

# A Beneš formula for the fractional Brownian storage

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## Abstract

The applicability of the Beneš approach to the “fractional Brownian storage”, i.e. a storage model where the net input process is a fractional Brownian motion (FBM) with drift, is studied. This requires the analysis of the last exit time probability density of a drifted FBM that, in turn, motivates the proof of a general “localization theorem” for FBM. The resulting Beneš formula contains a unknown function which can, however, be replaced by a constant with reasonable accuracy.

## 1 Introduction

Consider a storage system (unlimited fluid buffer) with constant leak rate  $c$  and input during the interval  $[s, t)$  given as

$$A(s, t) = m(t - s) + \sqrt{ma}(Z_t - Z_s), \quad (1.1)$$

where  $m < c$  and  $a$  are positive parameters and  $Z$  is a normalized *fractional Brownian motion* (FBM) with Hurst parameter  $H \in [\frac{1}{2}, 1)$ . The process  $(Z_t : t \in (-\infty, \infty))$  is characterized by the following properties:

- (i)  $Z$  has stationary increments;
- (ii)  $Z_0 = 0$ , and  $EZ_t = 0$  for all  $t$ ;
- (iii)  $EZ_t^2 = |t|^{2H}$  for all  $t$ ;
- (iv)  $Z_t$  has continuous paths;
- (v)  $Z$  is Gaussian, i.e. all its finite-dimensional marginal distributions are Gaussian.

The FBM is also a *self-similar* process, i.e., the processes  $Z_{\alpha t}$  and  $\alpha^H Z_t$  have the same finite-dimensional distributions for any  $\alpha > 0$ .

The amount  $X_t$  of fluid in the buffer at time  $t$  is then given by Reich’s formula [Ben63]

$$X_t = \sup_{s \leq t} (A(s, t) - c(t - s)). \quad (1.2)$$

$X$  is a stationary process since  $Z$  has stationary increments. This process was defined and analysed in [Nor94], where a lower bound for the probability  $P(X_0 > x)$  was given. In this paper, we derive a Beneš type formula for this probability. It turns out that this requires the study of general path properties of the FBM, and an interesting “localization theorem” is obtained as a by-product.

Throughout the paper, an expression of the type  $P(X \in a + bdx)$  refers to integration with respect to the distribution of  $(X - a)/b$ . This kind of infinitesimal notation is chosen for making the formulas easier to read and understand.

The paper is organized as follows. The general Beneš approach is outlined and the nature of the main problem for its applicability is identified in Section 2. A Beneš derivation of the well known exponential distribution in the classical Brownian case  $H = 1/2$  is provided in Section 3. This gives the motivation for a general “localization theorem” for FBM, which is addressed in Section 4. The Beneš formula for the fractional Brownian storage is then deduced in Section 5. Since the result still contains unknown quantities, a simplified approximate formula is given in Section 6, and results from simulation studies on the accuracy of the approximation are presented.

## 2 The Beneš approach and the problem of its applicability to a system with Brownian input

Consider an infinite fluid buffer with constant leak rate  $c$ , and denote by  $A(s, t)$  the amount of fluid that arrives during the interval  $[s, t)$ . We also denote  $A_t = A(0, t)$ . In the classical case of a FIFO queue, the “fluid” is working time which comes in instantaneous bursts with each customer,  $c = 1$  and  $X_t$  is the virtual waiting time, i.e., the time that a customer arriving at time  $t$  would have to wait for the beginning of his service. Beneš’ original formula (see [Ben63], or [Bor76]) gives a general expression for the probability  $P(X_t > x)$  through a partition of the time half-axis  $(-\infty, t)$  according to the first (earliest) time point  $s < t$  such that  $A(s, t) - c(t - s) = x$ . This first time  $s$  is “almost always” characterized by the additional condition that the system be empty at  $s$ , i.e.  $X_s = 0$  (see [Rob92], pp. 104–108).

In the case of a queue with constant service time 1,  $A(s, t)$  is simply the number of customers arriving during the interval  $[s, t)$ , and the Beneš formula can be written particularly simply as the sum

$$P(X_0 > x) = \sum_{n > x} P(A(x - n, 0) = n \text{ and } X_{x-n} = 0). \quad (2.1)$$

Simonian and Virtamo [SV91] generalized this result to the case where  $A_t$  is an absolutely continuous process, i.e., the integral of a finite arrival rate process  $\Lambda_t$ . Fixing  $c = 1$  for

convenience, their formula reads

$$\begin{aligned} & \mathbb{P}(X_0 > x) \\ &= \int_0^\infty du \int_0^1 d\lambda (1-\lambda) \frac{d^2}{dx d\lambda} \mathbb{P}(A(-u, 0) - u \leq x, \Lambda_{-u} \leq \lambda \text{ and } X_{-u} = 0). \end{aligned} \quad (2.2)$$

Let us now address situations when the arrival process  $A$  pertains to neither class for which the results (2.1) and (2.2) can be stated. Consider, in particular, the case when  $A_t = mt + \sqrt{ma}W_t$ , where  $W$  is a standard Brownian motion and  $m < 1 = c$  and  $a$  are positive parameters. Let us formally copy the formula (2.1) associated with an  $M/D/1$  system. From now on, we invert the time for convenience (the Beneš approach “looks backward in time”). Let us write the integral

$$\int_0^\infty \mathbb{P}(A_t - t \in x + dt, X_t = 0)$$

and proceed with heuristic reasoning. Since the Brownian motion has independent increments, the two events are independent. By analogy from an ordinary queueing system, let us guess that  $\mathbb{P}(X_t = 0) = 1 - m$ , since  $m = m/1$  is now also the load of the system. We then obtain the expression

$$(1 - m) \int_0^\infty \frac{1}{\sqrt{2\pi mat}} e^{-\frac{(x+(1-m)t)^2}{2mat}} dt. \quad (2.3)$$

This integral can be made explicit and reduces to the exponential

$$e^{-\frac{2(1-m)}{ma}x}. \quad (2.4)$$

This is exactly the correct result, which is immediately obtained from the well-known exponential distribution of the maximum of a Brownian motion with negative drift when applied to (1.2). However, the reasoning with which we deduced it was not correct. Indeed, the microstructure of the Brownian motion implies that  $\mathbb{P}(X_t = 0) = 0$ . This is somehow balanced by the fact that  $\mathbb{P}(\exists s \in [t, t+h] : A_s - cs = x)$  goes to zero with  $h$  much slower than  $\mathbb{P}(A_t - t \in [y, y+h])$ . We show in this paper how clarifying this “paradox” provides a non-usual, possibly new derivation of the last exit time distribution of a drifted Brownian motion. As the main result, we derive a Beneš formula for the fluid storage system with arrival process governed by a positively dependent FBM as introduced in Section 1. The resulting formula for the probability density of a last exit time resembles and is in fact partly inspired by Durbin’s result [Dur85]. Our technical approach is, however, different from his.

### 3 The Beneš formula in the Brownian case: a fresh look at a classical result

To simplify notation, consider first a standard Brownian motion  $W$  with negative drift  $-1$ . Denote by

$$T^x = \sup \{t : W_t = x + t\}$$

the last exit time of the process  $W_t - t$  from the level  $x$ . It is well known that

$$\mathbb{P}(T^x \in dt) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x+t)^2}{2t}} dt = \mathbb{P}(W_t \in x + t + dt), \quad (3.1)$$

hence  $\mathbb{P}(T^x < \infty) = e^{-2x}$ , from which the specific formula (2.4) for  $\mathbb{P}(X_0 > x)$  follows. Note also that (3.1) shows, with appropriate parameter scaling, that the integral (2.3) not only evaluates exactly to  $\mathbb{P}(X_0 > x)$  but also the integrand has the clear meaning  $\mathbb{P}(T^x \in dt)$ . Finding the reason behind this ‘‘coincidence’’ opens the possibility of generalizing it to the case of a fractional Brownian motion.

Let us consider a short interval  $[t, u]$  and decompose the probability of the event  $\{T^x \in [t, u]\}$  as follows:

$$\begin{aligned} & \mathbb{P}(T^x \in [t, u]) \\ &= \mathbb{P}(\exists s \in [t, u] : W_s = x + s) \mathbb{P}[W_v < x + v, \forall v > u \mid \exists s \in [t, u] : W_s = x + s]. \end{aligned}$$

The homogeneous Markovian structure of Brownian motion yields that the second factor is in fact independent of  $x$  and depends on  $t$  and  $u$  only through their difference  $u - t$ . Thus we have

$$\mathbb{P}(T^x \in [t, u]) = \mathbb{P}(\exists s \in [t, u] : W_s = x + s) \gamma(u - t), \quad (3.2)$$

where  $\gamma(\cdot)$  is some function. If we divide by  $u - t$  and let  $u \searrow t$ , the left hand side converges to the density of  $T^x$  at  $t$ . At the right hand side we have

$$\lim_{u \searrow t} \frac{1}{u - t} \mathbb{P}(\exists s \in [t, u] : W_s = x + s) = \infty, \quad \lim_{u \searrow t} \gamma(u - t) = 0,$$

so that the reasoning which applies to the case of smooth input would here give an indefinite  $\infty \cdot 0$  result. The trick which yields a meaningful product form limit is to divide the denominator  $u - t$  into two copies of  $\sqrt{u - t}$ . Let us call the event  $\{\exists s \in [t, u] : W_s = x\}$  the *localization* of  $W$  at  $x$  within  $[t, u]$ . This type of event is a counterpart, by changing the roles of time and place, to the type of event  $\{W_t \in [x, y]\}$ . The asymptotic probabilities of the two types of localization are connected by the following theorem, which is easy to derive using classical results for Brownian motion:

**Theorem 3.1**

$$\lim_{u \searrow t} \frac{1}{\sqrt{u - t}} \mathbb{P}(\exists s \in [t, u] : W_s = x) = \frac{2}{\pi\sqrt{t}} e^{-\frac{x^2}{2t}}.$$

In heuristic infinitesimal stenography, this could be written as

$$\mathbb{P}(\exists s \in [t, t + dt] : W_s = x) = \frac{1}{\sqrt{dt}} \sqrt{\frac{8}{\pi}} \mathbb{P}(W_t \in x + dt). \quad (3.3)$$

It is easy to see that the result remains the same if  $W$  is replaced by a drifted Brownian motion. It follows from (3.2) that there exists a finite positive limit

$$L = \lim_{u \searrow t} \frac{\gamma(u - t)}{\sqrt{u - t}},$$

and

$$\mathbb{P}(T^x \in dt) = \sqrt{\frac{8}{\pi}} \mathbb{P}(W_t \in x + t + dt) \cdot L.$$

Since  $T^0$  is finite and positive with probability 1, we have

$$1 = \sqrt{\frac{8}{\pi}} L \int_0^\infty \mathbb{P}(W_t \in t + dt) = \sqrt{\frac{8}{\pi}} L \int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{t^2}{2}} dt = \sqrt{\frac{8}{\pi}} L.$$

Thus, we have deduced as a side-product the following rather unobvious result about leaving an upper sector:

**Corollary 3.2**

$$\lim_{u \searrow t} \frac{1}{\sqrt{u-t}} \mathbb{P}[W_v < x + v, \forall v > u \mid \exists s \in [t, u]: W_s = x + s] = \sqrt{\frac{\pi}{8}}.$$

## 4 A localization theorem for the fractional Brownian motion

In this section, we derive a localization theorem for the FBM, corresponding to Theorem 3.1. First we need a rigorous form of the intuitive idea that “the FBM is locally FBM”. This is the message of Theorem 4.4 below. We start with three lemmas that are often useful when a conditionalized FBM is in the play.

In the rest of this paper we often use the notation

$$Z_t^* = \sup \{Z_s : s \in [0, t]\}.$$

**Lemma 4.1** Let  $t > s \geq 1$  and  $x$  be arbitrary. Then

$$\mathbb{E}[Z_t - Z_s \mid Z_1 = x] = \rho(s, t)(t - s)^H x$$

and

$$\mathbb{D}^2[Z_t - Z_s \mid Z_1 = x] = (1 - \rho(s, t)^2)(t - s)^{2H},$$

where

$$\rho(s, t) = \text{Corr}(Z_1, Z_t - Z_s) \leq Ht(t - s)^{1-H}.$$

( $\text{Corr}(\cdot, \cdot)$  means here the correlation coefficient.)

*Proof* The two formulas for the conditional mean and variance are standard. The bound for  $\rho(s, t)$  is obtained with the Mean Value Theorem.  $\square$

**Lemma 4.2** Let  $t > s \geq 1$  and  $x$  be arbitrary. Then

$$\mathbb{E} \left[ \sup_{u \in [s, t]} (Z_u - Z_s) \mid Z_1 = x \right] \leq (t - s)^H \mathbb{E} Z_1^* + Ht(t - s)|x|.$$

*Proof* Denote  $E(u) = \mathbb{E}[Z_u - Z_s \mid Z_1 = x]$ . By Lemma 4.1,

$$\sup_{u \in [s, t]} E(u) \leq Ht(t - s)|x|.$$

On the other hand, Lemma 4.1 also tells that for  $u, v \in [s, t]$

$$D^2[Z_v - Z_u \mid Z_1 = x] \leq D^2(Z_v - Z_u).$$

Given this condition, the Sudakov-Fernique inequality for the expectation of the maximum of centered Gaussian processes (see [Adl90], Theorem 2.9) yields the missing part of the asserted inequality:

$$\mathbb{E} \left[ \sup_{u \in [s, t]} (Z_u - Z_s - (E(u) - E(s))) \mid Z_1 = x \right] \leq \mathbb{E} \sup_{u \in [s, t]} (Z_u - Z_s) = (t - s)^H \mathbb{E} Z_1^*.$$

□

**Lemma 4.3** Let  $t > s \geq 1$  and  $x$  be arbitrary. Then

$$\mathbb{P} \left[ (t - s)^{-H} \sup_{u \in [s, t]} (Z_u - Z_s) > y \mid Z_1 = x \right] \leq 2e^{-y^2/2}$$

whenever  $y > \mathbb{E} Z_1^* + Ht(t - s)^{1-H}|x|$ .

*Proof* By inequality (2.6) of [Adl90] (a consequence of Borell's inequality), we have

$$\begin{aligned} & \mathbb{P} \left[ (t - s)^{-H} \sup_{u \in [s, t]} (Z_u - Z_s) \geq y \mid Z_1 = x \right] \\ & \leq 2 \exp \left( -y^2 / (2 \sup_{u \in [s, t]} D^2 \left[ (t - s)^{-H} (Z_u - Z_s) \mid Z_1 = x \right]) \right) \end{aligned} \quad (4.1)$$

whenever

$$y > \mathbb{E} \left[ (t - s)^{-H} \sup_{u \in [s, t]} (Z_u - Z_s) \mid Z_1 = x \right].$$

By Lemma 4.2, the latter condition is in turn implied by that appearing in the assertion. By Lemma 4.1,  $D^2 \left[ (t - s)^{-H} (Z_u - Z_s) \mid Z_1 = x \right] \leq 1$  for all  $u \leq t$ , and the right hand side of (4.1) is also bounded by  $2e^{-y^2/2}$ . □

The next theorem gives a rigorous form to the intuitively rather obvious principle that, for the FBM, “what happens in a short interval is asymptotically independent on the total increment in a preceding long interval”, or that “a conditionalized FBM is still locally a FBM”.

**Theorem 4.4** Let  $x(h)$  be any real-valued function such that

$$\lim_{h \searrow 0} h^{1-H} x(h) = 0.$$

Then

$$\mathcal{L} \left\{ (h^{-H}(Z_{1+ht} - Z_1) : t \in [0, 1]) \mid Z_1 = x(h) \right\} \xrightarrow{h \searrow 0} \mathcal{L} \{ Z_t : t \in [0, 1] \}$$

in the sense of weak convergence of probability measures on the space of continuous functions  $\mathcal{C}[0, 1]$  (equipped with the supremum norm), where  $\mathcal{L}\{\cdot\}$  and  $\mathcal{L}\{\cdot \mid \cdot\}$  denote the unconditional and conditional law, respectively, of a process having realizations in  $\mathcal{C}[0, 1]$ .

*Proof* As usually, the proof of weak convergence requires showing the convergence of the finite-dimensional distributions and tightness. As regards the former, it is sufficient to show that the means and the variances of all increments of the process converge as asserted, and this follows directly from Lemma 1:

$$|\mathbb{E} [h^{-H}(Z_{1+ht} - Z_1) \mid Z_1 = x(h)]| \leq h^{-H} H(1+h)h^{1-H}h^H|x(h)| \rightarrow 0$$

and

$$\mathbb{D}^2 [h^{-H}(Z_{1+ht} - Z_{1+hs}) \mid Z_1 = x(h)] = h^{-2H}(1 - \rho(1+hs, 1+ht)^2)(ht - hs)^{2H} \rightarrow (t - s)^{2H}$$

as  $h \searrow 0$ . As regards tightness, Theorem 8.3 of [Bil68] says that it is sufficient to show that for any  $\epsilon, \eta > 0$  there exist numbers  $\delta > 0$  and  $h_0 > 0$  such that

$$\frac{1}{\delta} \mathbb{P} \left[ h^{-H} \sup_{u \in [t, t+\delta]} |Z_{1+hu} - Z_{1+ht}| \geq \epsilon \mid Z_1 = x(h) \right] \leq \eta$$

for every  $t \in [0, 1]$  whenever  $h \in (0, h_0)$ . By Lemma 4.3, the left hand side is smaller than

$$\frac{4}{\delta} \exp \left( -\frac{\epsilon^2}{2\delta^{2H}} \right)$$

(an extra factor 2 comes from the elimination of the absolute value) when

$$\frac{\epsilon}{\delta^H} > \mathbb{E} Z_1^* + H(1 + h(1 + \delta))(h\delta)^{1-H}|x(h)|.$$

Choosing first  $\delta$  so that

$$\frac{4}{\delta} \exp \left( -\frac{\epsilon^2}{2\delta^{2H}} \right) < \eta$$

and  $\delta^H \mathbb{E} Z_1^* < \epsilon/2$ , and then  $h_0$  so that  $h \in (0, h_0)$  implies

$$H(1 + h(1 + \delta))\delta h^{1-H}|x(h)| < \frac{\epsilon}{2},$$

we are done. □

After all these preparations, the localization theorem now follows with relatively simple limit arguments.

**Theorem 4.5** For any real numbers  $\gamma$  and  $x$ ,

$$\lim_{h \searrow 0} h^{-H} \mathbb{P}(\exists s \in [0, h] : Z_{1+s} + \gamma(1+s) = x) = 2\mathbb{E}(Z_1^*) \phi(x), \quad (4.2)$$

where  $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ . and  $Z_t^* = \sup \{Z_s : s \in [0, t]\}$ .

*Proof* We give the proof in the case  $\gamma = 0$ . It is straightforward to check that adding a non-zero drift to  $Z$  does not alter the limit results of this section. The basic reason for this is that the drift of the scaled process  $h^{-H}(Z_{1+ht} - Z_1 + \gamma ht)$ ,  $t \in [0, 1]$ , goes to zero with  $h$ . Since the mapping  $f \mapsto \sup_{t \in [0, 1]} f(t)$  is continuous on  $\mathcal{C}[0, 1]$  and since  $Z_1^*$  has a continuous distribution, it follows from Theorem 4.4 that

$$\lim_{h \searrow 0} \mathbb{P} \left[ h^{-H} \sup_{t \in [0, 1]} (Z_{1+ht} - Z_1) \geq y \mid Z_1 = x(h) \right] = \mathbb{P}(Z_1^* \geq y) \quad (4.3)$$

for any  $y \geq 0$  and any bounded function  $x(h)$ . Moreover, by Lemma 4.3 we have for every  $h > 0$

$$\mathbb{P} \left[ h^{-H} \sup_{t \in [0, 1]} (Z_{1+ht} - Z_1) \geq y \mid Z_1 = x(h) \right] \leq 2e^{-y^2/2} \quad (4.4)$$

for  $y > \mathbb{E}Z_1^* + H(1+h)h^{1-H}|x(h)|$ .

Now the assertion follows by direct calculation, making a change of integration variable  $z = x - h^H y$  and applying the convergence result (4.3) with  $x(h) = x - h^H y$  under the integral sign, justified by a dominated convergence argument based on (4.4):

$$\begin{aligned} & h^{-H} \mathbb{P}(\exists s \in [0, h] : Z_{1+s} = x) \\ &= h^{-H} \int_{-\infty}^{\infty} \phi(z) \mathbb{P}[\exists s \in [0, h] : Z_{1+s} = x \mid Z_1 = z] dz \\ &= \int_{-\infty}^{\infty} \phi(x - h^H y) \mathbb{P}[\exists s \in [0, h] : Z_{1+s} = x \mid Z_1 = x - h^H y] dy \\ &= \int_{-\infty}^{\infty} \phi(x - h^H y) \mathbb{P}[\exists t \in [0, 1] : h^{-H}(Z_{1+ht} - Z_1) = y \mid Z_1 = x - h^H y] dy \\ &= \int_{-\infty}^0 \phi(x - h^H y) \mathbb{P} \left[ h^{-H} \inf_{t \in [0, 1]} (Z_{1+ht} - Z_1) \leq y \mid Z_1 = x - h^H y \right] dy \\ &\quad + \int_0^{\infty} \phi(x - h^H y) \mathbb{P} \left[ h^{-H} \sup_{t \in [0, 1]} (Z_{1+ht} - Z_1) \geq y \mid Z_1 = x - h^H y \right] dy \\ &\rightarrow_{h \searrow 0} 2\phi(x) \int_0^{\infty} \mathbb{P}(Z_1^* \geq y) dy \\ &= 2\mathbb{E}(Z_1^*) \phi(x). \end{aligned}$$

□



The generalization of the heuristic formula (3.3) thus reads:

$$\mathbb{P}(\exists s \in [t, t + dt] : Z_s = x) = \frac{2\mathbb{E}Z_1^*}{(dt)^H} \mathbb{P}(Z_t \in x + dt). \quad (4.5)$$

## 5 Last exit time density and Beneš formula in the fractional Brownian case

We now return to the “fractional Brownian storage” defined in the Introduction. It was shown in [Nor94] that the complementary distribution function of the buffer occupancy has an approximately Weibullian lower bound. Duffield and O’Connell [DO95] showed by a large deviation argument that this Weibullian bound is asymptotically accurate. In this section we present for the buffer occupancy distribution a Beneš type exact formula. Although it contains an unknown function, this formula probably yields a better approximation simply by replacing the unknown function by an appropriate constant.

We start again with the case of drift -1. Let  $Z$  be a normalized fractional Brownian motion with Hurst parameter  $H \in [1/2, 1)$  and denote

$$T^x = \sup \{t : Z_t = x + t\}.$$

We shall try to imitate the reasoning presented in Section 3 remembering, however, that  $Z$  is not a Markov process. Using Theorem 4.5, we obtain the following last exit time density.

### Theorem 5.1

$$\mathbb{P}(T^x \in dt) = 2\mathbb{E}(Z_1^*) L_H(x, t) \mathbb{P}(Z_t \in x + t + dt),$$

where

$$L_H(x, t) = \lim_{u \searrow t} \frac{1}{(u - t)^{1-H}} \mathbb{P}[Z_v < x + v, \forall v > u \mid \exists s \in [t, u] : Z_s = x + s]$$

is a finite positive function.

*Proof* We proceed as in Section 3, writing

$$\{T^x \in [t, u]\} = \{\exists s \in [t, u] : Z_s = x + s\} \cap \{Z_v < x + v, \forall v > u\}.$$

The two differences to the Brownian case are that the conditional probability of the second event given the first one now depends on  $x$ ,  $t$  and  $u$ , and that the denominator has to be divided into two unequal factors:

$$\begin{aligned} \mathbb{P}(T^x \in dt) / dt &= \lim_{u \searrow t} \frac{1}{u - t} \mathbb{P}(T^x \in [t, u]) \\ &= \lim_{u \searrow t} \left( \frac{1}{(u - t)^H} \mathbb{P}(\exists s \in [t, u] : Z_s = x + s) \right. \\ &\quad \cdot \left. \frac{1}{(u - t)^{1-H}} \mathbb{P}[Z_v < x + v, \forall v > u \mid \exists s \in [t, u] : Z_s = x + s] \right). \end{aligned} \quad (5.1)$$

Since the left side of (5.1) exists and is finite, and the same holds for the first factor of the right hand side by Theorem 4.5, it follows that

$$\lim_{u \searrow t} \frac{1}{(u-t)^{1-H}} \mathbb{P}[Z_v < x+v, \forall v > u \mid \exists s \in [t, u] : Z_s = x+s]$$

also exists and is finite. This concludes the proof.  $\square$

The corresponding result for a fractional Brownian motion with arbitrary negative drift is easily deduced with a scaling argument. Denote

$$T^{x,\theta} = \sup \{t : Z_t = x + \theta t\}.$$

**Corollary 5.2**

$$\mathbb{P}(T^{x,\theta} \in dt) = 2\theta \mathbb{E}(Z_1^*) \cdot L_H(\theta^{H/(1-H)}x, \theta^{1/(1-H)}t) \cdot \mathbb{P}(Z_t \in x + \theta t + dt),$$

where  $L_H(\cdot, \cdot)$  is the function defined in Theorem 5.1.

*Proof* By the self-similarity of  $Z$  we have

$$\begin{aligned} \mathbb{P}(T^{x,\theta} > t) &= \mathbb{P}(\exists u > t : Z_u = x + \theta u) \\ &= \mathbb{P}(\exists v > \alpha t : Z_{v/\alpha} = x + (\theta/\alpha)v) \\ &= \mathbb{P}(\exists v > \alpha t : \alpha^{-H} Z_v = x + (\theta/\alpha)v) \\ &= \mathbb{P}(\exists v > \alpha t : Z_v = \alpha^H x + \alpha^{H-1}\theta v) \\ &= \mathbb{P}(\exists v > \theta^{1/(1-H)}t : Z_v = \theta^{H/(1-H)}x + v) \quad (\text{choose } \alpha = \theta^{1/(1-H)}) \\ &= \mathbb{P}(T^{\theta^{H/(1-H)}x, 1} > \theta^{1/(1-H)}t), \end{aligned}$$

which yields, by Theorem 5.1,

$$\begin{aligned} \mathbb{P}(T^{x,\theta} \in dt) &= \mathbb{P}(T^{\theta^{H/(1-H)}x, 1} \in \theta^{1/(1-H)}(t + dt)) \\ &= 2\mathbb{E}(Z_1^*) \theta^{1/(1-H)} \mathbb{P}(Z_{\theta^{1/(1-H)}t} \in \theta^{H/(1-H)}x + \theta^{1/(1-H)}t + dt) L_H(\theta^{H/(1-H)}x, \theta^{1/(1-H)}t) \\ &= 2\mathbb{E}(Z_1^*) \theta^{1/(1-H)} \mathbb{P}(\theta^{H/(1-H)}Z_t \in \theta^{H/(1-H)}x + \theta^{1/(1-H)}t + dt) L_H(\theta^{H/(1-H)}x, \theta^{1/(1-H)}t) \\ &= 2\mathbb{E}(Z_1^*) \theta^{1/(1-H)} \mathbb{P}(Z_t \in x + \theta t + \theta^{-H/(1-H)}dt) L_H(\theta^{H/(1-H)}x, \theta^{1/(1-H)}t) \\ &= 2\mathbb{E}(Z_1^*) \theta \mathbb{P}(Z_t \in x + \theta t + dt) L_H(\theta^{H/(1-H)}x, \theta^{1/(1-H)}t). \end{aligned}$$

$\square$

Finally, we can write the Beneš formula for the fractional Brownian storage, i.e. the fluid storage system with input process  $A_t = mt + \sqrt{ma}Z_t$  and leak rate  $c$ , by replacing in the previous corollary  $\theta$  by  $(c - m)/\sqrt{ma}$  and  $x$  by  $x/\sqrt{ma}$  and integrating.

**Corollary 5.3** The stationary distribution of a fractional Brownian storage  $X_t$  is

$$\begin{aligned} \mathbb{P}(X_0 > x) = & \frac{2(c - m)\mathbb{E}(Z_1^*)}{\sqrt{2\pi ma}} \int_0^\infty t^{-H} \exp\left(-\frac{(x + (c - m)t)^2}{2mat^{2H}}\right) \\ & \cdot L_H\left(\left(\frac{c - m}{\sqrt{ma}}\right)^{H/(1-H)}x, \left(\frac{c - m}{\sqrt{ma}}\right)^{1/(1-H)}t\right) dt, \end{aligned}$$

where  $L_H(\cdot, \cdot)$  is the function defined in Theorem 5.1.

It follows from the Sudakov-Fernique inequality ([Adl90], Theorem 2.9) that

$$2\mathbb{E}Z_1^* \in [\sqrt{2/\pi}, \sqrt{8/\pi}],$$

the lower and upper bounds being achieved at  $H = 1$  (a trivial case not considered in this paper) and  $H = 1/2$ , respectively.

With  $H > 1/2$ , it is probably difficult to calculate the function  $L_H(x, t)$  exactly. At present, we can only make some speculative remarks.

**Remark 5.4** Since the meaning of  $L_H(x, t)$  is not only local, it cannot, regrettably, be a constant. It will, however, perhaps be found to consist of a constant asymptotic “microscopic” factor and a non-constant “macroscopic” factor.

**Remark 5.5** It can be expected that  $L_H(x, t)$  is not very variable in either of its arguments, except perhaps for small  $t$  when  $x > 0$ . It may turn out that  $\lim_{t \rightarrow \infty} L_H(x, t) > 0$ .

**Remark 5.6**  $L_H(0, t)$  is probably monotonically decreasing in  $t$ , since the condition in the definition of  $L_H(x, t)$  says that  $Z_s$  is bigger than its expectation (zero) and this is more exceptional with increasing  $t$ . By the positive correlations, the conditional probability of leaving the upper sector should decrease when  $t$  increases. With  $x > 0$ , the situation is more complicated, since  $\mathbb{P}(Z_t \in x + t + dt)$  then has its maximum at some positive value  $t_x$ . Replacing  $L_H(x, t)$  by  $L_H(x, t_x)$  would perhaps yield an upper bound.

## 6 Numerical considerations

As explained above, the exact values of the constant  $\mathbb{E}Z_1^*$  and the function  $L_H(x, t)$  are unknown. The best we can do for the numerical analysis at the moment is to replace them by a constant (cf. Remark 5.4). Its value is then uniquely fixed by the condition  $\mathbb{P}(X_0 > 0) = 1$ , due to the local ruggedness of the arrival process. Applying this condition we get

$$\mathbb{E}(Z_1^*) L_H = 1 - H.$$

Using this value, the expression of Corollary 5.3 for  $H = 1/2$  reduces to the exact formula (2.4). For a general value of  $H$ , the expression can be written in a neater form by a transform of variables,

$$P(X_0 > x) \approx 2 \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[(\frac{y}{x})^{H/(1-H)} + u]^2} du, \quad (6.1)$$

where  $u$  is a dimensionless integration variable and  $y$  is the dimensionless “buffer content”

$$y = x^{(1-H)/H} \frac{c - m}{(ma)^{1/(2H)}}.$$

It is immediately seen that with this expression we indeed have  $P(X_0 > 0) = 1$ .

In Figures 6.1 and 6.2 we see the general behaviour of (6.1) for two values of  $H$ , viz.  $H = 0.5$  and  $H = 0.7$ . The constant parameters are taken to be  $c = 1$  Mbit/s,  $m = 0.1$  Mbit/s and  $a = 30$  kbit·s $^{1-2H}$ . The horizontal units are bits and the vertical coordinate is the base 10 logarithm of the probability  $P(X_0 > x)$ . For reference, the lowerbound results of [Nor94] are also shown. Recall that the Beneš approximation is exact in the case  $H = 1/2$ , so that the figure shows exactly the error of the lower bound in this case.

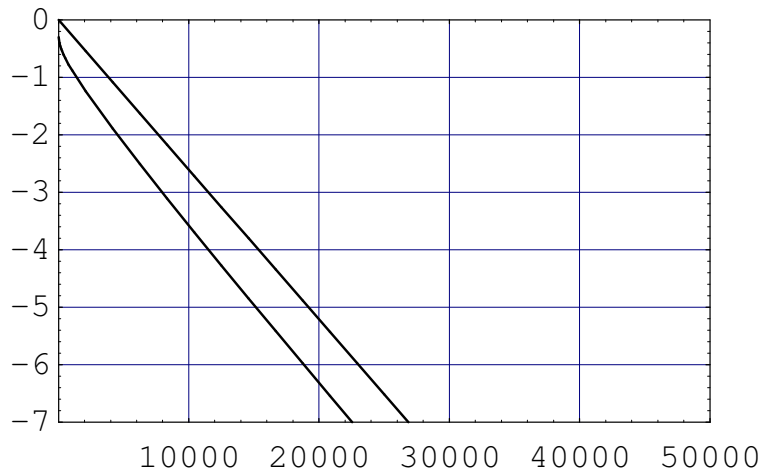


Figure 6.1: Queue length distribution (exact) and lower bound for  $H = 0.5$ .

## References

- [Adl90] R.J. Adler. *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*, volume 12 of *Lecture Notes-Monograph Series*. Institute of Mathematical Statistics, 1990.
- [Ben63] V.E. Beneš. *General Stochastic Processes in the Theory of Queues*. Addison Wesley, 1963.
- [Bil68] P. Billingsley. *Convergence of Probability Measures*. Wiley, New York, 1968.

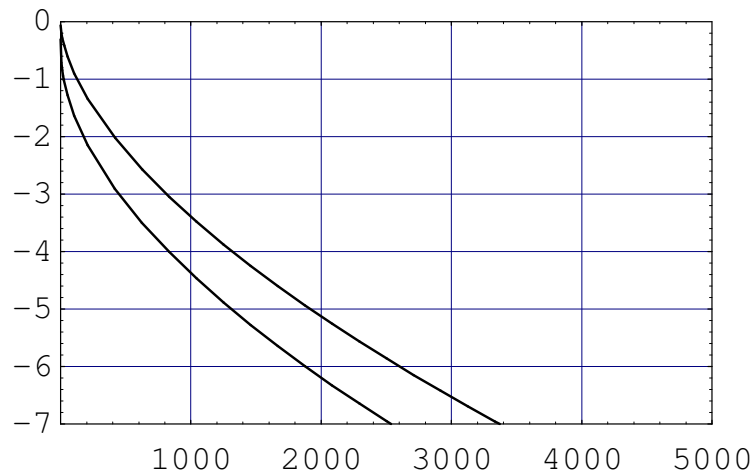


Figure 6.2: Approximate queue length distribution and lower bound for  $H = 0.7$ .

- [Bor76] A.A. Borovkov. *Stochastic Processes in Queueing Theory*. Springer Verlag, 1976.
- [DO95] N.G. Duffield and N. O’Connell. Large deviations and overflow probabilities for the general single-server queue, with applications. To appear in Proc. Cambridge Phil. Soc., 1995.
- [Dur85] J. Durbin. The first-passage density of a continuous Gaussian process to a general boundary. *Journal of Applied Probability*, 22:99–122, 1985. Corrections in J.A.P. 25(1988), p. 840.
- [Nor94] I. Norros. A storage model with self-similar input. *Queueing Systems*, 16:387–396, 1994.
- [Rob92] J.W. Roberts, editor. *Performance Evaluation and Design of Multiservice Networks. COST 224 Final Report*. CEC, 1992.
- [SV91] A. Simonian and J. Virtamo. Transient and stationary distributions for fluid queues and input processes with a density. *SIAM J. Appl. Math.*, 51(6), October 1991.