Pricing of Contingent Claims Under the Real-World Measure

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Certificate of Authorship

I certify that the work of this thesis has not previously been submitted for a degree nor has it been submitted as part of the requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition I certify that all information sources and literature used are indicated in the thesis.
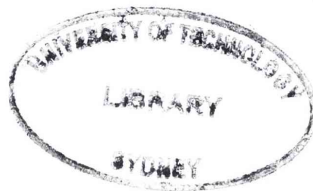
The work of Chapter 6 is based on the research article of Miller & Platen (2005) while three key results that appear in Hulley, Miller & Platen (2005) were based on work included in Chapters 2, 3 and 5. Hulley, Miller & Platen (2007) further extends computations within the previous paper. It is also expected that as a minimum, Chapters 4 and 5 will also form the basis of further journal articles.

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Signed   Date

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Notation and Abbreviations

Mathematical Notation

$(a, b)^+$	maximum of a and b;
$\mathbf{x}^\top$	transpose of a vector or matrix $\mathbf{x}$;
$|\mathbf{x}|$	absolute value of $\mathbf{x}$ or Euclidean norm;
$\mathbf{A} = [a^{i,j}]_{i,j=1}^{k,d}$	$(k \times d)$-matrix $\mathbf{A}$ with i, jth component $a^{i,j}$;
$\mathbf{A}^{-1}$	inverse of a matrix $\mathbf{A}$;
∞	infinity;
(a, b)	open interval $a < x < b$ in $\Re$;
$[a, b]$	closed interval $a < x < b$ in $\Re$;
$\Re = (-\infty, +\infty)$	set of real numbers;
$\Re^+ = [0, +\infty)$	set of non-negative real numbers;
$\Re^d$	d-dimensional Euclidean space;
Ω	sample space;
$A \cup B$	the union of sets A and B;
$\langle X, Y \rangle_t$	covariation of processes X and Y at time t;
$\langle X \rangle_t$	quadratic variation of process X at time t;
$n! = 1 \cdot 2 \cdot \ldots \cdot n$	factorial of n;
$\mathbb{1}_{\{A\}}$	indicator function for every event A to be true;
$\mathcal{A}$	collection of events, sigma-algebra;
$\underline{\mathcal{A}}$	filtration;
$E[X]$	expectation of X;
$E[X \,|\, \mathcal{A}]$	conditional expectation of X;
$P[A]$	probability of A;
$P[A \,|\, B]$	probability of A conditioned on B;
$\in$	element of;
$\neq$	not equal to;
$\mathbf{1}$	unit matrix;
$\ln(a)$	natural logarithm of a;
$\lim_{N \to \infty}$	limit as N tends to infinity.

Statistical Notation

$\Gamma(\cdot)$	gamma function;
$\Gamma(\cdot\,;\cdot)$	incomplete gamma function;
$\mathcal{Q}(\cdot\,;\cdot)$	regularised incomplete gamma function;
$\mathcal{Q}^{-1}(\cdot\,;\cdot)$	inverse of the regularised incomplete gamma function;
$\mathcal{N}(\cdot)$	standard Gaussian distribution function;
$p_{\mathcal{N}}(\cdot)$	standard Gaussian density;
$\mathcal{G}(\cdot\,;\cdot)$	two-parameter gamma complementary distribution function;
$p_{\mathcal{G}}(\cdot\,;\cdot)$	two-parameter gamma density;
$\chi^2(\cdot\,;\cdot)$	central chi-square distribution function;
$p_{\chi^2}(\cdot\,;\cdot)$	central chi-square density;
$\chi^2(\cdot\,;\cdot,\cdot)$	non-central central chi-square distribution function;
$p_{\chi^2}(\cdot\,;\cdot,\cdot)$	non-central chi-square density;
$\mathcal{I}(\cdot\,;\cdot,\cdot,\cdot;\cdot,\cdot,\cdot)$	doubly non-central Beta distribution function;
$p_{\mathcal{I}}(\cdot\,;\cdot,\cdot,\cdot;\cdot,\cdot,\cdot)$	doubly non-central Beta density;
$q(t,z_t;T,z_T)$	transition density to move from z_t at time t to z_T at time $T > t$; used in this text for the GOP;
$p(t,x_t;T,x_T)$	transition density to move from x_t at time t to x_T at time $T > t$; used in this text for the discounted GOP;
$I_\nu(\cdot)$	modified Bessel function of the first kind with index ν.

Abbreviations

GOP	growth optimal portfolio;
BSMM	Black-Scholes-Merton model;
MCEV	modified constant elasticity of variance;
SMMM	stylised minimal market model;
EMMM	extended minimal market model;
IRTS	interest rate term structure;
WSAI	world stock accumulation index;
SDE	stochastic differential equation;
OTM	out-of-the-money;
ATM	at-the-money;
ITM	in-the-money;
LHS	left-hand-side;
RHS	right-hand-side.

Thesis Specific Notation

$S_t^{i,j}$ primary security account for jth asset in the ith currency at time t;

B_t^i savings account in the ith currency at time t;

r_t^i short rate of the ith currency at time t;

ψ_t^i shadow short rate of the ith currency at time t;

$|\theta_t^i|$ market price of risk in the ith currency at time t; or volatility of the GOP in the ith currency at time t;

$X_t^{i,j}$ exchange price of the jth asset in units of the ith currency at time t;

$S_t^{i,\delta}$ non-negative wealth process (portfolio) in the ith currency at time t;

$S_t^{i,\delta*}$ Growth Optimal Portfolio (GOP) in the ith currency at time t;

$\bar{S}_t^{i,\delta*}$ discounted GOP in the ith currency at time t;

$\hat{S}_t^{\delta}$ benchmarked portfolio at time t;

$\Lambda_t^{i,\theta}$ candidate Radon-Nikodym derivative in the ith currency at time t;

$P_{i,\theta}$ candidate risk-neutral probability measure in the ith currency;

H_T^i contingent claim in the ith currency at time T;

U_t^{i,H_T^i} derivative price of contingent claim H_T^i in the ith currency at time t;

$P^i(t,T)$ zero-coupon bond price with maturity T in the ith currency at time t;

$M_T^i(t)$ discounted GOP contribution to the zero-coupon bond price with maturity T in the ith currency at time t; or, expected value of the candidate Radon-Nikodym derivative in the ith currency at time t;

$G_T^i(t)$ short rate contribution to the zero-coupon bond price with maturity T in the ith currency at time t;

$f^i(t,T)$ forward rate with maturity T in the ith currency at time t;

$m_T^i(t)$ discounted GOP contribution to the forward rate with maturity T in the ith currency at time t;

$g_T^i(t)$ short rate contribution to the forward rate with maturity T in the ith currency at time t;

U_t^i price process in the ith currency at time t;

$V_{T,K,U^i}^i(t)$ derivative price of the process U^i with expiry T and strike price K in the ith currency at time t;

$F_{T,U^i}^i(t)$ forward price of U^i with maturity T in the ith currency at time t;

Thesis Specific Notation (continued)

$A^{i,\pm}_{T,K,U^i}(t)$ European asset binary option on the process U^i with expiry T and strike price K in the ith currency at time t;

$B^{i,\pm}_{T,K,U^i}(t)$ European bond binary option on the process U^i with expiry T and strike price K in the ith currency at time t;

$c^i_{T,K,U^i}(t)$ European call option on the process U^i with expiry T and strike price K in the ith currency at time t;

$p^i_{T,K,U^i}(t)$ European put option on the process U^i with expiry T and strike price K in the ith currency at time t;

$\mathbf{zbc}^i_{\bar{T},T,K}(t)$ European call option on a zero-coupon bond with expiry $\bar{T}$, maturity $T \geq \bar{T}$ and strike price K in the ith currency at time t;

$\mathbf{zbp}^i_{\bar{T},T,K}(t)$ European put option on a zero-coupon bond with expiry $\bar{T}$, maturity $T \geq \bar{T}$ and strike price K in the ith currency at time t;

$\mathbf{cap}^i_{\mathcal{T},K,N}(t)$ interest rate cap with the set of dates $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$, strike rate K and notional N in the ith currency at time $t \leq T_0$;

$\mathbf{cpl}^i_{T_{\ell-1},T_\ell,K}(t)$ ℓth individual interest rate caplet within $\mathbf{cap}^i_{\mathcal{T},K,N}(t)$;

$\mathbf{flr}^i_{\mathcal{T},K,N}(t)$ interest rate floor with the set of dates $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$, strike rate K and notional N in the ith currency at time $t \leq T_0$;

$\mathbf{fll}^i_{T_{\ell-1},T_\ell,K}(t)$ ℓth individual interest rate floorlet within $\mathbf{flr}^i_{\mathcal{T},K,N}(t)$.

Abstract

The aim of this thesis is to price contingent claims under the real-world probability measure. Real-world pricing results naturally by selecting the numeraire as the growth optimal portfolio (GOP). Under this approach, the existence of an equivalent risk-neutral probability measure is not required. Furthermore, the GOP can be used to define other basic contingent claims, such as exchange prices, primary security accounts, and even zero-coupon bonds. We begin with application of the real-world pricing formula to derive forward prices for each of these financial quantities. The obtained formulae are model independent, yet reveal important differences between the real-world and classical risk-neutral approaches.

Real-world prices are systematically derived under each of the models studied within this thesis for the following contingent claims: zero-coupon bonds; options on the GOP; options on exchange prices; and interest rate caps and floors via options on zero-coupon bonds. We start with the classic Black-Scholes-Merton model, where the GOP follows a geometric Brownian motion. Under this model, real-world pricing recovers the results of classical risk-neutral pricing, since the corresponding Radon-Nikodym derivative is a martingale.

For each of the remaining models studied, the GOP is based on a time-transformed squared Bessel process. In each case, real-world prices may differ from classical risk-neutral prices because the candidate Radon-Nikodym derivative is a strict supermartingale. The second model considered proposes a modified form of the constant elasticity of variance model for the GOP. New analytic results for zero-coupon bonds and options on the GOP are derived that were previously analysed using numerical methods. Real-world prices for options on exchange prices and interest rate derivatives are also provided.

Three versions of the minimal market model are also examined. This model class overcomes some of the deficiencies of the aforementioned approaches since the dynamics for the GOP better reflect empirical market features, such as leptokurtic returns, the leverage effect and a stochastic yet stationary volatility structure. Under a stylised version of the minimal market model with a constant short rate, we derive analytic solutions to the complete suite of contingent claims examined within the thesis. We subsequently allow the short rate to be stochastic in order to accurately model the term structure of interest rates, with a focus on low interest rate environments. The proposed model provides a very good fit to interest rate

cap data from the Japanese market. Finally, we examine an extended version of the minimal market model, which contains one additional constant parameter and reverts to a constant short rate. Further analytic prices are derived under the real-world measure for zero-coupon bonds, options on the GOP and options on exchange prices while semi-analytic prices are available for interest rate caps and floors. The results for the extended minimal market model generalise those of the stylised model. They also possess a similar structural form to those derived for the modified constant elasticity of variance model. Hence the extended minimal market model summarises all of the features of models where the GOP is based on time-transformed squared Bessel processes.