Robust Output-feedback Discrete-Time Sliding Mode Control Utilizing Disturbance Observer

Ahmadreza Argha, Li Li, Steven W. Su* and Hung Nguyen

Abstract— This paper is devoted to the problem of designing a robust dynamic output-feedback discrete-time sliding mode controller (ODSMC) for uncertain discrete-time systems. The basic idea behind this scheme comes from the fact that output feedback discrete-time sliding mode control (ODSMC), unlike its continuous-time counterpart, does not require to exploit a discontinuous term including the sliding function. Therefore, it is not a vital requirement that the sliding function is expressed in terms of the system outputs only. Furthermore, our observer-based discrete-time sliding mode controller (DSMC) leads to a considerably larger region of applicability. Besides, with the assumption of dealing with slow exogenous disturbances, a methodology is developed which aims to reduce the thickness of the boundary layer around the sliding surface. Moreover, the boundedness of the obtained closed-loop system is analyzed and the bound on the underlying system state is derived.

I. INTRODUCTION

Mainly, sliding-mode control has been designed for the cases that the system states are assumed to be entirely available, which is not clearly very realistic for many of practical problems. Hence, this fact has motivated the researchers to design controllers which exploit only output information. The literature which have explored output-feedback discrete-time sliding mode control (ODSMC) includes both the dynamic and static output-feedback controllers [1] - [2]. Reference [3] proposes an observer-based sliding mode controller for continuous-time MIMO systems. Different frameworks and discussions for the design of static output-feedback sliding mode controller are given in [4], [5], [2]. Moreover, in order of designing direct torsion control of flexible shaft, [6] develops an observer-based discrete-time sliding mode control (DSMC) scheme.

The early DSMC publications have focused on creating a discrete-time counterpart to the continuous-time reachability condition [7] - [8]. However, it is stated that DSMC does not necessarily require the use of a variable structure discontinuous control (VSDC) strategy [9] - [10]. References [9], [11] have shown that using the pure linear control law can ensure that the state trajectories stay within a neighbourhood of the sliding surface in the presence of the bounded matched uncertainty. Moreover, according to the results presented in [9], [11], the use of a switching function in the control law may not necessarily improve the performance. Indeed, thanks to this fact that the sliding function is not required to be exploited in the ODSMC, this paper assumes a sliding

surface in the state space rather than state estimate space or state estimation error space [12] and [13]. This fact leads us to establish a considerably less conservative LMI condition. In other words, the feasibility region of the LMI condition of the proposed *scheme* or equivalently its applicability region is interestingly improved compared to that of presented in e.g. [12] and [13].

Specifically in the proposed *scheme*, with the smoothness and boundedness conditions of the external disturbance, the ODSMC exploits a disturbance estimator in the controller rather than VSDC. Note that the idea of using disturbance estimator in the DSMC has been developed in [14] in order to reduce the ultimate bound on the discrete-time system state. In other words, with this assumption that the maximum frequency component of the exogenous disturbance is slower than the sampling rate, a special controller can be designed with utilizing disturbance estimator in the sliding mode controller. This can considerably reduce the boundary layer thickness. However, the disturbance estimator in [14] has been designed for the cases that the system states are entirely available and the system does not involve unmatched uncertainties. A framework by exploiting output information only for discrete-time MIMO systems with unmatched disturbances and without uncertainties has been proposed in [15]. Indeed, the idea is to use an integral term of the estimation output error, in addition to the well-known Luenberger observer which observes the system state with a proportional loop, to make more degrees of freedom. This matter is referred to as *proportional integral observer* (PIO) in the literature [15]. Nevertheless, the underlying system in [15], unlike the system considered in this paper, does not involve unmatched uncertainties. The proposed *scheme* here extends the problem of utilizing disturbance observer in the ODSMC to the uncertain discrete-time systems using an innovative LMI based framework.

The rest of this paper is organized as follows: Section II describes the problem formulation. In Section III, the proposed *scheme* to design an observer-based ODSMC with disturbance estimator is given. Effectiveness of the proposed ODSMC is shown by a numerical example in Section IV. Finally, Section V concludes this paper.

II. PROBLEM FORMULATION

Consider the following uncertain linear discrete-time system,

$$\begin{cases} x(k+1) = [A + \Delta A(k)]x(k) + B[u(k) + f(k)] \\ y(k) = Cx(k), \end{cases} \quad (1)$$

Ahmadreza Argha, Li Li, Steven W. Su and Hung Nguyen are with Faculty of Engineering and Information Technology, University of Technology, Sydney, PO Box 123, Broadway, NSW 2007, Australia.
{Ahmadreza.Argha, Li.Li, Steven.Su, Hung.Nguyen}@uts.edu.au

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$. Without loss of generality, it is assumed that $m \leq p \leq n$, $\text{rank}(B) = m$, $\text{rank}(C) = p$. Besides, it is assumed that (A,B) is controllable and (A,C) is observable. The uncertain matrix $\Delta A(k)$ has the form of $\Delta A(k) = MR(k)N$, where matrices M and N are known and $R(k)$ is an unknown matrix satisfying $R^T(k)R(k) \leq I, \forall k \geq 0$; $f(k)$ denotes the external disturbance.

In what follows, it is assumed that the exogenous disturbance in the system (1) is smooth and bounded.

Assumption 1: The exogenous disturbance $f(k)$ in (1) satisfies the Lipschitz continuity condition,

$$\|f_d(k)\| \leq L_f T_s, \quad \forall k \geq 0, \tag{2}$$

where $f_d(k) = f(k) - f(k-1)$, $L_f > 0$ denotes Lipschitz constant and T_s is the sampling time.

Here, it is supposed that L_f has a small value. To this end, the sampling rate of the discrete signal processing system is assumed to be a big enough value compared to the maximum frequency component of the exogenous disturbance $f(k)$. Also, the following assumption is required to be considered in the sequel of this paper.

Assumption 2: The matrices A, B and C in the system (1) satisfies

$$\text{rank}\left(\begin{bmatrix} A - I_n & B \\ C & 0 \end{bmatrix}\right) = n + m.$$

Notice that the above assumption requires that $m \leq p \leq n$, which has already been assumed in this paper. Consider the following system state and disturbance observer

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L_1[y(k) - \hat{y}(k)] + B\hat{f}(k) \\ \hat{f}(k+1) = \hat{f}(k) + L_2[y(k) - \hat{y}(k)] \\ \hat{y}(k) = C\hat{x}(k), \end{cases}$$
$$\tag{3}$$

where $L_1 \in \mathbb{R}^{n \times p}$ and $L_2 \in \mathbb{R}^{m \times p}$ are observer gains. The following lemmas are useful in the sequel.

Lemma 1 ([16]): Let E, $F(k)$ and H be real matrices of appropriate dimensions with $F^T(k)F(k) \leq I, \forall k \geq 0$, then, for any scalar $\varepsilon > 0$, we have

$$EF(k)H + H^T F^T(k)E^T \leq \varepsilon EE^T + \varepsilon^{-1}H^T H.$$

Lemma 2: Let $\tilde{E}$ and $\tilde{H}$ be real matrices of appropriate dimensions, then, for any matrix $\Xi > 0$, we have

$$\tilde{E}^T \tilde{H} + \tilde{H}^T \tilde{E} \leq \tilde{E}^T \Xi \tilde{E} + \tilde{H}^T \Xi^{-1} \tilde{H}.$$

Proof: Note that $\Xi = \bar{\Xi}\bar{\Xi}^T > 0$, where $\bar{\Xi}$ is an invertible matrix. Then it can easily be proved by

$$[\tilde{E}^T \bar{\Xi} - \tilde{H}^T (\bar{\Xi}^T)^{-1}][\bar{\Xi}^T \tilde{E} - \bar{\Xi}^{-1} \tilde{H}] \geq 0. \qquad \blacksquare$$

Lemma 3 ([17]): The feasibility of

$$\Gamma(X) - J^T(X)\Psi^{-1}(X)J(X) < 0 \tag{4}$$

in the variable X, is equivalent to the feasibility of

$$\Gamma(X) + \mathcal{F}^T \Psi(X)\mathcal{F} + \mathcal{F}^T J(X) + J^T(X)\mathcal{F} < 0 \tag{5}$$

in the variables X and $\mathcal{F}$, where $\Gamma(X)$, $\Psi(X)$ and $J(X)$ are functions of X, $\Psi(X) > 0$, and $\mathcal{F}$ is an introduced auxiliary variable with appropriate dimension.

Lemma 4 ([18]): For a given $B \in \mathbb{R}^{n \times m}$ with $\text{rank}(B) = m$, and

$$B = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T,$$

where $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{m \times m}$ are two orthogonal matrices and $\Sigma := \text{diag}(\sigma_1, \cdots, \sigma_m)$, $\sigma_i, (i = 1, \cdots, m)$ denote nonzero singular values of B, suppose that $0 < P \in \mathbb{R}^{n \times n}$ is a real symmetric matrix, then there exists a real matrix $Z \in \mathbb{R}^{m \times m}$ such that

$$PB = BZ, \tag{6}$$

if and only if P has the following structure

$$P = U \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} U^T,$$

where $0 < P_{11} \in \mathbb{R}^{m \times m}$ and $0 < P_{22} \in \mathbb{R}^{(n-m) \times (n-m)}$. In the following of this paper, for simplification, we use the brief ΔA instead of $\Delta A(k)$.

III. DESIGNING OBSERVER-BASED OUTPUT-FEEDBACK DISCRETE-TIME SMC

In this section, the objective is to design a linear sliding function in the state space, such as

$$\sigma(k) = Sx(k), \tag{7}$$

where $S = B^T P_1 \in \mathbb{R}^{m \times n}$ and $P_1 > 0$ is a symmetric matrix that will be designed later. As seen, this structure of S would result in the non-singularity of SB. During the ideal sliding motion the sliding function satisfies

$$\sigma(k) = 0, \quad \forall k > k_s, \tag{8}$$

where $k_s > 0$ denotes the time that sliding motion starts.

Remark 1: In the case of CSMC, since the sliding function plays an important role in the discontinuous component of the controller, the switching function should be an entirely known one. Due to this fact, in the literature; e.g. [12], [13] and [19], the sliding function (7) has been supposed to satisfy

$$B^T P_1 = GC, \tag{9}$$

in which $G \in \mathbb{R}^{m \times p}$. Then, the sliding surface (7) can be rewritten as

$$\sigma(k) = GCx(k) = Gy(k),$$

which is in the output space. However, since this switching function would not be used in the discrete-time sliding mode controller, such an equality as (9) is unnecessary here. In fact, for the output-feedback DSMC, the sliding surface is not required to be a known one, so, it will only need to be proved that system state trajectories could be steered into a boundary layer around the sliding surface and be kept there thereafter. The same manner can be seen in [20] for the static ODSMC.

Note that reaching the ideal sliding surface (8) in one time step ($\sigma(k+1) = \sigma(k) = 0$) has frequently been used in the literature. However, this may cause excessive control action, which is not usually applicable. To overcome this, we exploit

a reaching law referred to as *linear reaching law* [10] as follows,

$$\sigma(k+1) = \Phi\sigma(k), \quad \forall k > 0, \qquad (10)$$

where $\Phi \in \mathbb{R}^{m \times m}$ is a stable matrix. Accordingly, the controller is assumed to be of the following structure

$$u(k) = -(SB)^{-1}(SA - \Phi S)\hat{x}(k) - \hat{f}(k). \qquad (11)$$

The term $(SB)^{-1}\Phi S\hat{x}(k)$ would govern the rate of convergence to the sliding manifold, in cooperation with nonlinear controller $u_n(k)$. Note that, unlike CSMC in which the so-called *equivalent controller* $-(SB)^{-1}SA\hat{x}(k)$ alone could not steer the closed-loop system state trajectories on the ideal sliding surface, in the case of discrete-time systems the equivalent controller is able to drive the state trajectories of the discrete-time system into a neighbourhood of the sliding manifolds and keeps them there thereafter [9]. However, with $\Phi = 0$ the control input aims at steering the system state to the sliding surface in one time step. In the case of a large initial distance from the sliding surface, this could lead to excessively large control input referred to as high-gain controller. Here, similar to [21], it is assumed that $\Phi = \lambda I_m$, where $0 \le \lambda < 1$ is a given constant value which would not belong to the spectrum of A. Due to the special form of Φ, it can commute with S and then the control law (11) could be written as

$$u(k) = -(SB)^{-1}SA_\lambda \hat{x}(k) - \hat{f}(k), \qquad (12)$$

where $A_\lambda = A - \lambda I_n$. Besides, we have

$$u_l(k) = -(SB)^{-1}SA_\lambda \hat{x}(k). \qquad (13)$$

Defining the state estimation error $e(k) = x(k) - \hat{x}(k)$ and disturbance estimation error $e_f(k) = f(k) - \hat{f}(k)$, the overall closed-loop system is obtained by applying the controller in (12) to (1), which is

$$\begin{cases} x(k+1) = [A + \Delta A - \hat{A}]x(k) + B(SB)^{-1}S[A_\lambda \ B]e_a(k) \\ e_a(k+1) = \begin{bmatrix} \Delta A \\ 0 \end{bmatrix} x(k) + (A_a - L_a C_a)e_a(k) + \bar{f}_d(k+1), \end{cases} \tag{14}$$

where $\bar{f}_d(k+1) = \begin{bmatrix} 0 \\ f_d(k+1) \end{bmatrix}$, $e_a(k) = \begin{bmatrix} e(k) \\ e_f(k) \end{bmatrix}$, $A_a = \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix}$, $L_a = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ and $C_a = [C \ 0]$. Then from (7) and (14) it can be found

$$\sigma(k+1) = \lambda\sigma(k) + S\Delta Ax(k) + S[A_\lambda \ B]e_a(k). \qquad (15)$$

Lemma 5 ([15]): If the matrix pair (A, C) is observable and A, B and C satisfies the rank condition in Assumption 2, then the matrix pair (A_a, C_a) is observable.

Remark 2: Note that exploiting the disturbance estimate in the ODSMC requires that the exogenous disturbances do not vary too much in one time step. This cannot only reduce the thickness of the boundary layer, but also relax the upper bound restriction on the exogenous disturbances, which can be seen in many references in the literature. Alternatively, this restriction is now on the maximum frequency component of the change of disturbance in terms of the sampling rate (see the continuity assumption in (2)).

The sequel of this section aims to consider the boundedness of the system (1) using the controller (12). The following lemma is given to characterize the boundedness of the system state.

Lemma 6 ([22]): Let $V(\zeta(k))$ be a Lyapunov candidate function. In the case that there exist real scalars $v \ge 0$, $\alpha > 0$, $\beta > 0$ and $0 < \rho < 1$ such that

$$\alpha \|\zeta(k)\|^2 \le V(\zeta(k)) \le \beta \|\zeta(k)\|^2,$$

and

$$V(\zeta(k+1)) - V(\zeta(k)) \le v - \rho V(\zeta(k)),$$

then $\zeta(k)$ will satisfy

$$\|\zeta(k)\|^2 \le \frac{\beta}{\alpha} \|\zeta(0)\|^2 (1 - \rho)^k + \frac{v}{\alpha\rho}.$$

The following theorem analyzes the boundedness of the overall closed-loop system in (14) and the sliding function in (15).

Theorem 1: The control law (12) can drive the system state into a boundary layer around the ideal sliding surface $\sigma(k) = 0$, where $\sigma(k)$ is defined in (7) and, thus, the system state is ultimately bounded if there exist symmetric matrices $P_1 := U \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} U^T > 0$, $Q_2 > 0$, real matrices X_1, X_2 and X_3, and scalars $\varepsilon > 0$, $\rho > 0$ satisfying the following LMI,

$$\begin{bmatrix} \mathcal{M}_{11} & \star & \star & \star & \star & \star & \star \\ 0 & -Q_2 + \rho I & \star & \star & \star & \star & \star \\ \sqrt{2}\lambda B^T P_1 & \sqrt{2}B^T P_1 [A_\lambda \ B] & -B^T P_1 B & \star & \star & \star & \star \\ 0 & Q_2 A_a - X_3 C_a & 0 & -Q_2 & \star & \star & \star \\ P_1 A + BX_1 & 0 & 0 & 0 & -P_1 & \star & \star \\ BX_2 & 0 & 0 & 0 & 0 & -P_1 & \star \\ 0 & 0 & \sqrt{2}M^T P_1 B & [M^T \ 0]Q_2 & M^T P_1 & 0 & -\varepsilon I \end{bmatrix} < 0, \tag{16}$$

where $0 < P_{11} \in \mathbb{R}^{m \times m}$, $0 < P_{22} \in \mathbb{R}^{(n-m) \times (n-m)}$, and $U \in \mathbb{R}^{n \times n}$ is defined in Lemma 4, $\mathcal{M}_{11} = -P_1 + X_2^T B^T + BX_2 + \rho I + \varepsilon N^T N$. Besides, $S = B^T P_1$ and the observer gains are given by

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = Q_2^{-1}X_3. \tag{17}$$

Proof: Define

$$V(\varpi(k)) = x^T(k)P_1 x(k) + e_a^T(k)Q_2 e_a(k) + \sigma^T(k)(SB)^{-1}\sigma(k), \tag{18}$$

where $\varpi(k) = \begin{bmatrix} x^T(k) & e_a^T(k) & \sigma^T(k) \end{bmatrix}^T$, $P_1 > 0$ and $Q_2 > 0$ are symmetric matrices and $S = B^T P_1$. Hence, we have

$$\begin{aligned} \Delta V(\varpi(k)) =& V(\varpi(k+1)) - V(\varpi(k)) \\ =& x^T(k+1)P_1 x(k+1) + e_a^T(k+1)Q_2 e_a(k+1) \\ & + \sigma^T(k+1)(SB)^{-1}\sigma(k+1) - x^T(k)P_1 x(k) \\ & - e_a^T(k)Q_2 e_a(k) - \sigma^T(k)(SB)^{-1}\sigma(k). \end{aligned} \tag{19}$$

It can be followed then

$$\begin{aligned} & x^T(k+1)P_1 x(k+1) \\ =& x^T(k)[A + \Delta A - B(SB)^{-1}S(A + \Delta A) + B(SB)^{-1}S(\lambda I_n + \Delta A)]^T \\ & \times P_1[A + \Delta A - B(SB)^{-1}S(A + \Delta A) + B(SB)^{-1}S(\lambda I_n + \Delta A)]x(k) \\ & + 2x^T(k)(\lambda I_n + \Delta A)^T S^T (SB)^{-1}S[A_\lambda \quad B]e_a(k) \\ & + e_a^T(k)[A_\lambda \quad B]^T S^T (SB)^{-1}S[A_\lambda \quad B]e_a(k) \end{aligned}$$

$$=x^T(k)[A+\Delta A-B(SB)^{-1}S(A+\Delta A)]^T P_1$$
$$\times [A+\Delta A-B(SB)^{-1}S(A+\Delta A)]x(k)$$
$$+x^T(k)(\lambda I_n+\Delta A)^T S^T(SB)^{-1}S(\lambda I_n+\Delta A)x(k)$$
$$+2x^T(k)(\lambda I_n+\Delta A)^T S^T(SB)^{-1}S\begin{bmatrix}A_\lambda & B\end{bmatrix}e_a(k)$$
$$+e_a^T(k)\begin{bmatrix}A_\lambda & B\end{bmatrix}^T S^T(SB)^{-1}S\begin{bmatrix}A_\lambda & B\end{bmatrix}e_a(k). \quad (20)$$

Besides,

$$e_a^T(k+1)Q_2 e_a(k+1) \quad (21)$$
$$=x^T(k)\begin{bmatrix}\Delta A\\0\end{bmatrix}^T Q_2\begin{bmatrix}\Delta A\\0\end{bmatrix}x(k)$$
$$+e_a^T(k)(A_a-L_aC_a)^T Q_2(A_a-L_aC_a)e_a(k)$$
$$+\bar f_d^T(k+1)Q_2\bar f_d(k+1)+2x^T(k)\begin{bmatrix}\Delta A\\0\end{bmatrix}^T Q_2(A_a-L_aC_a)e_a(k)$$
$$+2x^T(k)\begin{bmatrix}\Delta A\\0\end{bmatrix}^T Q_2\bar f_d(k+1)+2e_a^T(k)(A_a-L_aC_a)^T Q_2\bar f_d(k+1),$$

$$\sigma^T(k+1)(SB)^{-1}\sigma(k+1)$$
$$=x^T(k)(\lambda I_n+\Delta A)^T S^T(SB)^{-1}S(\lambda I_n+\Delta A)x(k)$$
$$+e_a^T(k)\begin{bmatrix}A_\lambda & B\end{bmatrix}^T S^T(SB)^{-1}S\begin{bmatrix}A_\lambda & B\end{bmatrix}e_a(k)$$
$$+2x^T(k)(\lambda I_n+\Delta A)^T S^T(SB)^{-1}S\begin{bmatrix}A_\lambda & B\end{bmatrix}e_a(k). \quad (22)$$

From (19)-(22), we have

$$\Delta V(\varpi(k))=\begin{bmatrix}x(k)\\e_a(k)\\\bar f_d(k+1)\end{bmatrix}^T\begin{bmatrix}\chi_{11} & \chi_{12} & \chi_{13}\\\chi_{12}^T & \chi_{22} & \chi_{23}\\\chi_{13}^T & \chi_{23}^T & \chi_{33}\end{bmatrix}\begin{bmatrix}x(k)\\e_a(k)\\\bar f_d(k+1)\end{bmatrix}, \quad (23)$$

where

$$\chi_{11}=(A+\Delta A)^T P_1(A+\Delta A)-(A+\Delta A)^T S^T(SB)^{-1}S(A+\Delta A)$$
$$-S^T(SB)^{-1}S-P_1+2(\lambda I_n+\Delta A)^T S^T(SB)^{-1}S(\lambda I_n+\Delta A)$$
$$+\begin{bmatrix}\Delta A\\0\end{bmatrix}^T Q_2\begin{bmatrix}\Delta A\\0\end{bmatrix},$$

$$\chi_{12}=2(\lambda I_n+\Delta A)^T S^T(SB)^{-1}S\begin{bmatrix}A_\lambda & B\end{bmatrix}+\begin{bmatrix}\Delta A\\0\end{bmatrix}^T Q_2(A_a-L_aC_a),$$

$$\chi_{13}=\begin{bmatrix}\Delta A\\0\end{bmatrix}^T Q_2,$$

$$\chi_{22}=2\begin{bmatrix}A_\lambda & B\end{bmatrix}^T S^T(SB)^{-1}S\begin{bmatrix}A_\lambda & B\end{bmatrix}$$
$$+(A_a-L_aC_a)^T Q_2(A_a-L_aC_a)-Q_2,$$

$$\chi_{23}=(A_a-L_aC_a)^T Q_2,$$

$$\chi_{33}=Q_2.$$

Letting

$$\Upsilon:=\begin{bmatrix}\chi_{11} & \chi_{12}\\\chi_{12}^T & \chi_{22}\end{bmatrix}<-\rho I, \quad (24)$$

where $\rho>0$ is a scalar variable. Then, defining $\bar x(k)=\begin{bmatrix}x^T(k) & e_a^T(k)\end{bmatrix}^T$ and using Lemma 2 it can be shown that

$$2\bar x^T(k)\begin{bmatrix}\chi_{13}\\\chi_{23}\end{bmatrix}\bar f_d(k+1)\leq \bar x^T(k)\begin{bmatrix}\chi_{13}\\\chi_{23}\end{bmatrix}\Pi^{-1}\begin{bmatrix}\chi_{13}\\\chi_{23}\end{bmatrix}^T\bar x(k)$$
$$+\bar f_d^T(k+1)\Pi\bar f_d(k+1), \quad (25)$$

where $\Pi>0$. It then follows from (23) and (25) that

$$\Delta V(\varpi(k))\leq -\bar x^T(k)\left\{\rho I-\begin{bmatrix}\chi_{13}\\\chi_{23}\end{bmatrix}\Pi^{-1}\begin{bmatrix}\chi_{13}\\\chi_{23}\end{bmatrix}^T\right\}\bar x(k)$$
$$+\bar f_d^T(k+1)[\Pi+\chi_{33}]\bar f_d(k+1). \quad (26)$$

Choosing $\Pi>0$ subject to

$$\hat\rho I<\rho I-\begin{bmatrix}\chi_{13}\\\chi_{23}\end{bmatrix}\Pi^{-1}\begin{bmatrix}\chi_{13}\\\chi_{23}\end{bmatrix}^T, \quad (27)$$

where $0<\hat\rho<\rho$, which is clearly always possible if $\eta>0$ exists, it follows from (26) that

$$\Delta V(\varpi(k))\leq -\hat\rho\bar x^T(k)\bar x(k)+\bar f_d^T(k+1)[\Pi+\chi_{33}]\bar f_d(k+1). \quad (28)$$

On the other hand, it can be seen that

$$V(\varpi(k))=\bar x^T(k)\begin{bmatrix}M_{P_1} & 0\\0 & Q_2\end{bmatrix}\bar x(k)$$
$$\triangleq \bar x^T(k)\mathbf{W}\bar x(k), \quad (29)$$

where $M_{P_1}=P_1B(B^T P_1 B)^{-1}B^T P_1+P_1$, then

$$\lambda_{\min}(\mathbf{W})\|\bar x(k)\|^2\leq V(\varpi(k))\leq \lambda_{\max}(\mathbf{W})\|\bar x(k)\|^2. \quad (30)$$

Furthermore, it can be shown that

$$\lambda_{\min}(\mathrm{diag}(P_1,Q_2,(SB)^{-1}))\|\varpi(k)\|^2\leq V(\varpi(k))$$
$$\leq \lambda_{\max}(\mathrm{diag}(P_1,Q_2,(SB)^{-1}))\|\varpi(k)\|^2.$$

Hence, from (28) and (30), also the continuity assumption in (2), we have

$$\Delta V(\varpi(k))\leq -\frac{\hat\rho}{\lambda_{\max}(\mathbf{W})}V(\varpi(k))+\gamma. \quad (31)$$

where $\gamma=\|\Pi+Q_2\|L_f^2 T_s^2$. Note that from (24) it is known that

$$\bar x^T(k)\Upsilon\bar x(k)=V(\varpi(k+1))\big|_{\bar f_d(k+1)=0}-V(\varpi(k))<-\rho\bar x^T(k)\bar x(k). \quad (32)$$

It is obvious that $V(\varpi(k+1))\big|_{\bar f_d(k+1)=0}>0$, and thus, from (32) and (30), we have $\rho<\lambda_{\max}(\mathbf{W})$. Therefore, $\frac{\hat\rho}{\lambda_{\max}(\mathbf{W})}<1$. Eventually, from Lemma 6 and (31), it can be illustrated that

$$\forall\varepsilon>0,\ \exists k^\star>0,\ \text{s.t. }\forall k>k^\star,$$
$$\|\varpi(k)\|^2\leq \frac{\lambda_{\max}(\mathbf{W})}{\hat\rho\lambda_{\min}(\mathrm{diag}(P_1,Q_2,(SB)^{-1}))}\gamma+\varepsilon. \quad (33)$$

Now it remains to consider the feasibility of $\Upsilon<-\rho I$ in (24). With the aid of Schur complement, (24) is equivalent to

$$\begin{bmatrix}\bar\chi_{11} & \star & \star & \star\\0 & \bar\chi_{22} & \star & \star\\\sqrt 2 B^T P_1(\lambda I_n+\Delta A) & \sqrt 2 B^T P_1\begin{bmatrix}A_\lambda & B\end{bmatrix} & -B^T P_1 B & \star\\Q_2\begin{bmatrix}\Delta A\\0\end{bmatrix} & Q_2(A_a-L_aC_a) & 0 & -Q_2\end{bmatrix}<0, \quad (34)$$

where

$$\bar\chi_{11}=(A+\Delta A)^T P_1(A+\Delta A)-(A+\Delta A)^T S^T(SB)^{-1}S(A+\Delta A)$$
$$-S^T(SB)^{-1}S-P_1+\rho I,$$
$$\bar\chi_{22}=-Q_2+\rho I.$$

Therefore, using Lemma 3 it can be shown that the feasibility of the inequality in (34) is equivalent to that of

$$\begin{bmatrix}\hat\chi_{11} & \star & \star & \star\\0 & \bar\chi_{22} & \star & \star\\\sqrt 2 B^T P_1(\lambda I_n+\Delta A) & \sqrt 2 B^T P_1\begin{bmatrix}A_\lambda & B\end{bmatrix} & -B^T P_1 B & \star\\Q_2\begin{bmatrix}\Delta A\\0\end{bmatrix} & Q_2(A_a-L_aC_a) & 0 & -Q_2\end{bmatrix}<0, \quad (35)$$

with

$$\hat{\chi}_{11} = (A + \Delta A + BF_3)^T P_1 (A + \Delta A + BF_3) - P_1 \\ + F_4^T (B^T P_1 B) F_4 + F_4^T B^T P_1 + P_1 BF_4 + \rho I, \quad (36)$$

where F_3 and F_4 are two auxiliary variables [17]. Hence, using Lemma 4, $\hat{\chi}_{11}$ in (36) can be rearranged as

$$\hat{\chi}_{11} = [P_1(A + \Delta A) + BZF_3]^T P_1^{-1} [P_1(A + \Delta A) + BZF_3] - P_1 \\ + F_4^T Z^T B^T P_1^{-1} BZF_4 + F_4^T Z^T B^T + BZF_4 + \rho I, \quad (37)$$

where Z satisfies $P_1 B = BZ$. Using the Schur complement it can be shown that (35) is equivalent to

$$\begin{bmatrix} \mathcal{M}_{11} & \star & \star & \star & \star & \star \\ 0 & \tilde{\chi}_{22} & \star & \star & \star & \star \\ \sqrt{2}B^T P_1(\lambda I_n + \Delta A) & \sqrt{2}B^T P_1 [A_\lambda\ B] & -B^T P_1 B & \star & \star & \star \\ Q_2 \begin{bmatrix} \Delta A \\ 0 \end{bmatrix} & Q_2 A_a - X_3 C_a & 0 & -Q_2 & \star & \star \\ P_1(A + \Delta A) + BX_1 & 0 & 0 & 0 & -P_1 & \star \\ BX_2 & 0 & 0 & 0 & 0 & -P_1 \end{bmatrix} < 0, \quad (38)$$

where $\mathcal{M}_{11} = -P_1 + X_2^T B^T + BX_2 + \rho I$, $X_1 = ZF_3$, $X_2 = ZF_4$ and $X_3 = Q_2 L_a$. With the help of Lemma 1 and the Schur complement, (38) is sufficed by the LMI in (16).

Besides, to find $\Pi > 0$ in (27), for given $P_1 > 0$, $Q_2 > 0$, L_a and $\rho > 0$, by utilizing Lemma 1 and the Schur complement, (27) is sufficed by,

$$\begin{bmatrix} (\hat{\rho} - \rho)I + \tilde{\varepsilon} N^T N & \star & \star & \star \\ 0 & (\hat{\rho} - \rho)I & \star & \star \\ 0 & Q_2 A_a - X_3 C_a & -\Pi & \star \\ 0 & 0 & [M^T\ 0]Q_2 & -\tilde{\varepsilon}I \end{bmatrix} < 0, \quad (39)$$

where $\tilde{\varepsilon} > 0$ is a scalar variable. ∎

Some remarks:

1) The solution of the LMI in (39) does not have direct influence on the controller design and the actual ultimate bound on the system state and/or sliding function, however, these parameters would lead us to determine a more accurate bound. Therefore, to obtain the minimum value of the bound in (33) the LMIs in (16) and (39) could be solved subject to a specific criteria. This issue is beyond the scope of this paper and remains for the future works.

2) Due to the full column rank of B, the columns of matrices B and $P_1 B$ are linearly independent if $P_1 > 0$. Consequently, if (6) holds for $P_1 > 0$ and Z, we have

$$rank(Z) \geq rank(BZ) = rank(P_1 B) \geq rank(B) = m, \quad (40)$$

which clearly denotes the non-singularity of Z. Also, it can easily be shown that

$$Z^{-1} = V \Sigma^{-1} P_{11}^{-1} \Sigma V^T. \quad (41)$$

3) Furthermore, unlike [23], [12] and [13] which use Lemma 2 to eliminate the cross terms between the system state (state estimate), the estimation error and even disturbance which obviously imposes some conservatism on the problem, here instead, it has been shown that the mentioned cross terms would not influence the feasibility region of the final LMI condition. Moreover, this paper, unlike [23] which uses Lemma 2 to deal with the negative terms in $\Delta V(\zeta(k))$ to make a convex problem, exploits Lemma 3 which is clearly a *lossless* technique and imposes no additional conservatism on the LMI condition.

4) It should be noticed that the parameter $\Phi = \lambda I$, $0 < \lambda <$

1 plays a significant role in the magnitude of the thickness of the boundary layer around the sliding surface [9]. From (15) it can be shown that

$$\sigma_i(k) = \lambda^k \sigma_i(0) + \sum_{j=0}^{k-1} \lambda^{k-1-j} \mathcal{D}_i(j), \ i = 1, \cdots, m, \quad (42)$$

where $\mathcal{D}(j) = S\Delta A x(j) + SA_\lambda e(j) + SBe_f(j)$. Supposing $\bar{\mathcal{D}}_i = \max(\mathcal{D}_i(k))$, it follows then from (42),

$$\forall \varepsilon_i > 0, \ \exists k^\dagger > 0, \ \text{s.t.} \ \forall k > k^\dagger, \ \sigma_i(k) < \frac{1}{1 - \lambda} \bar{\mathcal{D}}_i + \varepsilon_i, \ i = 1, \cdots, m. \quad (43)$$

Assuming $\gamma_{\sigma,i} \triangleq \frac{1}{1-\lambda} \bar{\mathcal{D}}_i + \varepsilon_i$, the boundary layer is

$$\gamma_\sigma = \sqrt{\sum_{i=1}^m \gamma_{\sigma,i}^2}. \quad (44)$$

As seen, the smallest boundary layer could be obtained by setting λ to zero. In that case, the discrete-time sliding mode controller steers the system state into the *quasi sliding mode band* only in one time step. As mentioned, this would result in a high-gain or excessively large control input which is not desirable for most of the practical systems since it can saturate the actuators of the control system. Hence, there is a tradeoff to be considered between the level of the control input and the thickness of the boundary layer.

5) The sliding surface in this scheme is set to be in the state space, this matter is significantly different from the sliding surface in [12] and [13] which is in the estimation error space or the state estimate space. The Lyapunov functional candidate also, in these references, contains the state estimate and the state estimation error. Here, instead we have used the system state directly in addition to the state estimation error and sliding function in the Lyapunov functional candidate. Roughly speaking, the main drawback of the schemes, given in [12] and [13], comes from the fact that in order to formulate an LMI problem, it is inevitable to use same positive definite decision variable P for both quadratic terms $x^T(k)Px(k)$ and $e^T(k)Pe(k)$, otherwise, a BMI problem would be arisen, which is not easy to handle. For example, [19] utilizes two different positive definite decision variables in its Lyapunov-based scheme for the design of a dynamic output-feedback CSMC (OCSMC), which naturally leads to a BMI problem. Note that, as mentioned earlier, since a variable structure discontinuous controller is not provided for the proposed ODSMC by the means of the sliding function, the introduced sliding function, here, can be defined to be in the state space. Furthermore, in this case we would not struggle with a BMI problem.

IV. SIMULATION RESULTS

Consider the system (1) with the following parameters:

$$A = \begin{bmatrix} 0.6 & 0 & 0.5 \\ 0 & 0.7 & 0.3 \\ 0 & 0 & 1.5 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0.3 \\ 0.6 & 0 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0.5 & 1 \\ 0.1 & 0 & 2 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.3 & 0.1 & -0.1 \end{bmatrix}^T, \ N = \begin{bmatrix} -0.2 & -0.2 & 0.3 \end{bmatrix},$$
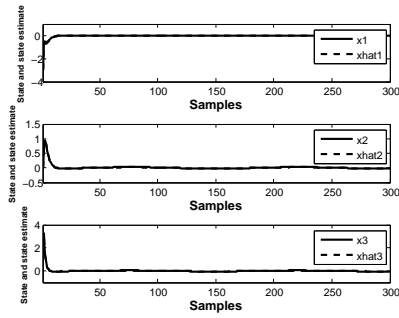
$$R(k) = 0.3\sin(k).$$

Fig. 1.   State and state estimate trajectories using the controller in (12)
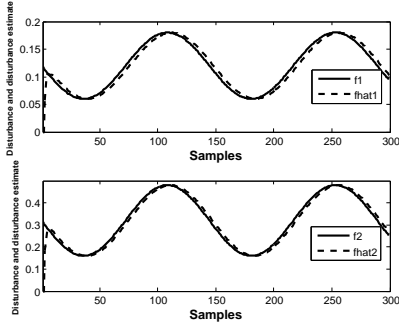

Fig. 2.   Disturbance and disturbance estimator output

Note that the open-loop system is unstable. Suppose $f(k) = \begin{bmatrix} 0.06 \\ 0.16 \end{bmatrix} [2 - \sin(\frac{1}{23})]$.   Solving the LMI (16), the following results are obtained:

$$P_1 = \begin{bmatrix} 0.83 & -0.88 & -0.27 \\ -0.88 & 2.2470 & 0.24 \\ -0.27 & 0.24 & 0.57 \end{bmatrix}, P_{11} = \begin{bmatrix} 0.32 & 0.02 \\ 0.02 & 0.61 \end{bmatrix}, P_{22} = 2.72,$$

$$Q_2 = \begin{bmatrix} 1.68 & -1.71 & 0.13 & -0.12 & -0.61 \\ -1.71 & 6.99 & 0.40 & -4.50 & 0.07 \\ 0.13 & 0.40 & 6.15 & 0.08 & -4.23 \\ -0.12 & -4.50 & 0.08 & 9.57 & 0.32 \\ -0.61 & 0.07 & -4.23 & 0.32 & 10.67 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 2.30 & -1.11 \\ 2.40 & -1.05 \\ -0.24 & 1.17 \end{bmatrix}, L_2 = \begin{bmatrix} 1.49 & -0.74 \\ -0.02 & 0.32 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.31 & 0.47 & -0.13 \\ -0.02 & -0.03 & 0.48 \end{bmatrix}, \rho = 0.08, \varepsilon = 2.52.$$

Applying the controller in (12) with given P_1 above to the system, the results are given in Figs. 1. Fig. 2 shows the performance of the disturbance estimator $\hat{f}(k)$ in (3).

V. CONCLUSIONS

In this note, a novel dynamic output-feedback LMI based robust DSMC for the systems involving unmatched uncertainties and matched disturbances has been developed. The proposed scheme is applicable to general systems including unstable systems. The boundedness of the obtained closed-loop system has been analyzed and a bound has been derived for the closed-loop system state, estimation error and also sliding function. With the assumption of dealing with slow exogenous disturbances, a unified scheme has been designed which includes an extra proportional integral estimator for estimating the disturbance. The framework presented in this paper are less conservative compared to the existing literature for the robust DSMC and also OCSMC. Additionally, the sliding mode controllers in this paper do not fall into the category of the high-gain controllers.

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