 3$ and $X_0 = 100$.

2.1.2 The Ruin Problem

Analysis of risk processes with respect to boundary crossing probabilities are referred by the *Ruin Problem* in actuarial mathematics. Ruin probability can be defined as a first boundary crossing probability by the risk process X_t. Formally, we define a stopping time τ_l as

$$\tau_l = \inf\{t \geq 0 : X_t \leq l\},$$

where l is a ruin level less than the initial value of the process X_t. Further, we define the finite and infinite ruin probabilities.

Definition 2.1.2 (Ruin Probability). The ruin probability in finite time (or with finite horizon) is defined by

$$\psi(T, x) = P(\tau_l < T), \quad 0 < T < \infty,\ x \geq 0.$$

The ruin probability in infinite time (or with infinite horizon) is defined by

$$\psi(x) = \psi(x, \infty), \quad x \geq 0.$$

2.1.3 Ornstein – Uhlenbeck (OU) Process

This section will define the OU process and therefore it is one of the most important points of this chapter. The insurance interpretation of this process in not included at this point however we will return to our discussion of the insurance model in order to fully describe its purpose in connection to the theory of risk.

GAUSSIAN ORNSTEIN – UHLENBECK PROCESS

Definition 2.1.3 (Gaussian Ornstein – Uhlenbeck Process). We say that a stochastic process X_t is a Gaussian Ornstein – Uhlenbeck process if X_t has the following representation in terms of a stochastic differential equation (SDE)

$$dX_t = \beta X_t dt + dW_t, \tag{2.1.1}$$

where W_t is a Wiener process defined by 1.3.4. Moreover, if $\beta < 0$ we say that this process is stable. Otherwise, in the case $\beta > 0$, this process is called unstable.

The following proposition provides a solution to the SDE 2.1.1. This new representation for X_t is further utilized in the later chapters of this thesis.

Proposition 2.1.1. *Let*

$$dX_t = \beta X_t dt + dW_t,$$

where W_t is a Brownian motion for all $t \geq 0$. The solution of this stochastic differential equation is given by

$$X_t = X_0 e^{\beta t} + e^{\beta t} \int_0^t e^{-\beta s} dW_s. \tag{2.1.2}$$

Proof:

In order to proof this proposition we use the Itô Lemma given by theorem 1.5.2 setting $g(t, X_t)$ to $X_t e^{-\beta t}$. Hence,

$$\begin{aligned} d(X_t e^{-\beta t}) &= -\beta X_t e^{-\beta t} dt + e^{-\beta t} dX_t + 0 \\ &= -\beta X_t e^{-\beta t} dt + e^{-\beta t}(\beta X_t dt + dW_t) \\ &= -\beta X_t e^{-\beta t} dt + \beta X_t e^{-\beta t} dt + e^{-\beta t} dW_t \\ &= e^{-\beta t} dW_t. \end{aligned}$$

Therefore, we obtain

$$d(X_t e^{-\beta t}) = e^{-\beta t} dW_t,$$

when written as an integral

$$X_t e^{-\beta t} = X_0 + \int_0^t e^{-\beta s} dW_s$$

and eventually

$$X_t = e^{\beta t} X_0 + e^{\beta t} \int_0^t e^{-\beta s} dW_s.$$

∎

Definition 2.1.4 (Generalized Ornstein – Uhlenbeck process). We say that a stochastic process X_t is a generalized Ornstein – Uhlenbeck process if X_t has the following representation in terms of a stochastic differential equation

$$dX_t = \beta X_t dt + dY_t, \tag{2.1.3}$$

where Y_t is a Lévy process introduced by definition 1.7.1. Moreover, if $\beta < 0$ we say that this process is stable. Otherwise, in the case $\beta > 0$, this process is called unstable.

Proposition 2.1.2. *Let*

$$dX_t = \beta X_t dt + dY_t,$$

where Y_t is a Lévy process for all $t \geq 0$. The solution of this stochastic differential equation is given by

$$X_t = X_0 e^{\beta t} + e^{\beta t} \int_0^t e^{-\beta s} dY_s.$$

Proof:

This result is a simple implication of the proposition 2.1.1.

■

2.1.4 Ornstein – Uhlenbeck Process generated by a Compound Poisson Process

Consider the Ornstein – Uhlenbeck process generated by a compound Poisson process

$$Y_t = \sum_{k=1}^{N_\lambda(t)} \xi_k - mt,$$

where $(\xi_k)_{k\geq 1}$ are i.i.d. random variables, represented by:

$$dX_t = (\beta X_t - m)dt + d(\sum_{k=1}^{N_\lambda(t)} \xi_k).$$

Moreover, the pulses $(\xi_k)_{k\geq 1}$ appear at arrival times $(T_k)_{k\geq 1}$ of a Poisson process $N_\lambda(t),\ t \geq 0$, with the intensity parameter $\lambda > 0$.

Proposition 2.1.3. *The explicit solution for an OU process in case of a Compound Poisson process with a drift is represented by the following equation:*

$$X_t = \frac{m}{\beta} + \Big(X_0 - \frac{m}{\beta}\Big)e^{\beta t} + \sum_{k=1}^{N_\lambda(t)} \xi_k e^{\beta(t-T_k)} \mathbb{I}(T_k \leq t).$$

Proof:

In order to prove this proposition we use the result of the calculation in section 2.1.3, namely the proposition 2.1.2. Hence,

$$X_t = X_0 e^{\beta t} + e^{\beta t} \int_0^t e^{-\beta s} dY_s,$$

where

$$Y_t = \sum_{k=1}^{N_\lambda(t)} \xi_k - mt.$$

Therefore,

$$\begin{aligned}
X_t &= X_0e^{\beta t} + e^{\beta t}\int_0^t e^{-\beta s} d\Big(\sum_{k=1}^{N_\lambda(t)} \xi_k - ms\Big) \\
&= X_0e^{\beta t} + e^{\beta t}\Big[\int_0^t e^{-\beta s} d\Big(\sum_{k=1}^{N_\lambda(t)} \xi_k\Big) - m\int_0^t e^{-\beta s} ds\Big] \\
&= X_0e^{\beta t} - \frac{m}{\beta}e^{\beta t}\Big(1 - e^{-\beta t}\Big) + e^{\beta t}\int_0^t e^{-\beta t} d\Big(\sum_{k=1}^{N_\lambda(t)} \xi_k\Big) \\
&= \frac{m}{\beta} + \Big(X_0 - \frac{m}{\beta}\Big)e^{\beta t} + \int_0^t e^{\beta(t-s)} d\Big(\sum_{k=1}^{N_\lambda(t)} \xi_k\Big) \\
&= \frac{m}{\beta} + \Big(X_0 - \frac{m}{\beta}\Big)e^{\beta t} + \sum_{k=1}^{N_\lambda(t)} \xi_k e^{\beta(t-T_k)}\mathbb{I}(T_k \le t).
\end{aligned}$$

■

SIMULATION OF THE OU – PROCESS GENERATED BY A COMPOUND POISSON PROCESS

To simulate the OU process generated by a Compound Poisson process we use its explicit representation quoted in the proposition 2.1.3. Two trajectories are simulated using the following representation of the OU process:

$$dX_t = \beta X_t dt + dY_t.$$

Figure 2.2 illustrates a trajectory of an OU process assuming $Y_t = \sum_{k=1}^{N_\lambda(t)} \xi_k - mt$, $\xi_k \sim N(30, 2)$, $\beta = -1$, $\lambda = 10$, $m = 30$ and $X_0 = 10$. Figure 2.3, however, illustrates a trajectory of an unstable OU process with parameters $\xi_k \sim N(30, 2)$, $\beta = 0.002$, $\lambda = 10$, $m = 30$, $X_0 = 10$ and $Y_t = -\sum_{k=1}^{N_\lambda(t)} \xi_k + mt$.

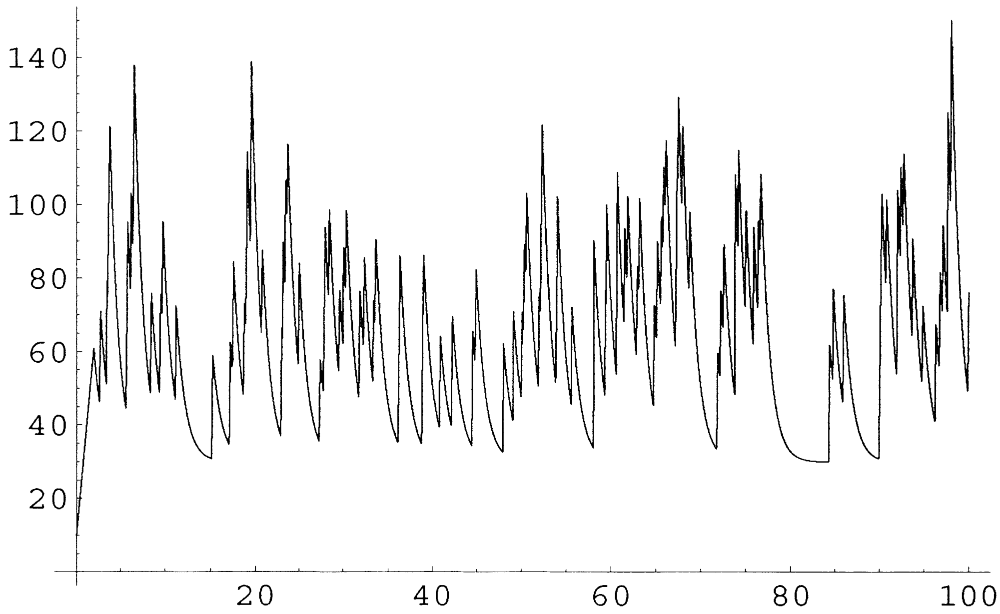

Figure 2.2: Trajectory of a Stable OU Process generated by a Compound Poisson Process with $\lambda = 10$. The parameters of the OU process are: $\beta = -1$ and $m = 30$.

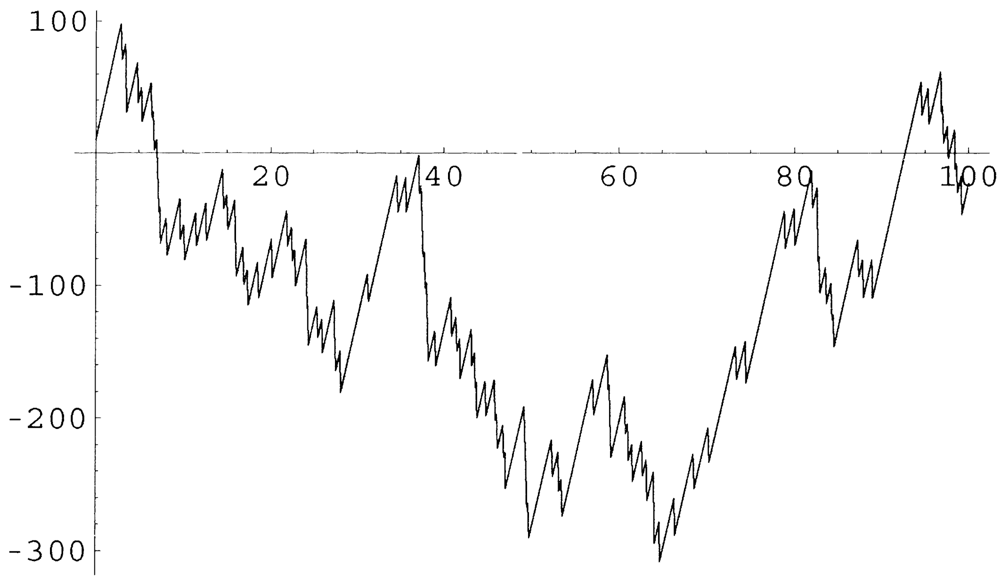

Figure 2.3: Trajectory of an Unstable OU Process generated by a Compound Poisson Process with $\lambda = 10$. The parameters of the OU process are: $\beta = 0.002$ and $m = 30$.

2.2 Continuous Time Markov Processes

This section is devoted to the theory of continuous time Markov processes. In particular, it focuses on one subfamily of this class of stochastic processes called Piecewise Deterministic Markov Processes (PDMP). The purpose of this section is to place the Ornstein – Uhlenbeck process in context of the PDMP theory. Such classification of the OU process allows to use the PDMP theory as a tool to obtain the first time boundary crossing probabilities in both finite and infinite time for the risk process X_t. This section is based on Rolski et al. [34].

We assume that the stochastic processes X_t considered in this section are *cádlág*, i.e. their sample paths belong to the set $\mathcal{D}(\mathbb{R}_+)$ of right – continuous functions $g : \mathbb{R}_+ \to E$ with left hand limits, where E denotes the state space. For our purposes we only consider the one dimensional case $E = \mathbb{R}$. Further, denote by $\mathcal{M}(\mathbb{R})$ the family of all real – valued measurable functions on $\mathbb{R}$.

2.2.1 Transition Kernels and Definition of a Markov Process

Denote by $\mathcal{B}(\mathbb{R})$ the σ – algebra of Borel sets in $\mathbb{R}$. Let, then $\mathcal{P}(\mathbb{R})$ be the set of all probability measures on $\mathcal{B}(\mathbb{R})$. A function

$$P : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$$

is said to be a transition kernel if the following four conditions are fulfilled for all $h, h_1, h_2 \geq 0$, $x \in \mathbb{R}$, $B \in \mathcal{B}(\mathbb{R})$:

$P(h, x, \cdot) \in \mathcal{P}(\mathbb{R})$

$P(0, x, \{x\}) = 1$

$P(\cdot, \cdot, B) \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R})$

$P(h_1 + h_2, x, B) = \int_{\mathbb{R}} P(h_2, y, B)P(h_1, x, dy)$.

Definition 2.2.1 (Markov process). A $\mathbb{R}$ – valued stochastic process X_t, $t \geq 0$ is called a (homogeneous) Markov process if there exists a transition kernel P and a probability measure $\alpha \in \mathcal{P}(\mathbb{R})$ such that

$\mathbb{P}(X(0) \in B_0, X(t_1) \in B_1, \ldots, X(t_n) \in B_n) =$

$= \int_{B_0} \int_{B_1} \cdots \int_{B_n} P(t_n - t_{n-1}, x_{n-1}, dx_n) \ldots P(t_1, x_0, dx_1)\alpha(dx_0)$,

for all $n = 0, 1, \ldots$, $B_0, B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$, $t_0 = 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$.

The probability measure α is called and *initial distribution* and we interpret $P(h, x, B)$ as the probability that, in time h, the stochastic process X_t moves from state x to a state in B.

2.2.2 Piecewise Deterministic Markov Processes (PDMP)

In general, PDMP can be described as Markov processes X_t whose trajectories have countably many jump epochs. The jump epochs and also the jump sizes are random in general. But, between the jumps epochs, the trajectories are governed by a deterministic rule.

Further, denote by $\mathbb{I}$ an arbitrary, finite non – empty set, $\{d_\nu, \nu \in \mathbb{I}\}$ a family of natural numbers, C_ν an open subset of $\mathbb{R}^{d_\nu}$ for each $\nu \in \mathbb{I}$ and $E = \{(\nu, z) : \nu \in \mathbb{I}, z \in C_\nu\}$. With this notation $\mathbb{I}$ is the set of possible different external states of the process. For example, $\mathbb{I} = \{$"*healthy*", "*sick*", "*dead*"$\}$. C_ν is the state space of the process if the external state is ν. This allows to consider different state spaces in different external states. Further we denote $X_t = (J_t, Z_t)$, where J_t describes the external states of X_t and Z_t indicates the evolution of the external component.

BEHAVIOR BETWEEN JUMPS

Between jumps, the process X_t follows a deterministic path, while the external component J_t is fixed, $J_t = \nu$. Starting at some point $z \in C_\nu$, the development of the deterministic path is completely specified by its velocities at all points of C_ν, i.e. through an appropriate function $c_\nu = (c_1, \ldots, c_{d_\nu}) : C_\nu \to \mathbb{R}^{d_\nu}$, called a *vector field.* If a sufficiently smooth vector field is given, then for every $z \in C_\nu$ there exists a path $\varphi_\nu(t, z)$, called an *integral curve*, such that

$$\varphi_\nu(0, z) = z$$

and

$$\frac{d}{dt}\varphi_\nu(t, z) = c_\nu(\varphi_\nu(t, z)).$$

Definition 2.2.2 (Vector Field Operator X). A vector field operator $\mathbf{X}$ is given by

$$\mathbf{X}g(z) = \sum_{i=1}^{d_\nu} c_i(z)\frac{\partial g}{\partial z_i}(z)$$

acting on differentiable functions g.

Furthermore, if g is continuously differentiable then the integral curve $\{\varphi_\nu(t, z), t < t^*(\nu, z)\}$, where

$$t^*(\nu, z) = \sup\{t > 0 : \exists\varphi_\nu(t, z) \in C_\nu\},$$

is solution to the differential equation

$$\frac{d}{dt}g(\varphi_\nu(t, z)) = (\mathbf{X}g)(\varphi_\nu(t, z)),$$

with

$$\varphi_\nu(0, z) = z.$$

THE ACTIVE BOUNDARY OF E

Denote by ∂C_ν the boundary of C_ν and let

$$\partial^* C_\nu = \{\tilde{z} \in \partial C_\nu : \tilde{z} = \varphi_\nu(t, z) \ for\ some\ (t, z) \in \mathbb{R}^+ \times C_\nu\}.$$

Additionally, let

$$\Gamma = \{(\nu, z) \in \partial E : \nu \in \mathbb{I}, z \in \partial^* C_\nu\}.$$

We will assume that $\varphi_\nu(t^*(\nu, z), z) \in \Gamma$ if $t^*(\nu, z) < \infty$. The set Γ is called the *active boundary* of E. We may interpret Γ as a set of those boundary points of E, that can be reached from E via integral curves within finite time and $t^*(\nu, z)$ is the time needed to reach the boundary from the point (ν, z). The condition $\varphi_\nu(t^*(\nu, z), z) \in \Gamma$ ensures that the integral curves cannot "disappear" inside E.

To fully define a PDMP on $(E, \mathcal{B}(E))$, we need more than a family of vector fields $\{c_\nu, \nu \in \mathbb{I}\}$. We also require a *jump intensity*, i.e. a measurable function $\lambda : E \to \mathbb{R}_+$, a *transition kernel* $Q : (E \cap \Gamma) \times \mathcal{B}(E) \to [0, 1]$, i.e. $Q(x, \cdot)$ is a probability measure for all $x \in E \cup \Gamma$ and $Q(\cdot, B)$ is measurable for all $B \in \mathcal{B}(E)$. In actuarial terminology, the jump intensity λ can be interpreted as a "force of transition", whereas $Q(x, \cdot)$ is the "after jump" distribution of a jump from state x (if $x \in E$) or from the boundary point x (if $x \in \Gamma$).

THE GENERATOR OF A PDMP

Consider a multivalued linear operator $\mathbf{A}$. This is simply a set

$$\mathbf{A} \subset \{(g, \tilde{g}) : g, \tilde{g} \in \mathcal{M}(\mathbb{R})\}$$

such that, if $(g_i, \tilde{g}_i) \in \mathbf{A}$ for $i \in \{1, 2\}$ then also $(ag_1 + bg_2, a\tilde{g}_1 + b\tilde{g}_2) \in \mathbf{A}$ for all $a, b \in \mathbb{R}$. The set $\mathcal{D}(\mathbf{A}) = \{g \in \mathcal{M}(\mathbb{R}) : (g, \tilde{g}) \in \mathbf{A}\ for\ some\ \tilde{g} \in \mathcal{M}(\mathbb{R})\}$ is called the domain of the operator $\mathbf{A}$.

Definition 2.2.3 (Full Generator of a Markov Process). The multivalued operator $\mathbf{A}$ that consists of all pairs $(g, \tilde{g}) \in \mathcal{M}(\mathbb{R}) \times \mathcal{M}(\mathbb{R})$ for which

$$\{g(X_t) - g(X_0) - \int_0^t \tilde{g}(X_v)dv, t \geq 0\}$$

becomes an $\mathcal{F}_t^X$ – martingale, is called the full generator of the Markov process X_t.

According to the definition of the full generator of a Markov process, in order to find martingales associated with a PDMP X_t, we have to find a function in the domain $\mathcal{D}(\mathbf{A})$ of the generator $\mathbf{A}$ of X_t. This problem is generally hard to solve, however solutions are possible if we restrict ourselfs to a subset of $\mathcal{D}(\mathbf{A})$, amply sufficient for the insurance setup.

The following theorem defines a generator of a PDMP used for actuarial purposes.

Theorem 2.2.1. *Let X_t be PDMP and let $g^* : E \cap \Gamma \to \mathbb{R}$ be a measurable function satisfying the following conditions:*

(a) *for each $(\nu, z) \in E$, the function $t \to g^*(\nu, \varphi(t, z))$ is absolutely continuous on $(0, t^*(\nu, z))$,*

(b) *for each x on the boundary Γ*

$$g^*(x) = \int_E g^*(y)Q(x, dy),$$

(c) *for each $t \geq 0$,*

$$\mathbb{E}(\sum_{i:T_i \leq t} |g^*(X_{T_i}) - g^*(X_{T_i-})| < \infty).$$

Then $g \in \mathcal{D}(\mathbf{A})$, where g denotes the representation of g^* to E, and $(g, \mathbf{A}g) \in \mathbf{A}$, where $\mathbf{A}g$ is given by

$$(\mathbf{A}g)(x) = (\mathbf{X}g)(x) + \lambda(x) \int_E (g(y) - g(x))Q(x, dy). \tag{2.2.1}$$

2.3 The Ornstein – Uhlenbeck Model revisited

In this section we reconsider the Ornstein – Uhlenbeck model of section 2.1.3 and demonstrate how to obtain the generator of this process. Let X_t be the risk reserve process described by the following SDE

$$dX_t = (\beta X_t + c)dt - d(\sum_{k=1}^{N_\lambda(t)} \xi_k). \tag{2.3.1}$$

It is easy to see that X_t is a PDMP with state space $E = \mathbb{R}$. The set $\mathbb{I}$ of external states consists of only one element and is therefore omitted. The characteristic of the PDMP X_t are given by

$$(\mathbf{X}g)(x) = (\beta x + c)\frac{\partial g}{\partial x}(x)$$

and the hazard along the integral curves is $\lambda(x) = \lambda$ and the Markov transition measure governing the stochastic evolution of the process equals $Q(x, dy) = dF(x-y)$. Hence, the generator of the OU – process 2.3.1 is represented by

$$\mathbf{A}[g(x)] = (\beta x + c)g'(x) + \lambda \int_0^\infty [g(x-y) - g(x)]dF(y), \tag{2.3.2}$$

where $F(x)$ is a distribution function of the jumps ξ_k.

2.4 Classification of the Ornstein – Uhlenbeck Models with respect to the Interest Rate

We classify the OU models into two groups: models with a constant interest rate and models with the interest rate dependent on time. Further, we divide the interest rate dependent on time models into models with a deterministic interest rate and a stochastic interest rate. As a result of this classification we obtain the following three models.

OU MODEL WITH A CONSTANT INTEREST RATE

This model has been previously mentioned and it is represented by the unstable generalized OU process of the form:

$$dX_t = \beta X_t dt + dL_t, \quad t \geq 0 \tag{2.4.1}$$

where $\beta > 0$ models the interest rate, and L_t is a Lévy process, in particular a compound Poisson process with a drift given by

$$L_t = ct - \sum_{k=1}^{N_\lambda(t)} \xi_k.$$

This model is analysed in chapter 3 of this thesis with respect to the ruin probabilities.

OU MODEL WITH A DETERMINISTIC INTEREST RATE DEPENDENT ON TIME

This model is analysed in chapter 4 of this thesis. It is also defined by the SDE 2.4.1, however the interest rate is a deterministic function of time. Hence, we write

$$dX_t = \beta_t X_t dt + dL_t,$$

where $t \geq 0$, β_t models the interest rate, and L_t is a Lévy process, in particular a compound Poisson process with a drift given by

$$L_t = ct - \sum_{k=1}^{N_\lambda(t)} \xi_k.$$

Further, the interest rate might be given by the following equation, Roberts & Shortland [32],

$$\beta_t = \beta + ae^{-t},$$

where $\beta, a = const$. This form of the interest rate can be derived by taking the expected value of the risk – free interest rate, I_t, under a Vasicek model satisfying

$$dI_t = (\beta - I_t)dt + \sigma dB_t,$$

with $I_0 = \beta + a$ and $\sigma = 1$. One may refer to Roberts & Shortland [32] for more details on this fact.

OU MODEL WITH A STOCHASTIC INTEREST RATE

The analysis of this model are beyond the scope of this thesis, however we mention it for the sake of completeness. The risk process X_t is defined by

$$dX_t = \beta_t X_t dt + dL_t,$$

where $t \geq 0$, β_t models the interest rate, and L_t is a Lévy process, in particular a compound Poisson process with drift given by

$$L_t = ct - \sum_{k=1}^{N_\lambda(t)} \xi_k.$$

Moreover, the interest rate β_t can be given by the Vasicek model:

$$d\beta_t = (\beta - \alpha\beta_t)dt + \sigma dB_t,$$

where B_t is a standard Brownian motion and β, α, σ are constant parameters, additionally $\alpha > 0$.

Name	Tail $\overline{F}$ or density f	Parameters
Exponential	$\overline{F} = e^{-\lambda x}$	$\lambda > 0$
Gamma	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{\beta x}$	$\alpha, \beta > 0$
Weibull	$\overline{F}(x) = e^{-cx^\tau}$	$c > 0, \tau \geq 1$
Truncated normal	$f(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2}$	-

Table 2.1: Claim size distribution functions: "small claims". All distribution functions have support $(0, \infty)$.

2.5 Claim Sizes Distributions

In actuarial modelling it is very common that the claim sizes distributions are classified into two groups: light tailed distributions and heavy tailed distributions. This classification stems from the fact that there are two types of claims in insurance "small claims" and "large claims", which can be described by light – tailed and heavy – tailed distributions, respectively.

This section contains a brief presentation of the most popular classes of distributions F which have been used to model the claims $S_1, S_2, \ldots$. These distributions are classified into two groups, *light - tailed distributions* and *heavy - tailed distributions*. The following two definitions are given by Asmussen [2].

Definition 2.5.1 (Light – tailed distributions). Light – tailed distributions means that the tail $\overline{F}(x) = 1 - F(x)$ satisfies $\overline{F} = O(e^{-sx})$ for some $s > 0$. Equivalently, the moment generating function $\hat{F}[s]$ is finite for some $s > 0$.

Definition 2.5.2 (Heavy – tailed distributions). F is heavy – tailed distribution if $\hat{F}[s] = \infty$ for all $s > 0$. In actuarial practice it is also assumed that F is heavy – tailed if 20% of the claims account for more than 80% of the total claims, i.e. if

$$\frac{1}{\mu_F} \int_{f_{0.2}}^{\infty} xF(dx) \geq 0.8,$$

Name	Tail $\overline{F}$ or density f	Parameters
Lognormal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-(\ln x - \mu)^2/(2\sigma^2)}$	$\mu \in \mathbb{R}, \sigma > 0$
Pareto	$\bar{F}(x) = \left(\frac{\kappa}{\kappa + x}\right)^\alpha$	$\alpha, \kappa > 0$
Burr	$\bar{F}(x) = \left(\frac{\kappa}{\kappa + x^\tau}\right)^\alpha$	$\alpha, \kappa, \tau > 0$
Benktander - type - I	$\overline{F}(x) = (1 + 2(\beta/\alpha) \ln x)$ $e^{-\beta(\ln x)^2 - (\alpha+1) \ln x}$	$\alpha, \beta > 0$
Benktander - type - II	$\overline{F}(x) = e^{\alpha/\beta} x^{-(1-\beta)} e^{-\alpha x^\beta / \beta}$	$\alpha > 0,\ 0 < \beta < 1$
Weilbull	$\overline{F}(x) = e^{-cx^\tau}$	$c > 0, 0 < \tau < 1$
Loggamma	$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} (\ln x)^{\beta-1} x^{-\alpha-1}$	$\alpha, \beta > 0$
Truncated α - stable	$\overline{F}(x) = P(\mid X \mid > x)$ where X is an α - stable rv	$1 < \alpha < 2$

Table 2.2: Claim size distribution functions: "large claims". All distribution functions have support $(0, \infty)$ except for the Benktander cases and the loggamma with $(1, \infty)$.

where $\overline{F}(f_{0.2}) = 0.2$ and μ_F is the mean of F.

Tables 2.1 and 2.2 present the most common distributions used for modelling of small and large claims, respectively.

Chapter 3

Infinite and Finite Ruin Probabilities for the Model with Constant Interest Rate

We consider generalisation of the classical Cramér – Lundberg model introduced in the section 2.1.1. It is assumed that the cumulative income of a firm is given by a process L. This process has stationary independent increments and can be represented as

$$L_t = (\xi_1 + \cdots + \xi_{N_\lambda(t)}) - ct, \quad t \geq 0. \tag{3.0.1}$$

Therefore, L_t is a compound Poisson process with drift c. It is often assumed that L_t is a more general process satisfying the stationary and independent increment property (definitions 1.2.5 and 1.2.6), together with a mild sample path regularity condition. Such processes are called Lévy processes. Formal definition of this class of processes is included in the section 1.7. Moreover, we call L_t *the income process.*

In this context, one interprets c as the rate at which premium payments are received form policyholders, $N_\lambda(t)$ as a cumulative number of claims incurred up to time t, and ξ_k as the size of the kth claim. Suppose now that the risk reserve of the company is invested in a savings account, continuously earning interest at rate β. This leads to the definition of the *assets process* X.

$$X_t = e^{\beta t}x - \int_0^t e^{\beta(t-s)}dL_s, \quad t \geq 0, \tag{3.0.2}$$

where x is the level of *initial assets* of the company. Additionally, combining equations 3.0.1 and 3.0.2 and denoting by $T_1, T_2, \ldots$ the times at which claims occur we may define process X_t as

$$X_t = e^{\beta t}x + \int_0^t e^{\beta(t-s)}ds - \sum_{k=1}^{N_\lambda(t)} e^{\beta(t-T_k)}\xi_k.$$

This representation of X_t also solves the following SDE

$$dX_t = \beta X_t dt - dL_t, \quad X_0 = x. \tag{3.0.3}$$

For proof of this fact one can refer to the proposition 2.1.2.

The primary objective of this chapter is to determine infinite and finite ruin probabilities for the model described above. The infinite horizon ruin probabilities were previously considered by Harrison [17]. We devote one section of this chapter to his results since the method used in order to determine the finite horizon ruin probabilities is, to some extent, based on Harrison's results. The main part of this chapter describes the martingale method and its application for our problem. The Laplace transform of the stopping time

$$\tau_l = \inf\{t \geq 0 : X_t \leq l\}, \quad l \leq x \tag{3.0.4}$$

is derived and a few special cases are considered. Finally, the martingale solution is compared with the PIDE solution for the case of exponential jumps. The results are also confirmed by Monte Carlo simulation.

3.1 Infinite Time Ruin Probabilities

This section is devoted to infinite time ruin probabilities and it is based on the paper by Harrison [17]. We slightly change the notation to suite our purposes and also formulate new propositions, which were not considered as separate theorems by Harrison. This change is implied by the easier generalization of such presented facts.

Let the process X_t satisfy the equation 3.0.3. We assume that the process L_t is a Lévy process and $\beta > 0$ is a constant interest rate. It can be shown that the solution of 3.0.3 is unique and it has the following representation

$$X_t = e^{\beta t}(X_s e^{-\beta s} - \int_s^t e^{-\beta y} dL_y), \quad t \geq s \geq 0. \tag{3.1.1}$$

We define a stopping time $\tau_l = \inf\{t \geq 0 : X_t \leq l\}, \quad l \leq x$, which can be of interest for insurance companies. It might be interpreted as an alarm time when the value of the capital becomes too low. This is also some form of generalisation of Harrison's results, since he considers just the case of $l = 0$. Our aim is to present Harrison's method and the final representation for $P(\tau_l < \infty)$.

Before we formulate the necessary propositions we need to introduce some notation. Denote by

$$Z(\beta, L) = \int_0^\infty e^{-\beta s} dL_s. \tag{3.1.2}$$

Furthermore, denote by $\hat{Z}$ a random variable independent of L_t such that $\hat{Z} = Z(\beta, L)$ by distribution. We also assume that

$$E \log(1 + |L_t|) < \infty. \tag{3.1.3}$$

Under this condition the random variable $Z(\beta, L)$ is finite with probability one and it has an absolute continuous distribution (definition 1.1.7). These facts are presented by

Harrison [17] in proposition 2.2 under the assumption of finiteness of second moment of L_t. We present this result and the proof below with more general assumption 3.1.3.

Lemma 3.1.1. *Let* $E\log(1+|L_t|)<\infty$. *Then the random variable* $Z(\beta, L)$ *is finite with probability one:*

$$P(Z(\beta,L)<\infty)=1.$$

Additionally, the random variable $Z(\beta, L)$ *has an absolutely continuous distribution.*

Proof:

To prove this lemma we use the Lévy – Itô decomposition 1.7.3. We represent the Lévy process L_t as follows

$$L_t = ct + M_t + Z_t,$$

where c is a constant, M_t is a square integrable martingale with stationary independent increments and Z_t is a Compound Poisson process. Additionally, M_t and Z_t are independent.

Denote $\Delta Y_t = Y_t - Y_{t-}$ as jumps of process Y_t. Then we may assume that $|\Delta M_t| < 1$ and $|\Delta Z_t| \geq 1$. Hence, using the Lévy – Itô decomposition, we obtain

$$\int_0^t e^{-\beta s} dL_s = \frac{c}{\beta}(1-e^{-\beta t}) + \int_0^t e^{-\beta s} dM_s + \int_0^t e^{-\beta s} dZ_s.$$

Further, denote by $\langle N\rangle_t$ the predictable square characteristic, defined by the theorem 1.6.1, of a martingale $N_t = \int_0^t e^{-\beta s} dM_s$. From the properties of the stochastic integrals with respect to square integrable martingales we know that N_t is a square integrable martingale (refer to proposition 1.6.3).

We need to now show that N_t has bounded jumps and finite and predictable square characteristic $\langle N\rangle_t$. Then using the theorem 1.6.4 we prove the finiteness of

$N_\infty = \int_0^\infty e^{-\beta s} dM_s$. Hence, by representation 1.6.3 we obtain the following

$$\langle N \rangle_t = \int_0^t e^{-2\beta s} d\langle M \rangle_s = E(M_1^2) \int_0^t e^{-2\beta s} ds \to E(M_1^2)/2\beta < \infty,$$

since M_t is a square integrable martingale with stationary independent increments and $\langle M \rangle_t = tE(M_1^2)$. Additionally, N_t has jumps bounded by 1. Therefore, by theorem 1.6.4, we get the finiteness of $\int_0^\infty e^{-\beta s} dM_s$ with probability 1.

To show a convergence of $\int_0^t e^{-\beta s} dZ_s$ to the limit (in probability) we use the fact that a compound Poisson process has the representation $Z_t = \sum_{k=1}^{N_\lambda(t)} \xi_k$ where $N_\lambda(t)$ is a Poisson process with rate λ and ξ_k are i.i.d. random variables. It is convenient to consider positive and negative jumps of Z_t separately. Therefore, we split Z_t into Z_t^+ and Z_t^- as follows

$$Z_t^+ = \sum_{k=1}^{N_\lambda(t)} \xi_k^+, \quad Z_t^- = \sum_{k=1}^{N_\lambda(t)} \xi_k^-.$$

By direct calculations, using propositions 1.7.4 and 1.4.2, we get for $u \geq 0$

$$\log E \exp(-u \int_0^t e^{-\beta y} dZ_y^\pm) = \lambda \int_0^t E(e^{-u\xi_k^\pm e^{-\beta y}} - 1) dy.$$

Further, integrating by parts

$$\lambda \int_0^t E(e^{-u\xi_k^\pm e^{-\beta y}} - 1) dy = \frac{1}{\beta} \int_0^\infty \frac{\lambda(e^{-uxe^{-\beta t}} - e^{-ux}) P(\xi_k^\pm > x)}{x} dx,$$

which tends to

$$\frac{1}{\beta} \int_0^\infty \frac{\lambda(1 - e^{-ux}) P(\xi_k^\pm > x)}{x} dx,$$

when $t \to \infty$. The last integral is finite under condition

$$E \log(1 + \xi_k^\pm) < \infty,$$

which is equivalent to the condition

$$E \log(1 + L_t^\pm) < \infty.$$

Concluding, both processes $\int_0^t e^{-\beta y} dZ_y^{\pm}$ converge in distribution to a compound Poisson distribution with the absolute continuous Lévy – Khinchin measure

$$\Pi^{\pm}(dx) = \frac{\lambda P(\xi_k^{\pm} > x)}{x} dx, \quad x \geq 1.$$

Therefore, the distribution of $\int_0^\infty e^{-\beta y} dZ_y^{\pm}$ are absolute continuous and infinite – divisible. Note that the both processes $\int_0^t e^{-\beta y} dZ_y^{\pm}$ are monotone increasing and so they converge with probability one to their limits. It implies that with probability one

$$\int_0^t e^{-\beta y} dZ_y = \int_0^t e^{-\beta y} dZ_y^{+} - \int_0^t e^{-\beta y} dZ_y^{-} \to \int_0^\infty e^{-\beta y} dZ_y,$$

where the limit random variable has an absolute continuous and infinite – divisible distribution.

■

Further, we use the Lévy – Khintchine representation 1.7.2 for the characteristic function of the Lévy process L_t. This theorem allows to represent the characteristic function of L_t in the following form

$$Ee^{iuL_t} = e^{t\psi(u)},$$

where the function $\psi(u)$ has a unique representation dependent on so called characteristic triplet (σ^2, μ, ν), where σ^2 and μ are determined by $EL_t = \mu t$ and $Var L_t = \sigma^2 t$. Additionally, $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$ and ν is a Lévy measure defined by 1.7.2.

Using those facts we may prove the next lemma.

Lemma 3.1.2.

$$e^{-\beta t}\hat{Z} + \int_s^t e^{-\beta y} dL_y = e^{-\beta s} Z(\beta, L) \quad (by\ distribution) \tag{3.1.4}$$

Proof:

We will show that the LHS and RHS have the same characteristic function. Let

$$\log(Ee^{iuL_t}) = t\psi(u).$$

Then, using representation 1.7.4 and introducing a new variable $z = ue^{-\beta y}$, we get the following

$$\log(E\exp(iu\int_s^t e^{-\beta y}dL_y)) = \int_{ue^{-\beta t}}^{ue^{-\beta s}} \frac{\psi(z)}{\beta z}dz, \quad t \geq s \geq 0.$$

Similarly,

$$\log(E\exp(iue^{-\beta t}\hat{Z})) = \log(E\exp(iue^{-\beta t}\int_0^\infty e^{-\beta y}dL_y)) = \int_0^{ue^{-\beta t}} \frac{\psi(z)}{\beta z}dz.$$

Hence,

$$\log(E(\exp(iue^{-\beta t}\hat{Z}) + iu\int_s^t e^{-\beta y}dL_y)) = \int_0^{ue^{-\beta t}} \frac{\psi(z)}{\beta z}dz + \int_{ue^{-\beta t}}^{ue^{-\beta s}} \frac{\psi(z)}{\beta z}dz =$$

$$= \int_0^{ue^{-\beta s}} \frac{\psi(z)}{\beta z}dz = \log Ee^{iue^{-\beta s}\hat{Z}} = \log Ee^{iue^{-\beta s}Z(\beta,L)},$$

which proves 3.1.4.

■

Recall that $M(\mathcal{F}_t, \mathbb{P})$ is a class of all martingales with respect to a natural filtration $\mathcal{F}_t$ and a probability measure $\mathbb{P}$. Refer also to definition of a martingale 1.2.8 and the optional stopping theorem 1.2.5.

Further denote

$$H(x) = E\mathbb{I}(Z(\beta, L) > x) = P(Z(\beta, L) > x).$$

Proposition 3.1.3. *Let $E\log(1+|L_t|)<\infty$. Then*

$$H(X_t)\in M(\mathcal{F}_t,\mathbb{P}).$$

Proof:

To check the martingale property we need to show that for any $t\geq s$

$$E(H(X_t)\mid\mathcal{F}_s)=H(X_s)$$

and that the expectation $E(H(X_t))$ is finite.

To show finiteness of $E(H(X_t))$ we need to just note that the function $H(x)=P(Z(\beta,L)>x)$ is bounded, which is of course true as $H(x)$ is a tail distribution function of the random variable $Z(\beta,L)$.

Now we need to show that for any $t\geq s$

$$E(H(X_t)\mid\mathcal{F}_s)=H(X_s).$$

Let random variable $\hat{Z}=Z(\beta,L)$ in distribution and let $\hat{Z}$ be independent of L_t. According to the Fubini theorem 1.2.2

$$E(H(X_t)\mid\mathcal{F}_s)=E(\mathbb{I}(\hat{Z}-X_t>0)\mid\mathcal{F}_s).$$

Further,

$$E(\mathbb{I}(\hat{Z}-X_t>0)\mid\mathcal{F}_s)=E(\mathbb{I}(\hat{Z}-e^{\beta t}(X_se^{-\beta s}-\int_s^t e^{-\beta y}dL_y)>0)\mid\mathcal{F}_s)=$$
$$=E(\mathbb{I}(e^{-\beta t}\hat{Z}+\int_s^t e^{-\beta y}dL_y-X_se^{-\beta s}>0)\mid\mathcal{F}_s).$$

According to the equation 3.1.4 the above coincides with

$$E(\mathbb{I}(e^{-\beta s}\hat{Z}-X_se^{-\beta s}>0)\mid\mathcal{F}_s)=E(\mathbb{I}(\hat{Z}-X_s>0)\mid\mathcal{F}_s)=E\mathbb{I}(\hat{Z}-X_s>0)=H(X_s),$$

which proves that $H(X_t)$ is a martingale.

This result leads to the following form of the infinite time probability of ruin.

Proposition 3.1.4.

$$P(\tau_l < \infty) = \frac{H(x)}{E[H(X_{\tau_l}) \mid \sigma(\tau_l)]},$$

where $\sigma(\tau_l)$ is a sigma – algebra generated by values of the random variable τ_l.

Proof:

By the optional stopping theorem 1.2.5,

$$EH(X_\tau) = H(x),$$

where x is the value of the initial assets and $\tau = \min(\tau_l, t)$. This implies that

$$E\mathbb{I}(\tau_l > t)H(X_t) + E\mathbb{I}(\tau_l \leq t)H(X_{\tau_l}) = H(x)$$

Consider this equation for $t \to \infty$. Note that on the set $(\tau_l > t)$ we have the bound $P(\hat{Z} - X_t > 0) \leq P(\hat{Z} - l > 0)$. Due to independency of τ_l and $\hat{Z}$ and the fact that $E|\hat{Z}| < \infty$ we have

$$E\mathbb{I}(\tau_l > t)H(X_t) \leq P(\tau_l > t)P(\hat{Z} - l > 0) \to 0.$$

Further, applying a monotone convergence theorem we obtain the following

$$E\mathbb{I}(\tau_l < \infty)H(X_{\tau_l}) = H(x),$$

which implies

$$P(\tau_l < \infty) = \frac{H(x)}{E[H(X_{\tau_l}) \mid \sigma(\tau_l)]},$$

where $\sigma(\tau_l)$ is a sigma – algebra generated by values of the random variable τ_l.

■

3.2 Examples

3.2.1 Wiener Process

This section will deal with the case when the Lévy process is a Wiener process. Hence,

$$L_t = W_t = ct + \sigma B_t,$$

where B_t is a standard Brownian motion.

The moment generating function of such defined Lévy process is equal to

$$Ee^{uW_t} = e^{t(cu - \frac{\sigma^2}{2}u^2)}.$$

Further, by proposition 1.7.4 we obtain the following

$$\log Ee^{u\hat{Z}} = \int_0^u \frac{cz - \frac{\sigma^2}{2}z^2}{\beta z} dz = \frac{cu}{\beta} - \frac{\sigma^2 u^2}{4\beta}.$$

Hence, the distribution of the random variable $\hat{Z}$ is $N(\frac{c}{\beta}, \frac{\sigma^2}{4\beta})$. Further, as W_t is a continuous process $P(\tau_l < \infty) = \frac{H(x)}{H(l)}$. This implies

$$P(\tau_l < \infty) = \frac{P(\hat{Z} > x)}{P(\hat{Z} > l)} = \frac{\int_x^\infty e^{-\frac{(z-c/\beta)^2}{\sigma^2}2\beta} dz}{\int_l^\infty e^{-\frac{(z-c/\beta)^2}{\sigma^2}2\beta} dz}.$$

3.2.2 Compound Poisson Process with Exponential Jumps

This example is concerned with the case of exponential jumps. Namely the Lévy process L_t is given by the compound Poisson process of the form

$$L_t = (\xi_1 + \cdots + \xi_{N_\lambda(t)}) - ct, \quad t \geq 0,$$

where ξ_k are i.i.d. random variables with exponential distribution with parameter μ. This choice of the distribution of jumps allows to represent the possible overshoot through level l by a random variable Δ_l, which turns out to have the same exponential distribution with parameter μ due to the memoryless property of the exponential distribution.

Proposition 3.2.1. *Let $\hat{Z}$ be a random variable independent of L_t and having the common distribution with $Z(\beta, L)$, which is defined by* 3.1.2. *Then*

$$\hat{Z} + \frac{c}{\beta} \sim \Gamma\Big(\mu, \frac{\lambda}{\beta}\Big) \tag{3.2.1}$$

and

$$\hat{Z} + \Delta_l + \frac{c}{\beta} \sim \Gamma\Big(\mu, \frac{\lambda}{\beta} + 1\Big).$$

Proof:

We consider exponential jumps and a Lévy process of the following form

$$L_t = \sum_{k=1}^{N_t(\lambda)} \xi_k - ct,$$

where $\xi_k \sim Exp(\mu)$. Then the characteristic function cumulant (proposition 1.4.2) is

$$\psi(u) = \lambda E(e^{ui\xi_k} - 1) - icu = \lambda\Big(\frac{1}{1 - iu/\mu} - 1\Big) - icu = \frac{\lambda iu}{\mu - iu} - icu.$$

Further we use 1.7.4 in order to determine the characteristic function of $Z(\beta, L)$. Hence,

$$\log E\exp(iuZ(\beta, L)) = \int_0^u \frac{\psi(z)}{\beta z} dz = \frac{\lambda}{\beta}\int_0^u \frac{i}{\mu - iz} dz - \frac{icu}{\beta} =$$
$$-\frac{icu}{\beta} - \frac{\lambda}{\beta}\log(\mu - iz)|_0^u = -\frac{icu}{\beta} - \frac{\lambda}{\beta}\log(1 - iu/\mu).$$

Therefore,

$$E\exp(iuZ(\beta, L)) = \exp(-\frac{icu}{\beta})(1 - iu/\mu)^{-\frac{\lambda}{\beta}},$$

which means that the random variable $Z(\beta, L) + \frac{c}{\beta} \sim Gamma(\mu, \frac{\lambda}{\beta})$, with the density function

$$f(y) = \mu e^{-\mu y}(\mu y)^{\frac{\lambda}{\beta}-1}/\Gamma(\lambda/\beta), \quad y > 0.$$

Because $\Delta_L \sim Exp(\mu)$ which is $\Gamma(\mu, 1)$ we also get

$$Z(\beta, L) + \Delta_l + \frac{c}{\beta} \sim \Gamma(\mu, \frac{\lambda}{\beta} + 1).$$

■

We may now use proposition 3.2.1 and rewrite the probability of ruin in infinite time as follows

$$P(\tau_l < \infty) = \frac{P(\hat{Z} > x)}{P(\hat{Z} + \Delta_l > l)} = \frac{\frac{\mu^{\lambda/\beta}}{\Gamma(\frac{\lambda}{\beta})} \int_{\frac{c}{\beta}+x}^{\infty} e^{-\mu z} z^{\lambda/\beta - 1} dz}{\frac{\mu^{\lambda/\beta+1}}{\Gamma(\frac{\lambda}{\beta}+1)} \int_{\frac{c}{\beta}+l}^{\infty} e^{-\mu z} z^{\lambda/\beta} dz} =$$

$$= \frac{\Gamma(\frac{\lambda}{\beta}+1) \int_{\frac{c}{\beta}+x}^{\infty} \mu e^{-\mu z} (\mu z)^{\lambda/\beta-1} dz}{\Gamma(\frac{\lambda}{\beta}) \int_{\frac{c}{\beta}+l}^{\infty} \mu e^{-\mu z} (\mu z)^{\lambda/\beta} dz} = \frac{\lambda}{\beta} \frac{\int_{(\frac{c}{\beta}+x)\mu}^{\infty} e^{-t} t^{\lambda/\beta-1} dt}{\int_{(\frac{c}{\beta}+l)\mu}^{\infty} e^{-t} t^{\lambda/\beta} dt}.$$

And eventually,

$$P(\tau_l < \infty) = \frac{\frac{\lambda}{\beta}\Gamma(\frac{\lambda}{\beta}, (x + \frac{c}{\beta})\mu)}{\Gamma(\frac{\lambda}{\beta} + 1, (l + \frac{c}{\beta})\mu)}. \tag{3.2.2}$$

3.3 Finite Time Ruin Probabilities

This section includes generalisation of Harrison's results presented in the previous section. We propose a family of martingales, which leads to an explicit representation for the Laplace transform of the stopping time τ_l. This Laplace transform can be eventually inverted numerically and the finite time ruin probability $P(\tau_l < T)$ can be determined. We also consider a special case when the Laplace transform can be inverted analytically. Further, we compare these analytical solutions with results obtained by Monte Carlo simulation.

Denote $x^{+v} = \mathbb{I}(x > 0)x^v$ for any v where $\mathbb{I}(A)$ is an indicator function of the set A. Then set

$$G_+(x, v) = E(Z(\beta, v) - x)^{+v}.$$

Note that when $v = 0$

$$G_+(x, 0) = H(x).$$

Now we may formulate the following proposition.

Proposition 3.3.1. *Let $v > 0$ and assume that $E|L_t|^v < \infty$. Then*

$$G_+(X_t, v)e^{-\beta vt} \in M(\mathcal{F}_t, \mathbb{P}).$$

Proof:

To check the martingale property 1.2.3 we need to show that for any $t \geq s$

$$E(G_+(X_t, v)e^{-\beta vt} \mid \mathcal{F}_s) = G_+(X_s, v)e^{-\beta vs}$$

and that the expectation $EG_+(X_t, v)$ is finite.

In order to show finiteness of $EG_+(X_t, v)$ we need to apply the following inequality

$$(|a| + |b|)^v \leq C_v(|a|^v + |b|^v).$$

According to this inequality

$$0 \leq EG_+(X_t, v) = E(\hat{Z} - X_t)^{+v} \leq C_v(E|\hat{Z}|^v + E|X_t|^v) < \infty,$$

were C_v is a constant and the RHS is finite because the condition $E|L_t|^v < \infty$ implies finiteness of both $E|\hat{Z}|^v$ and $E|X_t|^v$.

Now we need to show that for any $t \geq s$

$$E(G_+(X_t, v)e^{-\beta vt} \mid \mathcal{F}_s) = G_+(X_s, v)e^{-\beta vs}.$$

Let random variable $\hat{Z} = Z(\beta, L)$ in distribution and let $\hat{Z}$ be independent of L_t. According to the Fubini theorem 1.2.2

$$E(G_+(X_t, v) \mid \mathcal{F}_s) = E((\hat{Z} - X_t)^{+v} \mid \mathcal{F}_s).$$

Further,

$$E(G_+(X_t, v)e^{-\beta vt} \mid \mathcal{F}_s) = E((\hat{Z} - X_t)^{+v} \mid \mathcal{F}_s)e^{-\beta vt} =$$

$$E((\hat{Z} - e^{\beta t}(X_s e^{-\beta s} - \int_s^t e^{-\beta y} dL_y))^{+v} \mid \mathcal{F}_s)e^{-\beta vt} =$$

$$E((e^{-\beta t}\hat{Z} + \int_s^t e^{-\beta y} dL_y - X_s e^{-\beta s})^{+v} \mid \mathcal{F}_s).$$

By lemma 3.1.4, the above coincides with

$$E((e^{-\beta s}\hat{Z} - X_s e^{-\beta s})^{+v} \mid \mathcal{F}_s) = E((\hat{Z} - X_s)^{+v} \mid \mathcal{F}_s)e^{-\beta vs}.$$

Again by Fubini theorem we get $G_+(X_s, v)e^{-\beta vs}$, which proves that $G_+(X_s, v)e^{-\beta vs}$ is a martingale.

■

The next proposition uses the optional stopping theorem and leads to a formula for the Laplace transform of τ_l.

Proposition 3.3.2.

$$E\mathbb{I}(\tau_l < \infty)e^{-\beta v \tau_l} = \frac{G_+(x,v)}{E[G_+(X_{\tau_l},v) \mid \sigma(\tau_l)]}, \tag{3.3.1}$$

where $\sigma(\tau_l)$ is a sigma – algebra generated by values of the random variable τ_l.

Proof:

By the optional stopping theorem,

$$EG_+(X_\tau, v)e^{-\beta v \tau} = G_+(x,v)$$

for any bounded stopping time τ and so for $\tau = \min(\tau_l, t)$. It implies that

$$E\mathbb{I}(\tau_l > t)G_+(X_t,v)e^{-\beta v t} + E\mathbb{I}(\tau_l \le t)G_+(X_{\tau_l},v)e^{-\beta v \tau_l} = G_+(x,v).$$

Consider this equation for $t \to \infty$. Note that on the set $(\tau_l > t)$ we have the bound $(\hat{Z} - X_t)^{+v} \le (\hat{Z} - l)^{+v}$. Due to independency of τ_l and $\hat{Z}$ and the fact that $E|\hat{Z}| < \infty$ we have

$$E\mathbb{I}(\tau_l > t)G_+(X_t,v)e^{-\beta v t} \le P(\tau_l > t)E(\hat{Z} - l)^{+v}e^{-\beta v t} \to 0$$

Applying the monotone convergence theorem we get

$$E\mathbb{I}(\tau_l < \infty)G_+(X_{\tau_l},v)e^{-\beta v \tau_l} = G_+(x,v).$$

This implies that

$$E\mathbb{I}(\tau_l < \infty)e^{-\beta v \tau_l} = \frac{G_+(x,v)}{E[G_+(X_{\tau_l},v) \mid \sigma(\tau_l)]},$$

where $\sigma(\tau_l)$ is a sigma – algebra generated by values of the random variable τ_l.

■

3.4 Examples

3.4.1 Wiener Process

Similarly, as in the section 3.2.2 we consider $L_t = W_t$, where W_t is a Wiener process. However, this time we are concerned with the finite time ruin probabilities of ruin $P(\tau_l < T)$. As stated before, we derive the Laplace transform of τ_l and in this example, as we do not observe jumps down of the risk process X_t, we have $X_{\tau_l} = l$ on the set $\{\tau_l < \infty\}$ and therefore we have the explicit formula

$$E\mathbb{I}(\tau_l < \infty)e^{-\beta v \tau_l} = \frac{G_+(x,v)}{G_+(l,v)},$$

where $G_+(x,v) = E(Z(\beta,L) - x)^{+v} = \int_x^\infty (z-x)^v f(z)dz$. Hence, for the Gaussian case the distribution of $\hat{Z}$ is normal with parameters $\frac{c}{\beta}$ and $\frac{\sigma^2}{4\beta}$. Therefore,

$$E\mathbb{I}(\tau_l < \infty)e^{-\beta v \tau_l} = \frac{\int_x^\infty (z-x)^v e^{-\frac{(z-c/\beta)^2}{\sigma^2}2\beta} dz}{\int_l^\infty (z-x)^v e^{-\frac{(z-c/\beta)^2}{\sigma^2}2\beta} dz},$$

which coincides with the results of the section 3.2.2 for $v = 0$.

3.4.2 Compound Poisson Process with Exponential Jumps

Consider an example of exponential jumps which was previously described in section 3.2.2. Having the proposition 3.2.1 it is easy to determine the Laplace transform 3.3.1 for the exponential distribution of jumps. Note that 3.3.1 can be rewritten as follows

$$E\mathbb{I}(\tau_l < \infty)e^{-\beta v \tau_l} = \frac{E(\hat{Z} - x)^{+v}}{E(\hat{Z} + \Delta_l - l)^{+v}}.$$

Moreover,

$$E(\hat{Z} - x)^{+v} = E(\hat{Z} + \frac{c}{\beta} - (\frac{c}{\beta} + x))^{+v} =$$

$$E(\mathbb{I}(\hat{Z}+\frac{c}{\beta}-(\frac{c}{\beta}+x)>0)(\hat{Z}+\frac{c}{\beta}-(\frac{c}{\beta}+x))^{v}).$$

Using 3.2.1 the above becomes the following

$$\int_{\frac{c}{\beta}+x}^{\infty}(z-(\frac{c}{\beta}+x))^{v}dF(z),$$

where $F(z)$ is the distribution function of the random variable $\hat{Z}+\frac{c}{\beta}$. It follows from the proposition 3.2.1 that the above is equivalent to

$$\frac{\mu^{\lambda/\beta}}{\Gamma(\frac{\lambda}{\beta})}\int_{\frac{c}{\beta}+x}^{\infty}e^{-\mu z}z^{\lambda/\beta-1}(z-(\frac{c}{\beta}+x))^{v}dz.$$

Repeating the above calculations we get

$$E(\hat{Z}+\Delta_{l}-l)^{+v}=\frac{\mu^{\lambda/\beta+1}}{\Gamma(\frac{\lambda}{\beta}+1)}\int_{\frac{c}{\beta}+l}^{\infty}e^{-\mu z}z^{\lambda/\beta}(z-(\frac{c}{\beta}+l))^{v}dz,$$

which leads to the final formula

$$E\mathbb{I}(\tau_{l}<\infty)e^{-\beta v\tau_{l}}=\frac{\lambda}{\beta\mu}\frac{\int_{\frac{c}{\beta}+x}^{\infty}e^{-\mu z}z^{\lambda/\beta-1}(z-(\frac{c}{\beta}+x))^{v}dz}{\int_{\frac{c}{\beta}+l}^{\infty}e^{-\mu z}z^{\lambda/\beta}(z-(\frac{c}{\beta}+l))^{v}dz},$$

then after a slight change of variables we get

$$E\mathbb{I}(\tau_{l}<\infty)e^{-\alpha\tau_{l}}=\frac{\lambda}{\beta\mu}\frac{\int_{\frac{c}{\beta}+x}^{\infty}e^{-\mu z}z^{\lambda/\beta-1}(z-(\frac{c}{\beta}+x))^{\alpha/\beta}dz}{\int_{\frac{c}{\beta}+l}^{\infty}e^{-\mu z}z^{\lambda/\beta}(z-(\frac{c}{\beta}+l))^{\alpha/\beta}dz}. \tag{3.4.1}$$

Notice when $\alpha=0$, equation 3.4.1 becomes a probability of ruin in infinite time, which coincides with 3.2.2.

3.5 PIDE Solution

The aim of this section is to confirm the results of this chapter. We find an explicit representation for the Laplace transform of the stopping time τ_l in the integral form for the case of exponential jumps. We use the Theory of Piecewise Deterministic Markov Processes (PDMP) and the Itô formula in order to find the integro – differential equation for the first passage time under a given level l. Then we find a bounded solution of this integro – differential equation expressed by a special function. Furthermore, we change this form of the solution using integral representation and properties of this special function in order to match the solution 3.4.1.

3.5.1 The Dynkin's Formula

Recall the formula for the generator of the OU – process given by 2.3.2:

$$L[g(x)] = (\beta x + m)g'(x) + \lambda \int_0^\infty [g(x-u) - g(x)]dF(u), \tag{3.5.1}$$

where $F(x)$ is the distribution of the jumps ξ. Moreover, $g(x)$ is a continuously differentiable and bounded function.

Theorem 3.5.1 (Dynkin's Formula, [27]). *Let τ_l be a stopping time and assume that $q_\alpha(x)$ is bounded and twice continuously differentiable on $x \geq l$ with a bounded first derivative there, where we at $y = l$ mean the right - hand derivative. If $q_\alpha(x)$ solves*

$$L[q_\alpha(x)] - \alpha q_\alpha(x) = 0 \quad for\ x \geq l \tag{3.5.2}$$

together with the boundary conditions

$$q_\alpha(x) = 1 \quad for\ x < l,$$

and

$$\lim_{x\downarrow l} E_x(q_\alpha(x-\xi)) = 1, \quad where \quad P(\xi > 0) = 1,$$

$$\lim_{x\to\infty} q_\alpha(x) = 0.$$

Then

$$q_\alpha(x) = E(e^{-\alpha\eta}).$$

3.5.2 Exponentially Distributed Pulses

In this section we assume that ξ is exponentially distributed random variable with a positive parameter μ and the parameter $c = 0$ in 3.0.1.

Proposition 3.5.2. *With a natural hypothesis that $q_\alpha(x)$ is twice differentiable for $x > l$, equation 3.5.2 is equivalent to*

$$\beta x q_\alpha''(x) + (\mu\beta x + \beta - \lambda - \alpha) q_\alpha'(x) - \alpha\mu q_\alpha(x) = 0 \tag{3.5.3}$$

Proof:

Let ξ have the exponential distribution with parameter $\mu > 0$. Then
$dF(u) = \mu e^{-\mu u} du$, $u > 0$, and 3.5.2 becomes

$$\beta x q_\alpha'(x) + \lambda\left(\int_0^\infty q_\alpha(x-u)\mu e^{-\mu u} du - q_\alpha(x)\right) - \alpha q_\alpha(x) = 0, \tag{3.5.4}$$

setting $c = 0$.

Observe that

$$\left(\int_0^\infty q_\alpha(x-u)\mu e^{-\mu u} du\right)' = \int_0^\infty q_\alpha'(x-u)\mu e^{-\mu u} du.$$

Integrating by parts

$$\int_0^\infty q_\alpha'(x-u)\mu e^{-\mu u} du = -\mu\left(\int_0^\infty q_\alpha(x-u)\mu e^{-\mu u} du - q_\alpha(x)\right).$$

From 3.5.4

$$-\mu\left(\int_0^\infty q_\alpha(x-u)\mu e^{-\mu u}du - q_\alpha(x)\right) = \frac{\mu}{\lambda}(\beta x q_\alpha'(x) - \alpha q_\alpha(x)).$$

Differentiating 3.5.4 with respect to x and using the fact that

$$\left(\int_0^\infty q_\alpha(x-u)\mu e^{-\mu u}du\right)' = \frac{\mu}{\lambda}(\beta x q_\alpha'(x) - \alpha q_\alpha(x)),$$

we get

$$\beta x q_\alpha''(x) + (\mu\beta x + \beta - \lambda - \alpha)q_\alpha'(x) - \alpha\mu q_\alpha(x) = 0.$$

■

GENERAL SOLUTION TO THE INTEGRO – DIFFERENTIAL EQUATION 3.5.3

It is known that the solution of 3.5.3 is expressed in terms of second kind confluent hypergeometric function $\Psi(a, b, x)$, known also as Kummer function, and generalized Laguerre polynomials $L(a, b, x)$.

Moreover, the general solution of 3.5.3 is a linear combination

$$q_\alpha(x) = C_1\Psi_1(x) + C_2 L_1(x),$$

where

$$\Psi_1(x) = x^{\frac{\alpha+\lambda}{\beta}} e^{-x\mu}\Psi\left(\frac{\alpha+\beta}{\beta}, \frac{\alpha+\lambda+\beta}{\beta}, x\mu\right),$$

$$L_1(x) = x^{\frac{\alpha+\lambda}{\beta}} e^{-x\mu} l\left(\frac{-\alpha-\beta}{\beta}, \frac{\alpha+\lambda}{\beta}, x\mu\right).$$

are independent solutions of 3.5.3 and constants C_1, C_2 are defined from some additional conditions.

BOUNDARY CONDITIONS

In order to find the explicit solution we use the following two boundary conditions:

1. $\lim_{x\to\infty} q_\alpha(x) = 0$,

2. Dynkin's formula 3.5.2 together with $\lim_{x\downarrow l} E_x(q_\alpha(x-\xi)) = 1$, where $P(\xi > 0) = 1$.

The first boundary condition is limiting the solution to the following

$$q_\alpha(x) = Cx^{\frac{\alpha+\lambda}{\beta}} e^{-x\mu}\Psi\left(\frac{\alpha+\beta}{\beta}, \frac{\alpha+\lambda+\beta}{\beta}, x\mu\right),$$

which also limits the number of constants we need to find. In order to find C we use the following representation of the second boundary condition.

Proposition 3.5.3. *The Dynkin's formula 3.5.2 together with*

$$\lim_{x\downarrow l} E_x(q_\alpha(x-\xi)) = 1, \quad \textit{where} \quad P(\xi > 0) = 1.$$

is equivalent to

$$\beta l q'_\alpha(l) - (\lambda+\alpha)q_\alpha(l) + \lambda = 0.$$

Proof:

We know that the generator of the process X_t has the following form in case $m = 0$

$$L[q_\alpha(x)] = \beta x q'_\alpha(x) + \lambda \int_0^\infty [q_\alpha(x-u) - q_\alpha(x)]dF(u).$$

Therefore, the Dynkin's formula 3.5.2 takes the form

$$\beta x q'_\alpha(x) + \lambda \int_0^\infty [q_\alpha(x-u) - q_\alpha(x)]dF(u) - \alpha q_\alpha(x) = 0.$$

This can be written as

$$\beta x q_\alpha'(x) + \lambda E_x(q_\alpha(x-\xi)) - \lambda q_\alpha(x) - \alpha q_\alpha(x) = 0.$$

Passing to the limit when $x \to l$ we get

$$\beta l q_\alpha'(l) - (\lambda+\alpha) q_\alpha(l) + \lambda = 0.$$

■

Proposition 3.5.4. *The second boundary condition*

$$\beta l q_\alpha'(l) - (\lambda+\alpha) q_\alpha(l) + \lambda = 0.$$

is satisfied for

$$q_\alpha(x) = \frac{\lambda}{\mu\beta} \frac{x^{\frac{\alpha+\lambda}{\beta}} e^{-x\mu} \Psi\left(\frac{\alpha+\beta}{\beta}, \frac{\alpha+\lambda+\beta}{\beta}, x\mu\right)}{l^{\frac{\alpha+\lambda}{\beta}+1} e^{-l\mu} \Psi\left(\frac{\alpha+\beta}{\beta}, \frac{\alpha+\lambda+\beta}{\beta}+1, l\mu\right)}. \tag{3.5.5}$$

and it has the following integral representation:

$$q_\alpha(x) = \frac{\lambda}{\beta\mu} \frac{\int_x^\infty e^{-\mu z}(z-x)^{\alpha/\beta} z^{\lambda/\beta - 1} dz}{\int_l^\infty e^{-\mu z}(z-l)^{\alpha/\beta} z^{\lambda/\beta} dz}$$

Proof:

The second boundary condition

$$\beta l q_\alpha'(l) - (\lambda+\alpha) q_\alpha(l) + \lambda = 0$$

takes the following form

$$e^{l\mu} - C\beta l^{\frac{\alpha+\beta+\lambda}{\beta}} \mu \left(\Psi\left(\frac{\alpha+\beta}{\beta}, \frac{\alpha+\beta+\lambda}{\beta}, l\mu\right) - \Psi'\left(\frac{\alpha+\beta}{\beta}, \frac{\alpha+\beta+\lambda}{\beta}, l\mu\right)\right) = 0,$$

for

$$q_\alpha(x) = Cx^{\frac{\alpha+\lambda}{\beta}} e^{-x\mu} \Psi\left(\frac{\alpha+\beta}{\beta}, \frac{\alpha+\lambda+\beta}{\beta}, x\mu\right).$$

This immediately leads to the formula for the constant C and $q_\alpha(x)$ as in 3.5.5. This is due to the fact that for the second kind confluent hypergeometric function $\Psi(a, b, x)$ the following property holds:

$$\Psi(a, b, x) - \Psi'(a, b, x) = \Psi(a, b+1, x).$$

The integral representation is obtained by using the following representation:

$$\Psi(a, b, x) = \frac{e^x}{\Gamma(a)} \int_1^\infty e^{-xt}(t-1)^{a-1} t^{\,b-a-1} dt. \tag{3.5.6}$$

Using 3.5.6 we obtain

$$x^{\frac{\alpha+\lambda}{\beta}} e^{-x\mu} \Psi\left(\frac{\alpha+\beta}{\beta}, \frac{\alpha+\lambda+\beta}{\beta}, x\mu\right) =$$

$$= \frac{1}{\Gamma(\frac{\alpha+\beta}{\beta})} x^{\frac{\alpha+\lambda}{\beta}} e^{-x\mu} e^{x\mu} \int_1^\infty e^{-x\mu t}(t-1)^{\alpha/\beta} t^{\lambda/\beta-1} dt$$

$$= \frac{1}{\Gamma(\frac{\alpha+\beta}{\beta})} \int_1^\infty e^{-x\mu t}(xt-x)^{\alpha/\beta}(xt)^{\lambda/\beta-1} x dt.$$

Introducing a new variable $z = xt$ we get the following

$$\frac{1}{\Gamma(\frac{\alpha+\beta}{\beta})} \int_x^\infty e^{-\mu z}(z-x)^{\alpha/\beta} z^{\lambda/\beta-1} dz.$$

Similar calculation is valid for

$$l^{\frac{\alpha+\lambda}{\beta}+1} e^{-l\mu} \Psi\left(\frac{\alpha+\beta}{\beta}, \frac{\alpha+\lambda+\beta}{\beta}+1, l\mu\right).$$

The above becomes

$$\frac{1}{\Gamma(\frac{\alpha+\beta}{\beta})} \int_l^\infty e^{-\mu z}(z-l)^{\alpha/\beta} z^{\lambda/\beta} dz.$$

Eventually,

$$q_\alpha(x) = \frac{\lambda}{\beta\mu}\frac{\int_x^\infty e^{-\mu z}(z-x)^{\alpha/\beta} z^{\lambda/\beta-1}dz}{\int_l^\infty e^{-\mu z}(z-l)^{\alpha/\beta} z^{\lambda/\beta}dz},$$

which coincides with 3.4.1 when $c = 0$ and confirms the result obtained by the martingale method.

■

3.6 Special Case and Numerical Results

This section is devoted to the numerical confirmation of the results of this chapter. As it was shown above we cannot always find an explicit formula for the finite time ruin probability $P(\tau_l < T)$. However, the distribution of the stopping time τ_l can be expressed as a Laplace transform of τ_l. Further, the Laplace transform can be inverted numerically in order to obtain the probability of ruin in the finite time $P(\tau_l < T)$.

This section, however, identifies a special case when the solution can be obtained analytically. We derive the explicit formula through inversion of the Laplace transform and confirm the result with the Monte Carlo simulation.

3.6.1 Explicit Solution for the Case $\lambda = \beta$ and $\xi_k \sim Exp(1)$

Consider solution 3.4.1 with $\lambda = \beta$ and $\xi_k \sim Exp(1)$. Then

$$E\mathbb{I}(\tau_l < \infty)e^{-\alpha\tau_l} = \frac{\int_{\frac{c}{\beta}+x}^{\infty} e^{-z}(z-(\frac{c}{\beta}+x))^{\alpha/\beta}dz}{\int_{\frac{c}{\beta}+l}^{\infty} e^{-z}z(z-(\frac{c}{\beta}+l))^{\alpha/\beta}dz}. \tag{3.6.1}$$

Introducing new variables $u = z - (\frac{c}{\beta}+x)$ and $v = z - (\frac{c}{\beta}+l)$ we get the following

$$E\mathbb{I}(\tau_l < \infty)e^{-\alpha\tau_l} = \frac{\int_0^{\infty} e^{-u-(\frac{c}{\beta}+x)}u^{\alpha/\beta}du}{\int_0^{\infty} e^{-v-(\frac{c}{\beta}+l)}(v+\frac{c}{\beta}+l)v^{\alpha/\beta}dv}.$$

Further, this can be expressed by a gamma function $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt$. It follows that

$$E\mathbb{I}(\tau_l < \infty)e^{-\alpha\tau_l} = \frac{e^{-(\frac{c}{\beta}+x)}\Gamma(\frac{\alpha}{\beta}+1)}{e^{-(\frac{c}{\beta}+l)}\Big(\Gamma(\frac{\alpha}{\beta}+2)+(\frac{c}{\beta}+l)\Gamma(\frac{\alpha}{\beta}+1)\Big)}.$$

We may now utilize the fact that $\Gamma(a+1) = a\Gamma(a)$ and simplify the Laplace transform to the following

$$E\mathbb{I}(\tau_l < \infty)e^{-\alpha\tau_l} = \frac{e^{l-x}}{\frac{\alpha}{\beta}+1+\frac{c}{\beta}+l}.$$

Moreover, assuming that $\alpha = 0$ we get the probability of ruin in infinite horizon for this particular case.

$$P(\tau_l < \infty) = \frac{e^{l-x}}{1 + \frac{c}{\beta} + l}.$$

This further implies that

$$E(e^{-\alpha\tau_l} \mid \tau_l < \infty) = \frac{1 + \frac{c}{\beta} + l}{\frac{\alpha}{\beta} + 1 + \frac{c}{\beta} + l},$$

as $E\mathbb{I}(\tau_l < \infty)e^{-\alpha\tau_l} = E(e^{-\alpha\tau_l} \mid \tau_l < \infty)P(\tau_l < \infty)$. Furthermore,

$$E(e^{-\alpha\tau_l} \mid \tau_l < \infty) = \frac{\beta(1 + \frac{c}{\beta} + l)}{\alpha + \beta(1 + \frac{c}{\beta} + l)}. \tag{3.6.2}$$

We may use now the property of an exponential random variable. Namely, its form of the Laplace transform. It is known that if $\xi \sim Exp(\lambda)$, then $E(e^{-\alpha\xi}) = \frac{\lambda}{\alpha+\lambda}$. Therefore, expression 3.6.2 implies that the conditional distribution of τ_l given $\tau_l < \infty$ is exponential with parameter $\beta(1 + \frac{c}{\beta} + l)$. Hence,

$$P(\tau_l < T) = P(\tau_l < \infty)P(\tau_l < T \mid \tau_l < \infty) = \frac{e^{l-x}}{1 + \frac{c}{\beta} + l}(1 - e^{-T\beta(1+l+\frac{c}{\beta})}).$$

3.6.2 Monte – Carlo Method

Monte – Carlo method is widely applied to problems, which are too complex to solve analytically. This numerical method solves a problem by generating suitable random numbers and observing that fraction of the numbers obeying some property or properties. In our simulation we observe the trajectories of the Ornstein – Uhlenbeck process, which models the surplus of the insurance company. Further, we approximate the probability of ruin by the fraction of trajectories which cross the predefined ruin level. This method is based on the frequency approach to the definition of probability, as

$$P(A) = \lim_{n\to\infty} \frac{n(A)}{n}$$

or, for n large enough

$$P(A) \approx \frac{n(A)}{n},$$

where A is an event, $n(A)$ is a number of times the event A occurs and n is a total number of experiments. The fraction $\frac{n(A)}{n}$ is called relative frequency of the event A.

To speed up the Monte – Carlo simulation of τ_l and $X(\tau_l)$, in particular $l = 0$, we use the following approach. Note that the paths of the process X_t are determined by the jump values located at $(T_k)_{k\geq 1}$. The corresponding values of the process X_t at these moments of time are then defined by the following recursion:

$$X(T_0) = X(0) = x,$$

$$X(T_k) = -\frac{m}{\beta} + \Big(X(T_{k-1}) + \frac{m}{\beta}\Big)e^{\beta(T_k - T_{k-1})} - \xi_k, \quad k = 1, 2, \ldots.$$

Further, let X be an exponentially distributed random variable with parameter λ. Then

$$e^{\beta X} \sim U^{-\beta/\lambda}, \quad U \sim Unif(0,1).$$

In particular, when $\lambda = \beta = 1$, $e^{\beta X} \sim 1/Unif(0,1)$.

Using the above representation we conclude that the values of the process X_t at the jump times can be computed using the following recursion:

$$X(T_0) = X(0) = x,$$

$$X(T_k) = -\frac{m}{\beta} + \Big(X(T_{k-1}) + \frac{m}{\beta}\Big)U^{-\beta/\lambda} - \xi_k, \quad k = 1, 2, \ldots,$$

where $U \sim Unif(0,1)$.

3.6.3 Numerical Results

In order to confirm our analytical solutions we use Monte – Carlo simulation introduced in section 3.6.2. We consider the special case discussed in section 3.6.1.

Time	Explicit Solution	Monte – Carlo simulation
10	0.099053	0.09909870 (0.00008964)
20	0.164136	0.16411287 (0.00011111)
30	0.206898	0.20686806 (0.00012152)
40	0.234995	0.23505948 (0.00012721)
50	0.253456	0.25349827 (0.00013050)
60	0.265585	0.26566926 (0.00013251)
70	0.273555	0.27353233 (0.00013373)
80	0.278792	0.27877550 (0.00013452)
90	0.282232	0.28225939 (0.00013503)
100	0.284493	0.28450591 (0.00013535)

Table 3.1: Comparison of the numerical results and the explicit solution for $\lambda = \beta = 0.02$, $c = 0.002$, $x = 1.5$, $l = 1.0$ and $\xi_k \sim Exp(1)$.

Namely, we confirm that for $\lambda = \beta$ and $\xi_k \sim Exp(1)$ the solution for the finite time probability can be expressed as follows

$$P(\tau_l < T) = \frac{e^{l-x}}{1 + \frac{c}{\beta} + l}(1 - e^{-T\beta(1+l+\frac{c}{\beta})}).$$

We perform the computation for $\lambda = \beta = 0.02$, $c = 0.002$, $x = 1.5$, $l = 1.0$ and $\xi_k \sim Exp(1)$. The comparison of the numerical results and the explicit solution is included in Table 3.1 and also illustrated graphically in Figure 3.1.

To conclude, we can clearly see that Monte – Carlo simulation confirms the analytical results. The fit of the explicit solution to the results of Monte – Carlo simulation is very good, which can be seen in Figure 3.1.

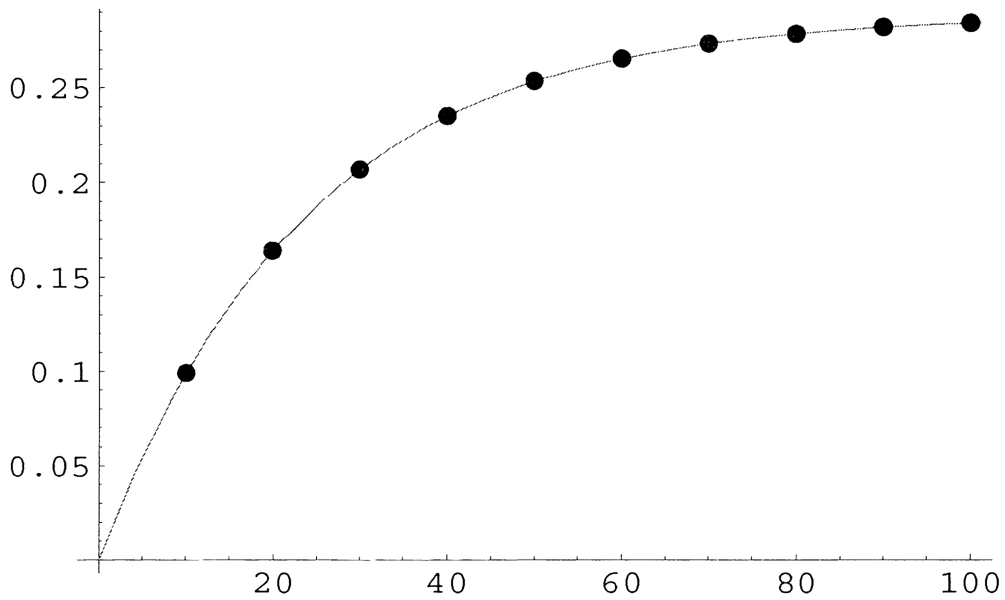

Figure 3.1: Comparison of the numerical results and the explicit solution for $\lambda = \beta = 0.02$, $c = 0.002$, $x = 1.5$, $l = 1.0$ and $\xi_k \sim Exp(1)$.

Chapter 4

Diffusion Approximation for the Model with Time Dependent Interest Rate

Consider a surplus model of an insurance company defined by the following Stochastic Differential Equation (SDE):

$$X_0 = x, \quad dX_t = \beta_t X_t dt + dL_t, \quad t \geq 0, \tag{4.0.1}$$

where β_t is a time dependent interest rate. Process L_t is a Lévy process, which models the income of the insurance company. In particular, L_t is a Compound Poisson process of the form

$$L_t = mt - \sum_{i=1}^{N_t(\lambda)} \xi_i,$$

where $N_t(\lambda)$ is a Poisson process with a positive parameter λ, while ξ_i are i.i.d. random variables such that $P(\xi_i > 0) = 1$. Parameter m is a positive drift parameter, which can be associated with incoming premiums.

We also define a stopping time

$$\tau_L = \inf\{t \geq 0 : X_t \leq L\},$$

which is a first crossing time through the general boundary $L < x$ by X_t.

The aim of this chapter is to find the distribution of the stopping time τ_L, namely $P(\tau_L < T)$, $0 \leq T < \infty$. This is achieved by a diffusion approximation of the model 4.0.1. Further, the first crossing time problem for the diffusion process is transformed into an equivalent first passage time problem for a standard Brownian motion through a moving boundary. Finally, the moving boundary is again approximated by a piecewise linear function and the distribution of the first crossing time through this barrier is found. Additionally, we focus on a special case of the problem. Namely, we consider the model 4.0.1 with the drift parameter $m = \frac{\lambda}{\alpha}$, $\xi_k \sim Exp(\alpha)$ and the barrier $L = 0$. It turns out that there exists an easy analytical formula for $P(\tau_L < T)$ in this particular case. This and other results of this chapter are confirmed by Monte Carlo simulation.

4.1 Diffusion Approximation

The idea behind diffusion approximation is simply to approximate the risk process by a Brownian motion (or a more general diffusion) by fitting the first and second moments, and use the fact that first passage probabilities are more readily calculated for diffusions than for the risk process itself.

Definition 4.1.1 (Diffusion Process, [33]). By the SDE

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW(t), \tag{4.1.1}$$

we mean the equation

$$X(t) = X(0) + \int_0^t a(v, X(v))dv + \int_0^t \sigma(v, X(v))dW(v), \quad t \geq 0,$$

and a stochastic process $X(t)$ fulfilling 4.1.1 is its solution. If the solution is unique, then the process $X(t)$ is called a diffusion process with infinitesimal drift function $a(t,x)$ and infinitesimal variance $\sigma^2(t,x)$ at (t,x), provided that $\sigma^2(t,x) > 0$ for all $t \geq 0$. We also say that $X(t)$ is an $(a(t,x), \sigma^2(t,x))$ - diffusion.

Theorem 4.1.1 (Existence of a Unique Solution, [33]). *Assume that $E(X^2(0)) < \infty$ and that for any $x > 0$ there exists a constant $c_x \in (0, \infty)$ such that for all $y, z \in \mathbb{R}$,*

$$\mid \sigma(t,z) - \sigma(t,y) \mid + \mid a(t,z) - a(t,y) \mid \leq c_x \mid z - y \mid$$

and

$$\sigma^2(t,y) + a^2(t,y) \leq c_x(1 + y^2)$$

whenever $0 \leq t \leq x$. Then there exists a unique solution $X(t)$ to 4.1.1.

The following proposition is a simple implication of theorem 4.1.1 and definition 4.1.1 as well as the Itô formula 1.5.2.

Proposition 4.1.2. *A stochastic process X_t described by the following SDE*

$$dX_t = (\beta_t X_t + m)dt + \sigma dB_t, \quad \text{where } B_t \text{ is a standard Brownian motion} \tag{4.1.2}$$

has a unique solution expressed by

$$X_t = e^{\int_0^t \beta(s)ds}(X_0 + m\int_0^t e^{-\int_0^u \beta(s)ds}du + \sigma \int_0^t e^{-\int_0^u \beta(s)ds}dB_u), \tag{4.1.3}$$

where B_t is a standard Brownian motion. Therefore, X_t is a well defined diffusion process.

Recall the definition of the weak convergence for stochastic processes.

Definition 4.1.2 (Weak Convergence, [37]). A sequence $(X^{(n)} : n \in \mathbb{N})$ of stochastic processes is said to converge weakly to a stochastic process X if for every bounded continuous functional f it follows that

$$\lim_{n\to\infty} E[f(X^{(n)})] = E[f(X)].$$

In this case we write $X^{(n)} \Longrightarrow X$.

Proposition 4.1.3. *A sequence of stochastic processes defined through the SDE*

$$dX^{(n)}(t) = \beta_t^{(n)} X_t^{(n)} dt + dL_t^{(n)}, \quad X_0^{(n)} = L_0,$$

converges weakly to the diffusion process Z satisfying the SDE

$$dZ_t = \beta_t Z_t dt + dW_t, \quad Z_0 = B_0. \tag{4.1.4}$$

Proof:

We may use the fact that $L^{(n)} \Longrightarrow W$ implies $X^{(n)} \Longrightarrow Z$, Schmidli [37]. Therefore, it is sufficient to prove that $L^{(n)} \Longrightarrow W$. This fact is an implication of two theorems, the Donsker's theorem for random walks 1.3.4 and theorem 1.7.1. We have a Lévy process

$$L_t = mt - \sum_{i=1}^{N_t(\lambda)} \xi_i.$$

Using the above mentioned theorems we get

$$\frac{L_t^{(n)} - E(L_t^{(n)})}{\sqrt{n}} \to N(0, \sigma^2 t).$$

Hence, the Lévy process L_t converges weakly to a Wiener process W_t such that $\sigma^2 = Var(W_t) = Var(L_t)$ and $\mu = E(W_t) = E(L_t)$. In our case $Var(L_t) = \lambda t E(\xi^2)$ and $E(L_t) = -\lambda t E(\xi) + mt$.

■

Define the ruin times $\tau_n = \inf\{t > 0 : X_t^{(n)} \leq 0\}$ and $\tau = \inf\{t > 0 : Z_t \leq 0\}$. We are interested in the convergence of the ruin probability $P[\tau_n \leq t]$ to the corresponding ruin probability of the diffusion.

Lemma 4.1.4. *Let $(V_n : n \in \mathbb{N})$ be a sequence of real random variables, and V be a random variable such that $V_n \Longrightarrow V$ and $Var(V) < \infty$. If*

$$\limsup_{n\to\infty} Var(V_n) < \infty$$

then

$$\lim_{n\to\infty} E[V_n] = E[V].$$

Proposition 4.1.5 (Convergence of Finite Time Ruin Probabilities, [37]). *Let* $X^{(n)}$ *be a sequence of stochastic processes with sample paths in* D, $X^{(n)} \Longrightarrow X$, *where* X *is a diffusion process, and define* $\tau_n^a = \inf\{t \geq 0 : X_t^{(n)} < a\}$, $\tau^a = \inf\{t \geq 0 : X_t < a\}$ *for a constant* $a \in \boldsymbol{R}$. *Then* $\tau_n^a \Longrightarrow \tau^a$, *in particular*

$$\lim_{n\to\infty} P[\tau_n^a \leq t] = P[\tau^a \leq t].$$

Proof:

To prove this theorem we use the definition 4.1.2 of weak convergence and the lemma 4.1.4. We may interchange the limit and expectation since $\limsup_{n\to\infty} Var(\tau_n^a) < \infty$. As a result of that we get the weak convergence $\tau_n^a \Longrightarrow \tau^a$ and in particular

$$\lim_{n\to\infty} P[\tau_n^a \leq t] = P[\tau^a \leq t].$$

■

4.2 First Passage Time Problem for a Gaussian OU – Process

As described in the previous section, any OU – process driven by a Levy process L_t can be approximated by a Gaussian OU – process Z_t. Furthermore, solving the first passage time problem for the diffusion process Z_t we get a proper approximation for the ruin probabilities of the general OU – process. Hence, we consider the following model

$$Z_0 = x, \quad dZ_t = (\beta_t Z_t + c)dt + \sigma B_t, \ t \geq 0 \tag{4.2.1}$$

where $c = m - \lambda E(\xi)$ and $\sigma = \sqrt{\lambda E(\xi^2)}$. We study the first passage time of the process Z_t under the level $L < x$. Therefore, the stopping time is defined as

$$\tau_L = \inf\{t \geq 0 : Z_t = L\}.$$

4.2.1 General Solution

Consider the model 4.2.1 with its explicit solution

$$Z_t = e^{\int_0^t \beta(s)ds}(Z_0 + c\int_0^t e^{-\int_0^v \beta(s)ds}dv + \sigma\int_0^t e^{-\int_0^v \beta(s)ds}dB_v).$$

We are interested in a distribution of the stopping time τ_L, hence we can express the non – ruin probability as follows:

$$P(\tau_L > T) = P(Z_t > L, \ t \leq T) =$$

$$= P(e^{\int_0^t \beta(s)ds}(Z_0 + c\int_0^t e^{-\int_0^v \beta(s)ds}dv + \sigma\int_0^t e^{-\int_0^v \beta(s)ds}dB_v) > L, \ t \leq T) =$$

$$= P(Z_0 + c\int_0^t e^{-\int_0^v \beta(s)ds}dv + \sigma\int_0^t e^{-\int_0^v \beta(s)ds}dB_v > Le^{-\int_0^t \beta(s)ds}, \ t \leq T) =$$

$$= P(\int_0^t e^{-\int_0^v \beta(s)ds} dB_v > \frac{L}{\sigma} e^{-\int_0^t \beta(s)ds} - \frac{Z_0}{\sigma} - \frac{c}{\sigma}\int_0^t e^{-\int_0^v \beta(s)ds} dv,\ t \le T).$$

At this point we perform a change of time for the Brownian motion. According to the theorem 1.5.1, $\int_0^t e^{-\int_0^v \beta(s)ds} dB_v$ has the same distribution as $\tilde{B}(\int_0^t e^{-2\int_0^v \beta(s)ds} dv)$, where $\tilde{B}$ is another standard Brownian motion, hence the problem is being reduced to the following

$$P(\tilde{B}(\int_0^t e^{-2\int_0^v \beta(s)ds} dv) > \frac{L}{\sigma} e^{-\int_0^t \beta(s)ds} - \frac{Z_0}{\sigma} - \frac{c}{\sigma}\int_0^t e^{-\int_0^u \beta(s)ds} du,\ t \le T),$$

where $\tilde{B}$ is a new standard Brownian motion. This is of course equivalent to

$$P(\tilde{B}(\int_0^t e^{-2\int_0^v \beta(s)ds} dv) < \frac{Z_0}{\sigma} - \frac{L}{\sigma} e^{-\int_0^t \beta(s)ds} + \frac{c}{\sigma}\int_0^t e^{-\int_0^v \beta(s)ds} dv,\ t \le T).$$

Let now

$$u = \int_0^t e^{-2\int_0^v \beta(s)ds} dv$$

If we denote the solution of $\int_0^t e^{-2\int_0^v \beta(s)ds} dv$ as $F(t)$ we may write that $t = F^{-1}(u)$. Hence, we now consider the following problem

$$P(\tilde{B}(u) < \frac{Z_0}{\sigma} - \frac{L}{\sigma} e^{-\int_0^{F^{-1}(u)} \beta(s)ds} + \frac{c}{\sigma}\int_0^{F^{-1}(u)} e^{-\int_0^v \beta(s)ds} dv,\ u \le \int_0^T e^{-2\int_0^v \beta(s)ds} dv).$$

For a simplicity denote

$$\kappa_T = \int_0^T e^{-2\int_0^v \beta(s)ds} dv$$

and

$$g(u) = \frac{Z_0}{\sigma} - \frac{L}{\sigma} e^{-\int_0^{F^{-1}(u)} \beta(s)ds} + \frac{c}{\sigma}\int_0^{F^{-1}(u)} e^{-\int_0^v \beta(s)ds} dv.$$

Therefore, the initial problem was transformed to the equivalent non – ruin probability

$$P(\tilde{B}(u) < g(u),\ u \le \kappa_T),$$

which implies

$$P(\tau_L > T) = P(\sigma_{g(u)} > \kappa_T)$$

for

$$\sigma_{g(u)} = \inf\{u \geq 0 : \tilde{B}(u) = g(u)\}.$$

Concluding, we have shown that it is possible to reduce a first crossing time problem for a Gaussian OU process 4.2.1 under the level $L < x$ to a first crossing time problem for a standard Brownian motion over a moving boundary $g(u)$. In the next section of this chapter we discuss an existing method, which provides an approximation of the distribution of $\sigma_{g(u)}$. However, before we move to this discussion consider the following two special cases of the problem.

4.2.2 Special Cases of the Problem

1) c = 0 and L = 0

Let $c = 0$ and consider a stochastic process described by the following SDE

$$dZ_t = \beta_t Z_t dt + \sigma B_t.$$

Let also $L = 0$ and consider the stopping time

$$\tau_0 = \inf\{t \geq 0 : Z_t = 0\}.$$

The above SDE has a unique solution expressed by

$$Z_t = e^{\int_0^t \beta(s)ds}(Z_0 + \sigma \int_0^t e^{-\int_0^v \beta(s)ds} dB_v).$$

Hence, the non – ruin probability can be expresses as follows:

$$P\{\tau_0 > T\} = P\{Z_t > 0,\ t \leq T\} = P\{e^{\int_0^t \beta(s)ds}(Z_0 + \sigma \int_0^t e^{-\int_0^v \beta(s)ds} dB_v) > 0,\ t \leq T\}.$$

We may now divide by $e^{\int_0^t \beta(s)ds}$, assuming that $\beta(t)$ is a positive function and as a result of that we get the following

$$P\{Z_0 + \sigma \int_0^t e^{-\int_0^v \beta(s)ds} dB_v > 0,\ t \leq T\} = P\{\int_0^t e^{-\int_0^v \beta(s)ds} dB_v > -\frac{Z_0}{\sigma},\ t \leq T\}.$$

According to the theorem 1.5.1 we may substitute $\int_0^t e^{-\int_0^v \beta(s)ds} dB_v$ for a standard Brownian motion of the form $\tilde{B}(\int_0^t e^{-2\int_0^v \beta(s)ds} dv)$. Hence, we have

$$P\{\tilde{B}(\int_0^t e^{-2\int_0^v \beta(s)ds} dv) > -\frac{Z_0}{\sigma},\ t \le T\}.$$

Furthermore, introducing a new variable $u = \int_0^t e^{-2\int_0^v \beta(s)ds} dv$ we get

$$P\{\tilde{B}(u) > -\frac{Z_0}{\sigma},\ u \le \int_0^T e^{-2\int_0^v \beta(s)ds} dv\} = P\{\tilde{B}(u) < \frac{Z_0}{\sigma},\ u \le \int_0^T e^{-2\int_0^v \beta(s)ds} dv\},$$

from the properties of a Browinan motion. Denoting $\kappa_T = \int_0^T e^{-2\int_0^v \beta(s)ds} dv$ and defining a new stopping time

$$\sigma_{\frac{Z_0}{\sigma}} = \inf\{u \ge 0 : \tilde{B} = \frac{Z_0}{\sigma}\},$$

we obtain the following:

$$P(\tau_0 > T) = P(\sigma_{\frac{Z_0}{\sigma}} > \kappa_T).$$

Therefore, according to the theorem 1.3.5 the density function of $\sigma_{\frac{Z_0}{\sigma}}$, which is equal to the density function of τ_0 can be expressed by:

$$p_{\sigma_{\frac{Z_0}{\sigma}}}(t) = \frac{\frac{Z_0}{\sigma}}{\sqrt{2\pi s^3}} \exp\{-\frac{(\frac{Z_0}{\sigma})^2}{2s}\},\ where\ \ s = \int_0^t e^{-2\int_0^v \beta(s)ds} dv.$$

2) Constant interest rate

Let $\beta_t = \beta = const$. In this case the problem is simplified to the following

$$P\Big(\tilde{B}_u < \frac{Z_0}{\sigma} + \frac{c}{\beta\sigma} - \Big(\frac{L}{\sigma} + \frac{c}{\beta\sigma}\Big)\sqrt{1 - 2\beta u},\ \ u \le \frac{1}{2\beta}\Big(1 - e^{-2\beta T}\Big)\Big),$$

which is illustrated in Figure 4.1.

Hence, it is a first crossing problem over the square root boundary by the standard Brownian motion. Square root boundaries were previously considered by Breiman [5], Sheep [38], Novikov [21] and Sato [35]. Mellin transforms for density functions of the

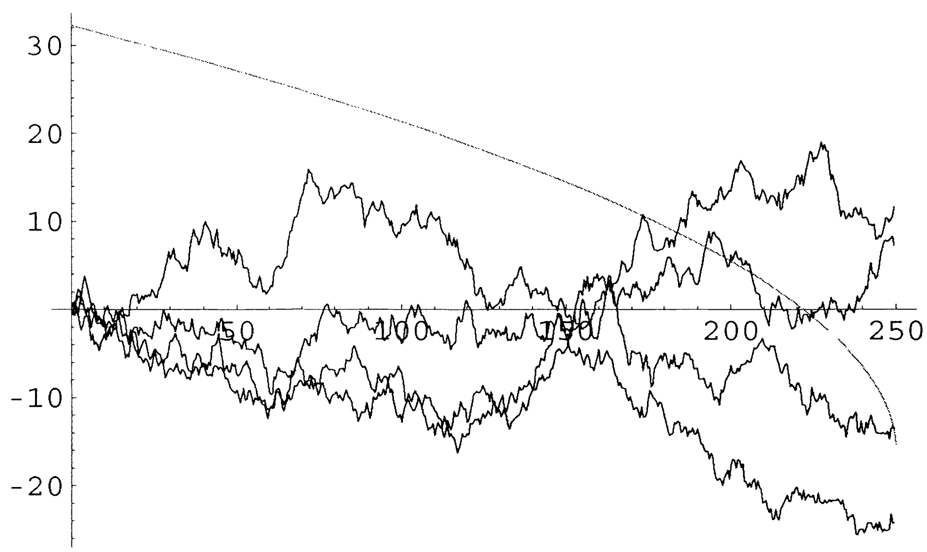

Figure 4.1: Boundary Crossing Problem for the Brownian motion.

stopping times $\tau_1 = \inf\{t \geq 0 : B_t \geq a + b\sqrt{t+c}\}$, $c \geq 0$ and $a + b\sqrt{c} > 0$ as well as $\tau_2 = \inf\{t \geq 0 : |B_t| \geq b\sqrt{t+c}\}$, $c > 0$ where found by Novikov [21] and Shepp [38], respectively. Unfortunately, these results cannot be utilized in our case. Hence, the following Piecewise Linear Approximation (PLA), which can be used to any general boundary.

4.3 Piecewise Linear Approximation (PLA)

In general we are interested in the calculation of probabilities of the form

$$P_t(g) := P\{B_s < g(s), 0 \leq s \leq t\},$$

where B_s is a standard Brownian motion.

In this section we consider approximation of $P_t(g)$ by $P_t(\hat{g}_n)$ where, in particular, the boundary $\hat{g}_n$ is a piecewise linear function. The probability $P_t(\hat{g}_n)$ can be calculated as an n – fold integral.

Consider $\hat{g}(s)$ as piecewise linear functions with nodes $t_i, t_0 = 0 < t_1 < \cdots < t_n = t$. Denote

$$p(i, \hat{g} \mid x, y) := P\{B_s < \hat{g}(s), t_i \leq s \leq t_{i+1} \mid B_{t_i} = x, B_{t_{i+1}} = y\}.$$

This conditional probability has a known explicit formula for the case of $\hat{g}$ a linear boundary on the interval $[t_i, t_{i+1}]$, which matches our needs and can be expressed as follows:

$$p(i, \hat{g} \mid x_i, x_{i+1}) = \mathbb{I}(\hat{g}(t_i) > x_i, \hat{g}(t_{i+1}) > x_{i+1})\Big[1 - \exp\Big\{-\frac{2(\hat{g}(t_i) - x_i)(\hat{g}(t_{i+1}) - x_{i+1})}{t_{i+1} - t_i}\Big\}\Big].$$

The following theorem, Novikov et al. [22], gives the representation for $P_t(\hat{g})$ as an n-fold integral of $p(i, \hat{g} \mid x, y)$.

Theorem 4.3.1.

$$P_t(\hat{g}) = E\Big[\prod_{i=0}^{n-1} p(i, \hat{g} \mid B_{t_i}, B_{t_{i+1}})\Big].$$

Hence, by theorem 4.3.1:

$$P_{\kappa_T}(g_u) = \int_{-\infty}^{g(t_1)} \int_{-\infty}^{g(t_2)} \ldots \int_{-\infty}^{g(t_n)=g(\kappa_T)} p(0, \hat{g} \mid B_{t_0} = x_0, B_{t_1} = x_1) \times \ldots$$
$$\times p(n-1, \hat{g} \mid B_{t_{n-1}} = x_{n-1}, B_{t_n} = x_n) dx_1 dx_2 \ldots dx_n.$$

4.4 Numerical Results

This section focuses on numerical computations. Firstly, it compares the explicit solution given in section 4.2.2 with the Monte Carlo simulation. Further, it shows how this solution can be used in order to approximate some finite – time probabilities of ruin for the general model with the deterministic, time dependent interest rate 4.0.1. Secondly, it compares the result of Monte Carlo simulation with the PLA approach described in this chapter. PLA is applied in order to approximate the probability of crossing the general boundary

$$g(u) = \frac{Z_0}{\sigma} - \frac{L}{\sigma} e^{-\int_0^{F^{-1}(u)} \beta(s)ds} + \frac{c}{\sigma} \int_0^{F^{-1}(u)} e^{-\int_0^v \beta(s)ds} dv$$

by a standard Brownian motion $\tilde{B}_u$ for $0 \leq u \leq \kappa_T$. This probability approximates the probability of crossing a constant boundary by a Gaussian OU – process. Refer also to Appendix B for further description of C++ programs used in this section. Consider the following numerical results:

COMPARISON OF THE EXPLICIT SOLUTION WITH MONTE CARLO SIMULATION

The explicit formula obtained in section 4.2.2 is compared with Monte Carlo simulation. Assume that the interest rate is a function given as follows:

$$\beta_t = \beta + ae^{-t},$$

where $\beta, a = const$. Such interest rate has been previously considered by Roberts and Shortland [32] and it is the expected value of the risk – free interest rate, I_t, under a Vasicek model satisfying

$$dI_t = (\beta - I_t)dt + \sigma dB_t,$$

Time	Explicit Solution	Diffusion MC
10	0.00221	0.00220(0.00044)
20	0.02947	0.02913(0.00160)
30	0.07466	0.07195(0.00245)
40	0.12196	0.11944(0.00308)
50	0.16592	0.16233(0.00350)
100	0.32434	0.32105(0.00443)
200	0.48130	0.47985(0.00471)
300	0.56140	0.55699(0.00471)
400	0.61150	0.60923(0.00463)
500	0.64640	0.64498(0.00454)

Table 4.1: Comparison of the explicit solution 4.4.1 with the Monte Carlo simulation for the Gaussian model. Non - constant interest rate $\beta_t = 0.0002 + 0.1e^{-t}$.

with $I_0 = \beta + a$ and $\sigma = 1$.

We set $\beta = 0.0002$ and $a = 0.1$ in the interest rate β_t, $t \geq 0$. Further, the probability of ruin in the finite – time horizon is given by

$$P(\sigma_0 \leq T) = 2 \int_{\frac{Z_0}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\} dx, \tag{4.4.1}$$

where $t := \int_0^T \exp\{-2\int_0^v \beta(s)ds\} dv$. Table 4.1 compares results obtained by this formula with the Monte Carlo simulation for $Z_0 = 5$ and $\sigma = 0.565685424$. This is also illustrated in Figure 4.2.

It shows a slight difference between the explicit solution and the Monte Carlo simulation with stems from the fact that in order to perform the Monte Carlo simulation it is necessary to change the time parameter from continuous to discrete. Therefore, the probabilities obtained by the Monte Carlo simulation are slightly lower then the explicit probabilities. In spite of this differences the explicit solution seems to be useful.

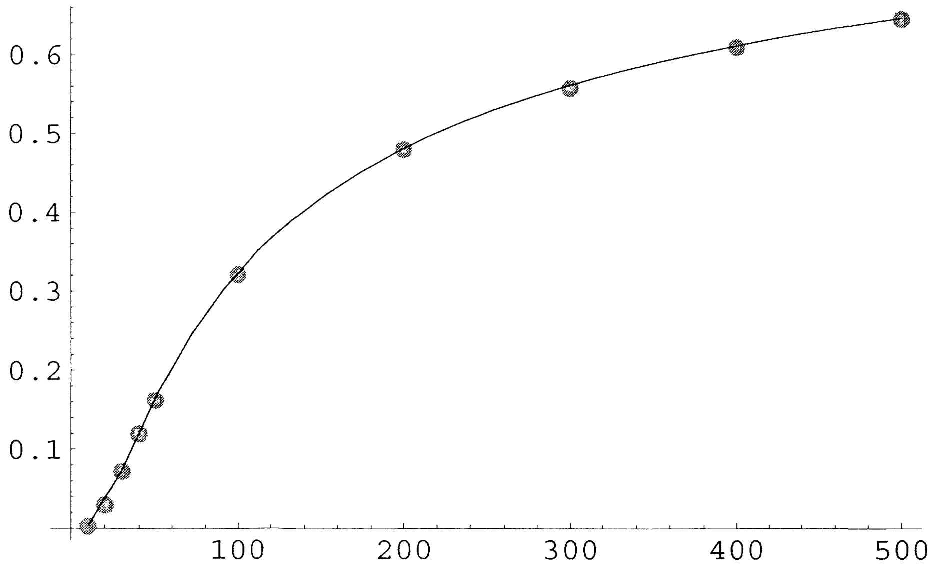

Figure 4.2: Comparison of the explicit solution 4.2.2 with the Monte Carlo simulation for the Gaussian model. Non – constant interest rate $\beta_t = 0.0002 + 0.1e^{-t}$.

Additionally, Table 4.2 shows a similar comparison, however the interest rate is assumed to be constant $\beta = 0.0002$. This is also illustrated in Figure 4.3.

APPROXIMATION OF THE INFINITE – TIME PROBABILITY OF RUIN FOR THE GENERAL MODEL 4.0.1

Results of this chapter, in particular the explicit solution introduced in section 4.2.2, can be used as a good approximation of the finite time ruin probabilities for the general model 4.0.1. This result, however can be just applied for the drift parameter m equal to $\frac{\lambda}{\alpha}$ and jumps $\xi_k \sim Exp(\alpha)$. The interest rate, nevertheless, can be any deterministic, bounded function of time. For the purpose of this illustration it is assumed that the interest rate is equal to

$$\beta_t = 0.0002 + 0.1e^{-t}.$$

Time	Explicit Solution	MC simulation
10	0.0051439	0.004991(0.0002114)
20	0.0476611	0.045453(0.0006249)
30	0.1055360	0.102405(0.0009095)
40	0.1605760	0.156796(0.0010908)
50	0.2090230	0.204257(0.0012095)
100	0.3719980	0.366295(0.0014454)
200	0.5237730	0.518531(0.0014990)
300	0.5991010	0.594370(0.0014730)
400	0.6457080	0.641031(0.0014391)
500	0.6779920	0.673463(0.0014068)

Table 4.2: Comparison of the explicit solution 4.2.2 with the Monte Carlo simulation for the Gaussian model. Constant interest rate $\beta = 0.0002$.

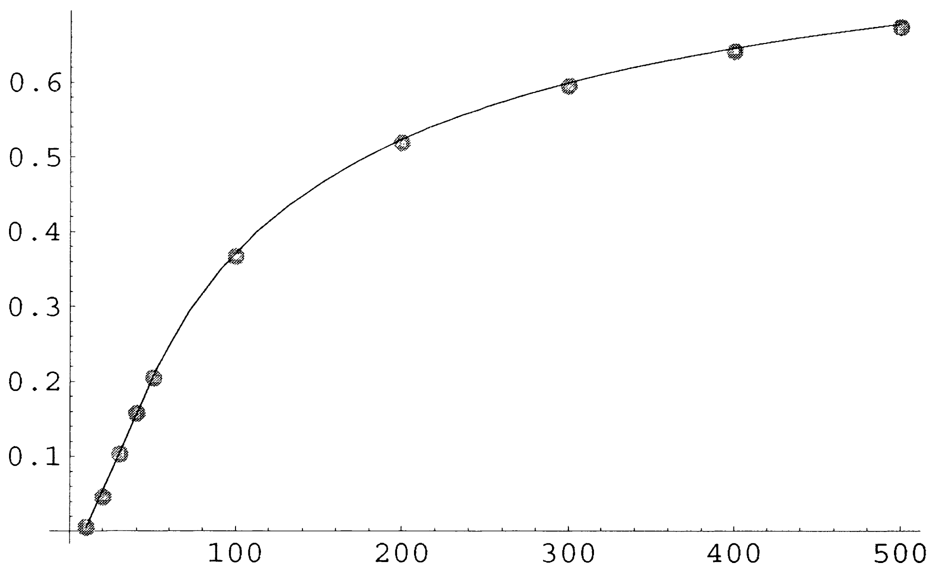

Figure 4.3: Comparison of the explicit solution 4.2.2 with the Monte Carlo simulation for the Gaussian model. Constant interest rate $\beta = 0.0002$.

Time	Explicit Solution	General Model MC
10	0.00221	0.01168(0.00032)
20	0.02947	0.04994(0.00065)
30	0.07466	0.09513(0.00088)
40	0.12196	0.13985(0.00104)
50	0.16592	0.18010(0.00115)
100	0.32434	0.32683(0.00141)
200	0.48130	0.47862(0.00150)
300	0.56140	0.55645(0.00149)
400	0.61150	0.60607(0.00147)
500	0.64640	0.64086(0.00144)

Table 4.3: Comparison of the explicit solution 4.2.2 with the Monte Carlo simulation for the general model. Non – constant interest rate $\beta_t = 0.0002 + 0.1e^{-t}$.

Further, $\lambda = 1$ and $\alpha = 2.5$, hence $m = 0.4$. The initial value of the process is equal to 5 and the parameter σ for the explicit solution is equal to $\frac{\sqrt{2\lambda}}{\alpha} = 0.565685424$. Table 4.3 shows the comparison of the explicit solution with Monte Carlo simulation for the general model 4.0.1, which is also illustrated in Figure 4.4.

This approximation seems to be not sufficiently accurate for small probabilities, however the accuracy increases with the increase of the probability of ruin.

PIECEWISE APPROXIMATION

Table 4.4 summarises results of three Monte Carlo simulations. The General Model is the model 4.0.1 with exponential jumps with parameter $\alpha = 5.0$. We also assume that $\lambda = 1.2$, $X_0 = 30$, $\beta_t = 0.0002 + 0.0001e^{-t}$ and (drift term) $m = 0$. Diffusion Approximation must be performed for the same $\beta_t = 0.0002 + 0.0001e^{-t}$ and $Z_0 = 30$ but c is assumed to be equal to $-\frac{\lambda}{\alpha} = -0.24$ and $\sigma = \sqrt{\frac{2\lambda}{\alpha^2}} = \sqrt{0.096} = 0.309839$. Piecewise Linear Approximation is valid for the same parameters as the Diffusion Approximation. Number of trials for General Model is 10^6, whereas it is 10^5 for both

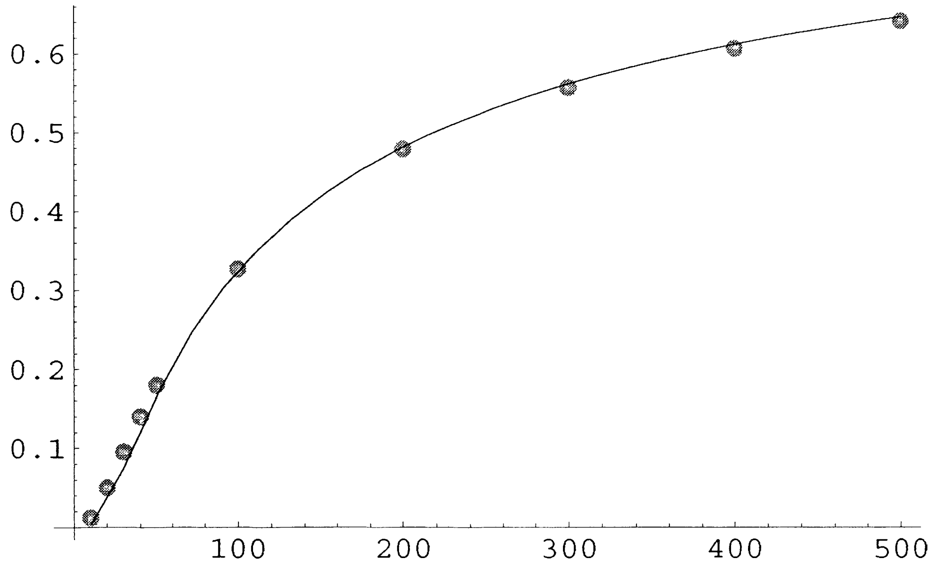

Figure 4.4: Comparison of the explicit solution 4.2.2 with the Monte Carlo simulation for the general model. Non – constant interest rate $\beta_t = 0.0002 + 0.1e^{-t}$.

Diffusion Approximation and Piecewise Linear Approximation.

Similarly, Table 4.5 summarizes results of another three Monte Carlo simulations. The General Model is the model 4.0.1 with exponential jumps with parameter $\alpha = 5$. We also assume that $\lambda = 1.2$, $X_0 = 30$, $\beta = 0.002$ and $m = 0$. Diffusion Approximation must be performed for the same $\beta = 0.002$ and $Z_0 = 30$ but c is assumed to be equal to $-\frac{\lambda}{\alpha} = -0.24$ and $\sigma = \sqrt{\frac{2\lambda}{\alpha^2}} = \sqrt{0.096} = 0.309839$. Piecewise Linear Approximation is valid for the same parameters as the Diffusion Approximation.

The probabilities obtained for both examples indicate that Piecewise Linear Approximation seems to be very closely related to the diffusion model. It confirms our calculations indicating that the PLA's ruin probabilities are greater then the ruin probabilities obtained for the diffusion model. We may notice, however that the diffusion approximation to the more general model is less accurate.

Time	General Model $\alpha = 5.0$, $\lambda = 1.2$ $X_0 = 30.0$, $m = 0.0$	Diffusion Model $c = -0.24$, $\sigma = 0.309839$ $Z_0 = 30.0$	Piecewise Approximation $c = -0.24$, $\sigma = 0.309839$ $Z_0 = 30.0$
90	0.00303(0.00016)	0.00176(0.00040)	0.00184(0.00012)
100	0.02554(0.00047)	0.02380(0.00145)	0.02387(0.00044)
110	0.11531(0.00096)	0.12363(0.00312)	0.12382(0.00096)
120	0.31494(0.00139)	0.33693(0.00448)	0.34299(0.00139)
130	0.58032(0.00148)	0.60694(0.00463)	0.61249(0.00143)
140	0.80576(0.00119)	0.82170(0.00363)	0.82225(0.00143)
150	0.93299(0.00075)	0.93341(0.00237)	0.93520(0.00072)
160	0.98274(0.00039)	0.98065(0.00131)	0.98140(0.00039)
170	0.99650(0.00018)	0.99550(0.00063)	0.99541(0.00019)
180	0.99949(0.00007)	0.99905 (0.00029)	0.99906(0.00009)
190	0.99995(0.00002)	0.99980 (0.00013)	0.99982(0.00004)

Table 4.4: Comparison of three methods used for calculation of ruin probabilities in a finite time horizon when the 'ruin level' is 0. Furthermore, the interest rate is dependent on time and equal to $\beta_t = 0.0002 + 0.0001e^{-t}$.

Time	General Model $\alpha = 5.0$, $\lambda = 1.2$ $X_0 = 30.0$, $m = 0.0$	Diffusion Model $c = -0.24$, $\sigma = 0.309839$ $Z_0 = 30.0$	Piecewise Approximation $c = -0.24$, $\sigma = 0.309839$ $Z_0 = 30.0$
100	0.003206(0.000169)	0.00220(0.00044)	0.00214(0.00043)
110	0.019697(0.000417)	0.01787(0.00126)	0.01804(0.00124)
120	0.077475(0.000802)	0.07951(0.00257)	0.07962(0.00253)
130	0.206143(0.001213)	0.21953(0.00393)	0.22374(0.00391)
140	0.404312(0.001472)	0.43079(0.00470)	0.43335(0.00465)
150	0.621796(0.001455)	0.65073(0.00452)	0.65083(0.00447)
160	0.800693(0.001198)	0.81304(0.00370)	0.81437(0.00365)
170	0.911935(0.000850)	0.91465(0.00265)	0.91539(0.00261)
180	0.967986(0.000528)	0.96573(0.00173)	0.96701(0.00167)
190	0.990154(0.000296)	0.98778(0.00104)	0.98753(0.00103)
200	0.997342(0.000154)	0.99647(0.00056)	0.99614(0.00058)

Table 4.5: Comparison of three methods used for calculation of ruin probabilities in a finite time horizon when the 'ruin level' is 0. Furthermore, the interest rate is constant and $\beta = 0.002$.

Appendix A

Simulation of Stochastic Processes

This chapter lists all the Mathematica codes used in simulation of trajectories of stochastic processes included in this thesis for illustration purposes.

A.0.1 Gaussian White Noise

```
<<Statistics‘Master‘

nd := Random[NormalDistribution[0, 1]];

T=500;

wn := Table[nd, {i, 0, T}];

ListPlot[wn,PlotJoined -> True,AspectRatio -> 1/3]
```

A.0.2 Gaussian Random Walk

```
<< Statistics‘Master‘

nd := Random[NormalDistribution[0, 1]];

wn=Table[nd, {i, 1, 1000}];

Y[0] = 0; Y[k_] := Y[k] = Y[k - 1] + wn[[k]];

rwalk[n_] := Table[Y[k], {k, 0, n}];

w = rwalk[1000];

ListPlot[w, PlotJoined -> True, AspectRatio -> 1/3]
```

A.0.3 Wiener Process

```
<< Statistics‘Master‘

T = 10;

delta = 1/50;

m = 0.5;
```

```
sigma = 2;

del[0] = 0;

del[k_] := Sum[delta, {i, 1, k}];

nd := Random[NormalDistribution[0, 1]];

wn = Table[nd, {i, 1, 1000}];

W[0] = 0;

W[k_] := W[k] = W[k - 1] + sigma Sqrt[delta] wn[[k]] + delta;

sbm = Table[{del[k], W[k]}, {k, 0, T (1/delta)}];

ListPlot[sbm, PlotJoined -> True, AspectRatio -> 1/3]
```

A.0.4 Poisson Process

```
<< Statistics`DiscreteDistributions`

<<Statistics`ContinuousDistributions`

lambda = 1;
```

```
T = 100;

poss = RandomArray[PoissonDistribution[lambda T], 1];

unif = Table[Random[Real, {0, T}], {i, 1, poss[[1]]}];

cond[i_, t_] := If[unif[[i]] < t, 1, 0];

X[t_] := Sum[cond[i, t], {i, 1, poss[[1]]}];

Plot[X[t], {t, 0,20}];
```

A.0.5 Compound Poisson Process

```
<< Statistics`DiscreteDistributions`

<<Statistics`ContinuousDistributions`

lambda = 1;

T = 100;

poss = RandomArray[PoissonDistribution[lambda T], 1];
```

```
unif = Table[Random[Real, {0, T}], {i, 1, poss[[1]]}];

norm = RandomArray[NormalDistribution[0.5, 2], poss[[1]]];

cond[i_, t_] := If[unif[[i]] < t, norm[[i]], 0];

X[t_] := Sum[cond[i, t], {i, 1, poss[[1]]}];

Plot[X[t], {t, 0, 20}];
```

A.0.6 Ornstein – Uhlenbeck Process

```
<< Statistics‘DiscreteDistributions‘

<< Statistics‘ContinuousDistributions‘

lambda = 10;

T = 100;

bet = 0.002;

m = 30;

x = 10;
```

```
poss = RandomArray[PoissonDistribution[lambda T], 1];

expon = RandomArray[ExponentialDistribution[1], poss[[1]]];

sumexpon[k_] := Sum[expon[[i]], {i, 1, k}];

suexponent = Table[sumexpon[j], {j, 1, poss[[1]]}];

norm = RandomArray[NormalDistribution[30, 2], poss[[1]]];

cond[i_, t_] := If[suexponent[[i]] < t, norm[[i]], 0];

X[t_] := -(m/bet) + (x + (m/bet)) Exp[bet t] -
      Sum[Exp[bet (t - suexponent[[i]])] cond[i, t], {i, 1, poss[[1]]}];

Plot[X[t], {t, 0, T}];
```

Appendix B

Documentation of C++ Programs used in the Thesis

B.1 General Model – Ruin Probabilities

B.1.1 Description of the Program

The program entitled 'General Ornstein – Uhlenbeck process – crossing probabilities' is written in the C++ programming language. Its aim is to compute level crossing probabilities for a General Ornstein – Uhlenbeck process. The OU – process is defined by the following stochastic differential equation:

$$dX_t = \beta X_t dt + dL_t, \quad t \geq 0,$$

where

$$L_t = mt - \sum_{k=1}^{N_\lambda(t)} \xi_k.$$

Here $(\xi_k)_{k\geq 1}$ are i.i.d. sequence of random variables appearing at arrival times $(T_k)_{k\geq 1}$ of a Poisson process $N_\lambda(t)$, $t \geq 0$, with the intensity parameter $\lambda > 0$. Then

$$X_t = -\frac{m}{\beta} + (X_0 + \frac{m}{\beta})e^{\beta t} - \sum_{k=1}^{N_\lambda(t)} \xi_k e^{\beta(t-T_k)} \mathbb{I}(T_k \leq t). \tag{B.1.1}$$

We write β but the program is easily adapted to the case when β depends on time β_t.

Input

From the formula $(B.1.1)$ we can see that we need to input a few parameters in order to get the probability of crossing level L. The values we need for the calculation are as follows:

- interest rate $\beta > 0$;
- intensity parameter of the Poisson process λ;
- drift parameter $m > 0$;
- distribution of jumps ξ_k;
- initial value (X_0) of the process X_t;
- value of the level $L < X_0$;
- time interval.

Output

The program results in the calculation of the probability of crossing level $L < X_0$ by the process X_t in the finite time horizon.

B.1.2 Method – Monte Carlo Simulation

The method used in order to obtain the ruin probabilities is Monte Carlo simulation. This simulation is based on the fact that the probability can be approximated by the proportion of events which resulted with a success. The success in our circumstances can be defined as a ruin. Hence, the proportion of simulations which result with the ruin is the Monte Carlo estimate of the ruin probabilities.

To speed up the Monte-Carlo simulation of τ_L and $X(\tau_L)$ (in particular $L = 0$) we use the following approach. Observe from $B.1.1$ that the paths of the process X_t are determined by the jump values located at $(T_k)_{k\geq 1}$. Hence, the values of the process X_t at those specific points of time are defined by the following recursion:

$$X(T_0) = X(0) = x,$$
$$X(T_k) = -\tfrac{m}{\beta} + \Big(X(T_{k-1}) + \tfrac{m}{\beta}\Big)e^{\beta(T_k - T_{k-1})} - \xi_k, \quad k = 1, 2, \ldots.$$

Proposition B.1.1. *Let X be an exponentially distributed random variable with parameter λ. Then*

$$e^{\beta X} \sim U^{-\beta/\lambda}, \quad U \sim Unif(0,1).$$

In particular, when $\lambda = \beta = 1$, $e^{\beta X} \sim 1/Unif(0,1)$.

This propositions is illustrated graphically. Figures $B.1$ and $B.2$ present histograms of two random variables $E_{\beta,\lambda}$ and $U_{\beta,\lambda}$ respectively.

$$E_{\beta,\lambda} = \exp\{-\beta E\},$$

where $E \sim Exp(\lambda)$ and

$$U_{\beta,\lambda} = U^{-\beta/\lambda},$$

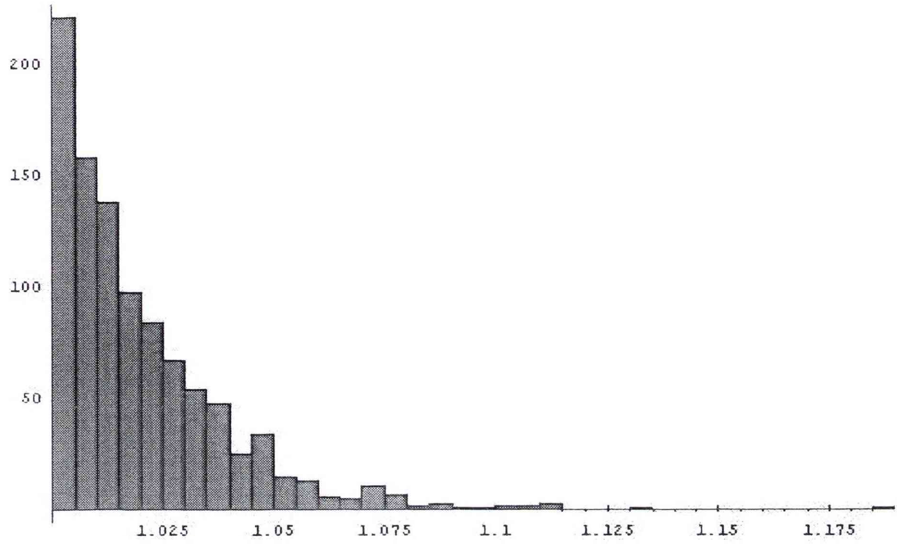

Figure B.1: Histogram of the random variable $E_{\beta,\lambda}$.

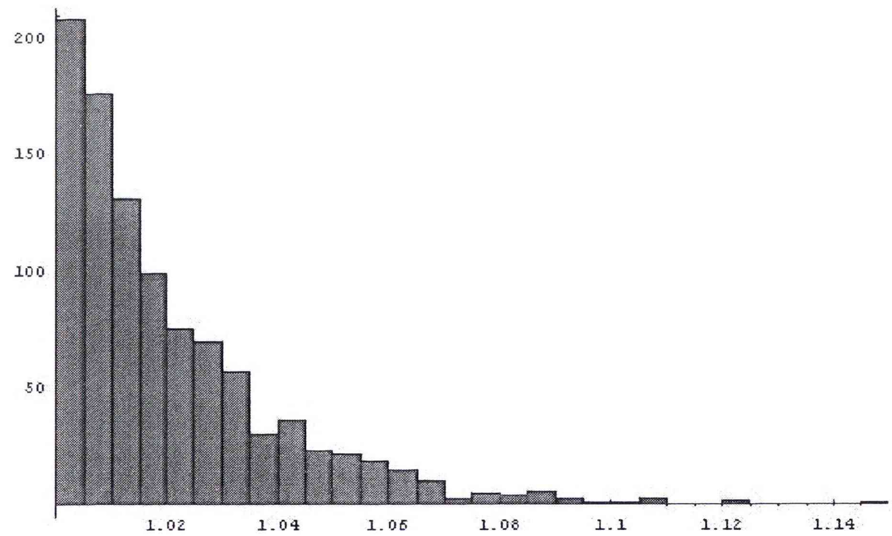

Figure B.2: Histogram of the random variable $U_{\beta,\lambda}$.

where $U \sim Unif(0,1)$.

Using the above proposition we conclude that the values of the process X_t at the jump times can be computed using the following recursion:

$$X(T_0) = X(0) = x,$$
$$X(T_k) = -\tfrac{m}{\beta} + \Big(X(T_{k-1}) + \tfrac{m}{\beta}\Big)U^{-\beta/\lambda} - \xi_k, \quad k = 1, 2, \ldots,$$

where $U \sim Unif(0,1)$.

B.1.3 Simulation Algorithm

In order to simulate the finite time, first crossing time probability under some known level we use the following algorithm:

1. Set RUIN = 0 and RUIN_COUNT = 0
2. BEGIN LOOP 1:

   FOR (k = 0; k < NUMBER_OF_TRIALS; k++)
3. Set ARRIVAL_TIME = 0.0
4. BEGIN LOOP 2:

   FOR (i = 0; ARRIVAL_TIME ≤ FINAL_TIME; i++)
5. IF (i == 0) Set OU_PROCESS = INITIAL_VALUE

   ELSE set OU_PROCESS = - (1/BETA) + (OU_PROCESS + 1/BETA)*

   EXP(BETA * EXPONENTIAL_LAMBDA) - DISTRIBUTION_OF_JUMPS
6. IF (OU_PROCESS ≤ LEVEL) set RUIN = 1 and BREAK

   ELSE set RUIN = 0

7. Set U to $Unif[0,1]$

8. Set EXPONENTIAL_LAMBDA = – (1/LAMBDA) * LOG(1 – U)

9. Set ARRIVAL_TIME = ARRIVAL_TIME + EXPONENTIAL_LAMBDA

10. END LOOP 2

11. Set RUIN_COUNT = RUIN_COUNT + RUIN

12. END LOOP 1

13. Set RUIN_PROBABILITY = RUIN_COUNT / NUMBER_OF_TRIALS

B.2 Diffusion Approximation – Ruin Probabilities

B.2.1 Description of the Program

The program entitled 'Diffusion Approximation – Ruin Probabilities' is very similar to the above program, the only difference lies in the risk process used. This time we consider a Gaussian Ornstein – Uhlenbeck process of the form:

$$dZ_t = \beta Z_t dt + dW_t,$$

where W_t is a Wiener process. We approximate this process by its discrete representation

$$\Delta Z_t = \beta Z_t \Delta + \Delta W_t$$

and use the following recursion formula

$$Z_0 = x,$$

$$Z_{t+1} = Z_t + \beta Z_t \Delta + \Delta W_t,$$

where

$$\Delta W_t = \Delta\mu + \sqrt{n}\sigma Z.$$

Z here is a standard normal random variable.

Input

As in the previous section we need to input the following parameters:

- interest rate $\beta > 0$;
- parameters of the normal distribution μ and σ;

- initial value (X_0) of the process X_t;
- value of the level $L < X_0$;
- time interval.

Output

The program results in the calculation of the probability of crossing level $L < X_0$ by the gaussian process X_t in the finite time horizon.

B.2.2 Simulation Algorithm

In order to simulate the finite time, first crossing time probability under some known level we use the following algorithm:

1. Set RUIN = 0, RUIN_COUNT = 0 and N = TIME / DELTA
2. BEGIN LOOP 1:
   FOR (k = 0; k < NUMBER_OF_TRIALS; k++)
3. Set OU_PROCESS = INITIAL_VALUE
4. BEGIN LOOP 2:
   FOR (i = 1; i ≤ (round down) N; i++)
5. Set OU_PROCESS = OU_PROCESS * (1 + BETA * DELTA) + SIGMA * SQRT(DELTA) * NORMAL(0,1) + MU * DELTA
6. IF (OU_PROCESS ≤ LEVEL) set RUIN = 1 and BREAK
   ELSE set RUIN = 0

7. END LOOP 2

8. Set RUIN_COUNT = RUIN_COUNT + RUIN

9. END LOOP 1

10. Set RUIN_PROBABILITY = RUIN_COUNT / NUMBER_OF_TRIALS

B.3 Piecewise Linear Approximation – Ruin Probabilities

B.3.1 Description of the Program

The program entitled 'Piecewise Linear Approximation' is based on the results described in chapter 4 of this thesis. Namely, it simulates the following expectation

$$P_t(\hat{g}) = E\Big[\prod_{i=0}^{n-1} p(i, \hat{g} \mid B_{t_i}, B_{t_{i+1}})\Big].$$

One can refer to section 4.3 for more details.

Input

- interest rate $\beta > 0$;
- parameters of the normal distribution μ and σ;
- initial value (X_0) of the process X_t;
- value of the level $L < X_0$;
- time interval.

Output

The program results in the calculation of the probability of crossing level $L < X_0$ by the gaussian process X_t in the finite time horizon.

B.3.2 Simulation Algorithm

In order to simulate the finite time, first crossing time probability under some known level we use the following algorithm:

1. Set COUNT = 0.0
2. Set KAPPA_T = $\int_0^T e^{-2\beta s} ds$
3. Set DELTA = KAPPA_T / NUMBER_OF_STEPS
4. BEGIN LOOP 1:
   FOR(k = 0; k < NUMBER_OF_EXPERIMENTS; k++)
5. Set BROWNIAN_MOTION = 0.0
6. Set P = 1.0
7. BEGIN LOOP 2:
   FOR(k = 0; k < NUMBER_OF_STEPS; k++)
8. Set NORMAL_RANDOM = NORMAL(0,1)
9. IF((G(k * DELTA) > BROWNIAN_MOTION) AND (G((k+1) * DELTA) > BROWNIAN_MOTION + NORMAL_RANDOM + SQRT(DELTA)))
10. Set FIRST_B = BROWNIAN_MOTION
11. Set SECOND_B = BROWNIAN_MOTION + NORMAL_RANDOM + SQRT(DELT.
12. Set P = P * (1 – EXP(– 2 * (G(k * DELTA) – FIRST_B)(G((k+1) * DELTA) – SECOND_B) / DELTA))

13. END IF

14. BEGIN ELSE

15. Set P = 0.0 and BREAK

16. END ELSE

17. Set BROWNIAN_MOTION = SECOND_B

18. END LOOP 2

19. Set COUNT = COUNT + P

20. END LOOP 1

21. Set PROBABILITY_OF_RUIN = 1 – COUNT / NUMBER_OF_EXPERIMENTS

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