L^∞-Bounded Robust Control of Nonlinear Cascade Systems*

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Abstract

In this paper, we consider the L^∞-bounded robust control problem for a class of nonlinear cascade systems with disturbances. Sufficient conditions are provided under which a hard bound is imposed on the system performance measure. The backstepping approach is used for controller design. A practical example is provided to illustrate the method.

Key-words. Nonlinear cascade systems, L^∞ bound, robust control, backstepping.

1 Introduction

In the design of robust control systems, L^∞-type (l^∞-type) criteria are used when a hard bound on the system performance measure is required. Some recent work in this area is described in the references [2, 3, 5, 7, 18]. For example, in our recent work [7], l^∞ robustness analysis and synthesis problems for general nonlinear systems were studied; in particular, necessary and sufficient conditions are provided, and a controller design procedure is given in terms of dynamic programming equations (or inequalities). However, solving dynamic programming equations for high order systems is computationally complex, and this motivates us to look for constructive controller design methods for nonlinear cascade systems with some special structure, as we discuss in this paper.

Recently, considerable attention has been paid to robust control problems for nonlinear cascade systems with strict-feedback form [12]. Some effective techniques for the construction of feedback control laws (e.g. backstepping) were developed exploiting the special structure of these systems. Different performance requirements have been considered, such as L_2 gain disturbance rejection with internal stability [14, 8, 9], input to state stability [11, 12], integral input to state stability [13], both local optimality and global inverse optimality [4, 6], etc.

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In this paper, we consider the L^∞-bounded robust control problem for nonlinear cascade systems with strict-feedback form. The disturbance inputs are assumed to be bounded. By assuming the L^∞*-bounded (LIB)* dissipation property [7] for the low order closed loop system, we provide a controller design method for the higher order cascade system such that the closed loop system is LIB dissipative. The popular backstepping technique [12] is adapted to this L^∞ context, and is used for the construction of the feedback controller.

The L^∞-bounded (LIB) dissipation property we considered here focuses more on the robustness with respect to magnitude-bounded disturbances. In the special case when the performance is the norm of the state, the LIB property is a much weaker property than the input to state stability (ISS) property [15]. It is also weaker than the asymptotic $\mathcal{L}_\infty$ bound property (ISS with restriction) [19] where the asymptotic behavior of the output with respect to the asymptotic behavior of the input is included.

This paper is organized as follows. In Section 2 the L^∞-bounded robust control problem for nonlinear cascade systems to be solved is formulated, and some preliminary results are given. In Section 3, the solution of the problem and its proof are given and the issue of asymptotic stability is considered. In Section 4 we present a physical example to illustrate the application of the backstepping method, and some concluding remarks are provided in Section 5.

2 Problem Statement

We consider a nonlinear cascade system of the form

$$\begin{cases} \dot{x} = f_1(x) + g_1(x)w + g_2(x)y \\ \dot{y} = u + f_2(x,y)w \\ z = g(x), \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n, y \in \mathbf{R}^m$ are the states, $u \in \mathbf{R}^m$ is the control input, $w \in \mathbf{W} \subset \mathbf{R}^r$ is the disturbance input, and $z \in \mathbf{R}$ is the performance quantity.

Assumption 2.1 *The set* $\mathbf{W} \subset \mathbf{R}^r$ *is bounded; functions* f_1, g_1, g_2, f_2 *are locally Lipschitz; for any locally Lipschitz control law* $u = \bar{\alpha}(x,y)$*, the closed-loop system is a forward complete system, meaning that for any initial state and any disturbance input, the solution is defined on the entire interval* $[0,+\infty)$*.*

Throughout this paper, we denote

$$d = \sup_{w \in \mathbf{W}} |w| < +\infty \tag{2}$$

where $|\cdot|$ is the Euclidean norm. Also, for $0 \le t_1 < t_2 \le +\infty$, we denote by $\mathcal{W}_{t_1,t_2}$ the class of $\mathbf{W}$-valued disturbance inputs defined on the time interval $[t_1, t_2]$.

Problem. Given a set $\bar{B}_0 \subset \mathbf{R}^{n+m}$, we wish to find, if possible, a state feedback controller

$$u = \bar{\alpha}(x,y)$$

such that the resulting closed loop system is L^∞ *bounded (LIB) dissipative* with respect to $\bar{B}_0$, i.e. there exists $\bar{\beta} : \bar{B}_0 \to \mathbf{R}$ such that for the closed loop system ((1)) with $u = \bar{\alpha}(x,y)$,

$$z(t) \le \bar{\beta}(x_0,y_0), \quad \forall (x_0,y_0) \in \bar{B}_0, \quad \forall w_{0,t} \in \mathcal{W}_{0,t}, \quad \forall t \ge 0. \tag{3}$$

Remark 2.2 The property (3) concerns the boundedness of a function of trajectories, and cover asymptotically stable, stable and limit cycle behavior. Combining the property (3) with asymptotic stability property is a stronger requirement and will be discussed in the later part of Section 3.

We solve this problem by using the popular backstepping technique [12] to construct the required state feedback controller $\bar{\alpha}(x, y)$. To this end, we consider the following subsystem

$$\left\{ \begin{array}{rcl} \dot{x} & = & f_1(x) + g_1(x)w + g_2(x)u \\ z & = & g(x). \end{array} \right. \tag{4}$$

In the spirit of backstepping, it is natural to assume that this subsystem enjoys the desired property, which here means the existence of a state feedback controller $\alpha(x)$, a set $B_0 \subset \mathbf{R}^n$, and a function $\beta : B_0 \to \mathbf{R}$ such that for the closed-loop system ((4)) with $u = \alpha(x)$,

$$z(t) \leq \beta(x_0), \quad \forall x_0 \in B_0, \quad \forall w_{0,t} \in \mathcal{W}_{0,t}, \quad \forall t \geq 0. \tag{5}$$

However, this is not enough, and in fact we need the following stronger assumption which, as we shall see (Lemma 2.4), implies (5) for the subsystem. The assumption is an extension of the dissipative system framework developed in [7] for LIB problems.

To specify this assumption, we need some notation. For a function $V : \mathbf{R}^n \to \mathbf{R}$ and a number $\delta \leq +\infty$, denote

$$S_\delta^V = \{x \in \mathbf{R}^n : V(x) < \delta\}, \quad \bar{S}_\delta^V = \{x \in \mathbf{R}^n : V(x) \leq \delta\}. \tag{6}$$

Assumption 2.3 *There exist a C^1 function $\alpha : \mathbf{R}^n \to \mathbf{R}^m$, a C^1 function $V : \mathbf{R}^n \to \mathbf{R}$, two numbers ρ, δ with $\delta < \rho \leq +\infty$, and two positive real numbers $\eta > 0, \Delta > 0$, such that the set $\bar{S}_\rho^V - S_\delta^V$ has non-empty interior and*

$$\begin{array}{rcll} V(x) & \geq & g(x), & \forall x \in \bar{S}_\rho^V, \\ \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] & \leq & \Delta, & \forall x \in \bar{S}_\rho^V, \forall w \in \mathbf{W}, \\ \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] & \leq & -\eta, & \forall x \in \bar{S}_\rho^V - S_\delta^V, \forall w \in \mathbf{W} \end{array} \tag{7}$$

where $\nabla V(x)$ is the gradient of V and $\bar{S}_\rho^V, S_\delta^V$ are defined by (6).

Lemma 2.4 *Under Assumptions 2.1 and 2.3, the closed-loop system ((4)) with $u = \alpha(x)$ is LIB dissipative with respect to $B_0 = \bar{S}_\rho^V$. In particular, (5) holds for $\beta : B_0 \to \mathbf{R}$ defined by*

$$\beta(x_0) = \left\{ \begin{array}{ll} \delta & \text{if } x_0 \in \bar{S}_\delta^V \\ V(x_0) & \text{if } x_0 \in \bar{S}_\rho^V - \bar{S}_\delta^V \end{array} \right. \tag{8}$$

Proof. By condition (7) in Assumption 2.3, it is easy to prove that $\bar{S}_\delta^V$ and $\bar{S}_\rho^V$ are both invariant sets in forward time. Furthermore, we can prove that for all $t \geq 0$,

$$z(t) = g(x(t)) \leq V(x(t)) \leq \delta, \quad \forall x_0 \in \bar{S}_\delta^V.$$

$$z(t) = g(x(t)) \leq V(x(t)) \leq V(x_0), \quad \forall x_0 \in \bar{S}_\rho^V - \bar{S}_\delta^V.$$

This shows that the closed-loop system is LIB dissipative with β defined by (8). $\square$

Remark 2.5 The Input-to-State Stability (ISS) [15] and the notion of is a stronger property than the LIB dissipation (when $g(x) = |x|$). Actually, we can show that an ISS Lyapunov function [17, 16] must satisfy the condition (7) in Assumption 2.3. In fact, from the Lyapunov characterization of ISS [17, 16], the ISS Lyapunov function V satisfies

$$\dot{V}(x) \le -\gamma_1(|x|) + \gamma_2(|w|), \quad \forall w \in \mathbf{R}^r, \ \forall x \in \mathbf{R}^n$$

for some class $\mathcal{K}_\infty$ functions γ_1, γ_2. If we only consider bounded disturbances with $|w| \le d$ (see (2)), then the ISS Lyapunov function V satisfies

$$\dot{V}(x) \le -\gamma_1(|x|) + \gamma_2(d), \quad \forall w \in \mathbf{W}, \ \forall x \in \mathbf{R}^n.$$

Hence if we choose $\delta_1 > 0$ such that $\eta = \gamma_1(\delta_1) - \gamma_2(d) > 0$, then we have

$$\begin{aligned} \dot{V}(x) &\le \gamma_2(d), & \forall w \in \mathbf{W}, \ \forall x \in \mathbf{R}^n, \\ \dot{V}(x) &\le -\eta, & \forall w \in \mathbf{W}, \ \forall |x| \ge \delta_1. \end{aligned}$$

Furthermore, choose $\delta > 0$ such that $V(x) \ge \delta \Rightarrow |x| \ge \delta_1$, then we have

$$\begin{aligned} \dot{V}(x) &\le \gamma_2(d), & \forall w \in \mathbf{W}, \ \forall x \in \mathbf{R}^n, \\ \dot{V}(x) &\le -\eta, & \forall w \in \mathbf{W}, \ \forall x \notin S_\delta^V. \end{aligned} \tag{9}$$

This is the condition in Assumption 2.3 (with $\rho = +\infty$). It is obvious that a function V satisfying the condition (7) in Assumption 2.3 does not have to be an ISS Lyapunov function. □

3 Solution to the Problem

The following theorem shows that the backstepping approach is successful in solving the LIB controller synthesis problem described in §2 for system (1). The following notation is used: for a function $\bar{V} : \mathbf{R}^{n+m} \to \mathbf{R}$ and a number $\delta \le +\infty$, denote

$$S_\delta^{\bar{V}} = \{(x,y) \in \mathbf{R}^{n+m} : \bar{V}(x,y) < \delta\}, \quad \bar{S}_\delta^{\bar{V}} = \{(x,y) \in \mathbf{R}^{n+m} : \bar{V}(x,y) \le \delta\}. \tag{10}$$

Theorem 3.1 *Given $\bar{B}_0 \subset \mathbf{R}^{n+m}$, assume Assumptions 2.1 and 2.3 hold. Then there exists a state feedback controller $u = \bar{\alpha}(x,y)$ such that the closed-loop system ((1)) with $u = \bar{\alpha}(x,y)$ is LIB dissipative with respect to $\bar{B}_0$ provided that*

$$\bar{B}_0 \subset \bar{S}_\rho^{\bar{V}}, \tag{11}$$

where $\bar{V} : \mathbf{R}^{n+m} \to \mathbf{R}$ is defined by

$$\bar{V}(x,y) \stackrel{\triangle}{=} V(x) + \frac{1}{2}[y - \alpha(x)]^T[y - \alpha(x)]. \tag{12}$$

i.e. there exists a function $\bar{\beta} : \bar{B}_0 \to \mathbf{R}$ such that (3) holds. Indeed, the items $\bar{\alpha}(x,y)$ and $\bar{\beta}$ are constructed from the functions $V, \bar{V}$, the subsystem feedback $\alpha(x)$ and several parameters in Assumption 2.3 as follows:

1. *Fix $0 < \varepsilon < \rho - \delta$, where ρ and δ are specified in Assumption 2.3, and define*

$$\begin{aligned} \bar{\alpha}(x,y) \stackrel{\triangle}{=} &-g_2^T(x)\nabla V^T(x) + \nabla\alpha(x)[f_1(x) + g_2(x)y] \\ &-[y - \alpha(x)](\tfrac{\Delta+\eta}{2\varepsilon} + \tfrac{d^2}{\eta}|\nabla\alpha(x)g_1(x)|^2 + \tfrac{d^2}{\eta}|f_2(x,y)|^2), \end{aligned} \tag{13}$$

where also Δ and η are specified in Assumption 2.3.

2. Define $\bar{\beta} : \bar{B}_0 \to \mathbf{R}$ *by*

$$\bar{\beta}(x,y) \triangleq \begin{cases} \delta+\varepsilon, & \text{if } (x,y) \in S^{\bar{V}}_{\delta+\varepsilon}, \\ \bar{V}(x,y), & \text{if } (x,y) \in \bar{S}^{\bar{V}}_{\rho} - S^{\bar{V}}_{\delta+\varepsilon}. \end{cases} \tag{14}$$

Remark 3.2 Notice that the maximal sets on which the closed-loop system ((4)) with $u = \alpha(x)$ and the closed-loop system ((1)) with $u = \bar{\alpha}(x,y)$ are LIB dissipative are $B_0 = \bar{S}^V_\rho$ and $\bar{B}_0 = \bar{S}^{\bar{V}}_\rho$, respectively. By (12), the projection of $\bar{S}^{\bar{V}}_\rho$ on the x subspace is $\bar{S}^V_\rho$.

In order to prove the above theorem, we use the following lemma.

Lemma 3.3 *Under Assumptions 2.1 and 2.3, for any* $0 < \varepsilon < \rho - \delta$*, there exist a function* $\bar{\alpha}(x,y)$ *such that the function* $\bar{V}(x,y)$ *defined by (12) satisfies*

$$\begin{aligned} \bar{V}(x,y) &\geq g(x), && \forall (x,y) \in \bar{S}^{\bar{V}}_\rho, \\ \nabla_x \bar{V}(x,y)[f_1(x) + g_1(x)w + g_2(x)y] & && \\ +\nabla_y \bar{V}(x,y)[\bar{\alpha}(x,y) + f_2(x,y)w] &\leq \Delta + \tfrac{\eta}{2}, && \forall (x,y) \in \bar{S}^{\bar{V}}_\rho, \forall w \in \mathbf{W}\ , \\ \nabla_x \bar{V}(x,y)[f_1(x) + g_1(x)w + g_2(x)y] & && \\ +\nabla_y \bar{V}(x,y)[\bar{\alpha}(x,y) + f_2(x,y)w] &\leq -\tfrac{\eta}{2}, && \forall (x,y) \in \bar{S}^{\bar{V}}_\rho - S^{\bar{V}}_{\delta+\varepsilon}, \forall w \in \mathbf{W}. \end{aligned} \tag{15}$$

Proof. By (12), we have

$$\bar{V}(x,y) \geq V(x) \geq g(x), \quad \forall x \in \bar{S}^V_\rho, \forall y \in \mathbf{R}^m, \tag{16}$$

proving the first line of (15).

Next, we evaluate the derivative of $\bar{V}(x,y)$ along the trajectory of system (1) with control u as follows:

$$\begin{aligned} \dot{\bar{V}}(x,y) &= \nabla V(x)\dot{x} + [y-\alpha(x)]^T[\dot{y} - \nabla\alpha(x)\dot{x}] \\ &= \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)y] \\ &\quad +[y-\alpha(x)]^T\{u + f_2(x,y)w - \nabla\alpha(x)[f_1(x) + g_1(x)w + g_2(x)y]\} \\ &= \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] + \nabla V(x)g_2(x)[y-\alpha(x)] \\ &\quad +[y-\alpha(x)]^T\{u + f_2(x,y)w - \nabla\alpha(x)[f_1(x) + g_1(x)w + g_2(x)y]\} \\ &= \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] + [y-\alpha(x)]^T g_2^T(x)\nabla V^T(x) \\ &\quad +[y-\alpha(x)]^T\{u + f_2(x,y)w - \nabla\alpha(x)[f_1(x) + g_1(x)w + g_2(x)y]\}. \end{aligned} \tag{17}$$

Now choose

$$\begin{aligned} u &= \bar{\alpha}(x,y) \\ &\triangleq -g_2^T(x)\nabla V^T(x) + \nabla\alpha(x)[f_1(x) + g_2(x)y] \\ &\quad -[y-\alpha(x)](c_1 + c_2|\nabla\alpha(x)g_1(x)|^2 + c_2|f_2(x,y)|^2), \end{aligned} \tag{18}$$

where c_1, c_2 will be decided shortly. Then we have

$$\begin{aligned} \dot{\bar{V}}(x,y) &= \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \\ &\quad +[y-\alpha(x)]^T\{-[y-\alpha(x)](c_1 + c_2|\nabla\alpha(x)g_1(x)|^2 + c_2|f_2(x,y)|^2) \\ &\quad -\nabla\alpha(x)g_1(x)w + f_2(x,y)w\} \\ &= \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] - c_1|y-\alpha(x)|^2 \\ &\quad -\{c_2|y-\alpha(x)|^2|\nabla\alpha(x)g_1(x)|^2 + [y-\alpha(x)]^T\nabla\alpha(x)g_1(x)w\} \\ &\quad -\{c_2|y-\alpha(x)|^2|f_2(x,y)|^2 - [y-\alpha(x)]^T f_2(x,y)w\} \\ &\leq \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] - c_1|y-\alpha(x)|^2 + \tfrac{1}{4c_2}|w|^2 + \tfrac{1}{4c_2}|w|^2 \\ &= \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] - c_1|y-\alpha(x)|^2 + \tfrac{1}{2c_2}|w|^2 \\ &\leq \nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] - c_1|y-\alpha(x)|^2 + \tfrac{1}{2c_2}d^2 \end{aligned} \tag{19}$$

Here, we have used the bound $|w| \le d$ in the last step (see (2)).

Now fix $0 < \varepsilon < \rho - \delta$, where ρ and δ specified in Assumption 2.3, and let $(x, y) \in \bar{S}^{\bar{V}}_{\rho} - S^{\bar{V}}_{\delta+\varepsilon}$. Then

$$\rho \ge \bar{V}(x, y) = V(x) + \frac{1}{2}[y - \alpha(x)]^T [y - \alpha(x)] \ge \delta + \varepsilon,$$

and hence either case (i) $\rho \ge V(x) \ge \delta$, or case (ii) $\rho \ge \frac{1}{2}[y - \alpha(x)]^T [y - \alpha(x)] \ge \varepsilon$.

Case (i) If $\rho \ge V(x) \ge \delta$, then $x \in \bar{S}^V_{\rho} - S^V_{\delta}$, and hence by Assumption 2.3 we have

$$\nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \le -\eta,$$

so we have

$$\dot{\bar{V}}(x, y) \le -\eta - c_1 |y - \alpha(x)|^2 + \frac{1}{2c_2} d^2.$$

If we choose

$$c_2 = \frac{d^2}{\eta}, \tag{20}$$

then we have

$$\dot{\bar{V}}(x, y) \le -\frac{\eta}{2} - c_1 |y - \alpha(x)|^2 \le -\frac{\eta}{2}.$$

Hence we choose c_2 as (20).

Case (ii) Now since $V(x) \le \bar{V}(x, y) \le \rho$, we have $x \in \bar{S}^V_{\rho}$, and so by Assumption 2.3,

$$\nabla V(x)[f_1(x) + g_1(x)w + g_2(x)\alpha(x)] \le \Delta,$$

and hence

$$\dot{\bar{V}}(x, y) \le \Delta - c_1 |y - \alpha(x)|^2 + \frac{1}{2c_2} d^2 = \Delta - c_1 |y - \alpha(x)|^2 + \frac{\eta}{2}.$$

If $\rho \ge \frac{1}{2}[y - \alpha(x)]^T [y - \alpha(x)] = \frac{1}{2}|y - \alpha(x)|^2 \ge \varepsilon$, and if we choose

$$c_1 = \frac{\Delta + \eta}{2\varepsilon}, \tag{21}$$

then we have

$$\dot{\bar{V}}(x, y) \le \Delta - 2c_1\varepsilon + \frac{\eta}{2} \le -\frac{\eta}{2}.$$

Hence we choose c_1 as (21).

Therefore with c_1 and c_2 as (21) and (20) we have shown that

$$\dot{\bar{V}}(x, y) \le -\frac{\eta}{2} \tag{22}$$

for $(x, y) \in \bar{S}^{\bar{V}}_{\rho} - S^{\bar{V}}_{\delta+\varepsilon}$. This proves the last line of (15).

Finally, suppose $(x, y) \in \bar{S}^{\bar{V}}_{\rho}$, so that $\bar{V}(x, y) \le \rho$. Then we have $V(x) \le \rho$, by (19) and hence

$$\dot{\bar{V}}(x, y) \le \Delta + \frac{\eta}{2}.$$

Thus $\bar{V}(x, y)$ satisfies the second line of (15), and the proof is complete. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: The function $\bar{\alpha}(x,y)$ in (13) is determined by (18) and (21), (20). By Lemma 3.3, if we denote $\bar{\rho} = \rho, \bar{\delta} = \delta + \varepsilon$, $\bar{\eta} = \frac{\eta}{2} > 0, \bar{\Delta} = \Delta + \frac{\eta}{2} > 0$, then the function $\bar{V}(x,y)$ satisfies

$$\begin{aligned} \bar{V}(x,y) &\geq g(x), \quad &&\forall (x,y) \in \bar{S}_{\rho}^{\bar{V}}, \\ \nabla_x \bar{V}(x,y)[f_1(x) + g_1(x)w + g_2(x)y] & && \\ +\nabla_y \bar{V}(x,y)[\bar{\alpha}(x,y) + f_2(x,y)w] &\leq \bar{\Delta}, \quad &&\forall (x,y) \in \bar{S}_{\rho}^{\bar{V}}, \forall w \in \mathbf{W}\ , \\ \nabla_x \bar{V}(x,y)[f_1(x) + g_1(x)w + g_2(x)y] & && \\ +\nabla_y \bar{V}(x,y)[\bar{\alpha}(x,y) + f_2(x,y)w] &\leq -\bar{\eta}, \quad &&\forall (x,y) \in \bar{S}_{\rho}^{\bar{V}} - S_{\bar{\delta}}^{\bar{V}}, \forall w \in \mathbf{W}. \end{aligned} \tag{23}$$

The proof now follows using similar arguments to the proof of Lemma 2.4. □

Remark 3.4 By Lemma 3.3 and the proof of Theorem 3.1, $\bar{V}(x,y)$ has a similar property as that in Assumption 2.3, so we can design recursively the controller achieving LIB dissipation for higher dimensional nonlinear cascade systems with strict-feedback form [12] using similar arguments to Theorem 3.1.

We now show that under some additional assumptions, we can obtain the asymptotic stability of the closed loop when $w = 0$.

Assumption 3.5 *In system (1), $f_1(0) = 0, g(0) = 0$; the C^1 function $\alpha : \mathbf{R}^n \to \mathbf{R}^m$ in Assumption 2.3 satisfies $\alpha(0) = 0$; the C^1 function $V : \mathbf{R}^n \to \mathbf{R}$ in Assumption 2.3 satisfies*

$$\begin{array}{c} V(0) = 0, \quad V(x) > 0, \quad \forall x \in \bar{S}_{\rho}^{V}, x \neq 0, \\ \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] < 0, \quad \forall x \in \bar{S}_{\rho}^{V}, x \neq 0, \end{array} \tag{24}$$

where ρ in given in Assumption 2.3.

Under Assumptions 2.1, 2.3 and 3.5, Lemma 2.4 can be strengthened as follows.

Lemma 3.6 *Under Assumptions 2.1, 2.3 and 3.5, the closed loop system ((4) and $u = \alpha(x)$) is LIB dissipative with respect to $B_0 = \bar{S}_{\rho}^{V}$. Moreover, when $w = 0$, the closed loop system is asymptotically stable provided that $x_0 \in \bar{S}_{\rho}^{V}$.*

Proof. The LIB dissipation property is given in Lemma 2.4. The asymptotic stability property can be obtained by a standard Lyapunov stability theorem. (e.g. Theorem 3.1 in [10]).

Remark 3.7 The LIB dissipation property plus the asymptotic stability when $w = 0$ is still weaker than ISS. The key difference is that here we only focus on two cases $|w|_\infty = 0$ and $|w|_\infty \leq d$, when $|w|_\infty = 0$, the asymptotic stability is guaranteed, when $|w|_\infty \leq d$ (for a single d), a bound on (a function of) the trajectories is guaranteed. But for the ISS case, the bound is a function of $|w|_\infty$ plus a function of the initial state. A simple example is $\dot{x} = -(1 + sin(w))x$, which is LIB dissipative (for any w) and asymptotic stability (when $w = 0$) but is not ISS. The LIB dissipation property plus the asymptotic stability when $w = 0$ is also different from the uniform ultimate boundedness (e.g.[1]) where the states are guaranteed to be bounded only when the time is large enough.

By Theorem 3.1, we have the following corollary.

Corollary 3.8 *Given $\bar{B}_0 \subset \mathbf{R}^{n+m}$, assume Assumptions 2.1, 2.3 and 3.5 hold. Then there exists a state feedback controller $u = \bar{\alpha}(x, y)$ such that the closed loop system ($u = \bar{\alpha}(x, y)$ and (1)) is LIB dissipative with respect to $\bar{B}_0$ provided that*

$$\bar{B}_0 \subset \bar{S}_\rho^{\bar{V}}, \tag{25}$$

where $\bar{V} : \mathbf{R}^{n+m} \to \mathbf{R}$ is defined by (12). Furthermore, when $w = 0$, the closed loop system is asymptotically stable provided $(x_0, y_0) \in \bar{S}_\rho^{\bar{V}}$.

Proof. The LIB dissipation property is proved in Theorem 3.1. Now we show the asymptotic stability when $w = 0$. By Lyapunov stability theorem (e.g. Theorem 3.1 in [10]), we only need to prove that the $\bar{V}(x, y)$ defined by (12) satisfies

$$\begin{array}{c} V(0,0) = 0, \quad V(x,y) > 0, \ \ \forall (x,y) \in \bar{S}_\rho^{\bar{V}}, (x,y) \neq (0,0), \\ \dot{\bar{V}}(x,y) = \nabla_x \bar{V}(x,y)[f_1(x) + g_2(x)y] + \nabla_y \bar{V}(x,y)\bar{\alpha}(x,y) < 0, \ \ \forall (x,y) \in \bar{S}_\rho^{\bar{V}}, (x,y) \neq (0,0). \end{array} \tag{26}$$

The first line of (26) is obvious since $\alpha(0) = 0$. When $w = 0$, the derivative of $\bar{V}(x, y)$ along the trajectory of system (1) with control u is:

$$\begin{array}{rcl} \dot{\bar{V}}(x,y) & = & \nabla V(x)\dot{x} + [y - \alpha(x)]^T[\dot{y} - \nabla\alpha(x)\dot{x}] \\ & = & \nabla V(x)[f_1(x) + g_2(x)y] \\ & & +[y - \alpha(x)]^T\{u - \nabla\alpha(x)[f_1(x) + g_2(x)y]\} \\ & = & \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] + \nabla V(x)g_2(x)[y - \alpha(x)] \\ & & +[y - \alpha(x)]^T\{u - \nabla\alpha(x)[f_1(x) + g_2(x)y]\} \\ & = & \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] + [y - \alpha(x)]^T g_2^T(x)\nabla V^T(x) \\ & & +[y - \alpha(x)]^T\{u - \nabla\alpha(x)[f_1(x) + g_2(x)y]\}. \end{array} \tag{27}$$

With the controller (18), we have

$$\begin{array}{rcl} \dot{\bar{V}}(x,y) & = & \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] \\ & & +[y - \alpha(x)]^T\{-[y - \alpha(x)](c_1 + c_2|\nabla\alpha(x)g_1(x)|^2 + c_2|f_2(x,y)|^2) \\ & = & \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] - (c_1 + c_2|\nabla\alpha(x)g_1(x)|^2 + c_2|f_2(x,y)|^2)|y - \alpha(x)|^2 \end{array} \tag{28}$$

Suppose $(x, y) \in \bar{S}_\rho^{\bar{V}}$, then $x \in \bar{S}_\rho^{V}$. If $x \neq 0$, then by (24),

$$\dot{\bar{V}}(x,y) \leq \nabla V(x)[f_1(x) + g_2(x)\alpha(x)] < 0.$$

If $x = 0$, $y \neq 0$, then

$$\dot{\bar{V}}(x,y) \leq -(c_1 + c_2|\nabla\alpha(x)g_1(x)|^2 + c_2|f_2(x,y)|^2)|y - \alpha(x)|^2 < 0.$$

Hence the second line of (26) holds and the proof is completed. □

4 Illustrative Example

Consider the task of controlling the speed of a fan driven by a DC motor ([6, section 8.3, page 223]). The system model is

$$\left\{ \begin{array}{rcl} J\dot{\nu} & = & \kappa_1 I - \tau_L - \tau_D(\nu) \\ L\dot{I} & = & v + \Delta v - \kappa_2\nu - RI \end{array} \right. \tag{29}$$

where ν is the fan speed, I is the armature current, v is the armature voltage (our control input), Δv is the actuator noise. τ_L is an uncertain constant load to torque, $\tau_D(\nu)$ is an uncertain, speed-dependent drag torque, J, κ_1, κ_2, L, and R are known positive constants.

We define $\xi_1 = \nu$ and $\xi_2 = (\kappa_1/J)I$ and apply a preliminary linear feedback

$$v = \kappa_2 \nu + RI + (JL/\kappa_1)u$$

to obtain

$$\begin{cases} \dot{\xi}_1 &= \xi_2 - \frac{1}{J}[\tau_L + \tau_D(\xi_1)] \\ \dot{\xi}_2 &= u + \Delta u \end{cases} \tag{30}$$

where u is the new control variable.

As in [6], we assume that

$$\frac{1}{J}|\tau_L| \le \tau$$

for some known constant τ.

We also assume that we want to keep the speed near a constant ν_0 and the drag torque $\tau_D(\nu)$ satisfies

$$\frac{1}{J}|\tau_D(\nu) - \tau_D(\nu_0)| \le b|\nu - \nu_0|, \quad \forall \nu \in [\nu_0 - a, \nu_0 + a]$$

for some constants $0 < a < \nu_0$ and $b \ge 0$.

We also assume that the actuator noise is bounded, i.e.

$$\Delta u \in [-\tau_u, \tau_u] \tag{31}$$

for some $\tau_u > 0$. Now we can express

$$\frac{1}{J}[\tau_L + \tau_D(\xi_1)] = \frac{1}{J}\tau_D(\nu_0) + (\tau + b|\xi_1 - \nu_0|)w_1 \tag{32}$$

$$\Delta u = \tau_u w_2 \tag{33}$$

where w_1, w_2 are bounded disturbances with $|w_1| \le 1, |w_2| \le 1$.

We further denote

$$x = \xi_1 - \nu_0, \quad y = \xi_2 - \frac{1}{J}\tau_D(\nu_0), \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{34}$$

and define the performance

$$z = (\nu - \nu_0)^2 = (\xi_1 - \nu_0)^2 = x^2. \tag{35}$$

Then the system becomes

$$\begin{cases} \dot{x} &= [\tau + b|x| \;\; 0]w + y \\ \dot{y} &= u + [0 \;\; \tau_u]w \\ z &= x^2 \end{cases} \tag{36}$$

with $|w| \le d = \sqrt{2}$.

This system takes the strict feedback form (1), in fact,

$$f_1(x) = 0, \;\; g_1(x) = [\tau + b|x| \;\; 0], \;\; g_2(x) = 1, \;\; f_2(x, y) = [0 \;\; \tau_u], \;\; g(x) = x^2. \tag{37}$$

The subsystem is

$$\begin{cases} \dot{x} = [\tau + b|x| \;\; 0]w + u \\ z = x^2 \end{cases} \tag{38}$$

Choose controller

$$u = \alpha(x) = -kx, \tag{39}$$

where $k > 0$ satisfies $0 < \frac{2\tau}{k-b} < a$, then the closed-loop subsystem is

$$\begin{cases} \dot{x} = -kx + [\tau + b|x| \;\; 0])w = -kx + (\tau + b|x|)w_1 \\ z = x^2 \end{cases} \tag{40}$$

Choose C^1 function $V : \mathbf{R} \to \mathbf{R}$ as

$$V(x) = x^2, \;\; \forall x \in \mathbf{R}. \tag{41}$$

Since $|w_1| \leq 1$, we have

$$\begin{aligned} \nabla V(x)[-kx + (\tau + b|x|)w_1] &= 2x[-kx + (\tau + b|x|)w_1] \\ &\leq -2(k-b)x^2 + 2\tau w_1 x \\ &= -2(k-b)[x - \tfrac{\tau w_1}{2(k-b)}]^2 + \tfrac{\tau^2 w_1^2}{2(k-b)} \end{aligned}$$

Now choose

$$\rho = (\frac{2\tau}{k-b})^2, \quad \delta = (\frac{1.5\tau}{k-b})^2,$$

then for any $x \in \bar{S}_\rho^V - S_\delta^V$, $x \geq \frac{1.5\tau}{k-b}$ and hence

$$\nabla V(x)[-kx + (\tau + b|x|)w_1] \leq -\frac{1.5\tau^2}{k-b}.$$

Also it is easy to see that for any $x \in \bar{S}_\rho^V$,

$$\nabla V(x)[-kx + (\tau + b|x|)w_1] \leq \frac{\tau^2}{2(k-b)}.$$

Above we have proved that $V(x)$ satisfies the condition in Assumption 2.3 with

$$\eta = \frac{1.5\tau^2}{k-b}, \quad \Delta = \frac{\tau^2}{2(k-b)}.$$

It is also easy to see that $V(x)$ satisfies the condition in Assumption 3.5. By Lemma 3.6, system (40) is LIB dissipative with respect to

$$B_0 = \bar{S}_\rho^V = [-\frac{2\tau}{k-b}, \frac{2\tau}{k-b}]$$

and it is asymptotically stable when $w_1 = 0$ provided $x_0 \in B_0$.

The function β defined by (8) is

$$\beta(x) = \max\{x^2, (\frac{1.5\tau}{k-b})^2\}, \quad \forall x \in [-\frac{2\tau}{k-b}, \frac{2\tau}{k-b}].$$

Now choose

$$\varepsilon = \frac{\tau^2}{(k-b)^2} < \rho - \delta$$

and the controller (13)

$$\begin{array}{rcl}\bar{\alpha}(x,y) & = & -g_2^T(x)\nabla V^T(x) + \nabla\alpha(x)[f_1(x) + g_2(x)y] \\ & & -[y-\alpha(x)](\frac{\Delta+\eta}{2\epsilon} + \frac{d}{\eta}c_2|\nabla\alpha(x)g_1(x)|^2 + c_2|f_2(x,y)|^2) \\ & = & -2x - ky - (y+kx)[\frac{\Delta+\eta}{2\epsilon} + \frac{\sqrt{2}}{\eta}k^2(\tau + b|x|)^2 + \frac{\sqrt{2}}{\eta}\tau_u^2] \\ & = & -2x - ky - (y+kx)[(k-b) + \frac{2\sqrt{2}(k-b)}{3\tau^2}k^2(\tau+b|x|)^2 + \frac{2\sqrt{2}(k-b)}{3\tau^2}\tau_u^2] \\ & = & -2x - ky - (y+kx)(k-b)[1 + \frac{2\sqrt{2}k^2}{3\tau^2}(\tau+b|x|)^2 + \frac{2\sqrt{2}\tau_u^2}{3\tau^2}]\end{array}$$

By Corollary 3.8, the obtained closed-loop system is L^∞-bounded dissipative on

$$\bar{B}_0 = \bar{S}_\rho^{\bar{V}}$$

and is asymptotically stable when $w = 0$ provided $(x_0, y_0) \in \bar{S}_\rho^{\bar{V}}$, where $\bar{V}(x,y)$ is defined by

$$\bar{V}(x,y) = V(x) + \frac{1}{2}[y-\alpha(x)]^T[y-\alpha(x)] = x^2 + \frac{1}{2}(y+kx)^2, \quad \forall (x,y) \in \mathbf{R}^2,$$

the solution of the closed-loop system

$$\left\{\begin{array}{rcl}\dot{x} & = & [\tau + b|x| \;\; 0]w + y \\ \dot{y} & = & \bar{\alpha}(x,y) + [0 \;\; \tau_u]w \\ z & = & x^2\end{array}\right. \tag{42}$$

satisfies

$$z(t) = x^2(t) \le \bar{\beta}(x_0, y_0), \quad \forall t \ge 0, \forall w_{0,t} \in \mathcal{W}_{0,t}, \forall (x_0, y_0) \in \bar{S}_\rho^{\bar{V}}, \tag{43}$$

where $\bar{\beta}(x,y)$ is defined by

$$\bar{\beta}(x,y) = \left\{\begin{array}{ll}\delta + \varepsilon, & \forall (x,y) \in S_{\delta+\varepsilon}^{\bar{V}}, \\ x^2 + \frac{1}{2}(y+kx)^2, & \forall (x,y) \in \bar{S}_\rho^{\bar{V}} - S_{\delta+\varepsilon}^{\bar{V}}.\end{array}\right.$$

Remark 4.1 The system model of the above example is almost the same as the one used in [6]. However, the design objectives are different. In our example, we want to find a static state feedback control law such that the closed loop system is LIB dissipative (when there are disturbances) and asymptotic stable (when there is no disturbance). While in [6], a dynamic state feedback is designed to achieve robust tracking including global boundedness, asymptotic tracking and finite gain properties.

5 Conclusion

In this paper we have demonstrated the feasibility of applying the backstepping method to the design of feedback controllers in the context of L^∞ performance criteria. For systems with the special cascade and strict-feedback form, the need to solve numerically high order dynamic programming equation is avoided. Future research will consider applications of these results, as well as the development of methods for the output feedback case (c.f. [9]).

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