

# Variational Inference for Heteroscedastic and Longitudinal Regression Models

by

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I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text. I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

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## **Abstract**

The focus of this thesis is on the development and assessment of mean field variational Bayes (MFVB), which is a fast, deterministic tool for inference in a Bayesian hierarchical model setting. We assess the performance of MFVB via the use of comprehensive comparisons against a Markov chain Monte Carlo (MCMC) benchmark. Each of the models considered are special cases of semiparametric regression. In particular, we focus on the development and assessment of the performance of MFVB for heteroscedastic and longitudinal semiparametric regression models. Generally, the new MFVB methodology performs well in its assessment of accuracy against MCMC for the semiparametric and nonparametric regression models considered in this thesis. It is also much faster and is shown to be applicable to real-time analyses. Several real data illustrations are provided. Altogether, MFVB proves to be a credible inference tool and a good alternative to MCMC, especially when analysis is hindered by time constraints.