Novel Bayesian Smoothing Algorithms for Improved Track Initiation and Maintenance in Clutter

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Doctor of Philosophy

2007

CERTIFICATE OF AUTHORSHIP OR ORIGINALITY

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text. I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

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Acknowledgment

I am thankful to **Prof. Subhash Challa** of Faculty of Engineering in University of Technology, Sydney (UTS), for the initiation of the project as well as continuous support and supervision. He was the source of enormous ideas and constantly helped me finding ways to solve problems from several angles. Without his guidance, both technical and at personal levels, it would have been difficult to keep myself focused on research.

I would like to express my deep gratitude to **Dr. Darko Mušicki** for his help toward my problems throughout my candidature. His original and practical insights into IPDA assisted me in understanding many key concepts involved at both theoretical and practical aspects of target tracking.

I am taking this opportunity to gratefully acknowledge the efforts of **Dr. Mark Morelande** and **Dr. Xueshi Wang** as they patiently gave me the time and helped me with research. Without their active help at different stages of my thesis, it would have been difficult for me to complete this work.

I would like to thank **Prof. G. Dissanayake and Faculty of Engineering at UTS** for providing me **Faculty of Engineering Robotics Interaction Scholarship** which helped me concentrate on my project.

On this occasion, I would also like to remember my parents. They were always by my side at difficult times inspiring me to pursue my higher degree. Without their moral support I would have never been in this stage.

I am also in great debt to all my friends who took lots of pain abandoning their own leisure time to discuss with me about my problems and encourage me all the time.

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ABSTRACT

Target tracking is a well established field with over fifty years of intense research. While in its core, it deals with estimating targets dynamic states, it is also a critical component of all "Situation Awareness" and threat assessment systems. These higher layer applications take decisions on important questions like number of targets, positions of them, the instant and position of their initiation, the instant and position of their maneuvers and above all, which of them are threatening and/or friendly. The lower level target tracking algorithms feed the necessary information to these decision taking systems.

There are a number of target tracking algorithms to cater for the need of such systems. Most of these available algorithms are based on filtering theory. But it is established that smoothing increases the accuracy of the systems at the expense of a slight lag between the instant of estimation and the instant at which the parameter of interest is being estimated. Hence smoothing is not widely used for practical target tracking applications.

However, the situation awareness system is expected to perform better if more precise information is obtained about initiation and termination of the targets along with improved discrimination of true/false targets.

This thesis addresses the problem of improved track initiation and maintenance with the smoothing framework to provide better information. It first reviews target tracking and filtering literature. It introduces the concept of random set smoother and derives the IPDA smoother under linear Gaussian assumption. IPDA smoother is also derived by extending the PDA smoother. Finally a theoretical link is established between Random Set smoothing and IPDA smoothing framework. To extend the domain into multiple sensor scenario, the problem of out-of-sequence measurements is also addressed in this thesis under target existence uncertainty.

Several realistic scenarios are simulated and the results are verified.

1. INTRODUCTION

1.1 *Background*

Multi-target tracking is one of the fundamental problems of all surveillance and monitoring problems with applications to homeland security, defense and others. Target tracking problems include, track initiation, maintenance, state estimation. merging, joining and termination. Solving these problems under difficult dynamic environmental conditions using sensors of limited capability poses a significant challenge. The problems are interrelated and errors in one stage have the potential to impact the errors in the other stages. For example, if a track was not properly initiated, it will not be confirmed and will lead to missed tracks. If a track is not terminated at the appropriate time, it leads to false estimates of the number of objects in the area of surveillance. These challenging problems have attracted a number of researchers over the last 2 decades to dedicate significant efforts to solve them. This thesis summarises the efforts to explore and address some of these challenges by developing new smoothing algorithms and demonstrating their performance in realistic target tracking scenarios.

1.2 *Target Tracking*

The target tracking can be defined as the problem of finding the "best guess" about the dynamic state of the target based on the observations received by a sensor within its surveillance region. The dynamic state of the target may consist of position, velocity, acceleration or any other parameter of interest. In its simplest form, a sensor observes a single non-maneuvering target and receives noisy measurements from it. The target tracker algorithm estimates the target's dynamic state which is the "best guess" based on all observations.

In practice, the sensor may not only receive target originated measurements, but also may receive measurements from uninteresting objects sharing the same surveillance space. Sometimes the actual target may even go undetected. The tracker needs to distinguish between the target originated measurements and those originated from "clutter" (the unwanted sources).

Moreover, the targets may maneuver (accelerate) at any random point of time which is unknown a-priori. Thus the tracker needs to determine the active dynamic model that the target is following.

Lastly even the assumption of existence of any target needs to be solved. The question of any target's presence within the sensor range is an uncertain event itself. Based on the observations received, a complete automatic tracker first needs to resolve this existence uncertainty. In these cases algorithms are to start without any prior knowledge of any track and "automatically" has to decide about the presence of any number of possible targets and then maintain them. This type of algorithms have been made to develop such complete algorithms.

The discussion above identifies the key issues of a typical target tracking problem. The different levels of uncertainties involved in the scenario are as follows:

- Sensor observations inherently contain random noises
- The number of received measurements may change from scan to scan because of the presence of "clutters"
- The sensor may miss true target altogether in some scans
- The target may maneuver at random instants
- The total number of targets within the surveillance region is unknown a priori
- Each target's initiation, confirmation or termination time are random and need to be estimated

The critical task of an automatic target tracker is to reduce uncertainties, mentioned above, by "suitably minimizing" certain cost functions. Significant research has been conducted in this field for more than five decades and several algorithms have been proposed in incremental manner to resolve each of the scenarios.

1.3 Scope and organization of the thesis

The thesis takes Bayesian approach as its basis for developing target tracking algorithm. Research has been carried out to devise algorithms to handle uncertainties and/or randomness of various levels. These can be brought under the same framework of Bayesian estimation, [20]. The thesis re-derives the established algorithms purely from Bayes' theorem as an illustration of the validity of the idea of taking the theorem as the fundamental one to attack a target tracking problem. In the process the thesis also establishes the approach of augmented state as a valid and robust platform for extending the established algorithms into smoothing under Bayesian formulation.

The available literature deals with deriving filter algorithms to handle the tracking related issues. In 1960 the filter proposed by R.E Kalman, [39, 40], which is now widely known as Kalman Filter, introduced a tool for engineers to deal with random uncertainty in real time. It is readily accepted as a core for target tracking algorithm. Later filter algorithms to deal with measurement origin uncertainty were proposed in the form of "Nearest Neighbor Filter" (NNF), [68, 71, 70] and Probabilistic Data Association Filter (PDAF), [10]. Among these PDAF proved to be the most effective one. The solution to target existence uncertainty was first proposed as a Markovian two-model ("observable" and "not observable") interacting algorithms in [5]. This approach was adopted due to the success of similar multiple model algorithm, Interacting Multiple Model Filter (IMMF) designed for maneuvering targets, [18]. Later a simpler algorithm of Integrated Probabilistic Data Association Filter (IPDAF) was proposed, [61, 59, 60]. It was an extension of PDAF for target existence uncertainty with probabilistic techniques to maintain tracks.

Smoothing, as opposed to filter algorithms, uses more observation and therefore provides better estimation. While the extension of basic filters like KF into smoothing was carried out within 1970, [52, 54, 55, 13, 53], serious research effort, addressing the tracking issues, was not present due to more computation and memory requirement for smoothing (than filters). Later in 1990, the popular PDAF was extended to smoothing in [45]. The well established IMMPDA was also extended to smoothing for improved maneuvering target tracking in clutter, [31, 75, 30, 32, 27, 26].

Smoothing improves estimation compared to filtering simply because of using more observation (or information) but at the cost of a time delay. In such a case, the improvements in estimating continuous variables like target dynamic state may not be that effective in decision taking application like "Situation Awareness" or "threat assessment". These higher level applications may improve their efficiency if a more accurate picture of the actual field scenario is provided to them. In that case, parameters representing the overall scenario, like number of targets, their initiation/termination instants and locations - may prove to be very useful ones for these applications. An effective smoothing algorithm can result in a better estimation of these "track maintenance" parameters and thus help increase the effectiveness of the critical applications like situation/threat awareness. In literature, even though filter algorithms are already available for automatic track maintenance, smoothing algorithms for the same purpose have not been investigated yet.

The scenario is even more critical when there are number of targets present and even more so when the dynamics of the targets are co-related. This gives rise to Random Set based tracking. Finite State Statistics (FISST), [50, 49, 48, 46], provides the basics for calculating such scenario under Bayesian recursion.

But again, while a Random Set based filter is available in the literature, the framework has not been extended to smoothing.

The thesis aims to develop a novel Bayesian smoothing algorithm for improving automatic target initiation, tracking and maintenance. The complete framework is developed using both the Bayesian and Random Set formalism. The associated modeling and assumptions (where applicable) are proposed and justified. The assumptions of the problems addressed in this thesis include:

- Both the sensor measurements and the target dynamics are subject to random noise
- The target(s) is(are) non-maneuvering
- The sensor receives "clutter" originated measurements along with the target originated measurements. The number of received measurements at any certain scan is randomly varying.
- The sensor may miss the target on some scan.
- The existence of the target is not known a priori.

Under the above mentioned conditions of single target tracking, IPDA algorithm has been proved to be the most successful and well established in literature. Therefore this thesis takes IPDA filter as the base and extends it to smoothing algorithm. The development of the random set smoother also builds a generalized platform which can be extended into smoothing for multiple target tracking scenario.

The thesis looks at the comprehensive formulation of the Bayesian smoothing methodologies for automatic target maintenance in clutter with the aim of extending to multiple target tracking environment. The thesis also investigates the improvement introduced by the application of the proposed algorithm compared to the standard filter algorithm. The improvement in the track maintenance parameters along with target dynamic states strengthens the usage of such algorithms and makes a significant difference for the decision layer applications like situation awareness and threat assessment. Thus the contribution of the thesis can be summarized as the development of a generic smoothing framework for automatic track maintenance (in clutter) and also establishing such a framework as the most suitable one by studying the improvement it provides in estimating several important performance parameters like number of targets, distinguishing true and false targets, deciding targets' termination time along with more accuracy in estimating the targets' dynamic state. The original contributions that this thesis make are as follows

- Derivation of an original augmented state IPDA smoother for scenario involving target existence uncertainty

- Formulation of a generalized random set smoother
- Proposing a random set smoother for target existence uncertainty
- Proving a theoretical connection between IPDA smoother and random set smoother
- Implementing the proposed smoothers and compare them against the filter algorithm
- Designing a flow for the proposed algorithm and simulating it for realistic scenarios
- Proposing a theoretical Bayesian modeling for out of sequence measurement problem under target existence uncertainty scenario and the developed smoothing algorithm as a solution to it. The corresponding modelings for IPDA algorithm and random set are also derived.

The thesis is organized as follows. In chapter 2, the general Bayesian approach for estimation along with smoothing will be outlined. Kalman Filter will be derived from Bayesian formulation. The Bayesian modeling will also be extended to smoothing and implemented via the demonstration of deriving Augmented state KF from the same Bayesian recursion.

Chapter 3 introduces the problem of measurement origin uncertainty. The corresponding Bayesian modeling to handle measurement origin uncertainty is investigated in this chapter. The model is also shown to reduce to PDAF, [10], (as an approximation) which is an effective algorithm for such scenario. Augmented state PDAF (AS-PDAF), [22], which is an extension to standard PDAF into smoothing estimation, has also been derived in the chapter from Bayesian modeling as as illustration.

In chapter 4, the original derivation of augmented state IPDA smoothing algorithm is carried out. First the problem of automatic track maintenance is introduced and Bayes' formulation for the scenario is proposed. The standard IPDA filter is also derived from the general Bayesian framework. The extension of the original Bayesian modeling to smoothing is then fully carried out in step by step manner. The necessary assumptions and models are mentioned in the course of deriving recursive update equations.

In chapter 5, the generalized random set smoother is derived. The target dynamic model and sensor model under the set formalism are proposed in this chapter. The belief mass functions for global transition densities and likelihood densities are also derived and the Bayesian recursive update for global target densities are calculated in the chapter. The specific scenario of target existence uncertainty is also addressed in the chapter and a resulting random set smoother for the specific scenario is formulated from the generalized version.

IPDA filer and the proposed AS-IPDA smoother are seen to be special cases of the generalized random set smoother.

In chapter 6, the issue of out of sequence measurement is modeled for automatic track initiation scenario. The generalized Bayesian model is first proposed in the chapter. The corresponding target dynamic motion model and sensor observation model to cater for delayed measurements (out of sequence measurements) for an automatic track maintenance algorithm is detailed in this chapter both for IPDA and random set formalism. The model is also shown to be following the principle of AS-IPDA and random set smoothing algorithm proposed in chapters (4) and (5) respectively.

In chapter 7, the proposed smoothers are then simulated under realistic and benchmark tracking environment and compared against established filter algorithms. The comparison has been carried out based on RMS error in target state estimation and various track maintenance statistics parameters like number of confirmed false/true targets, time of termination of a true target and etc. The implication of the achieved performance is also explained in the chapter. Finally conclusions of the work done so far in this thesis and future direction of research are pointed out in chapter 8.

1.4 Conclusion

In this chapter, a brief overview of general target tracking problems has been provided. A review of the literature and major milestones in the research of target tracking has been outlined. In the light of the literature review, the smoothing framework for a particular scenario has been identified as a valid significant and potential research area. The scope of the addressed problem of the thesis is also outlined in section 1.3. The original contributions of this thesis are also clearly defined the section 1.3. Then the organization of thesis is stated to clarify the systematic steps taken in the rest of the chapters to reach the solution of defined problem theoretically and validated.

2. INTRODUCTION TO BAYESIAN FILTERING AND SMOOTHING

2.1 Introduction

This chapter provides the theoretical background for estimation and smoothing algorithms from Bayesian perspective. It will formulate the problem of target tracking within the Bayesian reasoning framework. The well known Kalman Filter (KF) will also be derived from the Bayesian approach. The model will be extended to simple smoothing and Augmented State KF will be derived as an illustration of the validity of the model.

2.2 Background

Handling uncertainty has always been a challenging field. Any physical system inherently has random parameters and results in uncertainty. Probability theory is a powerful mathematical tool devised to handle such inherent randomness in systems. The most useful theorem put forward to resolve the problem of uncertainty is the one by Thomas Bayes, [12], and which is well known as Bayes' Theorem. Mathematicians, like Gauss, Legendre, then provided decision taking criterion based on probabilistic approach. Maximum likelihood(ML), Least square (LS) method, Maximum a posteriori (MAP) method and Minimum mean square error (MMSE) techniques - are statistical decision making mechanism to handle uncertainty based on various optimization criterion. The history of understanding the uncertainty in system started back in eighteenth century by Bayes and started to be more specific in nineteenth century when Gauss and others dealt with estimation of positions of heavenly bodies in sky.

Even though the probabilistic theory was well established, applying those into real time processing did not prove to be straightforward. The attempt posed problems like computational complexity, memory requirement and etc. Therefore even though the same technique like Least square or Minimum mean square estimation could be applied by Gauss to predict the position of heavenly bodies in an offline fashion, it could not be applied to a faster and real time engineering problems until around the mid twentieth century when computers were first used. Only then scientists and engineers saw a real opportunity to apply the classical statistical tools into more computationally extensive problems.

In his pioneering papers in 1960, R.E Kalman, and later in 1961 with R.S Bucy, [39, 40], proposed a linear filtering theory which is now well known as Kalman Filter and widely used in almost every engineering estimation problem. This new technique was welcomed by the engineers almost instantly because of the following reasons:

- the filter was recursive
- it was not computationally extensive
- it was optimal in the sense that it minimises mean squared estimator

Later in 1970, a discussion paper by H.W Sorenson, [73], illustrated that Kalman filter closely follows the same least square estimation technique introduced by Gauss back in eighteenth century. Due to its simplicity and accuracy, backed by the then recent development in electronics, Kalman filter grew in popularity among the engineers and mathematicians. It was later shown that Kalman filter could be obtained using Bayesian approach of stochastic estimation, [34]. By the end of 1970, the Kalman filter framework was extended to smoothing technique also, [52, 53, 54, 55]. For non-linear problem the Kalman filter was modified and named Extended Kalman filter (EKF) which was a sub-optimal solution. The other types of filters, presented in literature, that solve the basic problem of estimating target state from noisy measurements, are Particle filter (PF), Unscented Kalman filter (UKF), Point Mass filter (PMF) etc.

The following sections will introduce the Bayesian approach of target tracking from the basic Bayes' theorem and demonstrate the logical development of the theorem culminating into Kalman filter (KF) for linear Gaussian systems. The application of Bayes' theorem in smoothing perspective will also be illustrated through the development of augmented state KF.

2.3 Bayesian Estimation and Target Tracking

2.3.1 Bayes' Theorem

Introduced by Thomas Bayes' in 1763, [12], this theorem incorporates the necessary mathematical base to reconcile the prior knowledge and the present information into the estimation of a parameter which is co-related to the information. This philosophy consists a very suitable platform for solving any estimation including target tracking problem.

By definition, the Bayes' theorem calculates the probability of occurance of an event A with the condition that another correlated event B has occurred. Mathematically, this is expressed as

$$p(A|B) = \frac{p(A,B)}{P(B)} \tag{2.1}$$

where

- $p(A, B)$ is the joint probability of the events A and B.
- $p(B)$ is the probability of event B alone

Similarly, the reciprocal conditional probability of $p(B|A)$ can be given as

$$p(B|A) = \frac{p(A, B)}{P(A)} \tag{2.2}$$

If the event A is the one of interest, a more useful form of Bayes' theorem can be derived using (2.1) and (2.2). This form is given by

$$p(A|B) = \frac{p(B|A).p(A)}{P(B)} \tag{2.3}$$

The expression in (2.3) is of particular importance for basic estimation problem. Starting with prior knowledge for the event A, which is captured in $p(A)$, (2.3) provides the probability of event A utilizing information from another correlated event B. This simple encapsulation of correlation between two events can be extended to complex problems like target tracking. This is discussed in the next section (section 2.3.2).

2.3.2 Bayes' Theorem and Target Tracking

In the context of target tracking, the problem lies in estimating the target state based on the sensor observations. If the composite target state is denoted by S^{k-N} at time $t = k - N$ and the set of observations till time $t = k$ is denoted by y^k, the Bayes theorem proposes a solution to the estimation problem in probabilistic sense. If the events A and B in (2.3) are replaced by S^{k-N} and y^k, the posterior knowledge about the state can be summarized as

$$p(S^{k-N}|y^k) = \frac{p(y^k|S^{k-N}).p(S^{k-N})}{p(y^k)} \tag{2.4}$$

The state of the target, which is a random variable, can be estimated from the posterior density of (2.4). In (2.4), $N = 0$ for filtering, $N > 0$ for smoothing and $N < 0$ for prediction type of estimation. These three types of estimators are schematically illustrated in figure 2.1.

The thesis focuses on smoothing and therefore the types of smoothers are briefly revisited here

- *Fixed Lag Smoothing* "Fixed Lag Smoother" estimates the past state which lags by a fixed amount.

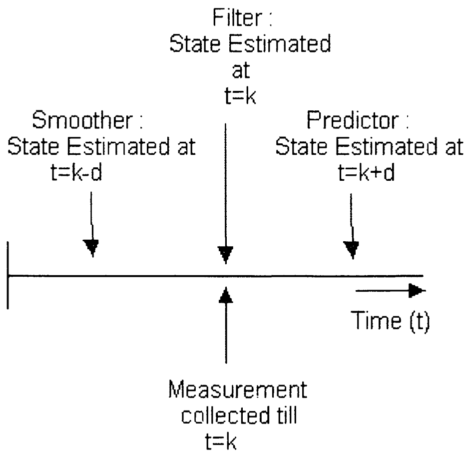

Fig. 2.1: Estimation : Filtering, Smoothing, Prediction

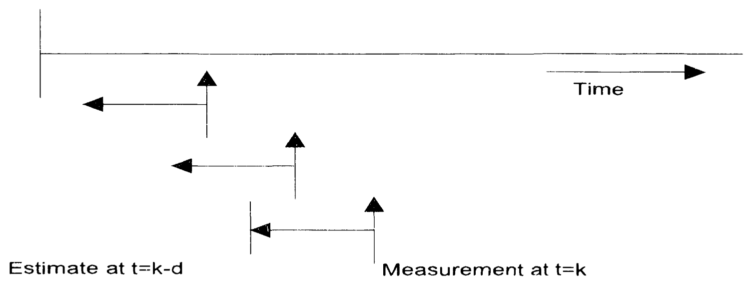

Fig. 2.2: Fixed Lag Smoothing

- *Fixed Interval Smoothing* In "Fixed Interval Smoothing", the states within time interval from $t = k-N$ to $t = k$ are estimated using the measurements within the window of the same interval from $t = k - N$ to $t = k$. The continuous smoothing is carried out by sliding the window while keeping the interval N fixed.

- *Fixed Point Smoothing* "Fixed Point Smoothing" estimates a state at particular point of time in the past. This is carried out generally to improve the system initial conditions.

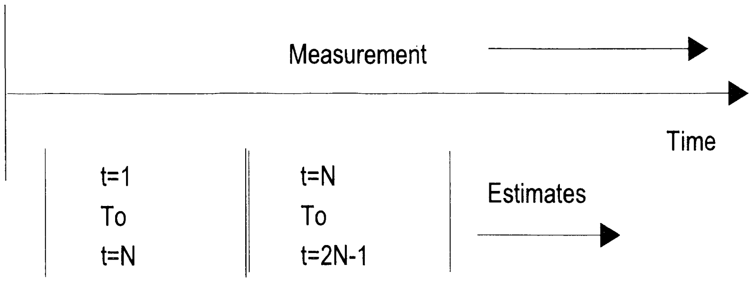

Fig. 2.3: Fixed Interval Smoothing

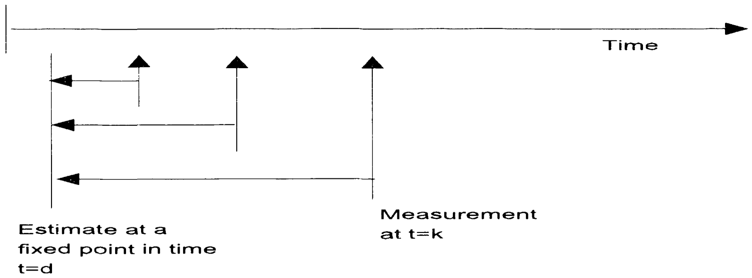

Fig. 2.4: Fixed Point Smoothing

2.3.3 *Recursive Bayes' Estimation*

The real time problem of target tracking demands the state estimation be a recursive process. From expression in (2.4), it is not evident that Bayes' theorem can provide a recursive estimation formula. But a careful observation and manipulation can result in a truly real time implementable recursive Bayesian formulation for estimation. The steps are described here in details.

The observation set y^k can be expanded as y_k, y^{k-1} where y_k denotes the observation at time $t = k$ only. Therefore the term $p(y^k|S^k)$ on the right hand side of (2.4) can be expanded as

$$\begin{aligned} p(y^k|S^k) &= p(y_k, y^{k-1}|S^k) \\ &= p(y_k|y^{k-1}, S^k).p(y^{k-1}|S^k) \end{aligned} \tag{2.5}$$

Hence using (2.5), the density in (2.4) can be formulated as

$$\begin{aligned} p(S^k|y^k) &= \frac{p(y_k|y^{k-1}, S^k).p(y^{k-1}|S^k).p(S^k)}{p(y^k)} \\ &= \frac{p(y_k|y^{k-1}, S^k).p(y^{k-1}|S^k).p(S^k)}{p(y_k.y^{k-1})} \end{aligned}$$

$$
\begin{aligned}
&= \frac{p(y_k|y^{k-1}, S^k).p(y^{k-1}|S^k).p(S^k)}{p(y_k|y^{k-1}).p(y^{k-1})} \\
&= \frac{p(y_k|y^{k-1}, S^k)}{p(y_k|y^{k-1})}.\frac{p(y^{k-1}|S^k).p(S^k)}{p(y^{k-1})} \\
&= \frac{p(y_k|y^{k-1}, S^k)}{p(y_k|y^{k-1})} \times p(S^k|y^{k-1}) \qquad (2.6)
\end{aligned}
$$

In (2.6), the term $p(S^k|y^{k-1})$ is the posterior PDF value of the target state for measurements upto time $t = k$ and through (2.6) it is incorporated with the new observation made at time $t = k$ while the state estimated is updated. Thus the relation in (2.6) gives a proper recursive estimation approach in Bayesian framework. This recursive method is useful in real time for its robustness and lies in the heart of every target tracking algorithm.

2.3.4 Discussion on Bayes' Theorem

The discussion in section 2.3.3 establishes that the classical Bayes' theorem provides a recursive estimator which can be implemented in solving the real time estimation problems. The state of interest can be continuous, i.e target dynamics etc., or discontinuous, i.e. the number of targets, the maneuvering model of the target, etc.. Thus any possible uncertainty related to the target tracking can be resolved using Bayesian approach with the full functionality retained in recursive nature of it.

2.4 Application of Bayes' Theorem To Target Tracking

Even though the theory suggests that the Bayes' theorem can solve target tracking problem, its practical applicability is not straightforward. In this section, several assumptions and prerequisites for the application of the theorem in target tracking will be discussed. The expression (2.6) will also be revisited to conceptualize the essence of target tracking.

As a first step, the composite target state and observation set can be expanded as

- $S^k = \{S_k, S_{k-1}, \ldots, S_0\} = \{S_k, S^{k-1}\}$
- $y^k = \{y_k, y_{k-1}, \ldots, y_0\} = \{y_k, y^{k-1}\}$

The expression of (2.6) can be re-written as

$$
\begin{aligned}
p(S^k|y^k) &= \frac{p(y_k|y^{k-1}, S^k)}{p(y_k|y^{k-1})} \times p(S^k|y^{k-1}) \\
&= \frac{p(y_k|y^{k-1}, S_k, S^{k-1})}{p(y_k|y^{k-1})} \times p(S_k, S^{k-1}|y^{k-1})
\end{aligned}
$$

$$= \frac{p(y_k|y^{k-1}, S_k, S^{k-1})}{p(y_k|y^{k-1})} \times p(S_k|S^{k-1}, y^{k-1})p(S^{k-1}|y^{k-1}) \tag{2.7}$$

To resolve (2.7) into more compact form, the following assumptions are made.

1. Under white noise assumption, measurements at a particular time $t = k$ depends only on the target state of interest S_k. Due to this, the term $p(y_k|y^{k-1}, S_k, S^{k-1})$ can be reduced to $p(y_k|S_k)$. On the other hand the term $p(y_k|y^{k-1})$ can be expanded through the total probability theorem as

$$p(y_k|y^{k-1}) = \int_{S_k} p(y_k|S_k, y^{k-1}).p(S_k|y^{k-1})dS_k \tag{2.8}$$

2. Target state transition is assumed to be Markov. Hence the term $p(S_k|S^{k-1}, y^{k-1})$ can be reduced to

$$p(S_k|S_{k-1}, \ldots, S_0, y^{k-1}) = p(S_k|S_{k-1}) \tag{2.9}$$

As a result the expression in (2.7) can be further simplified as

$$p(S^k|y^k) = \frac{p(y_k|S_k)}{p(y_k|y^{k-1})}p(S_k|S_{k-1})p(S^{k-1}|y^{k-1}) \tag{2.10}$$

For tracking a target, the state S_k at a particular time $t = k$ is more important than the composite state estimation of S^{k-1}. Therefore the expression $p(S_{k-1}|y^k)$ is more relevant to the solution of the problem. This can be obtained by integrating $p(S^{k-1})$ with respect to the states at other times. The relationship can be given as

$$p(S_k|y^k) = \int_{S_{k-1}} \ldots \int_{S_0} p(S^{k-1}|y^k)dS_0 \ldots dS_{k-1} \tag{2.11}$$

With the result obtained from (2.10), the definition in (2.11) is given by

$$\begin{aligned} & p(S_k|y^k) \\ = & \int_{S_{k-1}} \ldots \int_{S_0} \frac{p(y_k|S_k)}{p(y_k|y^{k-1})}p(S_k|S_{k-1})p(S^{k-1}|y^{k-1})dS_0 \ldots dS_{k-1} \\ = & \frac{p(y_k|S_k)}{p(y_k|y^{k-1})} \int_{S_{k-1}} p(S_k|S_{k-1}) \int_{S_{k-2}} \ldots \int_{S_0} p(S^{k-1}|y^{k-1})dS_0 \ldots dS_{k-1} \end{aligned} \tag{2.12}$$

According to the definition of (2.11), the integration series in (2.12) is reduced to

$$\int_{S_{k-2}} \dots \int_{S_0} p(S^{k-1}|y^{k-1})dS_0 \dots dS_{k-2} = p(S_{k-1}|y^{k-1}) \tag{2.13}$$

Replacing the expression in (2.12) by that in (2.13), the Bayesian estimation can be achieved. This is given by the density

$$\begin{aligned} & p(S_k|y^k) \\ = & \frac{p(y_k|S_k)}{p(y_k|y^{k-1})} \int_{S_{k-1}} p(S_k|S_{k-1}) \times p(S_{k-1}|y^{k-1})dS_{k-1} \end{aligned} \tag{2.14}$$

$$= \frac{p(y_k|S_k)}{p(y_k|y^{k-1})} \times p(S_k|y^{k-1}) \tag{2.15}$$

The forms derived in (2.14) or (2.15) are the ones used for standard target tracking application. It is also to be noted that (2.14) retains the recursive nature of the original approach. The density at time $t = k - 1$, denoted by $p(S_{k-1}|y^{k-1})$, is the a priori knowledge of the target state and is used to estimate the state at time $t = k$.

In literature, three terms in (2.15) are referred to with specific names. Those are

- $p(y_k|S_k)$ is called *Likelihood.*
- $p(S_k|y^{k-1})$ is called *Prediction.*
- $p(y_k|y^{k-1})$ is called *Normalization density*

Modeling the state S_k as an augmentation of past states from time $t = k-N$ to $t = k$ solves the smoothing estimation problem with the same Bayesian generalized framework given in (2.15).

In this way, a true recursive form of Bayes' theorem is achieved. The task of the target tracker is to calculate the above mentioned terms to solve the estimation problem. For these parameters to be evaluated, the algorithm needs to know specifically the target dynamic model and sensor model. In the next section this aspect will be described along with its relevance with the terms mentioned above. Consequently it will be shown that under linear Gaussian assumption the Bayesian approach results into Kalman Filter (KF) and eventually Augmented State KF (AS-KF).

2.4.1 Target Dynamic Model

The target dynamic model specifies the nature of the target state transition along the time. This model is necessary to determine the *Prediction* term stated in section 2.4. In general the target dynamic model is given by

$$S_k = f(k, S_{k-1}, u(k)) + v(k) \tag{2.16}$$

where

- $f(.)$ represents a generalized transition function. The function may be linear as well as non-linear.
- S_n refers to the target state at time $t = n$
- $u(k)$ denotes a particular input which indicates any maneuver of the target
- $v(k)$ is a random noise (generally called as "process noise") term. This term captures the naturally occuring random disturbances or uncertainty in the modelling itself.

  1. The random noise is drawn from a normally distribution that has a mean of zero and variance Q_k. In other words $E\{v(k)\} = 0$ and $Q_k = E\{v(k)v(k)^T\}$.
  2. The noise is not correlated in time. Mathematically $E\{v(i)v(j)^T\} = 0$ where $i \neq j$. (which means $Q_k = E\{v(i)v(j)^T\}$ is satisfied only when $i = j = k$)
  3. The noise is also independent of state, $E\{S_i v(j)^T\} = 0$ for any $i < j$.

The prediction term of (2.15) can be expanded applying the total probability theorem as (and as shown in (2.14))

$$p(S_k|y^{k-1}) = \int_{Sk-1} p(S_k|S_{k-1}) \times p(S_{k-1}|y^{k-1}) dS_{k-1} \tag{2.17}$$

The term $p(S_k|S_{k-1})$ of (2.15) denotes the transition in time and this distribution is given by the dynamic model of (2.16) (the exact procedure may vary due to the fact whether the function in non-linear or linear).

2.4.2 Sensor Observation Model

The sensor model captures what aspect of the target state is observed by the sensor along with its uncertainty level. It is dependent on the sensor type and working principle and therefore the model is said to be "sensor model". In general this is given by

$$y_k = h(k, S_k) + w(k) \tag{2.18}$$

where

- $h(.)$ is the state to observation transition function. It can be linear or non-linear depending on the type of sensor and also which aspect of target is observed.
- y_n and S_n refer to the measurement and target state respectively at time $t = n$
- $w(k)$ is the random measurement noise. This noise is assumed to have several properties as stated below
  1. The noise is normally distributed with mean zero and variance R_k. In other words, $E\{w(k)\} = 0$ and $R_k = E\{w(k)w(k)^T\}$.
  2. The noise is also not correlated in time, $E\{(w(i)w(j)^T)\} = 0$ for each $i \neq j$ (which means $R_k = E\{w(k)w(k)^T\}$ is satisfied only when $i = j = k$)
  3. The noise is also independent of state as well as observation. These can be mathematically states as $E\{y(i)w(j)^T\} = 0$, $E\{S_i w(j)^T\} = 0$ for any $i < j$.
  4. The measurement noise is also statistically independent of the process noise. Mathematically $E\{w(i)v(j)^T\} = 0$ for any i and j.

The likelihood term of (2.15) can be expanded as

$$p(y_k|S_k) = \int_{S_k} p(y_k|S_k).p(S_k|S_{k-1})dS_k \tag{2.19}$$

The second term within the integral is obtained from the target dynamic model while the first term within the integral is given by the sensor model (the exact derivation though may vary depending on the system).

The derivations of KF and AS-KF are presented next as an illustration of Bayesian approach to estimation.

2.5 The Kalman Filter

Kalman filter, the simplest of target trackers, deals with the estimation of target dynamic state from a set of noisy measurements. Attempts to establish connection between KF and Bayesian filter started by Ho and Lee, [34]. Finally

Koks and Challa, [43], completed the effort by solving each densities of likelihood, prediction and normalization under Gaussian assumption and showing that the resultant posterior density is achieved by the same equations of KF. The following derivation is the summary of that effort.

KF assumes linear target dynamics and sensor model given respectively by

$$x_k = Fx_{k-1} + v(k) \tag{2.20}$$

$$y_k = Hx_k + w(k) \tag{2.21}$$

where

- x_k denote the target state of interest at time k.
- F and $v(k)$ are the target transition matrix and process vectors respectively. The noises are samples drawn from the normal distribution with mean zero and variance Q_k.
- H models the sensor and $w(k)$ denotes sensor noise vectors respectively. The sensor noises are also drawn from a normal distribution with mean zero and variance R_k.

All other assumptions mentioned in section 2.4.1 and section 2.4.2 hold. KF starts with a priori density $p(x_{k-1}|y^{k-1})$ and assumes a normal distribution with mean $\hat{x}_{k-1|k-1}$ and variance $P_{k-1|k-1}$. The model specifications will be used to calculate prediction, likelihood and normalization terms which gives the recursive update of the state.

2.5.1 Prediction

The prediction term, $p(x_k|y^{k-1})$ is expanded as in (2.17)using the total probability theorem.

$$p(x_k|y^{k-1}) = \int_{x_{k-1}} p(x_k|x_{k-1}, y^{k-1})p(x_{k-1}|y_{k-1})dx_{k-1} \tag{2.22}$$

The integration on the right hand side of (2.22) is Chapman-Kolmogorov equation. Following target dynamic model (2.20), x_k depends on x_{k-1} and not on past measurements. Therefore $p(x_k|x_{k-1}, y^{k-1})$ reduces to $p(x_k|x_{k-1})$. Under Gaussian assumption, the density can be calculated as

$$p(x_k|x_{k-1}) = \mathcal{N}(x_k; F\hat{x}_{k-1}, Q_k) \tag{2.23}$$

(using the target dynamic model) where $\mathcal{N}(a; b, c)$ denotes normal pdf of random variable a with mean b and variance c.

Now the second term of the integration in (2.22) is the previous estimation of the state and is given by $\mathcal{N}(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1})$. Therefore the prediction term reduces to Chapman-Kolmogorov equation

$$p(x_k|y^{k-1}) = \int_{x_{k-1}} \mathcal{N}(x_k; F\hat{x}_{k-1}, Q_k)\mathcal{N}(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1})d_x k - 1 \quad . \tag{2.24}$$

The solution to the integration 2.24 (see [43, 36]) is a normal density given by

$$p(x_k|y^{k-1}) = \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}). \tag{2.25}$$

where

$$\hat{x}_{k|k-1} = F\hat{x}_{k-1|k-1} \tag{2.26}$$

$$P_{k|k-1} = FP_{k-1|k-1}F^T + Q_k \tag{2.27}$$

2.5.2 Likelihood

The likelihood term, $p(y_k|x_k)$ is obtained directly from the sensor model. The resultant density is normal given by

$$p(y_k|x_k) = \mathcal{N}(y_k; Hx_k, R_k) \tag{2.28}$$

2.5.3 Normalization

Normalization density $p(y_k|y^{k-1})$ is expanded as below in terms of prediction and likelihood terms.

$$\begin{aligned} p(y_k|y^{k-1}) &= \int_{x_k} p(y_k|x_k, y^{k-1})p(x_k|y^{k-1}) \\ &= \int_{x_k} \mathcal{N}(y_k; Hx_k, R_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})dx_k \\ &= \mathcal{N}(y_k; \hat{y}_k, S) \end{aligned} \tag{2.29}$$

where,

$$\hat{y}_k = H\hat{x}_{k|k-1} \tag{2.30}$$

$$S = HP_{k|k-1}H^T + R_k \tag{2.31}$$

(the full derivation is available in [43]).

2.5.4 Posterior Density Calculation

The calculation of posterior density is carried out by evaluating (2.15).

$$p(x_k|y^k) = \frac{\mathcal{N}(y_k; Hx_k, R_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})}{\mathcal{N}(y_k; \hat{y}_k, S)} \tag{2.32}$$

It is shown by Challa in [43] that the expression in (2.32) reduces to a normal distribution given by

$$p(x_k|y^k) = \mathcal{N}(x_k; \hat{x}_{k|k}, P_{k|k}) \tag{2.33}$$

where

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K(y_k - \hat{y}_k) \tag{2.34}$$

$$P_{k|k} = (I - KH)P_{k|k-1} \tag{2.35}$$

where

$$K = P_{k|k-1}H^T S^{-1} \tag{2.36}$$

The state update and associated co-variance are obtained from (2.34) and (2.35) respectively. Expressions (2.26), (2.27), (2.30),(2.31), (2.36), (2.34) and (2.35) are the steps of one iteration of KF.

2.5.5 The KF Equations

In summary, Kalman filter follows the following steps in one iteration.

$$\begin{aligned}
\hat{x}_{k|k-1} &= F\hat{x}_{k-1|k-1} \\
P_{k|k-1} &= FP_{k-1|k-1}F^T + Q_k \\
\hat{y}_k &= H\hat{x}_{k|k-1} \\
S &= HP_{k|k-1}H^T + R_k \\
K &= P_{k|k-1}H^T S^{-1} \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K(y_k - \hat{y}_k) \\
P_{k|k} &= (I - KH)P_{k|k-1}
\end{aligned}$$

It can also be mentioned here that KF follows a prediction-correction working principle as summarized in the figure 2.5.

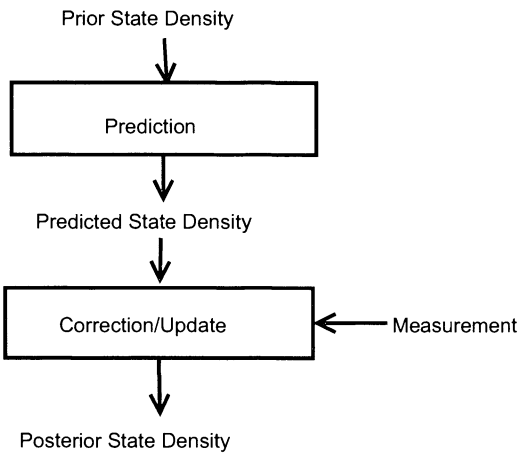

Fig. 2.5: KF Working Algorithm

2.6 The Kalman Smoother

Modeling the state vector S_k in (2.15) as the augmentation of past states from time $t = k - N$ to $t = k$ provides the basic definition of smoothing within itself. As there is no restriction of the value of N, by setting $N > 0$ the said expression will yield the smoothed estimation of the states S_{k-1} through to S_{k-N} as well the filtered estimation of state S_k based on the measurement received up to time $t = k$. In [36], it was noted that augmenting the state vector with past states builds a practical framework to implement the Bayesian smoothing in the same process as a standard filter.

This approach of augmenting the state vector is an easy appraoch to design a Bayesian smoothing. In the next section, first the Bayesian recursion model for augmented state approach will be proposed along with the demonstration that this approach follows the same densities as a standard Bayesian filter does with the minimal change of state vectors.

2.6.1 Augmented State Approach

In the augmented state approach, the target dynamic state is re-modeled as

$$\mathbf{S}_k = \begin{bmatrix} S_k & S_{k-1} & \dots & S_{k-d} \end{bmatrix}^T \tag{2.37}$$

which is basically the augmentation of state vectors of past time instants.

With the redefinition of target dynamic state, both the target dynamic model and sensor model need to be modified as

$$\mathbf{S}_k = \mathbf{f}(k, \mathbf{S}_k, u(k)) + \mathbf{v}(k) \tag{2.38}$$

$$\mathbf{Y}_k = \mathbf{h}((k, \mathbf{S}_k) + w(k)) \tag{2.39}$$

Therefore, for augmented state model, the Bayesian formulation of (2.15) results in

$$p(\mathbf{S}_k|\mathbf{Y}^k) = \frac{p(\mathbf{Y}_k|\mathbf{S}_k)}{p(\mathbf{Y}_k|\mathbf{Y}^{k-1})} \times p(\mathbf{S}_k|\mathbf{Y}^{k-1}) \tag{2.40}$$

It is evident from (2.40) that by introducing the state vector as an augmented one, the Bayesian formulation for estimating the target state has remained same as in standard filter. But in the process, the states $S_k, S_{k-1}, \ldots, S_{k-d}$ also get updated through the measurement at time $t = k$. Therefore, while S_k is the filtered estimate, the rest of the state vector elements are smoothed. Thus this approach gives a very rigorous mathematical formulation for smoothing while maintaining the basic filter algorithms almost unchanged. For the illustration purpose, a linear model for target dynamics and sensor model will be assumed for the derivation of AS-KF.

2.6.2 Augmented State Kalman Filter (AS-KF)

The linear target dynamics and sensor model are given by

$$\mathbf{X}_k = \mathbf{F}\mathbf{X}_{k-1} + \mathbf{V}(k) \tag{2.41}$$

$$y_k = \mathbf{H}\mathbf{X}_k + w_k \tag{2.42}$$

where

- $\mathbf{X}_k = \begin{bmatrix} x_k & x_{k-1} & \ldots & x_{k-N} \end{bmatrix}$ and x_k denote the target state of interest at time k.
- $\mathbf{F}$ is the state transition matrix given by

$$\mathbf{F} = \begin{bmatrix} F & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \vdots & \ddots & \ldots & \vdots \\ 0 & \ldots & I & 0 \end{bmatrix}$$

- $\mathbf{V}(k)$ is process noise vector given by

$$\mathbf{V}_k = \begin{bmatrix} v(k) & 0 & \ldots & 0 \end{bmatrix}^T$$

- $\mathbf{Q}_k$ is the noise covariance matrix

$$\mathbf{Q}_k = \begin{bmatrix} Q_k & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} \tag{2.43}$$

-

$$\mathbf{H} = \begin{bmatrix} H \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- w_k is the sensor noise vector. The sensor noises are drawn from a normal distribution with mean zero and variance R_k.

(where x_k, F_k, $v(k)$, Q_k are as defined in section 2.5 and 0's and I's refer to zero and identity matrix respectively with

All other assumptions mentioned in section 2.4.1 and section 2.4.2 hold.

Under the linear Gaussian assumption, the conditional mean of the distribution $p(\mathbf{X}_k|y_k)$ is the posterior density for the state $\mathbf{X}_k$. Starting from a prior knowledge of the state estimate $\hat{\mathbf{X}}_{k-1|k-1}$ and associated error covariance $\mathbf{P}_{k-1|k-1}$ at time $t = k-1$, a Kalman filter will calculate the desired conditional mean. It has been shown in [22] that the basic Kalman filter steps hold also for augmented state case. Therefore target state estimate $\hat{\mathbf{X}}_{k|k}$ can be achieved through following steps :

$$\hat{\mathbf{X}}_{k|k-1} = \mathbf{F}\hat{\mathbf{X}}_{k-1|k-1} \tag{2.44}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}\mathbf{P}_{k-1|k-1}\mathbf{F}^T + \mathbf{Q}_k \tag{2.45}$$
$$\hat{y}_k = \mathbf{F}\hat{\mathbf{X}}_{k|k-1} \tag{2.46}$$
$$\mathbf{S} = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^T +_k \tag{2.47}$$
$$\mathbf{K} = \mathbf{P}_{k|k-1}\mathbf{H}^T\mathbf{S}^{-1} \tag{2.48}$$
$$\hat{\mathbf{X}}_{k|k} = \hat{\mathbf{X}}_{k|k-1} + \mathbf{K}(y_k - \hat{y}_k) \tag{2.49}$$
$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}_{k|k-1} \tag{2.50}$$

Relations (2.49) and (2.50) are the recursive update equation of the Kalman filter. It is very much evident that the algorithm goes through the standard Kalman filter equations with the single state vector replaced by the augmented state one. The joint density update of $p(\mathbf{X}_k|y^k)$ also smooths the past states of $\begin{bmatrix} x_{k-1} & x_{k-2} & \dots & x_{k-N} \end{bmatrix}$.

2.6.3 Discussion on Augmented State modeling approach

The steps of AS-KF are identical to what standard KF follows. In the later chapters, tracking in clutter scenario will be modeled using Bayesian approach and its extension into smoothing through augmentation of state vectors will be derived. The similarity will be evident in those derivations also. Therefore augmentation of states provides an easy, but mathematically sound, platform for smoothing establishing it to be a powerful tool to develop smoothing algorithms. Introduction of additional state vectors, add to the memory requirement of the system.

2.7 Conclusion

In this chapter, the basic theories of estimation are pointed out. The fundamental definitions and how they relate to each other as well as target tracking theory are also described. Also in this chapter, the Bayes' theorem is introduced. The systematic reformulation of the theorem for target tracking application is detailed and derived. The iterative form of Bayes' theorem, which is the basis of every tracking algorithm, has been shown here. Derivation of standard KF from Bayes' theorem is given as an illustration of Bayesian approach. It is also shown that Bayesian modeling can be extended to smoothing and its implementation is performed through augmentation of states. As a part of extension AS-KF is also derived using the same approach.

3. BAYESIAN TRACKING IN CLUTTER

3.1 Introduction

This chapter focuses on the target tracking problem where the origin of measurement is uncertain. It is well known that under measurement origin uncertainty simple filters like KF will not perform well. The Bayesian definition of such problems and associated models are described in this chapter. It is also shown that the solution of Bayesian approach leads to Probabilistic Data Association Filter (PDAF). PDAF is a computationally efficient approximation of optimal Bayesian solution. The same Bayesian modeling can also be applied to an augmented state approach for smoothing. The effort results in Augmented State PDAF which is also shown to follow the standard PDAF steps. Therefore the validity of Bayesian modeling to handle measurement origin uncertainty is established in this chapter.

3.2 The problem of Tracking in Clutter

Kalman filter, presented in chapter 2, updates previous state estimate in a recursive way after the current measurement is available. In that process, it is presumed that in each sensor scan the tracker receives one measurement from the target. But in practice, in each scan a random number of measurements is gathered. The sources of the unwanted measurements (not originating from true targets) are called "clutters". These may be terrain, clouds or even thermal sources (present in the sensor surveillance region), electromagnetic disturbances and etc. The simple Kalman filter is not applicable under this scenario simply because of the problem of deciding which measurement to use for the state update in (2.34). The situation can be illustrated by figure 3.1.

Clutter received introduces the problem of "data association", the technique to associate obtained measurements to the target. In early 70's, Singer, Sea and Stein introduced Nearest Neighbor filter (NNF),[68, 70, 71]. NNF was a good technique to solve the "data association" problem. It takes the measurement, closest to the predicted one, as the valid measurement to update the state estimate using basic Kalman filter. Instead of taking one measurement, the need

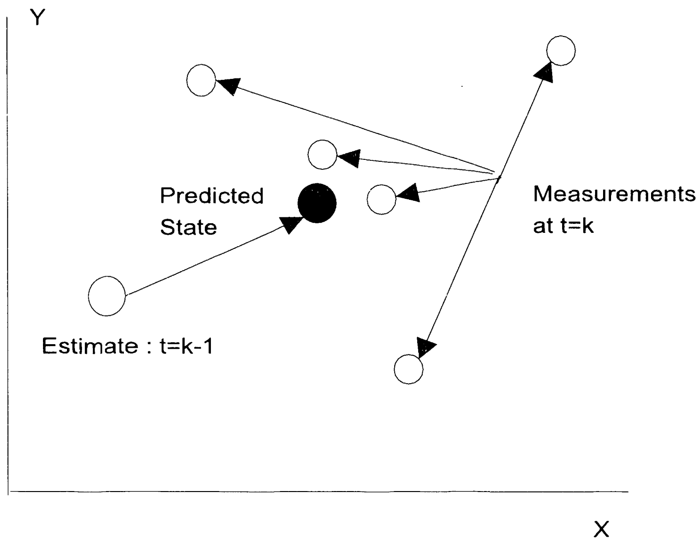

Fig. 3.1: Problems of Clutter

of taking all measurements in the "neighborhood" of the predicted measurement was first pointed out by Bar-Shalom and Jaffer, [7, 8]. Later in 1975, Bar-Shalom and Tse finally proposed a Probabilistic Data Association filter (PDAF), [10]. This filter validates a set of measurements and assigns a probability of it being "associated to the actual target". Finally PDAF updates the state by combining the validated measurements according to the weight given by the association probability. The method is approximately 1.5-2 times more computationally intensive than NNF, however it handles clutter much better than NNF.

PDAF is proved to be most computationally efficient algorithm to solve data association problem in cluttered environment. In the remaining sections of this chapter, the optimal Bayesian modeling of clutter will be discussed. The PDAF will be derived from the Bayesian model as a suboptimal algorithm. The augmented state PDAF will also be derived to demonstrate the application of Bayesian modeling into smoothing estimation.

3.3 Bayesian Modeling for Tracking in Clutter

At any given instant, the sensor gathers more than one measurement. The number of received measurement varies randomly from one scan to another. Therefore the measurement set in Bayesian model needs to cater for that possibility. In each scan, the measurement y_k is given by

$$y_k = \{y_k(1), y_k(2), \dots, y_k(m)\} \tag{3.1}$$

where $y_k(n)$ denotes $n-th$ measurement received at time $t=k$ and m is the total number of measurements received by the sensor. The posterior density of state is not only conditioned on the observations themselves, but also on the number of observations received. Therefore Bayesian recursion of (2.15) is modified to cater for "clutter" measurements as following:

$$\begin{aligned} & p(x_k|y^k, m^k) \\ = & \frac{p(y_k, m_k|x_k, y^{k-1}, m^{k-1})}{p(y_k, m_k|y^{k-1}, m^{k-1})} p(x_k|y^{k-1}, m^{k-1}) \end{aligned} \quad (3.2)$$

where m^n is the collection of all the measurements at every time instant till $t=n$ and m_n is the number of measurements at time $t=n$.

The formulation in (3.2) is similar to standard Bayesian filter in (2.15). The difference is in likelihood and normalization which are joint densities of all the observations and the number of received measurements. Under linear Gaussian model assumption of the system, the Bayesian filter reduces to PDA. The derivation will be discussed in the next section.

3.4 PDA Filter

Probabilistic data association filter (PDAF), follows the Bayesian modeling of target tracking in clutter for linear Gaussian system. The PDAF is a suboptimal implementation of standard Bayesian recursion presented in section 3.3. The sub-optimality is associated with the approximation of Gaussian mixtures to a single Gaussian.

The measurement validation process starts with previous state estimate $x_{k-1|k-1}$ and associated covariance matrix $P_{k-1|k-1}$. The standard KF prediction steps are then followed to derive the densities of predicted state and measurement. Based on the predicted measurement $\hat{y}_k$ and associated covariance S_k (from KF step (2.30) and (2.31) respectively of section 2.5.3), each obtained measurement undergoes a test given by

$$[y_k(n) - \hat{y}_k]^T S_k^{-1} [y_k(n) - \hat{y}_k] < \gamma \quad (3.3)$$

The chi-square test threshold λ is chosen to ensure a certain required probability of target originated measurement to be within the validation gate. This probability is called "gating probability" and is given by P_G. The region where the test is satisfied is called "validation region" or "gate". Under Gaussian assumption, theoretically this ellipsoidal region has a volume V_k and is given by

$$V_k = c_{n_z} |\gamma \sqrt{|S_k|}|^{1/2} \quad (3.4)$$

where n_z is the dimension of the measurement and c_{n_z} is the volume n_z dimensional unit hypersphere ($c_1 = 1, c_2 = \pi, c_3 = \frac{4\pi}{3}$, etc.). The measurements that satisfy the test, or in other words are within the gate, are considered as "valid" measurements and are used for state update. Therefore out of m received measurement m_k number of valid measurements are chosen. The scenario is depicted in figure 3.2.

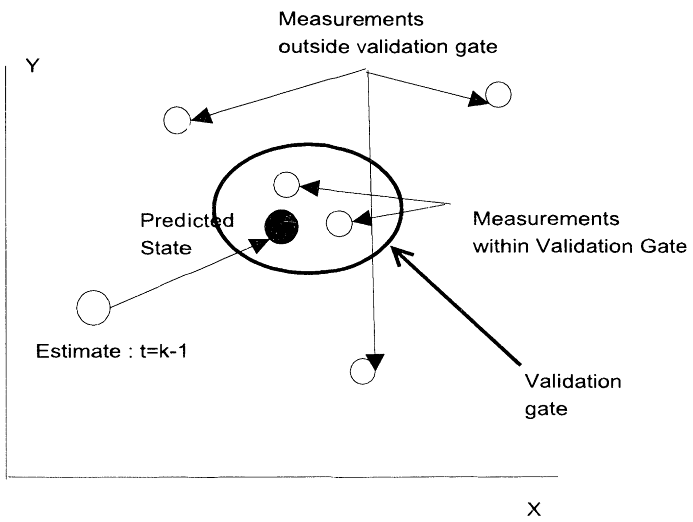

Fig. 3.2: Measurement Validation Process

This subset of m_k measurements are used for Bayesian recursion that result into PDA formulation of target tracking in clutter. This is given by

$$p(x_k|y^k, m^k) = \frac{p(y_k, m_k|x_k, y^{k-1}, m^{k-1})}{p(y_k, m_k|y^{k-1}, m^{k-1})} p(x_k|y^{k-1}, m^{k-1}) \tag{3.5}$$

Bayes' theorem applied on (3.5) provides the framework based on which the steps of PDAF can be derived. Expanding the expression provides familiar Bayesian recursion formula

$$\begin{aligned} & p(x_k|y_k, m_k, y^{k-1}, m^{k-1}) \\ = & \frac{p(y_k, m_k|x_k, y^{k-1}, m^{k-1}) p(x_k|y^{k-1}, m^{k-1})}{p(y_k, m_k|y^{k-1}, m^{k-1})} \\ = & \frac{Likelihood \times Prediction}{Normalization} \end{aligned} \tag{3.6}$$

Starting with previous state estimate $x_{k-1|k-1}$ and associated covariance $P_{k-1|k-1}$, solution of each of the terms - Likelihood, Prediction and Normalization - under linear Gaussian assumption of the system models (same as (2.20), (2.21) given in section 2.5) and their resultant density according to (3.6) completes the tracking in clutter problem. Here these densities will be revisited from Bayesian perspective.

3.4.1 PDAF Prediction

The predicted state does not depend on the number of the measurements received. Therefore, the predicted density $p(x_k|y^{k-1}, m^{k-1})$ is equivalent to $p(x_k|y^{k-1})$ and its is same as standard Bayesian filter. It reduces to KF prediction steps for linear Gaussian systems. The resultant estimates are deduced in section 2.5.1 and mentioned here for clarity.

$$p(x_k|y^{k-1}) = \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \tag{3.7}$$

where

$$\hat{x}_{k|k-1} = F\hat{x}_{k-1|k-1} \tag{3.8}$$

$$P_{k|k-1} = FP_{k-1|k-1}F^T + Q_k \tag{3.9}$$

3.4.2 PDAF Likelihood

For every one of the validated measurements an association event $\theta_k(.)$ is defined as :

1. $\theta_k(i)$: $i-th$ validated measurement is target originated and rest of them are from clutter, where $i = 1, 2, \ldots, m_k$.
2. $\theta_k(0)$: no measurement within the gate is target originated.

These association events $\theta_k(i)$, $i = 0, 1, 2, \ldots, m_k$ are mutually exclusive by definition. Using total probability theorem, the likelihood density can be expanded as

$$\begin{aligned} & p(y_k, m_k|x_k, y^{k-1}, m^{k-1}) \\ = & \sum_{i=0}^{m_k} p(y_k, \theta_k(i), m_k|x_k, y^{k-1}, m^{k-1}) \\ = & \sum_{i=0}^{m_k} p(y_k|x_k, \theta_k(i), m_k, y^{k-1}, m^{k-1}) \\ & \qquad \times P(\theta_k(i)|x_k, m_k, y^{k-1}, m^{k-1}) \end{aligned} \tag{3.10}$$

(capital $P(.)$ is used to denote probability of discrete random variable)

Due to white noise assumption, the first term within summation reduces to

$$p(y_k|x_k, \theta_k(i), m_k) \tag{3.11}$$

Since $\theta_k(i)$ is the association event and $P(\theta_k(i)|x_k, m_k, y^{k-1}, m^{k-1})$ is the probability of such event, it depends only the number of validated measurements. The definition of such an event is legitimate only if there exists a target. In other words, if there is no validated measurement, it is meaningless to associate the events to the target state. Therefore the explicit conditioning of the association probability on target state is not required. As a result, the second term within the summation reduces to

$$P(\theta_k(i)|x_k, m_k, y^{k-1}, m^{k-1}) = P(\theta_k(i)|m_k) \tag{3.12}$$

The likelihood density is then expressed as

$$\begin{aligned} & p(y_k, m_k|x_k, y^{k-1}, m^{k-1}) \\ = & \sum_{i=0}^{m_k} p(y_k|x_k, \theta_k(i), m_k) \times P(\theta_k(i)|m_k, y^{k-1}, m^{k-1}) \end{aligned} \tag{3.13}$$

The first term within the summation is the probability that $y_k(i)$-th measurement is the target originated and the rest of them are from clutter. Incorporating the validation region of volume V_k and gating probability P_G and assuming the clutter to be uniformly and independently distributed over the validation region, this probability can be easily evaluated as

$$\begin{aligned} & p(y_k|x_k, \theta_k(i), m_k) \\ = & \left\{ \begin{array}{cc} (\frac{1}{V_k})^{m_k-1} P_G^{-1} p(y_k(i)|x_k) & i = 1, 2, \ldots, m_k \\ (\frac{1}{V_k})^{m_k} & i = 0 \end{array} \right\} \end{aligned} \tag{3.14}$$

Now the second term within the summation in (3.13), $P(\theta_k(i)|m_k)$, is given by (see appendix D.4 in [6])

$$\begin{aligned} & P(\theta_k(i)|m_k) \\ = & \left\{ \begin{array}{cc} \frac{1}{m_k} P_D P_G \left[P_D P_G + (1 - P_D P_G) \frac{\mu_F(m_k)}{\mu_F(m_k - 1)} \right]^{-1} & i = 1, 2, \ldots, m_k \\ (1 - P_D P_G) \frac{\mu_F(m_k)}{\mu_F(m_k-1)} \left[P_D P_G + (1 - P_D P_G) \frac{\mu_F(m_k)}{\mu_F(m_k-1)} \right]^{-1} & i = 0 \end{array} \right. \end{aligned} \tag{3.15}$$

where $\mu_F(m_k)$ is the probability mass function (PMF) of the number of false measurements and P_D is the probability of detection of target originated measurement (probability that a correct measurement is available at all).

There are two models for PMF of number of false measurements. These are

1. *Parametric Model* : A Poisson density given by

$$\mu_F(m_k) = e^{\lambda V_k} \frac{(\lambda V_k)^{m_k}}{m_k!} \qquad m_k = 0, 1, 2, \ldots, m_k \tag{3.16}$$

   where

   λ is average number of false measurements per unit volume (λV_k is the average number of false measurements within gate).

2. *Nonparametric Model*

$$\mu_F(m_k) = \frac{1}{N} \qquad m_k = 0, 1, 2, \ldots, N-1 \tag{3.17}$$

The association probability for prior becomes

$$P(\theta_k(i)|m_k) = \begin{cases} \frac{P_D P_G}{P_D P_G m_k + (1 - P_D P_G)\lambda V_k} & i = 1, 2, \ldots, m_k \\ \frac{(1 - P_D P_G)\lambda V_k}{P_D P_G m_k + (1 - P_D P_G)\lambda V_k} & i = 0 \end{cases} \tag{3.18}$$

and the same for nonparametric model is

$$P(\theta_k(i)|m_k) = \begin{cases} \frac{P_D P_G}{m_k} & i = 1, 2, \ldots, m_k \\ 1 - P_D P_G & i = 0 \end{cases} \tag{3.19}$$

Moreover, according to the linear Gaussian sensor model (2.21), the measurement density $p(y_k(i)|x_k)$ is $\mathcal{N}(y_k(i); Hx_k, R_k)$. The likelihood density is therefore given by

$$p(y_k, m_k|x_k, y^{k-1}, m^{k-1}) = (\frac{1}{V_k})^{m_k} P(\theta_k(0)|m_k) + \sum_{i=1}^{m_k} (\frac{1}{V_k})^{m_k - 1} P_G^{-1} \mathcal{N}(y_k(i); Hx_k, R_k) P(\theta_k(i)|m_k) \tag{3.20}$$

where $P(\theta(.)|m_k)$ is either (3.18) or (3.19) depending non the choice between parametric and nonparametric model respectively for false measurement PMF.

3.4.3 PDAF Normalization

The normalization density $p(y_k, m_k|y^{k-1}, m^{k-1})$ can be expanded in terms of likelihood and prediction using total probability theorem.

$$\begin{aligned}
& p(y_k, m_k|y^{k-1}, m^{k-1}) \\
= & \int_{x_k} p(y_k, m_k|x_k, y^{k-1}, m^{k-1}) p(x_k|y^{k-1}, m^{k-1}) dx_k \\
= & \int_{x_k} \left[(\frac{1}{V_k})^{m_k} P(\theta_k(0)|m_k) + \sum_{i=1}^{m_k} P_G^{-1} \mathcal{N}(y_k(i); Hx_k, R_k) P(\theta_k(i)|m_k) \right] \\
& \times \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) dx_k \\
= & (\frac{1}{V_k})^{m_k} P(\theta_k(0)|m_k) + \sum_{i=1}^{m_k} P(\theta_k(i)|m_k) \\
& \times \left\{ \int_{x_k} [P_G^{-1} \mathcal{N}(y_k(i); Hx_k, R_k) \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) dx_k] \right\}
\end{aligned} \quad (3.21)$$

The term within integration results (excluding the constant P_G^{-1}) into a normal distribution (see [43]) given by

$$\int_{x_k} P_G^{-1} \mathcal{N}(y_k(i); Hx_k, R_k) \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) dx_k = P_G^{-1} \mathcal{N}(y_k(i); \hat{y}_k, S_k(i)) \quad (3.22)$$

where

$$\hat{y}_k = H\hat{x}_{k|k-1} \quad (3.23)$$

$$S_k(i) = HP_{k|k-1}H^T + R_k \quad (3.24)$$

As a result, the normalization density becomes,

$$\begin{aligned}
\delta &= p(y_k, m_k|y^{k-1}, m^{k-1}) \\
&= (\frac{1}{V_k})^{m_k} P(\theta_k(0)|m_k) + \sum_{i=1}^{m_k} P_G^{-1} P(\theta_k(i)|m_k) \mathcal{N}(y_k(i); \hat{y}_k, S(i))
\end{aligned} \quad (3.25)$$

3.4.4 PDAF Posterior Density

The posterior state density can now be calculated to complete one iteration of PDAF.

$$
\begin{aligned}
& p(x_k|y_k, m_k, y^{k-1}, m^{k-1}) \\
= & \frac{1}{\delta}\left[(\frac{1}{V_k})^{m_k} P(\theta_k(0)|m_k) + \sum_{i=1}^{m_k} P_G - 1\mathcal{N}(y_k(i); Hx_k, R_k)P(\theta_k(i)|m_k)\right] \\
& \times\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \\
= & \frac{1}{\delta}(\frac{1}{V_k})^{m_k} P(\theta_k(0)|m_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \\
& + \sum_{i=1}^{m_k} \frac{1}{\delta} P_G - 1P(\theta_k(i)|m_k)\mathcal{N}(y_k(i); Hx_k, R_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \quad (3.26)
\end{aligned}
$$

The term within summation can be evaluated as below to obtain a normal distribution.

$$
\begin{aligned}
& P_G - 1P(\theta_k(i)|m_k)\mathcal{N}(y_k(i); Hx_k, R_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \\
= & P_G - 1P(\theta_k(i)|m_k)\mathcal{N}(y_k(i); \hat{x}_{k|k-1}, S(i))\frac{\mathcal{N}(y_k(i); Hx_k, R_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})}{\mathcal{N}(y_k(i); \hat{x}_{k|k-1}, S(i))} \\
= & P_G - 1P(\theta_k(i)|m_k)\mathcal{N}(y_k(i); \hat{x}_{k|k-1}, S(i))\mathcal{N}(x_{k|k}(i); \hat{x}_{k|k}(i), P_{k|k}(i)) \quad (3.27) \\
& \text{[following [34]]}
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{x}_{k|k}(i) &= \hat{x}_{k|k-1} + K(y_k(i) - \hat{y}_k) && (3.28) \\
P_{k|k}(i) &= (I - KH)P_{k|k-1} && (3.29) \\
K_k &= P_{k|k-1}H^T S_k^{-1} && (3.30)
\end{aligned}
$$

Substituting (3.27) in (3.26), we have,

$$
\begin{aligned}
& p(x_k|y_k, m_k, y^{k-1}, m^{k-1}) \\
= & \frac{1}{\delta}(\frac{1}{V_k})^{m_k} P(\theta_k(0)|m_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \\
& + \sum_{i=1}^{m_k} \frac{1}{\delta} P_G - 1P(\theta_k(i)|m_k)\mathcal{N}(y_k(i); \hat{x}_{k|k-1}, S(i))\mathcal{N}(x_{k|k}(i); \hat{x}_{k|k}(i), P_{k|k}(i)) \\
= & \beta_k(0)\mathcal{N}(x_k; \hat{x}_{k|k}(0), P_{k|k}(0)) + \sum_{i=1}^{m_k} \beta_k(i)\mathcal{N}(x_{k|k}(i); \hat{x}_{k|k}(i), P_{k|k}(i)) \\
= & \sum_{i=0}^{m_k} \beta_k(i)\mathcal{N}(x_{k|k}(i); \hat{x}_{k|k}(i), P_{k|k}(i)) \quad (3.31)
\end{aligned}
$$

where

$$\beta_k(0) = \frac{1}{\delta}(\frac{1}{V_k})^{m_k} P(\theta_k(0)|m_k) \tag{3.32}$$

$$\beta_k(i) = \frac{1}{\delta} P_G - 1 P(\theta_k(i)|m_k)\mathcal{N}(y_k(i); \hat{x}_{k|k-1}, S(i)) \quad i = 1, 2, \ldots, m_k \tag{3.33}$$

$$\hat{x}_{k|k}(0) = \hat{x}_{k|k-1} \tag{3.34}$$

$$P_{k|k}(0) = P_{k|k-1} \tag{3.35}$$

The resultant posterior density is approximated by a single Gaussian mixture with first two moments as follows

$$\hat{x}_{k|k} = \sum_{i=0}^{m_k} \beta_k(i)\hat{x}_{k|k}(i) \tag{3.36}$$

$$P_{k|k} = \sum_{i=0}^{m_k} \beta_k i P_{k|k}(i) + \sum_{i=0}^{m_k} \beta_k(i)\hat{x}_{k|k}(i)\hat{x}_{k|k}(i)^T - \hat{x}_{k|k}\hat{x}_{k|k}^T \tag{3.37}$$

3.4.5 PDAF Algorithm : Summarized

The Bayesian approach has resulted in deriving the PDAF equations. One iteration of PDAF algorithm is summarized here in step by step manner.

1. **Step 1 : Prediction**

$$\begin{aligned}
\hat{x}_{k|k-1} &= F\hat{x}_{k-1|k-1} \\
P_{k|k-1} &= FP_{k-1|k-1}F^T + Q_k \\
\hat{y}_k &= H\hat{x}_{k|k-1} \\
S_k(i) &= HP_{k|k-1}H^T + R_k \\
K_k &= P_{k|k-1}H^T S_k^{-1}
\end{aligned}$$

2. **Step 2 : Gating**

   Of all the observations received, select those which satisfy the distance test

$$[y_k(n) - \hat{y}_k]^T S_k^{-1} [y_k(n) - \hat{y}_k] < \gamma \tag{3.38}$$

   where γ is the threshold to maintain P_G probability of gating and $n = 1, 2, 3, \ldots,$ number of received measurements.

3. **Step 3 : Update**

$$\begin{aligned}
\hat{x}_{k|k} &= \sum_{i=0}^{m_k} \beta_k(i)\hat{x}_{k|k}(i) \\
P_{k|k} &= \sum_{i=0}^{m_k} \beta_k i P_{k|k}(i) + \sum_{i=0}^{m_k} \beta_k(i)\hat{x}_{k|k}(i)\hat{x}_{k|k}(i)^T - \hat{x}_{k|k}\hat{x}_{k|k}^T
\end{aligned} \tag{3.39}$$

where m_k is the number of validated measurement in step 2 and

$$\begin{aligned}
\beta_k(0) &= \frac{1}{\delta}(\frac{1}{V_k})^{m_k} P(\theta_k(0)|m_k) \\
\beta_k(i) &= \frac{1}{\delta} P_G - 1P(\theta_k(i)|m_k)\mathcal{N}(y_k(i); \hat{x}_{k|k-1}, S(i)) \\
& \qquad i = 1, 2, \ldots, m_k \\
\hat{x}_{k|k}(0) &= \hat{x}_{k|k-1} \\
P_{k|k}(0) &= P_{k|k-1}
\end{aligned} \tag{3.40}$$

3.5 Augmented State PDA Filter (AS-PDAF)

3.5.1 Bayesian Model of AS-PDAF

Challa, Wang and Evans, [22], proposed Augmented State PDAF as a Bayesian solution of smoothing in clutter. It was observed by the authors that augmenting the state vector results in a Bayesian definition of smoothing for clutter problem and the resultant algorithm follows the standard PDA steps.

For AS-PDAF, the target dynamic and sensor model are same as given in (2.19) and (2.37) respectively. The assumptions on the noise vectors also hold true in the case of AS-PDA. Moreover, in a single scan, the sensor may miss the target and collect measurements from other sources. Therefore in general the sensor measurement vector consists of more than one elements at each scan. This is denoted as

$$y_k = \{y_k(1), y_k(2), \ldots, y_k(m_k)\} \tag{3.41}$$

where m_k is the number of validated measurements received at time $t = k$. (discussed in section 3.4).

The Bayesian formulation augmented state approach of target tracking in clutter is then given by

$$\begin{aligned} & p(\mathbf{X}_k|y^k, m^k) & (3.42) \\ = & \; p(\mathbf{X}_k|y_k, m_k, y^{k-1}, m^{k-1}) & (3.43) \end{aligned}$$

where $y_k(n)$ is the n-th validated measurement at time $t = n$, $\mathbf{Y}^k$ is the collection of all validated measurements up to $t = k$, m_k is the number of validated measurement at $t = k$ and m^k is the collection of number of validated measurements till $t = k$.

The Bayesian definition of smoothing in clutter in (3.42) is exactly similar to the definition of standard PDAF in (3.5) with the exception that the standard state vectors are replaced by augmented ones. As a result the derivation will also follow the same as standard PDAF. The result will be an algorithm for tracking in clutter with smoothing of the past states along with the filtering of the most recent one. The summary steps of such an algorithm is provided below.

In an iterative manner, starting from previous state estimate $\hat{\mathbf{X}}_{k-1|k-1}$ and associated error covariance $\mathbf{P}_{k-1|k-1}$ at time $t = k - 1$, the augmented state approach updates the state by following the steps similar to standard PDA filter. One iteration of AS-KF is as following:

1. **Step 1 : Prediction**

$$\begin{aligned} \hat{\mathbf{X}}_{k|k-1} &= \mathbf{F}\hat{\mathbf{X}}_{k-1|k-1} \\ \mathbf{P}_{k|k-1} &= \mathbf{F}\mathbf{P}_{k-1|k-1}\mathbf{F}^T + \mathbf{Q}_k \\ \hat{y}_k &= \mathbf{F}\hat{\mathbf{X}}_{k|k-1} \\ \mathbf{S}_k &= \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^T + R_k \\ \mathbf{K}_k &= \mathbf{P}_{k|k-1}\mathbf{H}^T\mathbf{S}^{-1} \end{aligned}$$

2. **Step 2 : Gating** Select the measurements that satisfy the distance test

$$[y_k(n) - \hat{y}_k]^T\mathbf{S}_k^{-1}[y_k(n) - \hat{y}_k] < \gamma$$

where γ is the threshold to maintain P_G probability of gating and $n = 1, 2, 3, \ldots,$ number of received measurements.

3. **Step 3 : Update**

$$\begin{aligned} \hat{\mathbf{X}}_{k|k} &= \sum_{i=0}^{m_k} \beta_k(i)\hat{\mathbf{X}}_{k|k}(i) \\ \mathbf{P}_{k|k} &= \sum_{i=0}^{m_k} \beta_k i\mathbf{P}_{k|k}(i) + \sum_{i=0}^{m_k} \beta_k(i)\hat{\mathbf{X}}_{k|k}(i)\hat{\mathbf{X}}_{k|k}(i)^T - \hat{\mathbf{X}}_{k|k}\hat{\mathbf{X}}_{k|k}^T \end{aligned} \quad (3.44)$$

where m_k is the number of validated measurement in step 2 and

$$\begin{aligned}
\beta_k(0) &= \frac{1}{\delta}(\frac{1}{V_k})^{m_k} P(\theta_k(0)|m_k) \\
\beta_k(i) &= \frac{1}{\delta} P_G - 1 P(\theta_k(i)|m_k)\mathcal{N}(\mathbf{Y}_k(i); \hat{\mathbf{X}}_{k|k-1}, \mathbf{S}(i)) \\
& \qquad\qquad i = 1, 2, \ldots, m_k \\
\hat{\mathbf{X}}_{k|k}(0) &= \hat{\mathbf{X}}_{k|k-1} \\
\mathbf{P}_{k|k}(0) &= \mathbf{P}_{k|k-1}
\end{aligned} \tag{3.45}$$

It is again evident that update equations of $\hat{\mathbf{X}}_{k|k}$ and $\mathbf{P}_{k|k}$ smooth the states of previous time instants as they are part of the augmented state vector. Therefore the augmented state approach implements the Bayesian formulation of target tracking in clutter.

3.6 Conclusion

In this chapter the Bayesian recursion for target tracking in clutter has been introduced. PDAF is a sub-optimal solution of Bayesian approach. PDAF has comparable performance over other similar algorithms, e.g particle filters etc, while being computationally efficient. In this chapter the PDAF formulation is derived from Bayesian modeling. Also the implementation of Bayesian smoothing algorithm through augmentation of states have been modeled and Augmented State PDAF is derived as an illustration. It is also noted that the steps of standard PDAF and AS-PDAF are exactly same. In conjunction with last chapters derivation of KF and AS-KF, this chapter provides a very rigorous platform based on Bayesian approach for target tracking solving continuous random variables like target's dynamic states. In the next chapter, a discrete event like "target existence" will be dealt with from Bayesian perspective and a smoother will be proposed.

4. BAYESIAN TRACKING FOR AUTOMATIC TRACK MAINTENANCE

4.1 Introduction

In this chapter, the problem of track maintenance in a cluttered environment will be introduced. The Bayesian definition of the same problem will also be discussed. The solution of the Bayesian model for resolving track maintenance issues gives rise to the well known Integrated Probabilistic Data Association Filter (IPDAF). The derivation will be carried out in this chapter as an illustration. Following the previous chapters, the Bayesian model will be re-defined for augmented state to develop a smoothing algorithm for automatic track maintenance. The solution of the augmented state approach, under linear Gaussian systems, reduces to the original derivation of Augmented State IPDA (AS-IPDA) smoother. The associated models and assumptions will also be detailed in the chapter.

4.2 Problem of Track Maintenance

The clutter, introduced in chapter 3, poses a problem of "data association" for tracking algorithms. The probabilistic solution to the problem is to propose a hypothesis that each of the validated measurement may or may not be from the target and assigns a probability to these hypothesis. Then each of the measurements is fed into simple KF whose output is then combined and weighted by the associated probability of each measurement hypothesis. The approach works with the assumption that the track exists and is already initiated by some prior knowledge about the target.

But in cluttered environment, it is almost impossible to detect a well defined target and then track it. Because in each scan a bunch of measurement is received and any combination of measurements from successive scans may look like a track. The problem is illustrated in figure 4.1.

It is clear from figure 4.1 above that after a track is initiated PDAF will continue tracking it without any verification. In that case, a target falsely started with clutter measurements (which in two successive scans looked like coming

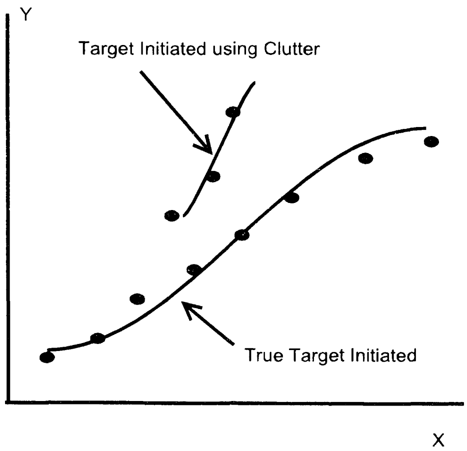

Fig. 4.1: Track Existence Problem

from a target by random chance), will be continued. Therefore a tracker may end up tracking a lot of unnecessary tracks resulting into high computational and memory requirements and low efficiency.

Generally in a cluttered scenario, tracks are initiated based on "two point differencing" method as described in [5]. The method actually looks for measurement in two successive scans that satisfy some defined velocity range. After initiation, each track goes through the confirmation and/or termination stages. The track evolution can be summarized as in figure 4.2.

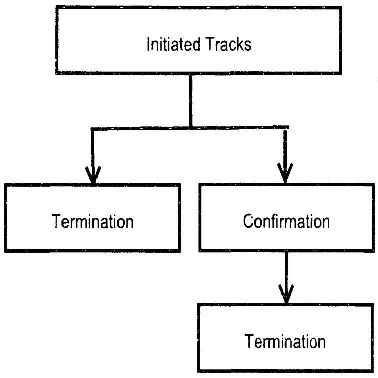

Fig. 4.2: Track evolution stages

Initial attempts to deal with the situation were based on heuristic. One of these attempts was to count "M out of N" scans where there is no measurement within the validation gate for certain track and if the condition was satisfied, the track was to be dropped. Another such attempt involved in checking the state error covariance and if it crossed certain threshold, the track would be dropped. But all these were not systematic and therefore performance of tracker depends significantly on the threshold chosen.

The Interacting Multiple Model Filter (IMMF), [18], was introduced for maneuvering target and it proved to be very successful. The working principal of IMMF is that it models target maneuver (acceleration) as discrete input to the system and runs separate filters for each of them. The output of each state is then combined to update the state. But in the process IMMF assigns probability to each model. This idea was then extended to deal with target existence uncertainty. In [5], the same IMM approach was taken with only two models defined as "observability" and "non-observability" of the target. The probabilities assigned to each of the model are then translated into "true track probability" based on which the decision on target maintenance is taken.

In early 90's another algorithm, Integrated Probabilistic Data Association Filter (IPDAF) was proposed [61, 59, 60] to resolve target maintenance issue. In contrast to IMMPDAF's two separate models, IPDA introduces a model termed as "target existence" with two possible possible events - "existence" and "non-existence" and assigns probability on such events to decide on the track status. The advantage of IPDA over IMMPDA was that it does not require two separate filters to run for each track and therefore is computationally less intensive. Moreover, theoretically it follows the steps of standard PDAF with the added condition of target existence.

IPDAF is now established as one of the most effective filter algorithm for target tracking in clutter with target existence uncertainty. With the help of existence probabilities, the tracker can easily differentiate between true and false targets and evaluate each track status as either confirmed, terminated and etc. In this chapter a Bayesian formulation for tracking to solve target existence problem will be proposed. It will also be shown that the proposed formulation results into IPDA steps. Taking that as the base, the formulation will be extended to smoothing and the original contribution of AS-IPDA smoother will be presented.

4.3 Bayes' Definition of Target Existence Uncertainty

Generally, the state update is carried out by calculating the posterior density $p(x_k|y^k)$ where y^k is the collection of all the validated measurements up to $t = k$. The definition, by itself, takes the existence of target as given. Therefore to resolve target existence, it needs to be introduced in the definition as an event and the resultant joint posterior density provides the Bayesian definition of tracking involving target existence uncertainty scenario. The joint posterior density is defined by

$$p(x_k, E_k = 1|y^k) \tag{4.1}$$

where

- $E_k = 1$ refers to the event that the target exists at $t = k$
- $E_k = 0$ refers to the event that the target does not exist at $t = k$

and y^k is the set of all validated measurement till $t = k$.
The joint density can be expanded as

$$p(x_k, E_k = 1|y^k) = p(x_k|E_k = 1, y^k)p(E_k = 1|y^k) \tag{4.2}$$

The two components on the right hand side of (4.2) capture the notion of target existence uncertainty problem. The second term $p(E_k = 1|y^k)$ is the probability of "target existence" event based on which the tracker decides the status of the target while the first term calculates the state density only if the target exists ensured by the condition on $E_k = 1$.

Expanding the state update part of (4.2) using Bayes' theorem, we get the familiar Bayesian recursion formula.

$$\begin{aligned} p(x_k|E_k = 1, y^k) &= p(x_k|E_k = 1, y_k, y^{k-1}) \\ &= \frac{p(y_k|x_k, E_k = 1, y^{k-1})p(x_k|E_k = 1, y^{k-1})}{p(y_k|E_k = 1, y^{k-1})} \\ &= \frac{Likelihood \times Prediction}{Normalization} \end{aligned} \tag{4.3}$$

where y_k refers to the set of validated measurements at $t = k$.

Solving (4.3) for linear Gaussian systems will yield IPDAF while the probability $p(E_k = 1|y^k)$ gives the tracker an automatic way to discriminate between true and false tracks. Both of these will be solved in the next sections.

4.4 IPDAF Algorithm

In this section each of the terms - likelihood, prediction and normalization - will be calculated separately and associated parameters and/or models will be introduced in due manner.

4.4.1 IPDAF Prediction

The prediction density $p(x_k|E_k = 1, y^{k-1})$ can be expanded using total probability theorem and is given by

$$\begin{aligned} & p(x_k|E_k = 1, y^{k-1}) \\ = \; & p(x_k, E_{k-1} = 1|E_k = 1, y^{k-1}) + p(x_k, E_{k-1} = 0|E_k = 1, y^{k-1}) \\ = \; & p(x_k|E_{k-1} = 1, E_k = 1, y^{k-1})P(E_{k-1} = 1|E_k = 1, y^{k-1}) \\ & + p(x_k|E_{k-1} = 0, E_k = 1, y^{k-1})P(E_{k-1} = 0|E_k = 1, y^{k-1}) \end{aligned} \tag{4.4}$$

The terms in $p(E_{k-1} = 1|E_k = 1, y^{k-1})$ and $p(E_{k-1} = 0|E_k = 1, y^{k-1})$ in (4.4) are backward transition probabilities of target existence event. To resolve these probabilities, IPDAF introduces a Markov transition probability matrix for two events - "target existence" and "target non-existence", as follows

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{10} \\ \Gamma_{01} & \Gamma_{00} \end{bmatrix} \tag{4.5}$$

where $\Gamma_{ij} = P(E_k = j|E_{k-1} = i)$ and $\Gamma_{ii} + \Gamma ji = 1$.

Now expanding (4.4) using the transition probabilities and using total probability, we get

$$\begin{aligned} & p(x_k|E_k = 1, y^{k-1}) \\ = \; & \left[\int_{x_{k-1}} p(x_k|x_{k-1}, E_{k-1} = 1, E_k = 1, y^{k-1}) p(x_{k-1}|E_{k-1} = 1, E_k = 1, y^{k-1}) dx_{k-1} \right] \\ & P(E_{k-1} = 1|E_k = 1, y^{k-1}) \\ + \; & p(x_k|E_{k-1} = 0, E_k = 1, y^{k-1})P(E_{k-1} = 0|E_k = 1, y^{k-1}) \\ = \; & \left[\int_{x_{k-1}} p(x_k|x_{k-1}, E_{k-1} = 1, E_k = 1, y^{k-1}) p(x_{k-1}|E_{k-1} = 1, E_k = 1, y^{k-1}) dx_{k-1} \right] \\ & \frac{P(E_k = 1|E_{k-1} = 1).P(E_{k-1} = 1|y^{k-1})}{P(E_k = 1|y^{k-1})} \\ + \; & p(x_k|E_{k-1} = 0, E_k = 1, y^{k-1}) \frac{P(E_k = 1|E_{k-1} = 0).P(E_{k-1} = 0|y^{k-1})}{P(E_k = 1|y^{k-1})} \\ = \; & \left[\int_{x_{k-1}} p(x_k|x_{k-1}, E_{k-1} = 1, E_k = 1, y^{k-1}) p(x_{k-1}|E_{k-1} = 1, E_k = 1, y^{k-1}) dx_{k-1} \right] \\ & \frac{\Gamma_{11} P(E_{k-1} = 1|y^{k-1})}{P(E_k = 1|y^{k-1})} \end{aligned}$$

$$+ \quad p(x_k|E_{k-1} = 0, E_k = 1, y^{k-1}) \frac{\Gamma_{01} P(E_{k-1} = 0|y^{k-1})}{P(E_k = 1|y^{k-1})} \tag{4.6}$$

Now going back to the first term in (4.4), it contains $p(E_{k-1} = 1|E_k = 1, y^{k-1})$ which denotes the probability of a track at $t = k - 1$ conditioned on its existence at current time $t = k$. This is a case of continuation of existing tracks. The second term however requires the probability $p(E_{k-1} = 0|E_k = 1, y^{k-1})$ that refers to the probability of a target that did not exist at $t = k-1$ conditioned on its existence at $t = k$. This clearly indicated a case of new track. IPDAF deals with the situation by modeling the Markov switching between two states - "target existence" and "target non-existence" as follows :

$$\Gamma = \begin{bmatrix} \Gamma_{11} & 1 - \Gamma_{11} \\ 0 & 1 \end{bmatrix} \tag{4.7}$$

which clearly indicates that once a target in out of existence, it will not come into existence. Taking that into account, the second term in (4.6) becomes zero. Moreover, the term $P(E_k = 1|y^{k-1})$ can be expanded and evaluated as

$$\begin{aligned} P(E_k = 1|y^{k-1}) &= \Gamma_{11} P(E_{k-1} = 1|y^{k-1}) + \Gamma_{01} P(E_{k-1} = 0|y^{k-1}) \\ &= \Gamma_{11} P(E_{k-1} = 1|y^{k-1}) + 0 \\ &= \Gamma_{11} P(E_{k-1} = 1|y^{k-1}) \end{aligned} \tag{4.8}$$

Therefore by substituting (4.8) in (4.6) and following the discussion above the prediction density becomes

$$\begin{aligned} & p(x_k|E_k = 1, y^{k-1}) \\ = & \int_{x_{k-1}} p(x_k|x_{k-1}, E_{k-1} = 1, E_k = 1, y^{k-1}) p(x_{k-1}|E_{k-1} = 1, E_k = 1, y^{k-1}) dx_{k-1} \\ = & \int_{x_{k-1}} \mathcal{N}(x_k; F x_{k-1}, Q_k) \mathcal{N}(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1}) dx_{k-1} \end{aligned} \tag{4.9}$$

where

- $\hat{x}_{k-1|k-1}$ and $P_{k-1|k-1}$ are respectively previous state estimate and associated error covariance.
- F and Q_k are target dynamic model parameters (discussed in section 2.5).

The expression in (4.9) is standard Chapman-Kolmogorov equation and results into a normal distribution as in standard KF prediction step (discussed in section 2.5.1). Therefore, the prediction density is given by

$$p(x_k|E_k = 1, y^{k-1}) = \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \tag{4.10}$$

where

$$\hat{x}_{k|k-1} = F\hat{x}_{k-1|k-1} \tag{4.11}$$
$$P_{k|k-1} = FP_{k|k-1}F^T + Q_k \tag{4.12}$$

4.4.2 IPDAF Likelihood

Similar to PDAF, the measurements are validated using gating technique explained in section 3.4. Therefore at any given instance, there are m_k number of validated measurements. The set of these validated measurements are referred as

$$y_k = \{y_k(1), y_k(2), \dots, y_k(m_k)\} \tag{4.13}$$

where $y_k(n)$ refers to n-th validated measurements at $t = k$.

Expanding the likelihood $p(y_k|x_k, E_k = 1, y^{k-1})$ into component measurements in the set, we get

$$\begin{aligned} & p(y_k|x_k, E_k = 1, y^{k-1}) \\ = \; & p(y_k(1), y_k(2), \dots, y_k(m_k)|x_k, E_k = 1, y^{k-1}) \end{aligned} \tag{4.14}$$

For the validated m_k number of measurements, the following hypotheses are assumed,

- $\theta_k(0)$: no validated measurement is from target
- $\theta_k(i)$: only i-th measurement is from target while the other validated measurements are from clutter where $i = 1, 2, \dots, m_k$.

The probabilities of these mutually exclusive events are given by

$$P(\theta_k(0)) = 1 - P_D P_G \tag{4.15}$$
$$P(\theta_k(i)) = \frac{P_D P_G}{m_k} \tag{4.16}$$

where P_D,P_G are detection and gating probabilities respectively (introduced in section 3.4.2).

Moreover, it is assumed that the number of clutter measurements received at certain scan conditioned on past measurements follow a Poisson distribution given by

$$P(m_k|y^{k-1}) = P_0(m_k) = \frac{e^{-\lambda}\lambda^{m_k}}{m_k!} \tag{4.17}$$

where λ is the mean of the distribution referring to the number of expected clutter measurements per scan within the validation gate. The expression is given by

$$\lambda = \begin{cases} 0 & m_k = 0 \\ m_k - P_D P_G P(E_k = 1|y^{k-1}) & m_k > 0 \end{cases} \tag{4.18}$$

where $P(E_k = 1|y^{k-1})$ is predicted probability of target existence and is given by (4.8).

Except the conditioning on the existence of the target, the likelihood in (4.14) is similar to the one for PDAF (3.6). Therefore, introducing the same logic as stated in standard PDAF, the number of validated measurement is also introduced as a condition. The likelihood then can be expanded as

$$\begin{aligned}
& p(y_k|x_k, E_k = 1, y^{k-1}) \\
= & \sum_{i=0}^{m_k} p(y_k, \theta_k(i), m_k|x_k, E_k = 1, y^{k-1}) \\
= & \sum_{i=0}^{m_k} p(y_k|\theta_k(i), x_k, E_k = 1, m_k, y^{k-1}) \\
& \qquad \times P(\theta_k(i), m_k|x_k, E_k = 1, y^{k-1}) \\
= & \sum_{i=0}^{m_k} p(y_k|\theta_k(i), m_k, x_k, E_k = 1, y^{k-1}) \\
& \qquad \times P(\theta_k(i)|x_k, E_k = 1, m_k, y^{k-1}) P(m_k|x_k, E_k = 1, y^{k-1}) \\
= & \sum_{i=0}^{m_k} p(y_k|\theta_k(i), x_k, E_k = 1) \\
& \qquad \times P(\theta_k(i)|x_k, E_k = 1, m_k, y^{k-1}) P(m_k|x_k, E_k = 1, y^{k-1}) \\
= & \sum_{i=0}^{m_k} p(y_k|\theta_k(i), x_k, E_k = 1) \\
& \qquad \times P(\theta_k(i)|x_k, E_k = 1, m_k) P(m_k|y^{k-1})
\end{aligned} \tag{4.19}$$

Now the first term in summation (4.19) refers to the probability that $i-th$ measurement is target originated while the rest is clutter. Assuming a uniform spatial clutter density, the probability is given by,

$$
\begin{aligned}
& p(y_k|\theta_k(i), x_k, E_k = 1) \\
= & \begin{cases} (\frac{1}{V_k})^{m_k} & i = 0 \\ (\frac{1}{V_k})^{m_k - 1} p(y_k(i)|x_k) & i = 1, 2, \ldots, m_k \end{cases}
\end{aligned} \tag{4.20}
$$

Substituting (4.15),(4.16),(4.17) and (4.20) in (4.19), the likelihood is calculated as

$$
\begin{aligned}
& p(y_k|x_k, E_k = 1, y^{k-1}) \\
= & (\frac{1}{V_k})^{m_k} P_0(m_k) P(\theta_k(0)|x_k, E_k = 1, m_k) \\
& + \sum_{i=1}^{m_k} (\frac{1}{V_k})^{m_k - 1} p(y_k|\theta_k(i), x_k, E_k = 1) P(\theta_k(i)|x_k, E_k = 1, m_k) P_0(m_k - 1) \\
= & (\frac{1}{V_k})^{m_k} P_0(m_k)(1 - P_D P_G) + \sum_{i=1}^{m_k} (\frac{1}{V_k})^{m_k - 1} p(y_k(i))|x_k) \frac{P_D P_G}{m_k} P_0(m_k - 1) \\
= & (\frac{1}{V_k})^{m_k} P_0(m_k)(1 - P_D P_G) + \frac{P_D P_G}{m_k} P_0(m_k) \frac{m_k}{\lambda} (\frac{1}{V_k})^{m_k - 1} \sum_{i=1}^{m_k} p(y_k(i))|x_k) \\
& [\text{from } (4.17) P_0(m_k - 1) = P_0(m_k) \frac{m_k}{\lambda}] \\
= & (\frac{1}{V_k})^{m_k} P_0(m_k) \left[1 - P_D P_G + P_D P_G \frac{V_k}{\lambda} \sum_{i=1}^{m_k} p(y_k(i))|x_k) \right]
\end{aligned} \tag{4.21}
$$

The density in the summation results into a normal (under linear Gaussian sensor model) distribution and is given as $p(y_k(i))|x_k) = \mathcal{N}(y_k(i); Hx_k, R_k)$. The likelihood density is thus given by

$$
\begin{aligned}
& p(y_k|x_k, E_k = 1, y^{k-1}) \\
= & (\frac{1}{V_k})^{m_k} P_0(m_k) \left[1 - P_D P_G + P_D P_G \frac{V_k}{\lambda} \sum_{i=1}^{m_k} \mathcal{N}(y_k(i); Hx_k, R_k) \right]
\end{aligned} \tag{4.22}
$$

4.4.3 IPDAF Normalization

The normalization density $p(y_k|E_k = 1, y^{k-1})$ can be expanded in terms of likelihood and prediction densities and calculated as

$$
\begin{aligned}
& p(y_k|E_k = 1, y^{k-1}) \\
= & \int_{x_k} p(y_k|x_k, E_k = 1, y^{k-1}) p(x_k|E_k = 1, y^{k-1}) dx_k \\
= & \int_{x_k} \left((\frac{1}{V_k})^{m_k} P_0(m_k) \left[1 - P_D P_G + P_D P_G \frac{V_k}{\lambda} \sum_{i=1}^{m_k} \mathcal{N}(y_k(i); Hx_k, R_k) \right] \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \right) dx_k
\end{aligned}
$$

$$
\begin{aligned}
&= (\frac{1}{V_k})^{m_k} P_0(m_k)(1 - P_D P_G) \int_{x_k} \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) dx_k \\
&+ (\frac{1}{V_k})^{m_k} P_0(m_k) P_D P_G \frac{V_k}{\lambda} \int_{x_k} \sum_{i=1}^{m_k} \mathcal{N}(y_k(i); Hx_k, R_k) \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) dx_k \qquad (4.23)
\end{aligned}
$$

The term in the integral is shown to result in a normal given by $\mathcal{N}(y_k(i); \hat{y}_k, S_k(i))$ where

$$
\begin{aligned}
\hat{y}_k &= H\hat{x}_{k|k-1} \qquad (4.24)\\
S_k(i) &= HP_{k|k-1}H^T + R_k \qquad (4.25)
\end{aligned}
$$

The resultant normalization density becomes

$$
\begin{aligned}
&p(y_k|E_k = 1, y^{k-1}) = \delta \\
&= (\frac{1}{V_k})^{m_k} P_0(m_k)(1 - P_D P_G) + (\frac{1}{V_k})^{m_k} P_0(m_k) P_D P_G \frac{V_k}{\lambda} \int_{x_k} \sum_{i=1}^{m_k} \mathcal{N}(y_k(i); \hat{y}_k, S_k(i)) dx_k \\
&= (\frac{1}{V_k})^{m_k} P_0(m_k)(1 - P_D P_G) + (\frac{1}{V_k})^{m_k} P_0(m_k) P_D P_G \frac{V_k}{\lambda} \sum_{i=1}^{m_k} \mathcal{N}(y_k(i); \hat{y}_k, S_k(i)) \qquad (4.26)
\end{aligned}
$$

4.4.4 IPDAF State Update

With the calculation of likelihood, prediction and normalization density in previous sections, the state recursion in (4.3) can be carried out as

$$
\begin{aligned}
&p(x_k|E_k = 1, y^k) \\
&= \frac{1}{\delta}(\frac{1}{V_k})^{m_k} P_0(m_k) \left[1 - P_D P_G + P_D P_G \frac{V_k}{\lambda} \sum_{i=1}^{m_k} \mathcal{N}(y_k(i); Hx_k, R_k)\right] \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \\
&= \frac{1}{\delta}(\frac{1}{V_k})^{m_k} P_0(m_k)(1 - P_D P_G) \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \\
&\quad + \frac{1}{\delta}(\frac{1}{V_k})^{m_k} P_0(m_k) P_D P_G \frac{V_k}{\lambda} \sum_{i=1}^{m_k} \left[\mathcal{N}(y_k(i); Hx_k, R_k) \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})\right] \\
&= \frac{1}{\delta}(\frac{1}{V_k})^{m_k} P_0(m_k)(1 - P_D P_G) \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \\
&\quad + \frac{1}{\delta}(\frac{1}{V_k})^{m_k} P_0(m_k) P_D P_G \frac{V_k}{\lambda} \sum_{i=1}^{m_k} \mathcal{N}(y_k(i); \hat{y}_k, S_k(i)) \mathcal{N}(x_k; x_{k|k}(i), P_{k|k}(i)) \\
&\quad \text{[following [34]]} \\
&= \sum_{i=0}^{m_k} \beta_k(i) \mathcal{N}(x_k; \hat{x}_{k|k}(i), P_{k|k}(0)) \qquad (4.27)
\end{aligned}
$$

where

$$x_{k|k}(0) = \hat{x}_{k|k-1} \tag{4.28}$$

$$P_{k|k}(0) = P_{k|k-1} \tag{4.29}$$

$$\beta_k(0) = \frac{1 - P_D P_G}{1 - \delta_k} \tag{4.30}$$

$$\beta_k(i) = \frac{P_D P_G \frac{V_k}{\lambda} \mathcal{N}(y_k(i); \hat{y}_k, S(i))}{1 - \delta_k} P_D P_G \frac{V_k}{\lambda} \mathcal{N}(y_k(i); \hat{y}_k, S(i)) \tag{4.31}$$

$$\delta_k = P_D P_G \frac{V_k}{\lambda} \sum_{i=0} m_k \mathcal{N}(y_k(i); \hat{y}_k, S_k(i)) \tag{4.32}$$

The resultant density is approximated as a normal distribution with first two moments

$$\hat{x}_{k|k} = \sum_{i=0}^{m_k} \beta_k(i) \hat{x}_{k|k}(i) \tag{4.33}$$

$$P_{k|k} = \sum_{i=0}^{m_k} \beta_k i P_{k|k}(i) + \sum_{i=0}^{m_k} \beta_k(i) \hat{x}_{k|k}(i) \hat{x}_{k|k}(i)^T - \hat{x}_{k|k} \hat{x}_{k|k}^T \tag{4.34}$$

The expressions in (4.33) and (4.34) complete the state update recursion of IPDA algorithm. The recursion of target existence probability is carried out in next section to complete the automation of track maintenance.

4.4.5 IPDAF : Target Existence Probability Update

The tracker starts with prior probability of existence $P(E_{k-1} = 1|y^k)$ and updates it with the expression derived here. The Bayes' theorem applied on the required probability expression $P(E_k = 1|y^k)$ yields

$$\begin{aligned} & P(E_k = 1|y^k) = P(E_k = 1|y_k, y^{k-1}) \\ = & \frac{p(y_k|E_k = 1, y^{k-1}) P(E_k = 1|y^{k-1})}{p(y_k|y^{k-1})} \\ = & \frac{p(y_k|E_k = 1, y^{k-1}) P(E_k = 1|y^{k-1})}{p(y_k|E_k = 1, y^{k-1}) P(E_k = 1|y^{k-1}) + p(y_k|E_k = 0, y^{k-1}) P(E_k = 0|y^{k-1})} \end{aligned} \tag{4.35}$$

The measurement density conditioned on target existence $p(y_k|E_k = 1, y^{k-1})$ is given by

$$\begin{aligned} p(y_k|E_k = 1, y^{k-1}) &= \int_{x_k} p(y_k|x_k, E_k = 1, y^{k-1}) p(x_k, E_k = 1, y^{k-1}) dx_k \\ &= \delta \end{aligned} \tag{4.36}$$

On the other hand the measurement density conditioned on target non-existence refers to the clutter density and therefore results in

$$p(y_k|E_k = 0, y^{k-1}) = (\frac{1}{V_k})^{m_k} P_0(m_k) \tag{4.37}$$

Noting that

$$\delta = (\frac{1}{V_k})^{m_k}(1 - \delta_k) \tag{4.38}$$

$$P(E_k = 0|y^{k-1}) = 1 - P(E_k = 1|y^{k-1}) \tag{4.39}$$

the existence probability update in (4.35) becomes

$$P(E_k = 1|y^k) = \frac{(1 - \delta_k)P(E_k = 1|y^{k-1})}{1 - \delta_k P(E_k = 1|y^{k-1})} \tag{4.40}$$

4.4.6 IPDAF Algorithm : Summarized

Following Bayesian approach for target existence model, the equations of IPDAF are derived in previous sections. One iteration of the algorithm is summarized in following steps:

1. **Step 1 : Prediction**

$$\begin{aligned}
\hat{x}_{k|k-1} &= F\hat{x}_{k-1|k-1} \\
P_{k|k-1} &= FP_{k-1|k-1}F^T + Q_k \\
P(E_k = 1|y^{k-1}) &= \Gamma_{11}P(E_{k-1} = 1|y^{k-1}) \\
\hat{y}_k &= H\hat{x}_{k|k-1} \\
S_k(i) &= HP_{k|k-1}H^T + R_k \\
K_k &= P_{k|k-1}H^T S_k^{-1}(i)
\end{aligned} \tag{4.41}$$

2. **Step 2 : Gating**

   Of all the observations received, select those which satisfy the distance test

$$[y_k(n) - \hat{y}_k]^T S_k^{-1}(i)[y_k(n) - \hat{y}_k] < \gamma \tag{4.42}$$

   where γ is the threshold to maintain P_G probability of gating and $n = 1, 2, 3, \ldots,$ number of received measurements.

3. **Step 3 : Update**

$$
\begin{aligned}
\hat{x}_{k|k} &= \sum_{i=0}^{m_k} \beta_k(i)\hat{x}_{k|k}(i) \\
P_{k|k} &= \sum_{i=0}^{m_k} \beta_k i P_{k|k}(i) + \sum_{i=0}^{m_k} \beta_k(i)\hat{x}_{k|k}(i)\hat{x}_{k|k}(i)^T - \hat{x}_{k|k}\hat{x}_{k|k}^T
\end{aligned}
\tag{4.43}
$$

where m_k is the number of validated measurement in step 2 and

$$
\begin{aligned}
\beta_k(0) &= \frac{1 - P_D P_G}{1 - \delta_k} \\
\beta_k(i) &= \frac{P_D P_G \frac{V_k}{\lambda}\mathcal{N}(y_k(i); \hat{y}_k, S_k(i))}{1 - \delta_k}\mathcal{N}(y_k(i); \hat{y}_k, S_k(i)) \\
\delta_k &= P_D P_G \frac{V_k}{\lambda} \sum_{i=0} m_k \mathcal{N}(y_k(i); \hat{y}_k, S_k(i)) \\
P(E_k = 1|y^k) &= \frac{(1 - \delta_k)P(E_k = 1|y^{k-1})}{1 - \delta_k P(E_k = 1|y^{k-1})} \\
\hat{x}_{k|k}(0) &= \hat{x}_{k|k-1} \\
P_{k|k}(0) &= P_{k|k-1} \\
\hat{x}_{k|k}(i) &= \hat{x}_{k|k-1} + K_k(y_k(i) - \hat{y}_k) \\
P_{k|k}(i) &= (I - K_k H)P_{k|k-1}
\end{aligned}
$$

The updated probability of target existence $P(E_k = 1|y^k)$ helps decide on the track status while $\hat{x}_{k|k}$ and $P_{k|k}$ give the updates state estimate.

4.5 Integrated Probabilistic Data Association (IPDA) Smoothing

The proposed IPDA smoothing in this thesis follows the same principle and philosophy defined by IPDA filter. In this section the models associated with the smoothing framework will be discussed and the formulation of the smoothing algorithm will be detailed.

4.5.1 Augmented State Target Dynamic Model

The target dynamic model is an augmentation of states from time $t = k$ to $t = k - N$ where N is the fixed lag. Therefore the smoothing uses target dynamic model of

$$
\mathbf{X}_k = \mathbf{F}_k \mathbf{X}_{k-1} + \mathbf{V}(k) \tag{4.44}
$$

where the parameters are as defined in 2.6.2.

4.5.2 Augmented State Sensor Model

The sensor observation model for the augmented state is modified as

$$y_k = \mathbf{H}_k \mathbf{X}_k + w_k \tag{4.45}$$

where the parameters are as defined in 2.6.2

4.5.3 Target Existence Model

As the states in the augmented vector correspond to the ones from time $t = k - N$ to $t = k$, the target itself may switch in between the possible events of "existence" and "non-existence" within that time interval. A simple transition matrix defined for switching between $\mathbf{X}_{k-1}$ and $\mathbf{X}_k$ does not capture the fact. For this reason, several hypotheses are proposed each consisting a transition between target "existence" and "non-existence".
According to the transition probability defined in (4.7), if a target goes out of "existence" (that is it was decided to be "non-existence" at some point of time), it cannot come into "existence" again and also if a target is decided to be "non-existing" once, it will remain that way. Under the above constraint, there exist $N + 2$ possible ways for target to switch between the "existence" and "non-existence" within the time interval from $t = k - N$ to $t = k$. Each of these ways are termed as possible "**Hypothesis**". Each of these hypotheses contains the joint state and existence of a target. The "**Hypotheses**" are defined mathematically as

- **Hypothesis $\mathbf{H}_k^m$** : *Target existed from time $t = k - N$ to $t = k - m$ but not from $t = k - m + 1$ to $t = k$*

$$\mathbf{H}_k^m = [\mathbf{X}_k^m, \mathbf{E}_k^m] = \begin{bmatrix} \phi_k, \overline{E}_k \\ \vdots \\ \phi_{k-m}, \overline{E}_{k-m} \\ x_{k-m-1}, E_{k-m} \\ \vdots \\ x_{k-N}, E_{k-N} \end{bmatrix}$$

  where $m = 0, 1, 2, \ldots, N$

- **Hypothesis $\overline{\mathbf{H}}_k$** : *Target did not exist anytime within the interval $t = k - N$ to $t = k$*

$$\begin{aligned}\overline{\mathbf{H}}_k &= [\mathbf{\Phi}_k, \overline{\mathbf{E}}_k] \\ &= \begin{bmatrix} \phi, \overline{E}_k \\ \vdots \\ \phi, \overline{E}_{k-N} \end{bmatrix}\end{aligned}$$

The above mentioned $N+2$ hypotheses cover all the possibilities of a target transition between the "existence" and "non-existence" event. The models will eventually be utilized for smoothing of the target existence probability and hence for the development of the smoother algorithm.

4.5.4 Bayesian Formulation of AS-IPDA Smoothing

With the models specifically defined as in section 4.5.1 through 4.5.3, the Bayesian definition of target tracking with target existence uncertainty for smoothing estimation can be defined as

$$p(\mathbf{X}_k, \mathbf{E}_k^0 | y^k) \tag{4.46}$$

Going along the same methodology of standard IPDA filter, the smoother tries to determine the joint density of augment state and existence. But the decision of target existence for smoother, unlike filter, does not depend only on the hypothesis $\mathbf{E}_k$ (that the target existed for whole duration of the specified fixed lag) and $\mathbf{E}_k^n$ (that the target did not exist at any time during the interval) but also on other possible existence hypotheses as mentioned in section 4.4.4. Therefore the smoother needs to determine the probability of each existence hypothesis $p(\mathbf{E}_k^0|y^k), \ldots, p(\mathbf{E}_k^N|y^k), p(\overline{\mathbf{E}}_k|y^k)$. Based on these hypotheses probabilities, the component existence probabilities can be calculated as

$$p(E_{k-m}|y^k) = \sum_{j=0}^{m} p(\mathbf{E}_k^j|y^k) \tag{4.47}$$

where $m = 0, 1, \ldots, N$.

Here the expression in (4.47) refers to the smoothed existence probabilities (except for $m = 0$ for which the expression is the filter estimate). Based on the re-estimated probability for target existence using more observation, a better estimate is possible and hence a better decision can be taken.

The state estimate is based on the decision of target existence. Hence, the state estimate, similar to IPDA filter, is given by

$$p(\mathbf{X}_k|\mathbf{E}_k^0, y^k) \tag{4.48}$$

The estimate (4.48) is conditioned on target existence hypothesis which is also consistence with standard IPDA technique.

4.6 Formulation of the Smoother

The development of the smoother requires the solution of two steps of the Bayesian definition of the problem :

1. Firstly : The probabilities of the existence hypotheses. These will in turn give the smoothed existence probabilities at each time within the interval through (4.47).
2. Secondly : If the target is decided to be existing, the conditional state estimate is given by $p(\mathbf{X}_k|\mathbf{E}_k^0, y^k)$. As the augmented state contains, besides current state x_k, all the states within the fixed interval of N, the Bayesian update will filter the current state and smooth the states at previous time instances.

The complete algorithm is devised based on the systematic interpretation of the above mentioned two stages. In the following discussions, the calculated probabilities will be presented and therefore a technique will emerge in an algorithmic form.

4.7 Smoothing of Existence Probabilities

Firstly, the probabilities of the augmented existence hypotheses are described. The definitions of these probabilities, from a Bayesian approach, are

$$p(\mathbf{E}_k^m|y^k) \tag{4.49}$$

$$(m = 0, 1, 2, \ldots, N)$$

$$p(\overline{\mathbf{E}}_k|y^k) \tag{4.50}$$

$$\tag{4.51}$$

where $p(\overline{\mathbf{E}}_k|y^k)$ is the probability of the hypothesis that the target did not exist at any time within the time interval. The probabilities are derived in appendix B. The resultant expressions are as given here

$$p(\mathbf{E}_k^0|y^k) = \frac{1}{\Delta}\delta\Gamma_{11}p(\mathbf{E}_{k-1}^0|y^{k-1}) \tag{4.52}$$

$$p(\mathbf{E}_k^1|y^k) = \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)\Gamma_{10}p(\mathbf{E}_{k-1}^0|y^{k-1}) \tag{4.53}$$

$$p(\mathbf{E}_k^m|y^k) = \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)\Gamma_{00}p(\mathbf{E}_{k-1}^{m-1})$$

$$(4.54)$$

$(m = 2, \ldots, N)$

where

$$\begin{aligned}\Delta &= p(y_k|\mathbf{E}_k, y^{k-1}).p(\mathbf{E}_k|y^{k-1}) \\ &+ \left[\sum_{m=0}^{N-1} p(y_k|\mathbf{E}_k^m, y^{k-1}).p(\mathbf{E}_k^m|y^{k-1})\right] \\ &+ p(y_k|\mathbf{E}_k^n, y^{k-1}).p(\mathbf{E}_k^n|y^{k-1})\end{aligned} \tag{4.55}$$

If the transition matrix is defined as in (4.7), with the specific value $\Gamma_{01} = 0$ and $\Gamma_{00} = 1$, the expressions (4.54) can be further simplified as

$$p(\mathbf{E}_k^m|y^k) = \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)p(\mathbf{E}_{k-1}^{m-1}) \tag{4.56}$$

Once these probabilities are available, it is straightforward to compute the probabilities of component target "existence" event at each instant of the interval through (4.47). The decision on target existence at any earlier time instant is then possible to be re-estimated using the smoothed existence probabilities derived in the above mentioned fashion.

4.8 AS-IPDA Smoothing Algorithm

The posterior density of augmented state conditioned on existence is the same as suggested by AS-PDAF algorithm described in section 3.4.5 and will closely follow the derivation steps of IPDAF as given in section 4.4 (and subsections therein). The resultant smoothing algorithm is therefore summarized in following steps.

The smoothing starts with the a priori state estimate $\hat{\mathbf{X}}_{k-1|k-1}$ and associated error covariance $\mathbf{P}_{k-1|k-1}$. Then the following steps are carried out in steps.

1. **Step 1:** An one step prediction is carried out on augmented state, measurement and their associated covariances

$$\begin{aligned}\hat{\mathbf{X}}_{k|k-1} &= \mathbf{F}\hat{\mathbf{X}}_{k-1|k-1} \\ \mathbf{P}_{k|k-1} &= \mathbf{F}\mathbf{P}_{k-1|k-1}\mathbf{F}^T + \mathbf{Q}_k \\ \hat{Y}_k &= \mathbf{H}\hat{\mathbf{X}}_{k|k-1} \\ \mathbf{S}_k &= \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^T + R_k\end{aligned}$$

2. **Step 2:** Kalman gain $\mathbf{K}_k$ is also calculated through standard Kalman filter procedure

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}^T\mathbf{S}_k^{-1}$$

3. **Step 3:** To cater for clutter measurement, a gate of volume V_k is formed and measurements within the gate are called as validated ones. Data association coefficients are calculated for each of the validated measurements along with the hypothesis that no measurement is target originated.

$$\beta_k(i) = \begin{cases} \frac{1-P_D P_G}{1-\delta_k} & i = 0 \\ \frac{P_D P_G \frac{V_k}{\lambda}}{1-\delta_k}\mathcal{N}(y_k(i); \mathbf{H}\hat{\mathbf{X}}_{k|k-1}, \mathbf{S}_k) & i = 1, 2, \ldots, m_k \end{cases}$$

4. **Step 4:** Probabilities for augment existence hypotheses $\mathbf{E}_k^0, \ldots, \mathbf{E}_k^N$ are calculated through

$$\begin{aligned} p(\mathbf{E}_k^0|y^k) &= \frac{1}{\Delta}\delta\Gamma_{11}p(\mathbf{E}_{k-1}^0|y^{k-1}) \\ p(\mathbf{E}_k^1|y^k) &= \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)\Gamma_{10}p(\mathbf{E}_{k-1}^0|y^{k-1}) \\ p(\mathbf{E}_k^m|y^k) &= \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)p(\mathbf{E}_{k-1}^{m-1}) \\ (m = 2, \ldots, N) & \\ p(\mathbf{E}_k^n|y^k) &= \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)\left[\Gamma_{00}^N.\Gamma_{10}p(E_{k-N-1}|y^{k-1})\right. \\ & \left.+\Gamma_{00}^{N+1}(1 - p(E_{k-N-1}|y^{k-1}))\right] \end{aligned} \tag{4.57}$$

where

$$\begin{aligned} \Delta &= p(y_k|\mathbf{E}_k, y^{k-1}).p(\mathbf{E}_k|y^{k-1}) \\ &+ \left[\sum_{m=0}^{N-1} p(y_k|\mathbf{E}_k^m, y^{k-1}).p(\mathbf{E}_k^m|y^{k-1})\right] \\ &+p(y_k|\mathbf{E}_k^n, y^{k-1}).p(\mathbf{E}_k^n|y^{k-1}) \end{aligned}$$

5. **Step 5:**

   From the probabilities of augmented existence hypotheses, target existence probabilities are obtained through

$$p(E_{k-m}|y^k) = \sum_{j=0}^{m} p(\mathbf{E}_k^j|y^k)$$

6. **Step 6:** If the target is decided to be existing, state and associated covariance are updated by

$$\hat{\mathbf{X}}_{k|k} = \sum_{i=0}^{m_k} \beta_k(i)\hat{\mathbf{X}}_{k|k}(i)$$
$$\mathbf{P}_{k|k} = \sum_{i=0}^{m_k} \beta_k(i)\left(\mathbf{P}_{k|k}(i) + [\hat{\mathbf{X}}_{k|k}(i) - \hat{\mathbf{X}}_{k|k}][\hat{\mathbf{X}}_{k|k}(i) - \hat{\mathbf{X}}_{k|k}]^T\right)$$

   where $\beta_k(i), \mathbf{X}_{k|k}(i), \mathbf{P}_{k|k}(i)$ can be obtained from the expressions in section 3.4.5 by replacing the state vectors with augmented ones wherever appropriate.

   Otherwise the target may be terminated without updating the state (as the target is decided to be non-existing, there is no necessity to estimate its state).

The steps stated above signify that $\hat{\mathbf{X}}_{k-1|k-1}$ and $\mathbf{P}_{k-1|k-1}$ evolve into $\hat{\mathbf{X}}_{k|k}$ and $\mathbf{P}_{k|k}$ respectively ensuring that the process is strictly recursive. One single recursion of a tracker that implements AS-IPDA goes through the flow shown in figure 4.3.
An important point to note is that if the target is decided to be non-existing at the current time $t = k$, based on the filtered existence probability (obtained in the same iteration of the AS-IPDA), none of the observations obviously contains information about the state of the target in previous time. This translated into the fact that under such scenario, smoothing should not be carried out. Appendix C proves the claim mathematically. The result shows that for hypotheses $\mathbf{H}_k^m$ and $\overline{\mathbf{H}}_k$ (both of which cater for the fact that the target does not exist at current time), the state is retained as the previous one instead of being updated with the current measurements.

4.9 Conclusion

In this chapter, an original smoothing algorithm for automatic track initiation in clutter from Bayesian first principle is derived. First Bayesian model for automatic track maintenance is detailed. Then the model is extended to smoothing and the solution of the proposed smoothing model results into augmented state IPDA smoother. Associated assumptions for augmented target dynamic and sensor models are also devised in this chapter. The proposed smoother is capable of reestimating the target state along with the existence probability at a fixed lag. Using more observations help smoother to take more accurate decision about the status of the target - confirmed, terminated and etc.. along with better state estimation. The principal features of the proposed smoother are

1. For state estimation, the smoother follows the IPDA steps almost identically with the difference of state and measurement vectors replaced by augmented ones.

2. For target existence probability estimation, it first calculates the probabilities of augmented hypotheses and then the probability at each time instant within the fixed lag interval.

3. The smoother is strictly recursive making it easily implementable.

4. The smoother is also a direct extension of AS-PDAF discussed in chapter 3.

Following the derivation, the algorithm steps are cleared. A flow chart for one iteration of a tracker that implements the smoother is also presented in the chapter.

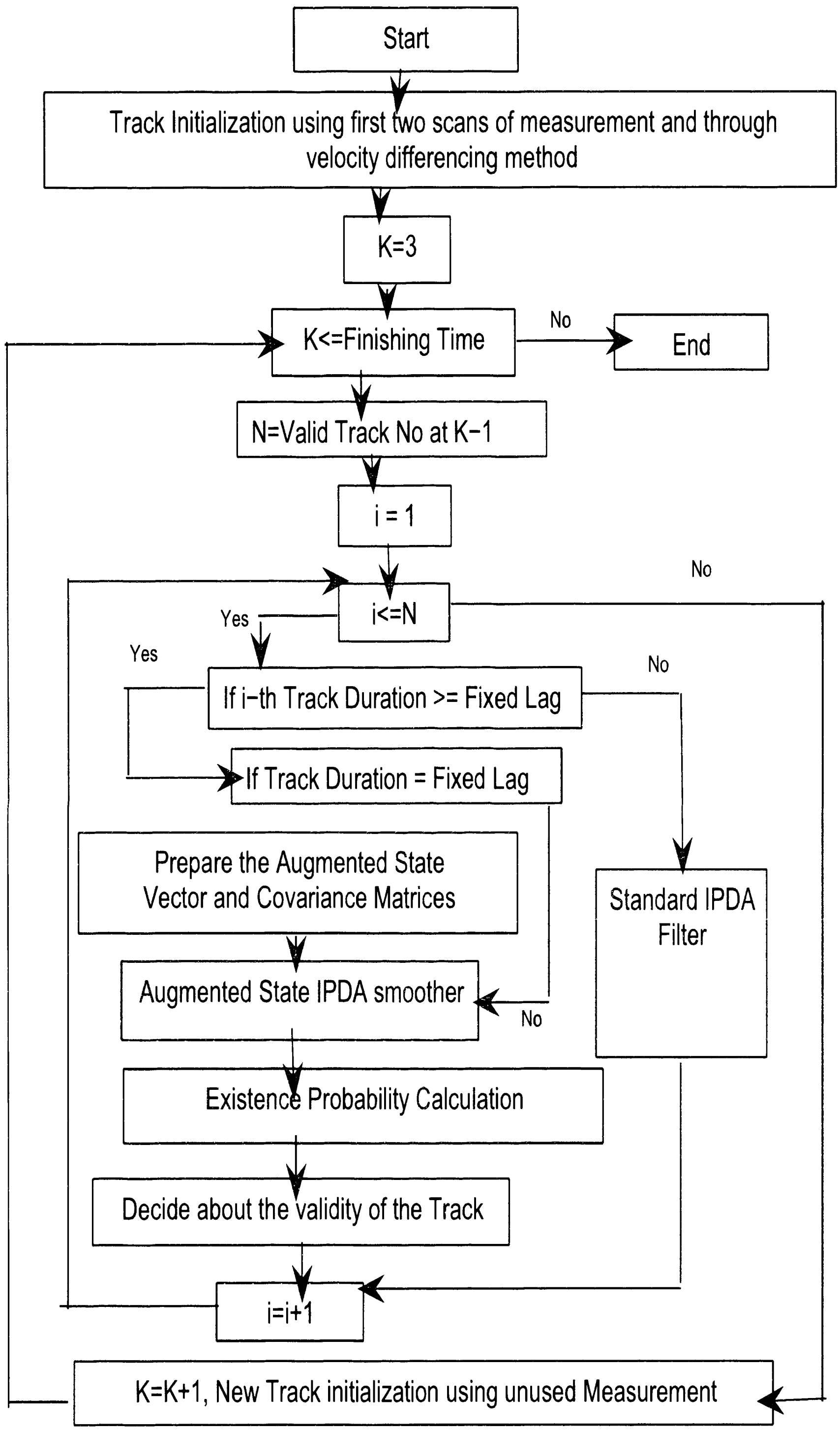

Fig. 4.3: Flow chart of IPDA smoothing

5. RANDOM SET SMOOTHER

5.1 Introduction

Random sets and in particular Finite Set Statistics (FISST) are recently proposed as the most appropriate framework for deriving Bayes' optimal, multi-target tracking algorithm. However most of the tracking filters that were derived from this framework are based on first and second order moments. The PHD filters proposed recently and their particle filter generalizations fall in this category.

One of the open questions was the fundamental connection between random set filters and the other Bayesian multi-target tracking algorithm. Establishing the connection is significant as the Random Set merely provides a framework for multi-target tracking, however its realization in practical systems is a result of a series of pragmatic approximations. Since such filters already exist and are known to perform well, it is imperative to find such a connection and establish its foundations. In [24], Challa et-al. have shown that the random set formalism leads to IPDA and its variants.

Here the result is extended to derive Generalized Random Set based smoothing algorithm. The significance of adopting random set formalism especially in multiple target tracking is reviewed. By modeling the target motion model as the union of the past states, a generalized smoother algorithm is devised. The associated random set smoother models are proposed and described. Finally the augmented state IPDA smoother of chapter 4 is derived from the random set formalism.

5.2 Background

The Bayesian probabilistic framework provides an adequate approach for dealing with Single Target Tracking. It is the most commonly accepted theoretical framework in the tracking community. The Bayesian updating paradigm is used in estimating the state of the single target in a recursive manner. These techniques generalize to the case of multiple targets when the number of targets is known [2] or is bounded by a known number [74, 41]. However, the fundamental difference between single target tracking and multitarget tracking is that the number of targets under consideration is not known in the latter. The number of target(s) is unknown a priori and is to be estimated. While the prescribed probabilistic framework can be used to model such a situation theoretically, the assessment of some of the components of the estimation model, such as the likelihood functions [47], is not easily done in practice. In 1994, Ronald Mahler introduced the Finite-Set Statistics (FISST) [50, 49], a multisensor-multitarget differential and integral calculus as a formalism to deal with the modeling difficulties in multi-target tracking. FISST [29, 35, 46] is based on the fact that belief-mass functions [69, 72, 43] are the multisensor-multitarget counterparts of probability-mass functions in the single target tracking models. Mahler's approach attempts to generalize the Bayesian framework into a formalism that can handle the multitarget, multisensor tracking problem. Using the FISST to develop implementable algorithms for multiple target tracking is not straightforward and approximations in the form of first order moments of the multitarget densities were proposed [48].

In [24], it is shown by Challa et. al. that the IPDA filter can be derived completely from the random set formalism for single target tracking. The focus was on establishing fundamental linkages between random set formalism and the IPDA algorithm. The IPDA algorithm is derived using FISST and insights are provided into the approximations of IPDA and their significance in addressing the target existence issue. The Integrated Probabilistic Data Association (IPDA) algorithm has been shown to out-perform other filters such as the IMM-PDA filter [16] and is therefore given more attention in the tracking community. IPDA explicitly models the target existence as a random variable and in so doing, it has made an attempt, albeit by accident, to capture the defining elements of random sets. The random set estimators that exist so far in literature, [24, 48, 49, 50, 56], consider only the filtered state estimation. In this chapter, we propose to continue the work done in [24] by deriving a random set approach for smoothing. Smoothing within the state estimation context is technically defined as a process where the current measurements are used to improve the estimates of the past states of the object of interest [66]. In the target tracking problem, this corresponds to estimating the past target states and associated tracker performance parameters.

Section 5.3 visits the random set filter formulation while section 5.4 discusses the random set filter models with target existence uncertainty. In section 5.6, target tracking with target existence uncertainty is described. In section 5.7 and 5.8, we derive the target dynamics and sensor model and deduce the Bayes update equation for the smoother to obtain the iterative global posterior density of the random set models. Finally, in section 5.9, we show that subject to certain simplifying assumptions and constraints, an IPDA smoother algorithm can be derived from the random set smoothing equations.

5.3 Formulation of Random Set Filter

Random set formulation requires the tool of Finite Set Statistics (FISST) ([50, 49]) and covered in appendix D. The key concept of FISST is the belief mass. The belief mass functions are non-additive generalizations of probability mass functions. Only on certain topological spaces, these belief masses behave the same as probability mass functions, [24]. In random set notation, the generalized multi-target dynamic and sensor models are Γ_k and Σ_k respectively while X_k and Y_k are realizations of them respectively. The models are defined as

- The target dynamic motion model

$$\Gamma_k = \Phi_k(X_{k-1}, V_{k-1}) \cup B_k(X_{k-1}) \tag{5.1}$$

  where $\Phi_k(.)$ models the dynamic transition of the target and $B_k(.)$ models the birth of the target.

- The sensor model

$$\Sigma_k = \Sigma_k^{'} \cup \Lambda_k \tag{5.2}$$

where $\Sigma_k^{'}$ denotes the target originated measurement and $\Lambda_k = \Lambda_k(1), \Lambda_k(2), \ldots, \Lambda_k(M)$ models the clutter measurements. Once the models for target dynamics and sensor measurements are available, the corresponding belief mass functions are calculated as

- The statistics of random set Γ_k is given by the belief mass

$$\beta_{\Gamma_{k|k-1}}(S|X_{k-1}) = Pr(\Gamma_k \subseteq S) \tag{5.3}$$

  This refers to the total probability of finding all targets in the region S given that they had a target state X_{k-1} at time $t = k - 1$.

- The belief mass of randomly varying finite measurement set Σ_k is given by

$$\beta_{\Sigma_k}(S|X_k) = \beta_{\Sigma_k^{'} \cup \Lambda_k}(S|X_k) = Pr(\Sigma_k^{'} \cup \Lambda_k \subseteq S) \tag{5.4}$$

  The belief mass function denotes the total probability of finding all the measurements is a given region S.

Differentiation of the belief masses in (5.3) and (5.4) above gives the Multi-target transition densities and likelihoods respectively. Using FISST, these densities can be obtained as below:

- Markov transition density

$$f_{k|k-1}(X_k|X_{k-1}) = \frac{\delta\beta_{\Gamma_{k|k-1}}(S|X_{k-1})}{\delta_{X_k}} \tag{5.5}$$

- Likelihood density

$$f_{\Sigma_k} = \frac{\delta\beta_{\Sigma_k}}{\delta Y_k} \tag{5.6}$$

The densities in (5.5) and (5.6) can now be used in standard Bayesian recursive equation to get the posterior target state density. The final posterior density is given by

$$f_{k|k}(X_k|Y^k) = \frac{1}{\Delta} f_{\Sigma_k} \times \int\limits_{X_{k-1}} f_{k|k-1}(X_k|X_{k-1}) f_{k-1|k-1}(X_{k-1}|Y^{k-1}) dX_{k-1} \tag{5.7}$$

The expressions in (5.3) through to (5.7) constitutes one iteration of a random set filter. In the next section random set model for tracking with target existence uncertainty will be developed leading to the derivation of IPDAF equations.

5.4 Random Set Filter Models for Target Existence Uncertainty

In random set notation, the target state is denoted as finite random set Γ_k with two realizations, $X_k = \{x_k\}$ (the target exists with state x_k) and $X_k = \phi$ (the target does not exist). The sensor reports the target originated measurement with probability P_D, while it picks up measurements from clutter with a probability P_{FA}. In random set notation, the sensor observation set Σ_k has realization $Y_k = \{Y_k(1), Y_k(2), \ldots, Y_k(m_k)\}$ where each $Y_k(i)$ denotes a measurement either from target or clutter and where $m_k = |Y_k|$ refers to the number of the collected measurements.

5.4.1 Markov Transition Density for Target Dynamics

From the general target dynamic model of (5.1), the birth process is referred to by $B_k(.)$. For the problem under consideration, at any given instant, only one target can be present. Therefore there is no birth process involved in target dynamics. In that case, the target dynamics reduces to

$$\Gamma_k = \Phi_k(X_{k-1}, V_{k-1}) \tag{5.8}$$

Under this constraint the realizations of Γ_k are

$$X_k = \{x_k\} \qquad X_k = \{\phi\} \tag{5.9}$$

where

- $X_k = \{x_k\}$ with probability p_v
- $X_k = \{\phi\}$ with probability $1 - p_v$

The discussion above establishes the fact that without the birth process, the process involves one target and it persists with probability p_v while it can vanish with probability $1 - p_v$. Also if there is no target present, it will continue to be case. Given the above mentioned model parameters, it is possible to calculate the belief mass function, $\beta_{\Gamma_{k|k-1}}(S|X_{k-1})$, which represents the total probability of finding the target in the region.

If $X_{k-1} = \{\phi\}$ (no target present), it will continue to be the case. Under that condition,

$$\beta_{\Gamma_{k|k-1}}(S|\phi) = Pr(\Gamma_k = \{\phi\} \subseteq S) = 1 \tag{5.10}$$

On the other hand, if $X_{k-1} = \{x_{k-1}\}$ (the target existed at $t = k-1$), it can either persist or vanish. Therefore the the belief mass function for this condition is

$$\begin{aligned}\beta_{\Gamma_{k|k-1}}(S|\{x_{k-1}\}) &= Pr(\Gamma_k = \{\phi\}|\{x_{k-1}\}) + Pr(\Gamma_k = \{x_k\}|\{x_{k-1}\}) \\ &= (1 - p_v) + p_v \int_S p(x_k|x_{k-1}) dx_{k-1} \end{aligned} \tag{5.11}$$

Differentiation of (5.10) and (5.11) give densities for each possible Markov transition in target states.

$$f(\phi|\phi) = 1 \tag{5.12}$$

$$f(\phi|\{x_{k-1}\}) = 1 - p_v \tag{5.13}$$

$$f(\{x_k\}|\{x_{k-1}\}) = p_v p(x_k|x_{k-1}) \tag{5.14}$$

5.4.2 Likelihood Densities

According to the sensor model (5.2), there are two parts in the finitely varying random set of Σ_k. These are the target originated measurement $\Sigma_k^{'}$ and the measurements from clutter Λ_k. Under the assumption that measurements from target and clutter are independent, the total belief mass function $\beta_{\Sigma_k}(S|Xk)$ can be denoted as a product of two component belief mass functions as

$$\begin{aligned}\beta_{\Sigma_k}(S|Xk) &= \beta_{\Sigma_k^{'}\cup\Lambda_k}(S|Xk) \\ &= Pr(\Sigma_k^{'} \subseteq S)P(\Lambda_k \subseteq S) \qquad (5.15)\end{aligned}$$

Analysis for each of them is done separately to deduce the complete belief mass function.

The realizations of $\Sigma_k^{'}$ are given by

- $\Sigma_k^{'} = \{y_k\}$ and
- $\Sigma_k^{'} = \{\phi\}$

Sensor received target originated measurement with a probability of P_D. Using this information, the belief mass function for the target originated measurement is given by

$$\begin{aligned}\beta_{\Sigma_k^{'}}(S|X_k) &= Pr(\Sigma_k^{'} \subseteq S|\{x_k\}) \\ &= Pr(\Sigma_k^{'} = \{\phi\}|\{x_k\}) + Pr(\Sigma_k^{'} = \{y_k\}|\{x_k\}) \\ &= 1 - P_D + P_D \int p(y_k|x_k)dy_k \qquad (5.16)\end{aligned}$$

Differentiation of (5.16) gives likelihood densities of measurements under the condition of target existence. These are

$$f(\phi|X_k) = 1 - P_D \qquad (5.17)$$

$$f(\{y_k\}|X_k) = P_D p(y_k|x_k) \qquad (5.18)$$

Now according to the model specification, the sensor receives false alarms with probability P_{FA}. Therefore with similar manipulation of the clutter model, the belief mass function of random set of observations generated by a clutter object is given by

$$\beta(\Lambda_k(i) \subseteq S) = \beta_C(S) = 1 - P_{FA} + P_{FA}p_c(S) \qquad (5.19)$$

with probability measure p for some spatial distribution c. Now, the total sensor model set $\beta_{\Sigma_k}(S|Xk)$ can be expanded using the belief masses of target and clutter generated measurements.

$$\begin{aligned}
& \beta_{\Sigma_k}(S|X_k) \\
= \; & Pr(\Sigma_k^{'}, \Lambda_k \subseteq S) \\
= \; & Pr(\Sigma_k^{'} \subseteq S|\{x_k\})P(\Lambda_k \subseteq S) \\
= \; & Pr(\Sigma_k^{'} \subseteq S)P(\Lambda_k(1), \Lambda_k(2), \ldots, \Lambda_k(M) \subseteq S) \\
= \; & Pr(\Sigma_k^{'} \subseteq S)P(\Lambda_k(1) \subseteq S)P(\Lambda_k(2) \subseteq S)\ldots P(\Lambda_k(M) \subseteq S) \\
= \; & \beta_{\Sigma_k'}(S|X_k)\beta_c(S)^M
\end{aligned} \tag{5.20}$$

The differentiation of (5.20) will give joint likelihood density. Using FISST product rule, the global likelihood density can be calculated as

$$\begin{aligned}
f_{\Sigma_k}(Y_k|X_k) &= f_{\Sigma_k' \cup \Lambda_k}(Y_k|X_k) \\
&= \sum_{Z_k \subseteq Y_k} f_{\Sigma_k'}(Z_k|Y_k) f_{\Lambda_k}(Y_k - Z_k)
\end{aligned} \tag{5.21}$$

Following [24], the global density of clutter process is a Poisson process and is given by

$$f_{\Lambda_k} = p_c(m_k) = (\frac{1}{V_k})^{m_k} \lambda^{m_k} e^{-\lambda} \tag{5.22}$$

Also it can be easily shown that

$$p_c(m_k - 1) = p_c(m_k)\frac{V_k}{\lambda} \tag{5.23}$$

In both (5.22) and (5.23), V_k refers to the volume of surveillance region, m_k is the number of clutter measurements and λ is the average number of clutter measurements within the validation gate.

The overall likelihood functions is obtained by substituting (5.17), (5.18) and (5.22) in (5.21).

$$\begin{aligned}
f(Y_k|X_k) &= f(\phi|X_k) f_{\Lambda_k}(y_k(1), y_k(2), \ldots, y_k(m_k)) \\
&+ \sum_i f_{\Sigma_k'}(y_k(i)|X_k) f_{\Lambda_k}(Y_k - \{y_k(i)\}) \\
&= p_c(m_k)(1 - P_D) + p_c(m_k - 1)\left(P_D \sum_{i=1}^{m_k} p(y_k(i)|x_k)\right) \\
&= p_c(m_k)\left[(1 - P_D) + \frac{V_k}{\lambda}\left(P_D \sum_{i=1}^{m_k} p(y_k(i)|x_k)\right)\right]
\end{aligned} \tag{5.24}$$

On the other hand the global likelihood function if the target does not exist $f(Y_k|\phi)$ denotes only clutter densities and therefore can be easily given by

$$f(Y_k|\phi) = p_c(m_k) \tag{5.25}$$

5.4.3 Global Update Density

In general the posterior density is given by standard Bayesian recursion.

$$\begin{aligned} & f_{k|k}(X_k|Y_k) \\ = \; & \frac{1}{\Delta} f_{\Sigma_k}(Y_k|X_k) \times \int f_{k|k-1}(X_k|X_{k-1}) f_{k-1|k-1}(X_{k-1}|Y^{k-1}) \delta X_{k-1} \end{aligned} \tag{5.26}$$

There are two realizations of target state $X_k = \{\phi\}$ and $X_k = \{x_k\}$. Therefore the posterior density comes in two forms

$$f_{k|k}(\{x_k\}|Y_k) = \text{Posterior probability that the target exists with state } \{x_k\} \tag{5.27}$$

$$f_{k|k}(\{\phi\}|Y_k) = \text{Posterior probability that the target does not exist} \tag{5.28}$$

and $f(X_k|Y_k)$ is a density in the sense that

$$\begin{aligned} & \int f_{k|k}(X_k|Y_k) \delta X_k \\ = \; & f_{k|k}(\{\phi\}|Y_k) + \frac{1}{1!} \int f(\{x_k\}|Y_k) \delta\{x_k\} \\ = \; & 1 \end{aligned} \tag{5.29}$$

To obtain the posterior state density, we need the solution of $f_{k|k}(\{x_k\}|Y_k)$. Therefore using the Bayesian update definition,

$$\begin{aligned} & f_{k|k}(\{x_k\}|Y_k) \\ = \; & \frac{1}{\Delta} f_{\Sigma_k}(Y_k|\{x_k\}) \times \int f_{k|k-1}(\{x_k\}|X_{k-1}) f_{k-1|k-1}(X_{k-1}|Y^{k-1}) \delta X_{k-1} \\ = \; & \frac{1}{\Delta} f_{\Sigma_k}(Y_k|\{x_k\}) \times \left[f_{k|k-1}(\{x_k\}|\{\phi\}) f_{k-1|k-1}(\{\phi\}|Y^{k-1}) \right] \\ + \; & \frac{1}{\Delta} f_{\Sigma_k}(Y_k|\{x_k\}) \times \left[f_{k|k-1}(\{x_k\}|\{x_{k-1}\}) f_{k-1|k-1}(\{x_{k-1}\}|Y^{k-1}) \right] \\ = \; & \frac{p_c(m_k)}{\Delta} \left[(1 - P_D) + \frac{V_k P_D}{\lambda} \left(\sum_{i=1}^{m_k} p(y_k(i)|x_k) \right) \right] \end{aligned}$$

$$
\begin{aligned}
&\times \quad \left[0 + p_v \int p(x_k|x_{k-1})p(x_{k-1}|Y^{k-1})dx_{k-1}\right] \\
&= \quad \frac{p_c(m_k)p_v}{\Delta}\left[(1-P_D) + \frac{V_k P_D}{\lambda}\left(\sum_{i=1}^{m_k} p(y_k(i)|x_k)\right)\right] \\
&\times \quad \left[\int p(x_k|x_{k-1})p(x_{k-1}|Y^{k-1})dx_{k-1}\right] \qquad (5.30)
\end{aligned}
$$

On the other hand, the posterior density $f_{k|k}(\{\phi\}|Y_k)$ can easily be obtained from the relation (5.29),

$$f_{k|k}(\{\phi\}|Y_k) \quad = \quad 1 - f_{k|k}(\{x_k\}|Y_k) \qquad (5.31)$$

The expressions in (5.30) and (5.31) complete the Bayesian filter steps for target tracking with track existence uncertainty. The posterior probability of target existence is given by

$$p_v(posterior) \quad = \quad \int f_{k|k}(\{x_k\}|Y_k) \qquad (5.32)$$

In the next section, these filter steps will be shown to deduce IPDAF under linear Gaussian assumptions.

5.4.4 Deriving IPDAF

In IPDAF, the target existence is modeled as an event with two possible events $p_{k,v}$ (target exists at $t = k$) and $p_{k,\overline{v}}$ (target does not exist at $t = k$). These events switch between themselves with defined Markov Transition probability matrix defined

$$
\begin{bmatrix} p_{k|k-1,v} \\ p_{k|k-1,\overline{v}} \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & 0 \\ 1-\Gamma_{11} & 1 \end{bmatrix} \begin{bmatrix} p_{k-1|k-1,v} \\ p_{k-1|k-1,\overline{v}} \end{bmatrix} \qquad (5.33)
$$

This transition also agrees with the random set model assumption that if a target exists, it may persist with probability p_v or may vanish with probability $1 - p_v$. On the other hand if a target does not exist, it will continue to be the case. Due to this transition model, the fixed probability of target existence p_v in random set filter equations need to be replaced by predicted probability of target existence $p_{k|k-1,v}$ which can be obtained iteratively from (5.33).

Another important factor to consider is that almost all practical algorithms use "gating" to choose a certain validated set of measurements instead of considering all obtained measurements. Therefore the detection probability P_D needs to replaced by $P_D P_G$ to cater for gating probability P_G of each validated measurements.

Introducing these aforementioned modifications the state update equations of random set filter (5.30) becomes

$$
\begin{aligned}
& f_{k|k}(\{x_k\}|Y_k) \\
= & \frac{p_c(m_k)p_{k|k-1,v}}{\Delta}\left[(1-P_DP_G)+\frac{V_kP_DP_G}{\lambda}\left(\sum_{i=1}^{m_k}p(y_k(i)|x_k)\right)\right] \\
\times & \left[\int p(x_k|x_{k-1})p(x_{k-1}|Y^{k-1})dx_{k-1}\right] \qquad (5.34)
\end{aligned}
$$

The integral in (5.34) is Chapman-Kolmogorov integral and for Gaussian linear densities of the component densities, it results into simple Kalman predictor and reduces to a normal distribution given by $\mathcal{N}(x_k;\hat{x}_{k|k-1},P_{k|k-1})$. Moreover, if the sensor model is assumed to be linear in states of the target and affected by white Gaussian noise the likelihood density $p(y_k(i)|x_k)$ is also given by a normal distribution $\mathcal{N}(y_k(i);Hx_k,R_k)$. Under these assumptions, the posterior density can be further reduced as

$$
\begin{aligned}
& f_{k|k}(\{x_k\}|Y_k) \\
= & \frac{p_c(m_k)p_{k|k-1,v}}{\Delta}\left[(1-P_DP_G)+\frac{V_kP_DP_G}{\lambda}\left(\sum_{i=1}^{m_k}\mathcal{N}(y_k(i);Hx_k,R_k)\right)\right] \\
\times & \left[\mathcal{N}(x_k;\hat{x}_{k|k-1},P_{k|k-1})\right] \\
= & \frac{p_c(m_k)p_{k|k-1,v}}{\Delta}(1-P_DP_G)\left[\mathcal{N}(x_k;\hat{x}_{k|k-1},P_{k|k-1})\right] \\
+ & \sum_{i=1}^{m_k}\frac{p_c(m_k)p_{k|k-1,v}}{\Delta}\frac{V_kP_DP_G}{\lambda}\mathcal{N}(y_k(i);Hx_k,R_k)\mathcal{N}(x_k;\hat{x}_{k|k-1},P_{k|k-1}) \\
= & \beta_{k,v}(0)\mathcal{N}(x_k;\hat{x}_{k|k-1},P_{k|k-1})+\sum_{i=1}^{m_k}\beta_{k,v}(i)\mathcal{N}(x_k;\hat{x}_{k|k}(i),P_{k|k}) \qquad (5.35)
\end{aligned}
$$

where $\beta_k(i)$'s, $i=0,1,2,\ldots,m_k$ are data association probabilities and defined as

$$\beta_{k,v}(0)=\frac{p_c(m_k)p_{k|k-1,v}}{\Delta}(1-P_DP_G) \qquad (5.36)$$

$$\beta_{k,v}(i)=\frac{p_c(m_k)p_{k|k-1,v}}{\Delta}\frac{V_kP_DP_G}{\lambda}\Lambda_k(i) \qquad (5.37)$$

$$[\text{where } \mathcal{N}(y_k(i);H\hat{x}_{k|k-1},S_k)=\Lambda_k(i)]$$

The summation of (5.35) is approximated by a normal distribution and with first two moments defined by

$$\hat{x}_{k|k} = \sum_{i=0}^{m_k}\beta_k(i)\hat{x}_{k|k}(0) \qquad (5.38)$$

$$P_{k|k} = \sum_{i=0}^{m_k} \beta_k i P_{k|k}(i) + \sum_{i=0}^{m_k} \beta_k(i)\hat{x}_{k|k}(i)\hat{x}_{k|k}(i)^T - \hat{x}_{k|k}\hat{x}_{k|k}^T \quad (5.39)$$

where

$$x_{k|k}(0) = \hat{x}_{k|k-1} \quad (5.40)$$

$$P_{k|k}(0) = P_{k|k-1} \quad (5.41)$$

The state update density defined by mean and associated variance (5.38) and (5.39) respectively from Random set formulation is exactly same as obtained for standard IPDAF in (4.33) and (4.34).

Now looking at the data association probabilities,

$$\beta_{k,\bar{v}} = \Delta^{-1} p_c(m_k) p_{k|k-1,\bar{v}} \quad (5.42)$$

$$\beta_{k,v}(0) = \Delta^{-1} p_c(m_k) p_{k|k-1,v}(1 - P_D P_G) \quad (5.43)$$

$$\beta_{k,v}(i) = \Delta^{-1} p_c(m_k) p_{k|k-1,v} \frac{P_D P_G V_k}{\lambda} \Lambda_k(i) \quad (5.44)$$

$$(5.45)$$

where $p_{k|k-1,v}$ and $p_{k|k-1,\bar{v}}$ are predicted probabilities of target existence and non-existence respectively and are obtained from (5.33).

From (5.29),

$$\beta_{k,\bar{v}} + \beta_{k,v}(0) + \sum_{i=1}^{m_k} \beta_{k,v}(i) = 1 \quad (5.46)$$

Solving (5.44) for Δ, we have

$$\begin{aligned} \Delta = p_c(m_k) p_{k|k-1,\bar{v}} + p_c(m_k) p_{k|k-1,v}(1 - P_D P_G) \\ + \sum_{i=1}^{m_k} p_c(m_k) p_{k|k-1,v} \frac{P_D P_G V_k}{\lambda} \Lambda_k(i) \\ = p_c(m_k)(1 - \delta_k p_{k|k-1,v}) \end{aligned} \quad (5.47)$$

because $p_{k|k-1,v} + p_{k|k-1,\bar{v}} = 1$ and assuming $\delta_k = P_D P_G - \sum_{i=1}^{m_k} \frac{P_D P_G V_k}{\lambda} \Lambda_k(i)$.

Posterior probability of target existence $p_{k|k,v}$ is given by the integral (5.32). This can be obtained easily and from [24], the probability can be directly given as

$$\begin{aligned} p_{k|k,v} &= \beta_{k,v}(0) + \sum_{i=1}^{m_k} \beta_{k,v}(i) \\ &= \frac{1 - \delta_k}{1 - \delta_k p_{k|k-1,v}} p_{k|k-1,v} \end{aligned} \quad (5.48)$$

This updated probability of target existence is what IPDAF gives in (4.40).

The state update given by (5.38) and (5.39) along with the existence probability recursion (5.48) completes the Random set filter based IPDA derivation.

In the next section, generalized smoothing algorithm for finitely varying random set is proposed. The resultant smoother will later be used to model target existence uncertainty scenario. This will result in AS-IPDA smoother (proposed and discussed in chapter 4) establishing the theoretical connection between random set smoothing approach and standard IPDA smoothing.

5.5 Generalized Random Set Smoother

This section looks at the original derivation of random set smoother. Also the assumptions and necessary modifications are detailed with explanation.

As stated previously, the tool for modeling a random set target tracking scenario in FISST involves the concept of belief mass function. Keeping the standard notation of random set

- Γ_k and Σ_k denote the augmented state random set model for target motion and sensor measurements at time $t = k$, respectively, while T_k refers to the target motion model at any particular time $t = k$.

- $\mathcal{X}_k$ and Y_k are the realizations of sets Γ_k and Σ_k respectively while X_k is the realization of T_k.

- Y^k denotes the collection of all the measurements up to time $t = k$.

For random set smoothing, the target dynamics and sensor measurements models are as follows:

- *Target Dynamics Modeling*

  At any particular time $t = k$, the random set model for target dynamics is given by

$$T_k = \Phi(X_{k-1}, V_{k-1}) \cup B_k(X_{k-1}) \tag{5.49}$$

  where

  - $\Phi(X_{k-1}, V_{k-1})$ refers to the transition dynamics of the existing targets

- $B_k(X_{k-1})$ refers to the target birth process which caters for the possibility of any new target detected

To model an augmented state smoother framework with N lag, the target dynamics model will contain the same target equations of (5.49) for time $t = k, k-1, \ldots, k-N$. Hence the target dynamic model of augmented state smoother will have the form of

$$\Gamma_k = \bigcup_{t=k-N}^{t=k} T_t \tag{5.50}$$

The set union operation of (5.50) suggests a stochastic dynamics for past states (included in the T_t) vector. This is not correct in the sense that stochastic dynamic is only applied for the current state. Therefore, for the rest of the derivation it is understood that the union operation is restrictive and takes care of the fact that uncertainty (modeled as V_{k-1}) affects only the current state while all the states undergo transition.

- *Sensor Measurement Models*

  The sensor model for random set formalization is given by

$$\Sigma_k = \Sigma_k^{'} \cup \Lambda_k \tag{5.51}$$

  where

  - $\Sigma_k^{'}$ denotes the model for target originated measurements
  - Λ_k refers to the clutter measurement model

Based on the above models, the belief mass functions of the associated set are obtained.

- The belief mass function of the target motion model is given by

$$\beta_{\Gamma_{k|k-1}}(S|\mathcal{X}_{k-1}) = Pr(\Gamma_k \subseteq S|\mathcal{X}_{k-1}) \tag{5.52}$$

  This refers to the total probability of finding a target within the space S at time $t = k$ provided that at time $t = t-1$ the target had an augmented state of $\mathcal{X}_{k-1}$.

- Similarly the belief mass function of the sensor model is given by

$$\beta_{\Sigma_{k|k}}(S|\mathcal{X}_k) = Pr(\Sigma_k \subseteq S|\mathcal{X}_k) \tag{5.53}$$

This is the total probability of the event that all the sensor observations at time $t = k$ will be within the set space S if the target has the augmented state of $\mathcal{X}_k$.

Multi-target Markov densities and multi-target likelihoods are obtained by differentiating the corresponding belief mass functions.

- The multi-target Markov density is the set derivative of the belief mass function stated in (5.52). The density is given by

$$f_{\Gamma_{k|k-1}}(\mathcal{X}_k|\mathcal{X}_{k-1}) = \frac{\delta\beta_{\Gamma_{k|k-1}}(S|\mathcal{X}_{k-1})}{\delta\mathcal{X}_k} \tag{5.54}$$

- Similarly the multi-target likelihood density is given by the set derivative of the belief mass function of (5.53) as

$$f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k) = \frac{\delta\beta_{\Sigma_{k|k}}(S|\mathcal{X}_k)}{\delta Y_k} \tag{5.55}$$

The updated multi-target state estimate follows using the standard Bayesian recursive approach

$$\begin{aligned} f_{\Gamma_{k|k}}(\mathcal{X}_k|Y^k) &= \frac{1}{\Delta} \times f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k) \times \\ \int f_{\Gamma_{k|k-1}}&(\mathcal{X}_k|\mathcal{X}_{k-1}) f_{\Gamma_{k-1|k-1}}(\mathcal{X}_{k-1}|Y^{k-1})\delta\mathcal{X}_{k-1} \end{aligned} \tag{5.56}$$

The proposed random set smoother models the target motion and sensor according to (5.50) and (5.51) respectively. The complete smoother follows the systematic solution of (5.54) through (5.56) which complete the smoothing algorithm in an iterative manner. In the next sections, we propose particular models for target dynamics and sensor measurements for problems involving target existence uncertainty. Based on the models, solutions for the smoothing steps are derived.

5.6 Target Existence Uncertainty in Random Set Domain

The scope of the thesis covers the single non-maneuvering target tracking in clutter with target existence uncertainty. The scenario considered is the one in which the target exists at time $t = k - 1$, and

- it can persist with probability p_v, or
- it can vanish with probability $1 - p_v$

The problem is to determine if a target exists and if it does, find the state of the target. At any particular time $t = k - d$, where $d = 0, 1, 2, \ldots, N$, X_{k-d}, which is the realization of T_{k-d} (defined by (5.49)), can be either of two possible events

- $X_{k-d} = \{\phi\}$
- $X_{k-d} = \{x_k\}$

where $\{x_k\}$ represents the state of the target if the target is present. The target dynamic model of Γ_k (defined in (5.50)) is the union set of any combination of different T_{k-d}. The measurements reported by the sensors originate from

- the actual target with probability P_D, and
- from a clutter source with probability P_{FA}

The instance of measurement model Σ_k at time $t = k$ is denoted by Y_k. The number of collected sensor report at any particular time $t = k$ is given by $|Y_k| = m_k$ and the clutters are uniformly distributed within the surveillance region of the sensor.

5.7 Random Set Models

The target dynamics and sensors measurement models specific to the problem defined in section 5.6 are developed in this section, using equations (5.50) and (5.51).

5.7.1 Markov Transition Densities for Random Set Smoother under Target Existence Uncertainty

At any single time instant, the random set model for target motion is given by (5.49). The problem defined in section 5.6 requires that at most one target be present in the surveillance region of the sensor. Given that constraint the birth process will be a null set and hence the target motion model of (5.49) is given by

$$T_k = \Phi(X_{k-1}, V_{k-1}) \tag{5.57}$$

The possible transition from X_{k-1} through the function $\Phi(.)$ gives rise to the current target state X_k.

where

- the realization of X_k can either be $\{x_k\}$ or $\{\phi\}$

The union operation of (5.50) gives rise to the realization of Γ_k which is denoted by $\mathcal{X}_k$.

At each time, the target state X_{k-d}, where $d = 0, 1, \ldots, N$, can have two realizations,

- $\{x_{k-d}\}$ - the target exists with state x_{k-d}.
- $\{\phi_{k-d}\}$ - the target does not exist.

$\mathcal{X}_k$ is then a union of all such possible states for the entire lag of N. In general, this union set can be given as

$$\mathcal{X}_k = \{X_k \cup X_{k-1} \cup \ldots \cup X_{k-N}\} \tag{5.58}$$

But under the assumption of no new target birth, the particular set of

$$\{x_i \cup \phi_j\} \tag{5.59}$$

where $i > j$, is impossible. Given this constraint, there are $N + 2$ number of set allowed for $\mathcal{X}_k$ as possible realizations of Γ_k. These can be summarized here as

- $\mathcal{X}_k^{x_k} = \{x_k, x_{k-1}, \ldots, x_{k-N}\}$ - target exists for the entire duration of the lag
- $\mathcal{X}_k^{m} = \{x_{k-m-1}, \ldots, x_{k-N}\}$, $m = 0, 1, \ldots, N-1$ - target existed between $t = k - N$ to $t = k - m - 1$ and disappeared afterwards
- $\mathcal{X}_k^{n} = \{\phi_k \ldots, \phi_{k-N}\}$ - target was not existing for the entire duration of the lag

These are exactly the same hypotheses developed in AS-IPDA algorithm in section 4.5.3. It is also noted that only $\mathcal{X}_k^{x_k}$ and $\mathcal{X}_k^0$ (two sets denoting the persistence of the target and disappearance of the target at the current time instant respectively) are evolved from the set $\mathcal{X}_{k-1}$ defined at previous time instant.

The belief mass function of the set Γ_k can be found using the definition of (5.52). For $N+2$ possible sets at time $t = k-1$, there will be $N+2$ possible belief masses. These are given by:

$$\begin{aligned} & \beta_{\Gamma_{k|k-1}}(S|X_{k-1}^{x_{k-1}}) = Pr(\Gamma_k = \mathcal{X}_k^{x_k} \subseteq S|\mathcal{X}_{k-1}^{x_{k-1}}) \\ & +Pr(\Gamma_k = \mathcal{X}_k^0 \subseteq S|\mathcal{X}_{k-1}^{x_{k-1}}) + \ldots + Pr(\Gamma_k = \mathcal{X}_k^{N-1} \subseteq S|\mathcal{X}_{k-1}^{x_{k-1}}) + \\ & Pr(\Gamma_k = \mathcal{X}_k^n \subseteq S|\mathcal{X}_{k-1}^{x_{k-1}}) \\ = \; & p_{x_k,x_{k-1}} \int p(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{x_{k-1}}) d\mathcal{X}_{k-1}^{x_{k-1}} + p_{x_k,1} \end{aligned} \tag{5.60}$$

In similar manner, the belief masses for other remaining sets are also calculated and given by

$$\beta_{\Gamma_{k|k-1}}(S|X_{k-1}^m) = p(m, m-1) \tag{5.61}$$

(where $m = 1, \ldots, N-2$).

$$\beta_{\Gamma_{k|k-1}}(S|X_{k-1}^{N-1}) = p_{n,N-1} \tag{5.62}$$

$$\beta_{\Gamma_{k|k-1}}(S|X_{k-1}^n) = 1 \tag{5.63}$$

$$\tag{5.64}$$

The complete Markov transition densities can be obtained by calculating the derivatives of (5.60) through (5.63). The calculated densities are

$$\begin{aligned} f(\mathcal{X}_k^n|\mathcal{X}_{k-1}^n) &= 1 && (5.65) \\ f(\mathcal{X}_k^n|\mathcal{X}_{k-1}^{N-1}) &= p_{n,N-1} && (5.66) \\ f(\mathcal{X}_k^m|\mathcal{X}_{k-1}^{m-1}) &= p(m, m-1) && (5.67) \\ & \quad \text{where } m = 0, 1, \ldots, N-2 \\ f(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{x_{k-1}}) &= p_{x_k,x_{k-1}} \int p(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{x_{k-1}}) d\mathcal{X}_{k-1}^{x_{k-1}} && (5.68) \\ f(\mathcal{X}_k^0|\mathcal{X}_{k-1}^{k-1}) &= p_{0,x_{k-1}} && (5.69) \end{aligned}$$

5.7.2 Likelihood density of Random Set Smoother

The random set model for sensor measurements along with clutter is given by (5.51) such that

- $\Sigma_k^{'}$ models the target originated measurement, and
- Λ_k models the clutter originated measurements

The belief mass function of the total measurement set Σ_k can then be expressed as

$$\beta_{\Sigma_{k|k}}(S|\mathcal{X}_k) = Pr(\Sigma_k^{'} \cup \Lambda_k \subseteq S|\mathcal{X}_k) \quad (5.70)$$

Assuming that the target originated measurements and clutter originated measurements are independent of each other, (5.70) can be re-written as

$$\begin{aligned} \beta_{\Sigma_{k|k}}(S|\mathcal{X}_k) &= \beta_{\Sigma_k^{'}}(S|\mathcal{X}_k) \times \beta_{\Lambda_k}(S|\mathcal{X}_k) \\ &= Pr(\Sigma_k^{'} \subseteq S|\mathcal{X}_k)\ Pr(\Lambda_k \subseteq S|\mathcal{X}_k) \quad (5.71) \end{aligned}$$

The sensor collects the report of the target with a detection probability of P_D. Hence, the measurement model for target originated data, provided that the target exists, can be expanded as

$$\begin{aligned} \beta_{\Sigma_k^{'}}(S|\mathcal{X}_k) &= Pr(\Sigma_k^{'} = \phi|\{\mathcal{X}_k^{x_k}\}) \\ &\quad + Pr(\Sigma_k^{'} \neq \phi, \Sigma_k^{'} \subseteq S|\{\mathcal{X}_k^{x_k}\}) \\ &= (1 - P_D) + P_D p_{\Sigma_k^{'}}(S|\mathcal{X}_k^{x_k}) \\ &= (1 - P_D) + P_D \int f(Y_k|\mathcal{X}_k^{x_k}) \\ &= f(\phi|\mathcal{X}_k^{x_k}) + +P_D \int f(Y_k|\mathcal{X}_k^{x_k}) \quad (5.72) \end{aligned}$$

For the clutter measurement model Λ_k, it is assumed that

- there are M number of clutters present at time $t = k$
- each of the clutter observations is independent of the other
- each clutter has the same probability measure with respect to spatial density

Based on these assumptions, the belief mass measure is

$$\begin{aligned}
&\beta_{\Lambda_k}(S|\mathcal{X}_k^{x_k}) = Pr(\Lambda_k \subseteq S|\mathcal{X}_k^{x_k}) \\
&= Pr(\Lambda_k(1) \subseteq S, \Lambda_k(2) \subseteq S, \ldots, \Lambda_k(M) \subseteq S|\mathcal{X}_k^{x_k}) \\
&\qquad \ldots Pr(\Lambda_k(M) \subseteq S|\mathcal{X}_k^{x_k}) \\
&= \prod_{i=1}^{M} Pr(\Lambda_k(i) \subseteq S|\mathcal{X}_k^{x_k}) \\
&= \beta_c(S)^M
\end{aligned} \tag{5.73}$$

Putting the expressions from (5.72) and (5.73) into (5.71), the complete belief mass measure for sensor model is

$$\beta_{\Sigma_{k|k}}(S|\mathcal{X}_k) = \beta_{\Sigma'_k}(S|\mathcal{X}_k) \;\times\; \beta_c(S)^M \tag{5.74}$$

The next task in deriving the smoothing algorithm is to find the derivative of the belief mass function in (5.74) to obtain the likelihood function $f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^{x_k})$. The belief mass function is a product of two separate belief measures, therefore the likelihood function is obtained using the product rule of FISST. It can be shown that

$$\begin{aligned}
& f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^{x_k}) \\
=\;& f_{\Sigma'_k \cup \Lambda_k}(Y_k|\mathcal{X}_k^{x_k}) \\
=\;& \sum_{\mathcal{Z}_k \subseteq Y_k} f_{\Sigma'_k}(\mathcal{Z}_k|\mathcal{X}_k^{x_k}) f_{\Lambda_k}(Y_k - \mathcal{Z}_k)
\end{aligned} \tag{5.75}$$

The global density of the clutter process $f_{\Lambda_k}\{\xi_1, \xi_2, \ldots, \xi_n\}$ is

$$n!\; C_{M,n} P_{FA}^n (1 - P_{FA}^{M-n}) c(\xi_1) \ldots c(\xi_n) \tag{5.76}$$

where $C_{M,n}$ denotes the number of combinations of n out of M, with the assumption of clutter being uniformly distributed in the surveillance volume V, $c(\xi_i) = \frac{1}{V}$. As a result, the density of clutter in (5.76) can further be reduced to

$$n! \left(\frac{1}{V}\right)^n C_{M,n} P_{FA}^n (1 - P_{FA}^n). \tag{5.77}$$

If M is large and P_{FA} is small the expression on the right hand side of (5.77) can be approximated by a poisson process. This further simplifies to

$$f_{\Lambda_k}\{\xi_1, \xi_2, \ldots, \xi_n\} = n! \left(\frac{1}{V}\right)^n \frac{\lambda^n e^{-\lambda}}{n!} \tag{5.78}$$

In literature, the number of clutter for the scan at time $t = k$ is denoted by m_k. Hence if n is replaced by m_k in expression (5.78), and the global density $f_{\Lambda_k}\{\xi_1, \xi_2, \ldots, \xi_n\}$ is denoted by $p_c(m_k)$, the clutter density becomes,

$$\begin{aligned} p_c(m_k) &= m_k! \left(\frac{1}{V}^{m_k}\right) \frac{\lambda^{m_k} e^{-\lambda}}{m_k!} \\ &= \left(\frac{1}{V}\right)^{m_k} \lambda^{m_k} e^{-\lambda} \end{aligned} \tag{5.79}$$

It is also evident that

$$p_c(m_k - 1) = p_c(m_k)\frac{V}{\lambda} \tag{5.80}$$

The overall likelihood function of (5.75), when the target exists, can now be derived using the relationships found in (5.72) and (5.79). This is given by

$$\begin{aligned} & f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^{x_k}) \\ = \; & f_{\Sigma_k^{'}}(\phi|\mathcal{X}_k^{x_k}) \times f_{\Lambda_k}(Y_k(1), \ldots, Y_k(m_k)) \\ & + \sum_i f_{\Sigma_k^{'}}(Y_k(i)|\mathcal{X}_k^{x_k}).f_{\Lambda_k}(Y_k - \{Y_k(i)\}) \\ = \; & p_c(m_k)\,(1 - P_D) \\ + \; & p_c(m_k - 1)\left(P_D \sum_i^{m_k} p(Y_k(i)|\mathcal{X}_k^{x_k})\right) \\ = \; & p_c(m_k)\left(1 - P_D + \frac{P_D V}{\lambda} \sum_i^{m_k} p(Y_k(i)|\mathcal{X}_k^{x_k})\right) \end{aligned} \tag{5.81}$$

When the target does not exist, the sensor model equation of (5.51) remains same with the difference that the target originated measurement can have only one realization, $\Sigma_k^{'} = \phi$. It is also easy to verify that the likelihood will have no representation of the target and only the clutter will contribute to the density. Therefore

$$\begin{aligned} f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^m) &= p_c(m_k) \\ & \quad \text{where } m = 0, 1, 2, \ldots, N-1 \\ f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^n) &= p_c(m_k) \end{aligned} \tag{5.82}$$

5.8 Calculation of Global Posterior Density of Target State and Existence

The Bayes recursion follows the same procedure as in single target single sensor case. Dropping the notation of Γ for clarity, the global posterior density is given by

$$\begin{aligned}
& f_{k|k}(\mathcal{X}_k|Y^k) \\
= & \frac{1}{\Delta} f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k) \times \\
& \int f_{\Gamma_{k|k-1}}(\mathcal{X}_k|\mathcal{X}_{k-1}) f_{\Gamma_{k-1|k-1}}(\mathcal{X}_{k-1}|Y^{k-1}) \delta\mathcal{X}_{k-1} \\
= & \frac{1}{\Delta} f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k) \\
& \Big\{ f_{k|k-1}(\mathcal{X}_k|\mathcal{X}_{k-1}^{\phi}) \; f_{k-1|k-1}(\mathcal{X}_{k-1}^{\phi}|Y^{k-1}) + \\
& \int f_{k|k-1}(\mathcal{X}_k|\mathcal{X}_k^{x_{k-1}}) \; f_{k-1|k-1}(\mathcal{X}_{k-1}^{x_{k-1}}|Y^{k-1}) d\mathcal{X}_{k-1}^{x_{k-1}} \Big\}
\end{aligned} \tag{5.83}$$

The posterior density $f_{k|k}(\mathcal{X}_k|Y^k)$ has following forms:

$$f(\mathcal{X}_k^n|Y^k) \tag{5.84}$$

$$f(\mathcal{X}_k^m|Y^k) \tag{5.85}$$

where $m = 0, 1, \ldots, N-1$

$$f(\mathcal{X}_k^{x_k}|Y^k) \tag{5.86}$$

The global posterior density is a probability density and

$$\begin{aligned}
& \int f_{k|k}(\mathcal{X}_k|Y^k)\delta\mathcal{X}_k \\
= & f_{k|k}(\mathcal{X}_k^n|Y^k) + \frac{1}{1!}\int f_{k|k}(\mathcal{X}_k^{x_k}|Y^k) d\mathcal{X}_k^{x_k} \\
+ & \sum_{m=0}^{N-1} \frac{1}{1!} \int f_{k|k}(\mathcal{X}_k^m|Y^k) d\mathcal{X}_k^m \\
= & 1
\end{aligned} \tag{5.87}$$

If the target exists, the state estimation will be obtained from $f_{k|k}(\mathcal{X}_k^{x_k}|Y^k)$. This is obtained through (5.83), where $\mathcal{X}_k$ is replaced by its realization $\mathcal{X}_k^{x_k}$.

$$f_{k|k}(\mathcal{X}_k^{x_k}|Y^k)$$

$$
\begin{aligned}
&= \frac{1}{\Delta_{x_k}} f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^{x_k}) \\
&\times \left[\sum_{m=0}^{N-1} f_{k|k-1}(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{m})\ f_{k-1|k-1}(\mathcal{X}_{k-1}^{m}|Y^{k-1})\right] \\
&+\frac{1}{\Delta} f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^{x_k}) \\
&\times\ f_{k|k-1}(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{n})\ f_{k-1|k-1}(\mathcal{X}_{k-1}^{n}|Y^{k-1}) \\
&+\frac{1}{\Delta}\ f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^{x_k}) \times \\
&\int f_{k|k-1}(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{x_{k-1}})\ f_{k-1|k-1}(\mathcal{X}_k^{x_{k-1}}|Y^{k-1})d\mathcal{X}_{k-1}^{x_{k-1}} \\
&= \frac{p_c(m_k)}{\Delta}\left(1-P_D+\frac{P_D V}{\lambda}\sum_{i}^{m_k} p(Y_k(i)|\mathcal{X}_k^{x_k})\right) \\
&\left(0+p_{x_k,x_{k-1}}\int p(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{x_{k-1}})\ p(\mathcal{X}_{k-1}^{x_{k-1}}|Y^{k-1})d\mathcal{X}_{k-1}^{x_{k-1}}\right) \\
&= \frac{p_c(m_k)p_{x_k,x_{k-1}}}{\Delta}\left(1-P_D+\frac{P_D V}{\lambda}\sum_{i}^{m_k} p(Y_k(i)|\mathcal{X}_k^{x_k})\right) \\
&\left(\int p(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{x_{k-1}})\ p(\mathcal{X}_{k-1}^{x_{k-1}}|Y^{k-1})d\mathcal{X}_{k-1}^{x_{k-1}}\right) \qquad (5.88)
\end{aligned}
$$

The posterior densities of the remaining realizations of $\mathcal{X}_k$ can also be obtained in a a similar manner and are given by (using (5.65) - (5.69))

$$
\begin{aligned}
& f_{k|k}(\mathcal{X}_k^0|Y^k) \\
=\ & \frac{1}{\Delta}\ f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^0) \times f_{k|k}(\mathcal{X}_k^0|Y^{k-1}) \\
=\ & \frac{1}{\Delta}\ f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^0) \times p_{0,x_k} f_{k|k}(\mathcal{X}_{k-1}^{x_k}|Y^{k-1}) \\
=\ & \frac{1}{\Delta}\ p_c(m_k) \times p_{0,x_k} f_{k|k}(\mathcal{X}_{k-1}^{x_k}|Y^{k-1}) \qquad (5.89)
\end{aligned}
$$

$$
\begin{aligned}
& f_{k|k}(\mathcal{X}_k^m|Y^k) \\
=\ & \frac{1}{\Delta}\ f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^m) \times f_{k|k}(\mathcal{X}_k^m|Y^{k-1}) \\
=\ & \frac{1}{\Delta}\ f_{\Sigma_{k|k}}(Y_k|\mathcal{X}_k^m) \times p_{m,m-1} f_{k|k}(\mathcal{X}_{k-1}^{m-1}|Y^{k-1}) \\
=\ & \frac{1}{\Delta}\ p_c(m_k) \times p_{m,m-1} f_{k|k}(\mathcal{X}_{k-1}^{m-1}|Y^{k-1}) \qquad (5.90)
\end{aligned}
$$

(where $m = 1, \ldots, N-2$).

$$
\begin{aligned}
& f_{k|k}(\mathcal{X}_k^{N-1}|Y^k) \\
=\ & \frac{1}{\Delta_{N-1}}\ p_c(m_k) \times p_{N-1,N-2} f_{k|k}(\mathcal{X}_{k-1}^{m-1}|Y^{k-1}) \qquad (5.91)
\end{aligned}
$$

The expressions in (5.89) and (5.91) are obtained through the realizations that by the definition of the random sets $\{\mathcal{X}_k^1, \ldots, \mathcal{X}_k^{N-1}\}$ at time $t = k$ are same as $\{\mathcal{X}_{k-1}^0, \ldots, \mathcal{X}_{k-1}^{N-2}\}$ at time $t = k-1$. The probability $p_{a,b}$ denotes the transition probability from $\mathcal{X}_{k-1}^b$ to $\mathcal{X}_{k-1}^a$. These expressions also prove the fact that under no target hypothesis the previous filtered and/or smoothed states are retained (which is proved in appendix C for AS-IPDA algorithm).

The state update under random set formalism for target existence uncertainty is carried out through expressions (5.88) through (5.91). The posterior probability of each of the possible sets are calculated as following:

$$p_{x_k} = \int f_{k|k}(\mathcal{X}_k^{x_k}|Y^k)d\mathcal{X}_k^{x_k} \tag{5.92}$$

$$p_m = \int f_{k|k}(\mathcal{X}_k^m|Y^k)d\mathcal{X}_k^m \tag{5.93}$$

$$m = 0, \ldots, N-1$$

$$p_n = \int f_{k|k}\mathcal{X}_k^n|Y^k d\mathcal{X}_k^n \tag{5.94}$$

The expressions, (5.88) through (5.94), solve the random set smoothing algorithm problem for target tracking with target existence uncertainty. The system uses the augmented set approach through the use of realizations for $\mathcal{X}_k$ and thus provide a smoothed state estimate of target's dynamic state.

5.9 Augmented State IPDA smoother Derivation

The original derivation of the IPDA filter, [61] takes the target existence as a random variable with two possible events:

- E_k refers to the event that the target exists at time $t = k$
- $\overline{E}_k$ refers to the event that the target does not exist at time $t = k$

The IPDA algorithm finds the probability of these two possible events, $p(E_k)$, $p(\overline{E_k})$.

A target can also switch between these two states with a predefined Markov transition probability matrix. The matrix is given by

$$\begin{bmatrix} p(E_k) \\ p(\overline{E}_k) \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} p(E_{k-1}) \\ p(\overline{E}_{k-1}) \end{bmatrix} \tag{5.95}$$

In the IPDA filter, the Markov transition probabilities from non-existence to existence is zero which is consistent with the fact that if a target goes out of existence, it will continue to be the case. In that context, the transition probability matrix of (5.95) can be expressed more specifically as

$$\begin{bmatrix} p(E_k) \\ p(\overline{E}_k) \end{bmatrix} = \begin{bmatrix} \gamma_{11} & 0 \\ 1-\gamma_{11} & 1 \end{bmatrix} \begin{bmatrix} p(E_{k-1}) \\ p(\overline{E}_{k-1}) \end{bmatrix} \tag{5.96}$$

Because of Markov transition, target existence evolves over time and therefore all the terms of static existence probabilities $p_{(.)}$ from section 5.7.1 need to be replaced by dynamic existence probabilities. The definitions of any such probabilities are as follows :

- $p_{x_k,k-1|k-1}$, $p_{m,k-1|k-1}$ and $p_{n,k-1|k-1}$ are the prior probabilities of random sets denoted by $\mathcal{X}_k^{x_k}$, $\mathcal{X}_k^m$ (where $m = 0, 1, 2, \ldots, N-1$) and $\mathcal{X}_k^n$ respectively.
- $p_{x_k,k|k-1}$, $p_{m,k|k-1}$ and $p_{n,k|k-1}$ are the predicted probabilities of random sets denoted by $\mathcal{X}_k^{x_k}$, $\mathcal{X}_k^m$ (where $m = 0, 1, 2, \ldots, N-1$) and $\mathcal{X}_k^n$ respectively.
- $p_{x_k,k|k}$, $p_{m,k|k}$ and $p_{n,k|k}$ are the posterior probabilities of random sets denoted by $\mathcal{X}_k^{x_k}$, $\mathcal{X}_k^m$ (where $m = 0, 1, 2, \ldots, N-1$) and $\mathcal{X}_k^n$ respectively.

In a single iteration of a fixed lag smoothing algorithm, the existence probabilities within the time from $t = K-N$ to $t = k$ are updated with the observation made at $t = k$. Therefore it is more appropriate to use the probability at time instant $t = k - N - 1$ for prediction of probabilities at $t = k$. Using the relationship in (5.96), the target existence (and non-existence) probabilities can be predicted as following:

$$p_{x_k,k|k-1} = \gamma_{11} p(E_{k-N-1}|Y^{k-1}) \tag{5.97}$$

$$p_{m,k|k-1} = \gamma_{10}\gamma_{11}^{N-m} p(E_{k-N-1}|Y^{k-1})$$

$$p_{n,k|k-1} = \gamma_{10} p(E_{k-N-1}||Y^{k-1}) + (1 - p(E_{k-N-1})) \tag{5.98}$$

Moreover, all practically implementable algorithms use a validation gate and consider only those measurements that are within the validation gate for state update. Therefore, only P_D does not suffice. The fact, that a detected target also has to be within the validation gate needs to be incorporated. This can be carried out by introducing a gating probability P_G and replacing P_D by $P_D P_G$ in the set of equations in section 5.8. Considering the changes states above, the Bayes' update equation for state estimation provided that the target exists simplifies to

$$
\begin{aligned}
&f_{k|k}(\mathcal{X}_k^{x_k}|Y^k)\\
&= \frac{p_c(m_k)p_{x_k,k|k-1}}{\Delta}\\
&\times \left(1 - P_D P_G + \frac{P_D P_G V}{\lambda}\sum_i^{m_k} p(Y_k(i)|\mathcal{X}_k^{x_k})\right)\\
&\times \left(\int p(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{x_{k-1}})\; p(\mathcal{X}_{k-1}^{x_{k-1}}|Y^{k-1})d\mathcal{X}_{k-1}^{x_{k-1}}\right)
\end{aligned}
\tag{5.99}
$$

The integration term on the last line of (5.99) is the well known Chapman-Kolmogorv integral. If both the transition probability density $p(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{x_{k-1}})$ and the prior density $p(\mathcal{X}_{k-1}^{x_{k-1}}|Y^{k-1})$ are Gaussian in nature, the solution of the Chapman-Kolmogorov integral is also Gaussian and follows the same form as Kalman Predictor step. We can write the resultant expression for the solution of the integral as

$$
\begin{aligned}
&\int p(\mathcal{X}_k^{x_k}|\mathcal{X}_{k-1}^{x_{k-1}})\; p(\mathcal{X}_{k-1}^{x_{k-1}}|Y^{k-1})d\mathcal{X}_{k-1}^{x_{k-1}}\\
&= \mathcal{N}(\mathcal{X}_k^{x_k}; \hat{\mathcal{X}}_{k|k-1}^{x_k}, \mathcal{P}_{k|k-1}^{x_k})
\end{aligned}
\tag{5.100}
$$

The target originated measurement obtained by the sensor are assumed to be linear in state of the target and are affected by white Gaussian noise. Under these assumptions the likelihood terms $p(Y_k(i)|\mathcal{X}_k^{x_k})$, for each i, are also Gaussian and are expressed as

$$
p(Y_k(i)|\mathcal{X}_k^{x_k}) = \mathcal{N}(Y_k(i); \mathcal{H}\mathcal{X}_k^{x_k}, \mathcal{R}_k) \tag{5.101}
$$

where

- $\mathcal{H}$ is the state to measurement transition matrix
- $\mathcal{R}_k$ is the noise co-variance matrix

If we put the derived relations from (5.100) and (5.101) into (5.99), the posterior density can be further simplified as

$$
\begin{aligned}
&f_{k|k}(\mathcal{X}_k^{x_k}|Y^k)\\
&= \frac{p_c(m_k)p_{x_k,k|k-1}}{\Delta}\\
&\times \left(1 - P_D P_G + \frac{P_D P_G V}{\lambda}\sum_i^{m_k} \mathcal{N}(Y_k(i); \mathcal{H}\mathcal{X}_k^{x_k}, \mathcal{R}_k)\right)
\end{aligned}
$$

$$\times\mathcal{N}(\mathcal{X}_k^{x_k};\hat{\mathcal{X}}_{k|k-1}^{x_k},\mathcal{P}_{k|k-1}^{x_k})$$
$$= \frac{p_c(m_k)p_{x_k,k|k-1}}{\Delta}(1-P_DP_G)\mathcal{N}(\mathcal{X}_k^{x_k};\hat{\mathcal{X}}_{k|k-1}^{x_k},\mathcal{P}_{k|k-1}^{x_k})$$
$$+\frac{p_c(m_k)p_{k|k-1,v}}{\Delta}\times\frac{P_DP_GV}{\lambda}\times$$
$$\sum_i^{m_k}\mathcal{N}(Y_k(i);\mathcal{H}\mathcal{X}_k^{x_k},\mathcal{R}_k)\mathcal{N}(\mathcal{X}_k^{x_k};\hat{\mathcal{X}}_{k|k-1}^{x_k},\mathcal{P}_{k|k-1}^{x_k}) \tag{5.102}$$

It can be easily shown that

$$\begin{aligned}
&\mathcal{N}(Y_k(i);\mathcal{H}\mathcal{X}_k^{x_k},\mathcal{R}_k)\mathcal{N}(\mathcal{X}_k^{x_k};\hat{\mathcal{X}}_{k|k-1}^{x_k},\mathcal{P}_{k|k-1}^{x_k})\\
&=\mathcal{N}(Y_k(i);\mathcal{H}\hat{\mathcal{X}}_{k|k-1}^{x_k},\mathcal{S}_k)\mathcal{N}(\mathcal{X}_k^{x_k};\hat{\mathcal{X}}_{k|k}^{x_k}(i),\mathcal{P}_{k|k}^{x_k}(i))\\
&=\Omega_k^i\mathcal{N}(\mathcal{X}_k^{x_k};\hat{\mathcal{X}}_{k|k}^{x_k}(i),\mathcal{P}_{k|k}^{x_k}(i))
\end{aligned} \tag{5.103}$$

where

$$\Omega_k^i = \mathcal{N}(Y_k(i);\mathcal{H}\hat{\mathcal{X}}_{k|k-1}^{x_k},\mathcal{S}_k) \tag{5.104}$$

- $\mathcal{S}_k$ is the innovation covariance
- $\hat{\mathcal{X}}_{k|k}^{x_k}(i)$ is the updated target state based on validated measurement $Y_k(i)$

The posterior target density can now be further simplified by using (5.103)

$$\begin{aligned}
&f_{k|k}(\mathcal{X}_k^{x_k}|Y^k)\\
&=\frac{p_c(m_k)p_{x_k,k|k-1}}{\Delta}(1-P_DP_G)\mathcal{N}(\mathcal{X}_k^{x_k};\hat{\mathcal{X}}_{k|k-1}^{x_k},\mathcal{P}_{k|k-1}^{x_k})\\
&+\frac{p_c(m_k)p_{x_k,k|k-1}}{\Delta}\times\\
&\left(\frac{P_DP_GV}{\lambda}\sum_i^{m_k}\Omega_k^i\mathcal{N}(\mathcal{X}_k^{x_k};\hat{\mathcal{X}}_{k|k}^{x_k}(i),\mathcal{P}_{k|k}^{x_k}(i))\right)\\
&=\epsilon_{k,x_k}(0)\mathcal{N}(\mathcal{X}_k^{x_k};\hat{\mathcal{X}}_{k|k-1}^{x_k},\mathcal{P}_{k|k-1}^{x_k})\\
&+\sum_i^{m_k}\epsilon_{k,x_k}(i)\mathcal{N}(\mathcal{X}_k^{x_k};\hat{\mathcal{X}}_{k|k}^{x_k}(i),\mathcal{P}_{k|k}^{x_k}(i))
\end{aligned} \tag{5.105}$$

The ϵ s are commonly known as data association probabilities and are given by

$$\epsilon_{k,\phi}(0) = \Delta^{-1}p_c(m_k)(1-p_{x_k,k|k-1}) \tag{5.106}$$
$$\epsilon_{k,x_k}(0) = \Delta^{-1}p_c(m_k)p_{x_k,k|k-1}(1-P_DP_G) \tag{5.107}$$
$$\epsilon_{k,x_k}(i) = \Delta^{-1}p_c(m_k)p_{x_k,k|k-1}\frac{P_DP_GV}{\lambda}\Omega_k^i \tag{5.108}$$

It can be stated that

$$\epsilon_{k,\phi}(0) + \epsilon_{k,x_k}(0) + \sum_{i=1}^{m_k} \epsilon_{k,x_k}(i) = 1 \tag{5.109}$$

The above expression in (5.109) helps us determining the normalizing constant Δ. If we put the respective expression of (5.106) to (5.108) in (5.109), Δ can be obtained as

$$\begin{aligned}
&\Delta^{-1}p_c(m_k)(1-p_{x_k,k|k-1}) + \Delta^{-1}p_c(m_k)p_{x_k,k|k-1}(1-P_DP_G) \\
&+\sum_{i=1}^{m_k}\Delta^{-1}p_c(m_k)p_{x_k,k|k-1}\frac{P_DP_GV}{\lambda}\Omega_k^i = 1 \\
&\Delta = p_c(m_k)(1-p_{x_k,k|k-1}) + p_c(m_k)p_{x_k,k|k-1}(1-P_DP_G) \\
&\qquad \sum_{i=1}^{m_k} p_c(m_k)p_{x_k,k|k-1}\frac{P_DP_GV}{\lambda}\Omega_k^i \\
&\Delta = p_c(m_k)\big((1-p_{x_k,k|k-1}) + p_{x_k,k|k-1}(1-P_DP_G) \\
&\qquad \sum_{i=1}^{m_k} p_{x_k,k|k-1}\frac{P_DP_GV}{\lambda}\Omega_k^i\Big)
\end{aligned} \tag{5.110}$$

Replacing $P_DP_G - \sum_{i=1}^{m_k}\frac{P_DP_GV}{\lambda}\Omega_k^i$ by δ_k, the normalization constant Δ can be obtained as follows:

$$\Delta = p_c(m_k)(1-\delta_k p_{x_k,k|k-1}) \tag{5.111}$$

Replacing Δ in (5.106)-(5.108) by the obtained expression in (5.111), the data association probabilities can be simplified as

$$\epsilon_{k,\phi}(0) = \frac{(1-p_{x_k,k|k-1})}{1-\delta_k p_{x_k,k|k-1}} \tag{5.112}$$

$$\epsilon_{k,x_k}(0) = \frac{p_{x_k,k|k-1}(1-P_DP_G)}{1-\delta_k p_{x_k,k|k-1}} \tag{5.113}$$

$$\epsilon_{k,x_k}(i) = \frac{p_{x_k,k|k-1}\frac{P_DP_GV}{\lambda}\Omega_k^i}{1-\delta_k p_{x_k,k|k-1}} \tag{5.114}$$

Substituting these values of data association probabilities in (5.105) completes the target state estimation problem of augmented state IPDA smoother. To obtain the posterior target existence probability, we use (5.92) to (5.94) . Hence the posterior target existence probability is:

$$p_{x_k,k|k} = \int f_{k|k}(\mathcal{X}_k^{x_k}|Y^k)d\mathcal{X}_k^{x_k}$$

$$= \int \Big(\epsilon_{k,x_k}(0) \mathcal{N}(\mathcal{X}_k^{x_k}; \hat{\mathcal{X}}_{k|k-1}^{x_k}, \mathcal{P}_{k|k-1}^{x_k})$$

$$+ \sum_i^{m_k} \epsilon_{k,x_k}(i) \mathcal{N}(\mathcal{X}_k^{x_k}; \hat{\mathcal{X}}_{k|k}^{x_k}(i), \mathcal{P}_{k|k}^{x_k}(i)) \Big) \, d\mathcal{X}_k^{x_k}$$

$$= \epsilon_{k,x_k}(0) + \sum_i^{m_k} \epsilon_{k,x_k}(i)$$

$$= \frac{p_{x_k,k|k-1}(1 - P_D P_G)}{1 - \delta_k p_{x_k,k|k-1}} + \frac{p_{x_k,k|k-1} \frac{P_D P_G V}{\lambda} \Omega_k^i}{1 - \delta_k p_{x_k,k|k-1}}$$

$$= \frac{1 - \delta_k}{1 - \delta_k p_{x_k,k|k-1}} p_{x_k,k|k-1} \tag{5.115}$$

The updated probabilities of the remaining sets can be found through the solution of (5.23) through (5.24) and using (5.89-5.90).

$$p_{0,k|k} = \int \frac{1}{\Delta} \, p_c(m_k) \times p_{0.x_k} f_{k|k}(\mathcal{X}_{k-1}^{x_k} | Y^{k-1}) d\mathcal{X}_{k-1}^{x_k}$$

$$= \frac{1}{\Delta} \, p_c(m_k) p_{0,x_k} \tag{5.116}$$

$$p_{m,k|k} = \int \frac{1}{\Delta} \, p_c(m_k) \times p_{m,m-1} f_{k|k}(\mathcal{X}_{k-1}^{m-1} | Y^{k-1}) d\mathcal{X}_{m-1}^{x_k}$$

$$= \frac{1}{\Delta} \, p_c(m_k) p_{m,m-1} \tag{5.117}$$

$$\tag{5.118}$$

The updated probability of null hypothesis $\mathcal{X}_k^n$ can be easily obtained as

$$p_{n,k|k} = 1 - p_{x_k,k|k} - \sum_{i=0}^{N-1} p_{m,k|k} \tag{5.119}$$

The derivations of (5.115) through (5.119) are the updated probabilities of each of the random set defined as the realizations of $\mathcal{X}_k$ at $t = k$. These are also same as "hypotheses probabilities" for AS-IPDA development given by (4.49) through (4.54). This completes the derivation of AS-IPDA from random set formalism.

5.10 Conclusion

In this chapter Random Set filter is first revisited and its calculation for global prediction and likelihood densities are calculated. The random set formalism provides a sound mathematical formulation to deal with multiple targets and especially if the target dynamics and/or their statistics are unknown but varying finitely. IPDAF is also proved to be a special case for Random set filter under linear Gaussian assumptions.

In this chapter the approach of Random set formulation for target tracking is extended to smoothing. The original derivation of generalized smoother for random but finitely varying set is derived. The appropriate target dynamic and sensor models are devised along with the belief mass functions for each of those. The developed generalized random set smoother is then modeled for target existence uncertainty resulting into an tracking algorithm for automatic track maintenance under random set domain. This proposed algorithm is then shown to be reducing into AS-IPDA smoother under linear Gaussian model.

6. TARGET EXISTENCE UNCERTAINTY WITH OUT-OF-SEQUENCE-MEASUREMENT

6.1 Introduction

In this chapter, a theoretical model will be proposed to handle out of sequence measurement (OOSM) problem. The problem of delayed arrival, which is termed as "out of sequence", of measurements is very common in a multi-sensor environment. The problem will be defined in this chapter and a corresponding theoretical framework will be proposed to address the issue under target existence uncertainty.

6.2 OOSM Problem

In a multi-sensor environment, sensor observations may arrive with a delay. Due to network congestion, blockage or simply because of the sensor property, this delayed arrival is a common feature for a large scale sensor network. Therefore the situation arises that observations at current time $t = k$ and at $t = k - d$ arrive at the same instant. The situation is depicted in diagram 6.2.

This problem is termed as "out of sequence measurement" problem in literature. A consistent modeling of the problem is required to use the delayed measurement without compromising the requirement of a real time tracking application.

An optimal solution to OOSM problem was first proposed as a filtering framework by Bar-Shalom in [4]. It was also noted by the author that extension

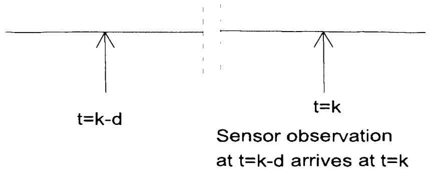

Fig. 6.1: Out of Sequence Measurement Problem

of such effort to longer time delays gives rise to "non-standard" smoothing algorithms. Such an approach was later considered by Mallick et al. in [51]. OOSM was then approximately solved in [33]. The key difference between the exact and approximate algorithms in [4] and [33] respectively is the effect of the process noise on the filtering scheme. Although both consider the effect of process noise, the optimal solution incorporates the non-zero conditional mean of the process noise into the filter update equations while the approximate filter does not. The OOSM problem in clutter was first considered in [21]. It was almost simultaneously considered by Orton and Marrs using particle filters in [63]. The authors concluded that target states from both the current time and delayed time need to be considered. They propose a method of using a forwardbackward sampling to address this issue.

So far the existing literature proposes the OOSM problem for track in clutter environment. In this chapter, a theoretical framework will be proposed to address the OOSM problem under target existence certainty. First the Bayesian modeling will be proposed and corresponding modifications in AS-IPDA modeling and also the random set modeling will be highlighted.

6.3 Bayes' model for OOSM for Target Existence Uncertainty

The Bayesian model for general target existence certainty is given by (4.46).

$$p(x_k, E_k|y^k)$$
$$p(x_k|E_k, y_k, y^{k-1}) \tag{6.1}$$

To accommodate delayed measurements, the measurement vector includes the delayed measurements. The redefinition of the measurement vector results in expression given by

$$Y_k = \{y_{k-l}\}, l \in \{0, \dots, d\} \tag{6.2}$$

where d denotes delay. Each measurement vector y_k contains the clutter at the corresponding time instant and therefore can be defined by

$$y_k = \{y_k(1), \dots, y_k(m_k)\} \tag{6.3}$$

where m_k is the number of validated measurement.

Correspondingly, the state vector also needs to be updated with the replacement of x_k, E_k with $x_k, E_k, \dots, x_{k-d}, E_{k-d}$. As a result the Bayesian model for OOSM problem under target existence uncertainty becomes

$$\begin{aligned} & p(x_k, E_k, \ldots, x_{k-d}, E_{k-d}|Y_k) \\ = \; & p(X_k, E_k|Y_k) \end{aligned} \tag{6.4}$$

The modeling in (6.4) is exactly same as the proposed Bayesian smoothing model in (4.46) with the lag being replaced by the delay parameter d. The only difference introduced by the OOSM problem is the augmentation of the sensor observation vector with the delayed measurements. In the next two sections, reflections of this modification to cater for OOSM issue will be detailed for both standard AS-IPDA sensor modeling and random set modeling for sensor observations.

6.4 Modification in AS-IPDA Sensor Model

The standard sensor observation model for the augmented state is given by (4.45) and is provided here for clarity.

$$Y_k = \begin{bmatrix} y_k & \ldots & y_{k-d} \end{bmatrix}^T = \mathcal{H}_k X_k + W_k \tag{6.5}$$

where

$$\mathcal{H}_k = \begin{bmatrix} H_k & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & H_{k-d} \end{bmatrix} \tag{6.6}$$

$$W_k = \begin{bmatrix} w(k) & \ldots & w(k-d) \end{bmatrix}^T \tag{6.7}$$

The sensor noise covariance matrix is defined as

$$\mathcal{R}_k = \begin{bmatrix} R_k & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & R_{k-d} \end{bmatrix} \tag{6.8}$$

(H_k, $w(k)$, R_k are as defined in section 2.5). In (6.6)-(6.8), 0's and I's refer to zero and identity matrix respectively with appropriate dimensions.

The target state augmentation is carried out in the same manner is defined in relations (4.44) through (2.43) with the lag parameter N replaced by the delay parameter d. As a result, the hypotheses defined in section 4.5.3, explaining the possible transitions between target existence and non-existing between time $t = k - d$ to $t = k$, also holds.

The redefined sensor observation model as (6.5) and the target motion model being the same as (4.44), the Bayesian model in (6.4) for handling OOSM suggests that the derivation follows exactly same as proposed in chapter 4. Therefore, the proposed AS-IPDA holds, in itself, a mechanism to solve OOSM problem with the sensor model redefined as in (6.5).

6.5 Modification in Random Set Sensor Model

The sensor model in random set domain is given by

$$\Sigma_k = \Sigma_k^{'} \cup \Lambda_k \tag{6.9}$$

where

- $\Sigma_k^{'}$ denotes the model for target originated measurements
- Λ_k refers to the clutter measurement model

Again to cater for delayed measurements, the sensor observation model has to include the measurements from the delayed instant. In random set formalism, the scenario can be modeled as

$$\Xi_k = \Sigma_k \cup \Sigma_{k-d} \tag{6.10}$$

The target motion dynamics is modeled in similar manner as in (5.50). The union of sets is taken over the entire delay period between $t = k - d$ to $t = k$.

The random set smoother derivation steps, described in chapter 5, can then be followed in exactly same manner replacing the sensor observation set Σ_k by Ξ_k.

6.6 Handling multiple delayed measurements

The models proposed in section 6.3 through 6.5 introduces a single delay of $t = k - d$. In reality measurements from different delayed time instants may arrive. This arrival of measurements from multiple delayed instants can be handled by introducing those delayed measurements in the generalized Bayesian formulation. This can be given by

$$Y_k = \{y_k, y_{k-d_1}, y_{k-d_2}, \cdots y_{k-d_e}\} \tag{6.11}$$

where d_e is the maximum delay among all the observations received.

In principle, this refers to the fact that as soon as a measurement arrives, it needs to be augmented in the measurement vector. Correspondingly, the target state and existence vector needs to be augmented covering the duration of the maximum delays.

$$X_k = x_k, E_k, \dots, x_{k-d_e}, E_{k-d_e} \tag{6.12}$$

The corresponding sensor model equations for catering multiple delayed measurements are to be modified in this manner

- The parameters H_k and R_k in (6.6) and (6.8) respectively have non-zero values where the time index corresponds to the delay for which observations are obtained and has zero values otherwise.

For random set formalism, the global observation set is the union of all the observation sets obtained from different delays.

$$\Xi_k = \Sigma_k \cup \Sigma_{k-d_1} \cup \Sigma_{k-d_2} \cup \dots \cup \Sigma_{k-d_e} \tag{6.13}$$

where e is the maximum of the delays of the measurements collected.

6.7 Conclusion

In this chapter, a theoretical modeling is proposed to address the problem of out of sequence measurements under target existence uncertainty. The Bayesian model is first proposed with a single delayed measurement. The modifications required in standard Bayesian approach is first identified. Then based on the model, the specific system models (both target motion model and sensor model) are introduced for both IPDA and random set approaches. Finally the model is generalized for any number of delayed measurements.

7. SIMULATION RESULTS

7.1 Introduction

This chapter provides the performance measures of the proposed algorithm through the results of various simulation scenarios. The smoother is compared with the standard filter algorithm against the major parameters like RMS error in state estimation, number of confirmed true and/or false tracks and detection of termination time of targets. The chapter provides details of the simulated scenario. The obtained results are also analyzed for conclusive deduction about the performance of the proposed algorithm.

7.2 Simulation Scenario

The basic simulated environment follows the benchmark scenario described in [61]. The two dimensional surveillance area is 1000m long and 400m wide. The sensor receives target originated measurement with a detection probability P_D (the value of which is specified in the description). The sensor also receives measurements from clutter. The number of these false measurements in a single scan is drawn randomly from a Poisson distribution with the mean of being $1 times 10^{-4}/scan/m^2$. But the false measurements, received in a single scan, are uniformly distributed within the whole surveillance region.

A single non-maneuvering target moves within the sensor surveillance region and follows the dynamic equation

$$x_{k+1} = Fx_k + v_k \tag{7.1}$$

The state of the target at time $t = k$, x_k, consists of position and velocity of the target in the direction of each co-ordinate.

$$x_k = \begin{bmatrix} x & \dot{x} & y & \dot{y} \end{bmatrix}^T \tag{7.2}$$

The transition matrix F encapsulates the target dynamics and is given by

$$F = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.3}$$

where $T = 1sec$ is the sampling period. The zero mean white Gaussian process noise v_k accounts for small target maneuvers. The variance of the distribution is known a priori and is given by

$$E[v_i v_j^{'}] = Q\delta(i,j) \tag{7.4}$$

where $\delta(i,j)$ is the Kronecker delta function and

$$Q = q \begin{bmatrix} \frac{T^4}{4} & \frac{T^3}{2} & 0 & 0 \\ \frac{T^3}{2} & T^2 & 0 & 0 \\ 0 & 0 & \frac{T^4}{4} & \frac{T^3}{2} \\ 0 & 0 & \frac{T^3}{2} & T^2 \end{bmatrix} \tag{7.5}$$

For the simulation purpose, q is chosen as 0.75.
The sensor reports the position of the target in the two co-ordinate system. The sensor model is therefore given by

$$y_k = Hx_k + w_k \tag{7.6}$$

where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{7.7}$$

The random noise w_k is assumed to be drawn from a Gaussian distribution with zero mean. The variance of the distribution is known and is given by

$$E[w_i w_j^{'}] = R\delta i, j \tag{7.8}$$

In the simulation scenario,

$$R = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} \tag{7.9}$$

which follows the benchmark environment of [61]. The elliptical gate volume is chosen to be 13 to ensure 99% probability of having target originated measurement within the gate.

For a meaningful comparison, the algorithms are executed through 1000 (unless otherwise specified) Monte Carlo runs. Again following the standard simulation environment, during the start of each iteration, one target appears with initial state $[\ 130 \quad 35 \quad 200 \quad 0\ ]^T$. The targets are initiated using two point differencing method as described in [5]. Both the algorithms use the Markov Chain One transition chain with probablities

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{01} \\ \Gamma_{10} & \Gamma_{00} \end{bmatrix} = \begin{bmatrix} 0.98 & 0 \\ 0.02 & 1 \end{bmatrix} \tag{7.10}$$

Each target is started with an initial probability of existence of 0.2. During each iteration, the existence probability is updated and the target evolves from remaining tentative either to confirmed or terminated targets. If the target existence probability is above "confirmation threshold", the target is confirmed. On the other hand if that is below the "termination threshold", the target is terminated. The chosen threshold values are given in the table below

	AS-IPDA Smoothing	IPDA
Confirmation Threshold	0.95	0.99
Termination Threshold	0.05	0.03

The reason for choosing the above mentioned thresholds is to make the false track statistics of both the algorithms almost the same while maximizing the true target confirmation statistics. The average number of false tracks were 164 for IPDA filter. The same for AS-IPDA smoothing was 172 (for lag 1), 175 (for lag 2 and 3), 179 (for lag 4) and 183 (for lag 5 and 6).

7.3 Simulation Results

7.3.1 Target Termination Time Detection

For this particular simulation, the algorithms are run for 50 Monte Carlo Runs with two specific values of transition probability Γ_{11} (refer to (7.10)). The duration of each run was 40 scans. But the true target was terminated at $30th$ scan and the observations in the next 10 scans contain only the clutter observations. Detection probability used was $P_D = 0.9$. The results are summarized in the tables 7.1 and 7.2.

Tab. 7.1: First Case : $\Gamma_{11} = 0.98$, Actual Termination Time = 30

Filter termination	**Smoother termination**			
	Lag 1	Lag 2	Lag 3	Lag 4
34	33	32	31	30

Tab. 7.2: Second Case : $\Gamma_{11} = 0.9$, Actual Termination Time = 30

Filter termination	**Smoother termination**			
	Lag 1	Lag 2	Lag 3	Lag 4
33	32	32	32	31

7.3.2 State Estimation

The simulation scenario for comparison of RMS error in state estimation follows the standard scenario as described in section 7.2. Two probabilities of detection $P_D = 0.9$ and $P_D = 0.8$ are used. The smoother was run on the same set of measurements for 1000 Monte Carlo runs for various fixed lags ranging from 1 to 6. The obtained comparison result is shown in the figures 7.1-7.8.

7.3.3 Number of Confirmed True targets

One of the major performance measures of automatic tracking algorithms is the number of true targets that it can confirm. The simulation is carried out to test the algorithms for that parameter and the obtained Monte Carlo result (1000 run) is shown in figure 7.9 and 7.10.

7.4 Performance Analysis

As expected theoretically, the application of smoothing algorithm reduces the state estimation error. In the case of smoothers, as opposed to filters, the inclusion of more measurements or, in other sense - information, results in a better

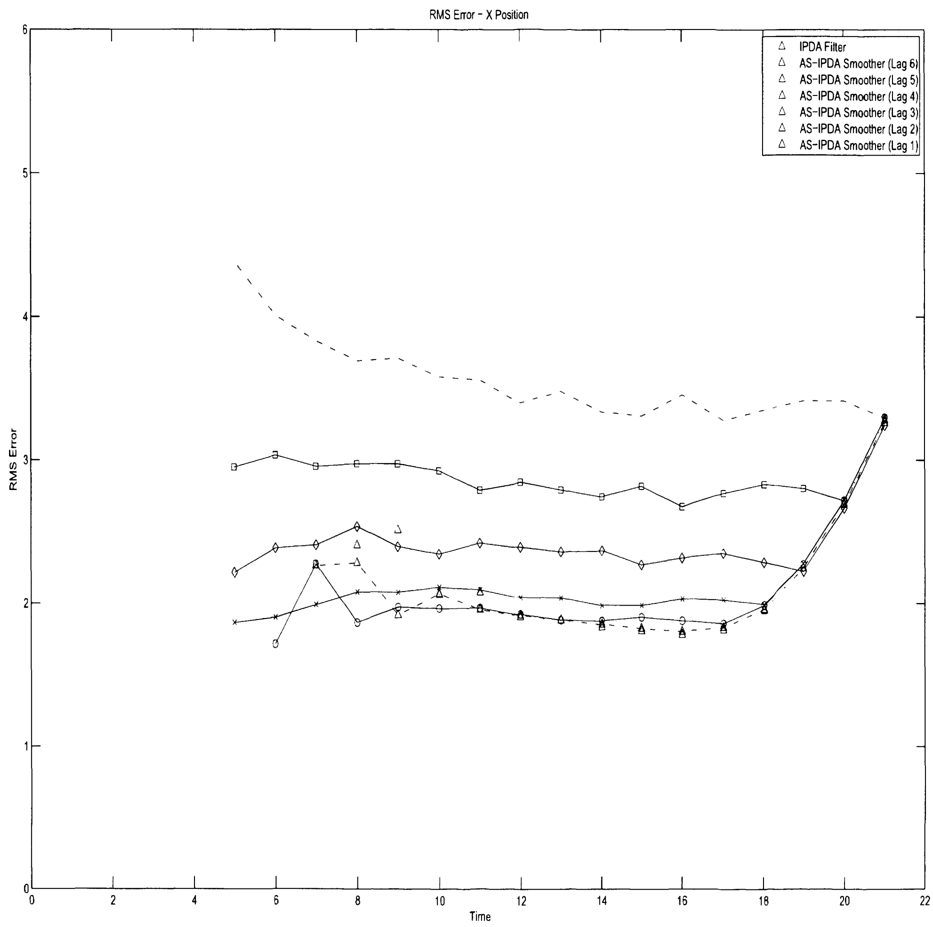

Fig. 7.1: RMS Error : X position, Detection Probability 0.9

target dynamic state estimation. The effect of smoother is also very evident in terms of number of confirmed true targets. The Monte Carlo result shows that more number of true targets are confirmed which translated into less percentage of lost targets. Finally it is also significant that the detection of termination time of a target can also be inferred better by introducing a fixed lag in estimation. This has a profound effect on applications like "Situation Awareness" where the improved result of the tracker may in turn make the decision processing more efficient.

Given that the AS-IPDA smoother provides better target state as well as maintenance statistics, its theoretical linkage to Random Set Formalism (as proved in chapter 5) provides a very suitable platform to contribute significantly in multi-target tracking environment.

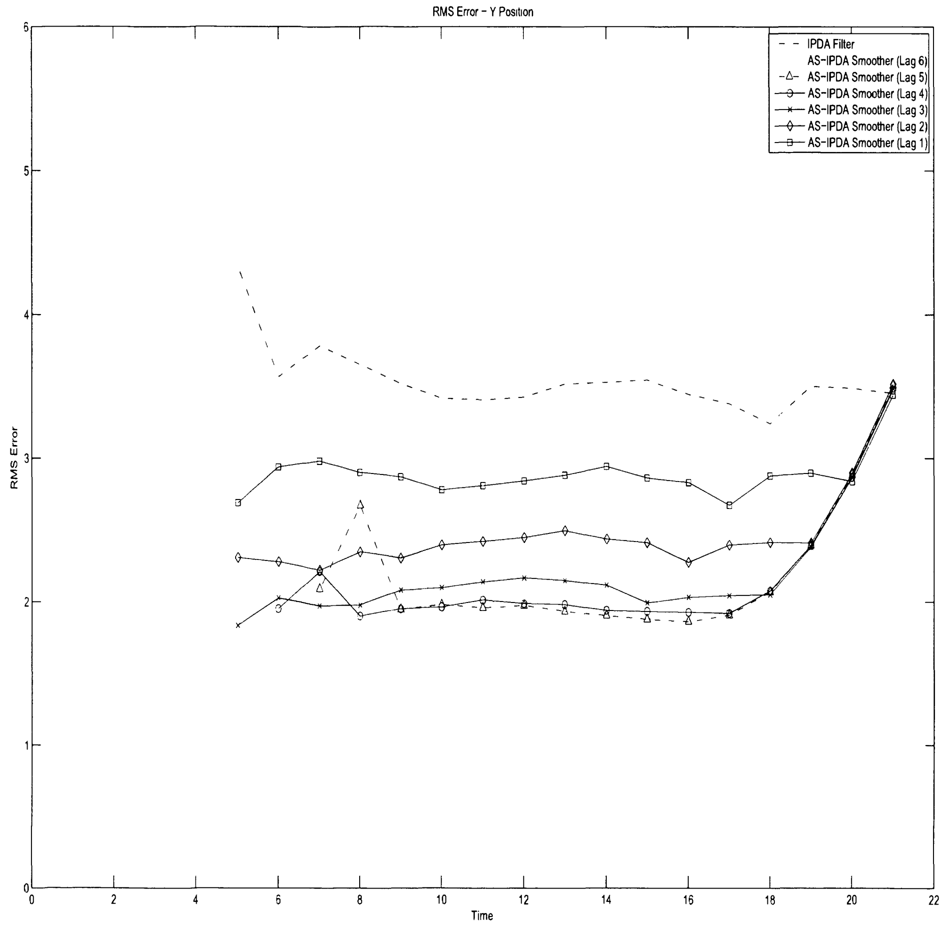

Fig. 7.2: RMS Error : Y position, Detection Probability 0.9

7.5 Conclusion

In this chapter, the simulation scenario of the proposed smoothing algorithm is described in detail along with particular values for various parameters. The proposed algorithm is applied with different fixed lags and is compared with standard IPDA filter against performance parameters like RMS error, target termination detection and confirmed number of true targets. Theoretically, smoother uses more observations, compared to filters, to deduce the estimate of the random variables of interest and therefore is expected to provide more accuracy. The simulated results in this chapters confirms this. The improvements observed due to application of the smoothing algorithm are significant in decision support system applications where a slight delay in information gathering can be allowed in return for improved accuracy.

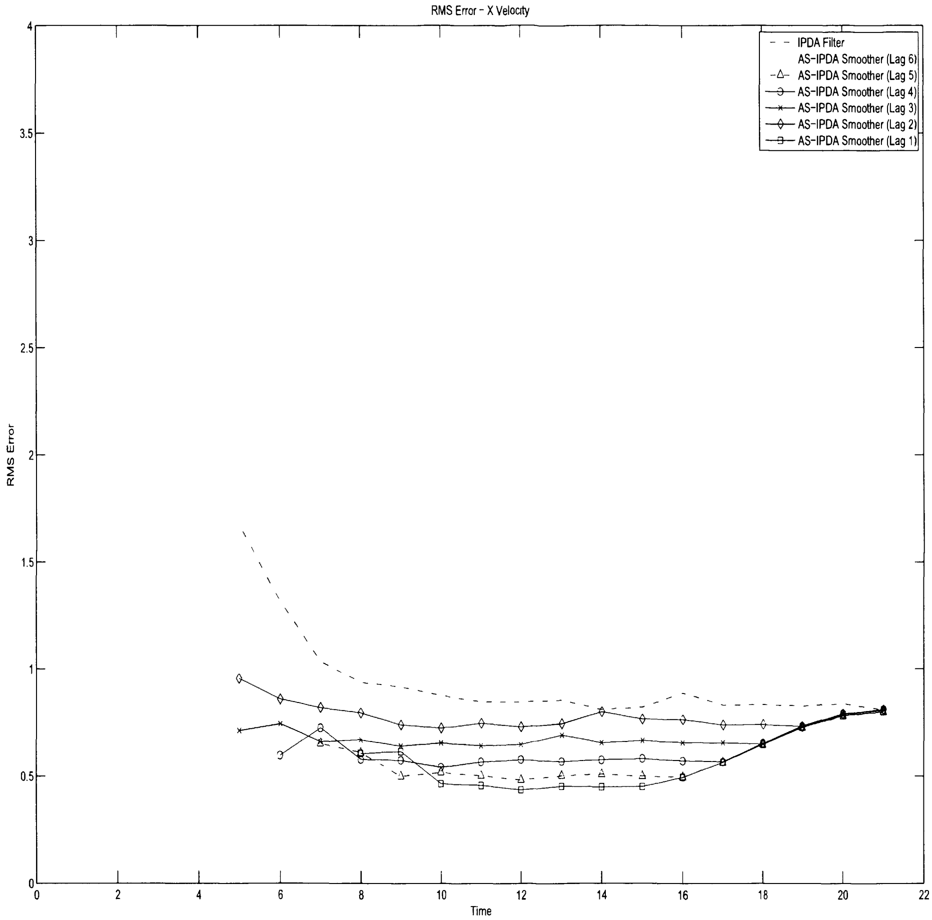

Fig. 7.3: RMS Error : X Velocity, Detection Probability 0.9

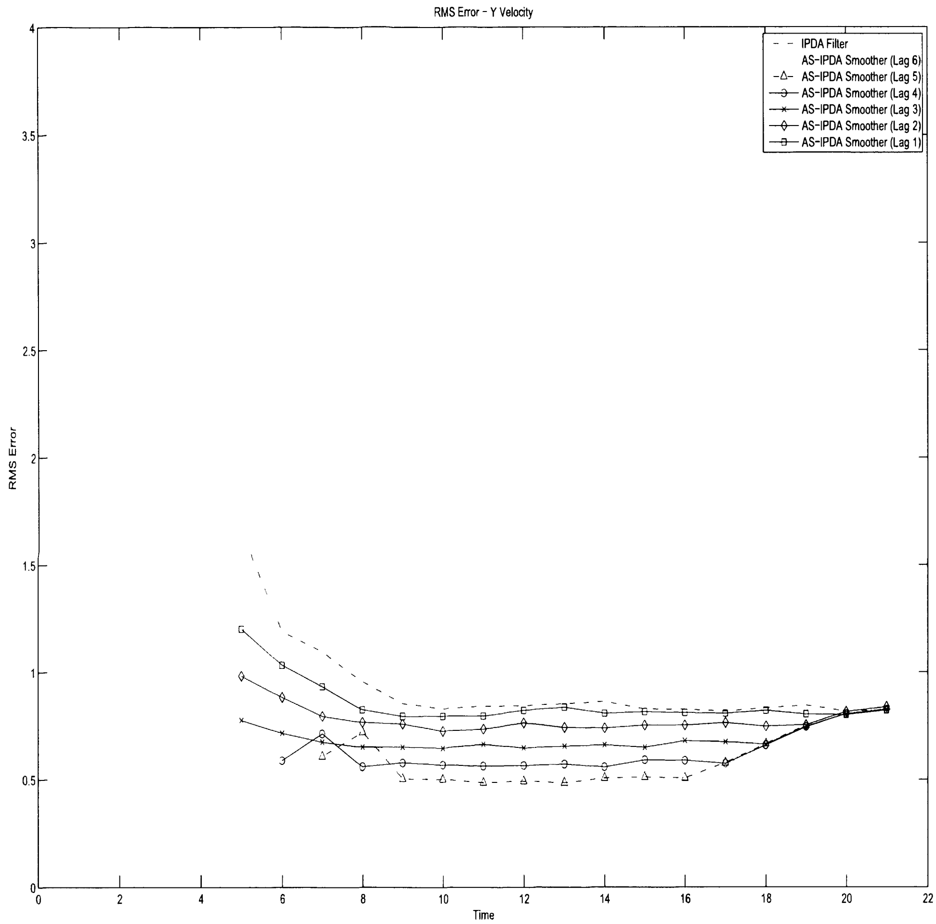

Fig. 7.4: RMS Error : Y Velocity, Detection Probability 0.9

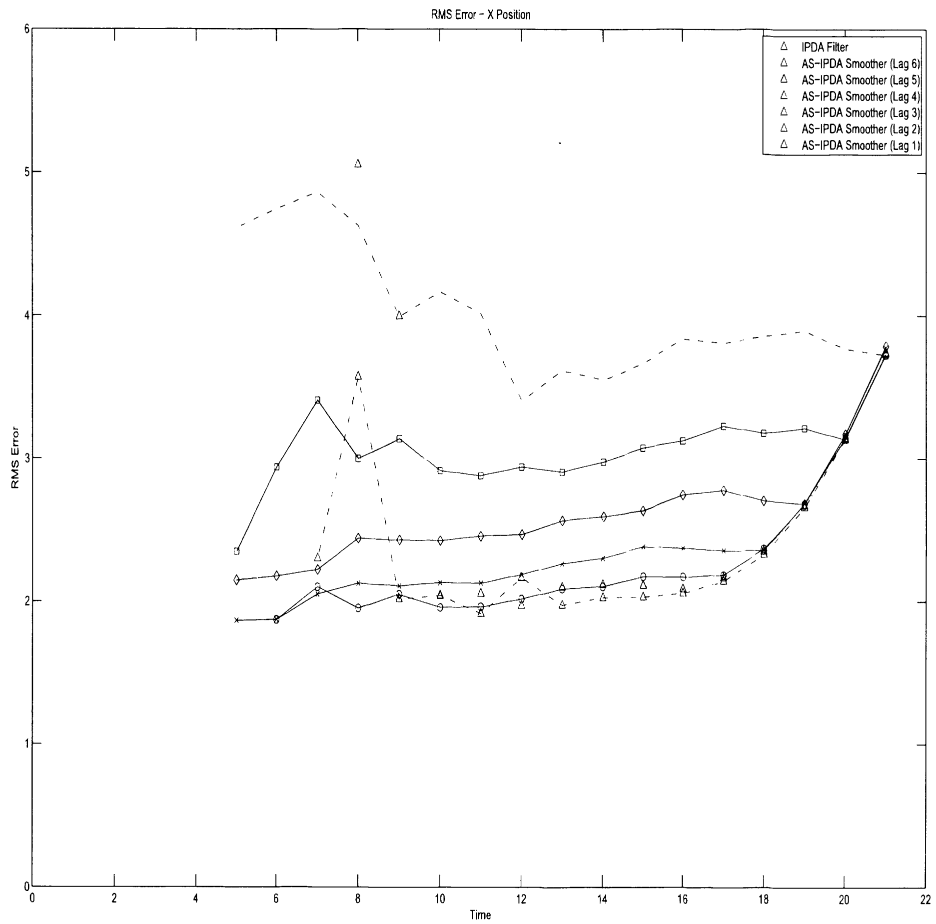

Fig. 7.5: RMS Error : X position, Detection Probability 0.8

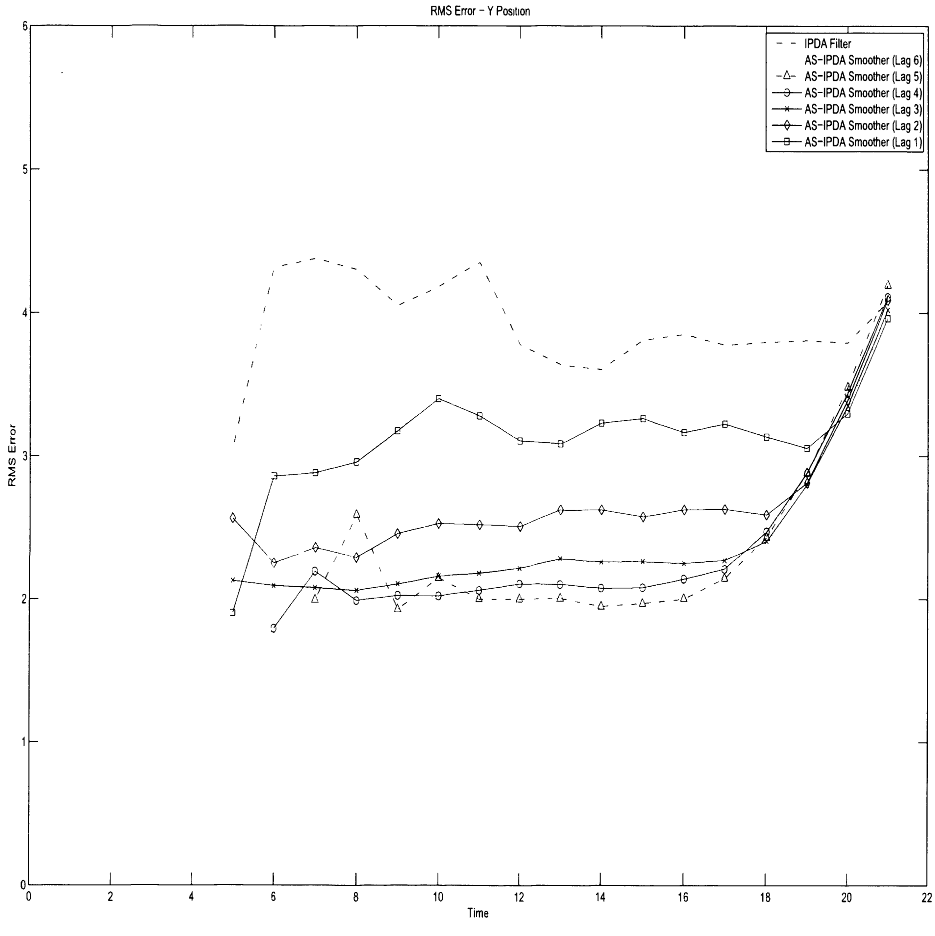

Fig. 7.6: RMS Error : Y position, Detection Probability 0.8

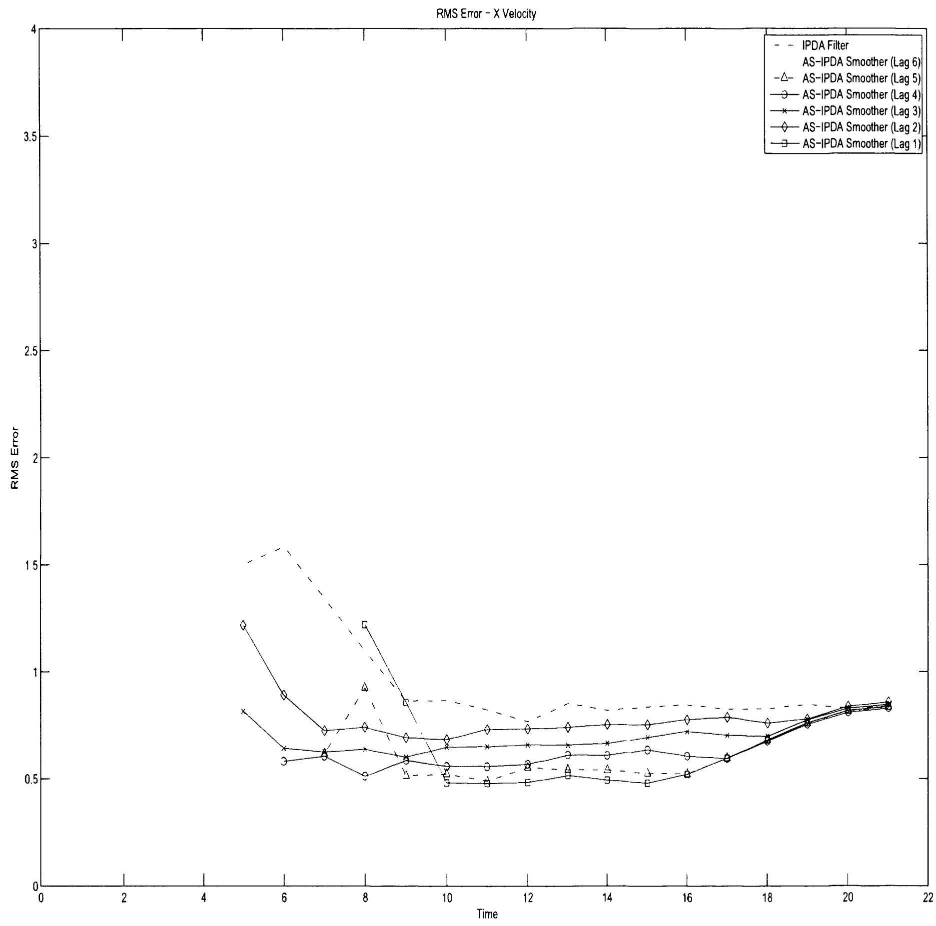

Fig. 7.7: RMS Error : X Velocity, Detection Probability 0.8

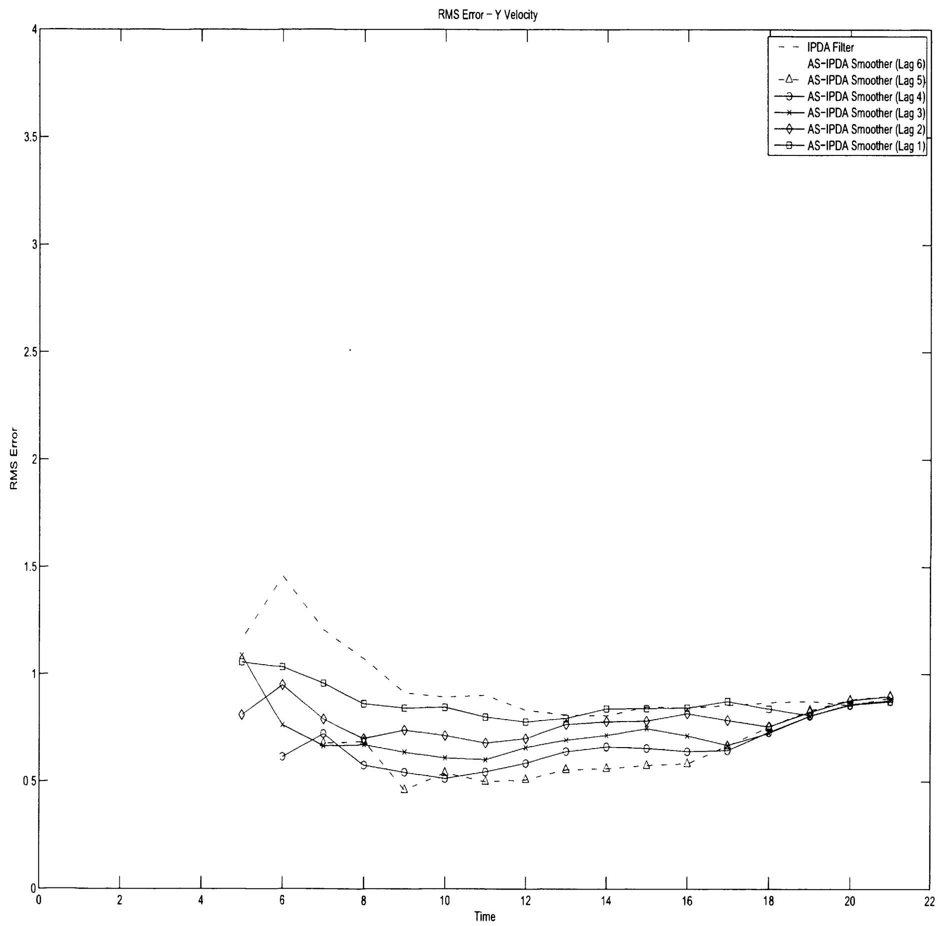

Fig. 7.8: RMS Error : Y Velocity, Detection Probability 0.8

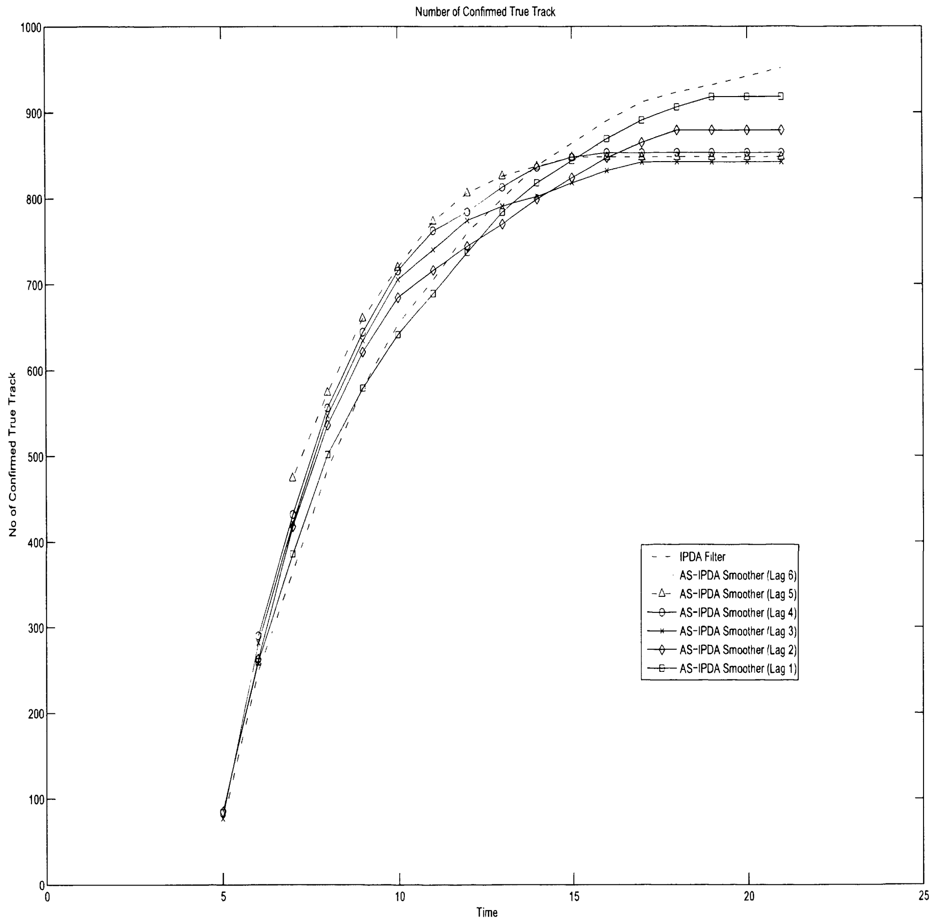

Fig. 7.9: Number of Confirmed True Tracks, Detection Probability 0.9

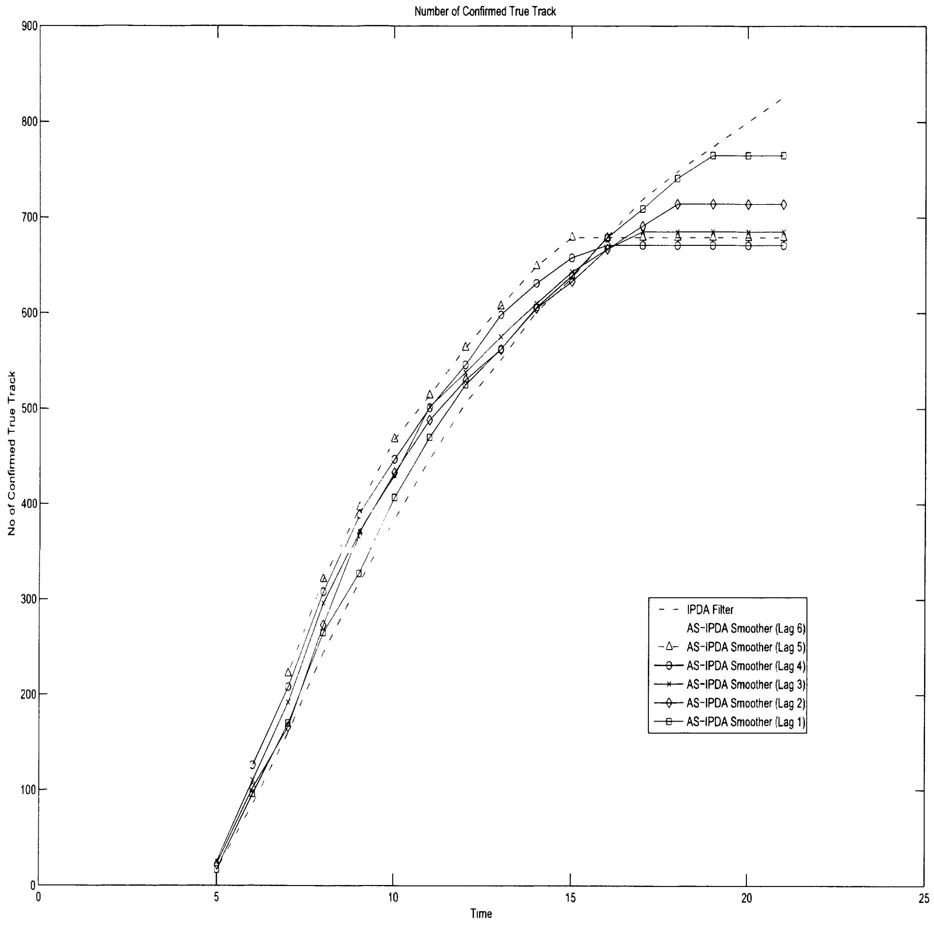

Fig. 7.10: Number of Confirmed True Tracks, Detection Probability 0.8

8. CONCLUSION AND FUTURE WORK

8.1 Introduction

This thesis investigated the effect of introducing smoothing estimation in target tracking scenario with target existence uncertainty. This chapter is aimed at summarizing the findings of the effort with conclusive remarks and a guide for future works.

8.2 Conclusion and Summary

In 1.3, the objectives and contributions of the thesis have been summarized. In light of that, the original contributions of the research, that have been achieved, are :

- Original formulation of Augmented State IPDA (AS-IPDA) smoother for improved track initiation and maintenance
- Original framework of generalized Random Set Smoother
- Establishing a link between the AS-IPDA algorithm and the generalized random set smoothing algorithm under target existence uncertainty scenario
- Theoretical framework for resolving out-of-sequence measurement problem in a multi-sensor tracking application with target existence uncertainty.

These proposed smoothers are also compared through simulation, using published benchmark scenarios [61, 59, 60], with standard IPDA filter for establishing the improvements through simulation. The thesis therefore also establishes the significance of applying smoothing through following advantages in performance measures:

- Improved state estimation for targets in clutter (less RMS error)
- Improved detection and confirmation of true targets (less percentage of lost tracks)
- Improved estimate of track termination times

The development of AS-IPDA provides a smoothing technique for a single target tracking in clutter for automatic track maintenance. The generalized random set smoothing algorithm, proposed in the thesis, lays the theoretical platform to tackle multi-target tracking problem with automatic track maintenance. The theoretical link between the AS-IPDA and the generalized random set smoothing algorithm also provides an insight about extending existing algorithms into the realm of multiple target tracking problem.In this thesis, Bayesian models are developed to incorporate target existence uncertainty problem for delayed measurements.

8.3 Extensions and Future Work

The research carried out was aimed at investigation of a smoothing algorithm that can enhance the performance of a tracker in terms of reduced error in state estimation and improved target maintenance in clutter. The theoretical work and formulation is carried out and was proved to be effective through simulation. The current task has also the potential for further exploitation and extensions.

The interacting multiple model algorithm IMM [18], was originally proposed for maneuvering target tracking. The IMM was then extended for problems involving clutter and to address the problem of target existence uncertainty, [5]. The resulting IMM-PDA algorithm is an alternative to IPDA algorithm for tracking with target existence uncertainty. In contrast to IPDA, IMM-PDA models the target state as "observable" ors "non-observable". The probabilities of these two interacting models give the "true target probability" to decide about the target existence. The IMM-PDA is extended to smoothing for maneuvering target tracking. Therefore an extension of the existing IMM-PDA to smoothing for target existence uncertainty scenario is still an open research question. Similar to AS-IPDA smoothing technique developed in this thesis, an IMM-PDA smoother can be derived where the target switches between "observable" and "non-observable" states.

Algorithm like IMM-IPDA, [28], considers automatic initiation of maneuvering target tracking in clutter. Extension of this into smoothing is also an interesting problem as it poses the problem of switching of targets among the various maneuvering models as well as existence models.

Established filtering algorithms like Multiple Hypothesis Algorithm can also be extended to smoothing for automatic track initiation.

AS-IPDA smoothing is shown to have improved track maintenance parameters. Extending the other algorithms into smoothing are also expected to have an improved affect on the track maintenance. These improvements provide a significant boost in performance for target trackers as well as higher level "Situation Awareness" applications to decide different parameters like instant of target appearance/confirmation/termination, maneuver, number of targets, classification of targets and etc. An effort of developing such algorithms and comparing the results in terms of track maintenance parameters, computational load requirement under both single target and multi-target environment is required.

A real time optimization of the algorithm and implementation of it is still a challenging task. While the core algorithm steps remain the same, optimization is possible in individual modules and its effect on performance - both in computation and in target tracking aspects - in real time is an interesting problem still to be undertaken.

Fusion of target existence uncertainties from different sensors and possible improvements thereby is also another aspect worthy of investigation. Also in the case of wireless sensors, the minimum data rate that will carry "useful" information is an important parameter because of limited band width. The consideration of other practical aspects like terrain, weather, power consumption, limited computation and some other factors affect the accuracy of the system as a whole. A systematic study of such a real time scenario is essential for deploying such systems in action.

Another multi disciplinary effort would be to apply the standard tracking applications in the realm of image processing, intelligent video processing and event detections and etc. In theory, the tracking algorithms are generic enough to be applicable wherever the estimation of some parameters are required. But the study of various problems to devise appropriate models for that specific problem is not completed yet, especially for the application of sophisticated algorithms like IPDA and etc. But this effort can have a very profound effect on the performance of some of the mostly used commercial application. More importantly, such an investigation can open a new horizon in applying advanced algorithms of "data fusion" for reduced uncertainty and false alarms.

APPENDIX

A. PUBLICATIONS

A.1 Refereed Journal Article

- **R.Chakravorty**, S.Challa, "Augmented State Integrated Probabilistic Data Association Smoothing for Multiple Target Tracking in Clutter", Journal of Advances in Information Fusion, vol 1, no 1, November 2006, pp. 63-74

A.2 Refereed Conference Article

- **R. Chakravorty**, D. Mušicki and S. Challa, "A fixed lag IPDA smoothing for target tracking in Clutter", Proceedings of 9th International Conference on Information Fusion, 2006
- B. Challa, S. Challa, D. Sharma, **R. Chakravorty**, Electricity Load Forecasting Using Real Time Information from Distributed Sensor Networks, Proceedings in Innovations and Commercial Applications of Distributed Sensor Networks Symposia, 2005
- **R.Chakravorty**, S.Challa, "Smoothing Framework for Automatic Track Initiation in Clutter", Proceedings of Eighth International Conference on Information Fusion, July 2005, pp. 54-61
- S. Challa, S. K. Deshpande, **R. Chakravorty**, A Novel Approach for Electrical Load Forecating Using Distributed Sensor Networks, Proceedings of Third International Conference on Intelligent Sensing and Information Processing, 2005, pp. 189-194
- **R.Chakravorty**, S.Challa ,"Fixed Lag Smoothing Technique for Track maintenance in clutter"', Proceedings of Intelligent Sensor, Sensor Networks and Information Processing, December, 2004, pp. 119-124
- **R. Chakravorty**, S.Challa, "A Single Lag Smoothing Technique for Track maintenance in clutter", IEEE proceedings on First Cybernatics and Intelligent System Conference, December, 2004, pp. 1266-1270

B. EXISTENCE HYPOTHESES PROBABILITY DERIVATION

B.1 Derivation of $p(\mathbf{E}_k^0|\mathbf{Y}^k)$

For $m = 0$

$$\begin{aligned}
p(\mathbf{E}_k^0|y^k) &= p(\mathbf{E}_k^0|y_k, y^{k-1}) \\
&= \frac{p(y_k|\mathbf{E}_k, y^{k-1}).p(\mathbf{E}_k|y^{k-1})}{\Delta} \\
&= \frac{1}{\Delta}\delta \times \left[p(E_k, \ldots, E_{k-N}, E_{K-N-1}|y^{k-1})\right] \\
&= \frac{1}{\Delta}\delta \times \left[p(E_k|E_{k-1}) \times p(E_{k-1}, \ldots, E_{k-N}, E_{K-N-1}|y^{k-1})\right] \\
&= \frac{1}{\Delta}\delta\Gamma_{11}p(\mathbf{E}_{k-1}^0|y^{k-1}) \qquad \text{(B.1)}
\end{aligned}$$

For $m = 1$

$$\begin{aligned}
p(\mathbf{E}_k^1|y^k) &= p(\mathbf{E}_k^1|y_k, y^{k-1}) \\
&= \frac{p(y_k|\mathbf{E}_k^1, y^{k-1}).p(\mathbf{E}_k^1|y^{k-1})}{\Delta} \\
&= \frac{1}{\Delta}p(y_k|\overline{E}_k, y^{k-1}).p(\overline{E}_k, E_{k-1}, \ldots, E_{k-N}|y^{k-1}) \\
&= \frac{1}{\Delta}p(y_k|\overline{E}_k, y^{k-1}).p(\overline{E}_k|E_{k-1}) \times p(E_{k-1}, \ldots, E_{k-N}|y^{k-1}) \\
&= \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)\Gamma_{10}p(\mathbf{E}_{k-1}^0|y^{k-1}) \qquad \text{(B.2)}
\end{aligned}$$

(B.3)

For $m = 2, \ldots, N$

$$\begin{aligned}
p(\mathbf{E}_k^m|y^k) &= p(\mathbf{E}_k^m|y_k, y^{k-1}) \\
&= \frac{p(y_k|\mathbf{E}_k^m, y^{k-1}).p(\mathbf{E}_k^m|y^{k-1})}{\Delta} \\
&= \frac{1}{\Delta}p(y_k|\overline{E}_k, y^{k-1}).p(\overline{E}_k, \ldots, \overline{E}_{k-m+1}, E_{k-m}, \ldots, E_{k-N}|y^{k-1}) \\
&= \frac{1}{\Delta}p(y_k|\overline{E}_k, y^{k-1}).p(\overline{E}_k|\overline{E}_{k-1}).p(\overline{E}_{k-1}, \ldots, \overline{E}_{k-m+1}, E_{k-m}, \ldots, E_{k-N}|y^{k-1}) \\
&= \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)\Gamma_{00}p(\mathbf{E}_{k-1}^{m-1}) \qquad \text{(B.4)}
\end{aligned}$$

(B.5)

B.2 Derivation of $p(\overline{\mathbf{E}}_k|y^k)$

$$
\begin{aligned}
p(\overline{\mathbf{E}}_k|y^k) &= p(\overline{\mathbf{E}}_k|y_k, y^{k-1}) \\
&= \frac{p(y_k|\overline{\mathbf{E}}_k, y^{k-1}).p(\overline{\mathbf{E}}_k|y^{k-1})}{\Delta} \\
&= \frac{1}{\Delta} p(y_k|\overline{E}_k, y^{k-1}).p(\overline{E}_k, \ldots, \overline{E}_{k-N}|y^{k-1}) \\
&= \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)\left[p(\overline{E}_k, \ldots, \overline{E}_{k-N}, E_{k-N-1}|y^{k-1}) + \right. \\
&\quad \left. p(\overline{E}_k, \ldots, \overline{E}_{k-N}, \overline{E}_{k-N-1}|y^{k-1})\right] \\
&= \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)\left[\Gamma_{00} p(\mathbf{E}^N_{k-1}|y^{k-1})\right. \\
&\quad \left. + \Gamma_{00} p(\overline{\mathbf{E}}_{k-1}|y^{k-1})\right] \\
&= \frac{1}{\Delta}\left(\frac{1}{V_k}\right)^{m_k} P_0(m_k)\Gamma_{00} \\
&\quad \left[p(\mathbf{E}^N_{k-1}|y^{k-1}) + p(\overline{\mathbf{E}}_{k-1}|y^{k-1})\right] \qquad \text{(B.6)}
\end{aligned}
$$

C. PROOF : HYPOTHESES H_K^M DOES NOT CONTRIBUTE TO AS-IPDA STATE UPDATE

$$\begin{aligned}
p(\mathbf{X}_k^m|\mathbf{E}_k^m,\mathbf{Y}^k) &= p(\mathbf{X}_k^m|\mathbf{E}_k^m,\mathbf{Y}_k,\mathbf{Y}^{k-1}) \\
&= \frac{p(\mathbf{Y}_k|\mathbf{X}_k^m,\mathbf{E}_k^m,\mathbf{Y}^{k-1}).p(\mathbf{X}_k^m|\mathbf{E}_k^m,\mathbf{Y}^{k-1})}{p(\mathbf{Y}_k|\mathbf{E}_k^m,\mathbf{Y}^{k-1})} \\
&= \frac{\mathit{Likelihood} \times \mathit{Prediction}}{\mathit{Normalization}} \qquad \text{(C.1)}
\end{aligned}$$

Under the assumption that the target does not exist at time $t = k$ (based on the hypothesis definition), the likelihood term in (C.1) reduces to,

$$\begin{aligned}
p(\mathbf{Y}_k|\mathbf{X}_k^m,\mathbf{E}_k^m,\mathbf{Y}^{k-1}) &= p(y_k|\overline{E}_k,y^{k-1}) \\
&= \left(\frac{1}{V_k}\right)^{m_k} P_0(m_k) \qquad \text{(C.2)}
\end{aligned}$$

The prediction term is

$$\begin{aligned}
&p(\mathbf{X}_k^m|\mathbf{E}_k^m,\mathbf{Y}^{k-1}) \\
&= p(x_{k-m-1},\ldots,x_{k-N}|E_{k-m-1},\ldots,E_{k-N},y^{k-1}) \qquad \text{(C.3)}
\end{aligned}$$

Lastly the normalization is

$$\begin{aligned}
&p(\mathbf{Y}_k|\mathbf{E}_k^m,\mathbf{Y}^{k-1}) \\
&= \int_{\mathbf{X}_k^m} p(\mathbf{Y}_k|\mathbf{X}_k^m,\mathbf{E}_k^m,\mathbf{Y}^{k-1}) \times p(\mathbf{X}_k^m,\mathbf{E}_k^m,\mathbf{Y}^{k-1}))d\mathbf{X}_k^m \\
&= \left(\frac{1}{V_k}\right)^{m_k} P_0(m_k) \qquad \text{(C.4)}
\end{aligned}$$

It is clear the likelihood and normalization terms are same and therefore (C.1) is simplified as

$$p(\mathbf{X}_k^m|\mathbf{E}_k^m, \mathbf{Y}^k)$$
$$= p(x_{k-m-1}, \ldots, x_{k-N}|E_{k-m-1}, \ldots, E_{k-N}, y^{k-1}) \tag{C.5}$$

Thus (C.5) shows that the other hypothesis, where the target does not exist at current time $t = k$, does not contribute to the update of the state and co-variance. So if the target does not exist at current time, the previous smoothed or filtered values are retained as it is.

D. CALCULUS OF FINITE-SET STATISTICS (FISST)

D.1 Introduction

The statistics of finitely varying random sets depend on the mathematics of finite sets. In this appendix, the finite set mathematics will be presented. This appendix is reconstructed principally from the text, [49].

D.2 Random Set Model for target dynamics and sensors

Random set notation for target motion model is give by

$$\Gamma_{k+1} = \Phi_k(X_k, V_k) \cup B_k(X_k) \tag{D.1}$$

$\Phi(.)$ represents the change of target dynamics from time $t = k$ to $t = k + 1$. $B(.)$ caters for target birth process in multiple target case.

Similarly, the sensor model in random set notation is given by

$$\Sigma = T(X) \cup C(X) \tag{D.2}$$

$T(.)$ defines the measurements originated from true targets while $C(.)$ accounts for clutter measurements.

The models (D.1) and (D.2) give rise to multi-target Markov transition densities and global likelihood for randomly varying set.

D.3 Belief Mass function of Sensor Model

The probability mass $p(S|x) = Pr(Z \in S)$ captures the statistical behavior of observation set Z. In random set domain, the statistics of Σ is characterized by its belief mass function $\beta(S|X)$. The belief mass measure is given by

$$\beta(S|X) = \beta_{\Sigma|\Gamma}(S|X) = Pr(\Sigma \subseteq S) \tag{D.3}$$

This belief mass measure is *total probability that all observations in a sensor (or multi sensor) scan* will be found in any region S. The belief mass function in (D.3) provides the global likelihood density.

D.4 Belief Mass function of target motion model

The probability mass $p(S|x_k) = Pr(X_{k+1} \in S)$, which is the probability that the target state X_{k+1} will be found in the region S conditioned on previous state x_k, provides the statistical measure of the target dynamics. For a randomly varying finite length set Γ_{k+1}, the sufficient statistics is captured in the belief mass function given by

$$\beta_{\Gamma_{k+1}}(S|X_k) = Pr(\Gamma_{k+1} \subseteq S) \tag{D.4}$$

Markov transition density is calculated from the belief mass measure in (D.4).

D.5 Basics of FISST Mathematics

Unlike probability mass, belief mass functions are non-additive measures. In general, if $S_1 \cap S_2 = \phi$, while joint probability mass $p(S_1 \cup S_2|x) = p(S_1|x) + p(S_2|x)$, the joint belief mass $\beta(S_1 \cup S_2|x) \geq \beta(S_1|x) + \beta(S_2|x)$. It is also noted in [49] that belief mass measure behaves like probability mass on certain abstract topological spaces. This additional property introduces some additional complexities in calculating densities from belief mass functions.

In general, the relation between belief mass measure of a certain randomly varying set (probability of the random set being in the region S), is given by

$$\beta(S|B) = \Pr(A \in S|B) \int_S f(A|B)\delta A \tag{D.5}$$

where A denotes either the target dynamic set Γ_k or sensor observation set Σ. The integration in (D.5) refers to the sum of densities or likelihoods of all possibilities suggested by random set.

D.6 Set Integral Rule

The integration in (D.5) follows FISST "set integeral" rule. An illustrative example (following [49]), will be useful to demonstrate "set integral".

Assuming a function $F(Y)$ is given for a finite-set variable Y, $F(Y)$ can have following forms

$$\begin{aligned} F(\phi) &= \text{probability that } Y = \phi \\ F(\{y\}) &= \text{likelihood that } Y = \{y\} \\ F(\{y_1, y_2\}) &= \text{likelihood that } Y = \{y_1, y_2\} \end{aligned}$$

$$\vdots$$

$$F(\{y_1, y_2, \ldots, y_j\}) \quad = \quad \text{likelihood that } Y = \{y_1, y_2, \ldots, y_j\} \tag{D.6}$$

In general, this $F(Y)$ can be a likelihood $F(Z) = f(Z|X)$ or Markov density $F(X) = f_{k+1|k}(X|X_k)$ and the forms refer to the possible sets. The "set integral" suggests that

$$\begin{aligned}\int_S F(Y)\delta Y &= F(\phi) + \sum_{j=1}^{\infty} \frac{1}{j!} \int_{\underbrace{S \times \ldots \times S}_{\text{j times}}} F(\{y_1, y_2, \ldots, y_j\}) dy_1 \ldots dy_j \\ &= F(\phi) + F_S(1) + F_S(2) + \ldots \end{aligned} \tag{D.7}$$

where
$F_S(j) = \frac{1}{j!} \int_{\underbrace{S \times \ldots \times S}_{\text{j times}}} F(\{y_1, y_2, \ldots, y_j\}) dy_1 \ldots dy_j$ denotes the total probability that Y contains j elements.

D.7 Set Derivative Rule

In (D.5), a procedure is suggested to build the belief mass measure of a finite-set from its density. For building density or likelihood from belief mass functions, an operation opposite to "set integral" is needed. This operation is termed as "set derivative" also follows FISST rules.

If $Y = \{y_1, y_2, \ldots, y_m\}$, the set derivatives are defined as

$$\begin{aligned}\frac{\delta\beta}{\delta y_j}(S) &= \frac{\delta}{\delta y_j}\beta(S) = \lim_{\lambda E_y} \frac{\beta(S \cup E_y) - \beta(S)}{\lambda(E_z)} \\ \frac{\delta\beta}{\delta Y}(S) &= \frac{\delta^m}{\delta y_1 \ldots \delta y_m}\beta(S) = \frac{\delta}{\delta y_1} \cdots \frac{\delta}{\delta y_m}\beta(S) \\ \frac{\delta\beta}{\delta\phi}(S) &= \beta(S)\end{aligned}$$

D.8 Calculating Likelihoods and Markov Densities

According to the "set derivative" and "set integral" rules, the belief mass measure and densities are related to each other as

$$\beta(S) = \int_S \frac{\delta\beta}{\delta X}(\phi)\delta X \tag{D.8}$$

$$F(X) = \left[\frac{\delta}{\delta X} \int_S F(Y)\delta Y\right]_{S=\phi} \tag{D.9}$$

The expressions in (D.8) and (D.9) are the key to calculating multi-target likelihoods and Markov densities under random set formalism.

- The true likelihood $f(Z|X)$ is given by

$$f(Z|X) = \frac{\delta\beta}{\delta Z}(\phi|X) \tag{D.10}$$

(D.11)

- The true Markov density is given by

$$f_{k+1|k}(X_{k+1}|X_k) = \frac{\delta\beta_{k+1|k}}{\delta X_{k+1}}(\phi|X_k) \tag{D.12}$$

Based on the target dynamic model and sensor model, the densities can be calculated after constructing the belief mass measure of the appropriate sets. After the likelihood and Markov density are obtained, standard Bayesian recursion updates the target state in usual manner.

D.9 Standard Rules of FISST calculus

Like ordinary differential calculus, FISST calculus follows some rules. These are summarized here for reference (for details, [49],pp. 31).

D.9.1 Sum Rule

$$\frac{\delta}{\delta Z}[a_1\beta_1(S) + a_2\beta_2(S)] = a_1\frac{\delta\beta_1}{\delta Z}(S) + a_2\frac{\delta\beta_2}{\delta Z}(S)$$
$$\int[a_1F_1(S) + a_2F_2(S)]\delta Z = a_1\int F_1(S)\delta Z + a_2\int F_2(S)\delta Z$$

D.9.2 Product Rule

$$\frac{\delta}{\delta z}[\beta_1(S)\beta_2(S)] = \frac{\delta\beta_1}{\delta z}(S)\beta_2(S) + \beta_1(S)\frac{\delta\beta_2}{\delta z}(S)$$
$$\frac{\delta}{\delta Z}[\beta_1(S)\beta_2(S)] = \sum_{W\subseteq Z}\frac{\delta\beta_1}{\delta W}(S)\frac{\delta\beta_2}{\delta(Z-W)}(S)$$

D.9.3 Constant Rule

$$\frac{\delta}{\delta Z}K = 0$$

D.9.4 Chain Rule

$$\frac{\delta}{\delta z} f(\beta(S)) = \frac{df}{dx}(\beta(S))\frac{\delta \beta}{\delta z}(S)$$
$$\frac{\delta}{\delta z} f(\beta_1(S)\ldots\beta_n(S)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\beta_1(S)\ldots\beta_n(S))\frac{\delta \beta_i}{\delta z}(S)$$

D.9.5 Power Rule

Let $Z = \{z_1, \ldots, z_k\}$ and let $n \geq 0$ be an integer. Let $p(S)$ be a probability mass function with density function $f_p(z)$. Then,

$$\frac{\delta}{\delta Z} p(S)^n = \begin{cases} \frac{n!}{(n-k)!} p(S)^{n-k} f_p(z_1)\ldots f_p(z_k) & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

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