David Goldbaum[1]

Conformity and Influence

[1] Economics Discipline Group, University of Technology Sydney, PO Box 123 Broadway, NSW 2007 Australia, E-mail: david.goldbaum@uts.edu.au

Abstract:
I model the behavior of decision-makers seeking conformity and influence in a connected population. The model allows for one-sided linking, with information flowing from the target to the link's originator. Conformity is achieved only with a social order, necessitating differentiated rewards despite *ex ante* homogeneity. The leader holds a strategic social location *ex post*, exerting influence independent of any leadership traits. A strong desire to influence produces non-conforming autonomous decision-makers. Socially detrimental multiple leaders can be sustained as well.

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JEL classification: C72, D83, D85

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My goal is to acquire works that great museums letch after.[1]

1 Introduction

Decision-makers enjoy conformity, particularly when pre-empting the popular choice. Thornton (2009) observes that an avant-garde collector's reputation is based on his or her success in being an early collector of a subsequently successful artist's works. At the same time, the success of an aspiring artist is driven, in part, by the reputation of the collectors acquiring the artist's works. The buying and selling of art is not conducted anonymously. The scenario is such that an individual benefits from acting in advance of a phenomenon, the emergence of which may be influenced by the individual's own actions.[2] Social considerations weigh on a variety of decisions, from conspicuous consumption to investment to political support.[3] Most decision-makers decide between conformity and early adoption, a tradeoff the influential can avoid.

The developed game captures the social aspect of decision making, including the tension between early adoption and conformity. Available actions create pathways for exerting or responding to social influence. Popularity arises from coordinating behavior, made possible by the flow of information over personal contacts. Actions manifest as a social structure in the form of selectively-employed directed links.

Prior to organizing, the *ex ante* homogeneous players face the same payment opportunities, there are no explicit costs for either moving or waiting, and they have equal access to when and how to choose among the options. Conformity in the decision-making environment requires *ex post* societal heterogeneity. A leader is not endowed with the traditional leadership attributes in the form of information or decision-making advantage. Rather, a leader comes by these attributes socially in order to serve the interests of all.

Equilibrium is reflected in the social structure. A population organized around a single leader is the prevailing equilibrium structure. The size of the following, and possible presence of multiple leaders, serve the interest of the followers.

The setting suggests a dynamic process for attaining or retaining a social structure. The present paper identifies equilibrium behavior for a simultaneous play game and the resulting social structures. The identified equilibrium structures are also stable in the dynamic setting for myopic players unwilling or unable to plan beyond the current period. While the current project employs equilibrium analysis to identify social structures consistent with socially influenced preferences, the computational analysis of Goldbaum (2017) and the experiments of Bostian and Goldbaum (2017) separately consider evolutionary behavior in pursuit of conformity and influence absent knowledge of the network.

Section 2 introduces the network structure, possible actions, payoffs, and an equilibrium concept of the model. Examples employing populations of two and three players illustrate that while a greater reward to leading undermines the interest in following, equilibrium always includes at least one follower. To identify how relative proximity to the leader alters behavior requires analysis with larger populations, an undertaking that starts in Section 3. Larger populations allow for multiple leaders with possible majority and minority

David Goldbaum is the corresponding author.

conforming populations. Section 4 considers the conditions necessary to allow multiple leaders to co-exist in equilibrium. Extensions of the model, including non-linear payoffs and possible best response cascades, are considered in Section 5.

Appendix A includes formal definitions of essential populations and social structures. Appendix B includes a formal statement and proof of the propositions and lemmas. Appendix C formally develops examples from Section 4.

Related Literature

The strategic complementarities found in Katz and Shapiro (1985) reward adopting a popular choice. Classic evidence of social influence in individual decisions, even in the absence of physical complementarities, can be found in Whyle (1954), Katz and Lazarsfeld (1955), and Arndt (1967). Hill, Provost, and Volinsky (2006) and Dwyer (2007) exploit modern technology to consider social connections as they develop in mobile phone friend networks and online chats. Early experiments, including those of Kelman (1961), Bearden and Rose (1990), and Lascu, Bearden, and Rose (1995), identify conformity in decision-making.

Some of the early examples exploring the influence of social networks model a bi-directional interaction between individual decision-making and global behavior, including Schelling (1971), Katz and Shapiro (1985), and Schelling (1973).[4]Cowan and Jonard (2004) document the impact of local and global connectivity on overall knowledge across a population.

Deutsch and Gerard (1955) differentiate between informational and normative conformity. The former is revealing of the underlying decision. The latter affects the decision-maker's relationship with others.[5] The conformity and influence arising in the social learning model of DeGroot (1974) are informational. The importance of network structure and individual behavior are further developed in such works as Acemoglu et al. (2013), Acemoglu and Ozdaglar (2011), Battiston and Stanca (2015), Buechel, Hellmann, and Klößner (2015), Corazzini et al. (2012), and Golub and Jackson (2010), where conformity and influence are the consequence of the social learning environment. Arifovic, Eaton, and Walker (2015) consider conformity as a motivator shaping beliefs and network formation in the context of social learning.

In the current investigation, players actively seek normative conformity and influence over the decisions of others. As such, they operate in a setting in which the actions of the population entirely define the state. There is no underlying exogenous truth to be discerned from the opinions of one's neighbors. All uncertainty is intrinsic. These features shape the nature of information gathering. Combining information from various sources does not necessarily serve the individual's objectives.

Coordination in adoption imparts a positive peer effect in the Brock and Durlauf (2001) model of utility-driven normative conformity. The Ali and Kartik (2012) preference for normative conformity, in the form of complimentary actions, motivates strategic exploitation of influence in the sequential decision-making of the Banerjee (1992) observational learning model.[6] The benefits of early adoption appear in models such as the Pesendorfer (1995) adoption of new fashion and in the Challet, Marsili, and Zhang (2001) model of investing.

The equilibrium concept adopted for the main analysis is that of a Nash equilibrium applied to the actions of the agents seeking conformity and influence. Equilibrium actions produce a social structure by which equilibrium can be defined. The resulting equilibrium notions of network structure are consistent with Haller and Sarangi (2005), Galeotti and Goyal (2010), Zhang, Park, and van der Schaar (2011), and Baetz (2015), that explicitly model the beneficial interaction that gives rise to network connectivity. In these examples, endogenously determined equilibrium network structures are the product of a static model or of simultaneous linking decisions. These models generate asymmetry in outcomes from *ex ante* homogeneity. This is in contrast to the possible heterogeneity produced by sequential play network formation games and equilibrium identification in Jackson and Wolinsky (1996), Watts (2001), and Jackson and Watts (2002).

Consistent with Arndt (1967), social connections form the foundation upon which the agents develop strategies to facilitate coordination. The static structures are relevant to populations seeking conformity when repeatedly confronted with a new set of options. Reliable social connections substitute for the inability to communicate or rely on the consistency of the choice option, as in Crawford and Haller (1990).

Multiple Nash equilibria exist in the present model. The asymmetry in the payoff means that the players have conflicting interests with regards to which equilibrium emerges. The two-player version reflects the endogenous heterogeneity that can emerge in research and development and duopoly games, as in Reinganum (1985), Sadanand (1989), Hamilton and Slutsky (1990), Amir and Wooders (1998), and Tesoriere (2008). Amir, Garcia, and Knauff (2010) generalize the issue of symmetry breaking, as is the case when a leader and follower emerge. The general n player game retains the issues regarding asymmetry in outcome while introducing new strategy possibilities. It also introduces the possibility of best response cascades, as in Dixit (2003) and Heal and Kunreuther (2010), refining the set of equilibrium structures.

2 Model

Let $N = \{1, ..., n\}$ be the set of players and let the $n \times n$ adjacency matrix g indicate contacts between players. If i can directly observe j then $g_{ij} = 1$ and $g_{ij} = 0$ otherwise. Let $g_{ii} = 1$ always. Write $N^d(i; g) = \{j \in N \backslash \{i\} | g_{ij} = 1\}$ for a set of players i can observe as contacts and let $n_i^d = |N^d(i; g)|$ indicate the number of contacts for player i. Let $O = \{O_1, \dots, O_m\}$ be a set of $m \geq 2$ options or alternatives.

Let a_i denote the action of player i. Players act simultaneously, with each player choosing (i) one of the m options autonomously or (ii) to imitate another player. The autonomous player chooses $a_i = o_i \in O$ where o_i is determined at random with uniform probability assigned to each option. Thus, $\Pr(o_i = o_j) = 1/m$ for i and j both acting autonomously, $i \neq j$. [7] To imitate another player, then $a_i = j$. A player who chooses an option autonomously is said to *lead* while a player who links to another is said to *follow*. The set of actions for player i is $A_i = O \cup N^d(i; g)$. Write $a = (a_1, \dots, a_n)$ for an action profile, where $a_i \in A_i$.

An action profile a induces an $n \times n$ adjacency matrix describing the paths of imitation between players as determined by their actions. If $a_i = j$ then $\sigma_{ij} = 1$ and if $a_i \in O_i$, such that i leads, then $\sigma_{ii} = 1$. Otherwise, $\sigma_{ij} = 0$. Thus, for the matrix σ, $\sigma \cdot \mathbf{1} = \mathbf{1}$, indicating that each player employs one and only one source to inform adoption, including possibly self-informed adoption. Imposing a single source is non-binding on the obtained solutions. Say that j is a **predecessor** of i if $\sigma_{ij} = 1$ or if there is a sequence of players $j_1, \dots, j_\tau$ such that $\sigma_{ij_1} = \dots = \sigma_{j_\tau j} = 1$. Write $N^P(i; \sigma)$ for the predecessors of i. Say that j is a **successor** of i if $\sigma_{ji} = 1$ or if there is a sequence of players $j_1, \dots, j_r$ such that $\sigma_{jj_1} = \dots = \sigma_{j_\tau i} = 1$. Write $N^S(i; \sigma)$ for the successors of i.

Let $N^L(\sigma) = \{i | \sigma_{ii} = 1\}$ denote the set of players who lead. A **leader** leads and has a non-empty set of successors. It is possible to *lead*, acting autonomously, without being a *leader*. If player i leads with player j as a successor, this makes player i player j's leader. Note that each player i has at most one player who leads as a predecessor, that is $|N^L(\sigma) \cap N^P(i; \sigma)| \in \{0, 1\}$ for each i. It is possible for a successor to be without a leader. Let L_i identify the predecessor of i who leads.

Define the distance from player i to her adopted alternative as the number of players between i and the alternative. This distance is relevant when determining payoffs. Using d_i to denote player i's distance,

$$d_i = \begin{cases} 0 & \text{if } i \in N^L(\sigma) \\ 1 & \text{if } \sigma_{ij} = 1, j \in N^L(\sigma) \\ \tau + 1 & \text{if } \sigma_{ij_1} = \dots = \sigma_{j_\tau j} = 1, j \in N^L(\sigma) \\ \infty & \text{otherwise.} \end{cases}$$

Use d_{ij} to denote the distance from successor i to predecessor j measured in the number of links connecting i to j. Observe that when $L_i = j$, $d_{ij} = d_i$.

Let $N^c(i; a)$ denote the set of "conforming" players adopting the same alternative as does player i (exclusive of i). Let $N^e(i; a)$ denote the set of "ensuing" conforming adopters who are of greater distance from the alternative than is i.[8] Observe $N^e(i; a) \subseteq N^c(i; a)$. Let $o_i \in O$ represent the alternative adopted by player i. Let μ_i^c and μ_i^e represent the cardinality of the respective populations, $\mu_i^c = |N^c(i; a)|$ and $\mu_i^e = |N^e(i; a)|$.

The payoff for player i rewards conformity and influence. Allowing for possible nonlinearity in the reward associated with each,[9]

$$\pi_{NL}(i; \sigma) = \phi(\mu_i^c) + \psi(\mu_i^e). \tag{1}$$

Each reward component should be increasing and continuously twice differentiable with $\phi(0) = \psi(0) = 0$. Appendix A develops a payoff function in μ_i^c and μ_i^e from a utility function valuing social interaction. As a function of μ_i^c, the conformity attribute of the payoff is similar in concept to the community effect of Blume and Durlauf (2001). The μ_i^e component rewards a player for the appearance of being influential whether or not the player actually informed the action of others.[10] Let $\Pi(a) = (\pi_1, \dots, \pi_n)'$ be the $n \times 1$ vector of payoffs according to a.

For illustration, consider the special case of linear reward components,

$$\pi(i; \sigma) = r_c \mu_i^c + r_e \mu_i^e, \tag{2}$$

with non-negative coefficients r_c and r_e.[11]

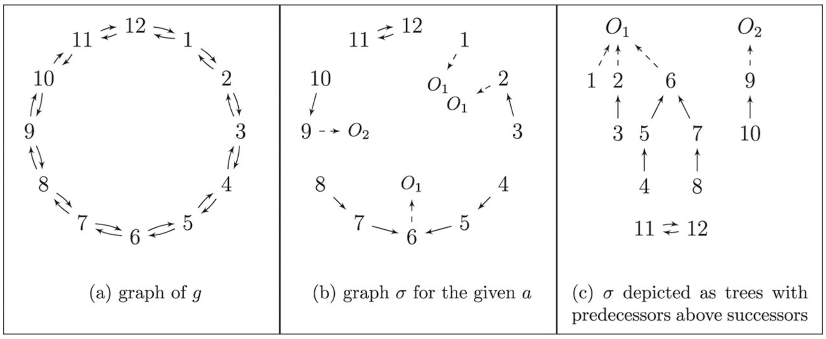


Figure 1: A g and illustrative feasible σ for a population of $n = 12$ players with $m = 2$ alternatives. The network structure produced by g is a ring. Frame (Figure 1a) is a network representation of the g matrix. Frame (Figure 1b) is a network representation of the σ matrix resulting from the actions $a = (O_1, O_1, 2, 5, 6, O_1, 6, 7, O_2, 9, 12, 11)$. Frame (Figure 1c) depicts the groupings implied by σ as trees (or "hierarchies") with predecessors positioned above successors and with the alternative above the trees. Dashed arrows indicate a leader's choice according to a. Followers 11 and 12, lacking a path to one of the alternatives, are placed at the bottom.

Example 1.

Consider a population of twelve players arranged in a ring with each player able to link to her nearest neighbor on either side. For $m = 2$, the set of feasible action profiles includes, as an illustrative example, the action

$$a = (O_1, O_1, 2, 5, 6, O_1, 6, 7, O_2, 9, 12, 11).$$

Figure 1 includes graphical representations of g and the σ induced by a. Here, $N^L(\sigma) = \{1, 2, 6, 9\}$ and for $i \in \{4, 5, 7, 8\}$, $L_i = 6$. The hierarchical presentation of σ in Figure 1c positions predecessors above successors in a tree structure rooted by the adopted alternative. Players 11 and 12 fail to adopt one of the alternatives as they are successors to each other and thus without a leader, a self-referencing loop.

Table 1 reports the payoff to each player based on the action a. For player $i \in \{1, \dots, 8\}$, $N^c(i; a) = \{1, \dots, 8\} \backslash \{i\}$ so that $\mu_i^c = 7$. In addition, for $i \in \{3, 5, 7\}$, $N^e(i; a) = \{4, 8\}$, reflecting that all players of equal distance from O_1 benefit equally from the players who are of greater distance. Player 9, having chosen differently than the other leading players, benefits only from her successor, player 10. For players $i \in \{4, 8, 10, 11, 12\}$, $N^e(i; a) = \emptyset$. Players 11 and 12, failing to adopt a choice, receive no payoff nor do they contribute to the payoff of any other player.

Table 1: Payoff for $a = (O_1, O_1, 2, 5, 6, O_1, 6, 7, O_2, 9, 12, 11)$ evaluated at $r_c = 1$ and $r_e = 3$.

Player	μ_i^c	μ_i^e	π_i	Player	μ_i^c	μ_i^e	π_i	Player	μ_i^c	μ_i^e	π_i
1	7	5	22	5	7	2	13	9	1	1	4
2	7	5	22	6	7	5	22	10	1	0	1
3	7	2	13	7	7	2	13	11	0	0	0
4	7	0	7	8	7	0	7	12	0	0	0

2.1 Strategic Behavior

Recall $\Pr\left(o_i = o_j\right) = 1/m$ for any two players i and j not in the same tree, producing a random element to pure strategy payoffs. A couple of small n examples illustrate the issues and outcomes inherent to the setting.

Table 2: The relevant action-dependent expected payoff matrix of Example 2 with $n = 2$, $m = 2$, $r_c = 1$, $r_e > 0$.

		Player 2 lead	follow
Player 1	lead	$\frac{1}{2}, \frac{1}{2}$	$r_e + 1, 1$
	follow	$1, r_e + 1$	0,0

Example 2.

$n = 2$, $m = 2$, $\phi(\mu^c) = \mu^c$, $\psi(\mu^e) = r_e \mu^e$, $r_e > 0$, and $g = \underset{2\times 2}{\mathbf{1}}$.

The Nash equilibrium strategy profile based on the rewards reported in Table 2 produces one leader and one follower. For the equilibrium with player 2 leading player 1, player 2 receives the higher payoff for being the leader. Player 1's lower payoff remains higher than the expected payoff obtained from also leading. The symmetry of the game means that there is also an equilibrium with player 1 leading player 2. The players want to avoid the strategy profile in which both lead. There is uncertainty in the payoff when both players lead. The low expected payoff reflects both the absence of an ensuing reward for each and an only $1/m = 1/2$ probability of matching on choice to receive the conformity reward. The players also want to avoid the outcome produced when each follows the other.[12]

A larger population introduces the possibility of adopting a minority option.

Table 3: Example 3 expected payoff table based on the actions of players 1 and 2 when player 3 leads. $n = 3$, $k = 2$, $r_c = 1$, $r_e > 0$.

		Player 2		
		lead	**follow 1**	**follow 3**
Player 1	lead	$1, 1, 1$	$\frac{3}{2} + r_e, \frac{3}{2}, 1 + \frac{r_e}{2}$	$1 + \frac{r_e}{2}, \frac{3}{2}, \frac{3}{2} + r_e$
	follow 2	$\frac{3}{2}, \frac{3}{2} + r_e, 1 + \frac{r_e}{2}$	$0, 0, 0$	$2, 2 + r_e, 2 + 2r_e$
	follow 3	$\frac{3}{2}, 1 + \frac{r_e}{2}, \frac{3}{2} + r_e$	$2 + r_e, 2, 2 + 2r_e$	$2, 2, 2 + 2r_e$
Player 3 leads				

Example 3.

$n = 3$, $m = 2$, $\phi(\mu^c) = \mu^c$, $\psi(\mu^e) = r_e \mu^e$, $r_e > 0$, and $g = \underset{3\times 3}{\mathbf{1}}$. Table 3 reports the expected payoff matrix for the actions for players 1 and 2 based on player 3 leading.

The set of equilibrium structures depends on r_e. Low r_e implies less emphasis on the ensuing reward so that conformity plays a larger role in driving decisions. The Nash equilibria for $r_e \leq 2$ have both 1 and 2 following player 3. The equilibrium set includes the structure in which both 1 and 2 directly imitate player 3, as in Figure 2c, as well as the vertical structures of Figure 2d and Figure 2e. As a Pareto improvement to Figure 2c, the middle agent in a vertical structure gains an ensuing reward without altering the rewards earned by the leader and more distant follower.[13]

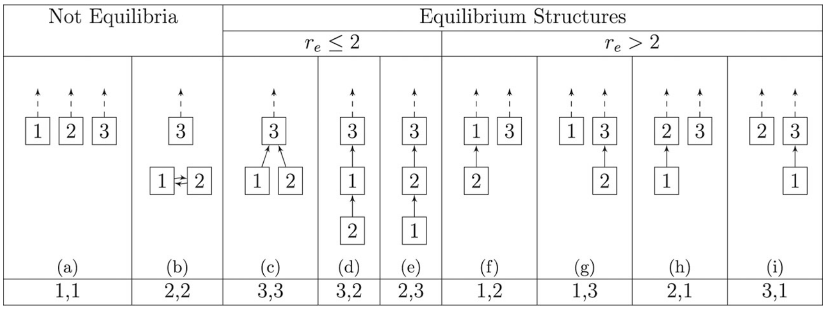


Figure 2: The structures generating the expected payoff produced for Example 3. Solid arrows indicates imitation. Dashed arrows represent direct selection of one of the options. The bottom row reports the corresponding Table 3 cell row and column.

For $r_e \leq 2$, players 1 and 2 each prefers following 3 to autonomy, independent of the action of the other follower. Given player 2 follows 1, for example, player 1 prefers following 3, producing Figure 2d, over being the population's unique leader, as in Figure 2f. The former ensures player 1's conformity with player 3 while preserving a distance advantage over player 2.

The strong ensuing reward of $r_e > 2$ undermines conformity by encouraging autonomy. The equilibrium structure consists of a leader with a single follower. The remaining autonomous player hopes to match the leader, thereby gaining the ensuing reward of the follower. The substantial premium to leading makes the expected value of this uncertain payoff greater than the certain reward of following in the presence of an existing follower.

For a network of directed links, a strongly connected network is one for which every player pair $\{i,j\}$ has either $g_{ij} = 1$ or there exists $j_1, \ldots, j_k$ such that $g_{ij_1} = \ldots = g_{j_k j} = 1$. As a consequence, for every $\{i,j\}$ pair there is a directed path from i to j. Let $G(n)$ be the universe of strongly connected networks based on a population size n. Both examples 2 and 3 are based on a g that is a complete graph (all players are able to link to any other player directly). For $n = 2$ the only strongly connected graph is the complete graph. There are 18 possible $g \in G(3)$ with five that are unique to a relabeling of the players. The benefit to coordinating on an alternative through imitation is the same for any $g \in G(3)$. For all $g \in G(3)$,

1. for $r_e \leq 2$, an action profile is a Nash equilibrium if and only if it produces one leader and two successors,
2. for $r_e > 2$, an action profile is a Nash equilibrium if and only if it produces one leader, one follower, and one autonomous player,
3. the non-empty set of Nash equilibria includes action profiles that produce i as the unique leader for all $i \in \{1,2,3\}$.

From 1 and 2 above, every equilibrium action produces one and only one non-trivial tree. From 3, it is always possible that any one of the players can hold the favorable position of leader.

As will be developed in the following sections, the features found in the $n = 3$ population generalize for any size population occupying a strongly connected network. These features are

- Pure strategy Nash equilibria exist.
- A unique leader, possibly in the presence of other autonomous players, is among the equilibrium social structures.
- Any player $i \in N$ can be the equilibrium leader of the non-trivial tree.
- The number of independent autonomous adopters depends primarily on r_c/r_e and m. As either r_c/r_e or m increases, the number of autonomous adopters decreases. Above a threshold there are no autonomous adopters, ensuring uniformity in choice as everyone follows the single leader.

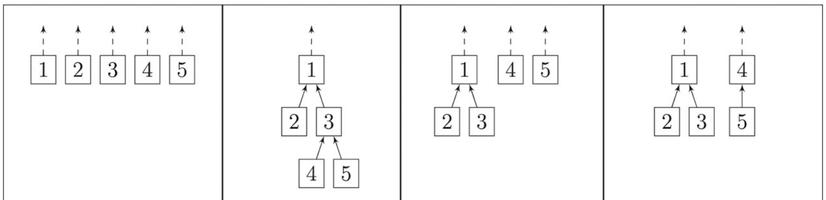
3 Single-Leader Equilibria

This section formally develops the behavior observed in the two examples of Section 2 while generalizing to a population of size n and a network of potential links $g \in G(n)$. Some additional aspects of the equilibrium actions only come to light when considering a larger n. For example, a population of $n > 3$ makes feasible coexisting multiple non-trivial trees. A social structure considered in Section 4 requires $n \geq 8$. The strongly connected graph allows for application to settings in which participants seek involvement in social phenomena without direct access to all members of the population. The limits to connectivity create scenarios that cannot otherwise be considered in the special case of the complete graph.

The equilibrium concept employed is that of pure strategy Nash equilibrium. As such, an equilibrium is an action profile in which each player's action is the optimal action given the actions of the other players. Such action profiles generate social structures such that equilibrium can be defined based on the attributes of the social structure produced. In support of establishing single leader structures as equilibria, the current section identifies the conditions ensuring followers want to follow, establishes that autonomous adopters can prefer their autonomy to following the leader, and identifies optimal behavior for those who follow.

3.1 Hierarchies

The term **hierarchy** refers to a non-trivial tree and thus a social order consisting of a leader and follower(s). Let $h(i;g)$ be the set of σ given g such that $i \in N^L(\sigma)$ with a non-trivial tree of successors. Let $H(i;g)$ represent the set of σ given g with $\{i\} = N^L(\sigma)$ and, necessarily, a successor population $N^S(i;\sigma) = N\backslash\{i\}$. For $g \in G(n)$, $H(i;g)$ is non-empty and, for $n_i^d > 1$ for at least one follower, then $\sigma \in H(i;g)$ is not unique. Let $h_L(i;g)$ represent the set of structures in which all $j \notin N^S(i;\sigma)$ lead (so that there is only one hierarchy and a population of players acting autonomously). With $n = 5$, the four frames of Figure 3 capure the principle social structure scenarios.



(a) No hierarchies (b) Single leader-follower hierarchy, $\sigma \in H(i;g)$ (c) One leader and autonomous adopters, $\sigma \in h_L(i;g)$ (d) Multiple leaders and hierarchies, $\sigma \in h(i;g)$

Figure 3: Example social structures with $n = 5$. Solid arrows indicate links in σ. Dashed arrows represent direct selection of one of the options.

Using follower $j \in N^S(i;\sigma)$ as a reference, classify the population based on their relative position to j in a structure σ. As identified in Figure 4, let $N^x(j;\sigma)$ be the set of successors of i who are of distance no greater than d_{ji}. Let $N^y(j;\sigma)$ be the set of successors of i with a distance greater than d_{ji} who are not successors of j. Recall that set $N^S(j;\sigma)$ identifies the population that succeeds player j. Let $\mu_j^x = \mu^x(j;\sigma) = |N^x(j;\sigma)|$, $\mu_j^y = \mu^y(j;\sigma) = |N^y(j;\sigma)|$, and $\mu_j^s = \mu^s(j;\sigma) = |N^S(j;\sigma)|$.[14] For any $\sigma \in H(i;g)$,

$$\mu_j^c \equiv 1 + \mu_j^x + \mu_j^y + \mu_j^s = n - 1 \text{ and} \tag{3}$$

$$\mu_j^e \equiv \mu_j^y + \mu_j^s. \tag{4}$$

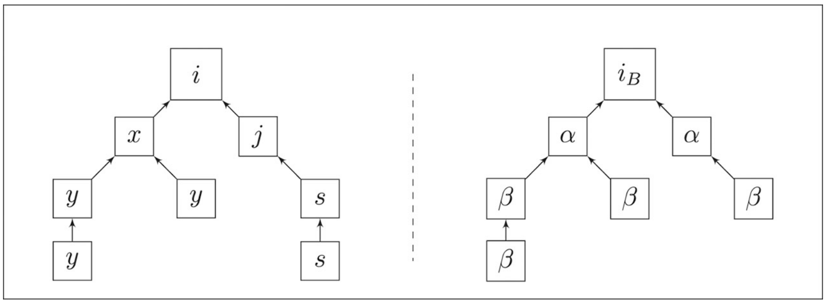


Figure 4: Labeled positions in relation to player j. In j's own tree are $x \in N^x(j;\sigma)$, $y \in N^y(j;\sigma)$, and $s \in N^S(j;\sigma)$ with $\mu_j^x = 1$, $\mu_j^y = 3$ and $\mu_j^s = 2$. In the presence of a second tree (considered in Section 4) are $\alpha \in N^\alpha(j;\sigma)$ and $\beta \in N^\beta(j;\sigma)$.

For $\sigma \in H(i;g)$, if j were to instead lead, the result has $\sigma' \notin H(i;g)$. Let $h^-(i,\sigma;g)$ be the set of alternate structures produced when each $j \in N^S(i;\sigma)$ individually leads rather than follows.

3.2 Follow the Leader

The structure in which the entire population follows a single leader offers parsimony from which to gain insight into behavior. This section develops conditions supporting such structures. The parsimony allows analysis with non-linear rewards expressed in eq. (1). A linear version based on eq. (2) rewards is introduced to transition to solutions involving an interior number of followers, considered in Section 3.5. The linearity comes at little cost, as will be demonstrated.

3.2.1 Nonlinear Rewards

Let,

$$A_{NL}(j;\sigma) = A_1 + A_2(\mu_j^y, \mu_j^s) + A_3(\mu_j^s) \tag{5}$$

$$A_1 = \tfrac{m-1}{m}(\phi(n-1) - \phi(n-2))$$
$$A_2(\mu_j^y, \mu_j^s) = \psi(\mu_j^y + \mu_j^s) - \psi(\mu_j^s)$$
$$A_3(\mu_j^s) = \tfrac{m-1}{m}(\phi(n-2) - \phi(\mu_j^s)) - \tfrac{1}{m}(\psi(n-2) - \psi(\mu_j^s))$$

for $\sigma \in H(i;g)$. Additionally, let

$$B_{NL}(n,m) := (m-1)\frac{\phi(n-1)}{\psi(n-2)} - 1. \tag{6}$$

As a reminder, $\varphi'(\mu) > 0$ and $\psi'(\mu) > 0$ with $\varphi(0) = \psi(0) = 0$. Let $\lambda(\mu) = \phi(\mu)/\psi(\mu)$ and let $\bar{j}$ represent a follower most distant from i in structure σ, possibly not uniquely so.

Proposition 1.
Given $\lambda'(\mu) \geq 0$, the $n-1$ followers in structure σ consisting of a single leader prefer following to leading if and only if $B_{NL} \geq 0$.

As a first step in establishing the social structures that prevail in equilibrium, Proposition 1 considers the preference of each follower and identifies the conditions under which everyone is content to follow leader *i*. The most distant follower of *i* prefers following to leading if and only if $B_{NL} \geq 0$. The condition $\lambda'(\mu) \geq 0$ is sufficient to ensure that the most distant follower is the marginal decision-maker of the population following *i*. For follower *j*, $A(j;\sigma)$ reflects the excess reward to following rather than leading. While $A(j;\sigma)$ is both structure- and position-dependent, Proposition 1 is independent of either structure- or position-specific conditions.

Corollary 1.
For a structure σ consisting of a single leader, a population of $n-1$ followers and the condition $B_{NL} \geq 0$, $\lambda'(\mu) < 0$ can produce a preference to lead among middle distance followers.

For $\lambda'(\mu) < 0$, the most distant follower need not be the marginal decision-maker, opening the possibility that some middle-distance follower prefers leading despite $B_{NL} \geq 0$.

The roles of $A_{NL}(j;\sigma)$, $B_{NL}(n,m)$, and $\lambda(\mu)$ in supporting Proposition 1 and its corollary warrant further explanation. The $A_{NL}(j;\sigma)$ is derived from $\mathbb{E}(\pi(j;\sigma) - \pi(j;\sigma'))$, reflecting player *j*'s excess reward to following an existing leader *i* over the value of leading in the presence of leader *i*. For $A_{NL}(j;\sigma) \geq 0$, player $j \in N^S(i;\sigma)$ prefers imitating over leading. For $\bar{j}$, for whom $\mu^x = n-2$ and $\mu^y = \mu^s = 0$, $B_{NL}(n,m) \geq 0$ arises as the condition producing $A_{NL}(\bar{j};\sigma) \geq 0$. By $\lambda'(\mu) \geq 0$, the relative strength of the ensuing premium gained by autonomously matching the leader is greater for $\bar{j}$ than for anyone else anywhere in the hierarchical structure. Consequently, if $A_{NL}(\bar{j};\sigma) \geq 0$ then $A_{NL}(j;\sigma) \geq 0$ for all $j \in N^S(i;\sigma)$.

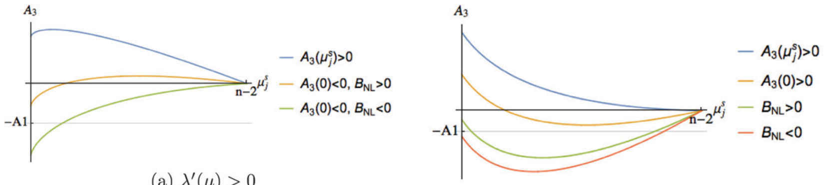
The three reference populations relevant to *j*'s decision are $\{i\}$, $N^x(j;\sigma)$, and $N^y(j;\sigma)$ with the contributions decomposed into the elements of $A_{NL}(j;\sigma)$. The net conformity contribution to a follower derived from matching with the leader, reflected in A_1, is strictly positive. Added separability in the two rewards allows independence between A_1 and player *j*'s position in the tree structure. The A_2 term captures the non-negative contribution of the $N^y(j;\sigma)$ population towards the ensuing reward when *j* follows. The remaining conformity contribution of the $N^y(j;\sigma)$ population is inseparable from that of the $N^x(j;\sigma)$ population, both accounted for in A_3. The net conformity reward of the combined $N^x \cup N^y$ adds to the draw towards following. The net ensuing reward offered by the $N^x \cup N^y$ population when *j* leads, with its negative coefficient, represents the draw towards leading. The $N^S(j;\sigma)$ population only indirectly impacts on *j*'s decision by affecting the reward contributions of the other populations.

The function $A_3(\mu^s_j)$ exists over the range $\mu^s_j \in [0, n-2]$. Follower $j = \bar{j}$, for whom $\mu^s_j = \mu^y_j = 0$, prefers following to leading when $A_3(0) \geq -A_1$, assured by the condition $B_{NL}(n,m) \geq 0$, thereby producing $A_{NL}(\bar{j};\sigma) \geq 0$. At the other extreme, $\mu^s_j = n-2$ is only possible if *j* is the sole direct imitator of *i* and thereby the predecessor of the remaining population. Let j_1 indicate a follower who is the unique direct follower of *i*. Whether leading or following, the entire $\mu^s_j = n-2$ population remains with j_1. With $\mu^x_{j_1} = \mu^y_{j_1} = 0$, there is no population within $N^S(i;\sigma)$ with whom j_1 can potentially gain a distance advantage over through leading, resulting in $A_3(n-2) = 0$. Assured of $n-2$ followers of her own, j_1 gains the additional certain conformity of *i* by making *i* the leader rather than leading herself, thus $A_{NL}(j,\sigma) = A_1 > 0$.

Between $\bar{j}$ and *i* (or j_1 if there is only one direct imitator of *i*) are followers who potentially exercise some direct influence, with $\mu^s_j > 0$, while also benefiting from conformity with other independent followers of *i*, with $\mu^x_j + \mu^y_j > 0$. The relative shapes of $\varphi(\mu)$ and $\psi(\mu)$ determine whether the balance between conformity and influence preserves the preference for following for all players above $\bar{j}$. The functions $\varphi(\mu)$ and $\psi(\mu)$ jointly shape $A_3(\mu^s_j)$, but with $\lambda'(\mu) \geq 0$, $\psi(\mu)$ alone establishes a lower bound that, due to the increasing $\psi(\mu)$, is monotonic in its progression from $A_3(0)$ (whether positive or negative) to $A_3(n-2) = 0$. Let $A^0_3(\mu)$ represent $A_3(\mu)$ as produced by $\lambda'(\mu) = 0$. For $\lambda'(\mu) \geq 0$, $A^0_3(\mu) \leq A_3(\mu)$,

$$A^0_3(\mu) = A_3(0)\left(1 - \frac{\psi(\mu)}{\psi(n-2)}\right).$$

With $A_3(n-2)$ anchored at zero, $\lambda'(\mu) \geq 0$ ensures the minimum of $A_3(\mu)$ is at one of the two endpoints, as determined by the sign of $A_3(0)$. The condition $\lambda'(\mu) \geq 0$ places no additional individual restrictions on φ or ψ. The possible $\lambda(\mu)$ remains quite rich, allowing for a broad combination of possible reward structures. Figure 5 illustrates the role of $\lambda(\mu)$. Frame Figure 5a, based on $\lambda'(\mu) > 0$, employs $\phi(\mu) = \mu^{1-a}/(1-a)$ and $\psi(\mu) = c\mu^{1-b}/(1-b)$ with $a < b$.



(a) $\lambda'(\mu) > 0$ (b) $\lambda'(\mu) < 0$

Figure 5: $A_3(j;\sigma)$ as shaped by $\lambda(\mu)$. The most distant follower of i has $\mu_j^s = \mu_j^y = 0$ so that $A_{NL} = A_1 + A_3(0)$. $B_{NL} \geq 0$ indicates that the most distant follower prefers to follow. For $\lambda'(\mu) \geq 0$, the minimum of $A_3(\mu_j^s)$ is at a boundary value $\mu_j^s \in \{0, n-2\}$. For $\lambda'(\mu) < 0$ the minimum can occur for an interior value of μ_j^s so that $B_{NL} \geq 0$ does not ensure $A(j;\sigma) \geq 0$ for all j.

For $\lambda'(\mu) < 0$, $A_3^0(\mu)$ becomes the upper bound on $A_3(\mu)$, introducing the possibility of an interior minimum. Consequently, the possibility for $A_{NL}(j;\sigma) \leq 0$ for some middle distance $j \in N^S(i;\sigma)$ arises despite $B_{NL} \geq 0$ ensuring $A_3(0) \geq -A_1$, as the example included in Figure 5b illustrates. It is based on $\phi(\mu) = ((1+\mu)^{1-a} - 1)/(1-a)$ and $\psi(\mu) = ((1+\mu)^{1-b} - 1)/(1-b)$ with $b < a$. A dip in $A_3(\mu_j^s)$ below $-A_1$ indicates that some possible middle-distance nodes of the hierarchical tree, if occupied, are inferior to leading. For $\lambda'(\mu) < 0$, the decision to lead or follow for all $j \in N^S(i;\sigma)$ cannot be identified from the preference of i's most distant follower.

Figure 6 presents the $A_{NL}(j;\sigma)$ surface, expressed as a function of μ_j^x, μ_j^y, and μ_j^s. The collection of nodes on the surface is the universe of possible follower rewards for positions in $\sigma \in H(i;g)$ from $g \in G(n)$. The near right and far corners, $A(0,0,n-2)$ and $A(1,n-3,0)$, are inherently positive. The condition $B_{NL} > 0$ indicates $A(n-2,0,0) > 0$ for the near left corner. The minimum $A(\mu^x,\mu^y,\mu^s)$ node is always to be found in the near edge with $\mu^y = 0$. The example illustrates a reward function with $\lambda'(\mu) < 0$ such that the leading edge includes nodes with $A(j;\sigma) < 0$, indicated in red. Any structure with one or more individuals occupying a node with $A(j;\sigma) < 0$ is not an equilibrium.

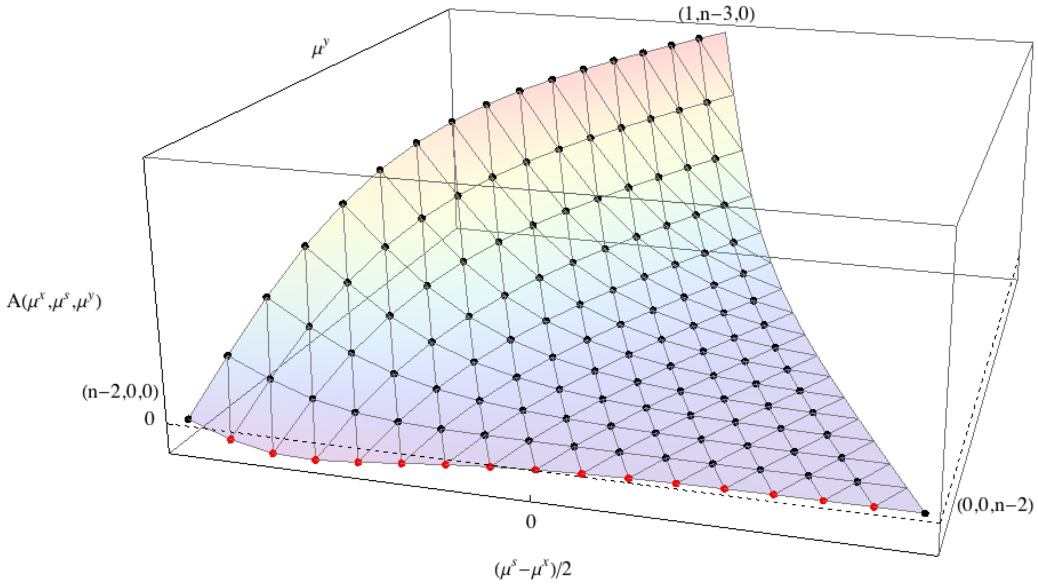


Figure 6: $A_{NL}(j;\sigma)$ surface produced for $n = 18$ with $\lambda'(\mu) < 0$ and $B_{NL} > 0$. Each point on the surface represents a unique feasible triplet $(\mu_j^x, \mu_j^y, \mu_j^s)$. The height of the point is $A_{NL}(j;\sigma)$. $A_{NL}(j;\sigma) < 0$ in red. For $\sigma \in H(i;g)$, $\mu_j^x + \mu_j^y + \mu_j^s = n-2$. Each follower occupies a point on the surface. As determined by $\sigma \in H(i;g)$, each point can be occupied by zero, one, or multiple players with the condition that at least one player occupies the lower left corner. The far corner is always the highest point. For $B_{NL} \geq 0$, all three corners are positive with only the near left corner at zero for $B_{NL} = 0$. For $B_{NL} < 0$ the near left corner is negative. For $\lambda'(\mu) \geq 0$, one of the near corners is the lowest point on the surface. For $\lambda'(\mu) < 0$, local convexity in the $\mu^y = 0$ plane allows a low point along the near edge. Any σ with a player located at a point with $A(j;\sigma) < 0$ cannot be an equilibrium. This $\lambda'(\mu) < 0$ surface is produced by $\varphi(\mu) = \tanh(a\mu)$ and $\psi(\mu) = 2\tanh(b\mu)$, $0 < b < a < 1$.

Examination of the social impact of $\lambda'(\mu) < 0$ resumes in Section 5. Until then, analysis will be dedicated to identifying the equilibrium structures supported by $\lambda'(\mu) \geq 0$.

3.2.2 Linear Rewards

The linear rewards of eq. (2) produce linearity in $A_{NL}(j;\sigma)$, $B_{NL}(n,m)$, and $A_3(\mu)$. Since linearity also generates $\lambda'(\mu) = 0$, $A_3(\mu) = A_3^0(\mu)$.

For $\pi(i;\sigma) = r_c\mu_i^c + r_e\mu_i^e$ for all $i \in N$, A_{NL} becomes

$$A(j;\sigma) = \frac{(m-1)}{m} r_c + r_e\mu_j^y + \frac{1}{m}((m-1)r_c - r_e)(n-2-\mu_j^s). \quad (7)$$

Each term of eq. (7) is the linear version of the corresponding A_1, A_2, and A_3 of eq. (5). Proposition 1 applies so that $B_{NL} \geq 0$ is a necessary and sufficient condition for $\pi(j,\sigma) \geq \pi(j,\sigma')$ for all $j \in N^S(i;\sigma)$ for any $\sigma \in H(i;g)$ and $\sigma' = \{h^-(i,\sigma;g)|\sigma'_{jj} = 1\}$. An evaluation of Proposition 1 based on the linear payoff function can be found in Appendix B.

Linearity in the reward function allows the role of each population, i, $N^x(j;\sigma)$, and $N^y(j;\sigma)$, to be considered in isolation,

$$A(j;\sigma) := \frac{(m-1)}{m} r_c + \frac{(m-1)}{m}(r_e + r_c)\mu_j^y + \frac{1}{m}\left((m-1)r_c - r_e\right)\mu_j^x. \quad (8)$$

The individual contributions of each population are included in Table 4. Also included are the $N^S(j;\sigma)$ and a potentially non-empty $N^l(i;\sigma)$ population, though the net contribution of each towards the decision to lead or follow negates to zero.

Table 4: Expected contribution by different populations within the hierarchy to j's payoff according to j's decision to lead or follow.

Population	j follows	j leads
$\{i\}$	r_c	$\frac{1}{m}r_c$
$N^x(j;\sigma)$	$r_c\mu^x$	$\frac{1}{m}(r_c + r_e)\mu^x$
$N^y(j;\sigma)$	$(r_c + r_e)\mu^y$	$\frac{1}{m}(r_c + r_e)\mu^y$
$N^S(j;\sigma)$	$(r_c + r_e)\mu^s$	$(r_c + r_e)\mu^s$
$N^l(j;\sigma)$	$\frac{1}{m}r_c(\mu^l - 1)$	$\frac{1}{m}r_c(\mu^l - 1)$

The linear reward functions in eq. (6) produce

$$B_{NL}(n,m) = (m-1)\frac{r_c(n-1)}{r_e(n-2)} - 1$$

for $r_e \neq 0$. Multiply $B_{NL}(n,m)$ by $(n-2)/(n-1)$ to obtain,

$$B(n,\theta) := \theta - \left(1 - \frac{1}{n-1}\right) \quad (9)$$

where

$$\theta = \frac{(m-1)r_c}{r_e}.$$

Throughout the paper, how θ compares to some threshold value determines whether all players prefer to follow an existing leader or whether there exists some player who prefers to lead in the presence of another leader. The condition $B \geq 0$ is just one expression of this threshold. Other threshold values for θ arise when analysis turns to more complicated social structures involving multiple leaders, as considered in Section 4.

A high θ indicates a strong inclination to follow. A preference that favors conformity, as indicated by a high r_c/r_e, induces following to exploit the certainty of conforming with i and i's other $\mu_i^s - 1$ followers. Similarly, a large m deters leading by reducing the likelihood of autonomously matching with the leader's choice. The threshold against which θ is measured reflects the size and relative positions of those populations important to the marginal decision-maker. For $\bar{j}$ in $\sigma \in H(i;g)$, the concern focuses on i and $N^x(j;\sigma)$ as the only non-empty populations.

3.3 How Best to Follow

Let $H'(i;g)$ be the set of $\sigma \in H(i;g)$ such that each $j \neq i$ minimizes μ_j^x. Let $H^*(i;g)$ be the non-empty subset of $H'(i;g)$ in which every follower minimizes d_{ji} (same as minimizing d_j through leader i).[15] Note that if σ' exists such that $\{\sigma,\sigma'\} \in H^*(i;g)$, then $d_j(\sigma) = d_j(\sigma')$ for all $j \in N$. As a result, all $\sigma \in H^*(i;g)$ offer exactly the same payoff profile. Similar to $H^*(i;g)$, let $h^*(i;g)$ be the set of strategies for which each successor of i imitates the player offering the shortest distance from i.

Lemma 1.
A player maximizes her own payoff as a follower of i by minimizing μ_j^x in the i-led hierarchy.

Lemma 1 identifies the optimal follow action. With $\pi_{NL}(j;\sigma) = \phi(n-1) + \psi(\mu_j^y + \mu_j^s)$ for $\sigma \in H(i;g)$, Lemma 1 emerges from the fact $\mu_j^y + \mu_j^s = n - 2 - \mu_j^x$ and that $\psi'(\mu) > 0$.[16] By Lemma 1, for $\sigma \in H'(i;g)$, no player can do better for herself as a follower.

Lemma 2.
The followers in a structure σ, consisting of a single leader, minimize μ_j^x if σ is an equilibrium.

A structure $\sigma \in H'(i;g)$, having each follower optimize against the available following options by minimizing μ_j^x, remains a candidate for Nash equilibrium. A follower who has not minimized μ_j^x has not achieved the personal maximum achievable payoff as a follower.

As a subset of the actions that minimizes μ_j^x, minimizing the distance to the leader is a sufficient action to achieve the minimum μ_j^x. When available, the option to increase d_j without causing the structure to exit $H'(i;g)$ is Pareto improving without opening exploitable position changes to other players. See the proof of Corollary 2 in Appendix B for details. The action by j to minimize μ_j^x but not d_j benefits some player $j' \in N^S(i;\sigma)\backslash\{j\}$ without cost to any player. The two structures in Figure 7 illustrates the opportunity for Pareto improvement.

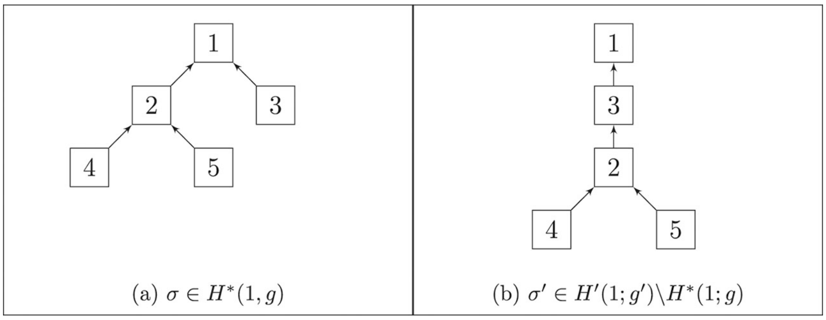


Figure 7: An example of $\sigma' \in H'(i;g)\backslash H^*(i;g)$ based on $N^d(2;g) = \{1,3\}$, $N^d(4;g) = \{2,5\}$, and $N^d(5;g) = \{2,4\}$, $\sigma \in H^*(1;g)$. The conditions for 2 to minimize μ_2^x without minimizing $d_{2,1}$ are present in σ. For players $i = 1,2,4,5$, $\pi(i;\sigma) = \pi(i;\sigma')$ while $\pi(3;\sigma) < \pi(3;\sigma')$.

3.4 A Structure as Equilibrium

With the followers optimally positioned within the tree structure according to Lemma 2, the conditions imposed in Proposition 1 ensure all followers prefer following to leading, making $\sigma \in H'(i;g)$ an equilibrium.

Proposition 2.
Given $\lambda'(\mu) \geq 0$, a structure consisting of a single leader and a population of $n-1$ followers, all of whom position themselves in the tree structure to minimize the size of their own $N^x(j;\sigma)$ population, is a Nash equilibrium if and only if $B_{NL} \geq 0$.

The only conditions needed to produce this set of structures as equilibria are $B_{NL} \geq 0$ and $\lambda'(\mu) \geq 0$, independent of the particular i or the characteristics of $\sigma \in H'(i;g)$ or $g \in G(n)$ for all i. The same conditions also identify $\sigma \in H^*(i;g)$ as an equilibrium, allowing Corollary 2.[17]

Corollary 2.
Given $\lambda'(\mu) \geq 0$, a structure consisting of a single leader and a population of $n-1$ followers, all of whom minimize their distance to the leader, is a Nash equilibrium if and only if $B_{NL} \geq 0$.

Let g^c represent the special case of a complete graph. The $\{\sigma^c\} = H^*(i;g^c)$ is a star network and $H'(i;g^c)$ additionally includes structures in which one player links indirectly to i through one of the $n-2$ direct successors.

Hereafter, optimizing followers are presumed to minimize d_{ji}.[18]

3.5 A Preference for Autonomy

A setting with $B<0$ excludes $\sigma \in H(i;g)$ as a potential equilibrium since not all members of the population wish to follow i. As with Proposition 1, $\lambda'(\mu) \geq 0$ ensures $\bar{j} \in N^S(i;\sigma)$ is the marginal decision-maker. As such, analysis can proceed employing linearity, preserving $\bar{j}$ as the marginal decision-maker while allowing summation over expected outcomes.[19]

Let $h_L(i, \mu_i^s; g)$ represent the set of structures in which i's successor population is of size $\mu_i^s < n-1$ and the $n-\mu_i^s-1$ most distant players from i on g lead rather than follow. For $\sigma \in h_L^*(i, \mu_i^s; g)$, each member of $N^S(i;\sigma)$ additionally minimizes her distance to i.

Observe that for $\sigma \in h_L(i, \mu_i^s; g)$ and $j \in N^S(i;\sigma)$,

$$\underbrace{1+\mu_j^x+\mu_j^y+\mu_j^s}_{=\mu_i^s}+\mu^l = n$$

where $\mu^l = |N^L(\sigma)|$ includes i. Let

$$C(\mu_i^s;\theta) = \theta - \left(1 - \frac{1}{\mu_i^s}\right). \tag{10}$$

Allow μ^* to represent the value of μ_i^s that solves $C(\mu_i^s;\theta)=0$,

$$\mu^* = \frac{1}{1-\theta}, \tag{11}$$

where $B<0$ ensures $\theta<1$. Defined below, $\bar{n}$ is an integer near μ^*, $|\bar{n}-\mu^*|<1$. Identify the most distant successor of i given σ as $\bar{j}(\mu_i^s)$ so that $A(\bar{j}(1);\sigma)$ is the value of $A(\bar{j};\sigma)$ for a σ in which $\mu_i^s=1$ and $A(\bar{j}(n-1);\sigma)$ is the value of $A(\bar{j};\sigma)$ for a $\sigma \in H(i;g)$.

Proposition 3.
Given $B<0$, the μ_i^s followers in σ prefer to follow the single leader i and the $n-\mu_i^s-1$ remaining players prefer leading to following i if $N^S(i;\sigma)$ is populated by the $\bar{n}$ players closest to i on g.

Regardless of j's action, each autonomous agent has a $1/m$ chance of contributing the j's conformity reward. Since the follower's decision is unaffected by the $N^l(\sigma)$ population, $A(j;\sigma) \geq 0$, for $A(j;\sigma)$ as expressed in eq. (8), remains the condition for j to follow. For $j \in N^S(i;\sigma)$, $A(j;\sigma)$ depends on the structure of the i-led hierarchy, including its overall size. Observe, $A(\bar{j}(1);\sigma) > 0$, $A(\bar{j}(\mu_i^s);\sigma)$ is decreasing in μ_i^s and, since $B<0$, $A(\bar{j}(n-1);\sigma) < 0$. The player $\bar{j}$ remains the marginal decision-maker.

The endogenously determined $\bar{n}$ is identified by the condition $A(\bar{j}(\bar{n});\sigma) \geq 0$ and $A(\bar{j}(\bar{n}+1);\sigma') < 0$. For $\sigma \in h_L^*(i,\bar{n};g)$, no $j \in N^S(i;\sigma)$ can improve her payoff within the i-led hierarchy nor by leading. Additionally, no $j \in N^l(\sigma)$ can improve her payoff by joining the i-led tree. The structure $\sigma \in h_L^*(i,\bar{n};g)$ precludes the existence of some $j \in N^L(\sigma)$ able to link to the i-led tree at a distance $d_j < d_{\bar{j}(\mu_i^s)}$ and thereby improve her reward. The individual optimality with regards to the size and membership in $N^S(i;\sigma)$ leaves structure $\sigma \in h_L^*(i,\bar{n};g)$ as a candidate Nash equilibrium.

The autonomy of leading is not pursued for the sake of individuality but rather a gambit of autonomously matching the choice of the leader and thereby earning the premium ensuing reward.

A non-trivial set of alternatives and a preference for conformity, meaning $m>1$ and $r_c>0$ so that $\theta>0$, are prerequisites for the existence of a non-trivial tree as a possible equilibrium. For $\theta=0$, $A(j;\sigma) \leq 0$. When due to $m=1$, the coordination problem is solved trivially without the leader-follower structure. When due to $r_c=0$, there is no conformity reward to be gained by delaying adoption.

Similar to the finding in Proposition 2, for $\mu_i^s = \bar{n} < n-1$, the candidate equilibrium size of the tree, according to Proposition 3, is determined by universal parameters, independent of the particular i or the characteristics of $\sigma \in H^*(i;g)$, $\sigma \in h^*(i;g)$, or the underlying $g \in G(n)$.

The conditions $B_{NL}<0$, $\lambda'(\mu) \geq 0$, and $\phi''(\mu) < 0$ preserve Proposition 3's primary feature of an interior $\bar{n}$ for nonlinear rewards, the outline of which is included in Appendix B.

4 Multiple Leaders

This section considers the viability of multiple hierarchies. For $B<0$, this entails examining whether those autonomous players not following i would prefer to organize behind some other leader. For $B \geq 0$, the issue is in identifying equilibrium structures not already excluded by Propositions 1 through 3.

Let $h(i_A, i_B; g)$ be the set of σ given $g \in G(n)$ such that $\{i_A, i_B\} \in N^L(\sigma)$ with successor populations $N^S(i_h;\sigma) \neq \emptyset$ for $h = A,B$. Let $H(i_A, i_B; g)$ represent the subset of $h(i_A, i_B; g)$ such that $\{i_A, i_B\} = N^L(\sigma)$. In $h^*(i_A, i_B; g)$ and $H^*(i_A, i_B; g)$ are structures σ in which each successor employs the shortest path to the chosen leader. Let $\mu_h^s = |N^S(i_h;\sigma)|$ indicate the number of successors in the i_h-led tree. Without loss of generality, assume $\mu_A^s \geq \mu_B^s$.

For $h = A,B$, let j_h represent $j \in N^S(i_h;\sigma)$. With two non-trivial trees, there is a need to identify and label populations in the i_{-h}-led tree based on their position relative to j_h. Let $N^\alpha(j_h;\sigma)$ be the set of successors of i_{-h} who are of distance no greater than d_{j_h,i_h} and let $N^\beta(j_h;\sigma)$ be the set of successors of i_{-h} who are of a distance greater than $d_{j_h i_h}$. Let $\mu_j^\alpha = \mu^\alpha(j;\sigma) = |N^\alpha(j;\sigma)|$ and $\mu_j^\beta = \mu^\beta(j;\sigma) = |N^\beta(j;\sigma)|$. The node labels in Figure 4 identify the agent's position relative to player j with $\mu_j^\alpha = 2$ and $\mu_j^\beta = 4$.

4.1 Maintaining Autonomy

For $B < 0$ and a given leader i_A with $\mu_A^i = \bar{n}$ followers, some or all of the $n - \bar{n} - 1$ individuals not in $N^S(i_A;\sigma)$ might prefer forming a second hierarchy to autonomy.[20] Recalling that $\theta = (m-1)r_c/r_e$, $B = \theta - (1-(n-1)^{-1})$, and $C = \theta - (1 - (\mu_i^s)^{-1})$, let

$$D(j;\sigma) := \frac{m-1}{m}\left(r_c + (r_e + r_c)\mu_j^y\right) + \frac{1}{m}\left((m-1)r_c - r_e\right)\mu_j^x - \frac{r_e \mu_j^\alpha}{m} \tag{12}$$

and

$$E(i_h;\theta,n,\sigma) := \theta - \left(1 - \frac{1}{\mu_h^s}\right) - \frac{\mu^\alpha(\bar{j}_h;\sigma)}{\mu_h^s}. \tag{13}$$

Proposition 4.
The set of σ in which the $\bar{n}$ players closest to i follow and the remaining $n - \bar{n} - 1$ players lead is the set of equilibrium structures if and only if $B < 0$.

By Proposition 4, $\sigma \in h_L^*(i, \mu_i^s; g)$ and only $\sigma \in h_L^*(i, \mu_i^s; g)$ structures are equilibria for $B < 0$. With conformity only weakly rewarded, the $n - \bar{n} - 1$ not following i_A are better served by leading than by organizing into a second hierarchy. The motivation for leading is the possibility of gaining a distance advantage over i_A's followers. The conformity of following i_B inadequately compensates for the sacrificed distance advantage against the $N^S(i_A;\sigma)$ population that would become j_B's $N^\alpha(j_B;\sigma)$ population in the event of a match.

The condition $D(j;\sigma) \geq 0$ indicates j prefers following to leading with $E(i_h;\sigma) \geq 0$ corresponding to $D(\bar{j}_h;\sigma) \geq 0$. That $D(j;\sigma) = A(j;\sigma) - r_e\mu_j^\alpha$ indicates that the presence of a second hierarchy makes leading more attractive for j relative to the second hierarchy's absence. Similarly, with $E(i_h;\sigma) = C(\mu_h^s;\theta) - \mu^\alpha(\bar{j}_h)/\mu_h^s$ the minimum threshold value on θ to maintain a μ_h^s-sized tree in the presence of an existing alternate tree is greater than the threshold necessary to maintain a μ_h^s-sized tree in the presence of a population of autonomous adopters. The greater $E(i_h;\sigma) > 0$ induced conformity reward necessarily compensates for the lost distance advantage over the $N^\alpha(j;\sigma)$ population when following.

Since $\mu_j^\alpha \geq 1$ for all $\sigma \in h(i_A, i_B; g)$,

$$\left(1 + \frac{\mu^\alpha(\bar{j}_h) - 1}{\mu_h^s}\right) \geq 1 > \left(1 - \frac{1}{n-1}\right). \tag{14}$$

Thus, $E(i_h;\sigma) \geq 0$ imposes a higher threshold for θ than does the condition $B \geq 0$. A multiple leader structure cannot be an equilibrium when $B < 0$. The set of Nash equilibria are drawn from $h_L(i, \mu_i^s; g)$ only. The set $h_L^*(i, \bar{n}; g)$ constitutes the set of Nash equilibria.

There are two features of this solution worth exploring. First, starting from the structure $\sigma \in h_L^*(i_A, \bar{n}; g)$ and $i_B \in N^L(\sigma)$, the condition $E(i_B;\sigma) < 0$ for all $\mu_B^s > 0$ makes it imprudent for any player not following i_A to instead follow i_B since following would lower the player's own expected payoff.

The conclusion applies to $\bar{j}_A$ as well. In the presence of an i_B-led hierarchy, the most distant followers of both trees prefer leading to following. Neither tree can persist in the presence of the other, independent of the size of either tree. The result points to a fragility of the $\sigma \in h_L^*(i, \bar{n}; g)$ equilibrium structures.

The conditions $B_{NL} < 0$, $\lambda'(\mu) \geq 0$, and $\phi''(\mu) < 0$ preserve Proposition 4's primary feature of a single leader and population of autonomous adopters, the outline of which is included in Appendix B.

Propositions 2 and 4 allow for the following observation.

Theorem 1.
Given a non-trivial choice, a preference for conformity, and a strongly connected population, for every $i \in N$ there exists an equilibrium structure with player i as the only leader.

A single-leader structure led by player i for all $i \in N$ is among the equilibrium set. The tree structure is supported by the presence of a meaningful choice between options and a desire to conform, reflected in $\theta > 0$. The only condition for a single leader is $\lambda'(\mu) \geq 0$.

As the culmination of Propositions 1 through 4, equilibrium structures exist under a broad range of permissible reward functions. Structure with a unique leader prevail among the equilibria. The single-leader structure as equilibrium extends to settings of weak conformity reward, where one might anticipate a single widespread norm is not of particular importance to the population. Additionally, leadership is supported by a population's desire for conformity and the appearance of influence, without the need for private advantage or individual leadership characteristics. Lastly, the network structure does not impinge upon an individual's ability to lead. Within the set of strongly connected networks, it is never the case that the particulars of the linking structure conspire to prevent an individual from leading in equilibrium.

4.2 Multiple Hierarchies

For a sufficiently strong conformity reward such that $E \geq 0$, multiple leaders do not lose followers defecting to instead lead. They may, however, still suffer defections and eventual dissolution by followers switching to a preferred hierarchy.

For $\sigma \in H(i_A, i_B; g)$, let $d\mu = \mu^s_A - \mu^s_B$ so that $d\mu$ captures the population size differential between the two trees. Let $N^{AB}(i_A, i_B; \sigma)$ represent the set of followers possessing potential links to predecessors in both trees. A strongly connected g ensures that each tree has at least one member able to link directly to a player in the other tree.

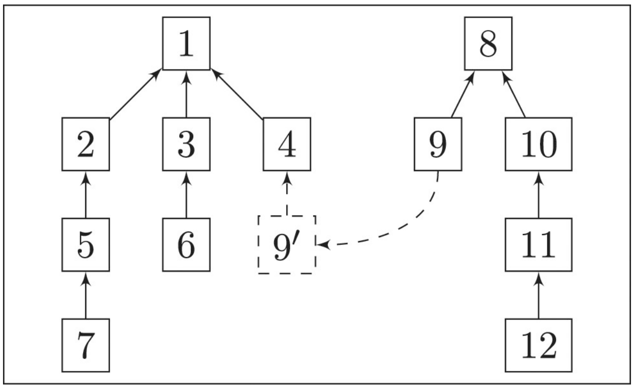


Figure 8: Example $\sigma \in H(1,8;g)$ with player $j_B = 9$ switching to join the player $i_A = 1$-led tree. The identified populations are $\{10\} = N^x(9;\sigma)$, $\{11, 12\} = N^y(9;\sigma)$, $\emptyset = N^S(9;\sigma)$, $\{2, 3, 4\} = N^\alpha(9;\sigma)$, $\{5, 6, 7\} = N^\beta(9;\sigma)$, $\{7\} = N^y_A(9)$, and $\{12\} = N^\beta_A(9)$.

Let $\sigma' = \sigma_{-j_h} \times \sigma'_{j_h}$ be the structure produced by j_h switching predecessors in order to become a member of the i_{-h}-led tree. The alternative structure identifies populations $N^\beta_{-h}(j_h) = N^\beta(j_h; \sigma')$ and $N^y_{-h}(j_h) = N^y(j_h; \sigma')$. The former is the population of players in j_h's current tree who are more distant from i_h than is j_h from i_{-h} in σ'. The latter is the population in the i_{-h} led tree more distant from i_{-h} than j_h in σ'. Let $\mu^\beta_{-h}(j_h) = |N^\beta_{-h}(j_h)|$ and $\mu^y_{-h}(j_h) = |N^y_{-h}(j_h)|$. Figure 8 illustrates the relative values.

Let

$$F_A(j_A;\sigma) := \theta - \frac{\mu^\beta_B(j_A) - \mu^\beta(j_A) - m(\mu^y(j_A) - \mu^y_B(j_A))}{d\mu - 1 - \mu^s(j_A))}, \quad (15)$$

$$F_B(j_B;\sigma) := \frac{\mu^\beta(j_B) - \mu^\beta_A(j_B) + m(\mu^y(j_B) - \mu^y_A(j_B))}{d\mu + 1 + \mu^s(j_B)} - \theta. \quad (16)$$

For $\sigma \in H(i_A, i_B; g)$, members of $N^{AB}(i_A, i_B; \sigma)$ have the option to switch leaders. All followers have the option to lead. Let $H^+(i_A, i_B; g)$ be the subset of $H^*(i_A, i_B; g)$ satisfying the three conditions of Proposition 5.

Proposition 5.

$B \geq 0$ allows multiple leader equilibrium structures under the condition that:

1. no leader is capable of linking directly with a member of another tree,
2. the most distant follower in each tree prefers following to leading despite the presence of other trees, and
3. all followers capable of linking to a member of another tree prefer their current position.

The first two conditions have previously been established. Proposition 1 identifies, for linear rewards, $B \geq 0$ as necessary and sufficient for following, rather than leading, in the presence of another leader. Thus, the absence of a link from one leader to any member of the other tree is a condition for an equilibrium $\sigma \in H(i_A, i_B; g)$.

Proposition 4 establishes that a sufficiently large θ, indicating a strong conformity reward such that $E(i_h;\sigma) \geq 0$ for $h = A,B$, indicates the followers $\sigma \in H(i_A, i_B; g)$ prefer following to leading. Consider $\sigma_1 \in H^*(i;g)$ and $\sigma_2 \in H^*(i_A, i_B; g)$. Though the reward to following in a multi-leader setting depends on the size of various position-specific relative populations, the structure-independent differential

$$\mathbb{E}(\pi(\bar{j}, \sigma_1) - \pi(\bar{j}_h, \sigma_2)) = \frac{m-1}{m}(n - 1 - \mu_h^s) r_c$$

reveals a reward to following in the multi-leader σ_2 that declines relative to following in the single-tree structure of σ_1 as the size of $\bar{j}_h$'s affiliated tree decreases.

In contrast, the attraction to lead depends only on the total size of the follower population and not on how the followers are distributed among leaders. Let $\sigma_h^{'}$, $h = 1,2$ represent the structure produced when $\bar{j}_h$ switches to leading. The differential

$$\mathbb{E}(\pi(\bar{j}; \sigma_1^{'}) - \pi(\bar{j}_h; \sigma_2^{'})) = r_e/m$$

is independent of n and μ_h^s. The non-zero value reflects that with two trees, there is one less follower, i_B. Thus, maintaining followers in a multi-leader setting requires a θ that more strongly penalizes autonomy.

The third condition of Proposition 5 requires $F_h(j_h;\sigma) \geq 0$ for $h = A,B$. The condition $F_h(j_h;\sigma) \geq 0$ indicates a greater expected reward to j_h for remaining in the i_h-led hierarchy than available from switching to the alternate hierarchy.

Two scenarios potentially satisfy the third condition of Proposition 5. Though the greater conformity reward of a larger hierarchy generally attracts the most distant followers from smaller and equal sized hierarchies, it is possible for a smaller hierarchy to have $F_B(\bar{j}_B;\sigma) \geq 0$ while still preserving $F_A(\bar{j}_A;\sigma) \geq 0$.[21] For this to occur requires (i) that the less populous tree has a population concentrated near i_B and (ii) that the more populous i_A-led tree has a bulge so that there is a large number of followers at a distance just below the most distant follower of i_B. The structure depicted in Figure 9 displays these features. Example 4 illustrates how this structure is advantageous to $\bar{j}_B$ while not attracting $\bar{j}_A$.

Alternatively, in the absence of a complete graph, it is possible that those with contacts enabling them to switch to the other hierarchy prefer the status quo. The remaining members, at least one of whom would prefer to switch, are without a link to the other hierarchy. In particular, a follower, even of a substantially smaller hierarchy, can be induced to stay by the ensuing reward of a non-empty $N^y(j;\sigma)$ population. The F_B conditions of this latter scenario are position specific, requiring confirmation for every follower in $N^{AB}(i_A, i_B;\sigma)$ in order to establish $\sigma \in H^*(i_A, i_B; g)$ as an equilibrium. Example 5 below illustrates this latter scenario.

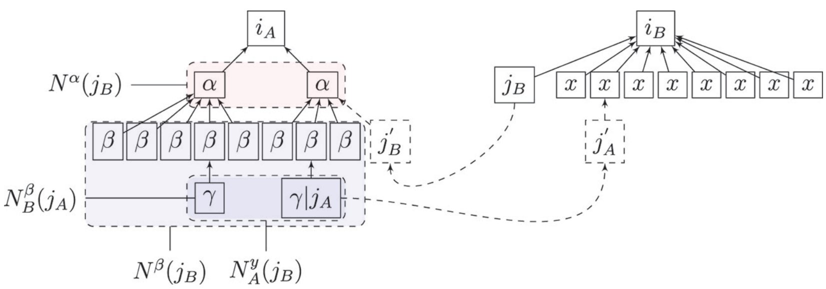


Figure 9: Example 4 of $\sigma \in H^+(i_A, i_B; g)$. The dashed link is the position available to j_h in the i_{-h} tree. γ identifies $j \in N_A^y(j_B;\sigma)$. $j_A \in N_A^y(j_B;\sigma)$. Here, $d\mu = 3$, $\mu_B^\beta(j_A) = 1$, $\mu^\beta(j_B) = 10$, $\mu_A^y(j_B) = 2$, $\mu_A^s = 12$, and $\mu_B^s = 9$. Let $m = 2$, then $F_A \geq 0$ implies $\theta \geq \frac{1}{2}$, $F_B \geq 0$ implies $\theta \leq \frac{3}{2}$, and $E(i_B) \geq 0$ implies $\theta \geq \frac{11}{9}$. There is nontrivial support $\theta \in [\frac{11}{9}, \frac{3}{2}]$ for which $\sigma \in H^+(i_A, i_B; g)$ is a Nash equilibrium.

Example 4.
With $N^{AB}(i_A, i_B; \sigma) = \{j_A, j_B\}$, the structure depicted in Figure 9 is in $H^+(i_A, i_B; g)$ for conforming values of θ. In the structure, player j_B benefits from holding a distance advantage over the large population of β-labeled players in the event that i_A and i_B match. She loses that advantage were she to switch to the larger i_A-led tree. Player j_A does not gain advantage over the β population with a switch to the i_B-led tree and thus prefers to stay with i_A for the greater conformity reward. The large $N^x(j_B, \sigma)$ population is needed to counter the benefits to j_B of the $N^\alpha(j_B, \sigma)$ and $N^y_A(j_B, \sigma)$ populations were she to switch.
In this example, with $\mu^y(j_B) = \mu^s(j_B) = 0$, $F_B(j_B; \theta) \geq 0$ reduces to

$$\theta \leq \bar{\theta} \equiv \frac{\mu^\beta(j_B) - m\mu^y_A(j_B)}{d\mu + 1}. \tag{17}$$

In addition, with $\mu^y(j_A) = \mu^s(j_A) = 0$, $F_A(j_A; \theta) \geq 0$ reduces to

$$\theta \geq \underline{\theta} \equiv \frac{\mu^\beta_B(j_A)}{d\mu - 1}. \tag{18}$$

The two conditions define upper and lower bounds on permissible θ to have $\sigma \in H^+(i_A, i_B; g)$. For a particular σ and g, the support producing $\sigma \in H^+(i_A i_B; g)$ may be empty, with $\bar{\theta} \leq \underline{\theta}$, or may have θ fall outside of the support. The lower bound on θ established by the condition $E(i_B) \geq 0$ can be greater than that produced by $F_A \geq 0$, in which case the most distant follower of i_B will lead before j_A considers switching to the i_B-led tree.

The second scenario, illustrated in Example 5, consists of structures in which a follower in the less populous i_B-led tree has a sufficiently large $\mu^y(j_B)$ such that the larger conformity reward offered by the i_A-led tree does not compensate for the loss of a distance advantage over the $N^y(j_B; \sigma)$ population.

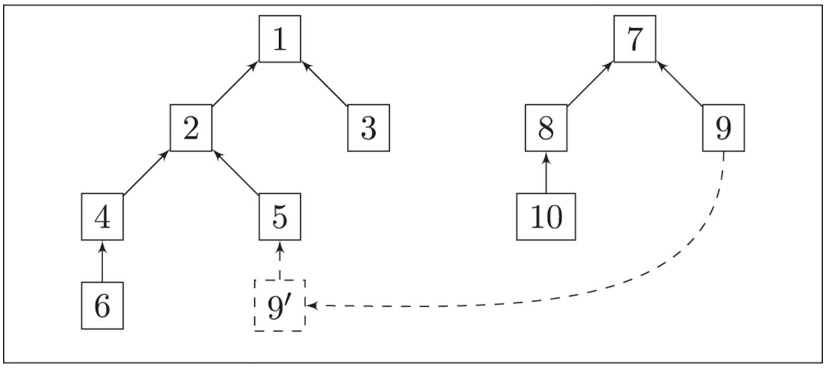


Figure 10: Example 5 of $\sigma \in H^+(i_A, i_B; g)$. With $N^{AB}(1, 7; \sigma) = \{6, 9\}$, $N^d(9; g) = \{5, 7\}$, and $N^d(10; g) = \{8, 9\}$, there is a nontrivial range for θ in which player 9 has a higher expected payoff following 7. Player 6 prefers following 4 over following 8, 9, or 10. Both 6 and 10 prefer following to leading.

Example 5.
The structure σ depicted in Figure 10 satisfies $F_B \geq 0$ with $\theta \leq 1 + \frac{m}{3}$. The $E(i_B) \geq 0$ condition is satisfied with $\theta \geq 2$. The condition $E(i_A) \geq 0$ is less stringent, requiring only that $\theta \geq 7/5$. To support $\sigma \in H^+(i_A, i_B; g)$ as an equilibrium requires $m \geq 3$. For, say, $m = 4$, then $r_c/r_e \in [6/9, 7/9]$ produces a non-empty $H^+(i_A, i_B; g)$. In this range the certainty of the conformity reward discourages player 10 from leading while the relatively high premium for leading pays enough to keep 9 from switching to the greater conformity reward offered by the larger player 1-led tree. In this example, $\sigma \in H^+(i_A, i_B; g)$ is preserved as the number of alternatives increases by a conforming r_c/r_e where $r_c/r_e \in (0, 1/3] = \lim_{m\to\infty}(\underline{\theta}/(m-1), \bar{\theta}/(m-1)]$.

5 Non-Conforming Environments

5.1 Interior Desire to Lead

Consider, again, the nonlinear payoff function of eq. (1),

$$\pi_{NL}(i; \sigma) = \phi(\mu^c_i) + \psi(\mu^e_i)$$

with $\lambda(\mu) = \phi(\mu)/\psi(\mu)$. For $\lambda'(\mu) \geq 0$, dissatisfaction with following, if present, originates with the most distant follower. In contrast, as seen in Figure 6, $\lambda'(\mu) < 0$ allows that middle distance followers may prefer leading while the most distant follower prefers following. For those players with $\mu_j^y = 0$, the decision between following and leading involves how to best position oneself to the $N^x(j;\sigma)$ population to supplement the $N^S(j;\sigma)$-assured reward.

For $\lambda'(\mu) < 0$, the importance of conformity relative to preemption decreases as μ_j^s increases so that the relative contribution of conformity is at its strongest when μ_j^s is small and becomes weaker as μ_j^s increases. A follower with $\mu_j^s > 0$ extracts the substantial component of the conformity reward from her successors. The marginal contribution of positioning the $N^x(j;\sigma)$ population to contribute to the conformity reward declines relative to the preemption reward as μ_j^s increases. Taking advantage of the certainty of her successors in establishing her own smaller hierarchy, the follower may find it beneficial to concede conformity with the larger population. With probability $1/m$ player j, as a leader of her own hierarchy, matches the choice of i thereby adding the $N^x(j;\sigma)$ population to her own $N^s(j;\sigma)$ successors for the large preemption reward.

Given $B_{NL} \geq 0$ settings with $\lambda'(\mu) < 0$ the population might tend to organize into multiple small conformity groups. Also in contrast to $\lambda'(\mu) \geq 0$, the connectivity structure of $g \in G(n)$ matters as does the particular $i \in N$ who is leader for possibly generating low μ_j^y positions for middle distance followers.

5.2 Excessive Popularity

A penalty for excessive popularity, in the nature of Arthur (1994), can leave an "excess" desire for conformity untapped, as illustrated in Example 6.

Example 6.

For $g \in G(n)$), consider a linear increasing conformity reward for a population not in excess of $n^\dagger$. Leaving in place $\psi(\mu) = r_e\mu^e$, let

$$\phi(\mu^c) = \begin{cases} r_c\mu^c & \text{for } \mu^c + 1 \leq n^\dagger \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

with $n/2 \leq n^\dagger < n - 2$. For $C(n^\dagger;\theta) > 0$, among the $n - n^\dagger$ excluded from an i_A-led tree, there remains an untapped desire to conform. For illustrative purposes, consider a second tree with an equal number of successors at each distance through the first $n_B - 1$ successors so that for $n_A > n_B$, the additional players are of greater distance from i_A than is the most distant follower of i_B. In order to have all $n - n^\dagger$ non-followers of i_A form into a second i_B structure, $\sigma \in \{H^*(i_A, i_B; g) | \mu_A^s = n^\dagger - 1\}$, requires

$$C(n^\dagger;\theta) < E(i_B;\theta) < \frac{m}{m-1}\left(1 - \frac{1}{n - n^\dagger - 1} + \frac{\mu^\alpha(j_B;\sigma)}{n - n^\dagger - 1}\right) \leq \theta. \tag{20}$$

Because there is no conformity reward when the two leaders match, the threshold on θ is higher than that produced by the condition $E(i_B) \geq 0$. When this condition does not hold, there is no interior value to $0 < \mu_B^s$ that is an equilibrium.[22]

5.3 Sequential Play

5.3.1 Subgame Perfect Hierarchies

The single leader structure can also be supported as a subgame perfect equilibrium (SPE) in a game with σ established through sequential moves. Typically, the first mover can establish herself as the leader and the remaining population adopts the following strategy to best accommodate this reality, generating the same single-leader Nash equilibrium structures produced by simultaneous play. There are instances, though, in which the first mover does not end up as the leader in the equilibrium structure, as illustrated in Example 7 below. The inability of player i to lead in a SPE indicates a susceptibility of the Nash equilibrium $\sigma \in H^*(i;g)$ to disruption by a player whose deviation from Nash equilibrium play, in a cascade of best responses, would lead to a different Nash equilibrium $\sigma' \in H^*(j;g)$ for $j \neq i$ favored by the original deviant player.

Example 7.

Figure 11 depicts three structures based on contact network g as presented in Figure 11a. Each member of the population has two contacts. Structures $\{\sigma^1\} = H^*(i;g)$ and $\sigma^3 \in H^*(j;g)$ are both Nash equilibria. Given i,

j, x, s_1, and s_2 as the order of play, σ^3, rather than σ^1, is the SPE despite i's first mover option to establish herself as a leader. The best response to player i leading produces $\sigma^2 \in h_L^*(j, \mu^s(j); g)$ in which both i and j lead but with j attracting all of the followers. Two features present in σ^1 and g are essential to exclude it from the set of SPE. First, player j has an advantage in attracting followers despite i moving first. Because s_1 and s_2 have no choice but to follow j, j has the larger population of followers regardless of x's decision regarding whom to follow. This compels x to follow j. Second, the potential defector from the actions producing σ^1 must be motivated to defect despite i's lead as is true here with $\pi(j; \sigma^2) > \pi(j; \sigma^1)$. The motivation in this example comes from player x. Player x best responds by following j when both i and j lead.

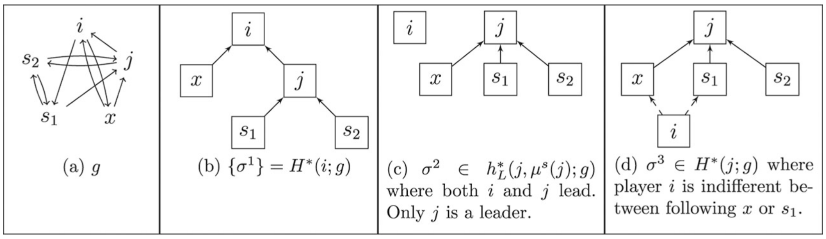


Figure 11: An example for which a $\sigma^1 \in H^*(i; g)$ is not a subgame perfect equilibrium. The tree structures of σ^1 and σ^3 are both Nash equilibrium structures based on g. For moves in the order i, j, x, s_1, then s_2, only σ^3 is a SPE. The structure σ^2 reflects the best response by players x, s_1, and s_2 if faced with both i and j leading. The fact that j prefers σ^2 to σ^1 undermines player i's leadership when considering a cascade of best responses.

Similar to the best response cascade discussed in Heal and Kunreuther (2010), were the population to start from σ^1, the cascade of best responses to the single deviation by player j transitions the population from σ^1 to σ^3. A Nash equilibrium can be susceptible to a best response cascade that just as easily reverses in direction to return to the original structure.[23] This does not contribute to identifying the more "fragile" Nash equilibria prone to transition to a more stable alternative Nash equilibrium. The SPE offers such a refinement for identifying fragile Nash equilibria subseptable to irreversible best-response cascades.

5.3.2 Multiple Hierarchies as Subgame Perfect

Recall Example 5 illustrating a multiple hierarchy structure as a Nash equilibrium with a single follower in the smaller hierarchy preferring the *status quo* in order to retain a distance advantage over the non-empty $N^y(j; \sigma)$ population. Refer to Figure 10. Reconsidered as a SPE, the same structure cannot be supported as an equilibrium. As the only conduit through which player 7 and the $N^S(7, \sigma)$ population can join $N^S(1, \sigma)$, player 9's switch to join the player 1-led tree enables the remainder of $N^S(7, \sigma)$ to also join in following $N^S(1, \sigma)$ in a cascade of best responses. The process is facilitated by player 10 who prefers the i-led tree. Irregardless of the order of play, player 10 imitates 9, allowing 9 to join the i-led hierachy. In addition to joining a larger hierarchy, player 9 now also precedes player 7 and the entire former $N^S(7, \sigma)$ population for a higher reward than the original $\sigma \in H^+(i_A, i_B; \sigma)$.

6 Conclusion

Katz (1957) identifies influence as being related to (1) a personification of certain values (who one is), (2) competence (what one knows), and (3) strategic social location (whom one knows). In the developed model, the leader/follower social structure is supported by the population's desire for conformity and influence such that a leader finds support among followers without employing specialized skills. Depending on the relative strength of these two desires, the social structure may produce full conformity or may support a central conforming population surrounded by unaffiliated autonomous decision-makers. Personal contacts provide scaffolding upon which the population-established information pathways facilitate both informed decisions and channels with which to exert influence. With the tacit support of the entire population, a leader identifies the choice for adoption. The choice disseminates to, and through, followers via a network of imitations.

A preference for appearing influential means that the number of followers and whether multiple leaders can be present in equilibrium depends on the tradeoff between following an existing leader or acting autonomously in the presence of that leader. The parameter θ captures this tradeoff in the linear rewards model. Interestingly, the term is relevant to the decisions of the population's followers, not its leader(s). Everyone wants to be the leader. It is the willing participation of the followers that makes the structure an equilibrium.

Left unresolved in the current analysis is the process by which the coordinating Nash equilibrium structure can emerge. The substantial coordination involved, confounded by the asymmetry of the equilibrium payoff, makes the realization of an equilibrium structure in a single round of play highly unlikely. The analysis developed here rests on the possibility that coordination can emerge as the consequence of building consistency in player relationships. Computational analysis points to processes by which the coordinating structure of the static Nash equilibrium solution can emerge as the consequence of reactive path-dependent repeated play.

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Appendix

A Foundations

Formally, define

- $h(i;g) = \{\sigma | i \in N^L, N^S(i;\sigma) \neq \emptyset\}$ as the set of structures in which i leads;
- $H(i;g) = \{\sigma | N^L(\sigma) = \{i\}, N^S(i;\sigma) = N \backslash \{i\}\}$ as the set of structures in which i uniquely leads;
- $h_L(i, \mu_i^s; g) = \{\sigma \in h(i;g) | N^L(\sigma) = N \backslash N^S(i;\sigma)\}$ as the set of structures in which i has μ_i^s followers and is the unique leader;
- $h(i_A, i_B; g) = \{\sigma \in h(i_A;g) \cap h(i_B;g)\}$ as the set of structures in which $\{i_A, i_B\}$ are leaders;
- $H(i_A, i_B; g) = \{\sigma \in h(i_A, i_B; g) | N^L(\sigma) = \{i_A, i_B\}\}$ as the set of structures in which only $\{i_A, i_B\}$ lead and are leaders;
- $N^c(i;a) = \{j \in N \backslash \{i\} | o_i = o_j\}$ as, for action profile a, the set of conforming adopters;
- $N^e(i;a) = \{j \in N^c(i;a) | d_j > d_i\}$ as, for action profile a, the set of ensuing adopters;
- $N^S(i;\sigma) = \{j \in N | \sigma_{ji} = 1 \text{or } \sigma_{jj_1} = \ldots = \sigma_{j_\tau i} = 1\}$ as, for structure σ, the set of players who are successors to i;
- $N^L(\sigma) = \{j \in N | \sigma_{jj} = 1\}$ as, for structure σ, the set of players who lead;
- $N^x(j;\sigma) = \{j_x \in N^S(i;\sigma) | d_{xi} \leq d_{ji}\}$ as, for structure σ, the set of players who are as close or closer to leader i as is j;
- $N^y(j;\sigma) = \{j_y \in N^S(i;\sigma) \backslash N^S(j,\sigma) | d_{yi} > d_{ji}\}$ as, for structure σ, the set of players who are farther from leader i than is j but not successor to j;
- $N^\alpha(j_h;\sigma) = \{j_\alpha \in N^S(i_{-h};\sigma) | d_{j_\alpha i_{-h}} \leq d_{j_h i_h}\}$ as, for structure σ, the set of players who are as close or closer to leader i_{-h} as is j to i_h;
- $N^\beta(j_h;\sigma) = \{j_\beta \in N^S(i_{-h};\sigma) \backslash N^S(j,\sigma) | d_{j_\beta i_{-h}} > d_{j_h i_h}\}$ as, for structure σ, the set of players who are farther from leader i_{-h} than is j to i_h;
- $N^{AB}(i_A, i_B;\sigma) = \{j | N^d(j;g) \cap \{i_A, N^S(i_A;\sigma)\} \neq \emptyset, N^d(j;g) \cap (i_B, N^S(i_B;\sigma)\} \neq \emptyset\}$ as, for structure σ, the set of players with potential links to members of both of the i_A-led tree and the i_B-led tree;

and recognize that for $g \in G(n)$, $N^d(i_h; g) \cap \{N^S(i_{-h}; \sigma), i_{-h}\} = \emptyset$ implies $N^{AB}(i_A, i_B; \sigma) \cap N^S(i_h; \sigma) \neq \emptyset$.

An $^{(*)}$ on the set of structures indicates that all followers imitate the contact offering the shortest distance to the leader, that is, $a_j = \arg\min_{N^d(j;g)} d_{ji}\ \forall j \in N^s(i; \sigma)$. The sets $h_L^*(i; g)$ and $h^*(i_A, i_B; g)$ have the additional condition that the $N^l(\sigma)$ population is at least as distant from the leader as is the most distant follower, measured on g, $d_{ij} \geq d_{i\bar{j}(\mu^s_{i_h})}$ for $j \in N^l(\sigma)$, $h = \emptyset, A, B$.

Utility of Interactions

Individuals face a discrete choice in which they receive utility from the interaction between their own choice and the choices of other members in the population. Let the $m \times \bar{d}$ matrix ω_i denote the adoption of an option with element $w_{i,o,d} = 1$ if player i adopts option $o_i \in O$ at distance $d_i = d$. Otherwise, $\omega_{i,o,d} = 0$. Let $\omega_{-i} = (\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_n)$ represent the choices of all agents other than i. Individual utility can be defined broadly as the sum of three elements:

$$V(\omega_i) = u(\omega_i) + S(\omega_i, \omega_{-i}) + \epsilon(\omega_i).$$

The current analysis considers only the social utility associated with a choice, $S(\omega_i, \omega_{-i})$, setting the innate preferences over the different options, $u(\omega_i)$, and the idiosyncratic random element of utility, $\epsilon(\omega_i)$, each to zero.[24]

Let the $n \times \bar{d}$ matrix Ω_i denote the possession of an option with element $\Omega_{i,o,d} = 1$ when player i adopts option $o_i \in O$ at distance $d_i \leq d$. Otherwise, $\Omega_{i,o,d} = 0$. Let

$$\mu_i = \sum_{j \neq i} \omega_j$$

and

$$\nu_i = \sum_{j \neq i} \Omega_j$$

so that μ_i denotes the aggregate choice for each option over all distances and ν_i denotes the cumulative aggregate choice at each distance.

The complementarities of the social choice depend only on the two measures of popularity,

$$\mu_i^c = \mathbf{1}'\mu_i'\omega_i\mathbf{1}$$

and

$$\mu_i^e = \mu_i^c - \mathbf{1}'\omega_i\nu_i'\omega_i\mathbf{1}.$$

Let

$$S(\omega_i, \mu_i^c, \mu_i^e) = \phi(\mu_i^c) + \psi(\mu_i^e),$$

then linearity with $\phi(x) = r_c x$ and $\psi(x) = r_e x$ produces constant cross partials

$$\frac{\partial^2 S(\omega_i, u_i^c, \mu_i^e)}{\partial \omega_{i,o,d} \partial \mu_{i,o,d}} = r_c \text{ and } \frac{\partial^2 S(\omega_i, u_i^c, \mu_i^e)}{\partial \omega_{i,o,d} \partial \nu_{i,o,d}} = r_e, \forall i, o, d$$

so that dependence across players is captured by the two constant coefficients.

B Propositions, Lemmas, and Proofs

Formal Statement and Proof of Proposition 1 and Corollary 1

Proposition 1.

For $\sigma \in H(i; g)$, $\sigma' \in h^-(i, \sigma; g)$, and $\lambda'(\mu) \geq 0$, then $\pi_{NL}(j, \sigma) \geq \pi_{NL}(j, \sigma')$ for all $j \in N^S(i; \sigma)$ if and only if $B_{NL} \geq 0$.

Proof.

Let σ_{-j} indicate the strategies of all players in $N \setminus \{j\}$. For $\sigma \in H(i;g)$, let $\sigma' = \sigma_j' \times \sigma_{-j}$ and $\sigma_{jj}' = 1$ producing $\sigma' \in h^-(i,\sigma;g)$. Let $\mu_j^h = \mu^h(j;\sigma) = |N^h(j;\sigma)|$ for $h = x, y, s$ so that relational populations are identified according to the structure σ. Recall $\phi'(\mu) > 0$ and $\psi'(\mu) > 0$. For player $j \in N^S(i;\sigma)$,

$$\pi(j;\sigma) = \phi(n-1) + \psi(\mu_j^y + \mu_j^s).$$

When leading, uncertainty in the outcome of whether $o_i = o_j$ generates uncertainty in j's payoff. Expectations are taken over the possible realization of o_i and o_j with

$$\mathbb{E}(\pi(j;\sigma')) \quad = \tfrac{1}{m}(\phi(n-1) + \psi(\mu_j^x + \mu_j^y + \mu_j^s)) + \tfrac{m-1}{m}(\phi(\mu_j^s) + \psi(\mu_j^s)). \tag{21}$$

The condition $A_{NL}(j;\sigma) \geq 0$, derived from $\mathbb{E}(\pi(j;\sigma) - \pi(j;\sigma')) \geq 0$, ensures that player $j \in N^S(i;\sigma)$ prefers her position as a follower of i over leading.

The condition $B_{NL} \geq 0$ is equivalent to $A(\bar{j};\sigma) \geq 0$ for $\bar{j} = \text{argmax}_{j \in N^S(i;\sigma)} d_{ji}$. For $\bar{j}$, $\mu^y(\bar{j}) = \mu^s(\bar{j}) = 0$, leaving $A_{NL}(\bar{j};\sigma) = ((m-1)/m)\phi(n-1) - (1/m)\psi(n-2) \geq 0$, or

$$A_{NL}(\bar{j};\sigma) \quad = A_1 + A_3(0) \geq 0.$$

The first term is strictly positive. $B_{NL} \geq 0$ implies $A_3(0) \geq -A_1$. For follower j, $A_{NL}(j;\sigma)$ is as defined in eq. (5).

That $A_3(\mu) \geq -A_1$ for all $\mu \in [0, n-2]$ is a necessary and sufficient condition for $A(j;\sigma) \geq 0$ for all j. Given $A(\bar{j};\sigma) \geq 0$, a sufficient condition is that $A_3(\mu)$ remain everywhere above a monotonic function passing through $A_3(0)$ and $A_3(n-2)$. Observe,

$$A_3(\mu) = \frac{1}{m}\left((m-1)\phi(n-2) - \psi(n-2) - ((m-1)\phi(\mu) - \psi(\mu))\right)$$

and

$$A_3(0) = \frac{1}{m}\left((m-1)\phi(n-2) - \psi(n-2)\right).$$

Since $\lambda'(\mu) \geq 0$ implies

$$\frac{\phi(\mu)}{\psi(\mu)} \leq \frac{\phi(n-2)}{\psi(n-2)},$$

for $\lambda'(\mu) < 0$,

$$\begin{aligned} A_3(\mu) \quad &= A_3(0) - ((m-1)\phi(\mu) - \psi(\mu)) \\ &= A_3(0) - \left((m-1)\tfrac{\phi(\mu)}{\psi(\mu)} - 1\right)\psi(\mu) \\ &> A_3(0) - \left((m-1)\tfrac{\phi(n-2)}{\psi(n-2)} - 1\right)\psi(\mu) = A_3^0(\mu). \end{aligned}$$

$A_3^0(\mu)$ is an afine transformation of $\psi(\mu)$

$$\begin{aligned} A_3^0(\mu) \quad &= A_3(0) - \left(((m-1)\phi(n-2) - \psi(n-2))\tfrac{\psi(\mu)}{\psi(n-2)}\right) \\ &= A_3(0) - \left(A_3(0)\tfrac{\psi(\mu)}{\psi(n-2)}\right) \\ &= A_3(0)\left(1 - \tfrac{\psi(\mu)}{\psi(n-2)}\right). \end{aligned}$$

Corollary 1.

For $\sigma \in H(i;g)$, $\sigma' = \{h^-(i,\sigma;g) | \sigma_{jj}' = 1\}$, and $B_{NL} \geq 0$, if $\lambda'(\mu) < 0$ then $\pi_{NL}(j,\sigma) < \pi_{NL}(j,\sigma')$ is possible for some $j \in N^S(i;\sigma) \setminus \{\bar{j}\}$.

Proof.

For $\lambda'(\mu) < 0$ so that

$$\frac{\phi(\mu)}{\psi(\mu)} > \frac{\phi(n-2)}{\psi(n-2)},$$

then $A_3(\mu) < A_3^0(\mu)$. While $A_3(\mu) \geq -A_1$ remains possible, it is no longer assured by $A_3(0) \geq -A_1$.

Evaluation of Proposition 1 with Linear Payoff

Proof.

Let σ_{-j} indicate the strategies of all players in $N\setminus\{j\}$. For $\sigma \in H(i;g)$, let $\sigma' = \sigma_j' \times \sigma_{-j}$ and $\sigma_{jj}' = 1$ producing $\sigma' \in h^{-}(i,\sigma;g)$. Let $\mu_j^h = \mu^h(j;\sigma) = |N^h(j;\sigma)|$ for $h = x, y, s$. For player $j \in N\setminus\{i\}$,

$$\mathbb{E}(\pi(j;\sigma)) = r_c(\mu_j^x + \mu_j^y + \mu_j^s + 1) + r_e(\mu_j^y + \mu_j^s). \tag{22}$$

The payoff to j when leading is uncertain due to the uncertainty in the outcome of whether $o_i = o_j$.

$$\mathbb{E}(\pi(j;\sigma')) \;\; = \tfrac{1}{m}((r_c + r_e)(\mu_j^x + \mu_j^y + \mu_j^s) + r_c) + \tfrac{m-1}{m}(r_c + r_e)\mu_j^s \tag{23}$$

$$= (r_c + r_e)\mu_j^s + \tfrac{1}{m}((r_c + r_e)(\mu_j^x + \mu_j^y) + r_c).$$

The condition $A(j;\sigma) \geq 0$, derived from $\mathbb{E}(\pi(j;\sigma) - \pi(j;\sigma')) \geq 0$, ensures that player $j \in N^S(i;\sigma)$ prefers her position as a follower of i over leading.

The first term of $A(j;\sigma)$ as expressed in eq. (8) is strictly positive. The coefficient on the second term is also positive. For $\theta = (m-1)r_c/r_e > 1$ the third coefficient is also positive making it a sufficient condition for $A(j;\sigma) > 0$ for all $j \in N\setminus i$. The necessary and sufficient condition ensuring $A(j;\sigma) \geq 0$ for all $j \in N\setminus i$ sets a lower threshold on θ, allowing the third term to be negative. For

$$(m-1)r_c \geq -((m-1)r_c - r_e)(n-2)$$

or equivalently, $\theta \geq 1 - \frac{1}{n-1}$, $A(j;\sigma) > 0$ for all j since $\mu^x(j) \leq \mu^x(\bar{j}) = n-2$ and $\mu^y(j) \geq \mu^y(\bar{j}) = 0$.

Formal Statement of Lemma 1

Lemma 1.

For $\{\sigma, \sigma'\} \in H(i;g)$ with $\sigma_{-j} = \sigma_{-j}'$ *and* $\{a_j, a_j'\} \in N^d(j;g)$, *then for* $\mu^x(j;\sigma) \leq \mu^x(j;\sigma')$,

$$\pi(j;\sigma) \begin{cases} = \pi(j;\sigma') & \text{if } \mu^x(j;\sigma) = \mu^x(j;\sigma'), \\ > \pi(j;\sigma') & \text{if } \mu^x(j;\sigma) < \mu^x(j;\sigma'). \end{cases}$$

Formal Statement and Proof of Lemma 2

Lemma 2.

$\sigma \in h'(i;g)$ is a necessary condition for $\sigma \in h(i;g)$ *to be a Nash equilibrium.*

Proof.

For player i, leading dominates following since to choose one's own successor as a predecessor pays zero. From $\mu^e(j) = \mu_j^y + \mu_j^s$ and $\mu_j^y + \mu_j^s = \mu_i^s - \mu_j^x$, decreasing μ_j^x increases $\pi_{NL}(j;\sigma)$ for any reward function that is increasing in μ^e. Among the following options, a player can do no better than to minimize μ_j^x. A player who is not minimizing μ_j^x is not optimizing against her available following options. Thus, any structure $\sigma \in h(i;g)\setminus h'(i;g)$ cannot be a Nash equilibrium.

The $B_{NL} \geq 0$ application of Lemma 2 is to $\sigma \in H'(i;g)$. For $\sigma \in H'(i;g)$ each player is optimizing from the set of strategies that preserve $\sigma \in H(i;g)$. Minimizing μ_j^x is also a necessary attribute of $h_L^*(i,\bar{n};g)$ for optimizing behavior under $B_{NL} < 0$.

Formal Statement and Proof of Proposition 2

Proposition 2.

Given $\lambda'(\mu) \geq 0$, $\{H'(i;g)\}_{i\in N}$ *a set of equilibrium structures if and only if* $B \geq 0$.

Proof.

From Proposition 1, given $B_{NL} \geq 0$, every player $j \in N \backslash \{i\}$ prefers any structure $\sigma \in H(i;g)$ over the structure produced by player j's deviation to lead. In combination with Lemma 2, $B_{NL} \geq 0$ implies that no follower in the population can do better for herself than to minimize her μ_j^x.

Corollary 2.

Given $\lambda'(\mu) \geq 0$, $\{H^*(i;g)\}_{i \in N}$ is a set of equilibrium structures if and only if $B_{NL} \geq 0$.

Proof

$H^*(i;g) \subseteq H'(i;g)$ implies that for $\sigma \in H(i;g)$ and $a_j \in N^d(j;g)$, if $a_j = \text{argmin}_{N^d(j;g)} d_{ji}$, then $a_j = \text{argmin}_{N^d(j;g)} \mu_j^x$. As further distinction between the strategies, structure $\sigma' \in H'(i;g)$ if $\sigma' \in H^*(i;g)$ or if $\sigma' = \sigma_{-j} \times \sigma_j^{'}$ with $a_j^{'} = j'$ where $\sigma \in H'(i;g)$ and where $j \in N^S(i;\sigma)$ satisfies the following three properties:

1. There exists $j' \in N^d(j,g)$ with $d_{j'i} = d_{ji}$, indicating that j' is equidistant to the leader as is j and that j has the option to imitate j',
2. $\mu^y(j;\sigma) = 0$, indicating that there are no successors to i of greater distance to i than j without also being a successor to j, and
3. either $\mu^s(j;\sigma) = 0$ or $\mu^s(j;\sigma) > 0$ with successors $N^S(j;\sigma)$ having no option to link to i but through j.

For $\{j_1, j_2\} \in N^d(j;g)$ with $d_{j_1 i} < d_{j_2 i}$, let $\sigma^h = \sigma|\sigma_{jj_h} = 1$, $h = 1, 2$, so that $\mu^x(j,\sigma^1) \leq \mu^x(j,\sigma^2)$. The condition that allows $\mu^x(j,\sigma^1) = \mu^x(j,\sigma^2)$ is $\mu_j^y = 0$. With $j_2 \notin N^S(j;\sigma)$, $\mu_j^y = 0$ implies $j_2 \in N^x(j;\sigma)$ and $d_{j_2 i} = d_{ji} = d_{j_1 i} + 1$. For $\sigma^1 \in H^{'}(i;g)$, a necessary and sufficient condition to have $\sigma^2 \in H^{'}(i;g)$ is that for all $j_s \in N^S(j;\sigma^1)$, $N^d(j_s;g) \subset \{N^S(j;\sigma) \cup \{j\}\}$. The condition establishes that no successor of j has the option to link to i without having the chain of links pass through j, a condition necessary to ensure that $\mu^x(j_s;\sigma^2)$ is minimized for all j_s.

From $H^*(i;g) \subseteq H'(i;g)$, $\sigma \in H^*(i;g)$ is an equilibrium if $B_{NL} \geq 0$ and $\sigma \in H^*(i;g)$ is not an equilibrium if $B_{NL} < 0$.

Formal Statement and Proof of Proposition 3

Proposition 3.

For $B < 0$ and $\sigma \in h_L(i, \mu_i^s; g)$, all $j \in N^S(i;\sigma)$ prefer following i to leading and the remaining population, $j \in N \backslash \{i, N^S(i;\sigma)\}$, prefer leading to following i if and only if $\sigma \in h_L^*(i, \bar{n}; g)$.

Proof.

Let $\mu_j^h = \mu^h(j;\sigma) = |N^h(j;\sigma)|$ for $h = x, y, s$ and $\mu^l = \mu^l(\sigma) = |N^L(\sigma)|$. For $\sigma \in h_L(i, \mu_i^s; g)$, let $\sigma^{'} = \sigma_j^{'} \times \sigma_{-j}$ and $\sigma_{jj}^{'} = 1$, $j \in N^S(i;\sigma)$. Uncertainty in the payoff to j when following stems from the uncertainty in whether $o_i = o_l$ for each $l \in N^L(\sigma) \backslash \{i\}$ with,

$$\mathbb{E}(\pi(j;\sigma)) = r_c(\mu_j^x + \mu_j^y + \mu_j^s + 1) + r_e(\mu_j^y + \mu_j^s) + \frac{1}{m} r_c(\mu^l - 1). \tag{24}$$

The payoff to j when leading is uncertain due to the uncertainty in the outcome of whether $o_j = o_l$ for each $l \in N^L(\sigma)$ with,

$$\mathbb{E}(\pi(j;\sigma')) = (r_c + r_e)\mu_j^s + \frac{1}{m}(r_c + r_e)(\mu_j^x + \mu_j^y) + \frac{1}{m} r_c(\mu^l(\sigma)). \tag{25}$$

$\mathbb{E}(\pi(j;\sigma) - \pi(j;\sigma'))$ from eqs. (24) and (25) is the same as from eqs. (22) and (23) when expressed in terms of μ_j^x, μ_j^y, and μ_j^s as in eq. (8).[25] The presence of a population of autonomous adopters does not alter the condition $A(j;\sigma) \geq 0$ for player j to prefer following to leading. Let $\bar{j}(\mu_i^s) = \text{argmax}_{j \in N^S(i;\sigma)} d_{ji}$, then $\mu^y(\bar{j}(\mu_i^s)) = \mu^s(\bar{j}(\mu_i^s)) = 0$ and $\mu^x(\bar{j}(\mu_i^s)) = \mu_i^s - 1$ so that

$$A(\bar{j}(\mu_i^s);\sigma) = \frac{1}{m}((m-1)r_c + ((m-1)r_c - r_e)(\mu_i^s - 1)) \tag{26}$$

and $C(\mu_i^s;\theta) = A(\bar{j}(\mu_i^s);\sigma)m/r_e\mu_i^s$. With $B < 0$, $((m-1)r_c - r_e) < 0$ so that $A(\bar{j}(\mu_i^s);\sigma)$ decreases as the size of the tree increases. For $\mu_i^s = 1$, $A(\bar{j}(1),\sigma) = (m-1)r_c > 0$ while $B < 0$ means that for $\mu_i^s = n-1$, $A(\bar{j}(n-1);\sigma) < 0$.

For $m = 1$, $A(j;\sigma) = -r_e\mu_j^x < 0$. For $r_c = 0$, $A(j;\sigma) = r_e((m-1)\mu_j^y - \mu_j^x)$ so that the most distant follower, with $A(\bar{j}(\mu_i^s);\sigma) = -r_e(\mu_i^s - 1) \leq 0$, prefers to lead in the presence of other followers. Player $\bar{j}$ is indifferent to leading only when she is the only follower, $m = 1$, and $r_c = 0$. With a non-trivial choice $(m < 1)$ and a preference for conformity $(r_c > 0)$, the equilibrium structure requires $\mu_i^s \geq 1$.

The value of μ_i^s that sets $C(\mu_i^s;\theta) = 0$ need not be an integer. There exists $\bar{n} \in \{\text{floor}(\mu^*), \text{ceil}(\mu^*)\}$ such that $A(j(\bar{n});\sigma) \geq 0$ and $A(j(\bar{n}+1);\sigma) < 0$. A structure $\sigma \in h_L(i,\bar{n};g)\backslash h_L^*(i,\bar{n};g)$ cannot be an equilibrium because either there are members of $N^S(i;\sigma)$ able to improve their payoff by choosing a different predecessor offering a shorter distance to i or there is a member of $N^L(\sigma)$ able to improve her payoff by choosing to follow a predecessor offering a shorter distance to i than the current $\bar{j}(\mu_i^s)$ player. For $\sigma \in h_L^*(i,\bar{n};g)$, no player is able to improve her payoff through unilateral deviation while preserving a single-leader structure.

To extend Proposition 3 to the nonlinear reward setting of eq. (1), let A_{NL}^l represent the expected payoff differential for following over leading in the presence of a non-empty autonomous $N^L(\sigma)\backslash\{i\}$ population. Then

$$A_{NL}^l(j;\sigma) = A_{NL}(j;\sigma) + A_4(\mu_j^s).$$

$A_4(\mu_j^s)$ is a term capturing the net following over leading expected contributions of the $N^L(\sigma)$ population for follower j. Because expectations are being taken over nonlinear functions, each possible outcome requires a separate term in a large $N^L(\sigma)$ population. As a simple illustration, consider a single autonomous adopter so that $\mu^l = 2$. Then,

$$A_4(\mu_j^s) = \frac{m-1}{m^2}((\phi(\mu_i^s + 1) - \phi(\mu_i^s)) - (\phi(\mu_j^s + 1) - \phi(\mu_j^s))). \tag{27}$$

Given leader i and follower j, let l identify the autonomous agent. The first inner parenthetical term of eq. (27) captures the value to j of matching with l when already adopting the same alternative as i, either as a follower of i or as a leader having also matched with i. The second inner parenthetical term is the value to j of matching with l when not adopting the same alternative as i. Here, and in general with $\mu^l \geq 3$ as well, $A_4(\mu_j^s)$ is positive and decreasing in μ_j^s for $\phi''(\mu) > 0$, zero for $\phi''(\mu) = 0$, and negative and increasing for $\phi''(\mu) < 0$. The condition $\phi''(\mu) \leq 0$ ensures that $\bar{j}$ remains the marginal decision-maker since $-A_4(\mu_j^s)$ is at its maximum at $\mu_j^s = 0$.

Formal Statement and Proof of Proposition 4

Proposition 4.
$\{h_L^*(i,\bar{n};g)\}_{i\in N}$ *is the set of equilibrium strategies if and only if* $B < 0$.

Proof.
Let $\mu_h^s = \mu^s(i_h)$. For $\sigma \in h^*(i_A, i_B;g)$, let $\sigma' = \sigma_{-j} \times \sigma_j'$, with $\sigma_{jj}' = 1$, $j \in N^S(i;\sigma)$. For j, the expected payoff for following and leading are, respectively,

$$\mathbb{E}(\pi(j;\sigma)) = r_c(1 + \mu_j^x + \mu_j^y + \mu_j^s) + r_e(\mu_j^y + \mu_j^s) \tag{28}$$

$$+\tfrac{1}{m}(r_c(\mu^l(\sigma) - 1 + \mu_j^\alpha + \mu_j^\beta) + r_e(\mu_j^\beta)),$$

$$\mathbb{E}(\pi(j;\sigma')) = (r_c + r_e)\mu_j^s \tag{29}$$

$$+\tfrac{1}{m}((r_c + r_e)(\mu_j^x + \mu_j^y + \mu_j^\alpha + \mu_j^\beta) + r_c\mu^l(\sigma)) \tag{29}$$

where $\mu_j^h = \mu^h(j;\sigma) = |N^h(j;\sigma)|$ for $h = x, y, s, \alpha, \beta$ and $\mu^l = \mu^l(\sigma) = |N^L(\sigma)|$. Observe, for $h = A, B$,

$$\underbrace{1 + \mu_j^x + \mu_j^y + \mu_j^s}_{=\mu_h^s} + \underbrace{\mu_j^\alpha + \mu_j^\beta}_{=\mu_{-h}^s} + \mu^l = n.$$

The condition $\mathbb{E}(\pi(j;\sigma) - \pi(j;\sigma')) \geq 0$ implies $D(j_h;\sigma) \geq 0$ as reported in eq. (12). For the most distant player(s) from i_h according to σ, $E(i_h;\sigma) = D(\bar{j}(\mu_h^s);\sigma)/r_e\mu_h^s$. With $\mu^y(\bar{j}(\mu_h^s)) = \mu^s(\bar{j}(\mu_h^s)) = 0$, $\mu^x(\bar{j}(\mu_h^s)) = \mu_h^s - 1$, and $\mu^{\alpha}(\bar{j}(\mu_h^s)) \geq \mu^{\alpha}(j_h)$ for all $j_h \in N^S(i_h;\sigma)$, $D(\bar{j}(\mu_h^s);\sigma) \geq 0$ implies $D(j_h;\sigma) \geq 0$ for all $j \in N^S(i_h;\sigma)$, so that $E(i_h;\sigma) \geq 0$ is necessary and sufficient to ensure $D(j_h;\sigma) \geq 0$ holds for all $j_h \in N^S(i_h;\sigma)$. Since

$$\left(1 + \frac{\mu_h^{\alpha}(j_{\mu_h^s}) - 1}{\mu_h^s}\right) \geq 1 > \left(1 - \frac{1}{n-1}\right),$$

the condition $E(i_h;\sigma) \geq 0$ violates $B > 0$. For $\sigma \in h_L^*(i,\bar{n};g)$, no player is able to improve her payoff through unilateral deviation.

To extend Proposition 4 to the nonlinear reward setting of eq. (1), let D_{NL} represent the expected payoff differential for following over leading in the presence of a two leaders, i_A and i_B. As reference for $j_h \in N^S(i_h;\sigma)$, let $\sigma' \in h_L(i;g)$ have a tree under i that matches the tree structure under i_h according to σ and where all non-members of $N^S(i_h;\sigma)$ adopt autonomously (rather than following i_{-h}). For follower j_h,

$$\begin{aligned}
D_{NL}(j;\sigma) &= A_{NL}(j;\sigma') + \tfrac{m-1}{m^2}D_1 + \tfrac{1}{m^2}D_2\\
D_1 &= \phi(\mu_h^s + \mu_{-h}^s + 1) - \phi(\mu_h^s) - (\phi(\mu_j^s + \mu_{-h}^s + 1) - \phi(\mu_j^s))\\
D_2 &= \psi(\mu_j^y + \mu_j^z + \mu_j^{\beta}) - \psi(\mu_j^y + \mu_j^z)\\
&\quad -\{\tfrac{m-1}{m}(\psi(\mu_j^z + \mu_{-h}^s) - \psi(\mu_j^z)) + \tfrac{1}{m}(\psi(\mu_{-h}^s + \mu_h^s - 1) - \psi(\mu_h^s - 1))\}
\end{aligned}$$

For j considering whether to lead or follow, D_1 is the conformity contribution of joining the i_{-h}-led hierarchy when already affiliated with the i_h-led hierarchy (as a follower or by independently matching) less conformity contribution of joining the i_{-h}-led hierarchy when not affiliated with the i_h-led hierarchy. D_2 is the ensuing contribution of matching with the i_{-h}-led hierarchy as a follower of i_h less the expected ensuing contribution of matching with the i_{-h}-led hierarchy when leading (made up of matching with just i_{-h} and matching with both i_{-h} and i_h). $\bar{j}(\mu_h^s)$ remains the marginal decision-maker in the i_h-led hierarchy for $\lambda'(\mu) \geq 0$ and $\phi''(\mu) \leq 0$ (conditions that combined to also require $\psi''(\mu) \leq 0$).

If linearized, $D_{NL}(j;\sigma)$ collapses to $A(j;\sigma') - r_e\mu_j^{\alpha}/m = D(j;\sigma)$.

Formal Statement and Proof of Proposition 5

Proposition 5.

For

$$\begin{aligned}
H^+(i_A, i_B; g) &= \{\sigma \in H^*(i_A, i_B; g)\}\\
\text{such that}\quad & N^d(i_h;g) \cap \{N^S(i_{-h};\sigma), i_{-h}\} = \emptyset,\\
& E(i_h, \mu_h^s, \theta, \sigma) \geqslant 0,\\
& F_h(j_h;\theta, m, \sigma) \geqslant 0 \text{ for all } j \in N^{AB}(i_A, i_B;\sigma),
\end{aligned}$$

a structure $\sigma \in H(i_A, i_B; g)$ is a Nash equilibrium if and only if $\sigma \in H^+(i_A, i_B; g)$. The set $H^+(i_A, i_B; g)$ is feasibly non-empty.

Proof.

For $\sigma \in H(i_A, i_B; g)$, without loss of generality, let $\mu_A^s \geq \mu_B^s$. With $g \in G(n)$, $\{i_h \cup N^{AB}(i_A, i_B;\sigma)\} \cap \{i_{-h} \cup N^S(i_{-h};\sigma)\}$, $h = A, B$ are both nonempty sets. The compliments, $\{i_h, N^S(i_h;\sigma)\}\backslash N^{AB}(i_A, i_B, \sigma)$ for $h = A, B$, can be nonempty, indicating that possibly i_h and some $j \in N^S(i_h;\sigma)$ have no direct potential link to $\{i_{-h}, N^S(i_{-h};\sigma)\}$ with the current σ.

For player $j_h \in N^S(i_h;\sigma)$, the expected payoff for remaining a follower in the i_h-led tree is $\mathbb{E}(\pi_h(j;\sigma))$ as expressed in eq. (28). Let $\sigma'_{h\to -h} = \sigma_{-jh} \times \sigma'_{jh}$, with $j_h \in N^S(i_{-h};\sigma'_{h\to -h})$. That is, $\sigma'_{h\to -h} \in H^*(i_A, i_B; g)$ represents the alternative to $\sigma \in H^*(i_A, i_B; g)$ based on a switch by player $j_h \in N^{AB}(i_A, i_B;\sigma) \cap N^S(i_h;\sigma)$ from the i_h-led tree to the i_{-h}-led tree. Compute

$$\begin{aligned}
\mathbb{E}(\pi(j_h;\sigma) - \pi(j_h;\sigma'_{h\to -h})) &= \tfrac{1}{m}((m-1)r_c(\mu_h^s - \mu_{-h}^s - 1 - \mu^s(j_h))\\
&\quad + r_e(\mu^{\beta}(j_h) - \mu_{-h}^{\beta}(j_h) + m(\mu^y(j_h) - \mu_{-h}^y(j_h))).
\end{aligned}$$

The condition $F_A \geq 0$ of eq. (15) corresponds to $\mathbb{E}(\pi(j_A;\sigma) - \pi(j_A;\sigma'_{A\to B})) \geq 0$ and the condition $F_B \geq 0$ of eq. (16) corresponds to $\mathbb{E}(\pi(j_B;\sigma) - \pi(j_B;\sigma'_{B\to A})) \geq 0$.

For leader i_h, the condition $F_h(i_h) \geq 0$ reduces to

$$-\left(\theta - \left(1 - \frac{1}{\mu^s_{-h} + 1}\right)\right) \geq 0.$$

Since $\mu^s_{-h} \leq (n-2)$, $B \geq 0$ ensures that $F_h(i_h) \leq 0$ for both leaders. The condition holds at equality only if $B = 0$ and $\mu^s_{-h} = (n-2)$, a condition that cannot hold for both leaders simultaneously. $F_h(j) > 0$ for all $j \in N^{AB}(i_A, i_B; \sigma)$ is feasible.

C Examples

Multiple-Leader Structures

Two scenarios allow for a multiple leader structure in equilibrium with linear payoff functions. Both feature a σ given $g \in G(n)$ such that a particular follower finds it advantageous and feasible to preserve the multiple leader structure.

Example 4

Let $\mu^h(j) = \mu^h(j;\sigma) = |N^h(j;\sigma)|$ for $h = x, y, s, \alpha, \beta$. For $h = A, B$, let $\sigma' = \sigma_{-j_h} \times \sigma'_{j_h}$ be the structure produced by j_h switching predecessors in order to become a member of the i_{-h}-led tree. The alternative structure identifies populations $N^\beta_{-h}(j_h) = N^\beta(j_h;\sigma')$ and $N^y_{-h}(j_h) = N^y(j_h;\sigma')$. Let $\mu^\beta_{-h}(j_h) = |N^\beta_{-h}(j_h)|$ and $\mu^y_{-h}(j_h) = |N^y_{-h}(j_h)|$.

The structure σ is as depicted in Figure 9. With $\mu^y(j_A) = \mu^s(j_A) = \mu^\beta(j_A) = \mu^y(j_B) = \mu^s(j_B) = \mu^\beta_A(j_B) = 0$, $F_A \geq 0$ and $F_B > 0$ of eqs. (17) and (18) jointly imply

$$\frac{\mu^\beta_B(j_A)}{\theta} + 1 \leq d\mu < \frac{\mu^\beta(j_B) - m\mu^y_A(j_B)}{\theta} - 1. \tag{30}$$

The four key features needed of σ to satisfy eq. (30) are

1. $\mu^s_B \geq 1 + \mu^\alpha(j_B) + (m\mu^y_A(j_B) - (\theta - 1)\mu^\beta(j_B))/\theta$ indicating that μ^s_B is larger than μ^s_A excluding the $N^S(i_A;\sigma)$ followers at distance $d_{j_B,i_B} + 1$. Each member of the $N^y_A(j_B)$ population requires m members of $N^S(i_B;\sigma)$ to keep j_B in $N^S(i_B;\sigma)$. $\theta = 1$ is the minimum possible threshold on θ derived from $E_h \geq 0$. The stronger condition $\mu^s_B \geq 1 + \mu^\alpha(j_B) + m\mu^y_A(j_B)$ ensures $F_B \geq 0$ over the entire feasible support for θ;
2. a concentration of the i_A-led population at the distance $d_{j_B,i_B} + 1$ is sufficiently large to have $\mu^s_A \geq \mu^s_B$ despite feature 1;
3. $d_{j_B,i_A} \geq d_{j_B,i_B} + 1$; and
4. $d_{j_A,i_B} = d_{j_B,i_B} + 1$.

Figure 9 is an equilibrium structure satisfying eq. (30). Feature 1 requires a large $N^x(j_B;\sigma)$ population based on the sizes of the $N^\alpha(j_B;\sigma)$ and $N^y_A(j_B;\sigma)$ populations. The $N^\beta(j_B;\sigma)$ population is sufficiently large to produce $\mu^s_A \geq \mu^s_B$ in accordance with feature 2. So that j_B prefers the i_B-led tree, she cannot benefit from the $N^\beta(j_B;\sigma)$ population were she to switch, which is captured by feature 3. Feature 4 puts j_A in a position where she fails to share in j_B's distance advantage over the β population from the i_B-led tree, thereby keeping $\mu^\beta_B(j_A)$ small. By feature 3, the β population exists within the distance range $d_{j_B,i_B} + 1$ and d_{j_B,i_A} (inclusive) but feature 4 constrains the population to have a distance of $d_{j_B,i_B} + 1$.

Example 5

The inequality $F_B(j_B) > 0$ supports follower $j_B \in \{N^S(i_B,\sigma) \cap N^{AB}(i_A, i_B;\sigma) | \mu^s_j = 0, \mu^y_j > 0\}$ in her current position, as illustrated in Example 5. The additional imposition of $\mu^y_A(j_B) = 0$ minimizes the attraction of the i_A-led tree to j_B as it implies player j_B must join the i_A-led tree at the maximum distance.

Notes

1 Avant-garde art collector David Teiger quoted in Thornton (2009) (p.100).
2 Of Teiger, Thornton (2009) comments, "He enjoys being a player in the power game of art, particularly at this level where patronage can have an impact on public consciousness" (p.100). Thornton also observes, "Unlike other industries, where buyers are anonymous and interchangeable, here, artists' reputations are enhanced or contaminated by the people who own their work" (p.88). Glazek (2014) profiles a patron who promotes emerging artists among collectors with little knowledge of art. Unresolved in the piece is whether the artists had no future among knowledgeable collectors or whether the artists' career were poisoned by their affiliation with the patron.
3 "Making a big bet on something before anyone else really grasps it. That is what success has in common in energy and in equities," political strategist Tim Phillips as cited in Confessore, Cohen, and Yourish (2015).
4 Watts (2001) and Jackson and Watts (2002) offer useful literature reviews of works on social influence.
5 The definition of conformity provided by Deutsch and Gerard (1955), and employed extensively in the psychology literature, applies to observed individual behavior in response to social influences present at the time of decision making. The current model allows that such social influences are present even when the decision-maker only learns the conforming action after having acted. In such instances, the desire for conformity still potentially shapes behavior, for example, by causing the decision-maker to attempt to anticipate the conforming action or to proactively alter the decision making process to gain relevant information before committing to an action.
6 One of the examples offered in the Ali and Kartik (2012) observational learning model roughly maps to the present setting. Allowing that political candidates value earlier contributors introduces a counterweight to the information advantage gained from delay. Freeing contributors to choose the timing of a contribution increases intrinsic uncertainty, particularly when contributions can be made simultaneously, which prevents contributors from knowing the value of their contribution on subsequent decision-makers.
7 Random assignment captures the absence of prior coordination between players or the collection of probability mass on a single option. The latter might arise as the consequence of being a focal point, for example. The same can be accomplished more formally with private object labels.
8 The notion of time and distance are isomorphic when adoption disseminates at a rate of one unit of time per link.
9 Since payoffs depend on the popularity of the adopted option and the relative time to adoption, in practice the information should eventually become available to the players. A report on the popularity of each alternative broken down by time is sufficient. Such information could be seen as emerging slowly over the network after all decisions have been made or as tabulated and published in a bulletin.
10 As a strategy, appearing influential is no substitute to being influential, recognized within the model once expectations are taken over outcomes. *Ex post* coincidental conformity and early adoption can be just as satisfying or financially rewarding. Early acquisition of a subsequently popular artist's works benefits the coincidental collector financially just as much as it does the influential collector. The coincidental collector may also benefit from social affirmation of the acquisition and reputation enhancement among outside observers not aware of the paths of influence.
11 The linear model can also be re-expressed as a model imposing cost or penalty to late adoption rather than rewarding early adoption, similar to the examples used in Brindisi, Çelen, and Hyndman (2014). Late adopters pay higher costs with payoffs $\pi(i;\sigma) = b_c(\mu_i^c) - b_e(\mu_i^c - \mu_i^e)$ where b_c is the per member conformity payoff and b_e is the cost associated with each player who acts concurrent or in advance of player i on the same alternative. With $b_c = r_c + r_e$ and $b_e = r_e$, the two scenarios are isomorphic.
12 The mixed strategy solution for this example has $\Pr_i(\text{lead}) = (1 + r_e)/(\frac{3}{2} + r_e)$. The value of the game in the mixed strategy solution is $v = (2 + 2r_e)/(3 + 2r_e)$. Since $v < 1$ for all $r_e \geq 0$, the value of the mixed strategy solution is always less than the follower's payoff in the pure strategy game.
13 The middle distance follower in a vertical structure benefits from the selfless act of the most distant follower choosing an indirect link to the leader. A coordination failure in which player 2 follows 1 to produce Figure 2d for player 1's benefit while player 1 follows 2 to produce Figure 2e for player 2's benefit results in the self-referencing loop of Figure 2b, the worst of all possible outcomes.
14 The possible $N^\alpha(i;\sigma)$ and $N^\beta(i;\sigma)$ populations of followers in an alternate i_B-led tree seen in Figure 4 are introduced and developed in Section 4.
15 That $H^*(i;g) \subseteq H'(i;g)$ is established formally in support of Corollary 2 in Appendix B.
16 It is straightforward to extend Lemma 1 to structures with a unique leader, μ_i^s followers, and $\mu^I = n - \mu_i^s - 1$ autonomous adopters considered in Section 3.5. For $\pi_{NL}(j;\sigma) = \phi(\mu_i^s) + \psi(\mu_j^y + \mu_j^s) + \sum_{\mu=1}^{\mu^I}(\phi(\mu_i^s + \mu) - \phi(\mu_i^s))f(\mu)$, $f(\mu) = \Pr(\text{leader matches with } \mu \text{ autonomous adopters})$, only the middle term is effected by j's decision about how to link to leader i.
17 Corollary 2 is consistent with Proposition 2. The corollary does not claim or imply that if $B_{NL} \geq 0$ and σ is an equilibrium then $\sigma \in H^*(i;g)$, which would violate the Proposition 2 stipulation that $\sigma \in H'(i;g)\backslash H^*(i;g)$ is also an equilibrium when $B_{NL} \geq 0$.
18 While $\sigma' \in H'(i;g)\backslash H^*(i;g)$ is socially preferred to $\sigma \in H^*(i;g)$, an argument in support of employing the strategy to minimize d_j rather than being content to minimizing μ_j^x includes the diminished position to exploit possible deviant behavior by other followers. Additionally, minimizing μ_j^x introduces a coordination problem if the option exists simultaneously for more than one follower, with the worst case resulting in a self-referencing loop.
19 Linearity allows for aggregation in expectations over possible states. Curvature in the reward components means accounting for each possible state separately, adding complexity to the equations without additional insight.
20 For any two $\{j_1, j_2\} \notin N^S(i_A;\sigma)$, the ability to form such a hierarchy is not assured by the assumption of strong connectivity since it may require the chain of links to pass through a member of $N^S(i_A;\sigma)$.
21 If equal in size, to the switcher's benefit, the act of switching makes the joined hierarchy larger, *ex-post*, to the departed hierarchy *ex-ante*.
22 For $\mu_B^s < n - n^\dagger - 1$, the necessary condition to have the most distant successor of i_B remain a follower is

$$\theta \geq \frac{m}{m-1}\left(1 + \frac{\mu^\alpha(j_B;\sigma) - 1}{\mu_B^s}\right) + \frac{(m-2)r_c}{(m-1)r_e}\left(\frac{n - n^\dagger - 1}{\mu_B^s} - 1\right)$$

for which, with $\mu^\alpha(j_B;\sigma) \geq 1$ and $\mu_B^s \leq n - n^\dagger - 1$, the threshold on θ declines as μ_B^s is increased.
23 As is trivially exemplified in the $n = 2$ example.
24 Also excluded from the utility function is a direct reward from early adoption. The coordination problem of interest is distinct from the utility some people might receive simply by being the first to try new products.
25 Equation (7) is, naturally, the same except having replaced $\mu_i^s - 1$ with its value of $n - 2$ for $\sigma \in H(i;g)$.

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