

# ON EXPONENTIAL STABILITY OF LINEAR NON-AUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

ABSTRACT. General linear non-autonomous functional differential equations of neutral type are considered. A novel approach to exponential stability of neutral functional differential equations is presented. Consequently, explicit criteria are derived for exponential stability of linear non-autonomous functional differential equations of neutral type. A brief discussion to the obtained results and illustrative examples are given.

## 1. INTRODUCTION

Functional differential equations of neutral type have numerous applications in science and engineering. They are used as models of lossless transmission lines, partial element equivalent circuits [2], steam or water pipes, heat exchangers [15] and control of constrained manipulators with delay measurements in mechanical engineering [19].

Problems of stability of functional differential equations of neutral type have been investigated intensively during the past decades, see e.g. [1]-[2], [4]-[10], [12]-[20] and references therein. In this paper, we investigate exponential stability of general linear neutral non-autonomous functional differential equations of the form

$$(1) \quad \frac{d}{dt} \mathcal{D}(t, x_t) = \mathcal{L}(t, x_t), \quad t \geq \sigma,$$

where, for each  $t \in \mathbb{R}$ ,  $x_t(\cdot) \in \mathcal{C} := C([-h, 0], \mathbb{R}^n)$  is defined by  $x_t(\theta) := x(t+\theta)$ ,  $\theta \in [-h, 0]$  with given  $h > 0$ ,  $\mathcal{D}(\cdot, \cdot) : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}^n$  and  $\mathcal{L}(\cdot, \cdot) : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}^n$  are given continuous functions such that  $\mathcal{D}(t, \cdot)$  and  $\mathcal{L}(t, \cdot)$  are bounded linear operators from  $\mathcal{C}$  to  $\mathbb{R}^n$  for each  $t \in \mathbb{R}$ .

In general, it is not easy to tackle the problem of stability analysis of time-varying differential equations of neutral type. The traditional approaches to problems of stability of time-varying differential equations of neutral type are based on the Lyapunov's method and its variants, see e.g. [4]-[10], [12]-[16], [18], [19]. Most existing results in the literature are given in terms of matrix inequalities and not straightforward to use.

There have been many works dedicated to stability of differential equations of neutral type, see [1]-[2], [4]-[10], [12]-[20]. However, problems of exponential stability of the linear neutral functional differential equation (1) has not yet been studied adequately and to

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date there have been no explicit stability criteria for (1) in the literature. The main purpose of this paper is to fill this gap.

In contrast to the traditional approaches, we present in this paper a novel approach to exponential stability of the linear neutral functional differential equation (1). Our approach is simple in that it relies only upon a system transformation, the comparison principle and spectral properties of Metzler matrices. Consequently, we can obtain new explicit criteria for exponential stability of (1). A discussion to the obtained results and a numerical example are given for illustration.

The organization of the paper is as follows. In the next section, we give notation and preliminary results to be used in our development. The main results are presented in Section 3. We derive new explicit criteria for exponential stability of the linear neutral functional differential equation (1). In Section 4, we make a comparison of the obtained results with some existing ones in the literature and give an illustrative example. Finally, a conclusion is drawn in Section 5.

## 2. PRELIMINARIES

Let  $\mathbb{N}$  be the set of all natural numbers. For given  $m \in \mathbb{N}$ , let  $\underline{m} := \{1, 2, \dots, m\}$ . For given integers  $l, q \geq 1$ ,  $\mathbb{R}^l$  denotes the  $l$ -dimensional vector space over  $\mathbb{R}$  and  $\mathbb{R}^{l \times q}$  stands for the set of all  $l \times q$ -matrices with entries in  $\mathbb{R}$ . For  $A = (a_{ij}) \in \mathbb{R}^{l \times q}$  and  $B = (b_{ij}) \in \mathbb{R}^{l \times q}$ ,  $A \geq B$  means that  $a_{ij} \geq b_{ij}$  for  $i = 1, \dots, l$ ,  $j = 1, \dots, q$ . In particular, if  $a_{ij} > b_{ij}$  for  $i = 1, \dots, l$ ,  $j = 1, \dots, q$ , then we write  $A \gg B$  instead of  $A \geq B$ . Denote by  $\mathbb{R}_+^{l \times q}$  the set of all nonnegative matrices. Similar notations are adopted for vectors.

For  $x \in \mathbb{R}^n$  and  $P \in \mathbb{R}^{l \times q}$  we define  $|x| = (|x_i|)$  and  $|P| = (|p_{ij}|)$ . Then one has

$$|PQ| \leq |P||Q|, \quad \forall P \in \mathbb{R}^{l \times q}, \quad \forall Q \in \mathbb{R}^{q \times r}.$$

Let  $I_n$  be the identity matrix in  $\mathbb{R}^{n \times n}$ . For any matrix  $M \in \mathbb{R}^{n \times n}$  the *spectral abscissa* (resp. the *spectral radius*) of  $M$  is defined by  $s(M) := \max\{\Re \lambda : \lambda \in \sigma(M)\}$  (resp.  $\rho(M) := \max\{|\lambda| : \lambda \in \sigma(M)\}$ ) where  $\sigma(M) := \{z \in \mathbb{C} : \det(zI_n - M) = 0\}$  is the spectrum of  $M$ . A matrix  $M \in \mathbb{R}^{n \times n}$  is said to be Hurwitz stable (resp. Schur stable) if,  $s(M) < 0$  (resp.  $\rho(M) < 1$ ). A norm  $\|\cdot\|$  on  $\mathbb{R}^n$  is said to be *monotonic* if  $\|x\| \leq \|y\|$  whenever  $x, y \in \mathbb{R}^n$ ,  $|x| \leq |y|$ . Every  $p$ -norm on  $\mathbb{R}^n$

$$\|x\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty; \quad \|x\|_\infty = \max_{i=1,2,\dots,n} |x_i|,$$

is monotonic. Throughout the paper, if otherwise not stated, the norm of vectors on  $\mathbb{R}^n$  is monotonic and the norm of a matrix  $P \in \mathbb{R}^{l \times q}$  is understood as its operator norm associated with a given pair of monotonic vector norms on  $\mathbb{R}^l$  and  $\mathbb{R}^q$ , that is

$$\|P\| = \max\{\|Py\| : \|y\| = 1\}.$$

A matrix  $M \in \mathbb{R}^{n \times n}$  is called a *Metzler matrix* if all off-diagonal elements of  $M$  are nonnegative. For given  $A := (a_{ij}) \in \mathbb{R}^{n \times n}$ , we associate the Metzler matrix  $M(A) := (\hat{a}_{ij})$  where  $\hat{a}_{ij} = |a_{ij}|$  if  $i \neq j$ , for  $i, j \in \underline{n}$  and  $\hat{a}_{ii} = a_{ii}$ , for  $i \in \underline{n}$ .

**Theorem 2.1.** [3]

- (i) Let  $A \in \mathbb{R}_+^{n \times n}$ . Then  $A$  is Schur stable if and only if  $Aq \ll q$ , for some  $q \in \mathbb{R}_+^n, q \gg 0$ .
- (ii) Let  $A \in \mathbb{R}^{n \times n}$  be a Metzler matrix. Then  $A$  is Hurwitz stable if and only if  $Ap \ll 0$ , for some  $p \in \mathbb{R}_+^n, p \gg 0$ .

The following lemma is used in what follows.

**Lemma 2.2.** Let  $A \in \mathbb{R}^{n \times n}$  be a Metzler matrix and  $B, C, D \in \mathbb{R}_+^{n \times n}$ . Then the following statements are equivalent

- (i)  $\rho(D) < 1$  and  $s(A + B(I_n - D)^{-1}C) < 0$ ;
- (ii) there exist  $p, q \in \mathbb{R}_+^n, p \gg 0, q \gg 0$  such that

$$(2) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \ll \begin{pmatrix} 0 \\ q \end{pmatrix}.$$

- (iii)  $s(A) < 0$  and  $\rho(C(-A)^{-1}B + D) < 1$ .

*Proof.* (i)  $\Rightarrow$  (ii) Since  $D \in \mathbb{R}_+^{n \times n}$  and  $\rho(D) < 1$ , it follows that  $(I_n - D)^{-1} = \sum_{k=0}^{\infty} D^k \geq 0$ . Furthermore, we have  $Dq_0 \ll q_0$ , for some  $q_0 \in \mathbb{R}_+^n, q_0 \gg 0$ , by Theorem 2.1 (i). Thus,  $B(I_n - D)^{-1}C \in \mathbb{R}_+^{n \times n}$  and  $A + B(I_n - D)^{-1}C$  is a Metzler matrix. Since  $s(A + B(I_n - D)^{-1}C) < 0$ , it follows that  $(A + B(I_n - D)^{-1}C)p \ll 0$  for some  $p \in \mathbb{R}_+^n, p \gg 0$ , by Theorem 2.1 (ii). This implies  $(A + B(I_n - D)^{-1}C)p + \epsilon Bq_0 \ll 0$ , for sufficiently small  $\epsilon$ . Let  $q := (I_n - D)^{-1}Cp + \epsilon q_0 \gg 0$ . Note that  $Ap + Bq = Ap + (B(I_n - D)^{-1}Cp + \epsilon Bq_0) \ll 0$  and  $Cp + Dq = q + \epsilon(Dq_0 - q_0) \ll q$ . Therefore, (i) holds.

(ii)  $\Rightarrow$  (i) Assume that  $Ap + Bq \ll 0$  and  $Cp + Dq \ll q$ , with  $p, q \in \mathbb{R}_+^n, p, q \gg 0$ . Note that  $Dq \leq Cp + Dq \ll q$ , with  $q \in \mathbb{R}_+^n, q \gg 0$ . Thus,  $\rho(D) < 1$ , by Theorem 2.1 (i). Furthermore,  $Cp \ll q - Dq$ . It follows that  $q = (I_n - D)^{-1}(I_n - D)q \geq (I_n - D)^{-1}Cp$ . Therefore,  $0 \gg Ap + Bq \geq Ap + B(I_n - D)^{-1}Cp = (A + B(I_n - D)^{-1}C)p$ . Thus,  $s(A + B(I_n - D)^{-1}C) < 0$ , by Theorem 2.1 (ii).

(ii)  $\Rightarrow$  (iii) Suppose  $Ap + Bq \ll 0$  and  $Cp + Dq \ll q$ , with  $p, q \in \mathbb{R}_+^n, p, q \gg 0$ . From  $B \in \mathbb{R}_+^{n \times n}$  and  $q \in \mathbb{R}_+^n$ , it follows that  $Bq \in \mathbb{R}_+^n$ . Thus,  $Ap \leq Ap + Bq \ll 0$ . Therefore,  $s(A) < 0$ , by Theorem 2.1 (ii). Since  $A$  is a Metzler matrix and  $s(A) < 0$ , one has  $(-A)^{-1} \in \mathbb{R}_+^{n \times n}$ , see e.g. [3]. From  $Ap + Bq \ll 0$ , it follows that  $Bq \ll -Ap$ . This gives  $(-A)^{-1}Bq \leq p$  and  $C(-A)^{-1}Bq \leq Cp$ . Thus,  $(C(-A)^{-1}B + D)q \leq Cp + Dq \ll q$ . By Theorem 2.1 (i),  $\rho(C(-A)^{-1}B + D) < 1$ .

(iii)  $\Rightarrow$  (ii) Assume that  $s(A) < 0$  and  $\rho(C(-A)^{-1}B + D) < 1$ . As mentioned above, it follows from  $s(A) < 0$  that  $(-A)^{-1} \in \mathbb{R}_+^{n \times n}$ . By Theorem 2.1,  $Ap_0 \ll 0$  and  $(C(-A)^{-1}B + D)q \ll q$ , for some  $p_0, q \in \mathbb{R}_+^n, p_0, q \gg 0$ . This implies,  $(C(-A)^{-1}B + D)q + \gamma Cp_0 \ll q$ ,

for sufficiently small  $\gamma > 0$ . Let  $p := (-A)^{-1}Bq + \gamma p_0 \gg 0$ . Then, we have  $Cp + Dq = (C(-A)^{-1}B + D)q + \gamma Cp_0 \ll q$  and  $Ap + Bq = A((-A)^{-1}Bq + \gamma p_0) + Bq = \gamma Ap_0 \ll 0$ . This completes the proof.  $\square$

To make a representation self contained and dynamic, we present here some basic facts on vector-valued functions of bounded variation and related topics. A matrix function  $\eta(\cdot) : [\alpha, \beta] \rightarrow \mathbb{R}^{m \times n}$  is called an increasing matrix function if

$$\eta(\theta_2) \geq \eta(\theta_1) \quad \text{for } \alpha \leq \theta_1 \leq \theta_2 \leq \beta.$$

A matrix-valued function  $\eta(\cdot) : [\alpha, \beta] \rightarrow \mathbb{R}^{m \times n}$  is said to be of bounded variation if

$$Var_{[\alpha, \beta]} \eta(\cdot) := \sup_{P[\alpha, \beta]} \sum_k \|\eta(\theta_k) - \eta(\theta_{k-1})\| < +\infty,$$

where the supremum is taken over the set of all finite partitions of the interval  $[\alpha, \beta]$ . The set  $BV([\alpha, \beta], \mathbb{R}^{m \times n})$  of all matrix functions  $\eta(\cdot)$  of bounded variation on  $[\alpha, \beta]$  satisfying  $\eta(\alpha) = 0$  is a Banach space endowed with the norm  $\|\eta\| = Var_{[\alpha, \beta]} \eta(\cdot)$ . Since all matrix norms on  $\mathbb{R}^{m \times n}$  are equivalent, it follows that the matrix function  $\eta(\cdot) = (\eta_{ij}(\cdot)) \in \mathbb{R}^{m \times n}$  is of bounded variation if and only if each  $\eta_{ij}(\cdot)$  is of bounded variation. Then  $\eta(\cdot) \in BV([\alpha, \beta], \mathbb{R}^{m \times n})$ , if and only if

$$\eta(\cdot) = \eta_2(\cdot) - \eta_1(\cdot),$$

where  $\eta_1(\cdot)$  and  $\eta_2(\cdot)$  are increasing matrix functions, see e.g. [11]. Therefore,  $\lim_{\theta \rightarrow \theta_0^-} \eta(\theta)$  exists for any  $\theta_0 \in [\alpha, \beta]$ , if  $\eta(\cdot) \in BV([\alpha, \beta], \mathbb{R}^{m \times n})$ .

Let us consider a subset of  $BV([\alpha, \beta], \mathbb{R}^{m \times n})$ :

$$NBV([\alpha, \beta], \mathbb{R}^{m \times n}) := \{\eta \in BV([\alpha, \beta], \mathbb{R}^{m \times n}); \eta \text{ is continuous from left on } (\alpha, \beta)\}.$$

Clearly,  $NBV([\alpha, \beta], \mathbb{R}^{m \times n})$  is closed in  $BV([\alpha, \beta], \mathbb{R}^{m \times n})$  and thus it is a Banach space with the norm  $\|\eta\| = Var_{[\alpha, \beta]} \eta(\cdot)$ . Given  $\eta(\cdot) \in NBV([\alpha, \beta], \mathbb{R}^{m \times n})$  then for any continuous functions  $\gamma \in C([\alpha, \beta], \mathbb{R})$  and  $\varphi \in C([\alpha, \beta], \mathbb{R}^n)$ , the integrals

$$\int_{\alpha}^{\beta} \gamma(\theta) d[\eta(\theta)] \quad \text{and} \quad \int_{\alpha}^{\beta} d[\eta(\theta)] \varphi(\theta)$$

exist and are defined respectively as the limits of  $S_1(P) := \sum_{k=1}^p \gamma(\zeta_k)(\eta(\theta_k) - \eta(\theta_{k-1}))$  and  $S_2(P) := \sum_{k=1}^p (\eta(\theta_k) - \eta(\theta_{k-1}))\varphi(\zeta_k)$  when  $d(P) := \max_k |\theta_k - \theta_{k-1}| \rightarrow 0$ , where  $P = \{\theta_1 = \alpha \leq \theta_2 \leq \dots \leq \theta_p = \beta\}$  is any finite partition of the interval  $[\alpha, \beta]$  and  $\zeta_k \in [\theta_{k-1}, \theta_k]$ . It is immediate from the definition that

$$\left\| \int_{\alpha}^{\beta} \gamma(\theta) d[\eta(\theta)] \right\| \leq \max_{\theta \in [\alpha, \beta]} |\gamma(\theta)| \|\eta\|,$$

$$\left\| \int_{\alpha}^{\beta} d[\eta(\theta)] \varphi(\theta) \right\| \leq \max_{\theta \in [\alpha, \beta]} \|\varphi(\theta)\| \|\eta\|.$$

Let  $\mathbb{R}^{m \times n}$  be endowed with the norm  $\|\cdot\|$  and let  $J$  be an interval of  $\mathbb{R}$ . Denote by  $C(J, \mathbb{R}^{m \times n})$ , the vector space of all continuous functions on  $J$  with values in  $\mathbb{R}^{m \times n}$ . In particular,  $C([\alpha, \beta], \mathbb{R}^{m \times n})$  is a Banach space endowed with the norm

$$\|\varphi\| := \max_{\theta \in [\alpha, \beta]} \|\varphi(\theta)\|.$$

In the following, the Banach space  $C([-h, 0], \mathbb{R}^n)$  is used frequently. For simplicity, we write  $\mathcal{C}$  instead of  $C([-h, 0], \mathbb{R}^n)$ .

### 3. EXPLICIT CRITERIA FOR EXPONENTIAL STABILITY

Consider the linear neutral functional differential equation (1), where  $\mathcal{D}(\cdot, \cdot) : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}^n$  and  $\mathcal{L}(\cdot, \cdot) : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}^n$  are given continuous functions such that  $\mathcal{D}(t, \cdot)$  and  $\mathcal{L}(t, \cdot)$  are bounded linear operators from  $\mathcal{C}$  to  $\mathbb{R}^n$  for each  $t \in \mathbb{R}$ .

Throughout, we assume that

$$(3) \quad \mathcal{D}(t, \varphi) := \varphi(0) - \mathcal{D}_0(t, \varphi), \quad t \in \mathbb{R},$$

and  $\mathcal{D}_0(t, \cdot)$  is a bounded linear operator from  $\mathcal{C}$  to  $\mathbb{R}^n$  for each  $t \in \mathbb{R}$ .

By the well-known Riesz representation theorem,

$$(4) \quad \mathcal{L}(t, \varphi) := \int_{-h}^0 d[\eta(t, \theta)]\varphi(\theta), \quad \varphi \in \mathcal{C}, \quad t \in \mathbb{R},$$

and

$$(5) \quad \mathcal{D}_0(t, \varphi) := \int_{-h}^0 d[\mu(t, \theta)]\varphi(\theta), \quad \varphi \in \mathcal{C}, \quad \forall t \in \mathbb{R},$$

where for each  $t \in \mathbb{R}$ ,  $\eta(t, \theta) : \mathbb{R} \times [-h, 0] \rightarrow \mathbb{R}^{n \times n}$  and  $\mu(t, \theta) : \mathbb{R} \times [-h, 0] \rightarrow \mathbb{R}^{n \times n}$  are of bounded variation in  $\theta$  on  $[-h, 0]$  and normalized so that  $\eta(t, \theta)$  and  $\mu(t, \theta)$  are continuous from the left in  $\theta$  on  $(-h, 0)$  and

$$\eta(t, \theta) = 0, \quad \theta \geq 0, \quad \eta(t, \theta) = \eta(t, -h), \quad \forall \theta \leq -h,$$

$$\mu(t, \theta) = 0, \quad \theta \geq 0, \quad \mu(t, \theta) = \mu(t, -h), \quad \forall \theta \leq -h.$$

Furthermore, suppose there exists a constant matrix  $D := (d_{ij}) \in \mathbb{R}_+^{n \times n}$  such that

$$(6) \quad \text{Var}_{[-h, 0]} \mu_{ij}(t, \cdot) \leq d_{ij}, \quad \forall t \in \mathbb{R}, \quad \forall i, j \in \underline{n},$$

where  $\mu(t, \theta) := (\mu_{ij}(t, \theta)) \in \mathbb{R}^{n \times n}$  for  $t \in \mathbb{R}, \theta \in [-h, 0]$  and  $\mu$  is *uniformly nonatomic* at zero, that is, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$(7) \quad \text{Var}_{[-\delta, 0]} \mu(t, \cdot) < \epsilon, \quad \forall t \in \mathbb{R}.$$

**Remark 3.1.** (i) Suppose  $\mu(t, \cdot)$ ,  $t \in \mathbb{R}$  has no singular part, that is

$$(8) \quad \int_{-h}^0 d[\mu(t, \theta)]\varphi(\theta) := \sum_{i=1}^m D_i(t)\varphi(-h_i) + \int_{-h}^0 E(t, s)\varphi(\theta)ds, \quad \varphi \in \mathcal{C},$$

where  $0 \leq h_i \leq h, i \in \underline{m}$  and  $D_i(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}, i \in \underline{m}$ ,  $E(\cdot, \cdot) : \mathbb{R} \times [-h, 0] \rightarrow \mathbb{R}^{n \times n}$ , are continuous functions. Furthermore, assume that there exists  $D_i \in \mathbb{R}_+^{n \times n}, i \in \underline{m}$  and a continuous function  $E_0(\cdot) : [-h, 0] \rightarrow \mathbb{R}_+^{n \times n}$  such that

$$(9) \quad |D_i(t)| \leq D_i, \forall t \in \mathbb{R}, \forall i \in \underline{m}; \quad |E(t, s)| \leq E_0(s), \quad \forall t \in \mathbb{R}, \forall s \in [-h, 0].$$

Clearly, (6) holds and  $\mu$  is uniformly nonatomic at zero.

(ii) Assume that  $\mu(t, \cdot) \equiv \mu(\cdot)$ , i.e., independent of  $t$ . Then  $\mu(\cdot)$  is uniformly nonatomic at zero if  $\text{Var}_{[-s, 0]}\mu(\cdot) \rightarrow 0$  as  $s \rightarrow 0$ .

For given  $\varphi \in \mathcal{C}$ , consider for (1) the initial condition

$$(10) \quad x(\theta + \sigma) = \varphi(\theta), \quad \theta \in [-h, 0].$$

**Definition 3.2.** Let  $\sigma \in \mathbb{R}$  and  $\varphi \in \mathcal{C}$  be given. A continuous function  $x(\cdot) : [-h + \sigma, \infty) \rightarrow \mathbb{R}^n$ , is said to be a solution of (1) with the initial value  $(\sigma, \varphi)$  (or simply, a solution of (1) through  $(\sigma, \varphi)$ ) if  $\mathcal{D}(t, x_t)$  is continuously differentiable on  $(\sigma, \infty)$  with a right hand derivative at  $\sigma$  and  $x(\cdot)$  satisfies (1) on  $[\sigma, \infty)$  and satisfies (10) on  $[-h, 0]$ .

It is well-known that for fixed  $\sigma \in \mathbb{R}$  and given  $\varphi \in \mathcal{C}$ , there exists a unique solution of (1) through  $(\sigma, \varphi)$ , denoted by  $x(\cdot; \sigma, \varphi)$ , provided (6) holds and  $\mu$  is uniformly nonatomic at zero, see e.g. [9, Theorem 1.1, page 256]

**Definition 3.3.** The equation (1) is said to be exponentially stable if there exist  $K, \beta > 0$  such that

$$\|x(t; \sigma, \varphi)\| \leq Ke^{-\beta(t-\sigma)}\|\varphi\|, \quad \forall t \geq \sigma, \quad \forall \varphi \in \mathcal{C}.$$

Let  $\hat{\eta}(t, \theta) := \eta(t, \theta), \theta \in [-h, 0)$ ;  $\hat{\eta}(t, 0) := \lim_{\theta \rightarrow 0^-} \eta(t, \theta)$  and let  $\tilde{\eta}(t, \theta) := 0, \theta \in [-h, 0)$ ;  $\tilde{\eta}(t, 0) := \eta(t, 0) - \lim_{\theta \rightarrow 0^-} \eta(t, \theta) = -\lim_{\theta \rightarrow 0^-} \eta(t, \theta) := A(t)$ . Clearly,

$$\eta(t, \theta) = \hat{\eta}(t, \theta) + \tilde{\eta}(t, \theta), \quad \theta \in [-h, 0],$$

and  $\hat{\eta}(t, \cdot)$  is continuous from the left on  $[-h, 0]$ . Therefore,

$$(11) \quad \mathcal{L}(t, \varphi) := \int_{-h}^0 d[\eta(t, \theta)]\varphi(\theta) = A(t)\varphi(0) + \int_{-h}^0 d[\hat{\eta}(t, \theta)]\varphi(\theta), \quad t \in \mathbb{R}, \quad \varphi \in \mathcal{C},$$

and thus (1) is reduced to

$$(12) \quad \frac{d}{dt}(x(t) - \int_{-h}^0 d[\mu(t, \theta)]x(t + \theta)) = A(t)x(t) + \int_{-h}^0 d[\hat{\eta}(t, \theta)]x(t + \theta), \quad t \geq \sigma.$$

In addition, we always assume that  $A(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and  $\mathcal{L}_1(\cdot; \cdot) : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}^n, (t, \varphi) \mapsto \mathcal{L}_1(t; \varphi) := \int_{-h}^0 d[\hat{\eta}(t, \theta)]\varphi(\theta)$ , are continuous functions.

We are now in the position to state the main result of this paper.

**Theorem 3.4.** *Let*

$$(13) \quad \bar{\mu}(t, \theta) := A(t)\mu(t, \theta), \quad t \in \mathbb{R}, \quad \theta \in [-h, 0].$$

Suppose there exist  $\beta > 0$  and  $p := (p_1, p_2, \dots, p_n)^T, q := (q_1, q_2, \dots, q_n)^T \in \mathbb{R}_+^n, p, q \gg 0$  such that

$$(14) \quad (a_{ii}(t)p_i + \sum_{j=1, j \neq i}^n |a_{ij}(t)|p_j) + \sum_{j=1}^n (Var_{[-h, 0]} \bar{\mu}_{ij}(t, \cdot) + Var_{[-h, 0]} \hat{\eta}_{ij}(t, \cdot)) e^{\beta h} q_j < -\beta p_i,$$

$$(15) \quad p_i + \sum_{j=1}^n Var_{[-h, 0]} \mu_{ij}(t, \cdot) e^{\beta h} q_j < q_i,$$

for any  $t \in \mathbb{R}$  and for any  $i \in \underline{n}$ . Then (1) is exponentially stable.

*Proof.* Let  $x(\cdot) := x(\cdot; \sigma, \varphi)$ , be the unique solution of (1) through  $(\sigma, \varphi)$ . Define

$$y(t) := \mathcal{D}(t, x_t) = x(t) - \int_{-h}^0 d[\mu(t, \theta)]x(t + \theta) \quad t \geq \sigma.$$

Then  $x(\cdot)$  and  $y(\cdot)$  satisfy the following system

$$(16) \quad \frac{dy(t)}{dt} = A(t)y(t) + \int_{-h}^0 d[\bar{\mu}(t, \theta)]x(t + \theta) + \int_{-h}^0 d[\hat{\eta}(t, \theta)]x(t + \theta), \quad t \geq \sigma,$$

$$(17) \quad x(t) = y(t) + \int_{-h}^0 d[\mu(t, \theta)]x(t + \theta), \quad t \geq \sigma.$$

Choose  $K > 0$  such that  $|\varphi(t)| \ll Kq$  and  $|\varphi(0)| + D|\varphi(t)| \ll Kp$ , for any  $t \in [-h, 0]$  and for any  $\varphi \in \mathcal{C}, \|\varphi\| \leq 1$ . Define  $u(t) := Ke^{-\beta(t-\sigma)}q, t \in [\sigma - h, \infty)$  and  $v(t) := Ke^{-\beta(t-\sigma)}p, t \in [\sigma, \infty)$ . It follows that  $|x(t)| \ll u(t), \forall t \in [\sigma - h, \sigma]$  and  $|y(\sigma)| \ll v(\sigma)$ . We claim that  $|x(t)| \leq u(t), \forall t \geq \sigma$  and  $|y(t)| \leq v(t), \forall t \geq \sigma$ .

Assume on the contrary that there exists  $t_0 > \sigma$  such that either  $|x(t_0)| \not\leq u(t_0)$  or  $|y(t_0)| \not\leq v(t_0)$ . Set  $t_1 := \inf\{t \in (\sigma, \infty) : (|x(t)|, |y(t)|) \not\leq (u(t), v(t))\}$ . By continuity,  $t_1 > \sigma$  and one of the following statements holds:

(i)  $|x(t)| \leq u(t), t \in [\sigma, t_1]$  and there is  $i_0 \in \underline{n}$  such that

$$(18) \quad |y(t)| \leq v(t), \forall t \in [\sigma, t_1]; |y_{i_0}(t_1)| = v_{i_0}(t_1), |y_{i_0}(\tau_k)| > v_{i_0}(\tau_k),$$

for some  $\tau_k \in (t_1, t_1 + \frac{1}{k}), k \in \mathbb{N}$ .

(ii)  $|y(t)| \leq v(t), t \in [\sigma, t_1]$  and there is  $k_0 \in \underline{n}$  such that

$$(19) \quad |x(t)| \leq u(t), \forall t \in [\sigma, t_1]; |x_{k_0}(t_1)| = u_{k_0}(t_1), |x_{k_0}(\xi_k)| > u_{k_0}(\xi_k),$$

for some  $\xi_k \in (t_1, t_1 + \frac{1}{k}), k \in \mathbb{N}$ .

Suppose (i) holds. Let  $A(t) := (a_{ij}(t)); \bar{\mu}(t, \theta) := (\bar{\mu}_{ij}(t, \theta)); \hat{\eta}(t, \theta) = (\hat{\eta}_{ij}(t, \theta))$ ,  $t \in \mathbb{R}$ ,  $\theta \in [-h, 0]$ . It follows from (16) that

$$\begin{aligned} \frac{d}{dt}|y_i(t)| &= \text{sgn}(y_i(t))\dot{y}_i(t) \leq a_{ii}(t)|y_i(t)| + \sum_{j=1, j \neq i}^n |a_{ij}(t)||y_j(t)| + \\ & \left| \sum_{j=1}^n \int_{-h}^0 d[\bar{\mu}_{ij}(t, \theta)]x_j(t + \theta) \right| + \left| \sum_{j=1}^n \int_{-h}^0 d[\hat{\eta}_{ij}(t, \theta)]x_j(t + \theta) \right|, \end{aligned}$$

for almost any  $t \in [\sigma, +\infty)$  and any  $i \in \underline{n}$ . It follows that for any  $t \in [\sigma, +\infty)$

$$\begin{aligned} D^+|y_i(t)| &:= \limsup_{h \rightarrow 0^+} \frac{|y_i(t+h)| - |y_i(t)|}{h} = \limsup_{h \rightarrow 0^+} \frac{1}{h} \int_t^{t+h} \frac{d}{d\zeta} |y_i(\zeta)| d\zeta \\ &\leq a_{ii}(t)|y_i(t)| + \sum_{j=1, j \neq i}^n |a_{ij}(t)||y_j(t)| + \\ & \sum_{j=1}^n \left| \int_{-h}^0 d[\bar{\mu}_{ij}(t, \theta)]x_j(t + \theta) \right| + \sum_{j=1}^n \left| \int_{-h}^0 d[\hat{\eta}_{ij}(t, \theta)]x_j(t + \theta) \right|, \end{aligned}$$

where  $D^+$  denotes the Dini upper-right derivative. In particular, it follows from (i) that

$$\begin{aligned} D^+|y_{i_0}(t_1)| &\leq a_{i_0 i_0}(t_1)K e^{-\beta(t_1 - \sigma)} p_{i_0} + \sum_{j=1, j \neq i_0}^n |a_{i_0 j}(t_1)| K e^{-\beta(t_1 - \sigma)} p_j + \\ & \sum_{j=1}^n (Var_{[-h, 0]} \bar{\mu}_{i_0 j}(t_1, \cdot) + \sum_{j=1}^n Var_{[-h, 0]} \hat{\eta}_{i_0 j}(t_1, \cdot)) K e^{-\beta(t_1 - \sigma)} e^{\beta h} q_j \\ &= K e^{-\beta(t_1 - \sigma)} \left( a_{i_0 i_0}(t_1) p_{i_0} + \sum_{j=1, j \neq i_0}^n |a_{i_0 j}(t_1)| p_j + \right. \\ & \left. \sum_{j=1}^n (Var_{[-h, 0]} \bar{\mu}_{i_0 j}(t_1, \cdot) + \sum_{j=1}^n Var_{[-h, 0]} \hat{\eta}_{i_0 j}(t_1, \cdot)) e^{\beta h} q_j \right) \\ &\stackrel{(14)}{<} -\beta K e^{-\beta(t_1 - \sigma)} p_{i_0} = D^+ v_{i_0}(t_1). \end{aligned}$$

On the other hand, (18) implies that

$$\begin{aligned} D^+|y_{i_0}(t_1)| &= \limsup_{t \rightarrow t_1^+} \frac{|y_{i_0}(t)| - |y_{i_0}(t_1)|}{t - t_1} \geq \overline{\lim}_{k \rightarrow \infty} \frac{|y_{i_0}(\tau_k)| - |y_{i_0}(t_1)|}{\tau_k - t_1} \\ &\geq \overline{\lim}_{k \rightarrow \infty} \frac{v_{i_0}(\tau_k) - v_{i_0}(t_1)}{\tau_k - t_1} = \lim_{k \rightarrow \infty} \frac{v_{i_0}(\tau_k) - v_{i_0}(t_1)}{\tau_k - t_1} = \frac{dv_{i_0}}{dt}(t_1) = D^+ v_{i_0}(t_1). \end{aligned}$$

This is a contradiction.

Assume that (ii) holds. It follows from (17) and (19) that

$$|x_{k_0}(t_1)| \stackrel{(17)}{\leq} |y_{k_0}(t_1)| + \sum_{j=1}^n \left| \int_{-h}^0 d[\mu_{k_0 j}(t_1, \theta)]x_j(t_1 + \theta) \right|$$



$$\begin{aligned}
&\leq |y_{k_0}(t_1)| + \sum_{j=1}^n \text{Var}_{[-h,0]} \mu_{k_0 j}(t_1, \theta) \sup_{\theta \in [-h,0]} |x_j(t_1 + \theta)| \\
&\stackrel{(ii)}{\leq} K e^{-\beta(t_1-\sigma)} p_{k_0} + \sum_{j=1}^n \text{Var}_{[-h,0]} \mu_{k_0 j}(t_1, \theta) K e^{-\beta(t_1-\sigma)} q_j e^{\beta h} \\
&= K e^{-\beta(t_1-\sigma)} \left( p_{k_0} + \sum_{j=1}^n \text{Var}_{[-h,0]} \mu_{k_0 j}(t_1, \theta) e^{\beta h} q_j \right) \\
&\stackrel{(15)}{<} K e^{-\beta(t_1-\sigma)} q_{k_0} = u_{k_0}(t_1).
\end{aligned}$$

This conflicts with (19). Thus, we have  $|x(t)| \leq u(t), \forall t \geq \sigma$  and  $|y(t)| \leq v(t), \forall t \geq \sigma$ . In particular, this yields  $|x(t; \sigma, \varphi)| \leq K e^{-\beta(t-\sigma)} q, \forall t \geq \sigma, \forall \varphi \in \mathcal{C}, \|\varphi\| \leq 1$ . Since (1) is linear, it follows that

$$|x(t; \sigma, \frac{\varphi}{\|\varphi\|})| = \frac{1}{\|\varphi\|} |x(t; \sigma, \varphi)| \leq K e^{-\beta(t-\sigma)} q, \quad \forall t \geq \sigma, \forall \varphi \in \mathcal{C}, \varphi \neq 0.$$

Therefore,

$$|x(t; \sigma, \varphi)| \leq K \|\varphi\| e^{-\beta(t-\sigma)} q, \quad \forall t \geq \sigma, \forall \varphi \in \mathcal{C}.$$

This yields

$$\|x(t; \sigma, \varphi)\| \leq K \|q\| e^{-\beta(t-\sigma)} \|\varphi\|, \quad \forall t \geq \sigma, \forall \varphi \in \mathcal{C}.$$

So (1) is exponentially stable. This completes the proof.  $\square$

**Remark 3.5.** (i) Let

$$(20) \quad V(t) := (V_{ij}(t)) \in \mathbb{R}_+^{n \times n}, t \in \mathbb{R},$$

where

$$V_{ij}(t) := \text{Var}_{[-h,0]} \bar{\mu}_{ij}(t, \cdot) + \text{Var}_{[-h,0]} \hat{\eta}_{ij}(t, \cdot), \quad t \in \mathbb{R}.$$

Note that (14) and (15) can be represented in the form

$$(21) \quad (M(A(t)))p + V(t)e^{\beta h}q \ll -\beta p$$

$$(22) \quad p + (\text{Var}_{[-h,0]} \mu(t, \cdot))e^{\beta h}q \ll q.$$

(ii) The proof of Theorem 3.4 also shows that  $\mathcal{D}(t, \varphi)$  exponentially decays. That is, there exist  $K_1, \beta > 0$  such that

$$\|\mathcal{D}(t, \varphi)\| \leq K_1 e^{-\beta(t-\sigma)} \|\varphi\|, \quad \forall t \geq \sigma, \forall \varphi \in \mathcal{C}.$$

**Theorem 3.6.** Let  $D \in \mathbb{R}_+^{n \times n}$  satisfy (6) and let  $V(t)$  be defined by (20). Suppose there exist  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}_+^{n \times n}$  such that

$$(23) \quad M(A(t)) \leq A, \quad \forall t \in \mathbb{R}; \quad V(t) \leq B, \quad \forall t \in \mathbb{R}.$$

Then (1) is exponentially stable if one of the following conditions holds:

(i) there exist  $\beta > 0$  and  $p, q \in \mathbb{R}_+^n, p \gg 0, q \gg 0$  such that

$$(24) \quad Ap + Be^{\beta h}q \ll -\beta p; \quad p + De^{\beta h}q \ll q.$$

(ii)  $\rho(D) < 1$  and  $s(A + B(I_n - D)^{-1}) < 0$ ;

(iii)  $s(A) < 0$  and  $\rho((-A)^{-1}B + D) < 1$ .

(iv) there exist  $p, q \in \mathbb{R}_+^n, p \gg 0, q \gg 0$  such that

$$(25) \quad \begin{pmatrix} A & B \\ I_n & D \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \ll \begin{pmatrix} 0 \\ q \end{pmatrix}.$$

*Proof.* Note that (ii), (iii) and (iv) of Theorem 3.6 are equivalent, by Lemma 2.2. By continuity, (iv) implies that (i) holds for sufficiently small  $\beta > 0$ . Thus, it remains to show that (1) is exponentially stable provided (i) holds.

Clearly, (14)-(15) follow from (23) and (31). Therefore, (1) is exponentially stable, by Theorem (3.4). This completes the proof.  $\square$

Consider the linear neutral differential system

$$(26) \quad \frac{d}{dt}(x(t) - \int_{-h}^0 d[\mu(t, \theta)]x(t+\theta)) = A(t)x(t) + \sum_{k=1}^r B_k(t)x(t-\tau_k) + \int_{-h}^0 C(t, \theta)x(t+\theta)d\theta, \quad t \geq \sigma,$$

where  $\mu(t, \theta)$  satisfies (8)-(9) and  $A(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}, B_k(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}, k \in \underline{r}$  and  $C(\cdot, \cdot) : \mathbb{R} \times [-h, 0] \rightarrow \mathbb{R}^{n \times n}$ , are given continuous functions and  $0 \leq \tau_k \leq h, k \in \underline{r}$ .

Let

$$(27) \quad V_0(t) := \sum_{i=1}^m |A(t)D_i(t)| + \int_{-h}^0 |A(t)E(t, s)|ds + \sum_{i=1}^r |B_k(t)| + \int_{-h}^0 |C(t, s)|ds, \quad t \in \mathbb{R}.$$

The following theorems are immediate from Theorem 3.4.

**Theorem 3.7.** Suppose there exist  $\beta > 0$  and  $p, q \in \mathbb{R}_+^n, p, q \gg 0$  such that

$$(28) \quad (M(A(t)))p + V_0(t)e^{\beta h}q \ll -\beta p, \quad \forall t \in \mathbb{R},$$

$$(29) \quad p + \left( \sum_{i=1}^m |D_i(t)| + \int_{-h}^0 |E(t, s)|ds \right) e^{\beta h}q \ll q, \quad \forall t \in \mathbb{R}.$$

Then (26) is exponentially stable.

**Theorem 3.8.** Let  $D_i \in \mathbb{R}_+^{n \times n}$  ( $i \in \underline{m}$ ) and  $E_0(\cdot) : [-h, 0] \rightarrow \mathbb{R}_+^{n \times n}$  satisfy (9). Assume that there exist  $A, B \in \mathbb{R}^{n \times n}$  such that

$$(30) \quad M(A(t)) \leq A, \quad t \in \mathbb{R}; \quad V_0(t) \leq B, \quad t \in \mathbb{R}.$$

Then (26) is exponentially stable if one of the following conditions holds:

(i) there exist  $\beta > 0$  and  $p, q \in \mathbb{R}_+^n, p \gg 0, q \gg 0$  such that

$$(31) \quad Ap + Be^{\beta h}q \ll -\beta p; \quad p + \left( \sum_{i=1}^m D_i + \int_{-h}^0 E_0(s) |ds \right) e^{\beta h} q \ll q;$$

(ii)  $\rho(\sum_{i=1}^m D_i + \int_{-h}^0 E_0(s) ds) < 1$  and  $s(A + B(I_n - \sum_{i=1}^m D_i - \int_{-h}^0 E_0(s) ds)^{-1}) < 0$ ;

(iii)  $s(A) < 0$  and  $\rho((-A)^{-1}B + \sum_{i=1}^m D_i + \int_{-h}^0 E_0(s) ds) < 1$ ;

(iv) there exist  $p, q \in \mathbb{R}_+^n, p \gg 0, q \gg 0$  such that

$$(32) \quad \begin{pmatrix} A & B \\ I_n & \sum_{i=1}^m D_i + \int_{-h}^0 E_0(s) ds \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \ll \begin{pmatrix} 0 \\ q \end{pmatrix}.$$

**Corollary 3.9.** Suppose  $A, B, D \in \mathbb{R}^{n \times n}$  are given and  $C(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  is a given continuous function. The linear neutral time-invariant differential system

$$(33) \quad \frac{d}{dt}(x(t) - Dx(t-r)) = Ax(t) + Bx(t-\tau) + \int_{-h}^0 C(\theta)x(t+\theta)d\theta,$$

is exponentially stable if

$$(34) \quad \rho(|D|) < 1; \quad s(M(A) + (|AD| + |B| + \int_{-h}^0 |C(\theta)|d\theta)(I_n - |D|)^{-1}) < 0.$$

**Corollary 3.10.** The linear differential system of retarded type

$$(35) \quad \frac{d}{dt}x(t) = A(t)x(t) + \int_{-h}^0 d[\hat{\eta}(t, \theta)]x(t+\theta), \quad t \geq \sigma.$$

is exponentially stable if there exists  $p \in \mathbb{R}^n, p \gg 0$  such that

$$(36) \quad (M(A(t)))p + (Var_{[-h,0]}\hat{\eta}_{ij}(t, \theta))e^{\beta h}p \ll -\beta p$$

Furthermore, if there exist  $A \in \mathbb{R}^{n \times n}$  and  $B = (b_{ij}) \in \mathbb{R}_+^{n \times n}$  such that

$$(37) \quad M(A(t)) \leq A, \quad t \in \mathbb{R}; \quad (Var_{[-h,0]}\hat{\eta}_{ij}(t, \cdot)) \leq B, \quad \forall t \in \mathbb{R},$$

and one of the following conditions holds:

(i)  $s(A + B) < 0$ ;

(ii)  $s(A) < 0$  and  $\rho((-A)^{-1}B) < 1$ .

(iii)  $(A + B)p \ll 0$  for some  $p \in \mathbb{R}_+^n$ .

then (35) is exponentially stable for any  $h > 0$ .

#### 4. DISCUSSION AND ILLUSTRATIVE EXAMPLES

As mentioned in the Introduction, problems of stability of linear neutral time-varying differential equations are not straightforward to solve. Up to our knowledge, there are no explicit criteria for exponential stability of linear neutral non-autonomous functional differential equations of the form (1). Some abstract criteria for asymptotic stability of (1) could be found in [9], [14] and recently in [8].

In [1], the authors considered the scalar linear neutral differential equation

$$(38) \quad \frac{d}{dt}(x(t) - d(t)x(t-r)) + a(t)x(t) + b(t)x(t-\tau) = 0,$$

where  $a(\cdot), b(\cdot), d(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+$  are continuous functions such that  $p_1 \leq a(t) \leq p_2; q_1 \leq b(t) \leq q_2, 0 \leq d(t) \leq d_1 < 1, \forall t \in \mathbb{R}$  and  $|d'(t)| \leq d_2, \forall t \in \mathbb{R}$  for some  $p_1, p_2, q_1, q_2, d_1, d_2 \in \mathbb{R}_+$ . Using a Lyapunov functional, it has been shown in [1, Theorem 2] that if

$$p_1 > q_2 + d_1(p_2 + q_2),$$

then every solution  $x(t)$  of (38) satisfies  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Some similar results can be found in [5], [9].

Note that  $-a(t) \leq -p_1, \forall t \in \mathbb{R}$  and  $|-b(t)| \leq q_2, \forall t \in \mathbb{R}$ . By Theorem 3.8 (ii), (38) is exponentially stable if  $-p_1 + (p_2d_1 + q_2)(1 - d_1)^{-1} < 0$ , or equivalently,

$$p_1 > q_2 + d_1(p_1 + p_2).$$

Note that the conditions  $|d'(t)| \leq d_2; a(t) \leq p_2; b(t) \geq q_1, \forall t \in \mathbb{R}$ , are redundant when we apply Theorem 3.8 (ii) to (38). Recall that, in general, exponential stability of linear neutral differential equations is stronger than their asymptotic stability, see [13].

Next, we consider a neutral logistic equation of the form

$$(39) \quad \frac{du(t)}{dt} = \gamma u(t) \left[ p \left( 1 - \frac{u(t)}{k} \right) + q \left( 1 - \frac{u(t-h)}{k} \right) + c \frac{d}{dt} \left( 1 - \frac{u(t-\tau)}{k} \right) \right],$$

where  $\gamma, p, q, c, k, h, \tau$  are positive numbers.

Using the change of variable,  $y(t) = \ln \frac{u(t)}{k}$ , it has been shown in [1] that if the linear equation

$$(40) \quad \frac{d}{dt}(y(t) + \gamma c y(t-\tau)) = -\gamma p y(t) - \gamma q y(t-h),$$

is exponentially stable, then the steady state value  $k$  of (39) is locally asymptotically stable. Applying Theorem 3.7 to (40), we conclude that (40) is exponentially stable if

$$\gamma c < \frac{p-q}{2p}.$$

To illustrate the advantage of the results of this paper, we present a numerical example for which the stability criteria given in [1], [7], [16], [18] cannot be applied.

Consider the linear time-varying differential system of neutral type defined by

$$\begin{aligned} \dot{\zeta}(t) &= -4\zeta(t) - (\sin t)\zeta(t) - \frac{1}{t^2 + 16}\eta(t-1) + \frac{1}{16}e^{-t^2} \int_{-1}^0 \eta(t+s)ds + \frac{1}{4}\dot{\zeta}(t-1) \\ \dot{\eta}(t) &= -3\eta(t) - e^{-t^2}\eta(t) + \frac{1}{6}e^{-t^2}\zeta(t-1) - \frac{1}{8}\dot{\eta}(t-1). \end{aligned}$$

Clearly, the given system can be rewritten as

$$(41) \quad \frac{d}{dt}(x(t) - Dx(t-1)) = A(t)x(t) + B(t)x(t-1) + \int_{-1}^0 C(t)x(t+s)ds,$$

where  $x(t) := (\zeta(t), \eta(t))^T \in \mathbb{R}^2, t \in \mathbb{R}$  and

$$D := \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{8} \end{pmatrix}; \quad A(t) := \begin{pmatrix} -4 - \sin t & 0 \\ 0 & -3 - e^{-t^2} \end{pmatrix},$$

$$B(t) := \begin{pmatrix} 0 & -\frac{1}{t^2+16} \\ \frac{1}{6}e^{-t^2} & 0 \end{pmatrix}; \quad C(t) := \begin{pmatrix} 0 & \frac{1}{16}e^{-t^2} \\ 0 & 0 \end{pmatrix}.$$

Let us define

$$A := \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}; \quad B := \begin{pmatrix} \frac{5}{4} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix}.$$

It is easy to check that  $\rho(|D|) < 1$  and

$$M(A(t)) \leq A, \quad \forall t \in \mathbb{R}; \quad |A(t)D| + |B(t)| + \int_{-1}^0 |C(t)|ds \leq B, \quad \forall t \in \mathbb{R}.$$

Thus, (30) holds. By simple computation, we have

$$A + B(I_2 - |D|)^{-1} = \begin{pmatrix} -\frac{4}{3} & \frac{1}{7} \\ \frac{2}{9} & -\frac{17}{7} \end{pmatrix}.$$

Because of

$$\begin{pmatrix} -\frac{4}{3} & \frac{1}{7} \\ \frac{2}{9} & -\frac{17}{7} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ll \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

the Metzler matrix  $A + B(I_2 - |D|)^{-1}$  is Hurwitz stable, by Theorem 2.1. Thus, the given system is exponentially stable, by Theorem 3.8 (ii).

Finally, a remarkable result on asymptotic stability of the linear neutral time-invariant differential systems can be found in [17]. It has been shown [17, Theorem 3.1] that (33) with  $C(\cdot) \equiv 0$  is asymptotically stable if

$$(42) \quad \|D\| < 1 \quad \text{and} \quad \mu(A) + \frac{\|B\| + \|A\|\|D\|}{1 - \|D\|} < 0,$$

where  $\mu(A) := \lim_{\epsilon \rightarrow 0^+} \frac{\|I_n + \epsilon A\| - 1}{\epsilon}$  is the matrix measure of  $A$ . It is quite easy to show that (42) is more conservative than (34) (with  $C(\cdot) \equiv 0$ ).

## 5. CONCLUDING REMARK

In this paper, we have addressed the challenging problem of stability analysis for linear non-autonomous systems described by functional differential equations of neutral type and presented some novel conditions for exponential stability of time-varying functional differential equations. By using a system transformation, the comparison principle and spectral properties of Metzler matrices, the resulting criteria are both generic and explicitly obtained. As such, the proposed approach can be extended to study problems of exponential stability of nonlinear and singular time-varying systems, which will be subject to our future work.

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