

On hybrid models of quantum finite automata

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Abstract

In the literature, there exist several quantum finite automata (QFA) models with both quantum and classical states. These models are of particular interest, as they show praiseworthy advantages over the fully quantum models in some nontrivial aspects. This paper characterizes these models in a uniform framework by proposing a general hybrid model consisting of a quantum component and a classical one which can interact with each other. The existing hybrid QFA can be naturally regarded as the general model with specific communication patterns (classical-quantum, quantum-classical, and two-way, respectively). We further clarify the relationship between these hybrid QFA and some other quantum models. In particular, it is shown that hybrid QFA can be simulated exactly by QFA with quantum operations, which in turn has a close relationship with two early proposed models: *ancilla QFA* and *quantum sequential machines*.

Keywords: Quantum computing, Automata theory, Quantum finite automata, Hybrid model

1. Introduction

Quantum finite automata (QFA), as a theoretical model for quantum computers with finite memory, have interested many researchers (see e.g. [1-7, 9-17, 19,20, 22-25,29]). So far, a variety of models of QFA have been introduced and explored to various degrees (one can refer to a review article [22] and references therein). Roughly speaking, these QFA models fall into the following two categories: one-way QFA (1QFA), where the tape head is required to move right on scanning each tape cell, and two-way QFA (2QFA), where the tape head is

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allowed to move left or right, and even to stay stationary. Notably, 2QFA are strictly more powerful than 1QFA: the former is able to recognize ¹ non-regular languages [11], while the latter only regular ones [5, 7, 11].

Another criterion which is used to classify different QFA is the state evolution type. In early references, the state evolution of a QFA is assumed to be unitary operators, in accord with the postulate of quantum mechanics that the state evolution of a closed quantum system is described by a unitary transformation. Later on, it was realized that a QFA does not need to be a closed system; it can interact with the environment. Thus the state evolution should be general quantum operations, i.e., trace-preserving completely positive mappings (see Hirvensalo [9, 10], Li et al [12] and Yakaryilmaz et al [29]). QFA with quantum operations have been thought to be a nice definition for QFA, since they possess nice closure properties and have a competitive computational power with their classical counterparts.

Another model worth mentioning is the *ancilla QFA* proposed in [20]. Actually, ancilla QFA represent the same model as 1QFA with quantum operations, but in different forms, which will become clear in later sections. Interestingly, ancilla QFA can also be regarded as *quantum sequential machines* [23, 13], assigned with some accepting states. The relationship between these models will be elaborated in this paper.

1.1. Hybrid models of QFA

In the literature, there is a class of QFA that differ from other QFA models by consisting of two interactive components: a quantum component and a classical one. We call them *hybrid models of QFA* in this paper. These hybrid models are of particular interest, as they show praiseworthy advantages over the fully quantum models in some nontrivial aspects.

The first hybrid model of QFA is the *two-way quantum finite automata with quantum and classical states* (2QCFA, for short) proposed by Ambainis and Watrous [4] in which on scanning an input symbol, the internal states evolve as follows: first the quantum part undergoes a unitary operator or a projective measurement that is determined by the current classical state and the scanned symbol, and then the classical part specifies a new classical state and a movement of the tape head (left, right, or stationary), which depends on the scanned symbol (along with the outcome if the quantum part makes a measurement). In [4], it was shown that 2QCFA are strictly more powerful than their classical

¹In this paper, recognizing a language always means recognizing it with bounded error.

counterparts—two-way probabilistic finite automata (2PFA). In this paper we will discuss a one-way variant of 2QCFA, denoted by 1QCFA (where the tape head is restricted to move towards only right).

Another hybrid model in the literature is the *one-way quantum finite automata with control language* (CL-1QFA, for short) proposed by Bertoni [6] in which on scanning an input symbol, a unitary operator followed by a projective measurement is applied to the current quantum state. Thus, a CL-1QFA fed with an input string x finally produces a sequence y of measurement results with a certain probability. The input string x is said to be accepted if y is in a given regular language (i.e., the control language). It was shown in [6, 17] that CL-1QFA recognize exactly the class of regular languages, and CL-1QFA can be more succinct (i.e., have less states) than deterministic finite automata (DFA) for certain languages.

Very recently, Qiu et al [25] proposed a new hybrid model called *1QFA together with classical states* (1QFAC, for short) which consists of a quantum component and a classical one. On scanning an input symbol, the two components interact in the following way: first the quantum component undergoes a unitary operator that is determined by the current classical state and the scanned symbol, and then the classical component evolves like a DFA. After scanning the whole input string, a projective measurement determined by the final classical state is performed on the final quantum state, giving the accepting and rejecting probabilities. In [25], it was proved that 1QFAC recognize all regular languages and are exponentially more concise than DFA for certain languages.

1.2. Motivation and contribution of the paper

The hybrid models mentioned above are of particular interest and worthy of further consideration, at least for the following reasons. First, hybrid models are easier to be physically implemented than fully quantum models. For example, while the position of the tape head in the 2QFA model introduced by Kondacs and Watrous [11] is a superposed quantum state and at least $O(\log n)$ qubits are required to store it (where n is the length of the input), in a 2QCFA the tape head position is merely a classical variable which is easy to store and manipulate. Second, hybrid models often save states by being augmented with a small number of quantum states. For example, CL-1QFA and 1QFAC have been proved to be much smaller than DFA when accepting the same language. Therefore, it is beneficial to construct a hybrid model for a practical problem (language) with an appropriate trade-off between quantum and classical states.

Last but not least, quantum engineering systems developed in the further will most probably have a classical human-interactive interface and a quantum processor, and thus they are hybrid models. In fact, hybrid models have already been encountered several times in quantum computing, varying from quantum Turing machines [27] and quantum finite automata [4, 6, 25] to multiple-prover quantum interactive proof systems (the model of complexity class MIP^* [8]) and quantum programming [28].²

This paper aims to characterize the structure of these hybrid models in a uniform framework and to clarify the relationship between these models and others. The contribution of this paper is as follows. (i) First, we characterize three hybrid models of QFA (1QCFA, CL-1QFA, and 1QFAC) in a uniform framework: each hybrid QFA is represented as a communication system consisting of a quantum component and a classical one with the communication between them modeled by controlled operations. The three models differ from each other only in the specific communication pattern. (ii) Second, we will show that 1QCFA, CL-1QFA, and 1QFAC can all be simulated exactly by 1QFA with quantum operations. 1QFA with quantum operations actually represent the same model as another early proposed model—ancilla QFA, which in turn can be regarded as quantum sequence machines assigned with some accepting states. We refer to Fig. 1 for their detailed relationship.

1.3. Organization of the paper

The reminder of this paper is organized as follows. Some preliminaries from quantum theory, automata theory and controlled operations are presented in Section 2. In Section 3 we characterize the structure of three hybrid models of QFA in a uniform framework. Section 4 clarifies the relationship between hybrid models of QFA and other models. Some results concerning the language recognition power and the equivalence problem of hybrid QFA follow as corollaries there. Some conclusions are made in Section 5.

²As stated by Selinger [28], quantum programs can be described by “quantum data with classical control flows”. Thus, a quantum program can be viewed as a hybrid model where quantum data are represented by states of a quantum component and classical control flows are implemented by a classical component.

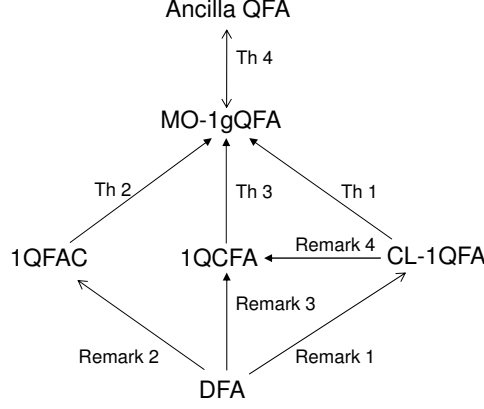


Figure 1: A diagram illustrating the relationship between the models mentioned in this paper. A one-directional arrow denotes that the model at the tail can be simulated by the one at the head; and a bidirectional arrow means that the two models can simulate each other. Note that all models in the diagram have the same language recognition power, recognizing the class of regular languages.

2. Preliminaries

2.1. Preliminaries from quantum theory

For convenience of the reader we briefly recall some basic notions from quantum theory. We refer to [18] for more details. According to von Neumann's formalism of quantum mechanics, a quantum system is associated with a Hilbert space which is called the state space of the system. In this paper, we only consider finite dimensional spaces. A (mixed) state of a quantum system is represented by a density operator on its state space. Here a density operator ρ on \mathcal{H} is a positive semi-definite linear operator such that $\text{Tr}(\rho) = 1$. When the rank of ρ is 1, that is, $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in \mathcal{H}$, then ρ is called a pure state. Let $L(\mathcal{H})$ and $D(\mathcal{H})$ be the sets of linear operators and density operators on \mathcal{H} , respectively.

The evolution of a closed quantum system is described by a unitary operator U on its state space. If the states of the system at times t_1 and t_2 are ρ_1 and ρ_2 , respectively, then $\rho_2 = U\rho_1 U^\dagger$ for some unitary operator U which depends only on t_1 and t_2 . Here U^\dagger is the complex conjugate and transpose of U .

In contrast, the evolution of an *open* quantum system is characterized by a quantum operation \mathcal{E} on its state space \mathcal{H} , which is a linear map from $L(\mathcal{H})$ to

itself that has an *operator-sum representation* as

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger, \quad (1)$$

where $\{E_k\}$, known as the *operation elements* of \mathcal{E} , are linear operators on \mathcal{H} . Furthermore, \mathcal{E} is said to be *trace-preserving* if the following completeness condition is satisfied:

$$\sum_k E_k^\dagger E_k = I. \quad (2)$$

Throughout the rest of this paper, when referring to a quantum operation \mathcal{E} , it is always assumed to be trace-preserving.

To extract information from a quantum system, a measurement has to be performed. A general measurement is described by a collection $\{M_m\}$ of measurement operators, where the index m refers to the potential measurement outcome, satisfying the completeness condition

$$\sum_m M_m^\dagger M_m = I.$$

If this measurement is performed on a state ρ , then the classical outcome m is obtained with the probability $p(m) = \text{Tr}(M_m^\dagger M_m \rho)$, and the post-measurement state is

$$\frac{M_m \rho M_m^\dagger}{\sqrt{p(m)}}.$$

For the case that ρ is a pure state $|\psi\rangle$, that is, $\rho = |\psi\rangle\langle\psi|$, we have $p(m) = \|M_m|\psi\rangle\|^2$, and the state $|\psi\rangle$ “collapse” into the state

$$\frac{M_m|\psi\rangle}{\sqrt{p(m)}}.$$

A special case of general measurements is the projective measurement $\{P_m\}$ where P_m ’s are orthogonal projectors.

Suppose we have physical systems A and B , whose state is described by a density operator $\rho^{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$. Then the state for system A is $\rho^A = \text{Tr}_B(\rho^{AB})$ where Tr_B , known as the *partial trace* over system B , is defined by a linear map from $\mathcal{H}_A \otimes \mathcal{H}_B$ to \mathcal{H}_A satisfying

$$\text{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle |a_1\rangle\langle a_2|$$

where $|a_1\rangle, |a_2\rangle \in \mathcal{H}_A$, and $|b_1\rangle, |b_2\rangle \in \mathcal{H}_B$.

2.2. Preliminaries from automata theory

For a non-empty set Σ , denote by Σ^k and Σ^* the sets of all strings over Σ with length k and with finite length, respectively. Let $|w|$ be the length of string w .

A DFA is a five-tuple $\mathcal{A} = (S, \Sigma, s_1, \delta, S_a)$ where S is a finite state set, Σ is a finite alphabet, $s_1 \in S$ is the initial state, $S_a \subseteq S$ is the accepting set, and $\delta : S \times \Sigma \rightarrow S$ is the transition function³: $\delta(s, \sigma) = t$ means that the current state s changes to t when scanning σ . Furthermore, δ can be extended to $\delta^* : S \times \Sigma^* \rightarrow S$ by defining: i) $\delta^*(s, \epsilon) = s$, and ii) $\delta^*(s, x\sigma) = \delta(\delta^*(s, x), \sigma)$ where $x \in \Sigma^*$ and $\sigma \in \Sigma$. \mathcal{A} is said to accept $x \in \Sigma^*$, if $\delta^*(s_1, x) \in S_a$.

We can also describe a DFA using the matrix notation. Let A_σ be an $|S| \times |S|$ matrix with $A_\sigma[i, j]$ being 1 if $\delta(s_j, \sigma) = s_i$, and 0 otherwise. Then each column of A_σ has a unique 1 and other entries are all 0. Let π be a 0-1 column vector with only the first entry being 1, and η the 0-1 row vector with $\eta_i = 1$ iff $s_i \in S_a$. Define a function $f_A : \Sigma^* \rightarrow \{0, 1\}$ as

$$f_A(x) = \eta A_{x_{|x|}} \cdots A_{x_2} A_{x_1} \pi.$$

Then $f_A(x) = 1$ iff A accepts x .

2.3. Controlled operations

Let $\mathcal{H}_A \otimes \mathcal{H}_B$ be the state space of a bipartite quantum system AB . Let $\{P_i\}_{i=1}^n$ be a projective measurement on \mathcal{H}_A with n outcomes, and \mathcal{E}_i , $1 \leq i \leq n$, quantum operations on \mathcal{H}_B . The *controlled operation* ‘**if subsystem A was measured in result i , then \mathcal{E}_i is performed on subsystem B** ’ can be defined by the following operation elements

$$\{P_i \otimes E_k^i : i = 1, \dots, n, k \in K_i\} \quad (3)$$

where $\{E_k^i\}_{k \in K_i}$ are operation elements of \mathcal{E}_i , that is, $\mathcal{E}_i(\rho) = \sum_{k \in K_i} E_k^i \rho E_k^{i\dagger}$. It is straightforward to check that the elements in (3) satisfy the completeness condition. Furthermore, for $\rho \otimes \varrho \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$, we have

$$\mathcal{E}(\rho \otimes \varrho) = \sum_{i=1}^n P_i \rho P_i \otimes \mathcal{E}_i(\varrho).$$

If the projective measurement $\{P_i\}_{i=1}^n$ in the above is replaced by a general measurement $\{M_i\}_{i=1}^n$, then we get a quantum operation \mathcal{E}' that has the

³Without loss of generality, δ is required to be a total function.

following effect

$$\mathcal{E}'(\rho \otimes \varrho) = \sum_{i=1}^n M_i \rho M_i^\dagger \otimes \mathcal{E}_i(\varrho).$$

3. Characterization of structures of hybrid models of QFA

In this section, structures of several existing hybrid models of QFA are characterized in a uniform framework: a hybrid QFA can be regarded as a communication system consisting of two blocks, with the communication between them modeled by controlled operations, which thus provides an intuitive insight into the structure of hybrid models.

3.1. Hybrid models: the general definition

Hybrid models discussed in this paper are depicted in Fig. 2. A hybrid model of QFA comprises a quantum component, a classical component, a classical communication channel, and a classical tape head (that is, the tape head is regulated by the classical component). On scanning an input symbol, the quantum and classical components interact to evolve into new states, during which communication may occur between them. In this paper, we focus on hybrid models with a one-way tape head, that is, on scanning an input symbol the model moves its tape head one cell right.

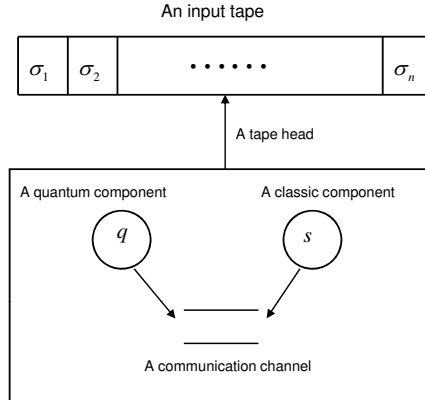


Figure 2: A diagram illustrating the idea behind hybrid models of QFA.

In the literature there are three models of QFA (CL-1QFA, 1QFAC, and 2QCFA) fitting into Fig. 2. For the last model, we will consider its on-way variant 1QCFA. We will show that these models can be characterized in a uniform

framework as in Fig. 2, with the only difference being the specific communication pattern:

- In CL-1QFA, only quantum-classical communication is allowed, that is, the quantum component sends its measurement result to the classical component, but no reverse communication is permitted.
- In 1QFAC, only classical-quantum communication is allowed, that is, the classical component sends its current state to the quantum component.
- In 1QCFA, two-way communication is allowed: (1) first, the classical component sends its current state to the quantum component; (2) second, the quantum component sends its measurement result to the classical component.

Throughout the remainder of this paper we let \mathcal{H}_Q , where $Q = \{q_1, q_2, \dots, q_n\}$, be the n -dimensional Hilbert space spanned by the orthonormal vectors $\{|q_i\rangle : i = 1, \dots, n\}$. Mathematically, $|q_i\rangle$ is an n -dimensional column vector having 1 as the i th entry and 0 else. Sometimes we abuse the notation slightly by writing q_i directly for $|q_i\rangle$.

3.2. CL-1QFA: a hybrid model with quantum-classical communication

Bertoni *et al* [6] introduced a QFA model called *one-way QFA with control language* (CL-1QFA), defined as follows.

Definition 1. A CL-1QFA is a 7-tuple

$$\mathcal{A} = (Q, \Sigma, \mathcal{C}, q_1, \{U_\sigma\}_{\sigma \in \Sigma}, \mathcal{M}, \mathcal{L}),$$

where Q is a finite state set, Σ is a finite alphabet, \mathcal{C} is a finite set of symbols (measurement outcomes), $q_1 \in Q$ is the initial quantum state, U_σ is a unitary operator for each $\sigma \in \Sigma$, \mathcal{M} is a projective measurement given by a collection $\{P_c\}_{c \in \mathcal{C}}$ of projectors, and $\mathcal{L} \subseteq \mathcal{C}^*$ is a regular language (called a control language).

In CL-1QFA \mathcal{A} , on scanning a symbol σ , a unitary operator U_σ followed by the projective measurement \mathcal{M} is performed on its current state. Thus, given an input string $x \in \Sigma^*$, the computation produces a sequence $y \in \mathcal{C}^*$ of measurement results with a certain probability $p(y|x)$ that is given by

$$p(y_1 \dots y_n | x_1 \dots x_n) = \left\| \prod_{i=1}^n (P_{y_i} U_{x_i}) |q_1\rangle \right\|^2 \quad (4)$$

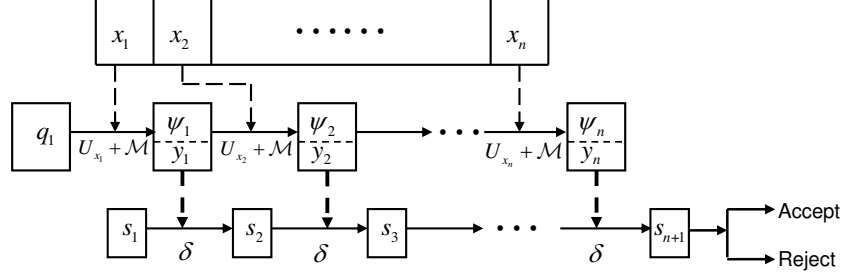


Figure 3: A CL-1QFA is a hybrid model with one-way communication allowed from the quantum component to the classical one, indicated by thick dashed lines. Here δ is the transition function of the DFA accepting \mathcal{L} .

where we define the ordered product $\prod_{i=1}^n A_i = A_n A_{n-1} \cdots A_1$. The computation is said to be *accepted* if y belongs to a fixed regular language $\mathcal{L} \subseteq \mathcal{C}^*$. Thus the probability of \mathcal{M} accepting x is

$$P_A(x) = \sum_{y_1 \dots y_n \in \mathcal{L}} p(y_1 \dots y_n | x_1 \dots x_n). \quad (5)$$

Obviously, a CL-1QFA can be regarded as a hybrid model comprising the following two components:

- A quantum component with state space \mathcal{H}_Q that undergoes unitary operators $\{U_\sigma\}$ and projective measurement \mathcal{M} .
- A classical component that is the DFA accepting the control language \mathcal{L} .

Note that the classical DFA accepting \mathcal{L} takes the measurement outcomes of the quantum component as input. Thus, communication occurs from the quantum component to the classical one. The structure of CL-1QFA is illustrated in Fig. 3.

Remark 1. It was proved in [17] that for each regular language, there exists a CL-1QFA recognizing it with certainty. The idea can be roughly described in Fig. 3 as follows. Given a regular language L , first the quantum component is elaborately designed to function as a bijective mapping from Σ^* to \mathcal{C}^* , such that $\mathcal{L} \subseteq \Sigma^*$ is mapped to a regular language $L' \subseteq \mathcal{C}^*$. Then the classical component is designed to be a DFA accepting L' . One can refer to [17] for more details.

3.3. 1QFAC: a hybrid model with classical-quantum communication

Recently, Qiu et al [25] proposed a new model named *1QFA together with classical states* (1QFAC), defined as follows.⁴

Definition 2. A 1QFAC \mathcal{A} is defined by a 8-tuple

$$\mathcal{A} = (Q, S, \Sigma, q_1, s_1, \{U_{s,\sigma}\}_{s \in S, \sigma \in \Sigma}, \delta, \{\mathcal{M}_s\}_{s \in S}),$$

where Q and S are finite sets of quantum states and classical states, respectively, Σ is a finite input alphabet, $q_1 \in Q$ and $s_1 \in S$ are initial quantum and classical states, respectively, $U_{s,\sigma}$ is a unitary operator on \mathcal{H}_Q for each s and σ , $\delta : S \times \Sigma \rightarrow S$ is a classical transition function, and for each s , \mathcal{M}_s is a projective measurement given by projectors $\{P_{s,a}, P_{s,r}\}$ where the two outcomes a and r denote accepting and rejecting, respectively.

The machine starts with the initial states s_1 and q_1 . On scanning an input symbol $\sigma \in \Sigma$, $U_{s,\sigma}$ is first applied to the current quantum state, where s is the current classical state; afterwards, the classical state s changes to $t = \delta(s, \sigma)$. Finally, when the whole input string is finished, a measurement \mathcal{M}_s determined by the final classical state is performed on the final quantum state, and the input is accepted if the outcome a is observed. Therefore, the probability of 1QFAC \mathcal{A} accepting $x = x_1x_2 \cdots x_n \in \Sigma^*$ is given by

$$P_{\mathcal{A}}(x) = \|P_{s_{n+1},a} U_{s_n,x_n} \cdots U_{s_2,x_2} U_{s_1,x_1} |q_1\rangle\|^2 \quad (6)$$

where $s_{i+1} = \delta(s_i, x_i)$ for $i = 1, \dots, n$.

Again, it is easy to see that a 1QFAC is actually a hybrid model comprising the follows two components:

- A quantum component with state space \mathcal{H}_Q , undergoing unitary operators $\{U_{s,\sigma}\}$.
- A classical component represented by a DFA $\mathcal{A}' = (S, \Sigma, s_1, \delta)$ without an accepting set.

Note that each unitary operator $U_{s,\sigma}$ is determined by the current classical state s . Thus, communication is required from the classical component to the quantum one. The structure of 1QFAC is illustrated in Fig. 4.

⁴In this paper we consider only the case that 1QFAC are language acceptors, and one can refer to [25] for a more general definition.

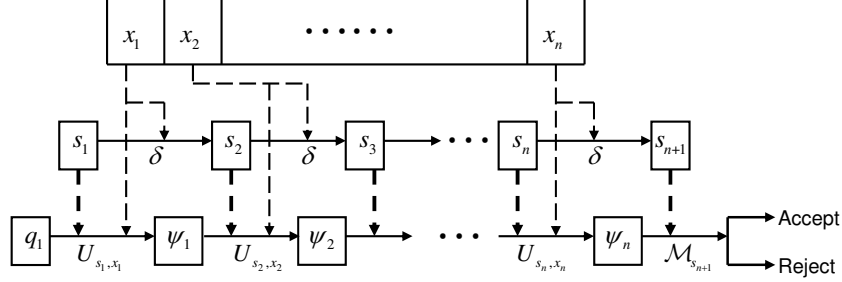


Figure 4: A 1QFAC is a hybrid model with one-way communication allowed from the classical component to the quantum one, indicated by thick dashed lines.

Remark 2. It follows straightforward from Fig. 4 that for each regular language \mathcal{L} , there exists a 1QFAC recognizing it with certainty. For that, the classical component is designed to be a DFA accepting \mathcal{L} , and the quantum component is set to be a qubit system with the orthonormal basis $|0\rangle$ and $|1\rangle$, with $|1\rangle$ being the initial state. Each operator $U_{s,\sigma}$ is simply set to be I . If s is an accepting state, then let $P_{s,a} = |1\rangle\langle 1|$ and $P_{s,r} = |0\rangle\langle 0|$; otherwise, let $P_{s,a} = |0\rangle\langle 0|$ and $P_{s,r} = |1\rangle\langle 1|$.

3.4. 1QCFA: a hybrid model with two-way communication

Ambainis and Watrous [4] proposed the model of *two-way QFA with quantum and classical states* (2QCFA). As proved in [4], 2QCFA can recognize non-regular language $L_{eq} = \{a^n b^n | n > 0\}$ in polynomial time and the palindrome language $L_{pal} = \{x \in \{a, b\}^* | x = x^R\}$ in exponential time, which shows the superiority of 2QCFA over their classical counterparts 2PFA. In the following we discuss 1QCFA, the one-way variant of 2QCFA.

Definition 3. A 1QCFA is specified by a 9-tuple

$$\mathcal{A} = (Q, S, \Sigma, \mathcal{C}, q_1, s_1, \{\Theta_{s,\sigma}\}_{s \in S, \sigma \in \Sigma}, \delta, S_a),$$

where Q and S are finite sets of quantum and classical states, respectively, Σ is a finite input alphabet, \mathcal{C} is a finite set of symbols (measurement outcomes), $q_1 \in Q$ and $s_1 \in S$ are initial quantum and classical states, respectively, $\Theta_{s,\sigma}$ for each s and σ is a general measurement on \mathcal{H}_Q with outcome set \mathcal{C} , $\delta : S \times \Sigma \times \mathcal{C} \rightarrow S$ specifies the classical state transition, and $S_a \subseteq S$ denotes a set of accepting states.

The notion of 1QCFA given above is slightly more general than the one-way version of 2QCFA in [4], where each $\Theta_{s,\sigma}$ is required to be either a unitary

operator or a projective measurement, both being special cases of general measurement considered here.

On scanning a symbol σ , first the general measurement $\Theta_{s,\sigma}$ determined by the current classical state s and the scanned symbol σ is performed on the current quantum state, producing some outcome $c \in \mathcal{C}$; then the classical state s changes to $s' = \delta(s, \sigma, c)$ by reading σ and c . After scanning all input symbols, \mathcal{A} checks whether its classical state is in S_a . If yes, the input is accepted; otherwise, rejected. Therefore, the probability of 1QCFA \mathcal{A} accepting $x = x_1x_2 \cdots x_n \in \Sigma^*$ is given by

$$P_{\mathcal{A}}(x) = \sum_{c_1 c_2 \cdots c_n \in \mathcal{C}^n} \chi_a(s_{n+1}) \|M_{s_n, x_n}^{c_n} \cdots M_{s_2, x_2}^{c_2} M_{s_1, x_1}^{c_1} |q_1\rangle\|^2 \quad (7)$$

where:

- (1) $\chi_a : S \rightarrow \{0, 1\}$ is defined by $\chi_a(s) = \begin{cases} 1, & \text{if } s \in S_a; \\ 0, & \text{otherwise.} \end{cases}$
- (2) $\{M_{s,\sigma}^c\}_{c \in \mathcal{C}}$ are measurement operators of $\Theta_{s,\sigma}$.
- (3) $s_{i+1} = \delta(s_i, x_i, c_i)$ for $i = 1, \dots, n$.

Obviously, a 1QCFA is a hybrid model comprising the following two components:

- A quantum component with state space \mathcal{H}_Q , undergoing general measurements $\{\Theta_{s,\sigma}\}$ with outcome set \mathcal{C} .
- A classical component represented by a DFA $\mathcal{A}' = (S, \Sigma \times \mathcal{C}, s_1, \delta, S_a)$.

Note that two-way communication occurs between the two components, since each $\Theta_{s,\sigma}$ is determined by the classical state s and the DFA \mathcal{A}' has to scan the outcome c produced by the quantum component. The structure of 1QCFA is illustrated in Fig. 5.

Remark 3. From Fig. 5 it is easy to see that for each regular language \mathcal{L} , there exists a 1QCFA recognizing it with certainty. Actually, the classical component alone is already sufficient for the task.

Remark 4. From Fig. 5 one can also see that a 1QCFA reduces to a CL-1QFA, if the following restrictions are made:

- (i) Each $\Theta_{s,\sigma}$ has the form of a unitary operator U_σ followed by a projective measurement \mathcal{M} . Thus no communication is required from the classical component to the quantum one.

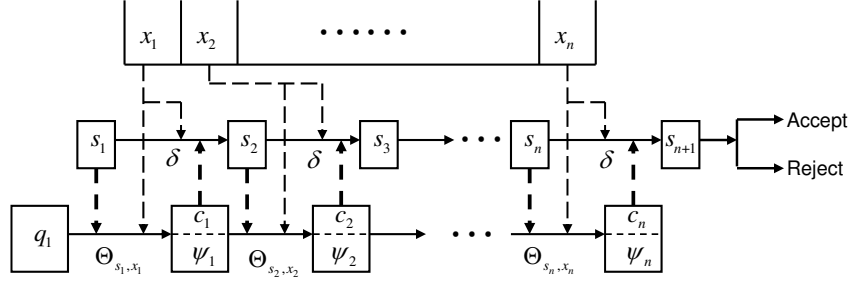


Figure 5: A 1QCFA is a hybrid model consisting of a quantum component and a classical one with two-way communication, indicated by thick dashed lines.

(ii) The classical transition δ is independent of σ .

As a result, CL-1QFA are included as a subset in 1QCFA. However, as will be shown in the next section, these two models actually accept the same set of languages. Thus, the communication from the classical component to the quantum one seems to be superfluous. We believe that two-way communication is useful for reducing the number of states, which, however, needs to be further explored.

In addition, 1QFAC illustrated in Fig. 4 can also be regarded as a variant of 1QCFA in Fig. 5 by restricting all $\Theta_{s,\sigma}$ to be unitary operator $U_{s,\sigma}$ (thus no communication is needed from the quantum part to the classical one) and changing the acceptance fashion.

4. Relationship between QFA models

Based on the characterization of structures of CL-1QFA, 1QFAC, and 1QCFA in the previous section, we will prove in this section that these models can all be simulated exactly⁵ by the model of 1QFA with quantum operations studied in [9, 10, 12, 29]. For simplicity, throughout the rest of this paper we will use the abbreviation MO-1gQFA (measured-once one-way general QFA) to denote the model of 1QFA with quantum operations, in accordance with the name used in [12]. After clarifying the relationship between hybrid models and MO-1gQFA, we will also reveal the relationship between MO-1gQFA and two early proposed models—ancilla QFA and quantum sequential machines.

⁵A QFA simulating another one exactly means that they have the same accepting probability for each input string.

4.1. Review of MO-1gQFA

In the following, we recall the definition of MO-1gQFA.

Definition 4. An MO-1gQFA is a five-tuple $\mathcal{A} = (Q, \Sigma, q_1, \{\mathcal{E}_\sigma\}_{\sigma \in \Sigma}, Q_a)$, where Q is a finite state set, Σ is a finite alphabet, $q_1 \in Q$ is the initial state, \mathcal{E}_σ for each $\sigma \in \Sigma$ is a quantum operation having operation elements $\{E_1^\sigma, \dots, E_k^\sigma\}$ ⁶ satisfying $\sum_{i=1}^k E_i^{\sigma\dagger} E_i^\sigma = I$, and $Q_a \subseteq Q$ denotes the accepting state set, associated with a projector $P_a = \sum_{q \in Q_a} |q\rangle\langle q|$.

On scanning a symbol σ , the quantum operation \mathcal{E}_σ is performed on the current state. After the operation corresponding to the last symbol is performed, a projective measurement is applied to determine acceptance. Thus, for the input string $x = \sigma_1 \sigma_2 \dots \sigma_n \in \Sigma^*$, \mathcal{A} produces the accepting probability given by

$$P_{\mathcal{A}}(x) = \text{Tr}(P_a \mathcal{E}_{\sigma_n} \circ \dots \circ \mathcal{E}_{\sigma_2} \circ \mathcal{E}_{\sigma_1}(|q_1\rangle\langle q_1|)),$$

where $\mathcal{E}_2 \circ \mathcal{E}_1(\rho)$ stands for $\mathcal{E}_2(\mathcal{E}_1(\rho))$.

MO-1gQFA have been thought to be a nice definition of QFA, since they possess nice closure properties and have a competitive computational power with their classical counterparts. The next lemma shows that any DFA can be simulated by a MQ-1gQFA.

Lemma 1. For any DFA $\mathcal{A} = (S, \Sigma, s_1, \delta, S_a)$, there exists an MO-1gQFA $\mathcal{A}' = (S, \Sigma, s_1, \{\mathcal{E}_\sigma\}_{\sigma \in \Sigma}, S_a)$ such that $|t\rangle\langle t| = \mathcal{E}_{\sigma_n} \circ \dots \circ \mathcal{E}_{\sigma_1}(|s\rangle\langle s|)$ iff $t = \delta^*(s, \sigma_1 \dots \sigma_n)$ where $s, t \in S$ and $\sigma_1 \dots \sigma_n \in \Sigma^*$.

Proof. Let $E_s^\sigma = |\delta(s, \sigma)\rangle\langle s|$ where $s \in S$ and $\sigma \in \Sigma$. Then for each $\sigma \in \Sigma$, we get a quantum operation \mathcal{E}_σ given by $\mathcal{E}_\sigma(\rho) = \sum_{s \in S} E_s^\sigma \rho E_s^{\sigma\dagger}$. By a direct calculation, it is easy to verify that $|t\rangle\langle t| = \mathcal{E}_\sigma(|s\rangle\langle s|)$ iff $t = \delta(s, \sigma)$. Then the claimed result can be obtained by induction on the length of the input string. \square

From the above lemma, it can be seen that $P_{\mathcal{A}'}(x) = 1$ if \mathcal{A} accepts x and $P_{\mathcal{A}'}(x) = 0$ if not. As a result, for each regular language there exists an MO-1gQFA which recognizes it with certainty. A full characterization of the languages recognized by MO-1gQFA is as follows.

⁶It can be assumed that for all σ , the numbers of operation elements of \mathcal{E}_σ are the same, since we can get a maximal k for all σ and then add zero operator to those whose number of elements is less than k .

Lemma 2 ([12]). (i) The languages recognized by MO-1gQFA are regular languages; (ii) for each regular language, there exists an MO-1gQFA recognizing it with certainty.

4.2. Simulation of CL-1QFA by MO-1gQFA

In the following, we show that for a given CL-1QFA, there exists an MO-1gQFA simulating it.

Theorem 1. Given a CL-1QFA \mathcal{A} with control language \mathcal{L} accepted by a DFA \mathcal{A}' , there exists an MO-1gQFA $\hat{\mathcal{A}}$ simulating it exactly, with state set $Q \times S$ where Q and S are state sets of \mathcal{A} and \mathcal{A}' , respectively.

Proof. Let $\mathcal{A} = (Q, \Sigma, \mathcal{C}, q_1, \{U_\sigma\}_{\sigma \in \Sigma}, \mathcal{M}, \mathcal{L})$ be a CL-1QFA. As shown in Section 3.2, \mathcal{A} is a hybrid model with a quantum component Q and a classical component being a DFA $\mathcal{A}' = (S, \mathcal{C}, s_1, \delta, S_a)$ accepting \mathcal{L} . The idea of the simulation is to encode the communication from Q to \mathcal{A}' into a controlled operation introduced in Section 2.3.

First by Lemma 1, there exists an MO-1gQFA $\mathcal{A}'' = (S, \mathcal{C}, s_1, \{\mathcal{F}_c\}_{c \in \mathcal{C}}, S_a)$, in which each \mathcal{F}_c on \mathcal{H}_S is represented by operation elements $\{F_c^s\}_{s \in S}$ with $F_c^s = |\delta(s, c)\rangle\langle s|$, such that $|t\rangle\langle t| = \mathcal{F}_c(|s\rangle\langle s|)$ iff $t = \delta(s, c)$.

Now we construct an MO-1gQFA

$$\hat{\mathcal{A}} = (\hat{Q}, \Sigma, \hat{q}_1, \{\hat{\mathcal{E}}_\sigma\}_{\sigma \in \Sigma}, \hat{Q}_a)$$

from \mathcal{A} and \mathcal{A}'' as follows:

- $\hat{Q} = Q \times S$;
- $\hat{q}_1 = (q_1, s_1)$;
- $\hat{Q}_a = Q \times S_a$, associated with the projector $\hat{P}_a = I_Q \otimes \sum_{s \in S_a} |s\rangle\langle s|$ where I_Q is the identity operator on \mathcal{H}_Q ;
- for each $\sigma \in \Sigma$, $\hat{\mathcal{E}}_\sigma$ has operation elements $\{\hat{E}_\sigma^{c,s}\}_{c \in \mathcal{C}, s \in S}$ where

$$\hat{E}_\sigma^{c,s} = P_c U_\sigma \otimes F_c^s = P_c U_\sigma \otimes |\delta(s, c)\rangle\langle s|.$$

It is easy to verify that the collection $\{\hat{E}_\sigma^{c,s}\}$ satisfies the completeness condition. Furthermore, for $\rho \otimes \varrho \in L(\mathcal{H}_Q \otimes \mathcal{H}_S)$, we have

$$\hat{\mathcal{E}}_\sigma(\rho \otimes \varrho) = \sum_{c \in \mathcal{C}} P_c U_\sigma \rho U_\sigma^\dagger P_c \otimes \mathcal{F}_c(\varrho).$$

Now let us check the behavior of $\hat{\mathcal{A}}$ on an input string. Suppose $\hat{\mathcal{A}}$ starts with the initial state (q_1, s_1) and scans a symbol σ . Then the result state is

$$\begin{aligned}\rho &= \hat{\mathcal{E}}_\sigma(|q_1\rangle\langle q_1| \otimes |s_1\rangle\langle s_1|) \\ &= \sum_{c \in \mathcal{C}} P_c U_\sigma |q_1\rangle\langle q_1| U_\sigma^\dagger P_c \otimes \mathcal{F}_c(|s_1\rangle\langle s_1|) \\ &= \sum_{c \in \mathcal{C}} P_c U_\sigma |q_1\rangle\langle q_1| U_\sigma^\dagger P_c \otimes |t_c\rangle\langle t_c|\end{aligned}$$

where $t_c = \delta(s_1, c)$. In this way, after scanning a string $x = x_1 x_2 \cdots x_n \in \Sigma^*$, the final state is

$$\rho_x = \sum_{y \in \mathcal{C}^n} |\phi_y\rangle\langle \phi_y| \otimes |s_y\rangle\langle s_y|$$

where $|\phi_y\rangle = \prod_{i=1}^n (P_{y_i} U_{x_i}) |q_1\rangle$ and $s_y = \delta^*(s_1, y)$. Note that $s_y \in S_a$ iff $y \in \mathcal{L}$. Thus the probability of $\hat{\mathcal{A}}$ accepting x is

$$\begin{aligned}P_{\hat{\mathcal{A}}}(x) &= \text{Tr}(\hat{P}_a \rho_x) = \sum_{y \in \mathcal{C}^n} \chi_a(s_y) \|\phi_y\|^2 \\ &= \sum_{y_1 y_2 \cdots y_n \in \mathcal{L}} \left\| \prod_{i=1}^n (P_{y_i} U_{x_i}) |q_1\rangle \right\|^2,\end{aligned}$$

where $\chi_a(s)$, defined after Eq. (7), indicates whether s is in S_a . Note that the above probability is equal to the one of \mathcal{A} given in Eq. (5). Therefore, we have completed the proof. \square

4.3. Simulation of 1QFAC by MO-1gQFA

By a similar idea as before, we can simulate 1QFAC by MO-1gQFA.

Theorem 2. *Given a 1QFAC with Q and S as its quantum and classical state sets, respectively, there exists an MO-1gQFA simulating it exactly, with $S \times Q$ as its state set.*

Proof. Let $\mathcal{A} = (Q, S, \Sigma, q_1, s_1, \{U_{s,\sigma}\}_{s \in S, \sigma \in \Sigma}, \delta, \{\mathcal{M}_s\}_{s \in S})$ be a 1QFAC. As shown in Section 3.3, \mathcal{A} is a hybrid model with a quantum component Q and a classical component being a DFA $\mathcal{A}' = (S, \Sigma, s_1, \delta)$ without an accepting set. The idea of the simulation is to encode the communication from \mathcal{A}' to Q into a controlled operation introduced in Section 2.3.

First by Lemma 1, there exists an MO-1gQFA $\mathcal{A}'' = (S, \Sigma, s_1, \{\mathcal{F}_\sigma\}_{\sigma \in \Sigma})$ (without an accepting set), such that $|t\rangle\langle t| = \mathcal{F}_\sigma(|s\rangle\langle s|)$ iff $t = \delta(s, \sigma)$. Then we construct an MO-1gQFA

$$\hat{\mathcal{A}} = (\hat{Q}, \Sigma, \hat{q}_1, \{\hat{\mathcal{E}}_\sigma\}_{\sigma \in \Sigma}, \hat{P}_a)$$

as follows:

- $\widehat{Q} = S \times Q$;
- $\widehat{q}_1 = (s_1, q_1)$;
- For each $\sigma \in \Sigma$, $\widehat{\mathcal{E}}_\sigma$ on $\mathcal{H}_S \otimes \mathcal{H}_Q$ is given by

$$\widehat{\mathcal{E}}_\sigma = (\mathcal{F}_\sigma \otimes \mathcal{I}) \circ \mathcal{E}_\sigma$$

where \mathcal{I} is the identity mapping on $L(\mathcal{H}_Q)$, and \mathcal{E}_σ is given by operation elements $\{E_s = |s\rangle\langle s| \otimes U_{s,\sigma}\}_{s \in S}$. Given $\rho \otimes \varrho \in L(\mathcal{H}_S \otimes \mathcal{H}_Q)$, we have

$$\widehat{\mathcal{E}}_\sigma(\rho \otimes \varrho) = \sum_{s \in S} \langle s | \rho | s \rangle \mathcal{F}_\sigma(|s\rangle\langle s|) \otimes U_{s,\sigma} \varrho U_{s,\sigma}^\dagger,$$

which captures the idea that if the current classical state is s , then $U_{s,\sigma}$ is performed on the quantum component, and furthermore s changes to another state according to \mathcal{F}_σ .

- Instead of specifying the set of accepting states,⁷ we specify here the final projective measurement $\{\widehat{P}_a, \widehat{P}_r\}$. Suppose $\mathcal{M}_s = \{P_{s,a}, P_{s,r}\}$. We let

$$\widehat{P}_a = \sum_{s_i \in S} |s_i\rangle\langle s_i| \otimes P_{s_i,a}$$

and

$$\widehat{P}_r = \sum_{s_i \in S} |s_i\rangle\langle s_i| \otimes P_{s_i,r}.$$

It is easily verified that \widehat{P}_a and \widehat{P}_r form a projective measurement.

In the following we verify that $\widehat{\mathcal{A}}$ and \mathcal{A} have the same accepting probability for any given input string. Let ρ_x be the state of $\widehat{\mathcal{A}}$ after scanning x and before the final measurement. Then by induction on the length of the input string, it is easy to show that for each $x = x_1 x_2 \cdots x_n \in \Sigma^*$, we have

$$\rho_x = |s_{n+1}\rangle\langle s_{n+1}| \otimes |\psi_x\rangle\langle \psi_x|$$

with

$$(1) \quad |\psi_x\rangle = U_{s_n, x_n} \cdots U_{s_2, x_2} U_{s_1, x_1} |q_1\rangle, \text{ and}$$

⁷Specifying an accepting state set is essentially equivalent to specifying a projective measurement.

(2) $s_{i+1} = \delta(s_i, x_i)$ for $i = 1, \dots, n$.

Therefore, the probability of $\hat{\mathcal{A}}$ accepting x is

$$P_{\hat{\mathcal{A}}}(x) = \text{Tr}(\hat{P}_a \rho_x) = \|P_{s_{n+1}, a} U_{s_n, x_n} \cdots U_{s_2, x_2} U_{s_1, x_1} |q_1\rangle\|^2,$$

which is equal to the accepting probability of \mathcal{A} given in Eq. (7). This completes the proof of Theorem 2. \square

4.4. Simulation of 1QCFA by MO-1gQFA

In this section we show how 1QCFA can be simulated by MO-1gQFA. This simulation is more complicated than previous simulations of CL-1QFA and 1QFAC, since now two-way communication occurs in 1QCFA. However, the ideals are similar.

Theorem 3. *Given a 1QCFA \mathcal{A} with Q and S as quantum and classical state sets, respectively, there exists an MO-1gQFA $\hat{\mathcal{A}}$ simulating it exactly, with $Q \times S$ as the state set.*

Proof. Let $\mathcal{A} = (Q, S, \Sigma, \mathcal{C}, q_1, s_1, \{\Theta_{s, \sigma}\}_{s \in S, \sigma \in \Sigma}, \delta, S_a)$ be a 1QCFA. As shown in Section 3.4, \mathcal{A} is a hybrid model comprising a quantum component Q and a classical component being a DFA $\mathcal{A}' = (S, \Sigma \times \mathcal{C}, s_1, \delta, S_a)$. The idea of the simulation is to encode the communication between Q and \mathcal{A}' into a controlled operation introduced in Section 2.3.

First it follows from Lemma 1 that for the DFA \mathcal{A}' , there exists an MO-1gQFA $\mathcal{A}'' = (S, \Sigma \times \mathcal{C}, s_1, \{\mathcal{F}_{\sigma, c}\}_{\sigma \in \Sigma, c \in \mathcal{C}}, S_a)$, such that $|t\rangle\langle t| = \mathcal{F}_{\sigma, c}(|s\rangle\langle s|)$ iff $t = \delta(s, \sigma, c)$ for $(\sigma, c) \in \Sigma \times \mathcal{C}$. Then we construct an MO-1gQFA

$$\hat{\mathcal{A}} = (\hat{Q}, \Sigma, \hat{q}_1, \{\hat{\mathcal{E}}_\sigma\}_{\sigma \in \Sigma}, \hat{Q}_a)$$

as follows:

- $\hat{Q} = Q \times S$;
- $\hat{q}_1 = (q_1, s_1)$;
- $\hat{Q}_a = Q \times S_a$, associated with the projector $\hat{P}_a = I_Q \otimes \sum_{s \in S_a} |s\rangle\langle s|$;
- Each $\hat{\mathcal{E}}_\sigma$ on $\mathcal{H}_Q \otimes \mathcal{H}_S$ is given by the following operation elements

$$\{M_{s, \sigma}^c \otimes F_{\sigma, c}^k E_s : s \in S, c \in \mathcal{C}, k \in K_{\sigma, c}\} \quad (8)$$

where:

- $E_s = |s\rangle\langle s|$ for $s \in S$, which is used to detect whether the classical state is s ;
- $\{M_{s,\sigma}^c\}_{c \in \mathcal{C}}$ are measurement operators of $\Theta_{s,\sigma}$;
- $\{F_{\sigma,c}^k\}_{k \in K_{\sigma,c}}$ are operation elements of $\mathcal{F}_{\sigma,c}$, that is, $\mathcal{F}_{\sigma,c}(\varrho) = \sum_{k \in K_{\sigma,c}} F_{\sigma,c}^k \varrho F_{\sigma,c}^{k\dagger}$.

For each $\sigma \in \Sigma$, $\hat{\mathcal{E}}_\sigma$ given by Eq. (8) satisfies the completeness condition

$$\begin{aligned}
& \sum_{s \in S} \sum_{c \in \mathcal{C}} \sum_{k \in K_{\sigma,c}} (M_{s,\sigma}^c \otimes F_{\sigma,c}^k E_s)^\dagger (M_{s,\sigma}^c \otimes F_{\sigma,c}^k E_s) \\
&= \sum_{s \in S} \sum_{c \in \mathcal{C}} M_{s,\sigma}^{c\dagger} M_{s,\sigma}^c \otimes E_s^\dagger \left(\sum_{k \in K_{\sigma,c}} F_{\sigma,c}^{k\dagger} F_{\sigma,c}^k \right) E_s \\
&= \sum_{s \in S} \left(\sum_{c \in \mathcal{C}} M_{s,\sigma}^{c\dagger} M_{s,\sigma}^c \right) \otimes E_s^\dagger E_s \\
&= I_Q \otimes \sum_{s \in S} E_s^\dagger E_s = I_Q \otimes I_S.
\end{aligned}$$

Furthermore, for $\rho \otimes \varrho \in L(\mathcal{H}_Q \otimes \mathcal{H}_S)$, by a direct calculation we have

$$\hat{\mathcal{E}}_\sigma(\rho \otimes \varrho) = \sum_{s \in S} \sum_{c \in \mathcal{C}} M_{s,\sigma}^c \rho M_{s,\sigma}^{c\dagger} \otimes \langle s|\varrho|s\rangle \mathcal{F}_{\sigma,c}(|s\rangle\langle s|).$$

The above equation intuitively captures the idea that if the current classical state is s , then $\Theta_{s,\sigma}$ is performed on the quantum component; furthermore, if the outcome of $\Theta_{s,\sigma}$ is c , then $\mathcal{F}_{\sigma,c}$ is applied on the classical component. For example, given the initial state $|q_1\rangle|s_1\rangle$, we have

$$\begin{aligned}
\hat{\mathcal{E}}_\sigma(|q_1\rangle\langle q_1| \otimes |s_1\rangle\langle s_1|) &= \sum_{c \in \mathcal{C}} M_{s_1,\sigma}^c |q_1\rangle\langle q_1| M_{s_1,\sigma}^{c\dagger} \otimes \mathcal{F}_{\sigma,c}(|s_1\rangle\langle s_1|) \\
&= \sum_{c \in \mathcal{C}} M_{s_1,\sigma}^c |q_1\rangle\langle q_1| M_{s_1,\sigma}^{c\dagger} \otimes |\delta(s_1, \sigma, c)\rangle\langle \delta(s_1, \sigma, c)|.
\end{aligned}$$

In the following, we verify that the constructed MO-1gQFA $\hat{\mathcal{A}}$ and the given 1QCFA \mathcal{A} have the same accepting probability for any given input string. Let ρ_x be the state of $\hat{\mathcal{A}}$ after scanning x and before the final measurement. Then for $x = x_1 x_2 \cdots x_n \in \Sigma^*$, we have

$$\rho_x = \sum_{y \in \mathcal{C}^n} |\psi_y\rangle\langle \psi_y| \otimes |s_{n+1}\rangle\langle s_{n+1}| \quad (9)$$

with

$$(1) \quad |\psi_y\rangle = M_{s_n, x_n}^{y_n} \cdots M_{s_2, x_2}^{y_2} M_{s_1, x_1}^{y_1} |q_1\rangle, \text{ and}$$

(2) $s_{i+1} = \delta(s_i, x_i, y_i)$ for $i = 1, \dots, n$.

Actually, Eq. (9) can be verified by induction on the length of x as follows.

The basis step. When $|x| = 0$, that is, $x = \epsilon$, the result holds obviously.

The induction step. Suppose the result holds for $x \in \Sigma^*$ with $|x| = n - 1$. Then for $\sigma \in \Sigma$, we obtain

$$\begin{aligned} \rho_{x\sigma} &= \hat{\mathcal{E}}_\sigma(\rho_x) = \sum_{y \in \mathcal{C}^{n-1}} \hat{\mathcal{E}}_\sigma(|\psi_y\rangle\langle\psi_y| \otimes |s_n\rangle\langle s_n|) \\ &= \sum_{y \in \mathcal{C}^{n-1}} \sum_{y_n \in \mathcal{C}} M_{s_n, \sigma}^{y_n} |\psi_y\rangle\langle\psi_y| M_{s_n, \sigma}^{y_n \dagger} \otimes \mathcal{F}_{\sigma, y_n}(|s_n\rangle\langle s_n|) \\ &= \sum_{yy_n \in \mathcal{C}^n} |\psi_{yy_n}\rangle\langle\psi_{yy_n}| \otimes |s_{n+1}\rangle\langle s_{n+1}| \end{aligned}$$

where $s_{n+1} = \delta(s_n, \sigma, y_n)$ and $|\psi_{yy_n}\rangle = M_{s_n, \sigma}^{y_n} |\psi_y\rangle$. Thus Eq. (9) holds.

Now the probability of $\hat{\mathcal{A}}$ accepting $x = x_1 x_2 \dots x_n \in \Sigma^*$ is

$$\begin{aligned} P_{\hat{\mathcal{A}}}(x) &= \text{Tr}(\hat{P}_a \rho_x) = \sum_{y \in \mathcal{C}^n} \chi_a(s_{n+1}) \| |\psi_y\rangle \|^2 \\ &= \sum_{y \in \mathcal{C}^n} \chi_a(s_{n+1}) \| M_{s_n, x_n}^{y_n} \dots M_{s_2, x_2}^{y_2} M_{s_1, x_1}^{y_1} |q_1\rangle \|^2, \end{aligned}$$

which is equal to the probability of \mathcal{A} accepting x given in Eq. (7). \square

4.5. Language recognition power and equivalence problem of hybrid QFA

In the study of QFA, an important problem is to characterize the language classes recognized by various models (e.g., [1, 5, 7]). This problem was considered for CL-1QFA in [6, 17] and for 1QFAC in [25]. Using our results presented in the previous sections, we can show that the three hybrid models indeed have the same language recognition power in the following sense.

Corollary 1. *The models CL-1QFA, 1QFAC, and 1QCFA all recognize exactly the class of regular languages.*

Proof. The proof is depicted in Fig. 6 \square

Two QFA over the same input alphabet are said to be equivalent if they have the same accepting probability for each input string. The equivalence problem of a QFA model is that given any two automata of the model, decide whether they are equivalent. This problem has been proven to be decidable for several quantum models [12, 13, 14, 24, 25]. Similar problems were also discussed for

alphabet, a finite output alphabet the initial state, and the accepting state set, and the transition function $\delta : Q \times \Sigma \times Q \times \Omega \rightarrow \mathbb{C}$ satisfies

$$\sum_{p \in Q, \omega \in \Omega} \delta(q_1, \sigma, p, \omega)^* \delta(q_2, \sigma, p, \omega) = \begin{cases} 1, & q_1 = q_2 \\ 0, & q_1 \neq q_2 \end{cases} \quad (10)$$

for all states $q_1, q_2 \in Q$ and $\sigma \in \Sigma$. Here \mathbb{C} is the set of complex numbers, and the $*$ operation in Eq. (10) denotes the complex conjugate.

An ancilla QFA has a one-way tape head that reads symbols in the input tape from left to right and has an output tape where an output symbol is written at each step. Given an input string $\sigma_1 \cdots \sigma_n \in \Sigma^*$, it starts with the initial state q_1 and reads the first symbol σ_1 . Then with amplitude $\delta(q_1, \sigma_1, p, \omega)$, it changes to state p and write ω on its output tape. After that, the automaton moves its tape head right to read the next symbol. The above procedure continues until the last symbol σ_n has been scanned. Finally, it checks whether the final state is in the set Q_a . If yes, then the input is accepted; otherwise, it is rejected.

Remark 5. Since the output tape is never read in the evolution of ancilla QFA, it is simply assumed that the output symbol is reset at every step, which thus results in the constant size of the output tape.

For $\sigma \in \Sigma$ and $\omega \in \Omega$, we define

$$V_{\sigma, \omega} = \sum_{q_i, q_j \in Q} \delta(q_j, \sigma, q_i, \omega) |q_i\rangle \langle q_j|$$

and

$$V_\sigma = \sum_{\omega \in \Omega} V_{\sigma, \omega} \otimes |\omega\rangle.$$

In matrix notations, $V_{\sigma, \omega}$ is a $|Q| \times |Q|$ matrix. Then, Eq. (10) is equivalent to

$$V_\sigma^\dagger V_\sigma = \sum_{\omega \in \Omega} V_{\sigma, \omega}^\dagger V_{\sigma, \omega} = I,$$

which means that V_σ is an isometric operator from \mathcal{H}_Q to $\mathcal{H}_Q \otimes \mathcal{H}_\Omega$.

Suppose now that the machine is in the current state ρ . Then its state ρ' after scanning σ can be obtained by tracing over Ω as follows:

$$\rho' = \text{Tr}_\Omega(V_\sigma \rho V_\sigma^\dagger) = \text{Tr}_\Omega \left(\sum_{\omega, \omega' \in \Omega} V_{\sigma, \omega} \rho V_{\sigma, \omega'}^\dagger \otimes |\omega\rangle \langle \omega'| \right) = \sum_{\omega \in \Omega} V_{\sigma, \omega} \rho V_{\sigma, \omega}^\dagger.$$

Therefore, for each $\sigma \in \Sigma$, the evolution of the machine is characterized by a quantum operation \mathcal{E}_σ that has operation elements $\{V_{\sigma,\omega} : \omega \in \Omega\}$. As a result, an ancilla QFA is an MO-1gQFA.

On the other hand, we can construct an ancilla QFA equivalent to a given MO-1gQFA. Let $\mathcal{A} = (Q, \Sigma, q_1, \{\mathcal{E}_\sigma\}_{\sigma \in \Sigma}, Q_a)$ be an MO-1gQFA, where $\mathcal{E}_\sigma(\rho) = \sum_{i=1}^k E_{\sigma,i} \rho E_{\sigma,i}^\dagger$. Now we construct an ancilla QFA $\mathcal{A}' = (Q, \Sigma, \Omega, q_1, \delta, Q_a)$ where Q, Σ, q_1, Q_a are identical to the ones in \mathcal{A} , $\Omega = \{1, 2, \dots, k\}$, and δ is defined by $\delta(q_j, \sigma, q_i, \omega) = \langle q_i | E_{\sigma,\omega} | q_j \rangle$ for $q_i, q_j \in Q$, $\sigma \in \Sigma$ and $\omega \in \Omega$. It is straightforward to verify that the states of \mathcal{A} and \mathcal{A}' are identical at every step for any given input string.

The above argument leads to the following theorem.

Theorem 4. *MO-1gQFA and ancilla QFA can simulate each other.*

Interestingly, the idea behind ancilla QFA was also used in [19] to construct quantum interactive proof systems (QIP systems, for short), although the authors did not define the notion of ancilla QFA explicitly. They proved each regular language can be recognized by a QIP system that takes a 1QFA as a verifier ([19], Proposition 4.2). The QIP system is essentially an ancilla QFA in that at each step the verifier simulates a DFA's behavior and writes its current state as output and then the prover erases the output.

To conclude this section, we would like to point out the relationship between ancilla QFA and the model quantum sequential machines (QSM) studied in [23, 13].

Definition 6. A QSM is a five-tuple $\mathcal{A} = (S, \Sigma, \Omega, s_1, \delta)$, where S is a finite set of internal states, Σ and Ω are finite input and output alphabets, respectively, $s_1 \in S$ is the initial state, and $\delta : \Sigma \times S \times \Omega \times S \rightarrow \mathbb{C}$ is a transition function satisfying

$$\sum_{\omega \in \Omega, t \in S} \delta(\sigma, s, \omega, t) \delta(\sigma, s', \omega, t)^* = \begin{cases} 1 & s = s', \\ 0 & s \neq s' \end{cases} \quad (11)$$

for all states $s, s' \in S$ and every $\sigma \in \Sigma$.

Intuitively we interpret $\delta(\sigma, s, \omega, t)$ as the transition amplitude that \mathcal{A} prints ω and enters state t after scanning σ in the current state s . Thus, given an input string $x = x_1 \dots x_n \in \Sigma^*$, QSM \mathcal{A} prints $y = y_1 \dots y_n \in \Omega^*$ with a certain probability denoted by $p(y|x)$. For the model of QSM, attentions are usually paid to the probability $p(y|x)$ instead of acceptance or rejection.

Remark 6. Now, if some accepting states are assigned to QSM \mathcal{A} , and we no longer care the output, but focus on the accepting probability of the input, then we get an ancilla QFA. In a word, an ancilla QFA is essentially a quantum sequential machine assigned with some accepting states.

5. Conclusions

In this paper we investigate three hybrid models of QFA—CL-1QFA, 1QFAC, and 1QCFA—which differ from other QFA models by consisting of two interactive components: a quantum one and a classical one. The contribution of this paper is twofold. (i) First, we characterize structures of these models in a uniform framework: each hybrid model can be seen as a two-component communication system with certain communication pattern. (ii) Second, we clarify the relationship between the hybrid models and other models. Specifically, we show that CL-1QFA, 1QFAC, and 1QCFA can all be simulated exactly by MO-1gQFA. Some results in the literature concerning the language recognition power and the equivalence problem of these hybrid models follow directly from these relationships. In addition, MO-1gQFA and another early proposed model called ancilla QFA are shown to be equivalent.

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