PRICING SWAPTIONS AND CREDIT DEFAULT SWAPTIONS IN THE QUADRATIC GAUSSIAN FACTOR MODEL

SAMSON ASSEFA

A DISSERTATION
SUBMITTED TO THE UNIVERSITY GRADUATE SCHOOL
AND THE FACULTY OF BUSINESS OF
THE UNIVERSITY OF TECHNOLOGY, SYDNEY,
IN FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
FINANCE AND ECONOMICS

MARCH 2007

CERTIFICATE OF AUTHORSHIP/ORIGINALITY

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Signature of Candidate

Production Note:
Signature removed prior to publication.

ACKNOWLEDGMENTS

I would like to thank my supervisor Associate Professor Erik Schlögl for the numerous fruitful discussions that made it possible for me to write this thesis. His support and encouragement has made it possible for me to have the confidence to complete my thesis work. I would also like to express my gratitude to my supervisor Professor Carl Chiarella for his support and interest in my research work. I would like to thank Professor Tony Hall for his support and encouragement especially during times of crisis which enabled me to continue to study for my PhD degree. I am also thankful for the support and encouragement of Professor Eckhard Platen who made it possible for me to become a PhD student at UTS. For being available to discuss problems about mathematical finance and for his support and encouragement during the first two years of my PhD studies, I would like to thank Professor Alex Novikov. For useful discussions regarding reduced form models of credit risk, I would like to thank Professor Marek Rutkowski. My special thanks go to my family for letting me pursue my dreams. I would like to express my deepest gratitude to the Quantitative Finance Research Center (QFRC) and the School of Finance and Economics of UTS for the financial support in the form of a scholarship for four years that made it possible for me to complete my PhD studies. I have received financial support from FIRN to attend important workshops and conferences in Australia. For this financial support I would like to thank the Financial Integrity Research Network (FIRN). I would like to thank the office staff at the School of Finance and Economics for the help and support they have provided me on numerous occasions. I would like to thank posthumously my friend Nicola Bruti-Liberati for the support and numerous discussions that he provided during my PhD studies.

CONTENTS

ABSTRACT

In this thesis we show how the multi-factor quadratic Gaussian model can be used to price default free and defaultable securities. The mathematical tools used include the theory of stochastic processes, the theory of matrix Riccati equations, the change of measure technique, Itô's formula, use of Fourier Transforms in swaption valuation and approximation methods based on replacing the values of some stochastic processes by their time zero values.

The first chapter of the thesis deals with the derivation of efficient closed form formulas for the price of zero coupon bonds in the multi-factor quadratic Gaussian model and the calibration of the multi-factor quadratic Gaussian model to the domestic and foreign forward rate term structures through closed form formulas.

In the second chapter of the thesis, we derive approximations for the price of default free swaptions which are based on log-quadratic Gaussian processes. Using numerical experiments, we show the limitations of these approximations. We also give some numerical results for the pricing of a default free swaption using moment-based density approximants of the probability density function of the swaption's payoff.

The third chapter of the thesis deals with the calibration of a quadratic Gaussian reduced form model of credit risk to the default free forward rate curve and to the survival probability of an obligor. We also consider different approximations for the price of credit default swaptions. Using numerical experiments, we show the limitations of the approximations.

The final chapter of this thesis considers a two country reduced form model of credit risk. We examine the relationship between the domestic forward credit spread and the foreign forward credit spread of an obligor and provide quanto adjustment formulas for the probability of survival of an obligor. In the final part of this chapter, we show that the valuation of a quanto default swap is tractable in a contagion type reduced form model of credit risk which assumes that underlying processes are modelled by quadratic Gaussian processes.

LIST OF FIGURES

LIST OF TABLES

ABBREVIATIONS AND NOTATION

Notation for Chapter 1:

- $\log_e(x)$=Natural Logarithm of x i.e. logarithm to the base e.
- $Tr[A]$=Trace of the rectangular matrix A.
- I_n=The identity matrix of dimension n.
- $0_{n\times n}$=The zero matrix which is the square matrix of dimension $n \times n$ which has all its elements equal to zero.
- 0_n=The zero column vector which is the column vector of length n such that all its elements are equal to zero.
- $\mathbf{1}_n := (1,\ldots,1)^\top$=Column vector of dimension n which has all its elements equal to the number one.
- SDE=Stochastic differential equation;
- $\dot{F}$=first order derivative of the time dependent (matrix) function F.
- RDE=Matrix Riccati differential equation;
- $\mathbb{Q}$: (Domestic) risk neutral measure for default free and defaultable economy.
- $\mathbb{Q}^f$: Foreign risk neutral measure for default free and defaultable economy.
- $\mathbb{E}^{\mathbb{Q}}$: Expectation under the probability measure $\mathbb{Q}$.
- $\mathbb{E}^{\mathbb{Q}^f}$: Expectation under the probability measure $\mathbb{Q}^f$.
- W_t: Standard multi-dimensional Brownian motion under the Risk Neutral Measure.
- $\mathbb{F} = (\mathcal{F}_t)_{(0\leqslant t\leqslant T^*)}$: Filtration generated by W_t representing default free market information.
- τ: Default time of an obligor or a corporation.
- H_t: Indicator function for default time τ.
- $\mathcal{H}_t = \sigma(H_u : u \leqslant t)$: Filtration generated by H_t.

- $\mathcal{G}_t = \sigma(G_u : u \leqslant t) = \sigma(F_u \vee H_u : u \leqslant t)$: Filtration generated by default free market information and whether default has taken place or not.
- r_t: The (domestic) default free instantaneous rate of interest rate.
- r_t^f: The foreign default free instantaneous rate of interest rate.
- I^d: Diagonal matrix with a one for row i if the ith factor is used to model r_t and a zero otherwise.
- I^f: Diagonal matrix with a one for row i if the ith factor is used to model r_t^f and a zero otherwise.
- Y_t: Gaussian Ornstein Uhlenbeck process with zero drift.
- Z_t: Gaussian Ornstein Uhlenbeck process used for modeling state variables.
- A: Constant diagonal matrix used to denote the speed of mean reversion matrix in the SDE of Y_t.
- Σ: Constant matrix used to denote the instantaneous volatility in the SDE of Y_t.
- $\alpha(t)$: A time dependent deterministic vector function used to calibrate Z_t to the term structure of default free zero coupon bonds .
- $\alpha^f(t)$: A time dependent deterministic vector function used to calibrate Z_t to the foreign term structure of default free zero coupon bonds.
- $\widehat{C}$: Constant symmetric matrix used to model the quadratic part of r_t in the quadratic Gaussian multifactor model.
- $\widehat{B}$: Time dependent deterministic vector function used to model the linear part of r_t in the quadratic Gaussian multifactor model.
- $\widehat{A}$: Time dependent deterministic scalar function used to model the scalar part of r_t in the quadratic Gaussian multifactor model.
- $D(t)$: The default free savings account.
- λ_t: The intensity of default in a reduced form model.
- $\widetilde{C}$: Constant symmetric matrix used to model the quadratic part of λ_t in the quadratic Gaussian multifactor model.
- $\widetilde{B}$: Time dependent deterministic vector function used to model the linear part of λ_t in the quadratic Gaussian multifactor model.
- $\widetilde{A}$: Time dependent deterministic scalar function used to model the scalar part of λ_t in the quadratic Gaussian multifactor model.

- $P(t,T)$: Price of the domestic default free zero coupon bond at time t for maturity T.
- $P^f(t,T)$: Price of the foreign default free zero coupon bond at time t for maturity T.
- S_t: The price of one unit of foreign currency in terms of domestic currency.
- $F(t,T)$: The (domestic) default free instantaneous forward rate.
- $F^f(t,T)$:The foreign default free instantaneous forward rate.
- $\mathbf{1}_{\tau>t}\bar{P}(t,T)$: Price of defaultable zero coupon bond at time t for maturity T.
- $C(t,T)$: Symmetric positive definite matrix used to express the quadratic part of $\log(P(t,T))$ in the quadratic Gaussian multifactor model.
- $B(t,T)$: Time dependent vector used to express the linear part of $\log(P(t,T))$ in the quadratic Gaussian multifactor model.
- $A(t,T)$: Time dependent scalar function used to express the scalar part of $\log(P(t,T))$ in the quadratic Gaussian multifactor model.
- $\bar{C}(t,T)$: Symmetric positive definite matrix used to express the quadratic part of $\log(\bar{P}(t,T))$ in the quadratic Gaussian multifactor model.
- $\bar{B}(t,T)$: Time dependent vector used to express the linear part of $\log(\bar{P}(t,T))$ in the quadratic Gaussian multifactor model.
- $\bar{A}(t,T)$: Time dependent scalar function used to express the scalar part of $\log(\bar{P}(t,T))$ in the quadratic Gaussian multifactor model.
- $C^S(t)$: Symmetric positive definite matrix used to express the quadratic part of $\log(S_t)$ in the quadratic Gaussian multifactor model.
- $B^S(t)$: Time dependent vector used to express the linear part of $\log(S_t)$ in the quadratic Gaussian multifactor model.
- $A^S(t)$: Time dependent scalar function used to express the scalar part of $\log(S_t)$ in the quadratic Gaussian multifactor model.
- $A^f(T) := A + 2\Sigma\Sigma^\intercal C^S(T)$: Time dependent speed of mean reversion matrix for the dynamics of Y_t under the foreign risk neutral measure.
- $\mathbb{T}$: Default free forward measure for maturity T corresponding to using $P(t,T)$ as the numeraire.
- $\mathbb{E}^{\mathbb{T}}$: Expectation under the (domestic) default free forward measure.

- $\mathbb{T}^f$: Foreign default free forward measure for maturity T corresponding to using $P^f(t,T)$ as the numeraire.
- $\mathbb{E}^{\mathbb{T}^f}$: Expectation under the foreign default free forward measure.
- $\Phi(\Omega, z)$: Characteristic function of the quadratic form Ω in Gaussian random variables.
- $M(t,T)$: The mean of Y_T under $\mathbb{T}$ conditional on $\mathcal{F}_t$.
- $V(t,T)$: The variance-covariance matrix of Y_T under $\mathbb{T}$ conditional on $\mathcal{F}_t$.
- $M^f(t,T)$: The mean of Y_T under $\mathbb{T}^f$ conditional on $\mathcal{F}_t$.
- $V^f(t,T)$: The variance-covariance matrix of Y_T under $\mathbb{T}^f$ conditional on $\mathcal{F}_t$.

Notation for Chapter 2:

- FFT: Fast Fourier Transform;
- DFT: Discrete Fourier Transform;
- ATM: At the money strike rate;
- ITM: In the money strike rate;
- OTM: Out of the money strike rate;
- bp: Basis points;
- CMS: Constant Maturity Swap;
- LLM: Lognormal Libor Market Model;
- PVBP01: Present Value of a Basis Point;
- $Re[c]$: Real part of the complex number c.
- $\mathbb{T}_\alpha$: The default free forward measure corresponding to using $P(t,T_\alpha)$ as the numeraire.
- $\mathbb{E}^{\mathbb{T}_\alpha}$: Expectation under $\mathbb{T}_\alpha$.
- $\mathcal{T} = \{T_{\alpha+1}, \ldots, T_\beta\}$: Payment dates for a forward swap starting on initial day T_α and ending on final day T_β.
- K: Fixed rate payed in a default free interest rate swap by the receiver.
- τ_i: Year fraction between T_{i-1} and T_i.

- $Swap_{\alpha,\beta}(t)$: The swap rate at time t which is the value of the fixed rate K that will make the value of a swap with starting date T_α and ending date T_β equal to zero.
- $Swaptn_{\alpha,\beta}(t)$: The exact price of a European payer swaption with maturity date T_α to enter a swap starting on date T_α and ending T_β.
- $\widetilde{P}_{\alpha,\beta}(t)$: The log-quadratic Gaussian process that is used to derive approximation to swaption prices.
- $w_i(t)$: The ith weight obtained by dividing $K\tau P(t,T_i)$ by $\widetilde{P}_{\alpha,\beta}(t)$ for $i = \alpha+1,\ldots,\beta-1$ and $(1+K\tau_\beta)P(t,T_\beta)$ by $\widetilde{P}_{\alpha,\beta}(t)$ for $i=\beta$.
- $\Phi(Q_1,\ldots,Q_N,z_1,\ldots,z_N)$: Joint characteristic function of N quadratic forms in Gaussian random variables dented by $Q_1,\ldots,Q_N$.
- $G_i(x,k)$: Type of payoff function which varies with i.
- $\hat{C}_{G_i}(z)$: Fourier transform with respect to strike of an option with payoff given by $G_i(x,k)$
- $\widetilde{Swaptn}_{\alpha,\beta}(t)$: The approximation of a swaption price that is obtained by approximating the exercise region through $\widetilde{P}_{\alpha,\beta}(t)$.
- $Swaptn1_{\alpha,\beta}(t)$: The approximation of a swaption price that is obtained by approximating the exercise region and the payoff through $\widetilde{P}_{\alpha,\beta}(t)$.
- $P_{\alpha,\beta}(t)$: The present value of a basis point.
- $\mathbb{Q}_{\alpha,\beta}$: The swap measure corresponding to using $P_{\alpha,\beta}(t)$ as the numeraire.
- $\mathbb{E}^{\mathbb{Q}_{\alpha,\beta}}$: Expectation under $\mathbb{Q}_{\alpha,\beta}$.
- $L(t,T_i)$: Forward libor rate for period $[T_i,T_{i+1}]$;
- K_{ATM}: The at the money strike rate of a default free swaption.
- K_{ITM}: An in the money strike rate of a default free swaption.
- K_{OTM}: An out of the money strike rate of a default free swaption.

Notation for Chapter 3:

- CIR: Cox-Ingersoll-Ross;
- CDS: Credit Default Swap;
- DPVBP: defaultable present value of a basis point;

- $\mathcal{T} = \{T_{n+1}, \ldots, T_N\}$: Payment dates for the premium leg of a credit default at time $T_n < T_{n+1}$.
- β_i: Year fraction between T_i and T_{i-1}.
- $\zeta(\tau)$: The last premium payment before default or the premium date on which default occurred if default coincided with the premium date.
- K: The premium rate of a CDS
- Z: Deterministic amount payed as default protection in case of default in CDS contract.
- δ: The recovery rate which is used to determine the default protection payment amount.
- $CDS(t, \mathcal{T}, T, K, Z)$: The value at time t of a payer forward credit default swap starting at time T with premium payment schedule $\mathcal{T}$, premium rate K and default protection payment Z.
- $CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)$: The value of a credit default swaption at time t which gives the owner of the swaption to enter into a CDS at time T paying a premium rate of K to get a default protection of Z.
- $R_f(T)$: The market CDS rate which is the value of the premium rate that would make the value of a CDS equal to zero.
- $G(0,T)$: The probability of survival of an obligor under the risk neutral measure.
- $\bar{G}(0,T)$: The probability of survival of an obligor under the default free forward measure.
- g_j: The conditional probability of default over (T_j, T_{j+1}).
- $H(0,T)$: The probability of default of an obligor under the risk neutral measure.
- $d(g_j, g_{j+1})$: Distance between the conditional probabilities of default.
- ν: Parameter used to determine smoothness of probability of survival.
- σ_k: Estimate of Gaussian error in market CDS quotes.
- $\overline{M}(t,T)$: The mean of Y_T under the defaultable forward measure.
- $\bar{V}(t,T)$: The variance-covariance matrix of Y_T under the defaultable forward measure.
- $U_{n,N}(T_n)$: The defaultable present value of a basis point.

- $\bar{w}_j(T_n)$: The ratio of the defaultable bond $\bar{P}(T_n, T_j)$ and the defaultable present value of a basis point.
- $\mathbb{U}$: The measure which is absolutely continuous to the risk neutral measure corresponding to using the defaultable present value of a basis point as the numeraire.
- $\mathbb{E}^{\mathbb{U}}$: Expectation under the measure $\mathbb{U}$.
- K_{DATM}: The at the money strike rate of a credit default swaption.
- K_{DITM}: An in the money strike rate of a credit default swaption.
- K_{DOTM}: An out of the money strike rate of a credit default swaption.

Notation for Chapter 4:

- $\mathbb{Q}^d$: The domestic risk neutral measure for default free and defaultable securities.
- $\mathbb{Q}^f$: The foreign risk neutral measure for default free and defaultable securities.
- $\mathbb{E}^{\mathbb{Q}^d}$: Expectation under the domestic risk neutral measure denoted by $\mathbb{Q}^d$.
- $\mathbb{E}^{\mathbb{Q}^f}$: Expectation under the foreign risk neutral measure denoted by $\mathbb{Q}^f$.
- W_t^d: Standard Brownian motion under the domestic risk neutral measure denoted by $\mathbb{Q}^f$.
- W_t^f: Standard Brownian motion under the foreign risk neutral measure denoted by $\mathbb{Q}^f$.
- τ: Default time of a reference entity(corporation or obligor).
- τ^d: Default time of a reference entity(corporation or obligor) in the domestic economy.
- τ^f: Default time of a reference entity(corporation or obligor) in the foreign economy.
- λ: Intensity of default for the default time τ.
- λ^d: Intensity of default for the default time τ^d.
- λ^f: Intensity of default for the default time τ^f.
- $H_t = \mathbf{1}_{\tau \leqslant t}$: Indicator function for default time τ.
- $H_t^d = \mathbf{1}_{\tau \leqslant t}$: Indicator function for default time τ^d.

- $H_t^f = \mathbf{1}_{\tau \leqslant t}$: Indicator function for default time τ^f.
- $\mathcal{F}_t$: Filtration generated by W_t^d.
- $\mathcal{H}_t$: Filtration generated by H_t.
- $\mathcal{G}_t$: Filtration generated by $\mathcal{F}_t \vee \mathcal{H}_t^d \vee \mathcal{H}_t^f$.
- Λ^d: The $(\mathcal{F}_t, \mathbb{Q})$-martingale hazard process of τ^d.
- Λ^f: The $(\mathcal{F}_t, \mathbb{Q})$-martingale hazard process of τ^f.
- Γ^d: The $\mathcal{F}_t$-hazard process of τ^d.
- Γ^f: The $\mathcal{F}_t$-hazard process of τ^f.
- $P^d(t,T)$: The price of a domestic default free zero coupon bond.
- $P^f(t,T)$: The price of a foreign default free zero coupon bond.
- $\mathbf{1}_{\tau > t}\bar{P}^d(t,T)$: The price of a domestic defaultable zero coupon bond when there is cross default of the entity(corporation or obligor).
- $\mathbf{1}_{\tau > t}\bar{P}^f(t,T)$: The price of a foreign defaultable zero coupon bond when there is cross default of the entity(corporation or obligor).
- $\mathbf{1}_{\tau^d > t}\bar{P}^d(t,T)$: The price of a domestic defaultable zero coupon bond when there is no cross default of the entity(corporation or obligor).
- $\mathbf{1}_{\tau^f > t}\bar{P}^f(t,T)$: The price of a foreign defaultable zero coupon bond when there is no cross default of the entity(corporation or obligor).
- S_t: The foreign exchange representing the price of one unit of currency in terms of domestic currency.
- σ_S: The instantaneous volatility of the foreign exchange rate.
- $X(t,T)$: The forward exchange rate.
- $\sigma_M(t,T)$: The instantaneous volatility of the forward exchange rate.
- $\bar{X}(t,T)$: The defaultable forward exchange rate.
- $\mathbb{T}^d$: The domestic forward measure corresponding to using $P^d(t,T)$ as the numeraire.
- $\mathbb{T}^d$: The foreign forward measure corresponding to using $P^f(t,T)$ as the numeraire.
- $\overline{\mathbb{T}}^d$: The defaultable forward measure for the domestic economy corresponding to using the domestic defaultable zero coupon bond as the numeraire.

- $\overline{\mathbb{T}}^f$: The defaultable forward measure for the foreign economy corresponding to using the foreign defaultable zero coupon bond as the numeraire.
- $f^d(t,T)$: The domestic continuously compounded default free instantaneous forward rates.
- $f^f(t,T)$: The foreign continuously compounded default free instantaneous forward rates.
- $\bar{f}^d(t,T)$: The domestic continuously compounded defaultable instantaneous forward rates.
- $\bar{f}^f(t,T)$: The foreign continuously compounded defaultable instantaneous forward rates.
- $s^d(t,T)$: The domestic continuously compounded instantaneous forward rate spread.
- $s^f(t,T)$: The foreign continuously compounded instantaneous forward rate spread.
- $\sigma^d(t,T)$: The volatility of domestic default free instantaneous forward rates.
- $\sigma^d_s(t,T)$: The volatility of the domestic continuously compounded instantaneous forward rate spread.
- $\sigma^f_s(t,T)$: The volatility of foreign continuously compounded instantaneous forward rate spread.
- $\bar{G}^d(t,T)$: The probability of survival of the reference entity(corporation or obligor) under the domestic forward measure.
- $\bar{G}^f(t,T)$: The probability of survival of the reference entity(corporation or obligor) under the foreign forward measure.
- σ^d_g: The instantaneous volatility of the probability of survival of the reference entity(corporation or obligor) under the domestic forward measure.
- σ^f_g: The instantaneous volatility of the probability of survival of the reference entity(corporation or obligor) under the foreign forward measure.
- $\eta^d(t,T)$: The instantaneous volatility of $P^d(t,T)$.
- $\eta^f(t,T)$: The instantaneous volatility of $P^f(t,T)$.
- $\bar{\eta}^d(t,T)$ The instantaneous volatility of $\bar{P}^d(t,T)$.
- $\bar{\eta}^f(t,T)$ The instantaneous volatility of $\bar{P}^f(t,T)$.
- η^i_t: Diffusion part of the intensity of default λ^i_t in a contagion type model.

- α^d: The jump in the intensity of default in the domestic market due to the default of the obligor in the foreign market.
- α^f: The jump in the intensity of default in the foreign market due to the default of the obligor in the domestic market.
- $s^i(t,T)$: Forward credit spread process for the obligor in the ith economy.
- η_{ij}: Deterministic function representing the jump in the forward credit spread $s^i(t,T)$ due to default of obligor in the jth market.

INTRODUCTION

In this thesis we extend some results that have been given for the default free single factor quadratic Gaussian model (see, e.g., Pelsser (2000) and Jamshidian (1996)) to the multi-factor quadratic Gaussian model for default free and defaultable market. We also consider the extension of the reduced form model for credit risk (see Bielecki and Rutkowski (2002)) to include a foreign economy besides a domestic one. The extension is made under the assumptions of cross default[1] and absence of cross default.

The Black and Scholes (BLSC) model (see Black (1976) and Black & Scholes (1973)) is used by market participants to quote the implied volatilities of caps, floors and swaptions. The BLSC model assumes that the libor (swap) rate has constant volatility contrary to the fact that the implied volatility of quoted caps as well as floors (swaptions) exhibit a term structure of volatilities that vary with the strike price and maturity of the interest rate option. This term structure of implied volatilities is referred to the smile or the skew implied by the interest rate option. In order to be consistent with the smile, various models have been proposed in the literature. The lognormal libor rate model (LLM) and the lognormal swap rate model (LSM) (see Brace, Gatarek and Musiela (1997), Musiela and Rutkowski (2005) and Brigo and Mercurio (2006) for a detailed discussion) have been extended by some researchers to account for the smile through the introduction of stochastic volatility and jumps in the dynamics of the libor or swap rate. However using a libor market model or a swap market model to price exotic interest rate contracts can be a computationally intensive task. Therefore some market participants prefer to use factor models to price interest rate contracts. Moreover there has been some suggestions by some practitioners to use models that require less computational effort (see, e.g, Mercurio and Pallavivini (2005) or Andreasen (2006)). The quadratic Gaussian model can be used as a basis for such a model similar to what has been done in Mercurio and Pallavivini (2005) using

[1] By cross default, we mean that the default of the obligor in one of the markets (domestic or foreign) will lead to the default of the obligor in the other market.

a Gaussian short rate model. The quadratic Gaussian model is also a tractable model that can be use for the pricing of credit sensitive contracts. In Brigo and Alfonsi (2004), a reduced form model of credit risk based on the Cox-Ingersoll-Ross (CIR) process (see Cox et al. (1985)) was introduced. However in this model, we do not have closed form formulas for defaultable zero coupon bonds if we assume that there is correlation between the short term of interest rate and the intensity of default. In contrast to the reduced form model of credit risk based on CIR processes, the multi-factor quadratic Gaussian model enables the calculation of the price of defaultable zero coupon bonds when there is correlation between the short term of interest and the intensity of default[2]. However the computation of the closed form price of the default free and defaultable zero coupon bonds can be computationally intensive. In chapter one, we give computationally efficient formulas for the computation of the price in the multi-factor quadratic Gaussian model extending the formulas that were given in Pelsser (2000) for the single factor quadratic Gaussian model. In addition to deriving efficient formulas for the price of zero coupon bonds, we also extend the formula which is used to calibrate the quadratic Gaussian model (see Pelsser (2000)) to the term structure of default free zero coupon bonds from the single factor model to the general multi-factor model. In this chapter, we also provide closed form formulas for the calculation of zero coupon bond prices in the foreign economy and a closed form formula for the calibration of the multi-quadratic Gaussian model to the the term structure of foreign default free zero coupon bonds. In the last section of chapter one, we consider a multi-factor quadratic Gaussian model where the Gaussian process Y_t which is used to model the state variables has a piecewise constant speed of mean reversion matrix and piecewise constant instantaneous volatility. For such a model, we provide a method for calculating the price of zero coupon bonds in closed form.

Chapter two considers the pricing of default free swaptions. In the multi-factor quadratic Gaussian model, one can price caps and floors accurately and efficiently using the Fourier transform technique. The exact pricing of swaptions on the other hand cannot be done through closed form formulas in the multi-factor quadratic Gaussian model. For pricing correlation sensitive interest rate contracts, it is important to calibrate to market information which is represented by the quotes of liquid swaption prices (see Brigo and Mercurio (2006) and Mercurio and Pallavivini

[2] See Grasselli and Tebaldi (2004) for a discussion regarding solvability of the CIR model and the quadratic Gaussian model.

(2005)). Since we do not have exact formulas for swaption prices in the multi-factor quadratic Gaussian model, it is important to have approximations which can make the calibration of the model to swaption prices easier. Hence in this chapter we investigate different approximations that are based on replacing the ratio of default free zero coupon price to the sum of default free zero coupon bond prices by their time zero values. This technique was introduced in Rebonato (1998) and is popular in deriving approximation to the price of interest rate options (see, e.g. Brigo and Mercurio (2006), d'Aspremont (2003) and Schrager and Pelsser (2006)). We try to investigate the limitations of this technique in the context of a multi-factor quadratic Gaussian model through numerical experiments.

Chapter three provides a method to extract the term structure of survival probabilities that is implied by the quotes of credit default swaps (CDS). This method is an extension of the method proposed in Martin et al. (2001) for a reduced form model with a deterministic intensity of default to a reduced form model with stochastic intensity of default. We then consider a quadratic Gaussian model for the intensity of default and provide closed form formulas for the calibration of the model to the term structure of survival probabilities when the intensity of default is independent of the instantaneous rate of interest. If the intensity of default and the instantaneous rate of interest are not independent, we provide a method for calibrating the intensity of default to the term structure of survival probabilities through the numerical solution of an ordinary differential equation (ODE). In the last section of this chapter, we provide various approximations for the price of a credit default swaption. These approximations are derived through a similar technique used to approximate the price of default free swaptions. In this case we replace the ratio of defaultable zero coupon bond prices to the sum of defaultable zero coupon bonds by their time zero values. We present some numerical results that indicate the limitations of such an approach.

The last chapter of this thesis considers the extension of the reduced form model of credit risk to a two country setting. We first show that if we use a Heath Jarrow Morton (HJM) model for the instantaneous defaultable forward rates (see Bielecki and Rutkowski (2002) and Schönbucher (2000)), we do not have the freedom to specify the volatilities of the domestic defaultable forward rates, the volatilities of the foreign defaultable forward rates and the volatilities of the foreign exchange rate independent of each other. We then give a quanto adjustment formula that can be used to obtain the foreign probability of survival from the domestic probability of survival or vice versa. This extends the result that is given in the Gaussian case (see, e.g. Anderson (2003)) to other factor models. We also consider a contagion type model for the intensity of default by assuming that an obligor has different intensity of default in the domestic and foreign market but default of the obligor in one market leads to a jump in the intensity of default in the other market. Assuming a quadratic Gaussian intensity of default in both markets, we show how we can price a quanto default swap in this contagion type model.

1. QUADRATIC GAUSSIAN FACTOR MODELS

In this section we review some basic properties of quadratic Gaussian factor models for the default free instantaneous rate of interest and the intensity of default. The quadratic Gaussian factor model was introduced by Beaglehole and Tenney (1991) and analyzed further in El Karoui et al. (1991) and Jamshidian (1996). There has been further work on applying the quadratic Gaussian factor model to price default free interest rate derivatives and empirical results on fitting the model to historical data (see Cherif and Durand (1995), Boyle and Tian (1999), Pelsser (2000), Ahn, Dittmar and Gallant (2002), Leippold and Wu (2002), Leippold and Wu (2003), Chen et al. (2003), Kim (2004) and Chen et al. (2006)). Quadratic Gaussian models for default free interest rates have been extended to a multi-country context in Cherif et al. (1994) and Leippold and Wu (2002). In this chapter, we first extend the result given in Pelsser (2000) regarding the efficient computation of bond prices in a single factor quadratic Gaussian model to the multifactor setting. We then provide a closed form calibration formula for the multifactor quadratic Gaussian model extending the result given in Pelsser (2000) for the single factor quadratic Gaussian model and the result given in Jamshidian (1996) for the separable[1] multifactor quadratic Gaussian model.

We now consider a model where besides default free assets, defaultable assets are traded. We assume that we have a filtered probability space $(\tilde{\Omega}, \mathcal{G}, \mathbb{Q})$ where $\mathbb{Q}$ is the risk-neutral measure. We assume the filtration $\mathbb{F} = (\mathcal{F}_t)_{(0 \leqslant t \leqslant T^*)}$ is generated by n independent Brownian motions $W(t) = W^i(t), i = 1, \ldots, n$ and satisfies the usual conditions. The time horizon is assumed to be finite so that $T^* > 0$ is some finite number. Let τ denote the default time of a corporate which has issued defaultable bonds. Let $H_t = \mathbf{1}_{\tau \leqslant t}$ represent the default indicator function. Let $\mathcal{H}_t = \sigma(H_u : u \leqslant t) = \sigma(\{\tau \leqslant u\} : u \leqslant t)$. We then define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. We

[1] By separable quadratic Gaussian models, it is meant that the state variables are independent.

denote by λ_t the $\mathcal{G}_t$-intensity of H_t which has the property that

$$H_t - \int_0^{t\wedge\tau} \lambda_u \, du \tag{1.1}$$

is a $\mathcal{G}$-martingale under the risk neutral measure $\mathbb{Q}$. Let the random vector

$$Y_t + \alpha(t) = (Y_{1t}, \ldots, Y_{nt})^\top + (\alpha_1(t), \ldots, \alpha_n(t))^\top$$

follow a Gaussian Ornstein-Uhlenbeck process:

$$dY_t = AY_t \, dt + \Sigma \, dW_t \tag{1.2}$$

where $Y_0 = (0, \ldots, 0)$ and A and Σ are constant square matrices. We assume that A is a diagonal matrix so that

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ldots & \vdots \\ 0 & \ldots & a_{ii} & \ldots & 0 \\ \vdots & 0 & \ldots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_{nn} \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \sigma_{11} & 0 & \ldots & \ldots & & 0 & 0 \\ \vdots & \ddots & \ldots & \ldots & \ldots & \ldots & \vdots \\ \vdots & \ldots & \ddots & 0 & 0 & \ldots & 0 \\ \rho_{i1}\sigma_{ii} & \ldots & \ldots & \sigma_{ii} & 0 & \ldots & 0 \\ \vdots & \ldots & \ldots & \ldots & \ddots & \ldots & \vdots \\ \rho_{n1}\sigma_{nn} & \ldots & \ldots & \ldots & \ldots & \ldots & \sigma_{nn} \end{bmatrix}$$

is chosen so that

$$\Sigma\Sigma^\top = \begin{bmatrix} \sigma_{11}^2 & \rho_{21}\sigma_{11}\sigma_{22} & \ldots & \ldots & \rho_{1n}\sigma_{11}\sigma_{nn} \\ \vdots & \ddots & & \ldots & \vdots \\ \rho_{i1}\sigma_{ii}\sigma_{11} & \ldots & \sigma_{ii}^2 & \ldots & \rho_{in}\sigma_{ii}\sigma_{nn} \\ \vdots & \ldots & \ldots & \ddots & \vdots \\ \rho_{n1}\sigma_{nn}\sigma_{11}\ldots & \ldots & \ldots & \ldots & \sigma_{nn}^2 \end{bmatrix}.$$

Hence Σ is the lower triangular matrix corresponding to the Cholesky decomposition of $\Sigma\Sigma^\top$. We now use quadratic forms in

$$Z_t = Y_t + \alpha(t) \tag{1.3}$$

as the state variables. Note there is no loss of generality in assuming (1.3) as we can write the general case of a Gaussian Ornstein Uhlenbeck process which has a drift term equal to a deterministic time dependent vector $\mu(t)$:

$$dx_t = (\mu(t) + Ax_t)dt + \Sigma\, dW_t \tag{1.4}$$

$$x_t = (x_{1t}, \ldots, x_{nt})^\top$$

as

$$x_t = Y_t + \exp(A\,t)x_0 + \int_0^t \exp(A\,(t-s))\mu(s)\,ds. \tag{1.5}$$

However it is better to use (1.3) as it makes it easier to derive calibration of the quadratic Gaussian model to the term structure of default free and defaultable zero coupon bonds through $\alpha(t)$. We now model the short term interest rate r_t as a quadratic form in $Z_t = Y_t + \alpha(t)$. If we are considering a reduced model of credit risk, we also assume that the intensity of default λ_t is a quadratic form in $Z_t = Y_t + \alpha(t)$. Therefore we can write r_t as

$$r_t = Y_t^\top \widehat{C} Y_t + \widehat{B}^\top(t) Y_t + \widehat{A}(t) \tag{1.6}$$

and in the defaultable case λ_t as

$$\lambda_t = Y_t^\top \widetilde{C} Y_t + \widetilde{B}^\top(t) Y_t + \widetilde{A}(t) \tag{1.7}$$

where $\widehat{C}$ and $\widetilde{C}$ are constant and symmetric matrices, $\widehat{B}(t)$ and $\widetilde{B}(t)$ are time dependent deterministic vectors, $\widehat{A}(t)$ and $\widetilde{A}(t)$ are time dependent deterministic scalars. We denote by $D(t)$ the default free savings account which is the value of investing one unit of currency at time $t = 0$ and rolling over the account at the default free instantaneous rate of interest r_t:

$$D(t) = \exp\left(\int_0^t r_s\,ds\right). \tag{1.8}$$

Suppose G_T is some stochastic process and we are considering the expectation G_T with under a given measure $\mathbb{M}$ and with respect to the sigma algebra $\mathcal{F}_t$ representing the information from the market at time t given by

$$\mathbb{E}^{\mathbf{M}}[G_T|\mathcal{F}_t].$$

To simplify the notation we write instead

$$\mathbb{E}_t^{\mathbf{M}}[G_T].$$

We denote the price of a default free zero coupon bond by $P(t,T)$ which is given by (see, e.g., Musiela and Rutkowski (2005))

$$P(t,T) = \mathbb{E}_t^{\mathbf{Q}}\left[\exp\left(-\int_t^T r_s\,ds\right)\right].$$

We denote the predefault price of a defaultable zero coupon bond by $\bar{P}(t,T)$ so that under the assumption of the reduced form model we are considering, the price of a defaultable zero coupon bond is given by[2] (see, e.g., Bielecki and Rutkowski (2002), p.245)

$$\mathbf{1}_{\tau>t}\bar{P}(t,T) = \mathbf{1}_{\tau>t}\mathbb{E}_t^{\mathbf{Q}}\left[\exp\left(-\int_t^T (r_s+\lambda_s)\,ds\right)\right].$$

If we model the default free interest rate r_t and the intensity of default λ_t by quadratic forms of multivariate quadratic Gaussian processes, default free zero coupon bond prices $P(t,T)$ and defaultable zero coupon bond prices $\bar{P}(t,T)$ are log-quadratic Gaussian. We state the following general result obtained in El Karoui et al. (1991) which is also valid even if the matrices A and Σ in (1.3) are time dependent.

Theorem 1.1 (El Karoui, Viswanathan and Myneni).

$$P(t,T) = \exp\left(-Y_t^\top C(t,T)Y_t - B(t,T)^\top Y_t - A(t,T)\right)$$

[2] Here we are assuming that we have a reduced form model where the intensity is modeled by a quadratic form in Gaussian random variables. The discussion given in the first section of Chapter 4 provides more detail regarding valuation formulas for the price of credit sensitive securities in a reduced form model such as the one we are considering here.

$$\bar{P}(t,T) = \exp\left(-Y_t^{\top}\bar{C}(t,T)Y_t - \bar{B}(t,T)^{\top}Y_t - \bar{A}(t,T)\right)$$

where $C(t,T), B(t,T), A(t,T)$ solve the following Riccati ordinary differential equations(ODE)

$$\partial_t C(t,T) = -A^{\top}C(t,T) - C(t,T)A + 2C(t,T)^{\top}\Sigma\Sigma^{\top}C(t,T) - \widehat{C},\ \ C(T,T) = 0_{n\times n} \tag{1.9}$$

$$\partial_t B(t,T) = -A^{\top}B(t,T) + 2C(t,T)\Sigma\Sigma^{\top}B(t,T) - \widehat{B}(t), \quad B(T,T) = 0_n \tag{1.10}$$

$$\partial_t A(t,T) = -Tr[\Sigma\Sigma^{\top}C(t,T)] + \frac{1}{2}B(t,T)^{\top}\Sigma\Sigma^{\top}B(t,T) - \widehat{A}(t), \quad A(T,T) = 0. \tag{1.11}$$

Similarly $\bar{C}(t,T), \bar{B}(t,T), \bar{A}(t,T)$ solve the following Riccati ODE

$$\partial_t \bar{C}(t,T) = -A^{\top}\bar{C}(t,T) - \bar{C}(t,T)A + 2\bar{C}(t,T)^{\top}\Sigma\Sigma^{\top}\bar{C}(t,T) - \widetilde{C},\ \ \bar{C}(T,T) = 0_{n\times n} \tag{1.12}$$

$$\partial_t \bar{B}(t,T) = -A^{\top}\bar{B}(t,T) + 2\bar{C}(t,T)\Sigma\Sigma^{\top}\bar{B}(t,T) - \widetilde{B}(t), \quad \bar{B}(T,T) = 0_n \tag{1.13}$$

$$\partial_t \bar{A}(t,T) = -Tr[\Sigma\Sigma^{\top}\bar{C}(t,T)] + \frac{1}{2}\bar{B}(t,T)^{\top}\Sigma\Sigma^{\top}\bar{B}(t,T) - \widetilde{A}(t), \quad \bar{A}(T,T) = 0. \tag{1.14}$$

In general if we consider the matrices A, Σ in (1.3) to be time dependent, we cannot guarantee the existence of a closed form solution of the Riccati equations. However if we assume A, Σ are constant, Freiling (2002) (see Appendix A) presents an analytic solution of matrix Riccati equations of type (1.9). If we assume the matrices A, Σ are constant, the solution of (1.9) can be written as a function of time to maturity $T - t$. Thus we have

$$\partial_t C(t,T) = -\partial_T C(t,T) = -\partial_T C(T-t).$$

Hence if we multiply the right hand side of the matrix Riccati equation (1.9) by -1 and write it as a function of time to maturity $T - t$, we get an initial value ODE

$$\frac{dC(T-t)}{dT} = A^{\top}C(T-t) + C(T-t)A - 2C(T-t)^{\top}\Sigma\Sigma^{\top}C(T-t) + \hat{C} \tag{1.15}$$

with initial condition $C(T-t)|_{T=t} = 0_{n\times n}$. In the following we apply Theorem A.2

(see Freiling (2002) for more detail) to solve[3] (1.15). Let I_n denote the identity matrix of dimension $n \times n$ and $0_{n\times n}$ denote the zero matrix[4] of dimension $n \times n$. According to Theorem A.2, we can solve (1.15), by considering the linear ODE associated with (1.15):

$$\begin{pmatrix} \dot{Q}(T-t) \\ \dot{P}(T-t) \end{pmatrix} = \begin{pmatrix} -A & 2\Sigma\Sigma^{\top} \\ \hat{C} & A^{\top} \end{pmatrix} \begin{pmatrix} Q(T-t) \\ P(T-t) \end{pmatrix}, \quad \begin{pmatrix} Q(0) \\ P(0) \end{pmatrix} = \begin{pmatrix} I_n \\ 0_{n\times n} \end{pmatrix}. \tag{1.16}$$

We can apply the second part of Theorem A.2 to conclude that the solution of (1.15) is given by

$$C(t,T) = P(T-t)Q^{-1}(T-t). \tag{1.17}$$

To find a closed form formula for $B(t,T)$ we use an integrating factor to solve the ODE given by (1.10). It is not difficult to show that (1.16) implies that $Q(t,T) := Q(T-t)$ satisfies

$$\partial_t Q(t,T) = (A - 2\Sigma\Sigma^{\top}C(t,T))Q(t,T)), \quad Q(T,T) = I_n. \tag{1.18}$$

Using (1.10) and (1.18), we have after some simplifications

$$\partial_t(Q(t,T)^{\top}B(t,T)) = -Q(t,T)^{\top}\widehat{B}(t). \tag{1.19}$$

Using direct integration and the boundary conditions for (1.10) and (1.18), we get an explicit analytic solution for $B(t,T)$:

$$B(t,T) = Q^{-1}(t,T)^{\top} \int_t^T Q(s,T)^{\top}\hat{B}(s)\, ds. \tag{1.20}$$

The solution of $A(t,T)$ can be found by direct integration of (1.11):

$$A(t,T) = \int_t^T \left(Tr[\Sigma\Sigma^{\top}C(s,T)] - \frac{1}{2}B(t,T)^{\top}\Sigma\Sigma^{\top}B(s,T) + \hat{A}(s) \right) ds \tag{1.21}$$

Let $\mathbb{T}$ be the default free T-measure which is given by the following Radon Nikodým

[3] Alternatively we can consider the ODE satisfied by $\frac{d}{dt}C(T-t)$ and apply Theorem A.3.

[4] The zero matrix is a matrix which has all its entries equal to zero.

density

$$\left.\frac{d\mathbb{T}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{P(T,T)}{D(T)}\frac{D(t)}{P(t,T)}. \tag{1.22}$$

Let $V(t,T)$ denote the variance-covariance matrix of Y_T under the measure $\mathbb{T}$ conditional on $\mathcal{F}_t$ and $M(t,T)$ denote the vector representing the mean of Y_T under the measure $\mathbb{T}$ conditional on $\mathcal{F}_t$. We can show how to get closed form formulas for $V(t,T)$ and $M(t,T)$. By Itô's formula, the stochastic differential equation (SDE) satisfied by the default free zero coupon bond is given by

$$dP(t,T) = P(t,T)(r_t - (2\,Y_t^{\mathsf{T}} C(t,T) + B(t,T))\Sigma\, dW_t). \tag{1.23}$$

Now using Girsanov's theorem (Karatzas and Shreve 1991, p. 191) we have that

$$W_t^{\mathbb{T}} = W_t + \int_0^t 2\,\Sigma^{\mathsf{T}}(C(s,T)Y_s + B(s,T))\, ds. \tag{1.24}$$

is a Brownian motion under the measure $\mathbb{T}$. Hence the dynamics of Y_t under the measure $\mathbb{T}$ is given by

$$dY_t = [(A - 2\Sigma\Sigma^{\mathsf{T}} C(t,T))Y_t - \Sigma\Sigma^{\mathsf{T}} B(t,T)]dt + \Sigma\, dW_t^{\mathbb{T}}. \tag{1.25}$$

Now using the fact that

$$\partial_t Q(t,T) = (A - 2\Sigma\Sigma^{\mathsf{T}} C(t,T))Q(t,T), \quad Q(T,T) = I_n, \tag{1.26}$$

we can solve the Ornstein Uhlenbeck equation (1.25) explicitly for Y_t by using $Q(t,T)$ as an integrating factor[5]. Thus we have

$$Y_T = Q^{-1}(t,T)Y_t - \int_t^T Q^{-1}(s,T)\Sigma\Sigma^{\mathsf{T}} B(s,T)\, ds + \int_t^T Q^{-1}(s,T)\Sigma\, dW_s^{\mathbb{T}}. \tag{1.27}$$

We can calculate $M(t,T)$ by taking the conditional expectation of (1.27). There-

[5] Consider the SDE that is satisfied by $Q(t,T)^{-1}Y_t$ which can be solved by direct integration.

fore we have:

$$
\begin{aligned}
M(t,T) =&\mathbb{E}_t^{\mathbf{T}}[Y_T] \\
=&\mathbb{E}_t^{\mathbf{T}}\left[Q^{-1}(t,T)Y_t - \int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds + \int_t^T Q^{-1}(s,T)\Sigma\,dW_s^{\mathbf{T}}\right]
\end{aligned}
\tag{1.28}
$$

$$
=Q^{-1}(t,T)Y_t - \int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds \tag{1.29}
$$

where we have used the fact that

$$
\int_0^t Q^{-1}(s,T)\Sigma\,dW_s^{\mathbf{T}}
$$

is a martingale (see, e.g., Karatzas and Shreve (1991), p. 139) in (1.28) to get (1.29). The conditional variance $V(t,T)$ is calculated using

$$
\begin{aligned}
V(t,T) =&\mathbb{E}_t^{\mathbf{T}}[Y_T Y_T^\top] - \mathbb{E}_t^{\mathbf{T}}[Y_T]\mathbb{E}_t^{\mathbf{T}}[Y_T]^\top \\
=&\mathbb{E}_t^{\mathbf{T}}\Bigg[\left(Q^{-1}(t,T)Y_t - \int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds + \int_t^T Q^{-1}(s,T)\Sigma\,dW_s^{\mathbf{T}}\right) \\
&\left(Q^{-1}(t,T)Y_t - \int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds + \int_t^T Q^{-1}(s,T)\Sigma\,dW_s^{\mathbf{T}}\right)^\top\Bigg] \\
&-\mathbb{E}_t^{\mathbf{T}}[Y_T]\mathbb{E}_t^{\mathbf{T}}[Y_T]^\top
\end{aligned}
\tag{1.30}
$$

Now using (1.27), (1.29) and $\int_0^t Q^{-1}(s,T)\Sigma\,dW_s^{\mathbf{T}}$ is a martingale on the expression given by (1.30), we get after some simplifications:

$$
V(t,T) = \mathbb{E}_t^{\mathbf{T}}\left[\left(\int_t^T Q^{-1}(s,T)\Sigma\,dW_s^{\mathbf{T}}\right)\left(\int_t^T Q^{-1}(s,T)\Sigma\,dW_s^{\mathbf{T}}\right)^\top\right]. \tag{1.31}
$$

Let

$$
F_t := \int_t^T Q^{-1}(s,T)\Sigma\,dW_s^{\mathbf{T}}. \tag{1.32}
$$

The quadratic variation of F_t which is denoted by $\langle F \rangle_t$ is the unique process such that (see, e.g., Karatzas and Shreve (1991), p. 138)

$$F_t^2 - \langle F \rangle_t \tag{1.33}$$

is a martingale. Moreover the quadratic variation of F_t is given by (see, e.g., Karatzas and Shreve (1991), p. 138)

$$\int_0^t Q^{-1}(s,T)\Sigma\Sigma^\top Q^{-1}(s,T)\, ds. \tag{1.34}$$

Using (1.33) is a martingale and (1.34), we can show that (1.31) is equal to

$$\int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top Q^{-1}(s,T)\, ds. \tag{1.35}$$

We therefore have proven the following lemma.

Lemma 1.2. *The conditional mean $M(t,T)$ and conditional variance-covariance matrix $V(t,T)$ of Y_T are given by the following formulas.*

$$M(t,T) = Q^{-1}(t,T)Y_t - \int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\, ds \tag{1.36}$$

$$V(t,T) = \int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top Q^{-1}(s,T)\, ds. \tag{1.37}$$

If we assume that r_t is given by

$$r_t = (Y_t + \alpha(t))^\top (Y_t + \alpha(t)).$$

We can show that

$$r_t = -\partial_T(Log_e(P(t,T)))|_{T=t} = Y_t^\top Y_t + 2\alpha(t)^\top Y_t + \alpha(t)^\top \alpha(t).$$

and $\hat{B}(s)$ in (1.20) will be equal to $2\alpha(s)$ (see Cherif et al. (1994)). We will show in Theorem 1.7 that $\alpha(t)$ can be used to calibrate the quadratic gaussian model to the data given by the prices of default free zero coupon bonds. Hence $\hat{B}(s)$ will be given by an integral and therefore the computation of $B(s,T)$ will involve a double integral. Thus computation of $V(t,T)$ involves a single integral while the computation of $M(t,T)$ involves an integrand that is a double integrand. This will be slower than an alternative method of computing $V(t,T)$ and $M(t,T)$ which is given in Cherif et al. (1994) (see Lemma 1.4), a result which is valid in the more general case of time dependent speed of mean reversion matrix $A(t)$ and a time dependent variance-covariance matrix $\Sigma(t)$ for the state variable Y_t. In the following we provide the result which is given in Cherif et al. (1994) as a lemma since it will be cited several times in this thesis.

Lemma 1.3. *The following ODE's are satisfied by $V(t,T)$ and $M(t,T)$*

$$\partial_T V(t,T) = AV(t,T) + V(t,T)\,A^{\mathsf{T}} - 2V(t,T)\,\widehat{C}\,V(t,T) + \Sigma\Sigma^{\mathsf{T}} \tag{1.38}$$
$$\partial_T M(t,T) = AM(t,T) - 2V(t,T)\,\widehat{C}\,M(t,T) - V(t,T)\,\widehat{B}(t) + \mu(T), \tag{1.39}$$

with initial conditions $V(t,t) = 0_{n\times n}$ and $M(t,t) = Y_t$.

The proof of lemma 1.3 is found in Cherif et al. (1994). Therefore $V(t,T)$ can be obtained by solving a matrix Riccati differential equation while $M(t,T)$ can be obtained by solving a system of ODE's. In lemma 1.3, $\mu(T)$ represents the drift term of the Ornstein Uhlenbeck process of the Gaussian factors underlying the quadratic Gaussian factor model. For our model Y_t has no drift and therefore $\mu(T) = 0_n$ but we will see later that when we consider a two economy model, Y_t will have a drift under the foreign risk neutral measure. We have in Lemma 1.3 a particular case of the more general result given by Cherif et al. (1994) as we are considering the case of a constant speed of mean reversion matrix A and a constant instantaneous volatility matrix Σ instead of the more general case of a time dependent A and a time dependent Σ. We provide new analytic formulas for $V(t,T)$ and $M(t,T)$ in the following lemma by solving the ODE's given in (1.38) and (1.39).

Lemma 1.4.

$$V(t,T) = \widetilde{P}(T-t)\widetilde{Q}^{-1}(T-t) \tag{1.40}$$
$$M(t,T) = \widetilde{Q}^{-1}(t,T)^{\mathsf{T}} Y_t + \widetilde{Q}^{-1}(t,T)^{\mathsf{T}} \int_t^T \widetilde{P}(t,s)^{\mathsf{T}} \hat{B}(s)\, ds. \tag{1.41}$$

where $(\tilde{Q}(T) \quad \tilde{P}(T))^\top$ *is the solution of the following system of linear ODE's*

$$\frac{d}{dT}\begin{pmatrix} \tilde{Q}(T) \\ \tilde{P}(T) \end{pmatrix} = \begin{pmatrix} -A^\top & 2\hat{C} \\ \Sigma\Sigma^\top & A \end{pmatrix}\begin{pmatrix} \tilde{Q}(T) \\ \tilde{P}(T) \end{pmatrix}, \quad \begin{pmatrix} \tilde{Q}(0) \\ \tilde{P}(0) \end{pmatrix} = \begin{pmatrix} I_n \\ 0_{n\times n} \end{pmatrix}. \tag{1.42}$$

Proof. Assuming time independent A and Σ enables us to find the closed form solution of (1.38) using Theorem A.2 which is given in Appendix A. This is similar to what we did to solve for $C(t,T)$. The system of linear ODE's given by (1.42) can be solved explicitly using matrix exponentiation (see, e.g. Leonard (2002) or Moler and Van Loan (1978)). Therefore applying the second part of Theorem A.2, we get (1.40). To solve (1.39), consider first the equation satisfied by $\tilde{Q}(t,T) := \tilde{Q}(T-t)$ which can be obtained from (1.42)

$$\partial_T\tilde{Q}(t,T) = -A^\top\tilde{Q}(t,T) + 2\,\hat{C}\tilde{P}(T-t), \quad \tilde{Q}(t,t) = I_n. \tag{1.43}$$

Using (1.40) we can rewrite (1.43) as

$$\partial_T\tilde{Q}(t,T) = (-A^\top + 2\,\hat{C}V(T-t))\tilde{Q}(t,T), \quad \tilde{Q}(t,t) = I_n. \tag{1.44}$$

We now use $\tilde{Q}(t,T)^\top$ as an integrating factor. Specifically differentiate

$$\tilde{Q}(t,T)^\top M(t,T)$$

with respect to T and simplify to get:

$$\partial_T(\tilde{Q}(t,T)^\top M(t,T)) = -\tilde{Q}(t,T)^\top\hat{B}(t). \tag{1.45}$$

We can solve (1.45) using direct integration and the boundary conditions given in (1.10) and (1.44) to get (1.41). □

The alternative formulas for $V(t,T)$ and $M(t,T)$ which are given by (1.40) and (1.41) respectively are more efficient than the previous formulas given by (1.36) and (1.37)[6]. In order to calculate $A(t,T)$, we have to integrate a nonlinear function of a double integral (specifically consider $B(s,T)^\top\Sigma\Sigma^\top B(s,T)$). Thus we cannot use cubature methods (see Cools and Haegemans (2003)) for three dimensional

[6] We will see later in this chapter that we can get $\hat{B}(t)$ through a single integral and therefore we can use cubature to perform the double integration which is necessary to calculate $M(t,T)$. In comparison in the integrand of (1.36), $B(t,T)$ requires a double integral such that we need a triple integral to calculate $M(t,T)$ this way.

integrals to gain some efficiency in integration. Hence it is important to find efficient ways of computing these values. For the single factor quadratic Gaussian model, Pelsser (2000) provides efficient formulas for the calculation of $B(t,T)$ and $A(t,T)$. We extend these formulas to the multifactor quadratic Gaussian model. First to simplify the discussion we assume that

$$r_t = (Y_t + \alpha(t))^{\top}(Y_t + \alpha(t)). \tag{1.46}$$

Hence the constant matrix $\hat{C} = I_n$ where I_n is the $n \times n$ identity matrix, $\hat{B}(t) = 2\,\alpha(t)$ and $\hat{A}(t) = \alpha(t)^{\top}\alpha(t)$ in (1.6). Let $F(t,T)$ denote the default free forward rate as seen from the date $t < T$ i.e.

$$F(t,T) := -\partial_T \log_e(P(t,T)). \tag{1.47}$$

Let $\mathbf{1}$ denote the column vector of dimension n which has all its elements equal to the number one:

$$\mathbf{1} := (1, \dots, 1)^{\top}.$$

The following lemma generalizes a result given in Pelsser (2000) which is given for the one factor case.

Lemma 1.5.

$$M(t,T) = Q^{-1}(t,T)Y_t + \frac{1}{2}Q(t,T)^{\top}\partial_T B(t,T) - \alpha(T) \tag{1.48}$$

and

$$B(0,T) = 2\int_0^T Q^{-1}(0,s)^{\top}\tilde{F}(s)\, ds \tag{1.49}$$

where

$$\tilde{F}(s) = \sqrt{\frac{F(0,s) - tr(V(0,s))}{n}}\mathbf{1}$$

Proof. First we use the following known result (see Corollary 11.3.1 in Musiela and Rutkowski (2005))

$$F(t,T) = \mathbb{E}_t^{\mathrm{T}}[r_T] \tag{1.50}$$

Note that

$$r_T = (Y_T + \alpha(T))^{\top}(Y_T + \alpha(T)) = Y_T^{\top}Y_T + 2\alpha(T)^{\top}Y_T + \alpha(T)^{\top}\alpha(T)$$

and

$$\mathbb{E}_t^{\mathrm{T}}[Y_T^{\top}Y_T] = tr(V(t,T)) + M(t,T)^{\top}M(t,T) \tag{1.51}$$

Using the formula given in (1.36), we have

$$M(t,T)^{\top}M(t,T) = \left(Q^{-1}(t,T)Y_t - \int_t^T Q^{-1}(s,T)\Sigma\Sigma^{\top}B(s,T)\,ds\right)^{\top} \left(Q^{-1}(t,T)Y_t - \int_t^T Q^{-1}(s,T)\Sigma\Sigma^{\top}B(s,T)\,ds\right) \tag{1.52}$$

and therefore we can use this expression to calculate

$$\mathbb{E}_t^{\mathrm{T}}[r_T] = \mathbb{E}_t^{\mathrm{T}}[Y_T^{\top}Y_T + 2\alpha(T)^{\top}Y_T + \alpha(T)^{\top}\alpha(T)]. \tag{1.53}$$

On the other hand from (1.47) we get

$$F(t,T) = Y_t^{\top}\partial_T C(t,T)Y_t + Y_t^{\top}\partial_T B(t,T) + \partial_T A(t,T) \tag{1.54}$$

Now we match quadratic terms with quadratic terms in equations (1.54) and (1.53) and get

$$\partial_T C(t,T) = Q^{-1}(t,T)^{\top}Q^{-1}(t,T). \tag{1.55}$$

We do not use (1.55) here but this result will be used in another lemma. Matching linear terms with linear terms in equations (1.54) and (1.53), we have

$$\partial_T B(t,T) = -2Q^{-1}(t,T)^{\top}\int_t^T Q^{-1}(s,T)\Sigma\Sigma^{\top}B(s,T)\,ds + 2Q^{-1}(t,T)^{\top}\alpha(T), \tag{1.56}$$

while matching constant terms gives us

$$\begin{aligned}\partial_T A(t,T) &= tr(V(t,T)) - 2\alpha(T)^{\top}\left(\int_t^T Q^{-1}(s,T)\Sigma\Sigma^{\top}B(s,T)\,ds\right) + \\ &+ \left(\int_t^T Q^{-1}(s,T)\Sigma\Sigma^{\top}B(s,T)\,ds\right)^{\top}\left(\int_t^T Q^{-1}(s,T)\Sigma\Sigma^{\top}B(s,T)\,ds\right) + \alpha(T)^{\top}\alpha(T).\end{aligned} \tag{1.57}$$

From (1.56) we get

$$-\int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds = \frac{1}{2}Q(t,T)^\top \partial_T B(t,T) - \alpha(T). \tag{1.58}$$

Since

$$M(t,T) = Q^{-1}(t,T)Y_t - \int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds$$

we can use (1.58) to write

$$M(t,T) = Q^{-1}(t,T)Y_t + \frac{1}{2}Q(t,T)^\top \partial_T B(t,T) - \alpha(T)$$

and this proves (1.48) given in the lemma. Note that this equation can be used to calculate $M(t,T)$ for $t > 0$ as we will later show in lemma 1.7 that we can find a formula[7] for $\partial_T B(t,T)$. Therefore the calculation of $M(t,T)$ can be done using one dimensional integration. Using (1.58) we can simplify (1.57) to obtain

$$\partial_T A(t,T) = tr(V(t,T)) + \frac{1}{4}\partial_T B(t,T)^\top Q(t,T)Q(t,T)^\top \partial_T B(t,T). \tag{1.59}$$

When $t = 0$, we have

$$F(0,T) = \partial_T A(0,T) = tr(V(0,T) + \frac{1}{4}\partial_T B(0,T)^\top Q(0,T)Q(0,T)^\top \partial_T B(0,T) \tag{1.60}$$

as $Y_0 = (0,\ldots,0)$. Therefore

$$\partial_T B(0,T)^\top Q(0,T)Q(0,T)^\top \partial_T B(0,T) = 4(F(0,T) - tr(V(0,T))) \tag{1.61}$$

Now one solution to (1.61) among many is to choose

$$Q(0,T)^\top \partial_T B(0,T) = 2\sqrt{\frac{F(0,T) - tr(V(0,T))}{n}}\mathbf{1}. \tag{1.62}$$

Thus we have

$$\partial_T B(0,T) = Q^{-1}(0,T)^\top 2\sqrt{\frac{F(0,T) - tr(V(0,T))}{n}}\mathbf{1}. \tag{1.63}$$

[7] The formula will involve only one dimensional integration.

Integrating both sides completes the proof of the lemma. □

Remark 1.6. *Note that in* (1.63) *we can choose the right hand side vector to be different than our choice as long as* $\vec{v}$ *in*

$$\partial_T B(0,T) = Q^{-1}(0,T)^{\top} 2\vec{v}$$

has the property that

$$\vec{v}^{\top}\vec{v} = \sqrt{F(0,T) - tr(V(0,T))}.$$

Hence there is no unique way of choosing $\alpha(t)$. *We have not investigated how this can be used to calibrate the quadratic Gaussian model in different ways.*

Lemma 1.7.

$$\begin{aligned} B(t,T) =& (I_n + 2C(t,T)V(0,t))Q(0,t)^{\top}(B(0,T) - B(0,t)) + \\ &+ 2C(t,T)(\alpha(t) - \tilde{F}(t)) \end{aligned} \tag{1.64}$$

$$\begin{aligned} \partial_T B(t,T) =& 2Q^{-1}(t,T)^{\top}Q^{-1}(t,T)(V(0,t)Q(0,t)^{\top}(B(0,T) - B(0,t)) + \\ &+ \alpha(t) - \tilde{F}(t)) + (I_n + 2C(t,T)V(0,t))Q(0,t)^{\top}2Q^{-1}(0,T)^{\top}\tilde{F}(T) \end{aligned} \tag{1.65}$$

$$\begin{aligned} A(t,T) =& \log_e\left(\frac{P(0,t)}{P(0,T)}\right) - \frac{1}{2}\log_e(\mid I_n + 2C(t,T)V(0,t) \mid) - \\ &- (\alpha(t) - \tilde{F}(t))^{\top}(I_n + 2C(t,T)V(0,t))^{-1}C(t,T)(\alpha(t) - \tilde{F}(t)) + \\ &+ (\alpha(t) - \tilde{F}(t) + \frac{1}{2}V(0,t)B(t,T))(I_n + 2C(t,T)V(0,t))^{-1}B(t,T) \end{aligned} \tag{1.66}$$

Proof. We prove the lemma using the approach used by Pelsser (2000) for the one factor quadratic Gaussian model. Specifically we use the fact that under the default free forward measure[1] which is denoted by $\mathbb{T}$, the values of contingent claims normalized by the default free bond of maturity T are martingales. Therefore if we assume $T < S$, we have

$$\frac{P(t,S)}{P(t,T)} = \mathbb{E}_t^{\mathbb{T}}[P(T,S)]. \tag{1.67}$$

[1] The default free forward measure corresponds to using the default free bond of maturity T as the numeraire.

First note that

$$\frac{P(t,S)}{P(t,T)} = \exp\Big(- Y_t^{\mathsf{T}}(C(t,S) - C(t,T))Y_t \\ - (B(t,S) - B(t,T))^{\mathsf{T}} Y_t - (A(t,S) - A(t,T))\Big). \quad (1.68)$$

Since we know the moment generating function of a quadratic form in Gaussian random variables (see Lemma B.1), we can calculate

$$\mathbb{E}_t^{\mathbb{T}}[P(T,S)] = \mathbb{E}_t^{\mathbb{T}}\Big[\exp\Big(- Y_T^{\mathsf{T}} C(T,S) Y_T - B(T,S)^{\mathsf{T}} Y_T - A(T,S)\Big)\Big]. \quad (1.69)$$

To do so note that

$$-Y_T^{\mathsf{T}} C(T,S) Y_T - B(T,S)^{\mathsf{T}} Y_T - A(T,S) \quad (1.70)$$

is a quadratic form in the Gaussian random vector Y_T with a known mean vector $M(t,T)$ and variance covariance matrix $V(t,T)$ under the forward measure $\mathbb{T}$ (note the mean and variance are with respect to the filtration $\mathcal{F}_t$). Thus (1.69) can be obtained by evaluating the moment generating function of the quadratic form (1.70) at the value $z = 1$.

Therefore a straightforward application of the moment generating function of a quadratic form in Gaussian variables gives:

$$\mathbb{E}_t^{\mathbb{T}}[P(T,S)] = \mathbb{E}_t^{\mathbb{T}}\Big[\exp\Big(- Y_T C(T,S) Y_T - B(T,S)^{\mathsf{T}} Y_T - A(T,S)\Big)\Big] \\ = \mid I + 2C(T,S)V(t,T) \mid^{-\frac{1}{2}} \exp\Big(- \frac{1}{2}(M(t,T)^{\mathsf{T}} V(t,T)^{-1} M(t,T) + 2A(T,S)) \\ + \frac{1}{2}(M(t,T) - V(t,T)B(T,S))^{\mathsf{T}} (I + 2C(T,S)V(t,T))^{-1} \\ V(t,T)^{-1}(M(t,T) - V(t,T)B(T,S))\Big). \quad (1.71)$$

We now use (1.41) from lemma 1.4 for the mean $M(t,T)$ in (1.71) and match quadratic terms with quadratic terms, linear terms with linear terms and constant terms with constant terms in (1.68) and (1.71). This procedure yields

$$-(C(t,S) - C(t,T)) = -\frac{1}{2} Q^{-1}(t,T)^{\mathsf{T}} V(t,T)^{-1} Q^{-1}(t,T) \\ + \frac{1}{2} Q^{-1}(t,T)^{\mathsf{T}} (I + 2C(T,S)V(t,T))^{-1} V(t,T)^{-1} Q^{-1}(t,T) \quad (1.72)$$

which can be rewritten as

$$(I + 2C(T,S)V(t,T))^{-1} = I - 2Q(t,T)^{\mathsf{T}}(C(t,S) - C(t,T))Q(t,T)V(t,T). \quad (1.73)$$

We do not use (1.73) here. For the linear terms we get

$$\begin{aligned}
-(B(t,S) - B(t,T)) &= Q^{-1}(t,T)^{\mathsf{T}}V(t,T)^{-1}\int_t^T Q^{-1}(s,T)\Sigma\Sigma^{\mathsf{T}}B(s,T)\,ds \\
&\quad - Q^{-1}(t,T)^{\mathsf{T}}(I + 2C(T,S)V(t,T))^{-1}V(t,T)^{-1}\int_t^T Q^{-1}(s,T)\Sigma\Sigma^{\mathsf{T}}B(s,T)\,ds \\
&\quad - Q^{-1}(t,T)^{\mathsf{T}}(I + 2C(T,S)V(t,T))^{-1}B(T,S) \quad (1.74)
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
B(T,S) &= (I + 2C(T,S)V(t,T))Q(t,T)^{\mathsf{T}}(B(t,S) - B(t,T)) \\
&\quad + 2C(T,S)\int_t^T Q^{-1}(s,T)\Sigma\Sigma^{\mathsf{T}}B(s,T)\,ds. \quad (1.75)
\end{aligned}$$

From the proof of Lemma 1.5, we have (1.58) and (1.62) which can be used to show that

$$\int_0^t Q^{-1}(s,t)\Sigma\Sigma^{\mathsf{T}}B(s,t)\,ds = \alpha(t) - \tilde{F}(t). \quad (1.76)$$

We can thus write (1.75) in terms of t and T by substituting 0 for t, t for T and T for S to get

$$\begin{aligned}
B(t,T) &= (I + 2C(t,T)V(0,t))Q(0,t)^{\mathsf{T}}(B(0,T) - B(0,t)) \\
&\quad + 2C(t,T)\int_0^t Q^{-1}(s,t)\Sigma\Sigma^{\mathsf{T}}B(s,t)\,ds. \quad (1.77)
\end{aligned}$$

and then substitute (1.76) in (1.77) which proves (1.64).

We can use the formula for $B(t,T)$ as given in (1.64) to find the derivative $\partial_T B(t,T)$ which can be used to calculate the mean $M(t,T)$ more efficiently for $t > 0$ as claimed in the proof of lemma 1.5. To do so we use (1.55) for $\partial_T C(t,T)$

and (1.49) for $\partial_T B(0,T)$ to get

$$\begin{aligned}\partial_T B(t,T) =& 2Q^{-1}(t,T)^\top Q^{-1}(t,T)V(0,t)Q(0,t)^\top(B(0,T)-B(0,t))+\\ &+(I_n+2C(t,T)V(0,t))Q(0,t)^\top 2Q^{-1}(0,T)^\top \widetilde{F}(T)+\\ &+2Q^{-1}(t,T)^\top Q^{-1}(t,T)(\alpha(t)-\widetilde{F}(t)).\end{aligned} \tag{1.78}$$

We now can simplify (1.78) to prove (1.65) stated in the lemma. For constant terms we get

$$\begin{aligned}&-(A(t,S)-A(t,T)) = -\frac{1}{2}\log_e | I+2C(T,S)V(t,T) |\\ &-\frac{1}{2}\Big(\int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds\Big)^\top V(t,T)^{-1}\Big(\int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds\Big)\\ &-A(T,S)+\frac{1}{2}\Big(\int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds\Big)^\top (I+2C(T,S)V(t,T))^{-1}V(t,T)^{-1}\\ &\quad\Big(\int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds\Big)+\Big(\int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds\Big)^\top\\ &\quad\quad (I+2C(T,S)V(t,T))^{-1}B(T,S)+\\ &\quad\quad +\frac{1}{2}B(T,S)^\top V(t,T)^\top (I+2C(T,S)V(t,T))^{-1}B(T,S).\end{aligned} \tag{1.79}$$

Thus for the special case of $t=0$ we have

$$\begin{aligned}A(T,S) =& \log_e\left(\frac{P(0,T)}{P(0,S)}\right)-\frac{1}{2}\log_e | I+2C(T,S)V(0,T) |\\ &-(\alpha(T)-\tilde{F}(T))^\top (I+2C(T,S)V(0,T))^{-1}C(T,S)(\alpha(T)-\tilde{F}(T))\\ &+\Big[(\alpha(T)-\tilde{F}(T))^\top+\frac{1}{2}B(T,S)^\top V(0,T)^\top\Big](I+2C(T,S)V(0,T))^{-1}B(T,S).\end{aligned} \tag{1.80}$$

Now substituting t for T and T for S in (1.80) proves (1.66). □

Therefore Lemma (1.5) and Lemma (1.7) can be used to compute $B(t,T)$ and $A(t,T)$ using analytic formulas and one dimensional integration. For pricing purposes, if we need $M(0,T)$, we can once again use

$$F(0,T) = \mathbb{E}^{\mathrm{T}}[r_T^{\mathrm{T}} r_T] = tr(V(0,T)) + (M(0,T) + \alpha(T))^{\mathrm{T}}(M(0,T) + \alpha(T)) \quad (1.81)$$

to get

$$M(0,T) = \tilde{F}(T) - \alpha(T). \quad (1.82)$$

We can calculate

$$\tilde{F}(T) = \sqrt{F(0,T) - tr(V(0,T))}$$

in closed form and therefore we only need one dimensional integration for calculating $\alpha(T)$ as the following theorem shows.

Theorem 1.8. *We can calibrate to the default free term structure at $t = 0$ given in terms of forward rates $F(0,T)$ by the following one dimensional integral*

$$\alpha(T) = \tilde{F}(T) + 2\exp(A\,T) \int_0^T \exp(-As) V(0,s) \tilde{F}(s)\, ds$$

where A is the speed of mean reversion diagonal matrix in the dynamics of Y_t given in (1.3).

Proof. Differentiating equation (1.82) with respect to T, we get

$$\frac{dM(0,T)}{dT} + \frac{d\alpha(T)}{dT} = \frac{d\tilde{F}(T)}{dT}. \quad (1.83)$$

From the assumption that we made in (1.46) we have that $\hat{C} = I_n$ where I_n is the $n \times n$ identity matrix, $\hat{B}(t) = 2\,\alpha(t)$ and $\hat{A}(t) = \alpha(t)^{\mathrm{T}}\alpha(t)$. Using the result of Cherif et al. (1994) which is given in lemma 1.3, we therefore have

$$A\,M(0,T) - 2V(0,T)M(0,T) - 2V(0,T)\alpha(T) + \frac{d\alpha(T)}{dT} = \frac{d\tilde{F}(T)}{dT} \quad (1.84)$$

Using (1.82), we can reduce (1.84) to

$$\frac{d\alpha(T)}{dT} = A\,\alpha(T) - A\,\tilde{F}(T) + 2V(0,T)\tilde{F}(T) + \frac{d\tilde{F}(T)}{dT}. \quad (1.85)$$

We now can solve (1.85) using the integrating factor $\exp(-A\,T)$ i.e consider the

ODE satisfied by $\exp(-A\,T)\alpha(T)$. Thus the solution of (1.85) is given by

$$\alpha(T) = \exp(A\,T)(\alpha(0) - \int_0^T \exp(-A\,s)(A - 2V(0,s))\tilde{F}(s) - \exp(-As)\frac{d\tilde{F}(s)}{ds}\,ds. \tag{1.86}$$

Integrating by parts the expression

$$\int_0^T \exp(-A\,s)\frac{d\tilde{F}(s)}{ds}\,ds$$

in (1.86) and noting that $\alpha(0)$ is equal to $\tilde{F}(0)$, we obtain the following simplified one dimensional integral

$$\alpha(T) = \tilde{F}(T) + 2\exp(A\,T)\int_0^T \exp(-A\,s)V(0,s)\tilde{F}(s)\,ds. \tag{1.87}$$

This completes the proof of the theorem which generalizes the result given in Pelsser (2000). □

1.1 A Quadratic Gaussian Model to include a Foreign Economy

In the following we consider a model for the default free term structures of two economies, a domestic economy and a foreign economy. To model the interest rate term structures of a domestic and foreign market and the movement of the exchange rate between the two economies, we now assume that we have two filtered probability spaces. Let $(\tilde{\Omega}, \mathcal{F}^i, \mathbb{Q}^i)$ represent the filtered probability space which is used to model the domestic(foreign) economy for $i = d(i = f)$ whereby $\mathbb{Q}^i$ is the domestic(foreign) risk-neutral measure for $i = d(i = f)$. We assume the filtration $\mathbb{F}^i = (\mathcal{F}^i{}_t)_{(0 \leqslant t \leqslant T^*)}$ corresponding to the domestic economy(foreign) economy for $i = d(i = f)$ is generated by n independent standard Brownian motions $W^i(t) = W^j(t), j = 1, \ldots, n$ under $\mathbb{Q}^i$ which is the domestic(foreign) risk neutral measure for $i = d(i = f)$. The n Brownian motions represent the information generated by a domestic economy, a foreign economy and the movement of the exchange rate between the economies. The exchange rate is defined to be the value of one unit of foreign currency in terms of domestic currency. Assuming that there is extra information in the movement of the exchange rate between the domestic and foreign market is supported by empirical results (see Leippold and Wu (2002)). Once again we assume the state of the international economy is represented by

$$Z_t = Y_t + \alpha(t) = (Y_{1t}, \ldots, Y_{nt}) + \alpha(t)$$

which is an n-dimensional Gaussian Ornstein-Uhlenbeck process under the domestic economy. Hence we assume as in the introduction of this section the dynamics of Y_t is given by (1.3) under the domestic risk neutral measure.

Definition 1.9. *Let S_t denote the foreign exchange rate between the domestic and foreign economy i.e S_t is the value of one unit of foreign currency in terms of units of domestic currency.*

We now denote by r^i the domestic(foreign) instantaneous rate of interest for $i = d(i = f)$. We assume that we use different factors to model r_t^d and r_t^f and

$$r_t^i = (\alpha(t) + Y_t)^\top I^i(\alpha(t) + Y_t), \quad i = d, f$$

where I^i is used to denote a matrix that has a one or a zero for the diagonal element of row $k, k \in \{1, \ldots, n\}$ depending on whether the kth factor is used to model r_t^i or not. All off-diagonal elements of $I^i(i = d, f)$ are taken to be equal to zero. The exchange rate is assumed to be log-quadratic Gaussian and we give the following theorem which is given in Cherif et al. (1994).

Theorem 1.10 (Chérif, El Karoui, Myneni, and Viswanathan). *A necessary and sufficient condition for the factors Z_t to be Gaussian under the domestic and foreign risk neutral probabilities is to assume that the dynamics of S_t is given by*

$$\frac{dS_t}{S_t} = (r_t^d - r_t^f)dt + (2C^S(t)\ Z_t + (B^S(t))^\top \Sigma\ dW_t^d \tag{1.88}$$

where $C^S(t)$ is a square matrix and $B^S(t)$ is a vector and both are assumed to be square integrable. Thus the instantaneous volatility of S_t is an affine function of the factors Z_t. If we assume that S_t is a regular function of Z_t then $C^S(t)$ is a symmetric matrix and there exists a deterministic function $A^S(t)$ such that:

$$S_t = \exp(Z_t^\top C^S(t)\ Z_t + B^S(t)^\top Z_t + A^S(t)). \tag{1.89}$$

and

$$r_t^d - r_t^f = Z_t^\top\ \hat{C}^S(t)\ Z_t + \hat{B}^S(t)^\top\ Z_t + \hat{A}^S(t) \tag{1.90}$$

where $C^S(t), B^S(t), A^S(t)$ satisfy the following equations

$$\frac{dC^S(t)}{dt} = -A^\top C^S(t) - C^S(t)A - 2C^S(t)\Sigma\Sigma^\top C^S(t) + \hat{C}^S(t) \quad (1.91)$$

$$\frac{dB^S(t)}{dt} = -A^\top B^S(t) - 2C^S(t)\Sigma\Sigma^\top B^S(t) + \hat{B}^S(t) \quad (1.92)$$

$$\frac{dA^S(t)}{dt} = -Tr[\Sigma\Sigma^\top C^S(t)] - \frac{1}{2}B^S(t)^\top\Sigma\Sigma^\top B^S(t) + \hat{A}^S(t) \quad (1.93)$$

Proof. See Cherif et al. (1994). □

The initial data is given by the value of the foreign exchange rate at time $t = 0$ such that $C^S(0), B^S(0)$ and $A^S(0)$ are known. Therefore the ODE given by (1.91) is an initial value symmetric matrix Riccati ODE which can be solved in closed form. Note however that we set up our model such that $Y_0 = (0, \dots, 0)$. Therefore we can take $C^S(0)$ to be equal to the zero matrix, $B^S(0)$ to be the zero vector of dimension n and use $A^S(0)$ to calibrate to the foreign exchange rate data at time $t = 0$. Similar to the solution of (1.15) we have the following lemma.

Lemma 1.11. *The solution of* (1.91) *is given by*

$$C^S(t) = P^S(t)Q^S(t)^{-1}$$

where $(Q^S(t), P^S(t))^\top$ is the solution of the linear system

$$\begin{pmatrix} \frac{d}{dt}Q^S(t) \\ \frac{d}{dt}P^S(t) \end{pmatrix} = \begin{pmatrix} A & 2\Sigma\Sigma^\top \\ I^d - I^f & -A^\top \end{pmatrix} \begin{pmatrix} Q^S(t) \\ P^S(t) \end{pmatrix}, \quad \begin{pmatrix} Q^S(0) \\ P^S(0) \end{pmatrix} = \begin{pmatrix} I \\ C^S(0) \end{pmatrix} \quad (1.94)$$

Proof. From the equality given by equation (1.90), it follows that $\widehat{C^S}$ is equal to the matrix corresponding to the quadratic term of

$$r_t^d - r_t^f = (\alpha(t) + Y_t)^\top I^d(\alpha(t) + Y_t) - (\alpha(t) + Y_t)^\top I^f(\alpha(t) + Y_t).$$

Thus $\hat{C}^S = I^d - I^f$. We can now apply Lemma (A.2) which is given in Appendix A(see Freiling (2002) for more detail) to get the result of the lemma. □

The following lemma gives closed form solutions for $B^S(t)$ and $A^S(t)$.

Lemma 1.12.

$$B^S(t) = 2\,(Q^S(t)^{-1})^\top \int_0^t Q^S(r)^\top (I^d - I^f)\alpha(r)\, dr \tag{1.95}$$

and

$$A^S(t) = A^S(0) + \int_0^t -Tr[\Sigma\Sigma^\top C^S(r)] - \frac{1}{2}B^S(r)^\top \Sigma\Sigma^\top B^S(r) + \alpha(r)^\top (I^d - I^f)\alpha(r)\, dr. \tag{1.96}$$

Proof. First note that from (1.90), we have

$$\hat{B}^S(t) = 2(I^d - I^f)\alpha(t)$$

and

$$\hat{A}^S(t) = \alpha(t)^\top (I^d - I^f)\alpha(t)$$

where $\alpha(t)$ is the vector corresponding to the vector $Y_t + \alpha(t)$ which is used as the state variables for the quadratic Gaussian model. From Lemma 1.11 it follows that $Q^S(t)$ satisfies the following ODE:

$$\frac{d}{dt}Q^S(t) = (A + 2\Sigma\Sigma^\top C^S(t))Q^S(t).$$

Using $Q^S(t)^\top$ as the integrating factor we can now solve for $B^S(t)$. Specifically we have

$$\frac{d}{dt}(Q^S(t)^\top B^S(t)) = Q^S(t)^\top 2(I^d - I^f)\alpha(t).$$

We can integrate the above equation to obtain the result (1.95). The solution of (1.93) can be found by direct integration which gives (1.96) using the initial condition for (1.93). □

To make the results given in this section easier to read we use the following notations:

- $P(t,T)$:=price at time t of the domestic default free zero coupon bond of maturity T;
- $P^f(t,T)$:=price at time t of the foreign default free zero coupon bond of

maturity T

- $\alpha^f(t) := I^f\alpha(t)$ time dependent function used for calibration to the foreign discount term structure;
- $F^f(t,T) := -\frac{d}{dT}\log_e(P^f(t,T))$ the foreign instantaneous forward rate;
- $V^f(t,T)$:=Variance of Y_T with respect to the filtration $\mathcal{F}_t$ under $\mathbb{Q}^f$
- $M^f(t,T)$:=Mean of Y_T with respect to the filtration $\mathcal{F}_t$ under $\mathbb{Q}^f$
- $A^f(T) := A + 2\Sigma\Sigma^\top C^S(T)$
- $W^{\mathbb{T}}$:=Standard Brownian motion under $\mathbb{T}$.
- $\mathbb{T}^f$:= the foreign default free forward measure
- $W^{\mathbb{T}^f}$:=Standard Brownian motion under $\mathbb{T}^f$.

Assuming that S_t is a log-quadratic Gaussian process ensures that the dynamics of Y_t remains Gaussian under the foreign risk neutral measure $\mathbb{Q}^f$ and is given by (see, e.g,Cherif et al. (1994))

$$\begin{aligned} dY_t =& (\Sigma\Sigma^\top B^S(t) + (A + 2\Sigma\Sigma^\top C^S(t))Y_t)\,dt + \Sigma\, dW_t^f && (1.97)\\ =& (\Sigma\Sigma^\top B^S(t) + A^f(t))\,dt + \Sigma\, dW_t^f. && (1.98)\end{aligned}$$

Since we assumed that the foreign instantaneous short rate of interest r_t^f is given by

$$r_t^f = (Y_t + \alpha(t))^\top I^f(Y_t + \alpha(t)) = Y_t^\top I^f Y_t + 2\alpha^f(t)Y_t + \alpha^f(t)^\top\alpha(t),$$

it now follows by Theorem 1.1 that the price of the foreign default free zero coupon bond is log-quadratic Gaussian[8]. Therefore

$$P^f(t,T) = \exp(-Y_t C^f(t,T)Y_t - B^f(t,T)Y_t - A^f(t,T)) \qquad (1.99)$$

where $C^f(t,T)$, $B^f(t,T)$ and $A^f(t,T)$ satisfy the following symmetric Riccati ODE's (see Cherif et al. (1994)):

[8] Note that, as mentioned earlier in this section, Theorem 1.1 is true in the general case of a time dependent speed of mean reversion matrix and a time dependent instantaneous volatility matrix even if Y_t has a non zero drift.

$$\frac{d}{dt}C^f(t,T) = -A^{f\top}C^f(t,T) - C^f(t,T)A^f + \\ + 2C^f(t,T)\Sigma\Sigma^\top C^f(t,T) - I^f, \quad C^f(T,T) = 0_{n\times n} \tag{1.100}$$

$$\frac{d}{dt}B^f(t,T) = -A^{f\top}B^f(t,T) + 2C^f(t,T)\Sigma\Sigma^\top B^f(t,T) - \\ - 2C^f(t,T)\Sigma\Sigma^\top B^S(t) - 2\alpha^f(t), \quad B^f(T,T) = 0_n \tag{1.101}$$

$$\frac{d}{dt}A^f(t,T) = -Tr[\Sigma\Sigma^\top C^f(t,T)] - B^f(t,T)^\top\Sigma\Sigma^\top B^S(t) + \\ + \frac{1}{2}B^f(t,T)^\top\Sigma\Sigma^\top B^f(t,T) - \alpha^f(t)^\top\alpha(t)^f, \quad A^f(T,T) = 0 \tag{1.102}$$

We now have a time dependent speed of mean reversion matrix $A^f(t)$. In (1.16) we used the fact that A is a constant matrix to get a closed form solution[9]. Here we cannot use matrix exponentiation to get a closed form solution of the associated ODE (see Theorem A.2). However we can give a closed form solution for $C^f(t,T)$ using Theorem B.3 which is given in the Appendix B.

Lemma 1.13. *The closed form solutions of $C^f(t,T), B^f(t,T)$ and $A^f(t,T)$ can be given by the following formulas:*

$$C^f(t,T) = \frac{1}{S_t}P^{\tilde{f}}(t,T)Q^{\tilde{f}}(t,T)^{-1} \tag{1.103}$$

where for fixed T, $(Q^{\tilde{f}}(t,T), P^{\tilde{f}}(t,T))^\top$ is the solution of the terminal value linear ODE given by

$$\begin{pmatrix} \frac{d}{dt}Q^{\tilde{f}}(t,T) \\ \frac{d}{dt}P^{\tilde{f}}(t,T) \end{pmatrix} = \begin{pmatrix} A & -2\Sigma\Sigma^\top \\ -I^d & -A^\top \end{pmatrix} \begin{pmatrix} Q^{\tilde{f}}(t,T) \\ P^{f}(t,T) \end{pmatrix}, \quad \begin{pmatrix} Q^{\tilde{f}}(T,T) \\ P^{\tilde{f}}(T,T) \end{pmatrix} = \begin{pmatrix} I_n \\ S_T \end{pmatrix}, \tag{1.104}$$

[9] We can use matrix exponentiation to solve (1.16).

$$B^f(t,T) = \frac{1}{S_t}\Big((Q^{\tilde{f}}(T,T)^\top)^{-1} B^S(T) + 2(Q^{\tilde{f}}(t,T)^\top)^{-1} \int_t^T Q^{\tilde{f}}(s,T)^\top I^d \alpha(s)\, ds \Big), \tag{1.105}$$

$$A^f(t,T) = \frac{1}{S_t}\Big(A^S(T) + \int_t^T Tr[\Sigma\Sigma^\top C^{\tilde{f}}(s,T)] - \frac{1}{2} B^{\tilde{f}}(s,T)^\top \Sigma\Sigma^\top B^{\tilde{f}}(s,T) + \\ + \alpha(s)^\top I^d \alpha(s)\, ds \Big). \tag{1.106}$$

Proof. Default free securities in the foreign economy can be converted into domestic default free securities using the foreign exchange rate S_t. Therefore the value of $P^f(t,T)$ which is one unit of foreign currency at time T is equal to S_T when converted to domestic currency using the foreign exchange rate. By the principle of risk neutral valuation (see, e.g. Harrison and Pliska (1981)) the value of a domestic security discounted by the domestic savings account is a martingale under $\mathbb{Q}$. Hence we first calculate

$$\begin{aligned} \widetilde{P}^f(t,T) :=& \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(- \int_t^T r_s^d\, ds \Big) S_T \Big] \\ =& \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(- \int_t^T r_s^d\, ds \Big) \exp(Y_T^\top C^S(T) Y_T + B^S(T)^\top Y_T + A^S(T)) \Big]. \end{aligned} \tag{1.107}$$

By the absence of arbitrage converting the domestic security back to a foreign security through the inverse of the foreign exchange, we can claim that the following equality must be true:

$$P^f(t,T) = \frac{1}{S_t} \widetilde{P}^f(t,T). \tag{1.108}$$

Since the payoff at time T is log-quadratic Gaussian in (1.107), we can now apply Theorem B.3 given in the Appendix B to get the following result:

$$\widetilde{P}^f(t,T) = \exp(Y_t^\top C^{\tilde{f}}(t,T) Y_t + B^{\tilde{f}}(t,T)^\top Y_t + A^{\tilde{f}}(t,T)) \tag{1.109}$$

where $C^{\tilde{f}}(t,T), B^{\tilde{f}}(t,T)$ and $A^{\tilde{f}}(t,T)$ solve the following ODE's

$$\begin{aligned}\frac{d}{dt}C^{\tilde{f}}(t,T) &= -A^{\top}C^{\tilde{f}}(t,T) - C^{\tilde{f}}(t,T)A + \\ &+ 2C^{\tilde{f}}(t,T)\Sigma\Sigma^{\top}C^{\tilde{f}}(t,T) - I^d, \quad C^{\tilde{f}}(T,T) = C^S(T)\end{aligned} \tag{1.110}$$

$$\begin{aligned}\frac{d}{dt}B^{\tilde{f}}(t,T) = -A^{\top}B^{\tilde{f}}(t,T) + 2\,C^{\tilde{f}}(t,T)\Sigma\Sigma^{\top}B^{\tilde{f}}(t,T) - 2\,I^d\alpha(t),\\ B^{\tilde{f}}(T,T) = B^S(T)\end{aligned} \tag{1.111}$$

$$\begin{aligned}\frac{d}{dt}A^{\tilde{f}}(t,T) = -Tr[\Sigma\Sigma^{\top}C^{\tilde{f}}(t,T)] + \frac{1}{2}B^{\tilde{f}}(t,T)^{\top}\Sigma\Sigma^{\top}B^{\tilde{f}}(t,T) - \alpha(t)^{\top}I^d\alpha(t)\\ A^{\tilde{f}}(T,T) = A^S(T)\end{aligned} \tag{1.112}$$

We can solve (1.110) by applying Theorem A.3 from Appendix A. Hence we need to solve the terminal value ODE given by (1.104). Using standard results for a system of linear ODE's with constant coefficients (see, e.g., Leonard (2002), we can give the solution of (1.104) by the following formula

$$\begin{pmatrix} Q^f(t,T) \\ P^f(t,T) \end{pmatrix} = \exp(-\widetilde{M}\,(T-t))(I \quad C^S(T)) \tag{1.113}$$

where

$$\widetilde{M} = \begin{pmatrix} A & -2\Sigma\Sigma^{\top} \\ -I^d & -A^{\top} \end{pmatrix}.$$

Since $\widetilde{M}$ is a constant matrix, we can find $\exp(-\widetilde{M}\,(T-t))$ in closed form. Therefore using Theorem A.3, we can conclude that the solution of (1.110) is given by

$$P^{\tilde{f}}(t,T)Q^{\tilde{f}}(t,T)^{-1}. \tag{1.114}$$

Using (1.108), we get (1.103) from (1.114). To solve for $B^f(t,T)$ first note that from (1.104), we get that $Q^{\tilde{f}}(t,T)$ is a solution of the following ODE for fixed T

$$\frac{d}{dt}Q^{\tilde{f}}(t,T) = (A - 2\Sigma\Sigma^{\top}C^{\tilde{f}}(t,T))Q^{\tilde{f}}(t,T). \tag{1.115}$$

We now use $Q^{\tilde{f}}(t,T)^\top$ as an integrating factor to solve for $B^{\tilde{f}}(t,T)$. Specifically differentiation with respect to t of $Q^{\tilde{f}}(t,T)^\top B^{\tilde{f}}(t,T)$ gives us after some simplifications the following ODE

$$\frac{d}{dt}Q^{\tilde{f}}(t,T)^\top B^{\tilde{f}}(t,T) = -2\,Q^{\tilde{f}}(t,T)^\top I^d\alpha(t). \tag{1.116}$$

The solution of (1.116) can be obtained using direct integration and the boundary condition $B^{\tilde{f}}(T,T) = B^S(T)$. Using (1.108) and the closed form solution for $B^{\tilde{f}}(t,T)$, we get (1.105). We can get $A^{\tilde{f}}(t,T)$ through direct integration of (1.112) and using the boundary condition, $A^{\tilde{f}}(T,T) = A^S(T)$. We then apply (1.108) to the closed form solution of $A^{\tilde{f}}(t,T)$ to get (1.106). This completes the proof of the lemma. □

There are other ways of solving for $P^f(t,T)$ which do not require us to solve a Riccati ODE's as in Lemma 1.13. For some of the formulas , we will use the following lemma which is given in Cherif et al. (1994).

Lemma 1.14. *Let*

$$\left.\frac{d\mathbb{N}}{d\mathbb{T}}\right|_{\mathcal{F}_T} = \frac{N_T}{P(T,T)}\frac{P(0,T)}{N_0} = \frac{\exp(Y_T^\top C^N(T)Y_T + B^N(T)^\top Y_T + A^N(T))P(0,T)}{\exp(Y_0^\top C^N(0)Y_0 + B^N(0)^\top Y_0 + A^N(0))} \tag{1.117}$$

represent the Radon Nikodým density for the change of measure from the forward measure $\mathbb{T}$ *to a new measure* $\mathbb{N}$ *which corresponds to the measure under which* $N_t = \exp(Y_t^\top C^N(t)Y_t + B^N(t)^\top Y_t + A^N(t))$ *is the numeraire i.e. discounting risk free assets by the numeraire* N_t *converts them into martingales under the corresponding measure* $\mathbb{N}$. *Let*

$$V^N(t,T) = \textit{ the variance-covariance matrix of } Y_t \textit{ under } \mathbb{N}$$

and

$$M^N(t,T) = \textit{ the mean vector of } Y_t \textit{ under } \mathbb{N}.$$

Then we have the following formulas relating $V^N(t,T)$ *and* $M^N(t,T)$ *to the variance-covariance matrix* $V(t,T)$ *and the mean vector* $M(t,T)$ *under the forward measure*

$\mathbb{T}$:

$$V^N(t,T) = \Big(I - 2V(t,T)C^N(T)\Big)^{-1} V(t,T) \tag{1.118}$$

$$M^N(t,T) = \Big(I - 2V(t,T)C^N(T)\Big)^{-1} \Big(M(t,T) + V(t,T)B^N(T)\Big). \tag{1.119}$$

Proof. The authors in Cherif et al. (1994) base the proof of the lemma on the direct calculation using multivariate Gaussian densities. We give a much shorter proof by using the moment generating function of a quadratic form of Gaussian random variables which is given in Lemma (B.1). We now calculate the moment generating function of Y_t under $\mathbb{N}$. Let $\Phi^N(z)$ denote the moment generating of Y_t under $\mathbb{N}$. Using the Radon Nikodým density (1.117) and the abstract Bayes formula we have

$$\Phi^N(z) = \mathbb{E}_t^{\mathbb{N}}[\exp(z^\top Y_T)] \tag{1.120}$$

$$= \mathbb{E}_t^{\mathbb{T}}\Big[\frac{\exp(Y_T^\top C^N(T)Y_T + B^N(T)^\top Y_T + A^N(T))}{\mathbb{E}_t^{\mathbb{T}}[\exp(Y_T^\top C^N(T)Y_T + B^N(T)Y_T + A^N(T))]}\exp(z^\top Y_T)\Big]. \tag{1.121}$$

We denote by $\Omega_N(T)$ the quadratic form

$$\Omega_N(T) := Y_T^\top C^N(T)Y_T + (B^N(T) + z)^\top Y_T + A^N(T)) \tag{1.122}$$

so that we can write (1.121) as

$$\Phi^N(z) = \frac{\mathbb{E}_t^{\mathbb{T}}\Big[\exp(\Omega_N(T))\Big]}{\mathbb{E}_t^{\mathbb{T}}[\exp(Y_T^\top C^N(T)Y_T + B^N(T)Y_T + A^N(T))]}. \tag{1.123}$$

If we assume that $C^N(T)$ is symmetric[10] and use the fact that $V^N(t,T)$ is invertible for $t \neq T$ (and therefore positive definite), it will follow that $\Phi^N(z)$ is defined everywhere (see Theorem 3.2a in Mathai and Provost (1992)). We can calculate

$$\mathbb{E}_t^{\mathbb{T}}\Big[\exp(\Omega_N(T))\Big]$$

[10] This is true for the case of zero coupon default free bonds and zero coupon defaultable bonds considered in this thesis because $C(t,T)$ and $\bar{C}(t,T)$ are solutions of symmetric matrix Riccati equations and will be true if we consider log quadratic Gaussian processes which can be obtained through the solution of symmetric matrix Riccati equations such that the initial or terminal condition is also a symmetric matrix.

using the moment generating function of a quadratic form in Gaussian random variables which is given in lemma B.1 (see (B.1)):

$$
\begin{aligned}
\Phi(\Omega_N,1) =& \mid I-2C^N(T)\,V(t,T)\mid^{-\frac{1}{2}} \exp\Bigg(-\frac{1}{2}(M(t,T)V(t,T)^{-1}M(t,T)- \\
&-2A^N(T))+\frac{1}{2}(M(t,T)+V(t,T)(B^N(T)+z))^{\top}(I-2C^N(T)\,V(t,T))^{-1} \\
&V(t,T)^{-1}(M(t,T)+V(t,T)(B^N(T)+z))\Bigg) \\
=& \mid I-2C^N(T)\,V(t,T)\mid^{-\frac{1}{2}} \exp\Bigg(-\frac{1}{2}(M(t,T)V(t,T)^{-1}M(t,T)-2A^N(T))+ \\
&\frac{1}{2}(M(t,T)+V(t,T)B^N(T))^{\top}(I-2\,C^N(T)V(t,T))^{-1}V(t,T)^{-1} \\
&(M(t,T)+V(t,T)B^N(T))\Bigg)\exp\Bigg(\frac{1}{2}z^{\top}V(t,T)(I-2C^N(T)V(t,T))^{-1}V(t,T)^{-1} \\
&(M(t,T)+V(t,T)(B^N(T)+z))+\frac{1}{2}(M(t,T)+V(t,T)B^N(T))^{\top} \\
&(I-2\,C^N(T)V(t,T))^{-1}z\Bigg) \\
=& \mathbb{E}_t^{\mathbb{T}}[\exp(Y_T^{\top}C^N(T)Y_T+B^N(T)Y_T+A^N(T))] \\
&\exp\Bigg(z^{\top}(I-2V(t,T)C^N(T))^{-1}(M(t,T)+V(t,T)B^N(T)) \\
&+\frac{1}{2}z^{\top}(I-2V(t,T)C^N(T))^{-1}V(t,T)z. \qquad (1.124)
\end{aligned}
$$

Hence inserting (1.124) into (1.123) and simplifying we get

$$
\begin{aligned}
\Phi^N(z) = \exp\Bigg(&\frac{1}{2}z^{\top}(I-2V(t,T)C^N(T))^{-1}(M(t,T)+V(t,T)C^N(T)) \\
&+\frac{1}{2}z^{\top}(I-2C^N(T)V(t,T))^{-1}V(t,T))^{-1}V(t,T)z \qquad (1.125)
\end{aligned}
$$

which is the moment generating function of a multivariate Gaussian random variable with mean vector and variance-covariance matrix as in the lemma. This completes the proof of the lemma. □

Before we state the next theorem we will state the following well known result which will be used in the proof of the theorem. The change of measure from the domestic forward measure $\mathbb{T}$ to the foreign forward measure $\mathbb{T}^f$ is done through the forward exchange rate (see, e.g., Cherif and El Karoui (1993) or Schlögl (2002)) which is denoted by $X(t,T)$ and is given by

$$X(t,T) = \frac{S_t\, P^f(t,T)}{P(t,T)}. \tag{1.126}$$

Since $\mathbb{T}$ is the domestic default free forward measure corresponding to using $P(t,T)$ as the numeraire, it follows from (1.126) that $X(t,T)$ is a martingale under $\mathbb{T}$. Therefore the dynamics of $X(t,T)$ is given by

$$\frac{dX(t,T)}{X(t,T)} = (2\,Y_t^{\top}(C^S(t) - C^f(t,T) + C(t,T)) + B^S(t) - B^f(t,T) + B(t,T))\Sigma^{\top}\, dW^{\mathbf{T}}. \tag{1.127}$$

Using Girsanov's theorem the Brownian motions under the different measures are related by

$$dW_t^{\mathbf{T}} = dW_t^{\mathbf{T}^f} + \Sigma^{\top}\Big(2\Big(C^S(t) - C^f(t,T) + C(t,T)\Big)Y_t + B^S(t) - B^f(t,T) + B(t,T)\Big)\, dt. \tag{1.128}$$

Thus the dynamics of Y_t under the foreign forward measure $\mathbb{T}^f$ can be obtained using (1.128) and (1.25):

$$dY_t = \Sigma\Sigma^{\top}(B^S(t) - B^f(t,T))\, dt + (A^f(t) - 2\Sigma\Sigma^{\top}C^f(t,T))Y_t\, dt + \Sigma\, dW_t^{\mathbf{T}^f}. \tag{1.129}$$

Theorem 1.15. *We can get $C^f(t,T)$ using the following formula:*

$$C^f(t,T) = \frac{1}{2}(\Sigma\Sigma^{\top})^{-1}\Big(A^f(t) - \Big(\frac{d}{dt}Q^f(t,T)\Big)Q^{-1}(t,T)\Big), \tag{1.130}$$

where

$$Q^f(t,T)^{\top} = Q(t,T) - 2\tilde{P}(t,T)^{\top}C^S(T). \tag{1.131}$$

Proof. Let $Q^f(t,T)$ be the solution of the following ODE

$$\frac{d}{dt}Q^f(t,T) = (A^f(t) - 2\Sigma\Sigma^{\top}C^f(t,T))Q^f(t,T), \quad Q^f(T,T) = I^f. \tag{1.132}$$

Then we can solve the SDE (1.129) exactly by considering the integrating factor $Q^f(t,T)^{-1}$, specifically consider

$$d(Q^f(t,T)^{-1}Y_t) = Q^f(t,T)^{-1}\Sigma\Sigma^{\top}(B^S(t) - B^f(t,T))\, dt + \Sigma\, dW_t^{\mathbf{T}^f} \tag{1.133}$$

which can be integrated directly to yield

$$Y_T = Q^f(t,T)^{-1}Y_t + \int_t^T Q^f(r,T)^{-1}\Sigma\Sigma^\top(B^S(r) - B^f(r,T))\,dr + \int_t^T Q^f(r,T)^{-1}\Sigma\,dW_r^{\mathbb{T}^f}.$$

Therefore the mean of Y_T under $\mathbb{T}^f$ conditional on $\mathcal{F}_t$ is given by

$$M^f(t,T) = Q^f(t,T)^{-1}Y_t + \int_t^T Q^f(r,T)^{-1}\Sigma\Sigma^\top(B^S(r) - B^f(r,T))\,dr. \quad (1.134)$$

On the other hand using the result of Cherif et al. (1994) given in Lemma 1.14, we can find the mean $M^f(t,T)$ directly. Recall from (1.36), $M(t,T)$ is given by

$$M(t,T) = Q^{-1}(t,T)Y_t - \int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds.$$

The change of measure from the domestic forward measure $\mathbb{T}$ to the foreign forward measure $\mathbb{T}^f$ is done through the forward exchange rate (see (1.126)) which is equal to

$$X(T,T) = S_T$$

at time T i.e. the spot exchange rate. Hence applying Lemma 1.14, we get

$$\begin{aligned} M^f(t,T) = \Big(I - 2V(t,T)C^S(T)\Big)^{-1}\Big(&Q^{-1}(t,T)Y_t \\ &- \int_t^T Q^{-1}(s,T)\Sigma\Sigma^\top B(s,T)\,ds + V(t,T)B^S(T)\Big). \quad (1.135)\end{aligned}$$

Therefore comparing the result in (1.135) with (1.134), we get that

$$Q^f(t,T)^{-1} = \Big(I - 2V(t,T)C^S(T)\Big)^{-1}Q^{-1}(t,T). \quad (1.136)$$

From (1.36) and (1.41), we get that

$$Q(t,T) = \tilde{Q}(t,T)^\top. \quad (1.137)$$

Thus using (1.40), (1.137) and $V(t,T) = V(t,T)^\top$ we get from (1.136):

$$Q^f(t,T) = Q(t,T)\Big(I - 2V(t,T)C^S(T)\Big) = Q(t,T) - 2\widetilde{P}(t,T)^\top\, C^S(T).$$

We can therefore solve (1.132) for $C^f(t,T)$ to obtain (1.130). □

Lemma 1.16. *Another way of calculating $P^f(t,T)$ is through*

$$\begin{aligned}P^f(t,T) = \frac{1}{S_t}\, & P(t,T)\mid I - 2C^S(T)V(t,T)\mid^{-\frac{1}{2}} \\ & \exp\Big(-\frac{1}{2}(M(t,T)^\top V(t,T)^{-1}M(t,T) - 2A^S(T)) \\ & + \frac{1}{2}(M(t,T) + V(t,T)B(T,S))^\top (I - 2C^S(T)V(t,T))^{-1} \\ & \qquad V(t,T)^{-1}(M(t,T) + V(t,T)B^S(T))\Big). \qquad (1.138)\end{aligned}$$

Proof. A foreign default free zero coupon bond can be converted into a domestic security by regarding it as a domestic security that pays S_T units of domestic currency at time T. In the domestic economy we can use the domestic risk neutral measure $\mathbb{Q}^d$ to calculate the arbitrage free price of this security at time $t \leqslant T$ and then convert this price using the prevailing exchange rate to find the price of the foreign default free zero coupon bond $P^f(t,T)$ (see the proof of Lemma 1.13 or Cherif and El Karoui (1993) for more detail). Hence we have

$$\begin{aligned}P^f(t,T) =& \frac{1}{S_t}\,\mathbb{E}_t^{\mathbb{Q}^d}\Big[\exp\Big(-\int_t^T r_r^f\,dr\Big)S_T\Big] \\ =& \frac{1}{S_t}\,P(t,T)\,\mathbb{E}_t^{\mathrm{T}}[\exp(Y_T C^S(T)Y_T + B^S(T)^\top Y_T + A^S(T))]. \qquad (1.139)\end{aligned}$$

We can now proceed as in Lemma 1.7(see (1.71)) to prove the lemma. Thus we use the moment generating function of a quadratic form in Gaussian random variables (see Lemma B.1 in Appendix B). We only have to consider the quadratic form in Y_T

$$Y_T C^S(T)Y_T + B^S(T)^\top Y_T + A^S(T).$$

Thus we have

$$\mathbb{E}_t^{\mathbb{T}}[\exp(Y_T C^S(T) Y_T + B^S(T)^\top Y_T + A^S(T))] = \mid I - 2C^S(T)V(t,T) \mid^{-\frac{1}{2}}$$
$$\exp\Big(- \frac{1}{2}(M(t,T)^\top V(t,T)^{-1} M(t,T) - 2A^S(T))$$
$$+ \frac{1}{2}(M(t,T) + V(t,T)B(T,S))^\top (I - 2C^S(T)V(t,T))^{-1}$$
$$V(t,T)^{-1}(M(t,T) + V(t,T)B^S(T))\Big). \quad (1.140)$$

Thus substituting (1.140) into (1.139) we get the result of the theorem. □

If we assume that $I^d Y_t$ is independent of $I^f Y_t$, we can use the proof method used to prove Lemma 1.5 and Lemma 1.7 to obtain similar results for the parameters of $P^f(t,T)$.

Lemma 1.17.

$$I^f M^f(t,T) = Q^f(t,T)^{-1} Y_t + \frac{1}{2}(Q^f(t,T)^\top)^{-1} \partial_T B^{\bar{f}}(t,T) - \alpha^f(T)$$

where

$$B^{\bar{f}}(t,T) = B^f(t,T) - I^f B^S(t).$$

$$B^f(0,T) = 2 \int_0^T (Q^f(0,s)^\top)^{-1} \tilde{F}^f(s)\, ds$$

where

$$\tilde{F}^f(s) := I^f \sqrt{\frac{F^f(0,s) - tr(I^f V^f(0,s) I^f)}{m}}\mathbf{1}_n,$$

and $F^f(0,T)$ denotes the foreign forward rate for maturity T at time $t = 0$, $m < n$ is the number of factors used to model r_t^f.

Proof. See the proof of Lemma 1.5. □

Lemma 1.18.

$$B^f(t,T) = (I^f + 2C^f(t,T) I^f V^f(0,t)) Q^f(0,t)^\top (B^f(0,T) - B^f(0,t))$$
$$+ 2C^f(t,T)(\alpha^f(t) - \tilde{F}^f(t)) \quad (1.141)$$

$$A^f(t,T) = \log_e\Big(\frac{P^f(0,t)}{P^f(0,T)}\Big) - \frac{1}{2}\log_e(|\,I^f + 2C^f(t,T)I^fV^f(0,t)\,|)$$
$$- (\alpha^f(t) - \tilde{F}^f(t))^\top(I^f + 2C^f(t,T)I^fV^f(0,t))^{-1}C^f(t,T)(\alpha^f(t) - \tilde{F}^f(t))$$
$$+ (\alpha^f(t) - \tilde{F}^f(t) + \frac{1}{2}I^fV^f(0,t)B^f(t,T))(I^f + 2C^f(t,T)I^fV^f(0,t))^{-1}B^f(t,T) \tag{1.142}$$

Proof. See the proof of Lemma 1.7. □

Lemma 1.17 and Lemma 1.18 give computationally efficient ways of computing $B^f(t,T)$ and $A^f(t,T)$ by reducing the dimensions of the integrals that are needed to calculate $B^f(t,T)$ and $A^f(t,T)$. This show we have a tractable model for the pricing of default free bond prices in both economies because default free zero coupon bond prices in the foreign economy are also log-quadratic Gaussian. Recall that the foreign short term interest rate is given by

$$(Y_t + \alpha(t))^\top I^f(Y_t + \alpha(t)).$$

The drift term of the factors that are used to model the foreign instantaneous rate of interest is denoted by $\alpha^f(t) := I^f\alpha(t)$. If we assume that the domestic and foreign short term interest rates have no common factors and the factors used to model them given by I^dY_t and I^fY_t are independent, we can calibrate the drift term $\alpha^f(t)$ to the foreign forward rate curve using the formula which is given in the following theorem (see Assefa (2007) for a closed form solution of $\alpha^f(t)$ where I^dY_t and I^fY_t can be correlated).

Theorem 1.19. *We can calibrate to the foreign term structure at time $t = 0$ by the following integral*

$$\alpha^f(T) = \tilde{F}^f(T) + 2Q^S(T)\int_0^T Q^S(r)^{-1}V^f(0,r)\tilde{F}^f(r)\,dr-$$
$$- Q^S(T)\int_0^T Q^S(r)^{-1}\Sigma\Sigma^\top I^fB^S(r)\,ds. \tag{1.143}$$

Proof. First note that similar to (1.50), we have

$$F^f(0,T) = \mathbb{E}^{\mathrm{T}^f}[r^f(T)]. \tag{1.144}$$

Substituting

$$r^f(T) = (Y_T + \alpha(T))^\top I^f(Y_T + \alpha(T)) = (Y_T + \alpha^f(T))^\top I^f(Y_T + \alpha^f(T))$$

into (1.144), and using the fact that Y_T is Gaussian under $\mathbb{T}^f$, we get

$$F^f(0,T) = Tr[I^f V^f(0,T) I^f] + (M^f(0,T) + \alpha^f(T))^\top I^f (M^f(0,T) + \alpha^f(T)). \quad (1.145)$$

Hence we get[11]

$$\tilde{F}^f(T)^\top \tilde{F}^f(T) = (M^f(0,T) + \alpha^f(T))^\top I^f (M^f(0,T) + \alpha^f(T)). \quad (1.146)$$

We can therefore linearize (1.145) as follows:

$$\tilde{F}^f(T) = I^f(\alpha^f(T) + M^f(0,T)) \quad (1.147)$$

Differentiating (1.147) with respect to T gives us

$$\frac{d}{dT}\tilde{F}^f(T) = I^f\left(\frac{d}{dT}\alpha^f(T) + \frac{d}{dT}M^f(0,T)\right) = \frac{d}{dT}\alpha^f(T) + I^f\frac{d}{dT}M^f(0,T). \quad (1.148)$$

The dynamics of Y_t under $\mathbb{Q}^f$ is obtained by using the fact that

$$dW_t^d = dW_t^f + \Sigma^\top(2C^S(t)Y_t + B^S(t))dt$$

(see Cherif et al. (1994)) which gives us

$$dY_t = (\Sigma\Sigma^\top B^S(t) + (A + 2\Sigma\Sigma^\top C^S(t))Y_t)dt + \Sigma\, dW_t^f.$$

Recall that we stated that the ODE for the mean of Y_T conditional on $\mathcal{F}_t$ which is given in (1.40) is valid for a general setting in which the speed of mean reversion matrix $A(t)$ and the instantaneous variance covariance matrix $\Sigma(t)$ can be time dependent. Thus $M^f(t,T)$ which denotes the mean of Y_T under the foreign risk

[11] We have defined $\tilde{F}^f(T)$ in the statement of Lemma 1.17

neutral measure $\mathbb{Q}^f$ and conditional on $\mathcal{F}_t$ satisfies

$$\frac{d}{dT}M^f(0,T) = A^f(T)M^f(0,T) - 2V^f(0,T)I^f M^f(0,T)- \\ - 2V^f(0,T)\alpha^f(T) + \Sigma\Sigma^\top B^S(T), \quad M(0,0) = (0,\ldots,0)^\top \quad (1.149)$$

when $t = 0$. Hence substituting (1.149) in (1.148) gives us:

$$\frac{d}{dT}\alpha^f(T) = -I^f A^f(T)M^f(t,T) + I^f 2V^f(t,T)I^f M^f(0,T)+ \\ + I^f 2V^f(0,T)\alpha^f(T) - I^f\Sigma\Sigma^\top B^S(T) + \frac{d}{dT}\tilde{F}^f(T). \quad (1.150)$$

Now note that $I^f = I^f I^f$ and therefore $\alpha^f(T)$ can be written as

$$I^f\alpha^f(T). \quad (1.151)$$

We now make the assumption that the $I^d Y_t$, which are the factors used to model the domestic short rate of interest r_t, are independent of $I^f Y_t$, which are the factors used to model the foreign short rate of interest r_t^f. Under this assumption

$$A^f(T)I^f = I^f A^f(T) \quad (1.152)$$

because in

$$A^f(T) = A + 2\Sigma\Sigma^\top C^S(T),$$

A is a diagonal matrix and $\Sigma\Sigma^\top C^S(T)$ commutes with I^f provided we choose the initial value[12] $C^S(0)$ to be equal to I^f. We also have under the assumption of independence between $I^d Y_t$ and $I^f Y_t$:

$$I^f\Sigma\Sigma^\top = \Sigma\Sigma^\top I^f, \quad V^f(0,T)I^f = I^f V^f(0,T) \quad (1.153)$$

Hence using (1.147) we can reduce (1.150) to

$$\frac{d}{dT}\alpha^f(T) = A^f(T)\alpha^f(T) - A^f(T)\tilde{F}^f(T) + 2V^f(0,T)\tilde{F}^f(T)- \\ - \Sigma\Sigma^\top I^f B^S(T) + \frac{d}{dT}\tilde{F}^f(T). \quad (1.154)$$

[12] Recall that we said we calibrate S_t through the initial value for $A^S(0)$ as we have $Y_0 = 0_n$ and therefore we are free to choose the value of $C^S(0)$.

From (1.94) given in Lemma 1.11, we can get

$$\frac{d}{dT}Q^S(T) = A^f(T)Q^S(T), \quad Q^S(0) = I. \tag{1.155}$$

Therefore

$$\frac{d}{dT}(Q^S(T)^{-1}\alpha^f(T)) = -Q^S(T)^{-1}A^f(T)\alpha^f(T) + Q^S(T)^{-1}(A^f(T)\alpha^f(T)- \\ - A^f(T)\tilde{F}^f(T) + 2V^f(0,T)\tilde{F}^f(T) - \Sigma\Sigma^{\mathsf{T}} I^f B^S(T) + \frac{d}{dT}\tilde{F}^f(T)). \tag{1.156}$$

which simplifies to

$$\frac{d}{dT}(Q^S(T)^{-1}\alpha^f(T)) = Q^S(T)^{-1}(-A^f(T)\tilde{F}^f(T) + 2V^f(0,T)\tilde{F}^f(T)- \\ - \Sigma\Sigma^{\mathsf{T}} I^f B^S(T) + \frac{d}{dT}\tilde{F}^f(T)). \tag{1.157}$$

We can solve (1.157) by direct integration to get

$$Q^S(T)^{-1}\alpha^f(T) = \alpha^f(0) - \int_0^T Q^S(r)^{-1}A^f(r)\tilde{F}^f(r)\, dr \\ + 2\int_0^T Q^S(r)^{-1}(V^f(0,r)\tilde{F}^f(r) - \Sigma\Sigma^{\mathsf{T}} I^f B^S(r))\, dr + \int_0^T Q^S(r)^{-1}\frac{d}{dT}\tilde{F}^f(r)\, dr. \tag{1.158}$$

Integrating by parts the last term of (1.158) we have

$$\int_0^T Q^S(r)^{-1}\frac{d}{dT}\tilde{F}^f(r)\, dr = Q^S(T)^{-1}\tilde{F}^f(T) - Q^S(0)^{-1}\tilde{F}^f(0)+ \\ + \int_0^T Q^S(r)^{-1}A^f(r)\tilde{F}^f(r)\, dr. \tag{1.159}$$

Using

$$Q^S(0)\tilde{F}^f(0) = \alpha^f(0),$$

we can simplify (1.158) to get

$$Q^S(T)^{-1}\alpha^f(T) = Q^S(T)^{-1}\tilde{F}^f(T) + 2\int_0^T Q^S(r)^{-1}V^f(0,r)\tilde{F}^f(r)\,dr - \\ - \int_0^T Q^S(r)^{-1}\Sigma\Sigma^\top I^f B^S(r)\,dr. \quad (1.160)$$

We can now obtain the statement of the theorem from (1.160). □

1.2 Explicit Solutions for Piecewise Constant Case

In this section, we give a method for explicitly solving the parameters of the default free zero coupon bond when we assume that the speed of mean reversion matrix A and the instantaneous variance-covariance matrix $\Sigma\Sigma^\top$ are assumed to be piecewise constant in the quadratic Gaussian model (1.3). The explicit solution for this case is not given in the literature.

Theorem 1.20. *Let $\mathcal{T} = \{0 = T_1, \ldots, T_m\}$ be a partition of the time interval $[0, T^*]$. Assume $A(t)$ and $\Sigma\Sigma^\top(t)$ in (1.3) are piecewise constant such that*

$$A(t) = A_i,\ \Sigma\Sigma^\top(t) = \Sigma_i\Sigma_i^\top,\ i = 1, \ldots, m.$$

Then we can solve for $C(t,T), B(t,T)$ and $A(t,T)$ by a backward recursion by first solving for the values of $C(t,T), B(t,T)$ and $A(t,T)$ in the last interval $[T_{m-1}, T^*]$.

Proof. Recall that we said Lemma (1.9) is true even if A and $\Sigma\Sigma^\top$ were assumed to be time dependent rather than constant(see Cherif et al. (1994)). We now assume that $A(t)$ and $\Sigma(t)\Sigma(t)^\top$ are piecewise constant as stated in the theorem and $\Sigma(t)\Sigma(t)^\top$ is positive definite[1]. Because $A(t)$ and $\Sigma(t)\Sigma(t)^\top$ are only piecewise contstant, we can only have a piecewise differentiable solution of the the following

[1] we can weaken this assumption to being semi- positive definite.

terminal value ODE's(see Filippov (1988))

$$\partial_t C(t,T) = -\left(A(t)^{\top} C(t,T) + C(t,T)A(t)\right) + 2C(t,T)^{\top}\Sigma(t)\Sigma^{\top}(t)C(t,T) - I \tag{1.161}$$

$$\partial_t B(t,T) = -A^{\top} B(t,T) + 2C(t,T)\Sigma(t)\Sigma(t)^{\top} B(t,T) - 2\alpha(t) \tag{1.162}$$

$$\partial_t A(t,T) = -Tr[\Sigma(t)\Sigma(t)^{\top} C(t,T)] - \frac{1}{2}B(t,T)^{\top}\Sigma(t)\Sigma(t)^{\top} B(t,T) - \alpha(t)^{\top}\alpha(t). \tag{1.163}$$

Clearly $A(t)$ and $\Sigma(t)\Sigma(t)^{\top}$ are piecewise continuous and locally bounded. Thus two of the assumptions of Theorem A.4 given in Appendix A (see for more detail Freiling (2002) or Filippov (1988)) are satisfied. Applying the conclusion of this theorem, we get that a unique solution of (1.161) exists and is positive semi-definite for all values t. These unique solutions of the terminal value ODE's (1.161),(1.162) and (1.163) can be found by the following procedure. For the last interval the values of $C_m(t,T), B_m(t,T)$ and $A_m(t,T)$ are obtained by solving explicitly the following ODE's

$$\partial_t C_m(t,T) = -\left(A^{\top} C_m(t,T) + C_m(t,T)A\right) + 2C_m(t,T)^{\top}\Sigma\Sigma^{\top} C_m(t,T) - I \tag{1.164}$$

$$\partial_t B_m(t,T) = -A^{\top} B_m(t,T) + 2C_m(t,T)\Sigma\Sigma^{\top} B_m(t,T) - 2\alpha(t) \tag{1.165}$$

$$\partial_t A_m(t,T) = -Tr[\Sigma\Sigma^{\top} C_m(t,T)] - \frac{1}{2}B_m(t,T)^{\top}\Sigma\Sigma^{\top} B_m(t,T) - \alpha(t)\top\alpha(t) \tag{1.166}$$

with the usual boundary conditions

$$C_m(T,T) = 0, B_m(T,T) = 0, A_m(T,T) = 0.$$

We now set $C(t,T) = C_m(t,T), B(t,T) = B_m(t,T)$ and $A(t,T) = A_m(t,T)$ for $T \in [T_{m-1}, T^*]$. For the values of $C(t,T), B(t,T)$ and $A(t,T)$ in the interval $[T_{m-2}, T_{m-1}]$ we can apply the results of Theorem A.4) to solve explicitly the following terminal value matrix Riccati ODE

$$\partial_t C_{m-1}(t,T) = -(A^{\mathsf{T}} C_{m-1}(t,T) + C_{m-1}(t,T)A) \\ + 2C_{m-1}(t,T)^{\mathsf{T}} \Sigma\Sigma^{\mathsf{T}} C_{m-1}(t,T) - I$$

with the boundary condition

$$C_{m-1}(T_{m-1},T_{m-1}) = C(T_{m-1},T_{m-1}).$$

Using the method of integrating factor and direct integration which were used to solve (1.10) and (1.11) respectively, we can solve the following terminal value linear ODE's

$$\partial_t B_{m-1}(t,T) = -\, A^{\mathsf{T}} B_{m-1}(t,T) + 2C_{m-1}(t,T)\Sigma\Sigma^{\mathsf{T}} B_{m-1}(t,T) - 2\alpha(t) \tag{1.167}$$

$$\partial_t A_{m-1}(t,T) = -\, Tr[\Sigma\Sigma^{\mathsf{T}} C_{m-1}(t,T)] - \frac{1}{2} B_{m-1}(t,T)^{\mathsf{T}} \Sigma\Sigma^{\mathsf{T}} B_{m-1}(t,T) - \alpha(t)\mathsf{T}\alpha(t) \tag{1.168}$$

with the boundary conditions

$$B_{m-1}(T_{m-1},T_{m-1}) = B(T_{m-1},T_{m-1}),$$

$$A_{m-1}(T_{m-1},T_{m-1}) = A(T_{m-1},T_{m-1}).$$

We continue this procedure until we solve for the values of $C(t,T), B(t,T)$ and $A(t,T)$ in the first interval $[T_1, T_2]$. This procedure insures that $C(t,T), B(t,T)$ and $A(t,T)$ are piecewise continuously differentiable and this is the smoothest solution that can be found. □

2. SWAPTIONS IN THE QUADRATIC GAUSSIAN MODEL

2.1 Introduction

There is some work in the pricing of default free swaptions in a factor model (see, e.g., Collin-Dufresne and Goldstein (2002), Tanaka et al. (2005) and Schrager and Pelsser (2006)) and in the context of libor market models (see, e.g., d'Aspremont (2003)). However none of these papers directly address the pricing of swaptions in a quadratic Gaussian factor framework. The paper by Tanaka et al. (2005) mentions a Gram-Charlier expansion framework that can be used also in quadratic Gaussian models but it does not go into the details of the implementation. The results presented in Schrager and Pelsser (2006) for affine factor models could be applied to quadratic Gaussian models by mapping the quadratic Gaussian model to an equivalent affine factor model with more factors than the original model through the procedure given in Cheng and Scaillet (2004). In this chapter we derive approximation to swaption prices in the multi-factor quadratic Gaussian model. Even though the use of libor models with jumps and/or stochastic volatility (see, e.g., Musiela and Rutkowski (2005)) is better for the pricing of interest rate derivatives because we can calibrate the models using libor rates and options on libor rates, such models can be computationally intensive. Therefore in Mercurio and Pallavivini (2005) (see also Mercurio and Pallavivini (2006)), a mixture of Gaussian short rate models is proposed for the pricing of exotic interest rate derivatives such as constant maturity swaps (CMS) and options on CMS. The quadratic Gaussian model is a better model than a Gaussian short rate model as the short rate in the Gaussian model is not guaranteed to be non-negative for all scenarios. Therefore one can improve the mixture model suggested in Mercurio and Pallavivini (2005) by using a mixture of quadratic Gaussian factor models. In order to price CMS and options on CMS in a manner consistent with market information, it is important that any model be calibrated to swaption prices. Even though caps and floors can be priced accurately in the multi-factor quadratic Gaussian model using numerical inversion of Fourier transforms, there is no such procedure for the accurate pricing of swaptions. In general we have to calibrate the quadratic Gaussian model to the prices of caps, floors and swaptions given by the market so that we can price options on CMS and other exotic interest rate options. Hence we try to find the parameters of the quadratic Gaussian model (i.e. A and Σ given in (1.3)) that minimize the difference between the prices given by the model and the prices given by the market. This procedure involves a non-linear optimization procedure where we search for the parameters by calculating the prices of caps, floors and swaptions for different values of the parameters. Hence the calibration of the quadratic Gaussian model can be computational intensive as the calculation of the exact price of a swaption requires multidimensional integration. The search can be made faster if we start with a set of parameters that are close to the local minimum or global minimum of the non-linear functional that is being minimized. Therefore it is useful to generate good starting points for the optimization procedure which is used to calibrate the model. Approximations to swaption prices can be used to generate good starting points through an initial less computationally intensive optimization procedure where instead of exact prices of swaptions, we would use the approximations. In this chapter we give different analytic approximations to the price of swaptions in a quadratic Gaussian factor framework. The formulas presented perform well for different strike values. We first give definitions of an interest rate swap and interest swaptions. We assume the notional amount is one unit of currency in all subsequent discussions.

Definition 2.1. *The spot libor rate at time t for maturity T denoted by $L(t,T)$ is the constant interest rate at which an amount of $P(t,T)$ (the price of a default free zero coupon bond of maturity T at time t) units of currencies invested at time t will give a payoff of one unit of currency at time T i.e.*

$$P(t,T)(1+(T-t)L(t,T)) = 1$$

Definition 2.2. *A standard interest rate payer swap over the period $[T_\alpha, T_\beta]$ is a contract where the payer pays a fixed rate denoted by K at the dates $\mathcal{T} = \{T_{\alpha+1}, \dots, T_{\alpha+n} = T_\beta\}$ to the receiver and receives in return the spot libor rate $L(T_{\alpha+i}, T_{\alpha+i+1})$ which is set at time $T_{\alpha+i}$ at the next date $T_{\alpha+i+1}$ for $i = 0, \dots, n-1$. In a standard interest rate swap, the fixed rate K is chosen such that the value of the contract is equal to zero at the start date of the contract T_α.*

If $T_\alpha = 0$, we have a standard interest rate swap for which we can get market

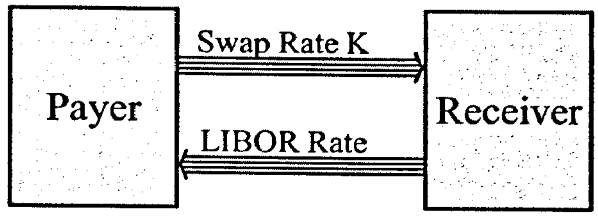

Fig. 2.1: interest rate swap over $[T_\alpha, T_\beta]$

quotes while for $T_\alpha > 0$, we have a forward starting swap. The value of a swap to the payer at time t is given by the difference between the cash flows received by the payer and the cash flows paid by the payer (see Chapter 13 in Musiela and Rutkowski (2005)). This value is given by

$$P(t, T_\alpha) - P(t, T_\beta) - K \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i) \tag{2.1}$$

where $\tau_i = T_i - T_{i-1}$. For a swap starting starting at time $t = 0$, the market practice is to choose the strike rate K such that the value of the swap is zero at time $t = 0$. Similarly for a swap starting at time $t > 0$, there is a strike rate such that the value of the swap is equal to zero at time $t = 0$. We call this strike rate as the market swap rate and denote it by $Swap_{\alpha,\beta}(t)$. We can calculate $Swap_{\alpha,\beta}(t)$ by setting (2.1) to zero. Thus

$$Swap_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}. \tag{2.2}$$

Definition 2.3. *A European payer swaption with maturity T_α gives the payer the right but not the obligation to enter a standard payer swap over the period $[T_\alpha, T_\beta]$ at time T_α by paying a fixed rate K which is agreed upon at the time $t < T_\alpha$ when the payer buys the swaption. If the rate K is lower than the prevailing market swap rate $Swap_{\alpha,\beta}(T_\alpha)$, then the payer will exercise the swaption but if $K \geqslant Swap_{\alpha,\beta}(T_\alpha)$, it is not in the interest of the payer to exercise the swaption.*

Under the absence of arbitrage, we can use risk neutral valuation theory (see, e.g., Harrison and Pliska (1981)) to calculate the price of a swaption. The price at time $t \leqslant T_\alpha$ of a payer swaption which we denote by $Swaptn_{\alpha,\beta}(t)$ is given by (see Chapter 13 of Musiela and Rutkowski (2005))

$$Swaptn_{\alpha,\beta}(t) = \mathbb{E}_t^{\mathbb{Q}}\Big[\exp\Big(-\int_t^{T_\alpha} r_s\, ds\Big)\Big(1 - P(t,T_n) - K\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)\Big)^+\Big]. \tag{2.3}$$

2.2 Pricing Swaptions under the Forward Measure

In this section we assume we have a quadratic Gaussian factor model based on a Gaussian Ornstein-Uhlenbeck multivariate factor Y_t as in chapter 1(see (1.3)). Thus in the SDE satisfied by Y_t we have a constant speed of mean reversion matrix A and a constant instantaneous volatility matrix Σ. The short term interest rate is assumed to be

$$r_t = Y_t^{\top}\hat{C}Y_t + \hat{B}^{\top}(t)Y_t + \hat{A}(t).$$

We assume for simplicity $\hat{C} = I, \hat{B} = 2\alpha(t)$ and $\hat{A}(t) = \alpha(t)^{\top}\alpha(t)$ (this is equivalent to assuming $r_t = Z_t^{\top}Z_t$ where $Z_t = Y_t + \alpha(t)$). To derive the first analytic approximation we write the price of a swaption as an option on a coupon bond (see, e.g., chapter 13 in Musiela and Rutkowski (2005))

$$Swaptn_{\alpha,\beta}(t) = \mathbb{E}_t^{\mathbb{Q}}\Big[\exp\Big(-\int_t^{T_\alpha} r_s\, ds\Big)\Big(1 - \sum_{i=\alpha+1}^{\beta} \tilde{\tau}_i P(T_\alpha, T_i)\Big)^+\Big]. \tag{2.4}$$

where $\tilde{\tau}_i = K\tau_i$ for $i = 1, \ldots, n-1$ and $\tilde{\tau}_n = 1 + K\tau_n$. Recall from chapter 1 the price of a default free bond is log-quadratic Gaussian and is denoted by

$$P(t,T) = \exp\big(-Y_t^{\top}C(t,T)Y_t - B(t,T)^{\top}Y_t - A(t,T)\big).$$

Let

$$w_i(t) = \frac{\tilde{\tau}_i P(t,T_i)}{\widetilde{P}_{\alpha,\beta}(t)} \tag{2.5}$$

for $i = \alpha+1, \ldots, \beta$ and let

$$\widetilde{P}_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tilde{\tau}_i P(t,T_i) \tag{2.6}$$

denote the present value of a basis point (PVBP01). Using Itô's formula we differentiate the SDE satisfied by (2.6) and obtain

$$d\widetilde{P}_{\alpha,\beta}(t) = \widetilde{P}_{\alpha,\beta}(t)\Big(r_t\,dt + \sum_{i=\alpha+1}^{\beta} w_i(t)(2\,C(t,T_i)Y_t + B(t,T_i))^{\mathsf{T}}\,\Sigma\,dW_t\Big). \quad (2.7)$$

We now replace $w_i(t)$ in the SDE by their values at time t_0 where $t_0 < t$ so that we can get analytic approximations to the price of swaptions in the quadratic Gaussian factor model. This technique has been used in the context of approximations in the lognormal libor market model (LLM) (see Rebonato (1998), d'Aspremont (2003)). Recently it has been used in the affine factor model in Schrager & Pelsser (2006) to approximate the swap rate as an affine factor. In this particular approximation we only freeze the weights $w_i(t)$ which gives an approximation of the dynamics of $P_{\alpha,\beta}(t)$. Hence we have

$$d\widetilde{P}_{\alpha,\beta}(t) \approx \widetilde{P}_{\alpha,\beta}(t)\Big(r_t\,dt + \sum_{i=\alpha+1}^{\beta} w_i(t_0)(2\,C(t,T_i)Y_t + B(t,T_i))^{\mathsf{T}}\,\Sigma\,dW_t\Big). \quad (2.8)$$

Consider the stochastic processes whose dynamics is given by

$$\frac{d\,\widetilde{aP}_{\alpha,\beta}(t,T)}{\widetilde{aP}_{\alpha,\beta}(t,T)} = \Big(Y_t^{\mathsf{T}} C^{\widetilde{aP}}(t,T_{\alpha+1},\dots,T_\beta)Y_t \;+\; B^{\widetilde{aP}}(t,T_{\alpha+1},\dots,T_\beta)^{\mathsf{T}}Y_t +$$
$$+\; A^{\widetilde{aP}}(t,T_{\alpha+1},\dots,T_\beta)\Big)\,dt - \sum_{i=\alpha+1}^{\beta} w_i(t_0)(2\,C(t,T_i)Y_t + B(t,T_i))^{\mathsf{T}}\,\Sigma\,dW_t \quad (2.9)$$

where

$$C^{\widetilde{aP}}(t,T_{\alpha+1},\dots,T_\beta) = -A^{\mathsf{T}}\sum_{i=\alpha+1}^{n} w_i(t_0)C(t,T_i) - \sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i)A -$$
$$-\,\partial_t\Big(\sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i)\Big) + 2\sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf{T}}\sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i)$$

$$B^{\widetilde{aP}}(t, T_{\alpha+1}, \ldots, T_\beta) = -A^\top \sum_{i=\alpha+1}^{\beta} w_i(t_0) B(t, T_i) - \partial_t \Big(\sum_{i=\alpha+1}^{\beta} w_i(t_0) B(t, T_i) \Big) + \\ + 2 \sum_{i=\alpha+1}^{n} w_i(t_0) C(t, T_i) \Sigma \Sigma^\top \sum_{i=\alpha+1}^{\beta} w_i(t_0) B(t, T_i)^\top$$

$$A^{\widetilde{aP}}(t, T_{\alpha+1}, \ldots, T_\beta)^\top = -Tr \Big[\Sigma \Sigma^\top \sum_{i=\alpha+1}^{\beta} w_i(t_0) C(t, T_i) \Big] - \\ - \partial_t \Big(\sum_{i=\alpha+1}^{\beta} w_i(t_0) A(t, T_i) \Big) + \frac{1}{2} \Big| \Sigma^\top \sum_{i=\alpha+1}^{\beta} w_i(t_0) B(t, T_i) \Big|^2 .$$

Theorem 2.4. *The log-quadratic Gaussian process*

$$\widetilde{aP}_{\alpha,\beta}(t) := \widetilde{P}_{\alpha,\beta}(t_0) \exp \Big(- Y_t^\top \sum_{i=\alpha+1}^{\beta} w_i(t_0)\, C(t, T_i)\, Y_t - \\ - \sum_{i=\alpha+1}^{\beta} w_i(t_0)\, B(t, T_i)^\top Y_t - \sum_{i=\alpha+1}^{\beta} w_i(t_0)\, A(t, T_i) \Big). \quad (2.10)$$

is a solution of (2.9) *and can be used to approximate the process which is the solution of* (2.8) *with initial value given by* $\widetilde{P}_{\alpha,\beta}(t_0)$.

Proof. Using Theorem B.2 given in Appendix B, we can show that $\widetilde{aP}_{\alpha,\beta}(t)$ is the solution of (2.9) with initial value given by $\widetilde{P}_{\alpha,\beta}(t_0)$.

We use the fact that $C(t, T_i), B(t, T_i), A(t, T_i), i = \alpha + 1, \ldots, \beta$, satisfy their corresponding Riccati equations, to obtain the following equalities

$$\partial_t \Big(\sum_{i=\alpha+1}^{\beta} w_i(t_0) C(t, T_i) \Big) = \sum_{i=1}^{n} w_i(t_0)(-A^\top C(t, T_i) - C(t, T_i)A + \\ + C(t, T_i) \Sigma \Sigma^\top C(t, T_i) - I)$$

$$\partial_t \Big(\sum_{i=\alpha+1}^{\beta} w_i(t_0) B(t, T_i) \Big) = \sum_{i=1}^{n} w_i(t_0)(-A^\top B(t, T_i) + 2C(t, T_i) \Sigma \Sigma^\top B(t, T_i) - 2\alpha(t))$$

$$\partial_t\Big(\sum_{i=\alpha+1}^{\beta} w_i(t_0)A(t,T_i)\Big) = \sum_{i=1}^{n} w_i(t_0)(-Tr[\Sigma\Sigma^{\mathsf T}C(t,T_i)]+\frac{1}{2}|\Sigma^{\mathsf T}B(t,T_i)|^2-|\alpha(t)|^2)$$

Using the above equalities in the SDE given by (2.9) and the fact that the weights $w_i(t_0)$ add up to one, we get

$$\begin{aligned}
\frac{d\widetilde{aP}_{\alpha,\beta}(t,T)}{\widetilde{aP}_{\alpha,\beta}(t,T)} = \Big(& Y_t^{\mathsf T}Y_t + 2\alpha(t)^{\mathsf T}Y_t + \alpha(t)^{\mathsf T}\alpha(t) + 2Y_t^{\mathsf T}\Big(\sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf T}\times \\
&\times \sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i) - \sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf T}C(t,T_i)\Big)Y_t+ \\
+2\Big(\sum_{i=\alpha+1}^{\beta} & w_i(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf T}\sum_{i=\alpha+1}^{n} w_i(t_0)B(t,T_i)-\sum_{i=1}^{\beta} w_i(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf T}B(t,T_i)\Big)^{\mathsf T} Y_t+ \\
&+\frac{1}{2}\Big|\Sigma^{\mathsf T}\sum_{i=\alpha+1}^{\beta} w_i(t_0)B(t,T_i)\Big|^2 - \sum_{i=\alpha+1}^{\beta} w_i(t_0)|\Sigma^{\mathsf T}B(t,T_i)|^2\Big)\,dt+ \\
&+ \sum_{i=\alpha+1}^{\beta} w_i(t_0)(2\,C(t,T_i)Y_t + B(t,T_i))^{\mathsf T}\,\Sigma\, dW_t \qquad (2.11)
\end{aligned}$$

First note that $r_t = Y_t^{\mathsf T}Y_t + 2\alpha(t)^{\mathsf T}Y_t + \alpha(t)^{\mathsf T}\alpha(t)$. We now show that the terms that remain in the drift do not cancel out. Since the weights $w_i(t_0), i = \alpha+1,\dots,\beta$ add up to one, we can write $C(t,T_i)$ as

$$C(t,T_i) = \sum_{j=\alpha+1}^{\beta} w_j(t_0)C(t,T_i).$$

Therefore the quadratic term in (2.11) can be rewritten as

$$\begin{aligned}
&\sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf T}\sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i) - \sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf T}\times \\
&\times \sum_{j=\alpha+1}^{\beta} w_j(t_0)C(t,T_i) = \sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf T} \sum_{j=\alpha+1, j\neq i}^{\beta} w_j(t_0)(C(t,T_j) - C(t,T_i)) \\
&= \sum_{i=\alpha+1}^{\beta}\sum_{j=\alpha+1, j\neq i}^{\beta} w_i(t_0)w_j(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf T}(C(t,T_j) - C(t,T_i)). \qquad (2.12)
\end{aligned}$$

Now the sum in the last line of (2.12) consists of terms that are multiplied by the product of the weights $w_i(t_0)w_j(t_0)$. Since the weights $w_i(t_0)$ are less than one, the product $w_i(t_0)w_j(t_0)$ is even smaller. Hence the error depends on the values:

$$C(t,T_i)\Sigma\Sigma^{\mathsf{T}}(C(t,T_j)-C(t,T_i))$$

in (2.12). Typical values of $C(t,T)$ which are obtained by calibrating the quadratic Gaussian model to market prices of options are of small order. Therefore the solution of (2.8) should be approximated by $\widetilde{P}_{\alpha,\beta}(t_0)$. This assertion is supported by numerical experiments conducted in section 2.3 where a similar approximation method is used and the most significant error is shown to be due to the freezing of the weights $w_i(t)$. The error that is introduced by using a log quadratic Gaussian process to approximate the solution of a SDE which is similar to (2.8) is shown to be not a significant source of error. A similar statement can be made with regards to the linear term in (2.11) as it can be written as

$$\sum_{i=\alpha+1}^{\beta} w_i(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf{T}}\sum_{i=\alpha+1}^{n} w_i(t_0)B(t,T_i)-\sum_{i=1}^{\beta} w_i(t_0)C(t,T_i)\Sigma\Sigma^{\mathsf{T}}B(t,T_i)$$

which has a similar structure to the quadratic term. We can also use the same argument for the constant term in (2.11):

$$\frac{1}{2}\left|\Sigma^{\mathsf{T}}\sum_{i=\alpha+1}^{\beta} w_i(t_0)B(t,T_i)\right|^2-\sum_{i=\alpha+1}^{\beta} w_i(t_0)|\Sigma^{\mathsf{T}}B(t,T_i)|^2$$

as it can be rewritten as

$$\frac{1}{2}\sum_{i=\alpha+1}^{\beta} w_i(t_0)B(t,T_i)^{\mathsf{T}}\Sigma\Sigma^{\mathsf{T}}\sum_{i=\alpha+1}^{\beta} w_i(t_0)B(t,T_i)-\sum_{i=\alpha+1}^{\beta} w_i(t_0)B(t,T_i)^{\mathsf{T}}\Sigma\Sigma^{\mathsf{T}}B(t,T_i).$$

The drift and volatility of

$$\exp\left(-Y_t^{\mathsf{T}}\sum_{i=\alpha+1}^{\beta} w_i(t_0)\,C(t,T_i)\,Y_t-\sum_{i=\alpha+1}^{\beta} w_i(t_0)\,B(t,T_i)^{\mathsf{T}}Y_t-\sum_{i=\alpha+1}^{\beta} w_i(t_0)\,A(t,T_i)\right) \tag{2.13}$$

are equal to that of $\widetilde{aP}_{\alpha,\beta}(t)$ but (2.13) is obtained by taking weighted averages of the quadratic forms: $\log_e(P(t,T_i)), i=\alpha+1,\dots,\beta$. Since the value of the zero coupon bond price $P(t,T_i)$ is between 0 and 1, $\log_e(P(t,T_i))<0$. Therefore the weighted average

$$\sum_{i=\alpha+1}^{\beta} w_i(t_0)\log_e(P(t,T_i))$$

has a value which is less than zero. Therefore (2.13) has a value that is between zero and one. To make (2.13) approximate $\widetilde{P}_{\alpha,\beta}(t)$ more accurately, we adjust the initial value of (2.13) by making it match with that of $\widetilde{P}_{\alpha,\beta}(t)$ at time t_0. Therefore multiplying (2.13) by

$$\widetilde{P}_{\alpha,\beta}(t_0) \tag{2.14}$$

we get $\widetilde{aP}_{\alpha,\beta}(t)$. □

Computations described later in this chapter show that for strike rates that are at the money or in the money and swap tenors less than five years using $\widetilde{aP}_{\alpha,\beta}(t,T)$ to approximate $\widetilde{P}_{\alpha,\beta}(t,T)$ given in (2.6) in the swaption payoff does not give much error in the swaption prices. However using $\widetilde{aP}_{\alpha,\beta}(t,T)$ to approximate only the exercise region of the swaption gives very good results for strike rates that are at the money, in the money and out of the money even when the swap tenors are equal to or greater than five years.

We first show how we can use $\widetilde{aP}_{\alpha,\beta}(t,T)$ to approximate the exercise region of $Swaptn_{\alpha,\beta}(t)$. In fact the swap underlying $Swaptn_{\alpha,\beta}(t)$ whose value is given by

$$1 - P(T_\alpha, T_\beta) - K\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)$$

will be exercised at time T_α if $\tilde{P}_{\alpha,\beta}(T_\alpha) < 1$. This exercise region can be approximated by $\widetilde{aP}_{\alpha,\beta}(t,T_\alpha) < 1$. Since $\widetilde{aP}_{\alpha,\beta}(t,T)$ is log-quadratic Gaussian, it will make it possible to use Fourier techniques to compute the approximate price of the default free swaption. The method of taking the Fourier transform of the option price with respect to the strike value and inverting the transform to get the option price was first introduced by Carr & Madan (1998). For a detailed explanation of this Fourier approach to option pricing, we refer the reader to Lee (2004). We use the results presented by Lee (2004) in this chapter. We will see in numerical experiments that this leads to an approximation that is accurate over a range of strikes because we only approximate the exercise region. First we give some definitions and notations. For simplicity we discuss valuation at time $t = 0$.

Definition 2.5. *The discounted characteristic function $\Phi(z)$ for $z \in \mathbb{R}^n$ of the* *random vector x is given by*

$$\Phi(z) = \mathbb{E}^{\mathbf{Q}}\Big[\exp - \Big(\int_0^T r_s\, ds\Big) \exp(iz.x)\Big] = P(0,T)\mathbb{E}^{\mathbf{T}}[\exp(iz.x)]. \tag{2.15}$$

Subsequently we will only have to deal with using

$$\Phi(z) = \mathbb{E}^{\mathbf{T}}[\exp(iz.x)]$$

to calculate $\mathbb{E}^{\mathbf{T}}[G(x_{T_\alpha})]$ for the appropriate payoff function $G(x)$, since we only need to multiply this value by $P(0,T)$ to get the option price. In general we need to dampen option prices by an appropriate factor $\exp(\hat{\alpha}K)$ for some $\hat{\alpha} > 0$ so that the fourier transform of the dampened option price is finite. If the characteristic function is defined everywhere, we have more choices for the value of $\hat{\alpha}$ and this yields additional formulas for inverting the transform (see Lee (2004)). For quadratic Gaussian random variables, the characteristic function is defined everywhere (see Mathai and Provost (1992)) and therefore we can use the additional formulas derived in Lee (2004) to calculate the option price including $\hat{\alpha} = 0$. We will not investigate the advantage of using different $\hat{\alpha}$ for calculating the option price in different ways. The approximate swaption pricing problem we are looking at is:

$$Swaptn_{\alpha,\beta}(0) \approx$$
$$\widetilde{Swaptn}_{\alpha,\beta}(0) := P(0,T_\alpha)\mathbb{E}_t^{\mathbf{T}_\alpha}\Big[\mathbf{1}_{\widetilde{aP}_{\alpha,\beta}(T_\alpha)<1} - \sum_{i=\alpha+1}^{\beta} \tilde{\tau}_i P(T_\alpha,T_i)\mathbf{1}_{\widetilde{aP}_{\alpha,\beta}(T_\alpha)<1}\Big] \tag{2.16}$$

The price of default free zero coupon bond prices in the quadratic Gaussian model are log-quadratic Gaussian and hence

$$Q_i(T_\alpha) := \log_e(P(T_\alpha,T_i)) = -Y_{T_\alpha}^{\top} C(T_\alpha,T_i) Y_{T_\alpha} - B(T_\alpha,T_i)^{\top} Y_{T_\alpha} - A(T_\alpha,T_i)$$

is a quadratic Gaussian random variable and $\widetilde{aP}_{\alpha,\beta}(T_\alpha) < 1$ is equivalent to

$$Q_{\alpha,\beta}(T_\alpha) := Y_t^{\top} \sum_{i=\alpha+1}^{\beta} w_i(0) C(T_\alpha,T_i) Y_t + \sum_{i=\alpha+1}^{\beta} w_i(0) B(T_\alpha,T_i)^{\top} Y_t +$$
$$+ \sum_{i=\alpha+1}^{\beta} w_i(0) A(T_\alpha,T_i) > \tilde{K}$$

where $\tilde{K} = 0$. In order to use the method of transforming the option price with respect to the strike price as introduced in Carr and Madan (1998) and extended in Lee (2004), we now regard the "strike price" to be $\tilde{K}$ in $\widetilde{Swaptn}_{\alpha,\beta}(0)$ and write

$$\widetilde{Swaptn}_{\alpha,\beta}(0, \tilde{K})$$

to make the approximate swaption price a function of $\tilde{K}$. As this is not the strike price of the swaption but a pseudo strike price which is always equal to zero, we do not have the computational advantage of using the discrete fourier transform (DFT) or the fast fourier transform (FFT) to simultaneously calculate the the prices of options at multiple strikes (see Lee (2004)) but this is a consequence of the fact that the approximation

$$\widetilde{aP}_{\alpha,\beta}(T_\alpha)$$

is strike dependent. However we can still use FFT to speed up the Fourier inversion needed to calculate the swaption price for each strike price rather than using a quadrature method to do the inversion. In Lee (2004), it is shown that we can have a better error control of the Fourier inversion by choosing to price the put or call option together with an optimum choice of $\hat{\alpha}$ to dampen the option price by $\exp(\hat{\alpha}\tilde{K})$. We do not make an investigation of the efficiency of the different formulas for the numerical Fourier inversion which are given in Lee (2004). Instead we choose $\hat{\alpha} = 0$ or $\hat{\alpha} = 1$ and use the integration routine *NIntegrate* of the commercial package ***Mathematica*** to do the Fourier inversion. The choice of $\hat{\alpha}$ is not motivated by any particular reason. We have not investigated the effect of choosing different values for $\hat{\alpha}$. Hence we can write the swaption pricing problem as

$$\widetilde{Swaptn}_{\alpha,\beta}(0, \tilde{K}) = P(0, T_\alpha)\mathbb{E}^{\mathbf{T}_\alpha}\left[\mathbf{1}_{Q_{\alpha,\beta}(T_\alpha) > \tilde{K}} - \sum_{i=\alpha+1}^{\beta} \tilde{\tau}_i P(T_\alpha, T_i)\mathbf{1}_{Q_{\alpha,\beta}(T_\alpha) > \tilde{K}}\right]. \tag{2.17}$$

The approximate option price $\widetilde{Swaptn}_{\alpha,\beta}(0, \tilde{K})$ is the sum of terms whose payoffs are of type

$$G_3(x, \tilde{K}) := \exp(b_1.x)\,\mathbf{1}_{b_0.x > \tilde{K}}, \quad x \in \mathbb{R}^n \tag{2.18}$$

where b_1 and b_0 are appropriately chosen (see Lee (2004)). In particular

$$b_0 = (0,1), b_1 = (1,0), x = (Q_i(T_\alpha), Q_{\alpha,\beta}(T_\alpha)) \tag{2.19}$$

for the terms corresponding to

$$P(T_\alpha, T_i)\mathbf{1}_{Q_{\alpha,\beta}(T_\alpha)>\tilde{K}},\ i = 1\dots n$$

while

$$b_0 = (0,1), b_1 = (0,0), x = (0, Q_{\alpha,\beta}(T_\alpha)) \tag{2.20}$$

for the term corresponding to

$$\mathbf{1}_{Q_{\alpha,\beta}(T_\alpha)>\tilde{K}}.$$

The Fourier transform with respect to strike $\tilde{K}$ of the dampened option price is

$$\widetilde{\widetilde{Swaptn}}_{\alpha,\beta}(z) = \int_{-\infty}^{\infty} \exp(\hat{\alpha}\tilde{K})\widetilde{Swaptn}_{\alpha,\beta}(0,\tilde{K}) \exp(iz\tilde{K})\, d\tilde{K} \tag{2.21}$$

where $i = \sqrt{-1}$. Notice that we need to choose $\hat{\alpha} > 0$ to dampen the option price in (2.21) so that the Fourier transform is defined and the swaption price can be obtained by inverting (2.21). In Lee (2004) (see Theorem 5.1 of this reference) a generalized formula is provided to calculate the option price for general $\hat{\alpha}$. This formula is derived using the residue theorem of complex analysis and hence it does not rely on inverting the Fourier transform for the chosen value of $\hat{\alpha}$ in (2.21). We refer the reader to Lee (2004) for the details of the derivation of the generalized formula[1]:

$$\frac{1}{\pi}\int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i} \mathrm{Re}\left[\frac{\Phi(Q_1, Q_2, zb_0 - b_1 i)}{iz}\exp(-iz\tilde{K})\right] dz \tag{2.22}$$

where Re[ ] means the real part of a complex number, $\tilde{K} = 0$ and $\Phi(Q_1, Q_2, zb_0 - b_1 i)$ is the joint characteristic function of two quadratic forms in Gaussian random variables which are denoted by Q_1 and Q_2 (i.e $\Phi(Q_1, Q_2, z_1, z_2)$ evaluated at $z_1 = 0, z_2 = z$ as we have $zb_0 - b_1 i = (0, z)$). We need to consider the joint characteristic function rather than the characteristic function for a single quadratic form in Gaussian random variables (see (B.1)) as we are considering payoffs of type

[1] There is also a detailed analysis of error control for the formula.

(2.18). The joint characteristic function of quadratic forms in Gaussian random variables is given in closed form in (Mathai and Provost 1992, p. 68). We first give the joint characteristic function of N quadratic forms in Gaussian random variables. Let $X = (X_1, \dots, X_M)$ be a multivariate random vector which is normally distributed with mean vector $MeanX$ and positive definite variance-covariance matrix of dimension $M \times M$ which is denoted by $VarX$. Also let I_M denote the $M \times M$ identity matrix and

$$Q_j = X^{\mathsf{T}} A_j X + a_j^{\mathsf{T}} X + d_j, \; A_j = A_j^{\mathsf{T}} \; j = 1, \dots, N$$

be N quadratic Gaussian random variables. Then the joint characteristic function $\Phi(Q_1, \dots, Q_N, z_1, \dots, z_N)$ of N quadratic forms in Gaussian random variables is given by

$$\begin{aligned}\Phi(Q_1, \dots, Q_N, z_1, \dots, z_N) &= |I_M - 2\sum_{j=1}^{N} iz_j A_j VarX|^{-\frac{1}{2}} \\ &\exp\Bigg\{ -\frac{1}{2} MeanX^{\mathsf{T}} VarX^{-1} MeanX + \sum_{j=1}^{N} iz_j d_j \\ &+ \frac{1}{2}(MeanX + \sum_{j=1}^{N} iz_j VarX a_j)^{\mathsf{T}} (I - 2\sum_{j=1}^{N} iz_j A_j VarX)^{-1} \\ &VarX^{-1}(MeanX + \sum_{j=1}^{N} iz_j VarX a_j)\Bigg\}. \qquad (2.23)\end{aligned}$$

For the pricing problem we are considering we only need to consider the joint characteristic function of two quadratic Gaussian random variables for any Gaussian random vector $M \geqslant 1$. Hence even if the number of factors used $X = (X_1, \dots, X_M)$ increases the dimension of the characteristic function does not increase. However we will have to deal with inverting matrices of dimension $M \times M$. In practice one only needs to use two or three factors to model term structures, therefore the additional computational burden is limited. Let 0_M denote the $M \times M$ matrix whose entries are all equal to zero i.e. the zero matrix. Therefore using the above formulas we can specifically give the Fourier transform of the $\widetilde{Swaptn}_{\alpha,\beta}(0, \tilde{K})$ with the respect to the strike $\tilde{K}$:

$$\widetilde{Swaptn}_{\alpha,\beta}(z) = \frac{\Phi(0_n, Q_{\alpha,\beta}(T_\alpha), 0, z)}{iz} + \sum_{j=\alpha+1}^{\beta} \frac{\Phi(Q_j(T_\alpha), Q_{\alpha,\beta}(T_\alpha), -i, z)}{iz}, \quad z \in \mathbb{R}. \tag{2.24}$$

For the particular case $x = (x_1, x_2)$ let us denote by $\mathcal{C}_{G_3}(\tilde{K})$ the option price which is normalized by the price of the zero coupon bond with maturity equal to the maturity of the option where the payoff is of type (2.18). Let $\hat{\mathcal{C}}_{G_3}(z)$ be defined[2] by:

$$\hat{\mathcal{C}}_{G_3}(z) := \frac{\Phi(x_1, x_2, b_0 z - b_1 i)}{iz}. \tag{2.25}$$

Then the problem of obtaining the option price through Fourier inversion for a specific strike $\tilde{K}$ is given by (see Lee (2004))

$$\mathcal{C}_{G_3}(\tilde{K}) = R_{\hat{\alpha},G_3} + \frac{1}{\pi} \int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i} \mathrm{Re}\left[\hat{\mathcal{C}}_{G_3}(z) \exp(-iz\tilde{K})\right] dz$$

where

$$R_{\hat{\alpha},G_3} = \begin{cases} \Phi(x_1, x_2, -b_1 i) = \Phi(x_1, x_2, -i, 0), & \hat{\alpha} < 0 \\ \dfrac{\Phi(x_1, x_2, -b_1 i)}{2} = \dfrac{\Phi(x_1, x_2, -i, 0)}{2}, & \hat{\alpha} = 0 \\ 0, & \hat{\alpha} > 0 \end{cases} \tag{2.26}$$

To calculate $\widetilde{Swaptn}_{\alpha,\beta}(0, \tilde{K})$ (for $\tilde{K} = 0$) by applying the above method, we need to use the formula several times since we have a payoff which is a sum of payoff type (2.18) (see also (2.19) and (2.20)). Thus the approximate price of the default

[2] In general the characteristic function of a random variable does not exist for all values of $z \in \mathbb{C}$ but we do not need to get into the details of the domain of existence for our problem since for quadratic forms in Gaussian random variables the joint characteristic function is defined everywhere.

free swaption can be written as

$$\widetilde{Swaptn}_{\alpha,\beta}(0,0) =$$
$$P(0,T_{\alpha,\beta})\Bigg(\frac{\Phi(0_M,Q_{\alpha,\beta}(T_\alpha),0,0)}{2}+\frac{1}{\pi}\int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i}\mathrm{Re}\Bigg[\frac{\Phi(0_M,Q_{\alpha,\beta}(T_\alpha),0,z)}{iz}\Bigg]dz$$
$$-\sum_{i=\alpha+1}^{\beta}\tilde{\tau}_i\frac{\Phi(Q_i(T_\alpha),Q_{\alpha,\beta}(T_\alpha),-i,0)}{2}+\frac{1}{\pi}\int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i}\mathrm{Re}\Bigg[\frac{\Phi(Q_i(T_\alpha),Q_{\alpha,\beta}(T_\alpha),-i,z)}{iz}\Bigg]dz\Bigg). \tag{2.27}$$

Hence when we use $\widetilde{aP}_{\alpha,\beta}(T_\alpha)$ to approximate the exercise region for the swaption price (2.3), we have to calculate several Fourier transforms and numerically invert several Fourier transforms since we are dealing with a sum of payoffs of type (2.18). Consequently even though this method is more accurate than using $\widetilde{aP}_{\alpha,\beta}(T_\alpha)$ instead of $\widetilde{P}_{\alpha,\beta}(T_\alpha)$ for the payoff in (2.3), it is takes more time to compute the swaption price. For strike values that are at the money or well in the money and swap tenors that are less than five years, a quicker way of calculating the swaption price is to substitute $\widetilde{aP}_{\alpha,\beta}(T_\alpha)$ for $\widetilde{P}_{\alpha,\beta}(T_\alpha)$ in (2.3) to get

$$Swaptn_{\alpha,\beta}(t) \approx Swaptn1_{\alpha,\beta}(t) = \mathbb{E}_t^{\mathbb{Q}}\Bigg[\exp\bigg(-\int_t^{T_\alpha} r_s\,ds\bigg)\Big(1-\widetilde{aP}_{\alpha,\beta}(T_\alpha)\Big)^+\Bigg]. \tag{2.28}$$

For (2.28) we have a single payoff function of type

$$G_1^{**}(x,k) := (\exp(k)-\exp(x))^+. \tag{2.29}$$

Let $C_{G_1^{**}}(k)$ denote the option price which has payoff type (2.29) and let $\hat{C}_{G_1^{**}}(k)$ denote the Fourier transform of $C_{G_1^{**}}(k)$ with respect to the strike price k. For such a payoff function, Lee (2004) gives the Fourier transform of the option price[3]. The Fourier transform with respect to the strike price is given by

$$\hat{C}_{G_1^{**}}(z) = \frac{\Phi(Q,z-i)}{iz-z^2}. \tag{2.30}$$

To obtain the price of the option from the transform by integrating along a contour

[3] Here it is meant as in previous sections $\mathbb{E}^{\mathrm{T}}[G_1^{**}(x,k)]$ i.e. the option price normalized by the zero coupon bond of the option's maturity

passing through $\hat{\alpha} < -1$ we use

$$C_{G_1^{**}}(k) = \frac{1}{\pi} \int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i} Re\Big[\frac{\Phi(z-i)}{iz - z^2} \exp(-izK)\Big]\, dz. \tag{2.31}$$

To minimize sampling and truncation error for the Fourier inversion Lee (2004) recommends which choices of $\hat{\alpha}$ to take and to use put-call parity type relationships to price the corresponding put or call option depending on the option type and the strike level. Here we do not investigate the various ways of calculating the swaption price which can minimize the error in the Fourier inversion as shown in Lee (2004). However since pricing the call type of payoff gives more flexibility in choosing $\hat{\alpha}$ as opposed to the put type payoff (2.29), we choose to value the swaption by writing the swaption payoff as:

$$1 - P(T_\alpha, T_\beta) - K \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) + \Big(K \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) - (1 - P(T_\alpha, T_\beta))\Big)^+. \tag{2.32}$$

Therefore the price of the swaption is given by

$$\begin{aligned} Swaptn_{\alpha,\beta}(0) = P(0, T_\alpha)\mathbb{E}_t^{\mathbb{T}_\alpha}\Big[1 - P(T_\alpha, T_\beta) - K \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) + \\ + \Big(K \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) - (1 - P(T_\alpha, T_\beta))\Big)^+\Big]. \end{aligned} \tag{2.33}$$

We can calculate

$$P(0, T_\alpha)\mathbb{E}_t^{\mathbb{T}_\alpha}[1 - P(T_\alpha, T_\beta) - K \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)] \tag{2.34}$$

exactly since

$$P(T_\alpha, T_i)/P(T_\alpha, T_\alpha)$$

is a martingale under $\mathbb{T}_\alpha$. Therefore (2.34) is equal to

$$P(0, T_\alpha) - P(0, T_\beta) - K \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i). \tag{2.35}$$

The part that we need to approximate is therefore

$$\mathbb{E}_t^{\mathbf{T}_\alpha}\left[\left(K\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)-(1-P(T_\alpha,T_\beta))\right)^+\right]=\mathbb{E}_t^{\mathbf{T}_\alpha}\left[\left(\sum_{i=\alpha+1}^{\beta}\tilde{\tau}_i P(T_\alpha,T_i)-1\right)^+\right]. \tag{2.36}$$

As already discussed we will use now $\widetilde{aP}_{\alpha,\beta}(T_\alpha)$ to approximate

$$\sum_{i=\alpha+1}^{\beta}\tilde{\tau}_i P(T_\alpha,T_i). \tag{2.37}$$

Therefore we get a call type option:

$$Swaptn_{\alpha,\beta}(0)\approx\widetilde{Swaptn1}_{\alpha,\beta}(0)=P(0,T_\alpha)\mathbb{E}_t^{\mathbf{T}_\alpha}\left[\left(\widetilde{aP}_{\alpha,\beta}(T_\alpha)-1\right)^+\right]. \tag{2.38}$$

Once again we introduce a pseudo strike price $\tilde{K}$ to use the Fourier technique to calculate (2.38) and write $Swaptn_{\alpha,\beta}(0,\tilde{K})$ to enable us to take the Fourier transform with respect to $\tilde{K}$:

$$Swaptn_{\alpha,\beta}(0,\tilde{K})\approx\widetilde{Swaptn1}_{\alpha,\beta}(0,\tilde{K})=P(0,T_\alpha)\mathbb{E}_t^{\mathbf{T}_\alpha}\left[\left(\widetilde{aP}_{\alpha,\beta}(T_\alpha)-\exp(\tilde{K})\right)^+\right]. \tag{2.39}$$

We now consider the call type payoff

$$G_1(x,k):=(\exp(x)-\exp(k))^+. \tag{2.40}$$

Let $\mathcal{C}_{G_1}(k)$ denote the option price normalized by the price of the default free zero coupon bond of maturity equal to the maturity of the option corresponding to the payoff type (2.40) and let $\hat{\mathcal{C}}_{G_1}(k)$ denote the Fourier transform of the dampened option price $\exp(\hat{\alpha})\mathcal{C}_{G_1}(k)$ with respect to the strike price k. The option price is then calculated through Fourier inversion(see (2.22)) as we can calculate $\hat{\mathcal{C}}_{G_1}(k)$ in closed form. Thus from Lee (2004) we have

$$\mathcal{C}_{G_1}(k)=R_{\hat{\alpha},G_1}+\frac{1}{\pi}\int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i}\mathrm{Re}\left[\hat{\mathcal{C}}_{G_1}(z)\exp\left(-iz\tilde{K}\right)\right]dz \tag{2.41}$$

where

$$R_{\hat{\alpha},G_1} = \begin{cases} \Phi(Q,-i) - \tilde{K}, & \hat{\alpha} < -1 \\ \Phi(Q,-i) - \tilde{K}/2, & \hat{\alpha} = -1 \\ \Phi(Q,-i), & -1 < \hat{\alpha} < 0 \\ \dfrac{\Phi(Q,-1)}{2}, & \hat{\alpha} = 0 \\ 0, & \hat{\alpha} > 0 \end{cases} \tag{2.42}$$

Since we now have to invert only one characteristic function of a quadratic form in Gaussian random variable, the calculation of $\widetilde{Swaptn1}_{\alpha,\beta}(0,\tilde{K})$ for $\tilde{K} = 0$ is much faster than calculating $\widetilde{Swaptn}_{\alpha,\beta}(0,\tilde{K})$ which requires the inversion of several joint characteristic functions of two quadratic forms in Gaussian random variables. One expects $\widetilde{Swaptn}_{\alpha,\beta}(0,\tilde{K})$ is more accurate than $\widetilde{Swaptn1}_{\alpha,\beta}(0,\tilde{K})$ as we only use $\widetilde{aP}_{\alpha,\beta}(T_\alpha)$ to approximate the exercise region of the swaption. Numerical experiments show that this is indeed true but the value $\widetilde{Swaptn1}_{\alpha,\beta}(0,\tilde{K})$ is close to the more accurate $\widetilde{Swaptn}_{\alpha,\beta}(0,\tilde{K})$ when the tenor of the swap underlying the swaption is less than or equal to five years. We now give the results of the numerical experiments conducted in a two factor quadratic Gaussian model where the relative errors of the approximations which are denoted by $\widetilde{Swaptn}_{\alpha,\beta}(0,0)$ and $\widetilde{Swaptn1}_{\alpha,\beta}(0,0)$ are calculated. The data set we use is the data given in chapter 7 of Pelsser (2000). This data set consists of of 36 cap and floor prices observed on January 4, 1994. We calibrate a two factor quadratic Gaussian model to this data. We assume that the mean reversion matrix A given in (1.2) in chapter 1 is a 2×2 diagonal matrix:

$$A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$$

and the instantaneous correlation matrix Σ is assumed to be constant such that

$$\Sigma\Sigma^\top = \begin{bmatrix} \sigma_{11}^2 & \rho\sigma_{11}\sigma_{22} \\ \rho\sigma_{11}\sigma_{22} & \sigma_{22}^2 \end{bmatrix}.$$

We find that the prices of caps and floors do not depend on the value of the instantaneous correlation between the two factors which is given by ρ as calibration to the cap-floor data using $\rho = 0.999, \rho = -0.999, \rho = 0$ gave the same cap and floor prices. This confirms the fact that the price of caps and floors are not sensitive to correlation (see chapter 6 in Brigo and Mercurio (2006)). We use the parameters

Cap Data				Floor Data			
T	K	Mid	QG2F	T	K	Mid	QG2F
1	0.0325	54.0	59.4	1	0.0375	11.0	11.2
1	0.035	38.0	44.5	1	0.035	4.5	5.4
1	0.0375	25.5	32.13	1	0.0325	3.0	2.16
2	0.05	47.0	46.0	2	0.045	74.0	74.8
2	0.055	28.0	27.8	2	0.04	32.0	33.9
2	0.06	16.55	16.5	2	0.035	8.5	10.2
3	0.05	142.0	143.3	3	0.045	93.5	92.3
3	0.055	99.5	100.8	3	0.04	41.0	41.8
3	0.06	69.5	70.1	3	0.035	11.5	12.9
4	0.05	277.0	275.5	4	0.045	110.0	106.6
4	0.055	210.0	206.8	4	0.04	50.0	48.5
4	0.06	158.5	154.0	4	0.035	16.0	15.4
5	0.065	205.0	201.2	5	0.055	346.0	323.2
5	0.07	160.0	154.4	5	0.05	226.0	209.8
5	0.075	127.0	118.0	5	0.045	131.0	119.2
10	0.065	661.0	711.2	10	0.055	510.0	495.8
10	0.07	549.0	591.5	10	0.05	335.0	327.8
10	0.075	457.0	491.3	10	0.045	196.0	194.0

Tab. 2.1: Cap and Floor Data(price in bp)

calibrated to the cap-floor data when $\rho = 0$:

$$\begin{aligned} a_{11} &= 0.1004623, \quad a_{22} = -0.0118329 \\ \sigma_{11} &= 0.0065293 \quad \sigma_{22} = 0.0341270. \end{aligned} \tag{2.43}$$

The following data lists the cap-floor data and the prices of the caps and floors under the two factor quadratic Gaussian model for the calibrated parameters given in (2.43). The values in the column with heading QG2F represent the cap and floor prices under the quadratic Gaussian model while the column with heading Mid is the average of the bid and ask quoted prices. The maturity in years of the cap or floor is given under the column with heading T and the corresponding strike price is given under the heading K. The discount curve for this date is obtained from quoted rates of the 1,3,6 and 12 month US-dollar money market rates and the swap-rates for maturities 2,3,4,5,7,and 10 years which is obtained from Datastream. Since we were not able to get data on swap-rates for maturities greater than 10 years, we extend the discount curve by extrapolation for the years 11 to 15. The

Zero Rates	
T	Discount
0.	1
0.083	0.997299
0.25	0.99163
0.5	0.982499
1.	0.962197
2.	0.916818
3.	0.866037
4.	0.81372
5.	0.761339
6.	0.712561
7.	0.664049
8.	0.619654
9.	0.576413
10.	0.534408
11.	0.492403
12.	0.448412
13.	0.400452
14.	0.346536
15.	0.28468

Tab. 2.2: Discount Curve

discount curve is obtained by interpolation using a piecewise cubic polynomial. To see the accuracy of the approximations $\widetilde{Swaptn}_{\alpha,\beta}(0,0)$ and $\widetilde{Swaptn1}_{\alpha,\beta}(0,0)$ to the swaption price $Swaptn_{\alpha,\beta}(0,0)$ in the two factor quadratic Gaussian model for the calibrated parameters given (2.43), we consider three different strike levels as in Schrager & Pelsser (2006). The swaption is said to be at the money (ATM) if the strike level K is chosen to be the current swap rate (see chapter 1 of Brigo and Mercurio (2006))i.e

$$K = \frac{P(0,T_\alpha) - P(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i)}.$$

We choose two other strike levels beside the ATM strike rate. A strike rate which makes the swaption in the money (ITM) and a strike rate that makes the swaption out of the money (OTM). The swaptions that have the at the money strike rate are denoted by ATM. The in the money swaptions are denoted by ITM and their strike levels are chosen to be 85% of the strike rate of the corresponding at the money swaption. The out of the money swaptions are chosen such that their strike level is 1.15% of the strike level of the corresponding ATM swaption. Since we can calculate the distribution of the two dimensional Gaussian factor under the forward measure $\mathbb{T}_\alpha$ (see lemma 1.40 in chapter 1), we can calculate the exact price of the swaption using the two dimensional integral given by:

$$Swaptn_{\alpha,\beta}(0,0) = P(0,T_\alpha) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(1 - P(T_\alpha,T_\beta) - K \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha,T_i)\right)^+ \frac{1}{2\pi |V(0,T_\alpha)|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(Y_{T_\alpha} - M(0,T_\alpha))^{\top} V(0,T_\alpha)^{-1}(Y_{T_\alpha} - M(0,T_\alpha))\right) dY_{T_\alpha} \tag{2.44}$$

where $M(0,T_\alpha)$ denotes the mean of Y_{T_α} under $\mathbb{T}$ and $V(0,T_\alpha)$ denotes the variance-covariance matrix of Y_{T_α} under $\mathbb{T}$.

We give this exact value in the following table together with the relative error in percentage of the approximate swaption price $\widetilde{Swaptn}_{\alpha,\beta}(0,0)$ next to it enclosed in parenthesis i.e. the value

$$\frac{Swaptn_{\alpha,\beta}(0,0) - \widetilde{Swaptn}_{\alpha,\beta}(0,0)}{Swaptn_{\alpha,\beta}(0,0)} * 100 \tag{2.45}$$

and in the line below it the relative error in percentage of the approximate swaption price $\widetilde{Swaptn1}_{\alpha,\beta}(0,0)$ i.e the value

$$\frac{Swaptn_{\alpha,\beta}(0,0) - \widetilde{Swaptn1}_{\alpha,\beta}(0,0)}{Swaptn1_{\alpha,\beta}(0,0)} * 100 \tag{2.46}$$

We see that from the numerical experiments that approximating the swaption

Mat.	Tenor			
T	1	3	5	10
1	39.80(0.00%) (0.01%)	118.57(0.00%) (0.09%)	188.43(0.00%) (0.26%)	320.83(0.00%) (1.03%)
3	68.88(0.00%) (0.00%)	194.12(0.00%) (0.16%)	301.14(0.00%) (0.48%)	505.5(0.00%) (1.96%)
5	78.88(0.00%) (0.00%)	221.02(0.00%) (0.21%)	342.44(0.00%) (0.61%)	568.68(0.00%) (3.38%)

Tab. 2.3: Relative Error(in %) for ATM Swaptions(price in bp)

Mat.	Tenor			
T	1	3	5	10
1	80.41(0.00%) (0.00%)	255.11(0.00%) (0.03%)	421.89(0.00%) (0.09%)	773.93(0.00%) (0.32%)
3	110.51(0.00%) (0.00%)	317.83(0.00%) (0.08%)	502.29(0.00%) (0.23%)	900.52(0.00%) (0.95%)
5	115.44(0.00%) (0.01%)	328.68(0.00%) (0.12%)	517.20(0.00%) (0.35%)	964.34(0.00%) (1.76%)

Tab. 2.4: Relative Error(in %) for ITM Swaptions(price in bp)

Mat.	Tenor			
T	1	3	5	10
1	16.83(0.00%) (0.02%)	45.13(0.00%) (0.28%)	66.86(0.00%) (0.91%)	98.81(0.01%) (4.22%)
3	40.77(0.00%) (0.03%)	111.85(0.00%) (0.32%)	169.56(0.00%) (0.94%)	258.53(0.00%) (4.42%)
5	52.47(0.00) (0.03%)	144.13(0.00%) (0.37%)	218.98(0.00%) (1.08%)	311.75(0.01%) (7.03%)

Tab. 2.5: Relative Error(in %) for OTM Swaptions(price in bp)

price by $\widetilde{Swaptn}_{\alpha,\beta}(0,0)$ performs well while approximating the swaption price by

$\widetilde{Swaptn1}_{\alpha,\beta}(0,0)$ does not perform well for strike rates that are out of the money and for swaptions on swaps of tenors that are equal to ten years whereby the relative errors in the swaption prices increases with the maturity of the swap. The relative errors of $\widetilde{Swaptn1}_{\alpha,\beta}(0,0)$ can come from two sources. The first source of error is that we have replaced the weights $w_i(t), \imath = \alpha+1, \ldots, \beta$ by their time zero values. The second source of error is that the drift of $\widetilde{aP}_{\alpha,\beta}$ is different from the drift of $\tilde{P}_{\alpha,\beta}$(see (2.8) and (2.9)). It appears from the results of the next section that the first source of error is more significant than the second source of error.

2.3 Pricing Swaptions under the Swap Measure

In this section, we shall give another approximation that will make it possible to approximate the price of a default free swaption. This approximation is based on a single payoff function and therefore the approximation formula of this section requires less computation and is faster but is generally less accurate than the approximation formulas given in the previous section.

As in the previous section, we assume we have a standard interest rate swap with a tenor structure given by $\mathcal{T} = \{T_{\alpha+1}, \ldots, T_{\alpha+n} = T_\beta\}$. We denote by $P_{\alpha,\beta}(t) = \sum\limits_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$ the present value of a basis point(PVBP01) and $\mathbb{Q}_{\alpha,\beta}$ denote the swap measure which is the measure corresponding to using $P_{\alpha,\beta}(t)$ as the numeraire. To lighten the notation, we assume in this section that $\alpha = 0$ and $\beta = n$. Using the change of numeraire technique (El Karoui, Geman & Rochet 1995), the following definition of the swap measure can be obtained (see also Musiela & Rutkowski (2005))

$$\frac{d\mathbb{Q}_{\alpha,\beta}}{d\mathbb{Q}} = \frac{\sum_{i=1}^{n} \tau_i P(T,T_i)}{D(T)\sum_{i=1}^{n} P(0,T_i)} = \frac{P_{\alpha,\beta}(T_\alpha)}{D(T)P_{\alpha,\beta}(0)}. \tag{2.47}$$

We give an approximation method which is similar to d'Aspremont (2003). In d'Aspremont (2003), the method is applied to the lognormal market model while here we are looking at log-quadratic Gaussian processes in a quadratic Gaussian factor model rather than a market model.

Theorem 2.6. *At time $t = 0$ the value of a swaption is given by*

$$Swaptn_{\alpha,\beta}(0) = P_{\alpha,\beta}(0)\mathbb{E}^{\mathbb{Q}_{\alpha,\beta}}[(Swap_{\alpha,\beta}(T_\alpha) - K)^+]. \tag{2.48}$$

Proof. The proof is straightforward and can be found in (Musiela and Rutkowski 2005) but we give a short proof for completeness. First note that we can write the price of the default free swaption as

$$
\begin{aligned}
&Swaptn_{\alpha,\beta}(0) \qquad (2.49)\\
=&\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_\alpha} r_s\,ds\Big)\sum_{i=1}^{n}\tau_i P(T_\alpha,T_i)\Big(\frac{1-P(T_\alpha,T_n)}{\sum_{i=1}^{n}\tau_i P(T_\alpha,T_i)}-K\Big)^+\Big]\\
=&P_{\alpha,\beta}(0)\mathbb{E}^{\mathbb{Q}}\Big[\frac{P_{\alpha,\beta}(T_\alpha)}{D(T)P_{\alpha,\beta}(0)}\Big(\frac{1-P(T_\alpha,T_n)}{P_{\alpha,\beta}(T_\alpha)}-K\Big)^+\Big]\\
=&P_{\alpha,\beta}(0)\mathbb{E}^{\mathbb{Q}}\Big[\frac{P_{\alpha,\beta}(T_\alpha)}{D(T)P_{\alpha,\beta}(0)}\big(Swap_{\alpha,\beta}(T_\alpha)-K\big)^+\Big]
\end{aligned}
$$

Now using the definition of the swap measure the proof is complete. □

Suppose we have a given set of discrete dates $\mathcal{T}_{\alpha,\beta}=\{T_\alpha=T_0,\ldots,T_n=T_\beta\}$. Let the forward libor rate $L(t,T_i)=L_i(t)$ be defined as follows

$$L_i(t) := \frac{P(t,T_i)-P(t,T_{i+1})}{\tau_{i+1}P(t,T_{i+1})}. \qquad (2.50)$$

The forward libor $L_i(t)$ is the simple interest rate that would apply over the period $[T_i,T_{i+1}]$ as seen from the date t under no arbitrage conditions. In the quadratic Gaussian factor model default free zero coupon bond prices are log-quadratic Gaussian and therefore ratios of zero coupon bonds can be written as log-quadratic processes. Recall the notation introduced in the previous sections $Q_i(t)=\log_e(P(t,T_i))$, then we can give a closed form expression for $L_i(t)$

$$L_i(t) = \frac{1}{\tau_{i+1}}\Big(\exp\big(Q_i(t)-Q_{i+1}(t)\big)-1\Big). \qquad (2.51)$$

From the above it follows that

$$P(t,T_i)-P(t,T_{i+1}) = \tau_{i+1}P(t,T_{i+1})L_i(t). \qquad (2.52)$$

Note that the difference $1-P(T_\alpha,T_n)$ can be written as follows

$$
\begin{aligned}
1-P(T_\alpha,T_n) = P(T_\alpha,T_\alpha)-P(T_\alpha,T_1)+\ldots+P(T_\alpha,T_i)-P(T_\alpha,T_i)+\ldots\\
+P(T_\alpha,T_{n-1})-P(T_\alpha,T_n). \qquad (2.53)
\end{aligned}
$$

Using (2.52) we can write

$$1 - P(T_\alpha, T_n) = \sum_{i=0}^{n-1} \tau_{i+1} P(T_\alpha, T_{i+1}) L_i(T_\alpha). \tag{2.54}$$

Hence as in Rebonato (1998) we can write the swap rate as a weighted sum of $L_i(t)$

$$Swap_{\alpha,\beta}(T_\alpha) = \sum_{i=0}^{n-1} \frac{\tau_{i+1} P(T_\alpha, T_{i+1}) L_i(T_\alpha)}{P_{\alpha,\beta}(T_\alpha)}. \tag{2.55}$$

Let $v_i(T_\alpha) = \frac{\tau_i P(T_\alpha, T_i)}{P_{\alpha,\beta}(T_\alpha)}$, $i = 1, \dots, n$. Then (2.55) can be written as

$$\begin{aligned} Swap_{\alpha,\beta}(T_\alpha) &= \sum_{i=0}^{n-1} v_{i+1}(T_\alpha) L_i(T_\alpha) \\ &= \sum_{i=0}^{n-1} v_{i+1}(T_\alpha) \frac{1}{\tau_{i+1}} \Big(\exp\big(Q_i(T_\alpha) - Q_{i+1}(T_\alpha)\big) - 1 \Big) \\ &= \sum_{i=0}^{n-1} v_{i+1}(T_\alpha) \frac{1}{\tau_{i+1}} \exp\big(Q_i(T_\alpha) - Q_{i+1}(T_\alpha)\big) - \sum_{i=0}^{n-1} \frac{1}{\tau_{i+1}} v_{i+1}(T_\alpha). \end{aligned} \tag{2.56}$$

We can now write the price of a swaption as

$$\begin{aligned} Swaptn_{\alpha,\beta}(K) = P_{\alpha,\beta}(0) \mathbb{E}^{Q_{\alpha,\beta}} \Bigg[\Bigg(& \sum_{i=0}^{n-1} v_{i+1}(T_\alpha) \frac{1}{\tau_{i+1}} \exp\big(Q_i(T_\alpha) - Q_{i+1}(T_\alpha)\big) \\ & - \Big(\sum_{i=0}^{n-1} \frac{1}{\tau_{i+1}} v_{i+1}(T_\alpha) + K \Big) \Bigg)^+ \Bigg] \end{aligned} \tag{2.57}$$

Similar to the method used in Rebonato (1998) and d'Aspremont (2003) for deriving analytic approximations to the price of swaptions, we replace the weights $v_n(T_\alpha)$ by their time $t = 0$ values.

$$\begin{aligned} Swaptn_{\alpha,\beta}(K) = P_{\alpha,\beta}(0) \mathbb{E}^{Q_{\alpha,\beta}} \Bigg[\Big(& \sum_{i=0}^{n-1} v_{i+1}(0) \frac{1}{\tau_{i+1}} \exp\big(Q_i(T_\alpha) - Q_{i+1}(T_\alpha)\big) \\ & - \Big(\sum_{i=0}^{n-1} \frac{1}{\tau_{i+1}} v_{i+1}(0) + K \Big)^+ \Bigg] \end{aligned} \tag{2.58}$$

We now have a basket pricing problem where the basket consists of a weighted sum of log-quadratic processes and a new strike

$$\tilde{K} = K + \sum_{i=0}^{n-1} \frac{1}{\tau_{i+1}} v_{i+1}(0).$$

Next we approximate the sum of log-quadratic processes through the method introduced in the first section. Let

$$F_{\alpha,\beta}(t) := \sum_{i=0}^{n-1} v_{i+1}(0) \frac{1}{\tau_{i+1}} \exp\big(Q_i(t) - Q_{i+1}(t)\big).$$

First note that

$$\exp\big(Q_i(t) - Q_{i+1}(t)\big) = \frac{P(t,T_i)}{P(t,T_{i+1})}.$$

Using Ito's lemma we get

$$\begin{aligned} d\Big(\frac{P(t,T_i)}{P(t,T_{i+1})}\Big) = \frac{P(t,T_i)}{P(t,T_{i+1})}\Big\{ &\Big(4Y_tC(t,T_{i+1})\Sigma\Sigma^\top C(t,T_{i+1})Y_t \\ &+ 2B(t,T_{i+1})^\top\Sigma\Sigma^\top C(t,T_{i+1})Y_t + B(t,T_{i+1})^\top\Sigma\Sigma^\top B(t,T_{i+1}) \\ &- 4Y_tC(t,T_{i+1})\Sigma\Sigma^\top C(t,T_i)Y_t - 2B(t,T_{i+1})^\top\Sigma\Sigma^\top C(t,T_i)Y_t \\ &- B(t,T_{i+1})^\top\Sigma\Sigma^\top B(t,T_i)\Big)\, dt + \\ &+ \Big(2(C(t,T_{i+1}) - C(t,T_i))Y_t + B(t,T_{i+1}) - B(t,T_i)\Big)^\top \Sigma\, dW_t\Big\} \end{aligned}$$

Therefore

$$\begin{aligned} dF_{\alpha,\beta}(t) = F_{\alpha,\beta}(t)\Big\{ &\sum_{i=0}^{n-1} \frac{\frac{v_{i+1}(0)P(t,T_i)}{\tau_{i+1}P(t,T_{i+1})}}{F_{\alpha,\beta}(t)} \Big(4Y_tC(t,T_{i+1})\Sigma\Sigma^\top C(t,T_{i+1})Y_t \\ &+ 2B(t,T_{i+1})^\top\Sigma\Sigma^\top C(t,T_{i+1})Y_t + B(t,T_{i+1})^\top\Sigma\Sigma^\top B(t,T_{i+1}) \\ &- 4Y_tC(t,T_{i+1})\Sigma\Sigma^\top C(t,T_i)Y_t - 2B(t,T_{i+1})^\top\Sigma\Sigma^\top C(t,T_i)Y_t \\ &- B(t,T_{i+1})^\top\Sigma\Sigma^\top B(t,T_i)\Big)\, dt \\ &+ \Big(2Y_t(C(t,T_{i+1}) - C(t,T_i)) + B(t,T_{i+1}) - B(t,T_i)\Big)\Sigma dW_t\Big\} \end{aligned}$$

Note we already used the empirical fact that the weights $\frac{\tau_i P(t,T_i)}{\sum_{i=1}^n \tau_i P(t,T_i)}$ can be replaced by their time zero values. The empirical fact that the weights can be frozen is supported by approximations to the price of swaptions in market models of interest rate (see, e.g., Rebonato (1998) or d'Aspremont (2003)). In the lognormal market model one can show that the weights have low volatility (see d'Aspremont (2003)). The case $\tau_i = \tau, i = 1, \ldots, n$ is a special case which gives us $\bar{v}_i(t) := \frac{P(t,T_i)}{\sum_{i=1}^{n} P(t,T_i)}$. Therefore we can freeze $\frac{P(t,T_i)}{\sum_{i=1}^{n} P(t,T_i)}$ when making approximations. Since

$$\begin{aligned}\frac{P(t,T_i)}{P(t,T_{i+1})} &= \frac{P(t,T_i)}{\sum\limits_{i=1}^{n} P(t,T_i)} \div \frac{P(t,T_{i+1})}{\sum\limits_{i=1}^{n} P(t,T_i)}\\ &= \frac{\bar{v}_i(t)}{\bar{v}_{i+1}(t)}\end{aligned}$$

we will replace $\frac{\bar{v}_i(t)}{\bar{v}_{i+1}(t)}$ by its time zero value when making approximations. It is therefore plausible to assume that for approximation purposes, we can also replace

$$u_{i+1}(t) := \frac{\frac{v_{i+1}(0)P(t,T_i)}{\tau_{i+1}P(t,T_{i+1})}}{F_{\alpha,\beta}(t)} = \frac{v_{i+1}(0)\bar{v}_i(t)}{\tau_{i+1}\bar{v}_{i+1}(t)} \frac{1}{\sum\limits_{i=0}^{n-1} \frac{v_i(0)\bar{v}_i(t)}{\tau_{i+1}\bar{v}_{i+1}(t)}} \tag{2.59}$$

by its time zero value.

We now replace $u_i(t)$ by its time $t = 0$ value $u_i(0)$ in the dynamics of $F_{\alpha,\beta}(t)$ to get:

$$\begin{aligned}dF_{\alpha,\beta}(t) \approx F_{\alpha,\beta}(t)\Bigg\{&\sum_{i=0}^{n-1} u_{i+1}(0)\Big(4Y_tC(t,T_{i+1})\Sigma\Sigma^{\mathsf{T}}C(t,T_{i+1})Y_t+\\ &+ 2B(t,T_{i+1})^{\mathsf{T}}\Sigma\Sigma^{\mathsf{T}}C(t,T_{i+1})Y_t + B(t,T_{i+1})^{\mathsf{T}}\Sigma\Sigma^{\mathsf{T}}B(t,T_{i+1})\\ &- 4Y_tC(t,T_{i+1})\Sigma\Sigma^{\mathsf{T}}C(t,T_i)Y_t - 2B(t,T_{i+1})^{\mathsf{T}}\Sigma\Sigma^{\mathsf{T}}C(t,T_i)Y_t\\ &- B(t,T_{i+1})^{\mathsf{T}}\Sigma\Sigma^{\mathsf{T}}B(t,T_i)\Big)\, dt\\ &+ \Big(2(C(t,T_{i+1}) - C(t,T_i))Y_t + B(t,T_{i+1}) - B(t,T_i)\Big)^{\mathsf{T}}\Sigma dW_t\Bigg\}\end{aligned} \tag{2.60}$$

As in the first section we look for a tractable process that approximates $F_{\alpha,\beta}(t)$. The log-quadratic Gaussian process defined by

$$\begin{aligned}\widetilde{F}_{\alpha,\beta}(t) := \exp\Big(&Y_t^{\mathsf{T}} \sum_{i=0}^{n-1} u_{i+1}(0)(C(t,T_{i+1}) - C(t,T_i))Y_t\\ &+ \sum_{i=0}^{n-1} u_{i+1}(0)(B(t,T_{i+1}) - B(t,T_i))^{\mathsf{T}}Y_t + \sum_{i=0}^{n-1} u_{i+1}(0)(A(t,T_{i+1}) - A(t,T_i))\Big).\end{aligned}$$

has a dynamics that is close to the SDE given by (2.60). This can be supported by looking at the dynamics of $\widetilde{F}_{\alpha,\beta}(t)$. Using Theorem B.2 given in Appendix B, the SDE satisfied by $\widetilde{F}_{\alpha,\beta}(t)$ is given by

$$\begin{aligned}\frac{d\widetilde{F}_{\alpha,\beta}(t,T)}{\widetilde{F}_{\alpha,\beta}(t,T)} = \Big(Y_t^{\top} C^{\tilde{F}}(t,T_1,\dots,T_n)Y_t \,+\, B^{\tilde{F}}(t,T_1,\dots,T_n)^{\top}Y_t + \\ A^{\tilde{F}}(t,T_1,\dots,T_n)\Big)\,dt + \Big(2\sum_{i=0}^{n-1} u_i(0)(C(t,T_{i+1})-C(t,T_i))Y_t \\ +\sum_{i=1}^{n} u_i(0)(B(t,T_{i+1})-B(t,T_i))\Big)^{\top}\Sigma\, dW_t \quad (2.61)\end{aligned}$$

where

$$\begin{aligned}C^{\tilde{F}}(t,T_0,\dots,T_n) := A^{\top}\sum_{i=0}^{n-1} u_{i+1}(0)(C(t,T_{i+1})-C(t,T_i)) \\ +\sum_{i=1}^{n} u_{i+1}(0)(C(t,T_{i+1})-C(t,T_i))A + \partial_t\Big(\sum_{i=0}^{n-1} u_i(0)(C(t,T_{i+1})-C(t,T_i))\Big) \\ +2\sum_{i=0}^{n-1} u_{i+1}(0)(C(t,T_{i+1})-C(t,T_i))\Sigma\Sigma^{\top}\sum_{i=0}^{n-1} u_{i+1}(0)(C(t,T_{i+1})-C(t,T_i)) \quad (2.62)\end{aligned}$$

$$\begin{aligned}B^{\tilde{F}}(t,T_0,\dots,T_n)^{\top} := A^{\top}\sum_{i=1}^{n} u_{i+1}(0)(B(t,T_{i+1})-B(t,T_i)) \\ +\partial_t\Big(\sum_{i=0}^{n-1} u_{i+1}(0)(B(t,T_{i+1})-B(t,T_i))\Big) \\ +2\sum_{i=0}^{n-1} u_{i+1}(0)(C(t,T_{i+1})-C(t,T_i))\Sigma\Sigma^{\top}\sum_{i=0}^{n-1} u_{i+1}(0)(B(t,T_{i+1})-B(t,T_i))^{\top} \\ (2.63)\end{aligned}$$

$$\begin{aligned}A^{\tilde{F}}(t,T_0,\dots,T_n)^{\top} := Tr\Big[\Sigma\Sigma^{\top}\sum_{i=0}^{n-1} u_{i+1}(0)(C(t,T_{i+1})-C(t,T_i))\Big] \\ +\partial_t\Big(\sum_{i=0}^{n-1} u_{i+1}(0)(A(t,T_{i+1})-A(t,T_i))\Big)+\frac{1}{2}\Big|\Sigma^{\top}\sum_{i=0}^{n-1} u_{i+1}(0)(B(t,T_{i+1})-B(t,T_i))\Big|^2 \\ (2.64)\end{aligned}$$

If we consider the quadratic term in (2.62), we can use the fact that $C(t,T_i), i = 1,\dots,n$, satisfy their corresponding Riccati equations to substitute the appropriate expressions for the term involving the partial derivatives with respect to t in (2.62) to get some cancelation of terms so that we have,

$$
\begin{aligned}
&Y_t^\top C^{\tilde{F}}(t,T_1,\dots,T_n)Y_t = \\
&\qquad Y_t^\top\Big(\sum_{i=0}^{n-1} u_{i+1}(0)(2C(t,T_{i+1})\Sigma\Sigma^\top C(t,T_{i+1}) - 2C(t,T_i)\Sigma\Sigma^\top C(t,T_i)\Big) \\
&\quad + 2\sum_{i=0}^{n-1} u_{i+1}(0)(C(t,T_{i+1}) - C(t,T_i))\Sigma\Sigma^\top \sum_{i=1}^{n} u_{i+1}(0)(C(t,T_{i+1}) - C(t,T_i)\Big)Y_t.
\end{aligned}
\tag{2.65}
$$

It appears that similar to what we have done in the proof of theorem 2.4 we could possibly argue that the drift terms in the SDE of $\widetilde{F}_{\alpha,\beta}(t)$ are approximately equal to the drift terms in the SDE of $F_{\alpha,\beta}(t)$. However, we have considered the dynamics of $\widetilde{F}_{\alpha,\beta}(t)$ under the risk neutral measure $\mathbb{Q}$ but what we need is the dynamics of $\widetilde{F}_{\alpha,\beta}(t)$ under the swap measure which we denoted by $\mathbb{Q}_{\alpha,\beta}$. The dynamics of $\widetilde{F}_{\alpha,\beta}(t)$ under $\mathbb{Q}_{\alpha,\beta}$ can be obtained by using Girsanov's theorem. We first use Itô's formula to derive the SDE satisfied by $P_{\alpha,\beta}(t)$ similar to what was done in (2.7). Thus we have

$$
dP_{\alpha,\beta}(t) = P_{\alpha,\beta}(t)\Big(r_t\,dt + \sum_{i=\alpha+1}^{\beta} w_i(t)(Y_t C(t,T_i) + B(t,T_i))\,\Sigma\,dW_t\Big). \tag{2.66}
$$

By Girsanov's theorem $W_t^{\alpha,\beta}$ given by

$$
dW_t^{\alpha,\beta} := W_t + \Sigma\Sigma^\top \sum_{i=1}^{n} \int_0^t w_i(t)(2C(u,T_i)Y_u + B(u,T_i))\,du \tag{2.67}
$$

is a standard Brownian motion under $\mathbb{Q}_{\alpha,\beta}$. Applying Girsanov's theorem we find that the drift term of $\widetilde{F}_{\alpha,\beta}(t)$ is different from zero. In comparison if we assume in

(2.56) $\tau_i = \tau, i = 1, \ldots, n$, we get

$$F_{\alpha,\beta}(t) = \sum_{i=0}^{n-1} v_{i+1}(0) \frac{1}{\tau_{i+1}} \exp\big(Q_i(t) - Q_{i+1}(t)\big) \tag{2.68}$$
$$\approx \sum_{i=0}^{n-1} v_{i+1}(t) \frac{1}{\tau_{i+1}} \exp\big(Q_i(t) - Q_{i+1}(t)\big) \tag{2.69}$$
$$= Swap_{\alpha,\beta}(t) + \frac{1}{\tau}. \tag{2.70}$$

Therefore as $Swap_{\alpha,\beta}(t)$ is a martingale under $\mathbb{Q}_{\alpha,\beta}$, the drift term of $F_{\alpha,\beta}(t)$ should be close to zero. However the drift of $\widetilde{F}_{\alpha,\beta}(t)$ under $\mathbb{Q}_{\alpha,\beta}$ is different from zero. We now assume that we can ignore the drift terms of $\widetilde{F}_{\alpha,\beta}(t)$ such that under this assumption the SDE of $\widetilde{F}_{\alpha,\beta}(t)$ will be approximately equal to the SDE of $F_{\alpha,\beta}(t)$. We also have to examine the dynamics of Y_t under $\mathbb{Q}_{\alpha,\beta}$. By Girsanov's theorem, the dynamics of Y_t is given by

$$dY_t = \Big(A - 2\Sigma\Sigma^{\mathrm{T}} \sum_{i=1}^{n} w_i(t) C(t, T_i)\Big) Y_t - \Sigma\Sigma^{\mathrm{T}} \Big(\sum_{i=1}^{n} w_i(t) B(t, T_i)\Big) + \Sigma \, dW^{\alpha,\beta} \tag{2.71}$$

where $W_t^{\alpha,\beta}$ is a standard Brownian motion under $\mathbb{Q}_{\alpha,\beta}$. Since $w_i(t), i = 1, \ldots, n$ are not deterministic, Y_t is different from a Gaussian Ornstein Uhlenbeck process. However if we replace $w_i(t), i = 1, \ldots, n$ by their time zero values, we get a Gaussian Ornstein Uhlenbeck process. Therefore the dynamics of Y_t under $\mathbb{Q}_{\alpha,\beta}$ is close to Gaussian if we use the empirical fact (see Rebonato (1998)) that weights $w_i(t), i = 1 \ldots, n$ can be assumed to be constant for deriving analytic approximations. However we can find the exact mean and variance-covariance matrix of Y_{T_α} under $\mathbb{Q}_{\alpha,\beta}$ conditional on $\mathcal{F}_t$ even though the exact dynamics of Y_t under $\mathbb{Q}_{\alpha,\beta}$ does not correspond to that of a Gaussian process. To do so first note that the mean of Y_{T_α} under $\mathbb{Q}_{\alpha,\beta}$ conditional on $\mathcal{F}_t$ can be written as

$$M_{\alpha,\beta}(t, T_\alpha) := \mathbb{E}_t^{\mathbb{Q}_{\alpha,\beta}}[Y_T] = P(t, T_\alpha) \mathbb{E}_t^{\mathbf{T}_\alpha}\Big[\frac{\sum_{i=1}^n P(T_\alpha, T_i)}{\sum_{i=1}^n P(t, T_i)} Y_{T_\alpha}\Big]$$
$$= \sum_{i=1}^{n} \frac{P(t, T_i)}{\sum_{i=1}^n P(t, T_i)} \mathbb{E}_t^{\mathbf{T}_\alpha}\Big[\frac{P(T_\alpha, T_i)}{\frac{P(t,T_i)}{P(t,T_\alpha)}} Y_{T_\alpha}\Big]$$
$$= \sum_{i=1}^{n} \frac{P(t, T_i)}{\sum_{i=1}^n P(t, T_i)} \mathbb{E}_t^{\mathbf{T}_\alpha}\Big[\frac{P(T_\alpha, T_i)}{\mathbb{E}_t^{\mathbb{Q}_{T_\alpha}}[P(T_\alpha, T_i)]} Y_{T_\alpha}\Big]. \tag{2.72}$$

In (2.72) the value

$$\frac{P(T_\alpha, T_i)}{\mathbb{E}_t^{\mathbb{Q}_{T_\alpha}}[P(T_\alpha, T_i)]} = \frac{P(T_\alpha, T_i)}{\frac{P(t,T_i)}{P(t,T_\alpha)}}$$

is the Radon-Nikodym density of $\mathbb{T}_i$ with respect to the measure $\mathbb{T}_\alpha$ so that we can just find the mean and variance of Y_{T_α} under this measure but it is computationally more efficient to use the result in Cherif et al. (1994). First we can find the distribution of the quadratic Gaussian factors under the measure $\mathbb{Q}_{T_\alpha}$ in closed form. Suppose $M(t, T_\alpha)$ denotes the mean of Y_{T_α} under $\mathbb{Q}_{T_\alpha}$ conditional on $\mathcal{F}_t$, and $V(t, T_\alpha)$ denotes the variance-covariance matrix of Y_{T_α} under $\mathbb{Q}_{T_\alpha}$ conditional on $\mathcal{F}_t$. Under the quadratic Gaussian model the zero coupon bond price $P(T_\alpha, T_i)$ is log-quadratic Gaussian such that

$$P(T_\alpha, T_i) = \exp\Big(-Y_{T_\alpha} C(T_\alpha, T_i) Y_{T_\alpha} - B(Y_{T_\alpha}, T_i)^\top Y_{T_\alpha} - A(Y_{T_\alpha}, T_i)\Big), \ i = 1, \ldots, n.$$

For $i = 1, \ldots, n$, let $M_i(t, T_\alpha)$ denote the mean and $V_i(t, T_\alpha)$ denote the variance-covariance matrix of Y_{T_α} under the measure whose Radon-Nikodym density with respect to the measure $\mathbb{T}_\alpha$ is given by

$$\frac{P(T_\alpha, T_i)}{\mathbb{E}^{\mathbb{Q}_{T_\alpha}}[P(T_\alpha, T_i)]}.$$

Applying Lemma 1.14 given in Chapter 1, we get

$$V_i(t, T_\alpha) = \Big[I + 2V(t, T_\alpha) C(T_\alpha, T_i)\Big]^{-1} V(t, T_\alpha) \tag{2.73}$$

$$M_i(t, T_\alpha) = \Big[I + 2V(t, T_\alpha) C(T_\alpha, T_i)\Big]^{-1} \Big[M(t, T_\alpha) - V(t, T_\alpha) B(T_\alpha, T_i)\Big]. \tag{2.74}$$

Therefore we can calculate $M_{\alpha,\beta}(t, T_\alpha)$ in closed form. Similarly the variance-covariance matrix of Y_{T_α} under $\mathbb{Q}_{\alpha,\beta}$ conditional on $\mathcal{F}_t$ which we denote by $V_{\alpha,\beta}(t, T_\alpha)$ can be calculated in closed form. Next we approximate Y_t by a Gaussian process with mean and variance-covariance matrix equal to $M_{\alpha,\beta}(t, T_\alpha)$ and $V_{\alpha,\beta}(t, T_\alpha)$ respectively. Under this approximation, $\tilde{F}_{\alpha,\beta}$ is a log-quadratic Gaussian process. We now define a new approximation of the price of the swaption $Swaptn_{\alpha,\beta}(K)$. Let the swaption price which is based on $\tilde{F}_{\alpha,\beta}(t)$ be given by,

$$\widetilde{Swaptn3}_{\alpha,\beta}(\tilde{K}) = P_{\alpha,\beta}(0) \mathbb{E}^{\mathbb{Q}_{\alpha,\beta}}\Big[\Big(\tilde{F}_{\alpha,\beta}(T_\alpha) - \tilde{K}\Big)^+\Big]. \tag{2.75}$$

To calculate $\widetilde{Swaptn3}_{\alpha,\beta}(\tilde{K})$, note that we only have a single payoff function of type

$$G_1(x,k) := (\exp(x) - \exp(k))^+. \tag{2.76}$$

Let $\mathcal{C}_{G_1}(k)$ denote the option price which has payoff type (2.76) and let $\hat{\mathcal{C}}_{G_1}(k)$ denote the Fourier transform of $\mathcal{C}_{G_1}(k)$ with respect to the strike price k. For such a payoff function, Lee (2004) gives the Fourier transform of the option price[4]. The Fourier transform with respect to the strike price is given by

$$\hat{\mathcal{C}}_{G_1}(z) = \frac{\Phi(z-i)}{iz - z^2} \tag{2.77}$$

where $\Phi(z)$ is the characteristic function of the random variable x. To obtain the price of the option from the transform by integrating along a contour passing through $\hat{\alpha}$ we use

$$\mathcal{C}_{G_1}(k) = \frac{1}{\pi} \int\limits_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i} Re\Big[\frac{\Phi(z-i)}{iz - z^2}\Big]\, dz. \tag{2.78}$$

For the quadratic Gaussian model we have the freedom to choose $\hat{\alpha}$ to be equal to zero or any other positive or negative number because the characteristic function is defined everywhere and depending on the choice of $\hat{\alpha}$, one can minimize the error of the Fourier inversion(see Lee (2004) for a detailed discussion). We do not investigate the effect of choosing different $\hat{\alpha}$ on the swaption pricing but use $\hat{\alpha} = 1$ for numerical tests. To calculate $\widetilde{Swaptn3}_{\alpha,\beta}(\tilde{K})$, we have to use the one dimensional characteristic function of the quadratic Gaussian random variable $Q_{\tilde{F}}(t) := \log_e(\tilde{F}_{\alpha,\beta}(t))$. Therefore we have

$$\widetilde{Swap3tn}_{\alpha,\beta}(\tilde{K}) = P_{\alpha,\beta}(0)\Big(\frac{\Phi(Q_{\tilde{F}}(T_\alpha), -i)}{2} + \frac{1}{\pi} \int\limits_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i} Re\Big[\frac{\Phi(Q_{\tilde{F}}(T_\alpha), z-i)}{iz - z^2}\Big]\Big). \tag{2.79}$$

In the following we conduct numerical tests to calculate the relative error of this approximation in a two factor quadratic Gaussian model. The parameters of the quadratic Gaussian model are obtained through calibration to the market data of the previous section(see Table 2.1 and (2.43) of section 2.2). We give the exact value of the swaption by using two dimensional integration(see (2.44) in the

[4] Here it is meant as in previous sections $\mathbb{E}^{\mathrm{T}}[G_1(x,k)]$ i.e. the option price normalized by the zero coupon bond of the option's maturity

following table together with the relative error in percentage of the approximate swaption price $\widetilde{Swaptn3}_{\alpha,\beta}(\tilde{K})$ next to it enclosed in parenthesis i.e. the value

$$\frac{Swaptn_{\alpha,\beta}(0,0) - \widetilde{Swaptn3}_{\alpha,\beta}(\tilde{K})}{Swaptn_{\alpha,\beta}(0,0)} * 100. \tag{2.80}$$

From the results given in Tables 2.6, 2.7 and 2.8, we see that we can get a good

T	Tenor		
	1	3	5
1	39.80(0.28%)	118.57(-0.22%)	188.43(-0.77%)
3	68.88(0.39%)	194.12(0.23%)	301.42(-0.03%)
5	78.88(0.17%)	221.02(0.32%)	342.44(0.00%)

Tab. 2.6: Relative Error(in %) for ATM Swaptions

T	Tenor		
	1	3	5
1	80.41(0.25%)	255.11(0.06%)	421.89(-0.13%)
3	110.51(0.35%)	317.84(0.26%)	502.29 (0.11%)
5	115.45(0.16%)	328.68(0.33%)	517.2(0.15%)

Tab. 2.7: Relative Error(in %) for ITM Swaptions

T	Tenor		
	1	3	5
1	16.83(0.30%)	45.12(-0.75%)	66.86(-1.93%)
3	40.77(0.42%)	111.85(0.19%)	169.55(-0.22%)
5	52.47(0.16%)	144.13(0.30%)	218.97(-0.18%)

Tab. 2.8: Relative Error(in %) OTM Swaptions

approximation of the swaption price using $\widetilde{F}_{\alpha,\beta}$ to approximate $F_{\alpha,\beta}$. Therefore for maturities that are less or equal to five years and swap tenors less or equal to five years, we can use $\widetilde{F}_{\alpha,\beta}$ to approximate the swap rate $Swap_{\alpha,\beta}(t)$ given by (2.2). The errors of the swaption approximation we considered in this section can be due to three reasons. The first source of error is that we replaced the weights in (2.57) by their time zero values in (2.58). The second source of error is that the drift of the approximation $\widetilde{F}_{\alpha,\beta}$ is not equal to the drift of $F_{\alpha,\beta}$. The third source of error

T	Tenor		
	1	3	5
1	39.80(-0.08%)	118.57(-0.59%)	188.43(-1.07%)
3	68.88(-0.05%)	194.12(-0.17%)	301.42(-0.41%)
5	78.88(-0.02%)	221.02(-0.133%)	342.44(-0.51%)

Tab. 2.9: Relative Error(in %) for ATM Swaptions using (2.81)

T	Tenor		
	1	3	5
1	80.41(-0.03%)	255.11(-0.2%)	421.89(-0.35%)
3	110.51(-0.03%)	317.84(-0.09%)	502.29 (-0.21%)
5	115.45(-0.01%)	328.68(-0.08%)	517.2(-0.31%)

Tab. 2.10: Relative Error(in %) for ITM Swaptions using (2.81)

is that we have approximated Y_{T_α} which is a non-Gaussian process under $\mathbb{Q}_{\alpha,\beta}$ by a Gaussian process which has mean and variance-covariance matrix that is equal to the exact mean and variance-covariance matrix of Y_{T_α} under $\mathbb{Q}_{\alpha,\beta}$. The fact that we get a good approximation of the swaption prices may be attributed to some error cancelation between the different sources of error. Hence we now consider removing the second source of error to see if we get a better approximation of swaption prices.

In order to find the contribution of the log-quadratic approximation $\widetilde{F}_{\alpha,\beta}$ to the error, we calculated the price of the swaption using $F_{\alpha,\beta}$. The price was calculated by integrating the payoff of the swaption using the probability density function of a two dimensional Gaussian random variable with mean and variance-covariance matrix equal to the exact mean and variance-covariance matrix of Y_{T_α} under $\mathbb{Q}_{\alpha,\beta}$. Assuming that the year fraction is a constant equal to $\tau_i = 0.25, i = \alpha + 1, \dots, \beta$, this price is calculated by the following formula:

$$P_{\alpha,\beta}(0) \iint_{-\infty-\infty}^{\infty\infty} \left(F_{\alpha,\beta} - (1+K)\right)^+$$
$$\frac{1}{2\pi|V_{\alpha,\beta}(0,T_\alpha)|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(Y_{T_\alpha} - M_{\alpha,\beta}(0,T_\alpha))^{\mathrm{T}} V(0,T_\alpha)^{-1}(Y_{T_\alpha} - M_{\alpha,\beta}(0,T_\alpha))\right) dY_{T_\alpha}. \tag{2.81}$$

T	Tenor		
	1	3	5
1	16.83(-0.15%)	45.12(-1.23%)	66.86(-2.31%)
3	40.77(-0.07%)	111.85(-0.27%)	169.55(-0.64%)
5	52.47(-0.03%)	144.13(-0.19%)	218.97(-0.72%)

Tab. 2.11: Relative Error(in %) OTM Swaptions using (2.81)

The results for the approximate swaption price based on (2.81) are presented in Table 2.9, Table 2.10 and Table 2.11. From the results, we see that, (2.81) is more accurate for the swaptions with swap tenor equal to one year while it has higher error for swaptions with swap tenor equal to five years. Hence it appears that removing the second source of error introduced by using $\widetilde{F}_{\alpha,\beta}$ instead of $F_{\alpha,\beta}$ gives less accurate results because the error cancelation between the different sources of error is reduced[5]. We also tested if we can improve (2.81) by using the exact probability density of Y_{T_α} under $\mathbb{Q}_{\alpha,\beta}$ using the formula:

$$P(0,T_\alpha)\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} P_{\alpha,\beta}(T_\alpha)\Big(F_{\alpha,\beta}-(1+K)\Big)^{+}$$
$$\frac{1}{2\pi|V(0,T_\alpha)|^{\frac{1}{2}}}\exp\Big(-\frac{1}{2}(Y_{T_\alpha}-M(0,T_\alpha))^{\mathrm{T}}V(0,T_\alpha)^{-1}(Y_{T_\alpha}-M(0,T_\alpha))\Big)\,dY_{T_\alpha}. \tag{2.82}$$

The results for the approximate swaption price based on (2.82) are presented in Table 2.12, Table 2.13 and Table 2.14. From the results, we see that the price based on (2.82) does not improve on the price based on (2.81). In fact for the OTM swaptions where the swap tenor is equal to five years, we have more error in the pricing. This shows that by replacing Y_{T_α} by a Gaussian process in (2.81), we actually have a better approximation of the distribution of the payoff of the swaption. It is clear from the numerical results given above that the first source of error which is the freezing of the weights in (2.58) is the significant source of error as removing the other sources of error does not improve the swaption prices. It appears that when approximations to swaption prices are calculated on the basis of the freezing of weights, we cannot ignore the volatility of the weights when the

[5] The third source of error does not appear to contribute significantly to the error of the approximation of swaption prices in numerical experiments conducted in Schrager and Pelsser (2006) for affine term structure models.

T	Tenor		
	1	3	5
1	39.80(-0.08%)	118.57(-0.59%)	188.43(-1.08%)
3	68.88(-0.05%)	194.12(-0.19%)	301.42(-0.47%)
5	78.88(-0.02%)	221.02(-0.17%)	342.44(-0.62%)

Tab. 2.12: Relative Error(in %) for ATM Swaptions using (2.82)

T	Tenor		
	1	3	5
1	80.41(-0.03%)	255.11(-0.21%)	421.89(-0.35%)
3	110.51(-0.03%)	317.84(-0.1%)	502.29 (-0.24%)
5	115.45(-0.01%)	328.68(-0.11%)	517.2(-0.38%)

Tab. 2.13: Relative Error(in %) for ITM Swaptions using (2.82)

maturity of the swaption is more than a year and the swap tenor underlying the swaption has a tenor of more than five years. The error becomes higher when considering OTM swaptions.

T	Tenor		
	1	3	5
1	16.83(-0.15%)	45.12(-1.24%)	66.86(-2.35%)
3	40.77(-0.08%)	111.85(-0.3%)	169.55(-0.74%)
5	52.47(-0.03%)	144.13(-0.25%)	218.97(-0.9%)

Tab. 2.14: Relative Error(in %) OTM Swaptions using (2.82)

2.4 Method of Moments Swaption Pricing

The use of Edgeworth expansions in option pricing was introduced by Jarrow and Rudd (1982) and has since been a subject of research. Some of the research has been to improve the Black and Scholes model (see Black and Scholes (1973)) by using Edgeworth type expansions (see, e.g., Corrado and Su (1997)). Even though the multi-factor quadratic Gaussian model allows the closed form pricing of caps, the price of swaptions cannot be obtained in analytically closed form. Therefore researchers have proposed different approximations based on approximating the risk neutral density of the swap rate using an Edgeworth expansion (Collin-Dufresne and Goldstein 2002) or using a Gram-Charlier expansion (Tanaka et al. 2005). In this section we conduct numerical experiments to compare approximations of swaption prices which are based on orthogonal series expansions of the risk neutral density of the swaptions' payoff. The method used is an application of Provost (2005) where a general approach to the expansion of probability density function using general polynomials is given. This method similar to the Edgeworth and Gram-Charlier approaches requires analytically closed form formulas for the moments of the swap rate. From the numerical results obtained, we will show that an approach based on the density of a beta random variable is better than one that is based on a standard normal variable which corresponds to using a Gram-Charlier series. We will consider the pricing of swaptions in the multivariate quadratic Gaussian factor model as we can calculate the prices of default free zero coupon bonds even when the factors are correlated. Instead of the swap rate, we choose an orthogonal series expansion of the probability density of $P_{\alpha,\beta}(T_\alpha)$(see (2.6)).

Lemma 2.7. *The process $\widetilde{P}_{\alpha,\beta}(T_\alpha)$ is bounded. Specifically the following holds in general*

$$0 < \widetilde{P}_{\alpha,\beta}(T_\alpha) < \sum_{i=1}^{n-1} \tilde{\tau}_{i+1}. \tag{2.83}$$

Proof. First recall that $\widetilde{P}_{\alpha,\beta}(T_\alpha) = \sum_{i=\alpha+1}^{\beta} \tilde{\tau}_i P(T_\alpha, T_i)$. We now use the property of default free zero coupon bonds in the quadratic Gaussian model. Specifically the zero coupon bond prices $P(T_\alpha, T_i), i = 1, \ldots, n$ satisfy

$$0 < P(T_\alpha, T_i) < 1, i = 1, \ldots, n.$$

□

For probability density approximations which are based on moments of random variables with finite support, the use of Legendre and Jacobi polynomials is recommended in Provost (2005). In this paper among the examples that were given, we see that for the finite support case that the use of beta densities and Jacobi polynomials is more efficient than using Legendre Polynomials. Motivated by this example we use a beta density which matches the first two moments of $P_{\alpha,\beta}(T_\alpha)$ and then a product of this beta density and Jacobi polynomials of increasing order to match the moments. One should note that we can calculate the moments of

$P_{\alpha,\beta}(T_\alpha)$ in closed form but for the mth positive moment we are looking at

$$(P_{\alpha,\beta}(T_\alpha))^m = \Big(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\Big)^m. \tag{2.84}$$

Typically for quarterly payments n becomes larger with the maturity of the swap. For a ten year with quarterly payments, we have 40 terms in $P_{\alpha,\beta}(T_\alpha)$. Using the multinomial formula we can expand the mth power of $P_{\alpha,\beta}(T_\alpha)$ but the number of terms in this expansion rises very rapidly. For example for a five year swap with quarterly payments, computing the 6th moment in *Mathematica* requires much more computational time than the 5th moment. One can get a better performance in terms of speed by using C++ but it clear that as m increases, calculating moments will be computationally intensive in any programming language. Hence getting a good approximation with fewer moments is important. However we will see that for orthogonal series expansions of the probability density of $P_{\alpha,\beta}(T_\alpha)$ (see Tanaka et al. (2005) for the Gram-Charlier case and the Cox-Ingersoll-Ross (CIR) factor model) increasing the moments can lead to a worse approximation for some strike price values because the approximation of the density of $P_{\alpha,\beta}(T_\alpha)$ through the orthogonal series leads to negative values for some parts of the domain of support. In fact using a large number of moments can lead to large errors in the approximation. We first briefly sketch the approximation method given in Provost (2005) and refer the reader to this reference for a detailed explanation. The expansion of the probability density of $Y = P_{\alpha,\beta}(T_\alpha)$ can be done as follows. Let us denote by (a, b) the support of $Y = P_{\alpha,\beta}(T_\alpha)$. We first make a transformation of Y to $X = \frac{Y-u}{s}$. The choice of $u \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$ is based on the support of the base density denoted by $\Psi_X[x]$. If $\Psi_X[x]$ has support (a_0, b_0), then we have

$$a_0 = \frac{a-u}{s}, \; b_0 = \frac{b-u}{s}.$$

Now the probability density function of X denoted by $f_X[x]$ can be approximated by

$$f_{X_n}[x] = \Psi_X[x] \sum_{l=0}^{n} \xi_l x_l. \tag{2.85}$$

The probability density function of Y denoted by $\Psi_Y[x]$ is therefore approximated by

$$f_{Y_n}[y] = \Psi_X\Big[\frac{y-u}{s}\Big]\sum_{l=0}^{n}\frac{\xi_l}{s}\Big(\frac{Y-u}{s}\Big)^l. \tag{2.86}$$

Let $\mu_X[k]$ denote the k^{th} raw moment of X which can be obtained from the k^{th} raw moment of Y denoted by $\mu_Y[k]$. Then the values ξ_l, for $l = 0, \ldots, n$ are calculated from the moments of X by first forming an $(n+1)\times(n+1)$ matrix[6] M whose ij^{th} entry is $\mu_X[i-1+j-1]$ and calculating

$$(\xi_0, \ldots, \xi_n)^{\mathsf{T}} = M^{-1}(\mu_Y[0], \ldots, \mu_Y[n])^{\mathsf{T}}. \tag{2.87}$$

Let $\Gamma[z]$ denote the Euler Gamma function

$$\Gamma[z] = \int_0^\infty t^{z-1}\exp(-t)\,dt$$

and let $Beta[y,z]$ denote the Euler Beta function

$$Beta[y,z] = \frac{\Gamma[y]\Gamma[z]}{\Gamma[y+z]} = \int_0^1 t^{y-1}(1-t)^{z-1}\,dt.$$

For the specific case of a beta density

$$\Psi_X[x] := \frac{1}{Beta[\bar{\alpha}+1,\bar{\beta}+1]}x^{\bar{\alpha}}(1-x)^{\bar{\beta}},\;\; 0<x<1 \tag{2.88}$$

as the base density, we have $a_0 = 0$ and $b_0 = 1$ so that $u = a$ and $s = b - a$. Moreover Provost (2005) recommends that the following modified form of the Jacobi polynomials be used

$$G_n[\sigma,\tau,x] := n!\frac{\Gamma[n+\sigma]}{\Gamma[2n+\sigma]}JacobiP[n,\sigma-\tau,\tau-1,2x-1] \tag{2.89}$$

where $JacobiP[n,\sigma-\tau,\tau-1,2x-1]$ is an nth-degree Jacobi polynomial(see Abramowitz and Stegun (1972)) and $\sigma = \bar{\alpha}+\bar{\beta}+1$ and $\tau = \bar{\alpha}$. The parameters $\bar{\alpha}$ and $\bar{\beta}$ are chosen such that the first central moments of the base density

[6] There are other ways of calculating the coefficients ξ_l, for a detailed discussion see Provost (2005).

$\Psi_X[x]$ match the moments of X. Furthermore we have

$$\int_0^1 Beta[\bar{\alpha}+1,\bar{\beta}+1]G_i[\sigma,\tau,x]G_h[\sigma,\tau,x]\,dx = \theta_h, \quad \text{when } i=h, h=0,1,\ldots,n$$

and zero otherwise (2.90)

where

$$\theta_h = \frac{h!\Gamma[h+\bar{\alpha}+1]\Gamma[h+\bar{\alpha}+\bar{\beta}+1]\Gamma[h+\bar{\beta}+1]}{(2h+\bar{\alpha}+\bar{\beta}+1)\Gamma[2h+\bar{\alpha}+\bar{\beta}+1]^2}.$$

Let δ_{hk} denote the coefficient of x^k in $G_h[\sigma,\tau,x]$. Then Provost (2005) gives the following density approximant of Y for the base density (2.88) and orthogonal polynomials (2.89) we have made:

$$f_{Y_n}[y] := \Psi\Big[\frac{y-u}{s}\Big]\sum_{l=0}^{n}\Big(\sum_{i=l}^{n}\frac{\delta_{il}}{s\,Beta[\bar{\alpha}+1,\bar{\beta}+1]\theta_i}\sum_{k=0}^{i}\delta_{ik}\mu_X[k]\Big)\Big(\frac{y-u}{s}\Big)^l. \tag{2.91}$$

In addition to using the beta density (2.88) as the base density, we also considered using the standard normal density

$$\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right), \ -\infty < x < \infty \tag{2.92}$$

as the base density and the modified Hermite polynomials

$$H_k^*[x] := (-1)^k 2^{-\frac{k}{2}} HermiteH[k,\frac{x}{\sqrt{2}}] \tag{2.93}$$

where $HermiteH[k,w]$ is the k^{th} degree Hermite polynomial (see Abramowitz and Stegun (1972) for the orthogonal polynomials). This corresponds to using the Gram Charlier A series similar to Tanaka et al. (2005) to approximate swaption prices. The swaption (2.3) that we considered for pricing is an option at time $t=0$ to enter a payer swap (2.2) starting after one year ($T_\alpha = 1$) and ending ten years after the start date ($T_{10} = T_\beta = 11$). We assume that the frequency of payments is annual ($T_{i+1} - T_i = 1, i = 1,9$). We refer to this swaption as the 1×10 swaption. We can calculate the conditional mean $M(0,T_\alpha)$ and variance-covariance matrix $V(0,T_\alpha)$ of Y_{T_α} under the default free forward measure $\mathbb{T}_\alpha$ in closed form (see Lemma 1.3 in chapter 1) and therefore we can use the probability density function of Y_T under $\mathbb{T}_\alpha$ to directly integrate the payoff function of the swaption. The exact price of this 1×10 swaption under the two factor quadratic Gaussian model is calculated using cubature[7]. Therefore we use the following double integral[8] to calculate the exact swaption price

$$
\begin{aligned}
Swaptn_{\alpha,\beta} &= P(0,T_\alpha) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(1 - P(T_\alpha,T_\beta) - K \sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\Big)^+ \\
&\frac{1}{2\pi|V(0,T_\alpha)|^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}(Y_{T_\alpha} - M(0,T_\alpha))^{\mathrm{T}} V(0,T_\alpha)(Y_{T_\alpha} - M(0,T_\alpha))\Big)\, dY_{1T_\alpha} dY_{2T_\alpha} \\
&= P(0,T_\alpha) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(1 - \sum_{i=\alpha+1}^{\beta} \tilde{K}_i P(T_\alpha,T_i)\Big)^+ \\
&\frac{1}{2\pi|V(0,T_\alpha)|^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}(Y_{T_\alpha} - M(0,T_\alpha))^{\mathrm{T}} V(0,T_\alpha)(Y_{T_\alpha} - M(0,T_\alpha))\Big)\, dY_{1T_\alpha} dY_{2T_\alpha}
\end{aligned}
\tag{2.94}
$$

where $\tilde{K}_i = K, i = 1,9$ and $\tilde{K}_{10} = 1 + K$.

Using the parameters that were obtained by calibrating a two factor quadratic Gaussian model to the discount and cap/floor price data given in Table 2.2 and Table 2.1, we computed the price of the 1×10 swaption. The results indicate that for these calibrated parameters the orthogonal series expansion based on the beta density has less error in pricing compared to the one based on a Gram Charlier A series. We considered the pricing performance of the two methods based on using the first three moments, the first five moments, and the first seven moments of $P_{\alpha,\beta}(T_\alpha)$. The error in the price of the swaption does not necessarily decrease when using additional moments in the orthogonal series approach(Tanaka et al. 2005). However for this particular case considered, the error in pricing decreased when using additional moments. The following tables (2.15),(2.16) and (2.17) give the exact price of the 1×10 swaption[9] with the relative error in % given in parenthesis next to the price. The values in the first column of tables (2.15),(2.16) and (2.17) are the difference between the at the money strike rate corresponding to

[7] For a description of algorithms for multidimensional integration based on cubature, see Cools and Haegemans (2003). Cubature is the method used by the **NIntegrate** function of *Mathematica*.

[8] Calculating the swaption price by directly integrating the payoff function using the probability density is feasible when the number of factors is 2 or even 3. However if the dimension of the vector Y_{T_α} increases, this method becomes less efficient though it is still a more accurate and faster way to compute the swaption price compared to using Monte Carlo simulations.

[9] Multiplying the prices by 10^4 will convert them into basis points.

the row with $\Delta K = 0$ and the strike rate obtained by adding ΔK value to the at the money strike rate. From the numerical results given in the tables, we conclude that using an orthogonal series approach based on the beta density is better than one based on a Gram Charlier A series for approximating the price of swaptions.

ΔK	Exact	Beta3	GC3
-0.025	0.17403	0.17404(-0.01%)	0.17405(-0.01%)
-0.02	0.13976	0.13978(-0.0.02%)	0.13981(-0.04%)
-0.015	0.10691	0.10694(-0.03%)	0.10698(-0.07%)
-0.01	0.07712	0.07712(0.00%)	0.07713(-0.02%)
-0.005	0.05210	0.05206(0.08%)	0.05202(0.16%)
0	0.03288	0.03281(0.2%)	0.03274(0.42%)
0.005	0.01938	0.01933(0.28%)	0.01927(0.59%)
0.01	0.01071	0.01069(0.21%)	0.01067(0.39%)
0.015	0.00557	0.00557(-0.16%)	0.00559(-0.42%)
0.02	0.00273	0.00276(-0.97%)	0.00279(-2.01%)
0.025	0.00128	0.00131(-2.4%)	0.00133(-4.39%)

Tab. 2.15: Swaption price and relative error in % using 3 moments

ΔK	Exact	Beta5	GC5
-0.025	0.17403	0.17403(-0.00%)	0.17403(-0.00%)
-0.02	0.13976	0.13976(-0.0.00%)	0.13977(-0.01%)
-0.015	0.10691	0.10691(-0.00%)	0.10693(-0.02%)
-0.01	0.07712	0.07712(0.00%)	0.07711(0.00%)
-0.005	0.05210	0.0521(0.00%)	0.05207(0.07%)
0	0.03288	0.03288(0.00%)	0.03284(0.11%)
0.005	0.01938	0.01939(-0.01%)	0.01938(0.04%)
0.01	0.01071	0.01071(-0.01%)	0.01073(-0.18%)
0.015	0.00557	0.00557(-0.01%)	0.00559(-0.49%)
0.02	0.00273	0.00273(0.02%)	0.00275(-0.68%)
0.025	0.00128	0.00128(0.06%)	0.00128(-0.45%)

Tab. 2.16: Swaption price and relative error in % using 5 moments

ΔK	Exact	Beta7	GC7
-0.025	0.17403	0.17402(-0.00%)	0.17402(0.00%)
-0.02	0.13976	0.13976(-0.0.00%)	0.13976(-0.00%)
-0.015	0.10691	0.10691(0.00%)	0.10691(-0.00%)
-0.01	0.07712	0.07712(0.00%)	0.07712(0.00%)
-0.005	0.05210	0.05210(0.00%)	0.0521(0.01%)
0	0.03288	0.03288(-0.00%)	0.03287(0.01%)
0.005	0.01938	0.01938(-0.00%)	0.01939(-0.02%)
0.01	0.01071	0.01071(0.00%)	0.01071(-0.05%)
0.015	0.00557	0.00556(0.01%)	0.00557(-0.05%)
0.02	0.00273	0.00273(0.03%)	0.00273(0.03%)
0.025	0.00128	0.00128(0.02%)	0.00127(0.17%)

Tab. 2.17: Swaption price and relative error % using 7 moments

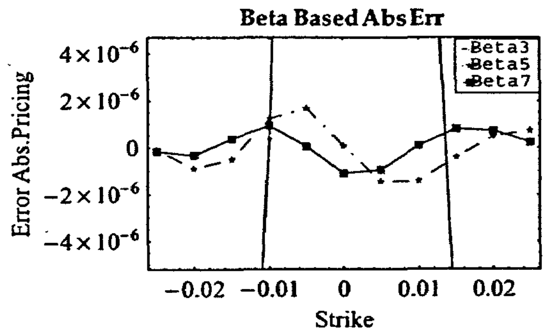

Fig. 2.2: Absolute Error in Swaption Price Beta base density

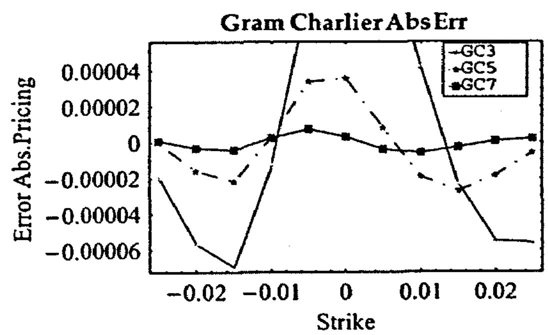

Fig. 2.3: Absolute Error in Swaption Price for Gram Charlier Series A

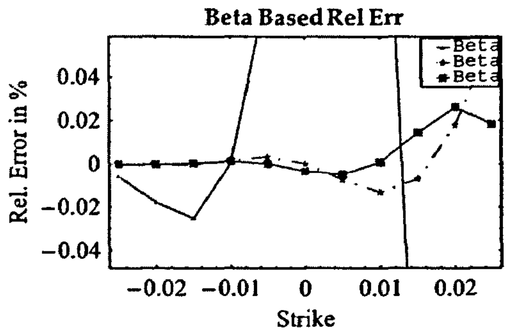

Fig. 2.4: Relative Error in % for Swaption Price using Beta base density

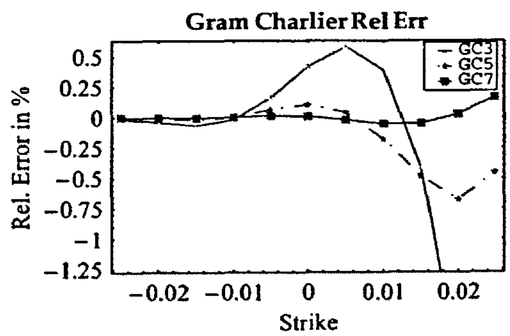

Fig. 2.5: Relative Error in % for Swaption Price using Gram Charlier Series A

3. CREDIT DEFAULT SWAPS AND CREDIT DEFAULT SWAPTIONS

There is some work in the pricing of credit default swaps and options on credit default swaps. In Brigo & Alfonsi (2004) an extension of the Cox-Ingersoll-Ross(CIR) model known as CIR++ is used to model the short term interest rate and the intensity of default. When the interest rate and the intensity of default are independent, closed form formulas for the price of single name credit default swaps are provided[1]. Moreover the independence enables the separation of the calibration of the short term interest rate to caps or swaptions from the calibration of the intensity of default to quotes of credit default swaps. However when there is correlation between the interest rate and intensity of default, the CIR++ reduced form model does not enable the calculation of the price of credit default swaps in closed form except for special cases, therefore the calibration of the model has to be done through the use of Monte Carlo simulation or through a Gaussian dependence mapping. In the reduced model adopted in this paper, the use of quadratic Gaussian processes enables us to calculate the price of credit default swaps in a closed form so that the calibration of the model to the default term structure and quotes of credit default swaps can be done through analytic formulas and solving numerically an ODE. In fact we can even derive closed form approximations to the price of credit default swaptions. The assumption of a correlation between the interest rate and the intensity of default does not prevent us from obtaining the analytic formulas. The first section provides details of the pricing formulas for credit default swaps and the calibration of the quadratic Gaussian model to credit default swap quotes. We first give new results on how to extract the probability of default under the assumption of a stochastic intensity. We then present a new result showing that the calibration of the quadratic Gaussian model to credit default swaps can be done analytically if we assume that we use different factors to model the interest

[1] For piecewise constant parameter case, closed form formulas for the price of default free bonds in the CIR model can be derived (see, e.g., Schlögl and Schlögl (2002), Meza, A. and Satzschneider (2002)).

rate and the intensity of default and assume no correlation between the factors. Furthermore we show that the calibration can be done through a solution of a non linear ODE in the general case of correlation between the factors. In the second section we derive three different ways of approximating the price of options on credit default swaps.

3.1 Pricing of Credit Default Swaps

In this section we first show how we can calculate the price of a domestic credit default swap in a closed form under the quadratic Gaussian factor model. We first extend the procedure to extract the probability of default from quotes of credit default swaps in a reduced form model of credit risk where the intensity of default is deterministic given in Martin et al. (2001) to a reduced form model of credit risk where the intensity of default is stochastic. We then give a calibration procedure that will be used to calibrate the drift term of the intensity of default to quotes of credit default swaps. This calibration can be done using closed form formulas if we assume that the short term interest rate r_t and the intensity of default λ_t are independent. If we assume that there is correlation between r_t and λ_t, the calibration can be carried out by solving an ordinary differential equation. We assume in this section that we have a quadratic Gaussian factor model for default free and defaultable securities[2] as described in chapter 1. Therefore the default free interest rate r_t and the intensity of default λ_t are modeled through quadratic Gaussian processes (see equations 1.6 and 1.7 in chapter 1.) In the following we describe a bilateral financial contract between two participants which we refer to as the payer and the receiver.

Definition 3.1. *A Credit Default Swap(CDS) is a contract that guarantees the payment of a deterministic fraction Z of a notional amount to the payer from the receiver at default time τ of a corporate if default occurs at or after an agreed time $T_n \geqslant 0$ known as the start time and before or at an agreed time $T = T_N > T_n$ known as the maturity time. We will call the payment Z at default time as the protection payment. In return the payer pays a constant premium K on the notional amount at specified dates $\mathcal{T} = \{T_{n+1}, T_{n+2}, \ldots, T_i, \ldots, T_{N-1}, T_N\}$ provided that default has not occurred before the premium payment date $T_i \in \mathcal{T}$. If the corporate defaults at time τ where $T_i < \tau \leqslant T_{i+1}$ and $n \leqslant i \leqslant N-1$, then the contract is terminated*

[2] See the discussion of the model at the beginning of this chapter

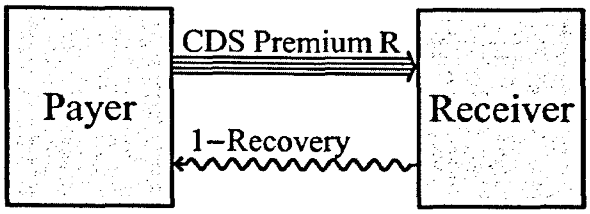

Fig. 3.1: CDS over $[T_\alpha, T_\beta]$

after a payment of $(\tau - T_i)K$ *of the notional amount by the payer to the receiver. This amount is referred to as the accrued premium.*

In the discussions that follow we will assume that the notional amount upon which the CDS contract is based is equal to one unit of currency. Let $\zeta(\tau) = \max[i : n+1 \leqslant i \leqslant N, T_i < \tau]$ and $\beta_i = T_i - T_{i-1}$. Then the price of a CDS at time $t \leqslant T_n$ is given by the following formula (see Bielecki and Rutkowski (2002), p. 224)

$$CDS(t,T,T,K,Z) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s\,ds\Big)Z\,\mathbf{1}_{T_n<\tau\leqslant T} -$$

$$\exp\Big(-\int_t^T r_s\,ds\Big)(\tau - T_{\zeta(\tau)})K\,\mathbf{1}_{T_n<\tau\leqslant T} - \sum_{i=n+1}^{N}\exp\Big(-\int_t^{T_i} r_s\,ds\Big)\beta_i\,K\,\mathbf{1}_{\tau>T_i}\Big|\mathcal{G}_t\Big]$$

Under the proper assumptions of a reduced model of default where the default time τ is the first jump time of a conditional Poisson process, we obtain

$$CDS(t,T,T,K,Z) = \mathbf{1}_{\tau>t}Z\int_{T_n}^{T}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^s r_k + \lambda_k\,dk\Big)\lambda_s\Big|\mathcal{F}_t\Big]\,ds-$$

$$-\mathbf{1}_{\tau>t}K\int_{T_n}^{T}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^s r_k + \lambda_k\,dk\Big)\lambda_s\Big|\mathcal{F}_t\Big](s - T_{\zeta(s)})\,ds-$$

$$-\mathbf{1}_{\tau>t}K\sum_{i=n+1}^{N}\beta_i\,\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^{T_i} r_k + \lambda_k\,dk\Big)\Big|\mathcal{F}_t\Big]$$

Using the defaultable bond $\mathbf{1}_{\tau>t}\bar{P}(t,s)$,

$$\mathbf{1}_{\tau>t}\bar{P}(t,s) = \mathbf{1}_{\tau>t}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^s r_k + \lambda_k\,dk\Big)\Big|\mathcal{F}_t\Big]$$

as the numeraire and denoting the corresponding defaultable forward measure by $\bar{\mathbb{T}}_s$ the price of a CDS is given by[3]

$$CDS(t,\mathcal{T},T,K,Z) = \mathbf{1}_{\tau>t}Z \int_{T_n}^{T_N} \bar{P}(t,s)\, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_t]\, ds -$$

$$-\mathbf{1}_{\tau>t}K \int_{T_n}^{T_N} \bar{P}(t,s)\ \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_t]\,(s - T_{\zeta(s)})\, ds - \mathbf{1}_{\tau>t}K \sum_{i=n+1}^{N} \beta_i\, \bar{P}(t,T_i). \quad (3.1)$$

As explained in chapter 1 we can get $\bar{P}(t,T)$ in closed form[4]. We only need to know how to calculate $\mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_t]$. This can also be obtained in closed form under the quadratic Gaussian model using Lemma 1.36 of chapter 1 as the defaultable bond $\bar{P}(t,s)$ is a log-quadratic Gaussian process.

We now show how we can calibrate the drift term of λ_t using credit default swap quotes for the corporation whose default time is denoted by τ. Under independence of the interest rate and the intensity of default, the probability of default is given by

$$\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^s \lambda_k\, dk\Big)\Big|\mathcal{F}_t\Big]. \quad (3.2)$$

In order to calibrate λ_t, we need the term structure of default probabilities. Let c and d represent positive integers such that $\{i_1,\ldots,i_c\}$ and $\{j_1,\ldots,j_d\}$ represent disjoint subsets of $\{1,\ldots,n\}$ where here n refers to the dimension of Y_t. Now let

$$Z_t^1 = \alpha^1(t) + Y_t^1 = (\alpha_{i_1}(t),\ldots,\alpha_{i_c}(t)) + (Y_{i_1 t},\ldots,Y_{i_c t})$$

and

$$Z_t^2 = \alpha^2(t) + Y_t^2 = (\alpha_{j_1}(t),\ldots,\alpha_{j_d}(t)) + (Y_{j_1 t},\ldots,Y_{j_d t})$$

be used to model r_t and λ_t respectively where we assume that the instantaneous correlation matrix of Y_t^1 and Y_t^2 is a diagonal matrix. Therefore r_t and λ_t are assumed to have zero correlation and this means Y_t^1 and Y_t^2 are independent. If there is a liquid market for defaultable zero coupon bonds for a range of maturities as in the case of a default free bond, then we can extract the default probabilities

[3] See Bielecki and Rutkowski (2002), p.224 or the discussion given in Chapter 4 for more detail.

[4] The discussion in Chapter 1 is for the case of default free bonds but it equally applicable for defaultable bonds by just considering the instantaneous rate to be $r_t + \lambda_t$.

and calibrate the drift term of Z_t^2 through $\alpha^2(t)$. In general the market for defaultable bonds issued by a corporate is not liquid. The market for credit default swaps where a corporate is the reference name underlying the credit default swap contract has better liquidity. However the maturities of credit default swaps traded are usually one year, three years, five years and ten years. Practitioners assume a piecewise constant intensity and use a bootstrapping procedure to extract the term structure of default probabilities. We can use these term structure of default probabilities to calibrate our model but this is in contradiction to the assumed stochastic intensity of default. Moreover this bootstrapping procedure has some disadvantages such as not being robust to unreliable quotes for some maturities. A better method for extracting the term structure of default probabilities is given in Martin et al. (2001). We give a brief description of this method. The method described in Martin et al. (2001) is for a time dependent deterministic λ_t. Here we extend this method to the case of a stochastic intensity of default λ_t under the assumption of independence between the default free short rate of interest and the intensity of default. We later show how we can still modify this method to the case where the short rate of interest and the intensity of default are not independent. Ignoring the accrued premium, the value of a credit default swap of maturity T to the seller at time $t = 0$ is given by

$$K \sum_{i=0}^{N} \beta_i \, \bar{P}(0, T_i) - Z \int_0^{T_N} \bar{P}(0, s) \, \mathbb{E}^{\mathbf{T}_s}[\lambda_s | \mathcal{F}_t] \, ds$$

where $0 = T_0, \ldots, T_N = T$ are the premium payment dates. The quoted CDS rates are chosen so that the value of the CDS is equal to zero at time $t = 0$. Therefore the premium K is chosen to be

$$R_f(T) := \frac{Z \int_0^{T_N} \bar{P}(0, s) \, \mathbb{E}^{\mathbf{T}_s}[\lambda_s | \mathcal{F}_t] \, ds}{\sum_{i=0}^{N} \beta_i \, \bar{P}(0, T_i)} \tag{3.3}$$

where we assume a notional of one unit of currency and a constant recovery rate δ such that $Z = 1 - \delta$. Assuming independence between r_t and λ_t, we can write

$R_f(T)$ as

$$R_f(T) = \frac{Z \int_0^{T_N} \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^s r_k + \lambda_k \, dk \Big) \lambda_s \Big| \mathcal{F}_t \Big] \, ds}{\sum_{i=0}^{N} \beta_i \, \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_i} r_k + \lambda_k \, dk \Big) \Big| \mathcal{F}_t \Big]}$$
$$= \frac{Z \int_0^{T_N} P(0,s) \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^s \lambda_k \, dk \Big) \lambda_s \Big| \mathcal{F}_t \Big] \, ds}{\sum_{i=0}^{N} \beta_i \, P(0,T_i) \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_i} \lambda_k \, dk \Big) \Big| \mathcal{F}_t \Big]}. \tag{3.4}$$

As described in Martin et al. (2001), we can discretize the interval $[0, T_N]$ by choosing $\Delta > 0$ and considering a set of dates $(0 = T_0, \ldots, T_j, \ldots, T_M = T)$, $\Delta = T_j - T_{j-1}$, j=1,...,M. We can now approximate $R_f(T)$ by

$$R_f(T) \approx \frac{Z \sum_{j=0}^{M} \frac{1}{2}(P(0,T_j) + P(0,T_{j+1})) \mathbb{E}^{\mathbb{Q}}\Big[\int_{T_j}^{T_{j+1}} \exp -\Big(\int_0^s \lambda_u \, du \Big) \lambda_s \, ds \Big]}{\sum_{i=0}^{N} \beta_i \, P(0,T_i) \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_i} \lambda_k \, dk \Big) \Big| \mathcal{F}_t \Big]}. \tag{3.5}$$

Let $G(0,t)$ denote the risk neutral probability of survival i.e. the probability there is no default between time zero and time t as seen with respect to the trivial filtration $\mathcal{F}_0$. Under the reduced form model we are considering, we can express $G(0,t)$ by the following formula

$$G(0,t) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^t \lambda_s \, ds \Big) \Big]. \tag{3.6}$$

Let the probability of default over (T_j, T_{j+1}) conditional on survival up to time T_j be denoted by g_j which is given by

$$g_j := \frac{\mathbb{E}^{\mathbb{Q}}[\tau > T_j] - \mathbb{E}^{\mathbb{Q}}[\tau > T_{j+1}]}{\mathbb{E}^{\mathbb{Q}}[\tau > T_j]} = 1 - \frac{G(0,T_{j+1})}{G(0,T_j)}. \tag{3.7}$$

Then we have

$$G(0,T_{j+1}) = G(0,T_j) - G(0,T_j) g_j, \quad G(0,0) = 1 \tag{3.8}$$

or as originally formulated in Martin et al. (2001) in terms of the risk neutral probability of default

$$H(0,T) := \mathbb{E}^{\mathbb{Q}}[\tau < T] = 1 - \mathbb{E}^{\mathbb{Q}}\Big[\exp - \Big(\int_0^T \lambda_s \, ds\Big)\Big] \tag{3.9}$$

we have

$$H(0,T_{j+1}) = H(0,T_j) + (1 - H(0,T_j))g_j, \quad H(0,0) = 0. \tag{3.10}$$

Therefore the probability of default between T_j and T_{j+1} can be expressed in terms of $G(0,T_j)$ and g_j

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_{T_j}^{T_{j+1}} \lambda_s \exp\Big(-\int_0^s \lambda_u \, du\Big)\Big] = G(0,T_j) - G(0,T_{j+1}) = G(0,T_j)g_j. \tag{3.11}$$

Therefore

$$R_f(T_N) = \frac{Z \sum_0^M \frac{1}{2}(P(0,T_j) + P(0,T_{j+1}))G(0,T_j)g_j}{\sum_{i=0}^N \beta_i \, \bar{P}(0,T_i)}. \tag{3.12}$$

The ratio in the right hand side of (3.12) is a function of g_j such that for a given maturity T_k,

$$F(g_0,\ldots,g_{k-1}) := \frac{Z \sum_0^{k-1} \frac{1}{2}(P(0,T_j) + P(0,T_{j+1}))G(0,T_j)g_j}{\sum_{i=0}^k \beta_i \, P(0,T_i)G(0,T_i)} \tag{3.13}$$

If we assume that the CDS quotes $R_f(T_k)$, $k = 1,\ldots,r$ are subject to a Gaussian error of σ_k, $k = 1,\ldots,r$, the procedure suggested in Martin et al. (2001) is to minimize

$$W(g_0,\ldots,g_{M-1}) = \nu \sum_{j=0}^{M-1} d(g_{j+1};g_j)^2 + \frac{1}{2}\sum_{k=1}^{r}\left(\frac{R_f(T_k) - F(g_0,\ldots,g_{k-1})}{\sigma_k}\right)^2 \tag{3.14}$$

where

$$d(g';g) = \sqrt{(g'-g)\log_e\left(\frac{g'}{g}\right) + (g-g')\log_e\left(\frac{1-g'}{1-g}\right)} \tag{3.15}$$

and ν is a positive constant which gives more smoothness for the default probability curve for higher values. The authors Martin et al. (2001) consider the case $\nu = 10, \nu = 10,000$ and $\sigma_k = 10^{-4}$, $k = 1,..r$ and time discretisations of $\Delta = 0.5$ and $\Delta = \frac{1}{6}$ corresponding to 6 months and 2 months. Thus we can extract the term structure of the probability of survival given by $G(0,T)$. Now under the assumption of independence between r_t and λ_t, we have

$$\mathbb{E}^{\bar{\mathrm{T}}}[r_T] = \mathbb{E}^{\mathrm{Q}}\Bigg[\frac{\exp\Big(-\int\limits_0^T r_s + \lambda_s\, ds\Big)}{\mathbb{E}^{\mathrm{Q}}\Big[\exp\Big(-\int\limits_0^T r_s + \lambda_s\, ds\Big)\Big]} r_T\Bigg] \tag{3.16}$$

$$= \frac{\mathbb{E}^{\mathrm{Q}}\Big[\exp\Big(-\int\limits_0^T r_s\, ds\Big) r_T\Big]\mathbb{E}^{\mathrm{Q}}\Big[\exp\Big(-\int\limits_0^T \lambda_s\, ds\Big)\Big]}{\mathbb{E}^{\mathrm{Q}}\Big[\exp\Big(-\int\limits_0^T r_s\, ds\Big)\Big]\mathbb{E}^{\mathrm{Q}}\Big[\exp\Big(-\int\limits_0^T \lambda_s\, ds\Big)\Big]} \tag{3.17}$$

$$= \mathbb{E}^{\mathrm{Q}}\Bigg[\frac{1}{P(0,T)}\exp\Big(-\int\limits_0^T r_s\, ds\Big) r_T\Bigg] \tag{3.18}$$

$$= \mathbb{E}^{\mathrm{T}}[r_T]. \tag{3.19}$$

Similar to the default free case(see equation 1.50 and theorem 1.8 in chapter 1) we calibrate the drift term of λ_t using

$$-\partial_T \log_e \bar{P}(0,T) = \mathbb{E}^{\bar{\mathrm{T}}}[r_T + \lambda_T]. \tag{3.20}$$

Under the assumption of independence equation (3.20) can be reduced to

$$-\partial_T \log_e P(0,T) - \partial_T \log_e G(0,T) = \mathbb{E}^{\mathrm{T}}[r_T] + \mathbb{E}^{\bar{\mathrm{T}}}[\lambda_T]. \tag{3.21}$$

The drift term of r_t is calibrated using a closed formula starting from the equation

$$-\partial_T \log_e P(0,T) = \mathbb{E}^{\mathrm{T}}[r_T]. \tag{3.22}$$

Therefore (3.21) simplifies to

$$\partial_T \log_e G(0,T) = \mathbb{E}^{\bar{\mathrm{T}}}[\lambda_T]. \tag{3.23}$$

Now the conditional mean and the conditional variance of the multivariate Gaussian factor Y_t under the defaultable forward measure $\bar{T}$ which are denoted by $\overline{M}(t,T)$ and $\bar{V}(t,T)$ respectively satisfy the ODE's given in lemma 1.3 as the defaultable bond $\bar{P}(t,T)$ is a log-quadratic Gaussian process. Under the assumption of independence we can rearrange the factors Y_t^1 and Y_t^2 used to model r_t and λ_t respectively such that $\bar{V}(t,T)$ is a block diagonal matrix. Therefore the ODE's given in lemma 1.3 can be solved separately for the conditional mean and conditional variance(under $\bar{\mathbb{T}}$) of Y_t^1 and Y_t^2 which we denote by $\overline{M}^1(t,T), \bar{V}^1(t,T)$ and $\overline{M}^2(t,T), \bar{V}^2(t,T)$. Note that under the assumption of independence $\overline{M}^1(t,T)$ is the same as the conditional mean of Y_t^1 under $\mathbb{T}$ and $\bar{V}^1(t,T)$ is the same as the conditional variance of Y_t^1 under $\mathbb{T}$. Here we only need to use the fact that we can solve $\overline{M}^2(t,T)$ independently from $\overline{M}^1(t,T)$. Thus if we assume for ease of exposition that

$$r_t = (Y_t^1 + \alpha^1(t))^\top (Y_t^1 + \alpha^1(t)) \tag{3.24}$$

$$\lambda_t = (Y_t^2 + \alpha^2(t))^\top (Y_t^2 + \alpha^2(t)), \tag{3.25}$$

we can proceed to calibrate the drift term of λ_t as in the proof of lemma 1.8 in chapter 1.

Assuming correlation between the factors used to model r_t and the factors used to model λ_t, will require additional approximations to extract the term structure of probabilities of default under the forward measure $\mathbb{T}$ corresponding to using the default free bond of maturity T as the numeraire. Moreover each of the system of ODE's given in lemma 1.3 do not separate into two independent systems and we shall see that we have to resort to numerically solving a first order system of non-linear ODE's in the general case. First we assume that we have a discretisation as in the above so that the time interval T is divided into M subintervals of equal length Δ. Moreover if default occurs between $T_j < \tau \leqslant T_{j+1}$ payment of the constant amount Z is made at T_{j+1} for $j = 0, \dots, M-1$. Under this assumption we can write (see Bielecki and Rutkowski (2002), p.224)

$$
\begin{aligned}
Z\,\mathbb{E}^{\mathbb{Q}}\Big[\exp-\Big(\int_0^{\tau} r_u\,du\Big)\mathbf{1}_{0<\tau\leqslant T}\Big] &= Z\sum_{j=0}^{M-1}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u\,du\Big)\mathbf{1}_{T_j<\tau\leqslant T_{j+1}}\Big] \\
&= Z\sum_{j=0}^{M-1}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u\,du\Big)\mathbf{1}_{\tau>T_j}\Big]-\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u\,du\Big)\mathbf{1}_{\tau>T_{j+1}}\Big] \\
&= Z\sum_{j=0}^{M-1}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_j} r_u+\lambda_u\,du\Big)\exp\Big(-\int_{T_j}^{T_{j+1}} r_u\,du\Big)\Big]- \\
&\qquad -\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u+\lambda_u\,du\Big)\Big] \\
&= Z\sum_{j=0}^{M-1}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_j} r_u+\lambda_u\,du\Big)\exp\Big(-\int_{T_j}^{T_{j+1}} r_u\,du\Big)\Big]- \\
&\qquad -\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u+\lambda_u\,du\Big)\Big] \\
&= Z\sum_{j=0}^{M-1}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_j} r_u+\lambda_u\,du\Big)\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_{T_j}^{T_{j+1}} r_u\,du\Big)\Big|\mathcal{F}_{T_j}\Big]\Big]- \\
&\qquad -\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u+\lambda_u\,du\Big)\Big] \qquad (3.26)
\end{aligned}
$$

$$
\begin{aligned}
&= Z\sum_{j=0}^{M-1}P(0,T_j)\mathbb{E}^{\mathrm{T}_j}\Big[\exp\Big(-\int_0^{T_j}\lambda_u\,du\Big)P(T_j,T_{j+1})\Big]- \\
&\qquad -P(0,T_{j+1})\mathbb{E}^{\mathrm{T}_{j+1}}\Big[\exp\Big(-\int_0^{T_{j+1}}\lambda_u\,du\Big)\Big]. \qquad (3.27)
\end{aligned}
$$

The value $P(T_j,T_{j+1})$ is stochastic when seen from time $t=0$, however if Δ is close to zero, $P(T_j,T_{j+1})$ is close to 1. Therefore it is close to being deterministic, and hence we would not introduce much error by assuming $\exp\Big(-\int_0^{T_{j+1}}\lambda_u\,du\Big)$ and

$P(T_j, T_{j+1})$ are independent under the $\mathbb{T}_j$. We therefore approximate (3.27) by

$$
\begin{aligned}
&Z \sum_{j=0}^{M-1} P(0,T_j)\mathbb{E}^{\mathbf{T}_j}\Big[\exp\Big(-\int_0^{T_j}\lambda_u\,du\Big)\Big]\mathbb{E}^{\mathbf{T}_j}\Big[P(T_j,T_{j+1})\Big]- \\
&\qquad\qquad - P(0,T_{j+1})\mathbb{E}^{\mathbf{T}_{j+1}}\Big[\exp\Big(-\int_0^{T_{j+1}}\lambda_u\,du\Big)\Big] \\
&= Z \sum_{j=0}^{M-1} P(0,T_{j+1})\Big(\mathbb{E}^{\mathbf{T}_j}\Big[\exp\Big(-\int_0^{T_j}\lambda_u\,du\Big)\Big] - \mathbb{E}^{\mathbf{T}_{j+1}}\Big[\exp\Big(-\int_0^{T_{j+1}}\lambda_u\,du\Big)\Big]\Big)
\end{aligned}
\tag{3.28}
$$

Let

$$\bar{G}(0,T) := \mathbb{E}^{\mathbf{T}}\Big[\exp\Big(-\int_0^{T}\lambda_u\,du\Big)\Big]$$

denote the probability of survival up to time T under the default free forward measure $\mathbb{T}$. Then

$$R_f(T) = \frac{Z\sum_{j=0}^{M-1} P(0,T_{j+1})(\bar{G}(0,T_j) - \bar{G}(T_{j+1}))}{\sum_{i=0}^{k} \beta_i\, P(0,T_i)\bar{G}(T_i)}. \tag{3.29}$$

Let the conditional default probability of default over (T_j, T_{j+1}) under the forward measure $\mathbb{T}$ be denoted by $\bar{g}_j$ which is given by

$$\bar{g}_j := \frac{\mathbb{E}^{\mathbf{T}_j}[\tau > T_j] - \mathbb{E}^{\mathbf{T}_{j+1}}[\tau > T_{j+1}]}{\mathbb{E}^{\mathbf{T}_j}[\tau > T_j]} = 1 - \frac{\bar{G}(0,T_{j+1})}{\bar{G}(0,T_j)}. \tag{3.30}$$

Then we have

$$\bar{G}(0,T_{j+1}) = \bar{G}(0,T_j) - \bar{G}(0,T_j)\bar{g}_j, \quad \bar{G}(0,0) = 1 \tag{3.31}$$

and $R_f(T)$ can be seen as a function of $(\bar{g}_0, \ldots, \bar{g}_M)$ and we can use an optimization procedure similar to the one that was used to extract $G(0,T)$ in case of independence between r_t and λ_t to extract $\bar{G}(T)$.

Once we have extracted $\bar{G}(T)$, we can obtain $\bar{P}(0,T)$ by using $\bar{P}(0,T) = P(0,T)\bar{G}(0,T)$. We can now use $\bar{P}(0,T)$ to calibrate the whole of $\alpha(t)$. Let us assume that r_t and λ_t are given by equations (3.24),(3.25) where we now assume that $Y1_t$ and $Y2_t$ are not independent. Then as in the case of the default free market(see (1.50) we have

$$\begin{aligned} -\partial_T \log_e \bar{P}(0,T) =& \mathbb{E}^{\bar{\mathbb{T}}}\Big[r_T + \lambda_T] \\ =& tr(\bar{V}(0,T)) + (\bar{M}(0,T) + \alpha(T))^{\mathsf{T}}(\bar{M}(0,T) + \alpha(T)) \end{aligned} \tag{3.32}$$

where $\bar{M}(0,T)$ is the mean vector under the defaultable forward measure[5] $\bar{\mathbb{T}}$ of $Y_t = (Y1_t, Y2_t)$ used to model r_t and λ_t and $\bar{V}(0,T)$ is the corresponding variance-covariance matrix. Let

$$\bar{F}(0,T) := -\partial_T \log_e \bar{P}(0,T) \tag{3.33}$$

denote the defaultable forward rate for maturity T at time $t = 0$. Hence (3.32) can be written as

$$\bar{F}(0,T) = tr(\bar{V}(0,T)) + (\bar{M}(0,T) + \alpha(T))^{\mathsf{T}}(\bar{M}(0,T) + \alpha(T)) \tag{3.34}$$

which is equivalent to

$$\tilde{F}(T) := \sqrt{\bar{F}(0,T) - Tr(\bar{V}(0,T))} = (\bar{M}(0,T) + \alpha(T))^{\mathsf{T}}(\bar{M}(0,T) + \alpha(T)). \tag{3.35}$$

To simplify the discussion we will now consider a two factor quadratic Gaussian model i.e. $Y_t = (Y_{1t}, Y_{2t})$ in (1.3). The first factor is used to model the short term rate of interest r_t so that we have $r_t = (Y_{1t} + \alpha_1(t))^2$. The second factor is used to model the intensity of default λ_t so that we have $\lambda_t = (Y_{2t} + \alpha_2(t))^2$. However the discussion that follows below can be easily extended to a multifactor quadratic Gaussian model where more than two factors are used for r_t or λ_t provided that the factors used to model r_t are not used to model λ_t and vice versa. In chapter 1, we were able to calibrate a multifactor quadratic Gaussian factor model to the default free forward rate term structure in closed form. Hence we can first calibrate $\alpha_1(t)$ to the default free forward rate term structure at time $t = 0$ using a closed form formula(see lemma 1.8. Next we need to calibrate to the defaultable forward rate term structure at time $t = 0$ through $\alpha_2(t)$. To continue our calibration procedure, we now have to consider how to linearize (3.35). Recall that in the case of the

[5] this is measure that corresponds to using $\bar{P}(0,T)$ as the numeraire

default free quadratic Gaussian factor model, we stated that in (1.62) that one way of defining $\tilde{F}(T)$ was to assume

$$\tilde{F}(T) = \sqrt{\frac{F(0,T) - V(0,T)}{n}}\mathbf{1}.$$

We cannot do a similar procedure when we try to linearize (3.35). Specifically let

$$\tilde{\tilde{F}}(T) = \begin{pmatrix} \tilde{\tilde{F}}_1(T) \\ \tilde{\tilde{F}}_2(T) \end{pmatrix}$$

denote a vector such that

$$\tilde{\tilde{F}}(T)^{\mathsf{T}}\tilde{\tilde{F}}(T) = \sqrt{\bar{F}(0,T) - Tr(\bar{V}(0,T))}$$

is true.

If we define

$$\tilde{\tilde{F}}(T) = \begin{pmatrix} \tilde{\tilde{F}}_1(T) \\ \tilde{\tilde{F}}_2(T) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\bar{F}(0,T)-Tr(\bar{V}(0,T))}{2}} \\ \sqrt{\frac{\bar{F}(0,T)-Tr(\bar{V}(0,T))}{2}} \end{pmatrix} \tag{3.36}$$

and proceed to derive the associated ODE for $\alpha(t)$ using the same techniques as in the proof of Theorem 1.8, we get the following system of linear ODE's

$$\begin{aligned} \frac{d}{dT}\alpha_1(T) &= a_{11}\,\alpha_1(T) + a_{12}\,\alpha_2(T) - a_{11}\,\tilde{\tilde{F}}_1(0,T) - a_{12}\,\tilde{\tilde{F}}_2(0,T) \\ &\quad +2\bar{V}_{11}(0,T)\tilde{\tilde{F}}_1(0,T) + 2\bar{V}_{12}(0,T)\tilde{\tilde{F}}_2(0,T) + \frac{d}{dT}\tilde{F}_1(0,T) \qquad (3.37) \\ \frac{d}{dT}\alpha_2(T) &= a_{21}\,\alpha_1(T) + a_{22}\,\alpha_2(T) - a_{21}\,\tilde{\tilde{F}}_1(0,T) - a_{22}\,\tilde{\tilde{F}}_2(0,T) \\ &\quad +2\bar{V}_{12}(0,T)\tilde{\tilde{F}}_1(0,T) + 2\bar{V}_{22}(0,T)\tilde{\tilde{F}}_2(0,T) + \frac{d}{dT}\tilde{F}_2(0,T) \qquad (3.38) \end{aligned}$$

whose solution is given by

$$\alpha_1(T) = \tilde{\tilde{F}}_1(T) + 2\,\exp(a_{11}\,T) \int_0^T \exp(-a_{11}\,r)(\bar{V}_{11}(0,r)\,\tilde{\tilde{F}}_1(r) + \bar{V}_{12}(0,r)\,\tilde{\tilde{F}}_2(r))\,dr \tag{3.39}$$

$$\alpha_2(T) = \tilde{\tilde{F}}_1(T) + 2\,\exp(a_{22}\,T) \int_0^T \exp(-a_{22}\,r)(\bar{V}_{11}(0,r)\,\tilde{\tilde{F}}_1(r) + \bar{V}_{12}(0,r)\,\tilde{\tilde{F}}_2(r))\,dr. \tag{3.40}$$

Using the result of Theorem 1.8, we can find the value of $\alpha_1(T)$ that can be used to calibrate the quadratic Gaussian model to the default free forward rate term structure. This formula is given by

$$\alpha_1(T) = \tilde{F}(T) + 2\,\exp(a_{11}\,T) \int_0^T \exp(-a_{11}\,r)V_{11}(0,r)\,\tilde{F}(r)\,dr \tag{3.41}$$

where $\tilde{F} = \sqrt{F(0,T) - V(0,T)}$. The problem with defining $\tilde{\tilde{F}}(T)$ as in (3.36) is that the solution of (3.37) given in (3.39) does not guarantee that $\alpha_1(t)$ is equal to (3.41). Therefore we have to define $\tilde{\tilde{F}}$ in such a way that the $\alpha_1(t)$ obtained through the calibration procedure to the default free forward rate term structure does not change. In fact this leads to a unique way of defining a vector $\tilde{\tilde{F}}(T)$:

$$\tilde{\tilde{F}}(T) := \begin{pmatrix} \sqrt{H(T)} \\ \sqrt{F(0,T) - Tr(\bar{V}(0,T)) - H(T)} \end{pmatrix} \tag{3.42}$$

for some function $H(T)$ which we need to solve for in the following. Using the definition given in (3.42), we can linearize (3.35) and proceed to derive the associated ODE for $\alpha(t)$ by using procedure that was used in the proof of Theorem 1.8. This gives us the ODE given in (3.37) whereby we now use the new definition (3.42) for $\tilde{\tilde{F}}(T)$. If $H(T)$ was a known function at this point, we can then proceed to solve

(3.37) using the same steps used in the proof of Theorem 1.8 to obtain

$$\begin{aligned}\alpha_1(T) =&\sqrt{H(T)} + 2\,\exp(a_{11}\,T)\int_0^T \exp(-a_{11}\,r)(\bar{V}_{11}(0,r)\,\sqrt{H(r)}+\\ &+\bar{V}_{12}(0,r)\sqrt{\bar{F}(0,r)-Tr(\bar{V}(0,r))-H(r)}\,dr \qquad (3.43)\\ \alpha_2(T) =&\sqrt{\bar{F}(0,T)-Tr(\bar{V}(0,T))-H(T)}+\\ &+2\,\exp(a_{22}\,T)\int_0^T \exp(-a_{22}\,r)(\bar{V}_{21}(0,r)\,\sqrt{H(r)}+\\ &+\bar{V}_{22}(0,r)\sqrt{\bar{F}(0,r)-Tr(\bar{V}(0,r))-H(r)}\,,dr. \qquad (3.44)\end{aligned}$$

Since $H(T)$ is not known but $\alpha_1(t)$ is known, we can regard (3.43) as an integral equation for $H(T)$. Once we solve this equation we can give the solution for $\alpha_2(t)$ using (3.44). Since a solution of (3.43) is also a solution of the ODE given in (3.37) with $\tilde{\bar{F}}(T)$ defined now as in (3.42), we will obtain an exact calibration to both the default free and defaultable term structures. The equation given in (3.43) is a nonlinear Volterra integral equation of the second kind in the unknown $H(T)$. We can convert (3.43) into a nonlinear first order differential equation using differentiation as the proof of the following theorem shows.

Theorem 3.2. *In a two factor quadratic Gaussian factor model, we have a perfect calibration to the default free term structure given by the price of default free zero coupon bonds through a closed form formula as given in theorem 1.8 and a perfect calibration to the term structure of survival probabilities which are extracted from CDS quotes through the numerical solution of the following first order non-linear ODE:*

$$\begin{aligned}\frac{1}{2}\frac{d}{dT}H(T) + (2\,\bar{V}_{11}(0,T)-a_{11})H(T) + \Big((a_{11}-2V(0,T))\tilde{\bar{F}}(T) - \frac{d}{dT}\tilde{\bar{F}}(T)+\\ +2\,\bar{V}_{12}(0,T)\sqrt{\bar{F}(0,T)-Tr(\bar{V}(0,T))-H(T)}\Big)\sqrt{H(T)} = 0 \quad (3.45)\end{aligned}$$

$H(T) > 0, \quad T \in [0,T^*], \quad H(0) = \alpha_1(0)^2.$

If we assume independence between r_t and λ_t, the exact solution of (3.45) *is given by*

$$H(T) = \tilde{F}(T) = \sqrt{F(0,T) - V(0,T)}$$

where $F(0,T)$ is default free instantaneous forward rate and $V(0,T)$ is the default free variance of Y_{1t} under $\mathbb{T}$.

Proof. As the discussion preceding this theorem shows, we only need to solve the Volterra integral equation of the second kind given in (3.43). We now show that (3.43) is equivalent to the solution of the first order non-linear ODE given by (3.45). From (3.43) we get the following equality

$$\alpha_1(T) - \sqrt{H(T)} = 2\exp(a_{11}T)\int_0^T \exp(-a_{11}r)(\bar{V}_{11}(0,r)\sqrt{H(r)} + \bar{V}_{12}(0,r)\sqrt{\bar{F}(0,r) - Tr(\bar{V}(0,r)) - H(r)}\,dr. \quad (3.46)$$

We now differentiate both sides of the integral equation (3.43) to obtain

$$\frac{d}{dT}\alpha_1(T) = \frac{\frac{d}{dT}H(T)}{2\sqrt{H(T)}} + 2a_{11}\exp(a_{11}T)\int_0^T \exp(-a_{11}r)(\bar{V}_{11}(0,r)\sqrt{H(r)} + \bar{V}_{12}(0,r)\sqrt{\bar{F}(0,r) - Tr(\bar{V}(0,r)) - H(r)})\,dr + 2V_{11}(0,T)\sqrt{H(T)} + 2V_{12}(0,T)\sqrt{\bar{F}(0,T) - Tr(V(0,T)) - H(T)}. \quad (3.47)$$

Using (3.46) to simplify (3.47), we get

$$\frac{\frac{d}{dT}H(T)}{2\sqrt{H(T)}} + a_{11}(\alpha_1(T) - \sqrt{H(T)}) + 2\bar{V}_{11}(0,T)\sqrt{H(T)} + 2\bar{V}_{12}(0,T)\sqrt{\bar{F}(0,T) - Tr(\bar{V}(0,T)) - H(T)} - \frac{d}{dT}\alpha_1(T) = 0. \quad (3.48)$$

We can further simplify (3.48) by using

$$\frac{d}{dT}\alpha_1(T) = \frac{d}{dT}\tilde{F}(T) + a_{11}(\alpha_1(T) - \tilde{F}(T)) + 2V(0,T)\tilde{F}(T) \quad (3.49)$$

which can be obtained from theorem 1.8 by differentiation and using the fact that

$$\alpha_1(T) - \tilde{F}(T) = 2\exp(a_{11}T)\int_0^T \exp(-a_{11}s)V(0,s)\tilde{F}(s)\,ds. \quad (3.50)$$

This gives us

$$\frac{\frac{d}{dT}H(T)}{2\sqrt{H(T)}} - a_{11}\sqrt{H(T)} + 2\,\bar{V}_{11}(0,T)\sqrt{H(T)} +$$
$$+ 2\,\bar{V}_{12}(0,T)\sqrt{\bar{F}(0,T) - Tr(\bar{V}(0,T)) - H(T)} - \frac{d}{dT}\tilde{F}(T) +$$
$$+ a_{11}\tilde{F}(T) - 2\,V(0,T)\tilde{F}(T) = 0. \quad (3.51)$$

Assuming $H(T)$ is a positive function, we now multiply both sides of (3.51) by $\sqrt{H(T)}$ and rearrange the terms to get (3.45) given in the lemma. In general (3.45) can be solved efficiently using numerical methods for first order ODE's. If we assume that the instantaneous correlation ρ between Y_{1t} and Y_{2t} is equal to zero i.e. the Brownian motions used to model Y_{1t} and Y_{2t} are independent, then

$$\bar{V}_{11}(0,T) = V(0,T) \quad (3.52)$$
$$\bar{V}_{12}(0,T) = 0. \quad (3.53)$$

is true. Therefore (3.45) becomes a simpler equation given by

$$\frac{1}{2}\frac{d}{dT}H(T) + (2\,\bar{V}_{11}(0,T) - a_{11})H(T) +$$
$$+ \left((a_{11} - 2V(0,T))\tilde{F}(T) - \frac{d}{dT}\tilde{F}(T)\right)\sqrt{H(T)} = 0 \quad (3.54)$$

It is easy to verify that

$$H(T) = F(0,T) - V(0,T) = \tilde{F}(T)^2$$

is a solution of (3.54) by substituting this value into the ODE. Substituting $\tilde{F}(T)^2$ for $H(T)$ in (3.44), we obtain $\alpha_2(T)$. Note that in this case

$$\bar{F}(0,T) = F(0,T) + G(0,T)$$

and

$$\sqrt{\bar{F}(0,T) - V(0,T) - \bar{V}_{22}(0,T) - \tilde{F}(T)^2} = \sqrt{G(0,T) - \bar{V}_{22}(0,T)}.$$

Hence $\alpha_1(T)$ is calibrated to the default free forward rate term structure while

$\alpha_2(T)$ is calibrated to the term structure of survival probabilities $G(0,T)$. $\square$

Remark 3.3. *Assuming that we use different factors to model r_t and λ_t, we can extend Theorem 3.2 to a quadratic Gaussian model where more than two factors are used to model r_t and λ_t. From the discussion preceding Theorem 3.2, we can see that we have to consider a vector $H(T)$ and using the method used in the proof of Theorem 3.2, we obtain a system of non-linear ODE's which have to be solved numerically.*

We now give numerical results to show that we can calibrate the drift terms of r_t and λ_t using CDS quotes and the default free term structure extracted from zero coupon bonds and default free swap rates. We still consider the two factor model where $r_t = (Y_{1t}+\alpha_1(t))^2$ and $\lambda_t = (Y_{2t}+\alpha_2(t))^2$ to carry out the numerical work. In the case of independence between r_t and λ_t we can use closed form formulas to do this calibration. Therefore we consider the case when r_t and λ_t are not independent where we have to numerically solve the nonlinear scalar ODE given by (3.45) in lemma 3.2. We use the default free zero coupon bond data given in Table 3.1 and the CDS quotes data given in Table 3.2 which is obtained from Martin et al. (2001) to test the calibration. We assume that the recovery rate is 30% and therefore the default payment Z is equal to $1 - 0.3 = 0.7$. For example the first row of Table 3.2 states that the market CDS rate for a one year protection against default is 0.0045 which is equivalent to a quote of 45 basis points. Using the optimization procedure described in this section(see (3.26) and the following paragraphs), we can extract the probability of default under the default free forward measure which we denoted by $\bar{G}(0,T)$ by using just the default free zero coupon data and the CDS quotes. The probability of default under $\mathbb{T}$ is obtained by just assuming correlation between r_t and λ_t without having to specify a specific value. We assume that the premium payments are made annually and therefore $\beta_i = 1, i = 1,\ldots,r$ for a CDS with maturity of r years. Since requiring a smoother $\bar{G}(0,T)$ will make the optimization procedure favor smoothness instead of matching exactly the CDS quotes, we need to have a higher level of discretization of the integral corresponding to the default leg of the CDS(see (3.3) and the paragraph before (3.26)). Therefore we chose $\Delta = 0.0625$ which is smaller than the values used for Δ in Martin et al. (2001) where λ_t is assumed to be deterministic. The value of $\nu = 10$ was chosen for the smoothness parameter and $\sigma_k = 10^{-4}, k = 1,\ldots,r$ in the objective function to minimize which is given in equations (3.14) and (3.15). In the case of correlation between r_t and λ_t, we have to make more approximations to the CDS rate as demonstrated in (3.26). Therefore what we are calibrating to is not the exact CDS rate but an approximation. In Table 3.3 we give this approximation next to the exact CDS quotes under the column with heading "Calib CDS". In the last column of table 3.3, we give the value of the CDS rate in a two factor quadratic Gaussian model whereby we use the formula given in (3.3) to calculate this value. To use (3.3) we have to assume specific values for the speed of mean reversion matrix A, the instantaneous volatility Σ and the correlation ρ in the two factor model we are considering:

$$dZ_t = (\alpha(t) + AY_t)dt + \Sigma dW_t \tag{3.55}$$

where A and Σ are constant square matrices given by

$$A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix},$$

$$\Sigma\Sigma^{\mathsf{T}} = \begin{bmatrix} \sigma_{11}^2 & \rho\sigma_{11}\sigma_{22} \\ \rho\sigma_{11}\sigma_{22} & \sigma_{22}^2 \end{bmatrix}.$$

In practice these parameters are obtained by calibrating to default free option data such as caps, floors or swaptions as well as options on credit sensitive securities such as options on credit default swaps. However this task is made difficult by the fact that options on credit sensitive securities is not liquidly available. Here we would like to show that for a high value of instantaneous correlation between Y_{1t} and Y_{2t}, we can still calibrate well to CDS quotes. We know that we have exact calibration to the default free term structure under the quadratic Gaussian model and hence this shows that we can also calibrate to the defaultable term structure even if r_t and λ_t are not independent. The specific parameters chosen are:

$$a_{11} = 0.01, a_{22} = 0.09, \sigma_{11} = 0.04, \sigma_{22} = 0.04, \rho = 0.9. \tag{3.56}$$

Unlike the multifactor affine factor model for default free and defaultable markets where we do not have closed form formulas for calibration, the multi factor quadratic Gaussian model for default free and defaultable markets inable us to calibrate exactly to the default free and defaultable term structures as the results in Table 3.3 and Figure 3.5 show. However for a numerical procedure of calibrating multifactor affine model to default free bonds and CDS quotes see Brigo and Alfonsi (2004).

T	$P(0,T)$
0.	1
1	0.93182
2	0.866762
3	0.806772
4	0.750876
5	0.699114
6	0.650255
7	0.604807
8	0.562855
9	0.523594
10	0.487314

Tab. 3.1: Zero Rates

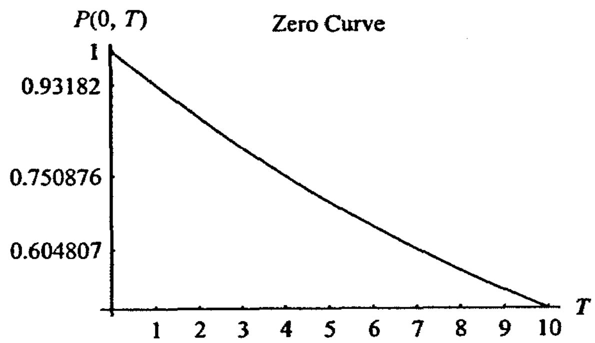

Fig. 3.2: Discount curve

T	CDS rate
1	0.0045
2	0.0055
3	0.0065
4	0.0070
5	0.0095
7	0.0105
10	0.0115

Tab. 3.2: CDS quotes

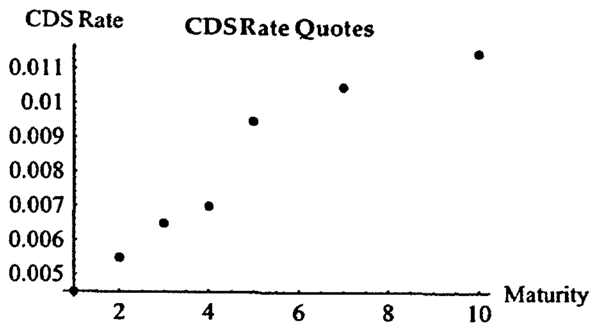

Fig. 3.3: CDS quotes given as basis points $* 10^{-4}$

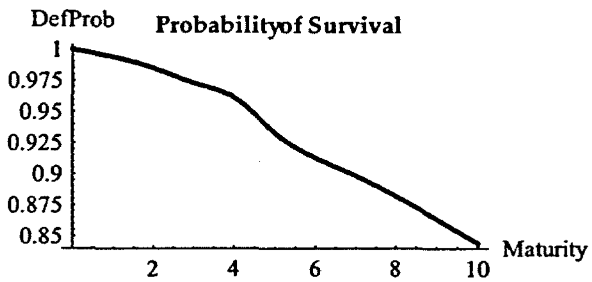

Fig. 3.4: Extracted Survival Probability Under Correlation $\Delta = 0.0625$

Mat.	CDS Quote	Calib CDS	QG CDS
1	0.0045	0.00448017	0.00448906
2	0.0055	0.00547543	0.00548332
3	0.0065	0.00647092	0.0064748
4	0.007	0.00696875	0.00696499
5	0.0095	0.00945755	0.00944627
7	0.0105	0.010453	0.0104067
10	0.0115	0.0114484	0.0113335

Tab. 3.3: Calibration results to CDS quotes

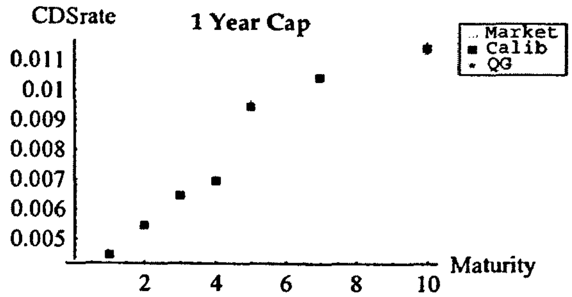

Fig. 3.5: Calibration results to CDS quotes

3.2 Pricing Credit Default Swaptions

In this section we discuss the pricing of an option on a CDS also known as a credit default swaption by some members of the financial market. First we show how we can calculate the price of a credit default swaption accurately in a multivariate quadratic Gaussian factor model. We then derive different analytic approximations for the price of a credit default swaptions. In order to get closed form formulas, we will assume as in previous sections that the dynamics of the Gaussian Ornstein Uhlenbeck process Y_t is given by

$$dY_t = AY_t\,dt + \Sigma\,dW_t \tag{3.57}$$

where A is a constant diagonal matrix and Σ is a constant matrix and the state variables are given by $Z_t = \alpha(t) + Y_t$.

Definition 3.4. *An option to enter a credit default swap at a future time T_n gives the buyer of the option the right but not the obligation to enter into a CDS agreement with the receiver at time T_n by paying a premium of K at times $\mathcal{T}_{n,N} = T_{n+1}, \ldots, T_N$ in return for a protection payment of Z if the referenced credit defaults before the maturity $T_N > T_n$ of the credit default swap. At default time $T_n < \tau \leqslant T_N$ the contract is terminated and the receiver receives the accrued amount $\tau - T_i$ where T_i is the payment immediately preceding τ. This option contract is only valid if the reference credit does not default until time T_n. If the reference credit defaults by time T_n, the option contract is terminated with no exchange of payments.*

Only if at the time of maturity of the T_n the prevailing market CDS rate $R_f(T_n)$ is above K will the buyer of the option find it beneficial to exercise this option.

The market CDS rate is set in such a way such that

$$CDS(T_n, \mathcal{T}_{n,N}, T, R_f(T_n), Z) = 0.$$

Hence at time T_n, the value of the CDS underlying the option which is given by

$$CDS(T_n, \mathcal{T}_{n,N}, T, K, Z) \tag{3.58}$$

will be positive if $K < R_f(T_n)$. Therefore the payoff of the option at time T_n

$$CDS(T_n, \mathcal{T}_{n,N}, T, K, Z) \tag{3.59}$$

will need to be positive so that the buyer of the credit default swaption finds it beneficial to exercise the option. Let the value of the credit default swaption be denoted by $CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)$, then at time $t < T_n$

$$CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) := \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^{T_n} r_s\, ds\Big) \\ (CDS(T_n, \mathcal{T}_{n,N}, T, K, Z))^{+}|\mathcal{F}_t\Big] \tag{3.60}$$

Ignoring the accrued premium term this becomes

$$\begin{aligned} CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) &= \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^{T_n} r_s\, ds\Big) \\ &\mathbf{1}_{\tau > T_n}\Big(Z\int_{T_n}^{T_N} \bar{P}(T_n, s)\, \mathbb{E}^{\bar{\mathbf{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\, ds - K\sum_{i=n+1}^{N} \beta_i\, \bar{P}(T_n, T_i)\Big)^{+}\Big|\mathcal{F}_t\Big] \\ &= \mathbf{1}_{\tau > t}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^{T_n} r_s + \lambda_s\, ds\Big)\Big(Z\int_{T_n}^{T_N} \bar{P}(T_n, s)\, \mathbb{E}^{\bar{\mathbf{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\, ds - \\ &- K\sum_{i=n+1}^{N} \beta_i\, \bar{P}(T_n, T_i)\Big)^{+}\Big|\mathcal{F}_t\Big] = \mathbf{1}_{\tau > t}\bar{P}(t, T_n)\mathbb{E}^{\bar{\mathbf{T}}_n}\Big[\Big(Z\int_{T_n}^{T_N} \bar{P}(T_n, s)\, \mathbb{E}^{\bar{\mathbf{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\, ds - \\ &- K\sum_{i=n+1}^{N} \beta_i\, \bar{P}(T_n, T_i)\Big)^{+}\Big|\mathcal{F}_t\Big] \end{aligned} \tag{3.61}$$

where $\bar{\mathbb{T}}_n$ is used to denote $\bar{\mathbb{T}}_{T_n}$.

The value of the CDS underlying the credit default swaption in (3.61) is given by

$$Z \int_{T_n}^{T_N} \bar{P}(T_n, s)\, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_N}]\, ds - K \sum_{i=n+1}^{T_N} \beta_i\, \bar{P}(T_n, T_i). \tag{3.62}$$

The value (3.62) is random as seen with respect to the filtration $\mathcal{F}_t$ since Y_{T_n} is part of the formulas for $\bar{P}(T_n, s), \bar{P}(T_n, T_i)$ and $\mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]$. However given Y_{T_n}, we can calculate (3.62) to a great degree of accuracy in the quadratic Gaussian factor model. Since Y_{T_n} is a Gaussian Ornstein Uhlenbeck process and we know the mean and variance-covariance matrix of Y_{T_n} under the defaultable forward measure $\bar{\mathbb{T}}_n$(see lemma 1.4), we can use Monte Carlo simulation to calculate $CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)$. In Brigo & Alfonsi (2004), it is indicated that we need a large number of Monte Carlo simulations because a CDS has a large variance. The conditional distribution of Y_{T_n} under $\bar{\mathbb{T}}_n$ and conditional on $\mathcal{F}_t$ is multivariate Gaussian and we know the mean and variance in closed form. Therefore if the dimension of the Y_{T_n} is of low order, we can calculate $CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)$ much faster using multidimensional integration by directly integrating the payoff (3.62) times the multivariate normal distribution representing the probability density function of Y_{T_n} under $\bar{\mathbb{T}}_n$:

$$\begin{aligned} CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) = \bar{P}(t, T_n) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Bigg(Z \int_{T_n}^{T_N} \bar{P}(T_n, s)\, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\, ds \\ - K \sum_{i=n+1}^{T_N} \beta_i\, \bar{P}(T_n, T_i)\Bigg)^{+} \\ \frac{1}{2\pi|\bar{V}(0, T_\alpha)|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(Y_{T_\alpha} - \overline{M}(0, T_\alpha))^{\mathsf{T}} \bar{V}(0, T_\alpha)^{-1} (Y_{T_\alpha} - \overline{M}(0, T_\alpha))\right) dY_{T_\alpha} \end{aligned} \tag{3.63}$$

where we have to discretize the integral

$$\int_{T_n}^{T_N} \bar{P}(T_n, s)\, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\, ds. \tag{3.64}$$

The multidimensional integral (3.63) can be done efficiently by using cubature techniques as shown in Cools and Haegemans (2003).

We have indicated how we can calculate the price of a credit default swaption to a great degree of accuracy if we use multidimensional integration or Monte Carlo simulation. However this task can be computationally demanding for a multifactor model where the number of factors used to model the interest rate and intensity is more than two. Even in the case of a two factor model it is better to find an analytic approximation for the price of a credit default swaption instead of using multidimensional integration or Monte Carlo simulation. We now show how to approximate the price of a credit default swaption using closed formulas involving the numerical inversion of Fourier transforms. To derive the first analytic approximation, we first rewrite (3.61) as

$$CDS_{OP}(t, T_n, T_{n,N}, T, K, Z) =$$

$$\mathbf{1}_{\tau>t}\bar{P}(t,T_n)\mathbb{E}^{\bar{\mathbb{T}}_n}\left[\sum_{i=n+1}^{N}\beta_i\,\bar{P}(T_n,T_i)\left(\frac{Z\int_{T_n}^{T_N}\bar{P}(T_n,s)\,\mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\,ds}{\sum_{i=n+1}^{T_N}\beta_i\,\bar{P}(T_n,T_i)}-K\right)^+\middle|\mathcal{F}_t\right]. \tag{3.65}$$

We now approximate the integral

$$\int_{T_n}^{T_N}\bar{P}(T_n,s)\,\mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\,ds \tag{3.66}$$

using a right Riemann sum. Hence we divide the interval $[T_n, T_N]$ into M subintervals by choosing $\delta = \frac{T_N - T_n}{M}$ and $T_{n+j} = T_n + \delta j, j = 1, \dots, M$ to get .

$$\frac{\delta Z\sum_{j=n+1}^{N}\bar{P}(T_n,T_j)\,\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]\,ds}{\sum_{i=n+1}^{T_N}\beta_i\,\bar{P}(T_n,T_i)} \tag{3.67}$$

where $\bar{\mathbb{T}}_j$ is used to denote $\bar{\mathbb{T}}_{T_j}$. Using

$$\bar{w}_j(T_n) := \frac{\bar{P}(T_n,T_j)}{\sum_{i=n+1}^{T_N}\beta_i\,\bar{P}(T_n,T_i)}, j = n+1, \dots, N \tag{3.68}$$

and ignoring the error introduced by the discretization (3.67), we can now express (3.65) as

$$
\begin{aligned}
&CDS_{OP}(t, T_n, T_{n,N}, T, K, Z) = \\
&\mathbf{1}_{\tau>t}\bar{P}(t,T_n)\mathbb{E}^{\bar{\mathbb{T}}_n}\left[\sum_{i=n+1}^{N}\beta_i\,\bar{P}(T_n,T_i)\left(\delta Z\sum_{j=n+1}^{N}\bar{w}_j(T_n)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]-K\right)^+\Bigg|\mathcal{F}_t\right] \\
&= \mathbf{1}_{\tau>t}\mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(-\int_t^{T_n} r_s+\lambda_s\,ds\right)\sum_{i=n+1}^{N}\beta_i\,\bar{P}(T_n,T_i) \\
&\qquad\left(\delta Z\sum_{j=n+1}^{N}\bar{w}_j(T_n)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]-K\right)^+\Bigg|\mathcal{F}_t\Bigg]. \qquad (3.69)
\end{aligned}
$$

Let $D(t)$ denote the default free savings account (see (1.8)) which is given by

$$D(t) = \exp\left(\int_0^t r_s\,ds\right).$$

In the following we consider a change of measure that would simplify the calculation of (3.69). First we define the defaultable present value of a basis point(DPVBP) by

$$U_{n,N}(t) := \mathbf{1}_{\tau>T_n}\sum_{i=n+1}^{N}\beta_i\bar{P}(t,T_i). \qquad (3.70)$$

Consider the measure that is absolutely continuous to $\mathbb{Q}$ which is defined by the Radon-Nikodým density:

$$\frac{d\mathbb{U}}{d\mathbb{Q}}\bigg|_{\mathcal{G}_{T_n}} = \frac{U_{n,N}(T_n)}{D(T_n)}\frac{D(0)}{U_{n,N}(0)} = \mathbf{1}_{\tau>T_n}\frac{\sum_{i=n+1}^{N}\beta_i\bar{P}(T_n,T_i)}{D(T_n)}\frac{D(0)}{\sum_{i=n+1}^{N}\beta_i\bar{P}(0,T_i)}. \qquad (3.71)$$

Now using the abstract Bayes formula and (3.71) we can rewrite (3.69) as

$$
\begin{aligned}
&CDS_{OP}(t, T_n, T_{n,N}, T, K, Z) = \\
&\qquad U_{n,N}(t)\,\mathbb{E}^{\mathbb{U}}\left[\left(\delta Z\sum_{j=n+1}^{N}\bar{w}_j(T_n)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]-K\right)^+\Bigg|\mathcal{F}_t\right]. \qquad (3.72)
\end{aligned}
$$

We now show how we can calculate

$$\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]. \tag{3.73}$$

Under the assumption of a multifactor quadratic Gaussian model, the interest rate and the intensity of default are quadratic forms in Y_T. We partition the vector Y_T into two disjoint parts consisting of Y_T^1 and Y_T^2 and for ease of exposition assume r_T and λ_T are given by (3.24) and (3.25) respectively. We denote by $\overline{M}^i(t,T)$ and $\bar{V}^i(t,T), i = 1, 2$ the conditional mean and variance-covariance matrix of $Y_T^i, i = 1, 2$ under the measure $\bar{\mathbb{T}}_T$ and with respect to the sigma field $\mathcal{F}_t$. Using lemma 1.4, we can find the mean and variance-covariance matrix of Y_T under the measure $\bar{\mathbb{T}}_T$ and conditional on $\mathcal{F}_t$ which are denoted by $\overline{M}(t,T)$ and $\bar{V}(t,T)$ respectively:

$$\bar{V}(t,T) = \overline{Pv}(T-t)\overline{Qv}^{-1}(T-t) \tag{3.74}$$

$$\overline{M}(t,T) = \overline{Qv}^{-1}(t,T)^{\top}Y_t + 2\overline{Qv}^{-1}(t,T)^{\top}\int_t^T \overline{Pv}(t,s)^{\top}\alpha(s)\, ds \tag{3.75}$$

where $(\overline{Qv}(T), \overline{Pv}(T))^{\top}$ is the solution of the following system of linear differential equations

$$\begin{pmatrix} \partial_T\overline{Qv}(T) \\ \partial_T\overline{Pv}(T) \end{pmatrix} = \begin{pmatrix} -A^{\top} & 2I \\ \Sigma\Sigma^{\top} & A \end{pmatrix}\begin{pmatrix} \overline{Qv}(T) \\ \overline{Pv}(T) \end{pmatrix}.$$

So if in particular $t = T_n$ and $T = T_j$, we can find the mean and variance-covariance matrix of $Y_{T_j}^2$ under the measure $\bar{\mathbb{T}}_j, T_j > T_n$ and with respect to the sigma field $\mathcal{F}_{T_n}$ which we denoted by $\overline{M}^2(T_n, T_j)$ and $\bar{V}^2(T_n, T_j)$ from $\overline{M}(T_n, T_j)$ and $\bar{V}(T_n, T_j)$. This shows that we can find the value of (3.73) in closed form. The mean and variance-covariance matrix of Y_{T_n} under $\mathbb{U}$ can be calculated explicitly. In fact similar to (2.72) in chapter 2, using (3.71) and the abstract Bayes formula we have

$$\mathbb{E}_t^{\mathbb{U}}[Y_{T_j}] = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^{T_n} r_s + \lambda_s\,ds\right)\frac{\sum\limits_{i=n+1}^{N}\beta_i\bar{P}(T_n,T_i)}{\sum\limits_{k=n+1}^{N}\beta_k\bar{P}(t,T_k)}Y_{T_j}\right] \tag{3.76}$$

$$= \mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\sum\limits_{i=n+1}^{N}\beta_i\bar{P}(T_n,T_i)}{\sum\limits_{k=n+1}^{N}\beta_k\frac{\bar{P}(t,T_k)}{\bar{P}(t,T_n)}}Y_{T_j}\right]$$

$$= \sum_{i=n+1}^{N}\frac{\beta_i\bar{P}(t,T_i)}{\sum_{k=n+1}^{N}\beta_k\bar{P}(t,T_k)}\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n,T_i)}{\frac{\bar{P}(t,T_i)}{\bar{P}(t,T_n)}}Y_{T_j}\right]$$

$$= \sum_{i=n+1}^{N}\frac{\beta_i\bar{P}(t,T_i)}{\sum_{k=n+1}^{N}\beta_k\bar{P}(t,T_k)}\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n,T_i)}{\mathbb{E}^{\bar{\mathbb{T}}_n}[\bar{P}(T_n,T_i)]}Y_{T_j}\right]. \tag{3.77}$$

We can calculate

$$\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n,T_i)}{\mathbb{E}^{\bar{\mathbb{T}}_n}[\bar{P}(T_n,T_i)]}Y_{T_j}\right] \tag{3.78}$$

by applying lemma 1.14 (see (1.118) and (1.119)) as $\bar{P}(T_n,T_i)$ is log-quadratic Gaussian and therefore (3.76) can be calculated in closed form. Note however as in (2.71) in chapter 2, we can use Girsanov's theorem to show that the dynamics of Y_t does not correspond to a Gaussian process under the measure $\mathbb{U}$. We can get the mean and variance-covariance matrix of Y_t under $\mathbb{U}$ using the weighted mean and variance-covariance matrix of Y_t under each $\bar{\mathbb{T}}_i$ because Y_t is a Gaussian vector under each $\bar{\mathbb{T}}_i$. To facilitate the derivation of an analytic approximation to (3.72), we now replace the weights $\bar{w}_j(T_n)$ by their time zero values. This freezing of the weights has been used to derive analytic approximations in the valuation of default free securities (see Rebonato (1998)) and in the valuation of defaultable securities (see Brigo and Mercurio (2006)). Therefore

$$\bar{Q}_\lambda(T_n) := \delta Z\sum_{j=n+1}^{N}\bar{w}_j(0)\bar{Q}_{\lambda_i} = \delta Z\sum_{j=n+1}^{N}\bar{w}_j(0)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] \tag{3.79}$$

is a quadratic form in $Y_{T_n} = (Y^1_{T_n}, Y^2_{T_n})$. We now made an additional approximation by replacing Y_t under $\mathbb{U}$ by a Gaussian process $\widetilde{Y}_t$ which has mean and variance-covariance matrix equal to the exact mean and variance-covariance matrix of Y_t

under $\mathbb{U}$. Therefore an approximation to the credit default swaption can be given by

$$CDS_{OP}(t,T_n,T_{n,N},T,K,Z) \approx \widetilde{CDS}_{OP}(t,T_n,T_{n,N},T,K,Z) = \\ U_{n,N}(t)\mathbb{E}^{\mathbb{U}}\left[\left(\delta Z \sum_{j=n+1}^{N} \bar{w}_j(0)\, Q_F(\widetilde{Y}_{T_n}) - K\right)^+ \middle| \mathcal{F}_t\right] \quad (3.80)$$

where the $Q_F(\widetilde{Y}_{T_n})$ denotes the quadratic form in $\widetilde{Y}_{T_n}$ which is obtained from (3.80) by replacing Y_{T_n} in (3.79) by $\widetilde{Y}_{T_n}$. This approximation is an efficient one since we only have to work with the characteristic function of a single quadratic form in Gaussian random variables[6]. Under this assumption (3.79) can be approximated by a quadratic form in Gaussian random variables. Since we can calculate the characteristic function of quadratic forms in Gaussian random variables(see lemma B.1) we can now apply the Fourier transform method which is based on transforming the approximate price of the credit default swaption which is given by (3.80) with respect to the strike price(see Carr and Madan (1998),Lee (2004)). This is similar to what we have done in approximating the price of swaptions in chapter 2 but here we do not have to approximate a sum of log-quadratic Gaussian processes by a log-quadratic process but only freeze the weights $\bar{w}_j(t)$ and approximate the non Gaussian dynamics of Y_t under $\mathbb{U}$ by a Gaussian process with matching mean and variance-covariance matrix. For ease of exposition we now discuss how to calculate (3.80) for $t = 0$. Let Ω denote a quadratic form in Gaussian random variables $\Phi^{\mathbb{U}}(\Omega, z)$ denote the characteristic function of Ω under $\mathbb{U}$

$$\Phi^{\mathbb{U}}(\Omega, z) := \mathbb{E}^{\mathbb{U}}[\exp(iz\Omega)]. \quad (3.81)$$

The payoff function associated with (3.80) is of type

$$G_2(x,k) := (x-k)^+. \quad (3.82)$$

For such a payoff function Lee (2004) gives a method to calculate the inverse transform with error bounds. First we give some definitions and notations similar to the ones given for pricing default free swaptions. Let $\hat{\alpha} > 0$ and $C_{\hat{\alpha},G_2}(K)$ denote

[6] When discussing the numerical results later in this section, we will show that we can find the exact characteristic function of (3.79) under $\mathbb{U}$ but it turns out in addition to be computationally less efficient, the approximation tends to have generally more error.

the dampened price of the option[7] with payoff $G_2(x,k)$

$$\mathcal{C}_{\hat{\alpha},G_2}(K) := \exp(\hat{\alpha})\mathbb{E}^{U}[G_2(x,k)]. \tag{3.83}$$

Let $\hat{\mathcal{C}}_{G_2}(z)$ denote the Fourier transform of the damped option price with respect to the strike price K

$$\hat{\mathcal{C}}_{G_2}(z) := \int_{-\infty}^{\infty} \exp(izK)\, \mathcal{C}_{\hat{\alpha},G_2}(K)\, dK = \frac{-\Phi^U(x,z)}{z^2}. \tag{3.84}$$

Then from Lee (2004) the option price can be obtained by the following Fourier inversion

$$\mathcal{C}_{G_2}(K) = R_{\hat{\alpha},G_2} + \frac{1}{\pi}\int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i} \mathrm{Re}\Big[\hat{\mathcal{C}}_{G_2}(z)\exp(-izK)\Big]\, dz$$

where

$$R_{\hat{\alpha},G_2} = \begin{cases} -\bar{\Phi}'(0) - K\bar{\Phi}(0), & \hat{\alpha} < 0 \\ \frac{-\bar{\Phi}'(0)-K\bar{\Phi}(0)}{2}, & \hat{\alpha} = 0 \\ 0, & \hat{\alpha} > 0 \end{cases} \tag{3.85}$$

If $\hat{\alpha} = 0$, Lee (2004) suggests that we use

$$\mathcal{C}_{G_2}(K) = R_{\hat{\alpha},G_2} + \frac{1}{\pi}\int_{0}^{\infty}\left(\mathrm{Re}\Big[\hat{\mathcal{C}}_{G_2}(z)\exp(-izK)\Big] + \frac{1}{z^2}\right) dz$$

to avoid convergence problems[8]. The characteristic function of a quadratic Gaussian variable exists everywhere and therefore we can choose to dampen the option price or not. This enables us a choice of methods to minimize the error in the numerical inversion of the Fourier transform. For further details see Lee (2004). Thus we can calculate (3.80) in closed form up to an inversion of a Fourier transform.

We now derive another approximation which uses the quadratic form given in (3.79) to approximate the exercise boundary of the credit default swaption. From the first approximation to the price of a credit default swaption which is given by

[7] We mean here the price of the credit default swaption divided by the predefault value of the defaultable present value of a basis point.

[8] Numerical experiments show that there is still some difficulty when using $\hat{\alpha} = 0$.

(3.80), we can see that the credit default swaption is exercised if

$$\bar{Q}_{n,N}(T_n) := \delta Z \sum_{j=n+1}^{N} \bar{w}_j(0)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] \tag{3.86}$$

is greater than the exercise price K. We have already seen in the derivation of the first approximation that $\bar{Q}_{n,N}(T_n)$ is a quadratic form in T_n which can be calculated in closed form(see the discussion after (3.80)). Ignoring the accrued term, the exact price of the credit default swaption is given by (3.60). If we now discretize the integral corresponding to the default leg using a Riemann sum as in (3.67), we can approximate[9] the exact price of the credit default swaption(see (3.60)) under $\bar{\mathbb{T}}_n$ by

$$\begin{aligned} &CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) \approx \\ &\quad \mathbf{1}_{\tau>t}\bar{P}(t,T_n)\mathbb{E}^{\bar{\mathbb{T}}_n}\Bigg[\delta Z \sum_{j=n+1}^{N} \bar{P}(T_n,T_j)\, \mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] - \\ &\qquad\qquad - K \sum_{i=n+1}^{N} \beta_i\, \bar{P}(T_n,T_i)\Bigg)^{+}\Bigg|\mathcal{F}_t\Bigg]. \end{aligned} \tag{3.87}$$

As (3.87) is close to the exact price of the credit default swaption, we can approximate the exact price of a credit default swaption with maturity $T_n = T$ by approximating the exercise boundary through (3.86). Therefore we get a second approximation to the price of the credit default swaption which is given by:

$$\begin{aligned} &CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) \approx \widetilde{CDS2}_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) \\ &= \mathbf{1}_{\tau>t}\bar{P}(t,T_n)\mathbb{E}^{\bar{\mathbb{T}}_n}\Bigg[\delta Z \sum_{j=n+1}^{N} \bar{P}(T_n,T_j)\, \mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]\mathbf{1}_{\bar{Q}_{n,N}(T_n)>K} - \\ &\qquad - K \sum_{i=n+1}^{N} \beta_i\, \bar{P}(T_n,T_i)\mathbf{1}_{\bar{Q}_{n,N}(T_n)>K}\Bigg)^{+}\Bigg|\mathcal{F}_t\Bigg]. \end{aligned} \tag{3.88}$$

We have already seen that we can calculate the exact value of

$$\bar{Q}_{\lambda_i}(T_n) := \mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] \tag{3.89}$$

[9] We can assume that this error can be ignored as we can approximate the integral with high degree of accuracy for a value of the discretization level δ which is close enough to zero.

which is a quadratic form in Y_{T_n}. The price of defaultable zero coupon bond prices in the quadratic Gaussian model are log-quadratic Gaussian and therefore we can write

$$\bar{P}(T_n, T_k) = \exp(\bar{Q}_k(T_n)) \tag{3.90}$$

where

$$\bar{Q}_k(T_n) := \log_e(\bar{P}(T_n, T_k)) = -Y_{T_n}^{\mathsf{T}} C(T_n, T_k) Y_{T_n} - B(T_n, T_k)^{\mathsf{T}} Y_{T_n} - A(T_n, T_k).$$

Hence in (3.88) the value

$$\delta Z \sum_{j=n+1}^{N} \bar{P}(T_n, T_j) \, \mathbf{1}_{\bar{Q}_{n,N}(T_n) > K} \, \mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j} | \mathcal{F}_{T_n}] \tag{3.91}$$

is a sum of payoffs of type

$$G_4(x, K) := b_2.x \exp(b_1.x) \, \mathbf{1}_{b_0.x > K}, \quad x \in \mathbb{R}^n \tag{3.92}$$

where b_0, b_1 and b_2 are appropriately chosen. In particular

$$\begin{aligned} b_0 &= (0,0,1),\ b_1 = (1,0,0),\ b_2 = (0,1,0), \\ x &= (x_1, x_2, x_3) = (Q_j(T_n), \bar{Q}_{\lambda_i}(T_n), \bar{Q}_{n,N}(T_n)). \end{aligned}$$

Since we know the characteristic function of a quadratic form in Gaussian random variables under $\bar{\mathbb{T}}_n$ in closed form(see (2.23)), we can now calculate (3.91) using the Fourier transform method given in Lee (2004). For a payoff of type (3.92), Lee (2004) gives different ways of calculating the price of the corresponding option. The method is based on transforming the dampened option price with respect to the strike price K. The most efficient way of numerically inverting (3.96) is to use the result given in Lee (2004). For the particular case $x = (x_1, x_2, x_3)$ let us denote by $C_{G_4}(K)$ the option price which is normalized by the price of the defaultable zero coupon bond with maturity equal to the maturity of the option where the payoff is of type (3.92). Let $\hat{C}_{G_4}(z)$ denote the Fourier transform of $C_{G_4}(K)$(see Lee (2004) for details)

$$\hat{C}_{G_4}(z) = \frac{-b_2.\nabla\Phi(x_1, x_2, x_3, b_0 z - b_1 i)}{z} \tag{3.93}$$

where $\Phi(x_1, x_2, x_3, w_1, w_2, w_3)$ represents the joint characteristic function of three quadratic Gaussian random variables x_1, x_2, x_3(see (2.23)) evaluated at

$$(w_1, w_2, w_3) = b_0 z - b_1 i.$$

The problem of obtaining the option price through Fourier inversion is given by Lee (2004) for the payoff of type (3.92) and it is:

$$\mathcal{C}_{G_4}(K) = R_{\hat{\alpha},G_4} + \frac{1}{\pi} \int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i} \mathrm{Re}\left[\hat{\mathcal{C}}_{G_4}(z) \exp(-izK)\right] dz \tag{3.94}$$

where

$$R_{\hat{\alpha},G_4} = \begin{cases} -ib_2.\nabla\Phi(x_1, x_2, -b_1 i) = \nabla\Phi(x_1, x_2, x_3, -i, 0, 0), & \hat{\alpha} < 0 \\ \dfrac{-ib_2.\nabla\Phi(x_1, x_2, x_3, -b_1 i)}{2} = \dfrac{-ib_2.\nabla\Phi(x_1, x_2, x_3, -i, 0, 0)}{2}, & \hat{\alpha} = 0 \\ 0, & \hat{\alpha} > 0 \end{cases} \tag{3.95}$$

The choice of the dampening factor $\hat{\alpha}$ is not restricted by the domain of existence of the characterisitc function since the characteristic function of a quadratic form in Gaussian random variables exists everywhere. Hence we can choose not to dampen the option price by choosing $\hat{\alpha} = 0$. However there are advantages in dampening the option price by different values of $\hat{\alpha}$ depending on the strike price in order to minimize the error in the Fourier inversion which is needed to calculate the price of the option (see Lee (2004) for a detailed discussion). We do not investigate the error differences obtained by choosing different $\hat{\alpha}$ for the dampening factor but use a uniform value of $\hat{\alpha} = 1$ for the different range of strike prices in our numerical experiments to be presented later in this section. Therefore we can find the Fourier transform of the dampened value

$$\int_{-\infty}^{\infty} \exp(\hat{\alpha}K)\mathbb{E}^{\bar{\mathbf{T}}_n}\left[\delta Z \sum_{j=n+1}^{N} \bar{P}(T_n, T_j)\, \mathbf{1}_{\bar{Q}_{n,N}(T_n)>K} \mathbb{E}^{\bar{\mathbf{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]\middle|\mathcal{F}_t\right] \exp(izK)\, dK \tag{3.96}$$

with respect to the strike price K in closed form and numerically invert the Fourier transform using (3.94) and (3.95). The calculation of the dampened value

$$\int_{-\infty}^{\infty} \exp(\hat{\alpha}K)\mathbb{E}^{\bar{T}_n}\left[K \sum_{i=n+1}^{N} \beta_i \bar{P}(T_n, T_i)\, \mathbf{1}_{\bar{Q}_{n,N}(T_n)>K} \middle| \mathcal{F}_t\right] \exp(izK)\, dK \quad (3.97)$$

involves payoffs of type

$$G_3(x, K) := \exp(b_1.x)\, \mathbf{1}_{b_0.x>K}, \quad x \in \mathbb{R}^n \qquad (3.98)$$

where

$$b_0 = (0,1), b_1 = (1,0), x = (\bar{Q}_i(T_n), \bar{Q}_{n,N}(T_n)).$$

We have already discussed in chapter 2 in the context of the approximation of the price of default free swaptions how to calculate option prices involving payoffs of type (3.98). Therefore we refer the reader to the discussion following (2.17) in chapter 2. Thus we have shown how to calculate the numerical inversion of the Fourier transform with respect to the strike price K of (3.88) which is formally given by:

$$\widehat{\widetilde{CDS2}}_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)(z) =$$
$$\int_{-\infty}^{\infty} \exp(izK) \exp(\hat{\alpha}K)\, \widetilde{CDS2}_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)\, dK. \quad (3.99)$$

The approximation of the price of the credit default swaption given by (3.88) is given by the numerical Fourier inversion and removal of the dampening[10] through the following formula:

$$\widetilde{CDS2}_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) =$$
$$\int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i} \mathrm{Re}[\widehat{\widetilde{CDS2}}_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)(z) \exp(-iKz)]\, dz. \quad (3.100)$$

We have discussed above how we can calculate (3.100) through the numerical inversion of several Fourier transforms and therefore we can now claim that we can calculate (3.88) in closed form[11]. Note however that we have to invert several

[10] For a detailed discussion see Lee (2004)
[11] By closed form, we mean up to a numerical inversion of the closed form Fourier transform

Fourier transforms depending on the discretization level δ of the integral given in (3.87) and on the coverage β_i for the premium leg in (3.97). Therefore this method takes more time to compute the approximate price of a credit default swaption. For numerical experiments we took a discretization level of $\delta = 0.0625$ in (3.67) which requires a larger number of numerical Fourier inversions. We can take $\delta = 0.25$ to obtain a price that differs from that of a finer discretization by a couple of basis points and therefore the computation can be speeded up. Moreover we believe that instead of using quadrature to do the numerical inversion of the Fourier transforms, we can use the discrete Fourier transform or the fast Fourier transform to obtain significant speed up of the inversion. Moreover as the number of factors increases to more than 2, the dimension of the multidimensional integral in (3.63) also increases by the same amount and cubature methods are slower while the approximation (3.88) can still be implemented efficiently through the numerical Fourier inversion. From numerical experiments given later in this section, we can see that the implementation of (3.88) through the Fourier technique discussed is much more accurate than (3.80) especially when the maturity of the credit default swaption is far from the present date $t = 0$.

We now give a third approximation for the exact price of a credit default swaption as given in (3.60) by using the formulation given in (3.80). Instead of making the assumption that we can approximate (3.79) by a quadratic form in a Gaussian random vector such that the Gaussian random vector has a mean and variance-covariance matrix equal to the exact mean and variance-covariance matrix of Y_{T_n} under $\mathbb{U}$, we calculate

$$CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) = \\ U_{n,N}(t)\,\mathbb{E}^{\mathbb{U}}\left[\left(\delta Z \sum_{j=n+1}^{N} \bar{w}_j(0)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] - K\right)^{+}\Big|\mathcal{F}_t\right]. \quad (3.101)$$

using a Gram Charlier series (see the discussion given in section 2.4 of chapter 2). The Gram Charlier series is used to approximate the density of (3.79). Therefore we calculate the exact higher moments of (3.79) as in (3.76) through a weighted

of the option price which can be done efficiently.

sum as follows

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{U}}[(\bar{Q}_\lambda(T_n))^k] =& \mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\sum\limits_{i=n+1}^{N}\beta_i\bar{P}(T_n,T_i)}{\sum\limits_{r=n+1}^{N}\beta_r\frac{\bar{P}(t,T_r)}{\bar{P}(t,T_n)}}(\bar{Q}_\lambda(T_n))^k\right]\\
=& \sum_{i=n+1}^{N}\frac{\beta_i\bar{P}(t,T_i)}{\sum_{r=n+1}^{N}\beta_r\bar{P}(t,T_r)}\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n,T_i)}{\frac{\bar{P}(t,T_i)}{\bar{P}(t,T_n)}}(\bar{Q}_\lambda(T_n))^k\right]\\
=& \sum_{i=n+1}^{N}\frac{\beta_i\bar{P}(t,T_i)}{\sum_{r=n+1}^{N}\beta_r\bar{P}(t,T_r)}\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n,T_i)}{\mathbb{E}^{\bar{\mathbb{T}}_n}[\bar{P}(T_n,T_i)]}(\bar{Q}_\lambda(T_n))^k\right] \qquad (3.102)
\end{aligned}
$$

where we can calculate

$$
\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n,T_i)}{\mathbb{E}^{\bar{\mathbb{T}}_n}[\bar{P}(T_n,T_i)]}(\bar{Q}_\lambda(T_n))^k\right] \qquad (3.103)
$$

by applying lemma 1.14 (see (1.118) and (1.119)) and the discussion given in Mathai and Provost (1992) regarding the efficient calculation of higher moments of quadratic forms in Gaussian random variables. Note that (3.103) corresponds to finding the higher moments of (3.79) under $\bar{\mathbb{T}}_i$ which is the defaultable forward measure[12] for maturity T_i(see the proof of lemma 1.14 for another way to show (3.79) is a quadratic form in Gaussian random variables under the change of measure given by the log-quadratic Gaussian process $\bar{P}(T_n,T_i)$ in (3.103)). Even though (3.79) is not a quadratic form under $\mathbb{U}$, it is a quadratic form under each forward measure $\bar{\mathbb{T}}_i$ for $i = n+1,\dots,N$.

We now present numerical results for the different approximations given in this section. We assume the default free discount and CDS data is as given in Tables 3.1 and 3.2. Since we would like to test the performance of the approximations for the price of credit default swaptions with maturities up to five years with an underlying CDS of maturity that can be fifteen years, we extended the discount data given by Table 3.1 through extrapolation. We give this additional default free discount data in Table 3.4. We extracted the probability of survival under the default free forward measure $\mathbb{T}$ which we denoted by $\bar{G}(0,T)$ as described in the previous section (see Figure 3.4). The survival probabilities have also to be extended by extrapolation for years 10 to 15 since we extracted the survival probability based

[12] We use $\bar{\mathbb{T}}_i$ to denote $\bar{\mathbb{T}}_{T_i}$ to lighten the notation.

T	$P(0,T)$
11	0.451034
12	0.410806
13	0.362682
14	0.302714
15	0.226954

Tab. 3.4: Zero Rates for years 10 to 15

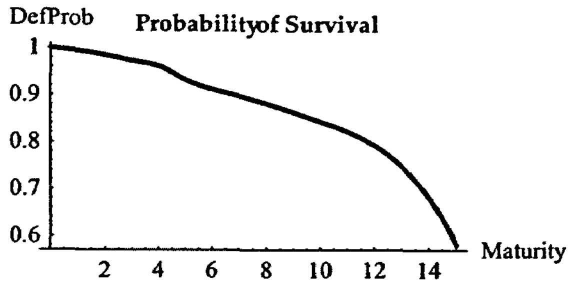

Fig. 3.6: Extracted Survival Probability with Extrapolation for years 10 to 15

on ten years default free discount data and CDS quotes of maximum maturity ten years. We give the figure for the extrapolated $\bar{G}(0,T)$ in Figure 3.6. As in the previous section we assume a two factor quadratic Gaussian model(see (3.55) and use the parameters given (3.56). There is no particular reason we use these parameters, our objective is to test the performance of the approximations for these parameters. The exact value of the price of the credit default swaption is calculated based on the double integral given in (3.63). In tables (3.5) to (3.13), the maturities for the credit default swaption are given in the rows and range from $T = 1$ to $T = 5$ years. The tenor of the CDS underlying the credit default swaption range from 1 to 10 years and the price of the corresponding credit default swaption are given along the columns. For each maturity tenor pair, we give next to the exact value the value obtained through the approximation (3.80) in parentheses and below it the relative error for the approximation expressed as a percentage. For each of the maturity tenor pair, we consider three strike prices. The first strike price is the at the money strike price which is the strike price that would make the value of a forward starting CDS equal to zero and is obtained by solving for K in

$$\mathbb{E}^{\mathbb{Q}}\Bigg[\exp\Big(-\int_0^{T_n} r_s+\lambda_s\,ds\Big)\Big(\delta Z\sum_{j=1}^{N}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_{T_n}^{T_j} r_s+\lambda_s\,ds\Big)\lambda_{T_j}|\mathcal{F}_{T_n}\Big]- \\ -K\sum_{i=1}^{N}\beta_i\,\bar{P}(T_n,T_i)\Big)\Bigg]=0. \quad (3.104)$$

Therefore the at the money strike price for the credit default swaption of maturity T_n with an underlying CDS of tenor equal to $T_N - T_n$ years is given by

$$K_{DATM} := \frac{\delta Z\sum_{j=1}^{N}\bar{P}(0,T_j)\mathbb{E}^{\bar{\mathbb{T}}_n}[\lambda_{T_j}]}{\sum_{i=1}^{N}\beta_i\bar{P}(0,T_i)}. \quad (3.105)$$

We then consider an in the money strike which is taken to be $0.85 \times K_{DATM}$ and an out of the money strike which is taken to be $1.15 \times K_{DATM}$.

Tables 3.5, 3.6 and 3.7 give the results for the approximation of the price of credit default swaptions based on (3.80). As we can see from these results the approximation has an error of a few basis points when the maturity of the credit default swaption is one year. For the five year maturity credit default swaption, we have an error that increases with the tenor of the underlying CDS such that for a CDS of tenor length equal to 10 years, we have a large error. There are two assumptions that could be the cause for this error. The first possible cause for the error is that we approximate Y_t by a Gaussian process with the same mean and variance-covariance matrix when deriving the approximation (3.80). The second possible reason is that we replace the weights $\bar{w}_j(t)$ by their time zero values. The first error can be eliminated by using the exact characteristic function of (3.79) as it can be calculated as the weighted sum of the characteristic functions of Gaussian quadratic forms similar to what we did in (3.102). Numerical tests using the exact characteristic function of (3.79) had in general more error in the approximate prices of the credit default swaptions obtained in comparison to the first approximation. We therefore can conclude that the error is mainly due to the fact that we replaced the weights $\bar{w}_j(T_n)$ in (3.79) by their time zero values. It appears replacing Y_t which has a non-Gaussian dynamics under $\mathbb{U}$ by a Gaussian process with the same mean and variance-covariance matrix has the effect of canceling some part of the error introduced by the freezing of the weights $\bar{w}_i(t)$. This observation is also supported by the results obtained for the approximation based on a Gram Charlier series which does not replace Y_t by a Gaussian process but calculates the exact moments of (3.79) using (3.102). Since the only assumption when calculating the higher moments is that the weights $\bar{w}_j(t)$ are frozen, the source of error can come from this assumption or from the lack of accuracy by the Gram Charlier series in approximating the probability density of (3.79). The more likely source of error is the freezing of the weights since from the numerical experiments conducted for the approximation of swaption prices through the Gram Charlier series method, we found out that the Gram Charlier series method approximates the swaption price well if a limited number of moments are used. As the results of Tables 3.11, 3.12 and 3.13 show the Gram Charlier series approach, where the density of (3.79) under $\mathbb{U}$ is approximated through an orthogonal series expansion, has less error compared to the first approximation but has similarly large errors for maturities of the credit default swaption equal to five years when the CDS tenor is equal to 10 years. This shows that freezing of the weights $\bar{w}(t)$ by replacing them by their time zero values leads to large errors for the prices of credit default swaptions when the underlying CDS tenors are more than five years. This observation is also supported by the results of Tables 3.8, 3.9 and 3.10 which are based on the approximation given by (3.88). The results for this approximation are very accurate as we only approximate the exercise region of credit default swaption. However the errors for the longer maturity credit default swaptions are relatively larger.

Mat.	Tenor			
T	1	3	5	10
1	17.50(16.81) (3.92%)	51.23(49.29) (3.78%)	95.3(94.72) (0.61%)	152.73(155.93) (-2.1%)
3	27.65(25.85) (6.51%)	107.97(102.03) (5.50%)	156.2(146.10) (6.46%)	254.79(268.09) (-5.22%)
5	45.77(41.07) (10.26%)	110.5(99.40) (10.04%)	170.9.44(157.92) (7.59%)	262.82(365.60) (-39.11%)

Tab. 3.5: Relative Error of Approximation given by (3.80) for $K = K_{DATM}$

Mat.	Tenor			
T	1	3	5	10
1	21.18(20.49) (3.25%)	64.09(62.17) (2.98%)	126.23(126.91) (-0.54%)	217.72(220.75) (-1.39%)
3	31.30(29.5) (5.76%)	127.98(122.04) (4.64%)	185.5(175.44) (5.42%)	321.50(335.02) (-4.20%)
5	51.94(47.24) (9.05%)	125.27(114.15) (8.88%)	195.84(182.99) (6.57%)	351.45(455.58) (-29.63%)

Tab. 3.6: Relative Error of Approximation given by (3.80) for $K = 0.85 \times K_{DITM}$

Mat.	Tenor			
T	1	3	5	10
1	14.4(13.72) (4.68%)	40.69(38.79) (4.66%)	70.25(69.65) (0.86%)	104.99(107.87) (-2.74%)
3	24.44(22.65) (7.31%)	90.94(85.06) (6.46%)	131.38(121.42) (7.58%)	201.22(213.58) (-6.14%)
5	40.34(35.69) (11.55%)	97.59(86.58) (11.28%)	149.24(136.32) (8.66%)	196.07(292.21) (-49.03%)

Tab. 3.7: Relative Error of Approximation given by (3.80) for $K = 1.15 \times K_{DOTM}$

Mat.	Tenor			
T	1	3	5	10
1	17.50(17.47) (0.18%)	51.23(51.39) (-0.32%)	95.3(95.27) (0.03%)	152.73(152.59) (0.09%)
3	27.65(27.71) (-0.2%)	107.97(108.0) (-0.03%)	156.2(155.99) (0.14%)	254.79(254.69) (0.04%)
5	45.77(45.81) (-0.08%)	110.5(110.67) (-0.15%)	170.9.44(171.46) (-0.33%)	262.82(260.79) (0.77%)

Tab. 3.8: Relative Error of Approximation given by (3.88) for $K = K_{DATM}$

Mat.	Tenor			
T	1	3	5	10
1	21.18(21.19) (-0.08%)	64.09(64.09) (-0.00%)	126.23(127.52) (-1.03%)	217.72(217.46) (0.12%)
3	31.30(31.26) (0.13%)	127.98(127.85) (0.10%)	185.5(185.35) (0.08%)	321.50(321.40) (0.03%)
5	51.94(52.06) (-0.23%)	125.27(125.05) (0.17%)	195.84(195.44) (0.20%)	351.45(349.46) (0.57%)

Tab. 3.9: Relative Error of Approximation given by (3.88) for $K = 0.85 \times K_{DITM}$

Mat.	Tenor			
T	1	3	5	10
1	14.4(14.5) (0.72%)	40.69(40.65) (0.09%)	70.25(70.29) (-0.05%)	104.99(104.83) (0.15%)
3	24.44(24.44) (-0,01%)	90.94(90.81) (-0.14%)	131.38(131.94) (0.43%)	201.22(201.1) (0.06%)
5	40.34(40.51) (-0.42%)	97.59(97.57) (0.02%)	149.24(149.4) (-0.11%)	196.07(192.59) (1.78%)

Tab. 3.10: Relative Error of Approximation given by (3.88) for $K = 1.15 \times K_{DOTM}$

Mat.	Tenor			
T	1	3	5	10
1	17.50(17.35) (0.84%)	51.23(50.89) (0.66%)	95.3(96.95) (-1.72%)	152.73(160.32) (-4.96%)
3	27.65(28.2) (-1.98%)	107.97(107.43) (0.5%)	156.2(155.83) (0.24%)	254.79(286.88) (-12.59%)
5	45.77(44.12) (3.61%)	110.5(109.49) (0.92%)	170.9.44(174.76) (-2.26%)	262.82(397.54) (-51.26%)

Tab. 3.11: Relative Error of Approximation based on a Gram Charlier series(based on 3 moments) for $K = K_{DATM}$

Mat.	Tenor			
T	1	3	5	10
1	21.18(21.07) (0.47%)	64.09(63.95) (0.21%)	126.23(129.46) (-2.56%)	217.72(225.84) (-3.72%)
3	31.30(31.96) (-2.09%)	127.98(127.86) (0.09%)	185.5(185.87) (-0.20%)	321.50(355.26) (-10.5%)
5	51.94(50.44) (2.89%)	125.27(124.64) (0.50%)	195.84(200.52) (-2.39%)	351.45(489.76) (-39.35%)

Tab. 3.12: Relative Error of Approximation based on a Gram Charlier series(based on 3 moments)for $K = 0.85 \times K_{DITM}$

Mat.	Tenor			
T	1	3	5	10
1	14.4(14.26) (0.93%)	40.69(40.40) (0.69%)	70.25(71.95) (-2.42%)	104.99(112.43) (-7.08%)
3	24.44(24.97) (-2.19%)	90.94(90.42) (0.57%)	131.38(131.06) (0.24%)	201.22(232.28) (-15.43%)
5	40.34(38.70) (4.08%)	97.59(96.60) (1.02%)	149.24(153.05) (-2.55%)	196.07(323.62) (-65.05%)

Tab. 3.13: Relative Error of Approximation based on a Gram Charlier series(based on 3 moments) for $K = 1.15 \times K_{DOTM}$

4. A TWO-COUNTRY REDUCED FORM MODEL

There is some work in the pricing of default free bond options in a two country setting or more generally in an international economy. The treatment of options under an international setting was first formalized in Amin and Jarrow (1991) and Amin and Jarrow (1992). Since then, there has been some work on the valuation of options whose underlying is a foreign default free bond (see for example Andreasen (1995), Bensaid and Bottazzi (2001), Cherif and El Karoui (1993), Frachot (1995), Frey and Sommer (1996), Jamshidian (1993) and Mellios and Poncet (2001)). While there is also a lot of literature that considers the pricing of credit default swaps in a single economy(see for example Brigo and Alfonsi (2004) and Schönbucher (2000)), the literature on the valuation of a quanto credit default swap or more generally credit sensitive securities involving currency risk is rare. Using hedging arguments Vaillant (2001) considers the valuation of a foreign defaultable bond. A numeraire independent framework for the valuation of credit derivatives is provided in Jamshidian (2004) and subsequently used to value credit default swaptions. The most relevant work on default in a two country setting[1] has been Levy and Levin (2002), Finkelstein (2000), Anderson (2003) and Ehlers and Schönbucher (2006). In this chapter we consider a two country model of default which accounts for currency risk. In the first section, we consider a default model involving a single corporation which has issued defaultable bonds in both the domestic and foreign country. Assuming cross default[2], we show how the domestic forward credit spread and the foreign credit spread are related with each other. We also derive a quanto adjustment formula that can be used to determine the probability of default of the corporation in the foreign economy from the probability of default of the corporation in the domestic economy or vice versa. This generalizes the quanto adjustment formula given in Finkelstein (2000) and Vail-

[1] I would like to thank Dr. Leif Anderson for the private communication regarding quanto adjustments to probabilities of default.

[2] Cross default means here that the default of the corporation in one economy triggers its default in the other economy.

lant (2001) for the case of an intensity of default that is of the Gaussian type. In subsequent sections we assume no cross default and give an extension of an HJM drift condition for the forward credit spread in a contagion model of default. We also discuss how we can tractably value a quanto default swap under a contagion model of default assuming a quadratic Gaussian factor model.

4.1 The Framework

We assume that we have a filtered probability space $(\tilde{\Omega}, \mathcal{F}, \mathbb{P}^d)$ where $\mathbb{P}^d$ is the risk-neutral measure for the domestic economy. Thus $\mathbb{P}^d$ is a measure under which any domestic tradeable security which does not pay coupons or dividends follows an $\mathcal{F}$-martingale process. We assume the filtration $\mathcal{F} = (\mathcal{F}_t)_{(0 \leqslant t \leqslant T^*)}$ is generated by n independent Brownian motions $W_t^d = W_{it}^d, i = 1, \ldots, n$ and satisfies the usual conditions. The time horizon is assumed to be finite so that $T^* > 0$ is some finite number. We now assume that there is a corporate that has issued debt in the form of zero coupon defaultable bonds in the domestic and foreign economy. Let τ denote the default time of the corporate defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q}^d)$ where $\mathbb{Q}^d$ denotes the domestic risk neutral measure for the extended domestic market which now includes defaultable securities. Let $H_t = \mathbf{1}_{\tau \leqslant t}$ and $\mathcal{H} = (\mathcal{H}_t)_{0 \leqslant t \leqslant T^*}$ be the filtration generated by H i.e. $\mathcal{H}_t = \sigma(H_u : u \leqslant t)$ which is completed by the null sets of $(\mathbb{Q}^d, \mathcal{G})$. We assume that $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$. For a specific construction of filtered probability space satisfying this property see chapter 13, section 13.1.5 of Bielecki and Rutkowski (2002). We denote by $P^d(t,T)$ the price of a domestic default free zero coupon bond and by $P^f(t,T)$ the price of a foreign default free zero coupon bond. We denote by $\mathbf{1}_{\tau > t} \bar{P}^d(t,T)$ the price of a domestic defaultable zero coupon bond and by $\mathbf{1}_{\tau > t} \bar{P}^f(t,T)$ the price of a foreign defaultable zero coupon bond. The values $\bar{P}^i(t,T)$ are referred to as the predefault values of the i^{th} economy defaultable zero coupon bond where $i = d\,(f)$ corresponds to the domestic (foreign) economy. We define the domestic continuously compounded default free instantaneous forward rates $f^d(t,T)$ by

$$f^d(t,T) := -\frac{d}{dT} \log_e P^d(t,T). \qquad (4.1)$$

Similarly we define the foreign continuously compounded default free instantaneous forward rates $f^f(t,T)$ by

$$f^f(t,T) := -\frac{d}{dT}\log_e P^f(t,T). \tag{4.2}$$

The domestic continuously compounded instantaneous defaultable forward rates $\bar{f}^d(t,T)$ are defined by

$$\bar{f}^d(t,T) := -\frac{d}{dT}\log_e \bar{P}^d(t,T), \tag{4.3}$$

while the foreign continuously compounded instantaneous defaultable forward rates $\bar{f}^f(t,T)$ are defined by

$$\bar{f}^f(t,T) := -\frac{d}{dT}\log_e \bar{P}^f(t,T). \tag{4.4}$$

Given the default free and defaultable forward rates we define the domestic continuously compounded instantaneous forward credit spread $s^d(t,T)$ by

$$s^d(t,T) := \bar{f}^d(t,T) - f^d(t,T), \tag{4.5}$$

and the foreign continuously compounded instantaneous forward credit spread $s^f(t,T)$ by

$$s^f(t,T) := \bar{f}^f(t,T) - f^f(t,T). \tag{4.6}$$

The $\mathcal{G}_t$-intensity of H_t in the i^{th} economy is denoted by λ_t^i and has the property that

$$H_t - \int_0^{t\wedge\tau} \lambda_u^i\,du \tag{4.7}$$

is a $\mathcal{G}$-martingale under $\mathbb{Q}^i$ for $i = d\,(f)$ corresponding to the domestic (foreign) economy. Again we refer the reader to p.394 of Bielecki and Rutkowski (2002) for a specific construction of a default time τ in an HJM model of default such that (4.7) holds for $i = d$ i.e. the domestic economy. An alternative construction of τ such that (4.7) is true can be based on a Cox process approach (see chapter 8, section 8.6.1 of Bielecki and Rutkowski (2002)). We now assume that $\mathcal{F}_t$ martingales are also $\mathcal{G}_t$ martingales. This is known as the martingale invariance property or the **H** hypothesis (see chapters 6 and 8 of Bielecki and Rutkowski (2002)). We also assume that the intensity λ_t is an $\mathcal{F}$- measurable process. Thus the $(\mathcal{F}, \mathbb{Q}^i)$ martingale hazard process in the *ith* economy (see Chapter 6,6.1 of Bielecki and Rutkowski (2002)) which is given by

$$\Lambda_t^i = \int_0^t \lambda_u^i \, du \tag{4.8}$$

is absolutely continuous. The $\mathcal{F}$-hazard process of τ under $\mathbb{Q}^i$ which is denoted by Γ_t satisfies the following equality

$$\mathbb{Q}^i\{\tau > t | \mathcal{F}_t\} = \exp(-\Gamma_t) \tag{4.9}$$

and is useful in calculating expectations involving H_t. Thus it is important to identify this process whenever possible. Using the **H** hypothesis and the fact that the filtration $\mathcal{F}$ supports only continuous martingales, we have by Proposition 6.2.2 of Bielecki and Rutkowski (2002) that the hazard process is equal to the martingale hazard process i.e $\Gamma = \Lambda$. For brevity of notation we will use the following:

$$\mathbb{E}^{\mathbf{M}}[Y|\mathcal{F}_t] =: \mathbb{E}_t^{\mathbf{M}}[Y]$$

for any $\mathcal{G}$ measurable random variable Y and $\mathbb{M}$ in $\mathbb{E}^{\mathbf{M}}$ is the measure under which we are calculating the expectation. Using the fact that $\Gamma = \Lambda$ we have the following valuation formula (see Bielecki and Rutkowski (2002), p. 230).

Proposition 4.1. *Let Y be an $\mathcal{F}$ measurable random variable, then we have:*

$$\mathbb{E}^{\mathbb{Q}^i}\left[\exp\left(-\int_t^T r_u \, du\right)\mathbf{1}_{\tau > T} Y \middle| \mathcal{G}_t\right] = \mathbf{1}_{\tau > t}\mathbb{E}^{\mathbb{Q}^i}\left[\exp\left(-\int_t^T r_u + \lambda_u \, du\right) Y \middle| \mathcal{F}_t\right]. \tag{4.10}$$

Thus calculations involving $\mathbf{1}_{\tau > T}$ can be reduced to one involving

$$\exp\left(-\int_t^T \lambda_u \, du\right).$$

Assume that the foreign exchange rate S_t which is the value of one unit of foreign currency in terms of the domestic currency is an $\mathcal{F}_t$-measurable continuous process which solves the following SDE

$$dS_t = S_t(r_t^d - r_t^f)\,dt + \sigma_t^S\,dW_t^d \tag{4.11}$$

where σ_t^S is bounded and adapted to the filtration $\mathcal{F}_t$. In our discussions in this chapter we assume that we are given the domestic filtered probability space $(\tilde{\Omega}, \mathcal{F}, \mathbb{P}^d)$. We obtain the foreign filtered probability space $(\tilde{\Omega}, \mathcal{F}, \mathbb{P}^f)$ using a change of measure technique. We define the foreign risk neutral measure $\mathbb{Q}^f$ by

$$\left.\frac{d\mathbb{Q}^f}{d\mathbb{Q}^d}\right|_{\mathcal{G}_{T*}} = \exp\left(\int_0^{T*} \sigma^S(u).W_u^d - \frac{1}{2}\int_0^{T*} ||\sigma^S(u)||^2\,ds\right). \tag{4.12}$$

Applying Proposition 5.3.1 of Bielecki and Rutkowski (2002) we get

$$W_t^f = W_t^d - \int_0^t \sigma_u^S\,du \tag{4.13}$$

is a standard Brownian motion under the foreign risk neutral measure $\mathbb{Q}^f$ and

$$H_t - \int_0^{t\wedge\tau} \lambda_u^d\,du \tag{4.14}$$

is a $\mathcal{G}_t$-martingale under $\mathbb{Q}^f$. But we also have by the property of the intensity λ_t^f:

$$H_t - \int_0^{t\wedge\tau} \lambda_u^f\,du \tag{4.15}$$

is a $\mathcal{G}_t$-martingale under $\mathbb{Q}^f$. Now by the uniqueness of the predictable versions of the $\mathcal{G}$-compensator, we must have $\lambda_t^f = \lambda_t^d$ and thus we drop the exponent[3] and write simply λ_t.

Definition 4.2. *The forward exchange rate which is denoted by $X(t,T)$ is the forward price in domestic currency of one unit of foreign currency in a forward contract with maturity T which is initiated at time t. Using a replication argument one can show that $X(t,T)$ is given by the following formula (see Musiela and*

[3] Note here we have a single default time τ and hence we are assuming that default in one economy implies default in the other economy.

Rutkowski (2005) or Schlögl (2002)):

$$X(t,T) = \frac{S_t P^f(t,T)}{P^d(t,T)}. \tag{4.16}$$

The change of measure between the domestic forward measure $\mathbb{T}^d$ and the foreign forward measure $\mathbb{T}^f$ (see Schlögl (2002)) is given by

$$\left.\frac{d\mathbb{T}^f}{d\mathbb{T}^d}\right|_{\mathcal{G}_t} = \frac{X(t,T)}{X(0,T)}. \tag{4.17}$$

Thus given a $\mathcal{G}$ measurable random variable Y, we have by the Bayes formula

$$\mathbb{E}^{\mathbb{T}^f}[Y|\mathcal{G}_t] = \mathbb{E}^{\mathbb{T}^d}\left[\frac{X(T,T)}{X(t,T)}Y|\mathcal{G}_t\right]. \tag{4.18}$$

Let $\bar{\mathbb{T}}^i$ denote the defaultable forward measure for the i^{th} economy (see Schönbucher (2000) and Bielecki and Rutkowski (2002), p. 471).

Definition 4.3. *The defaultable forward measure $\bar{\mathbb{T}}^i$ is given by the following Radon-Nikodým density*

$$\left.\frac{d\bar{\mathbb{T}}^i}{d\mathbb{T}^i}\right|_{\mathcal{G}_{T*}} = \mathbf{1}_{\tau>T}\frac{P^i(0,T)\bar{P}^i(T,T)}{P^i(T,T)\bar{P}^i(0,T)}. \tag{4.19}$$

Definition 4.4. *The defaultable forward exchange rate which is denoted by $\bar{X}(t,T)$ is the forward price in domestic currency of one unit of foreign currency in a defaultable forward contract with maturity T which is initiated at time t. Using a replication argument involving only defaultable bonds, one can show that $\bar{X}(t,T)$ is given by the following formula:*

$$\bar{X}(t,T) := \frac{S_t\bar{P}^f(t,T)}{\bar{P}^d(t,T)} \tag{4.20}$$

Using (4.19) and (4.17) we can show that the change of measure between the domestic and foreign defaultable forward measures is given by the following Radon-Nikodým density

$$\left.\frac{d\bar{\mathbb{T}}^f}{d\bar{\mathbb{T}}^d}\right|_{\mathcal{G}_{T*}} := \left.\frac{d\bar{\mathbb{T}}^f}{d\mathbb{T}^f}\right|_{\mathcal{G}_{T*}}\left.\frac{d\mathbb{T}^f}{d\mathbb{T}^d}\right|_{\mathcal{G}_{T*}}\left.\frac{d\mathbb{T}^d}{d\bar{\mathbb{T}}^d}\right|_{\mathcal{G}_{T*}} = \mathbf{1}_{\tau>T}\frac{\bar{X}(T,T)}{\bar{X}(0,T)}. \tag{4.21}$$

We first state a result on the dynamics of $s^i(t,T)$ which is similar to the drift condition for default free forward rates (see Heath et al. (1992)). Assume that the dynamics of the forward credit spread $s^i(t,T)$ under $\mathbb{Q}^i$ for $i = d, f$ is given by the following SDE:

$$ds^i(t,T) = \alpha_s^i(t,T)\,dt + \sigma_s^i(t,T)\,dW_t^i. \tag{4.22}$$

The following proposition can be found in Bielecki and Rutkowski (2002) (see Proposition 13.1.1) and Schönbucher (2000).

Proposition 4.5. *Under the risk neutral measure for the i^{th} economy, the drift term of the forward credit spread $s^i(t,T)$ in the SDE* (4.22) *satisfies the following equality:*

$$\alpha_s^i(t,T) = \sigma_s^i(t,T)\sigma^{i*}(t,T) + (\sigma_s^i(t,T) + \sigma^i(t,T))\sigma_s^{i*}(t,T) \tag{4.23}$$

where $\sigma^i(t,T)$ denotes the volatility of $f^i(t,T)$, $\sigma_s^i(t,T)$ denotes the volatility of $s^i(t,T)$,

$$\sigma^{*i}(t,T) := \int_t^T \sigma^i(t,u)\,du, \quad \text{and} \quad \sigma_s^{*i}(t,T) := \int_t^T \sigma_s^i(t,u)\,du \quad \text{for } i = d, f.$$

We shall refer to the drift condition for $s^i(t,T)$ given in (4.23) as the HJM condition when it is clear from the context that we are referring to the forward credit spread. For computational tractability or empirical work, it is sometimes convenient to have an HJM model of default where the default free forward rates and forward credit spreads satisfy the default free HJM drift condition and the HJM condition (4.23) respectively under the corresponding risk neutral measure of the economy to which the rates belong to. However we will see in the following that we are not free to specify the different forward rates and the foreign exchange rate independently from each other. We now consider the relationship between the forward credit spreads $s^d(t,T)$ and $s^f(t,T)$ for the special case $s^d(t,t) = \lambda_t^d$ and $s^f(t,t) = \lambda_t^f$ to show the link between the volatilities. To simplify the discussion, we assume that the short term interest rates r_t^d and r_t^f are independent of λ_t. Under this assumption we have

$$\mathbb{E}^{\mathbb{T}^i}[r_T^i|\mathcal{G}_t] = \mathbb{E}^{\mathbb{T}^i}[r_T^i|\mathcal{F}_t] = f^i(t,T). \tag{4.24}$$

Under the assumption of independence between the default free forward rates

$f^i(t,T)$ and the forward credit spreads $s^i(t,T)$, the HJM condition given by (4.23) simplifies to

$$\alpha_s^i(t,T) = \sigma_s^i(t,T) \int_t^T \sigma_s^i(t,u)\,du.$$

Therefore assuming independence (4.22) is given by

$$ds^i(t,T) = \sigma_s^i(t,T) \int_t^T \sigma_s^i(t,u)\,du + \sigma_s^i(t,T)\,dW_t^i \tag{4.25}$$

for $i = d\,(f)$ corresponding to the domestic (foreign) economy. Under the measure $\mathbb{Q}^i$ we have

$$\lambda_t^i = s^i(t,t) = s^i(0,t) + \int_0^t \sigma_s^i(u,t)\sigma_s^{*i}(u,t)\,du + \int_0^t \sigma_s^i(u,t)\,dW_u^i \tag{4.26}$$

for $i = d, f$. Now using $dW_t^d = dW_t^f + \sigma^S(t)\,dt$ we get from (4.26) for $i = d$ the following

$$\lambda_t^d = s^d(0,t) + \int_0^t \sigma_s^d(u,t)\sigma^S(u)\,du + \int_0^t \sigma_s^d(u,t)\sigma_s^{*d}(u,t)\,du + \int_0^t \sigma_s^d(u,t)\,dW_u^f. \tag{4.27}$$

Under the assumption that the foreign exchange rate S_t is a diffusion we have $\lambda_t = \lambda_t^d = \lambda_t^f$ (see equation (5.39) of Bielecki and Rutkowski (2002)). Therefore we can equate (4.27) to the equation we get from (4.26) for $i = f$ to get the following equality

$$\begin{aligned} s^d(0,t) + \int_0^t \sigma_s^d(u,t)\sigma^S(u)\,du + \int_0^t \sigma_s^d(u,t)\sigma_s^{*d}(u,t)\,du + \int_0^t \sigma_s^d(u,t)\,dW_u^f \\ = s^f(0,t) + \int_0^t \sigma_s^f(u,t)\sigma_s^{*f}(u,t)\,du + \int_0^t \sigma_s^f(u,t)\,dW_u^f. \end{aligned} \tag{4.28}$$

Now we can collect the terms involving the Brownian motion on one side of the equation to get

$$s^d(0,t) - s^f(0,t) + \int_0^t (\sigma_s^d(u,t)\sigma_s^{*d}(u,t) - \sigma_s^f(u,t)\sigma_s^{*f}(u,t))\,du + \int_0^t \sigma_s^d(u,t)\sigma^S(u)\,du$$
$$= \int_0^t (\sigma_s^f(u,t) - \sigma_s^d(u,t))\,dW_u^f. \quad (4.29)$$

Now the on the left side of the above equation we have predictable terms while the right hand side is a martingale. The only predictable martingales are constants. As the martingale on the right hand side of (4.29) has value zero at time $t = 0$, the constant in question must be equal to zero. This implies that the quadratic variation of the term on the right hand side of the equality in (4.29) is zero:

$$\int_0^t (\sigma_s^f(u,t) - \sigma_s^d(u,t))^2\,du = 0.$$

This in turn implies that

$$\sigma_s^f(u,t) = \sigma_s^d(u,t).$$

Therefore (4.29) can be simplified to the following condition:

$$s^f(0,t) - s^d(0,t) + \int_0^t \sigma_s^d(u,t)\sigma^S(u)\,du = 0. \quad (4.30)$$

Hence we see that volatilities for the instantaneous forward rate spreads $s^i(t,T)$ in the domestic and foreign market and the exchange rate volatility $\sigma^S(t)$ cannot be chosen independently from each other.

In the following discussion we assume that we have a zero rate of recovery. Let the probability of survival of the obligor in the ith market under $\mathbb{T}^i$ be defined by

$$\bar{G}^i(t,T) := \mathbb{E}_t^{\mathbb{T}^i}\Big[\exp\Big(-\int_t^T \lambda_u^i\,du\Big)\Big]$$
$$= \frac{1}{P^i(t,T)}\mathbb{E}_t^{\mathbb{Q}^i}\Big[\exp\Big(-\int_t^T r_u^i + \lambda_u\,du\Big)\Big] = \frac{\bar{P}^i(t,T)}{P^i(t,T)}. \quad (4.31)$$

For most corporations who are operating in both the domestic and foreign economies, it is usually the case that liquid quotes of CDS rates are not available in both economies. Hence if we extract the default probabilities in the economy with the more liquid quotes, it is useful to know how to modify these default probabilities in order to find the default probabilities in the other economy. Such an adjustment factor is known as a quanto adjustment. In the context of a Gaussian factor model, these adjustments can be derived in analytically closed form (see Anderson (2003), Vaillant (2001) and Finkelstein (2000)). We assume without loss of generality that the domestic economy has CDS quotes that are liquid. Hence using the method described in Chapter 3, we can extract the term structure of default probabilities $\bar{G}^d(t,T)$ in a model independent way. Hence we are looking for a quanto adjustment of $\bar{G}^d(t,T)$ which would give us $\bar{G}^f(t,T)$.

Theorem 4.6. *Assume we can calculate or approximate the following covariances:*

1. $Cov_t^{\bar{\mathbb{T}}^d}[\mathbf{1}_{\tau>T}S_T, \lambda_T]$ *which represents the covariance*[4] *between the value of a defaultable foreign zero coupon bond at time T measured in domestic currency i.e. $\mathbf{1}_{\tau>T}S_T$ and the intensity of default λ_T .*
2. $Cov_t^{\mathbb{T}^i}[\exp(-\Lambda_T), r_T^i]$ *which represents covariance between the $\mathcal{F}$-survival process of τ*
$$G_T := \mathbb{Q}^d[\tau > T|\mathcal{F}_T] = \exp(-\Lambda_T) \tag{4.32}$$
and the domestic(foreign) short term interest rate under the domestic(foreign) forward measure for $i = d(i = f)$.

Then

$$\frac{1}{\bar{G}^f(t,T)}$$

is the exact solution of the following Bernoulli ODE:

$$\begin{aligned}\frac{d}{dT}\left(\frac{1}{\bar{G}^f(t,T)}\right) - s^d(t,T)\left(\frac{1}{\bar{G}^f(t,T)}\right)+ \\ + \frac{1}{\bar{G}^d(t,T)}\exp(\Lambda_t)\, Cov_t^{\mathbb{T}^d}[\exp(-\Lambda_T), r_T^d]\left(\frac{1}{\bar{G}^f(t,T)}\right) = \\ \exp(\Lambda_t)\, Cov_t^{\mathbb{T}^f}[\exp(-\Lambda_T), r_T^f]\frac{1}{\bar{G}^f(t,T)^2}+ \\ \frac{\bar{G}^d(t,T)\exp(\Lambda_t)}{X(t,T)} Cov_t^{\bar{\mathbb{T}}^d}[\exp(-\Lambda_T)S_T, \lambda_T]\frac{1}{\bar{G}^f(t,T)^2}.\end{aligned} \tag{4.33}$$

[4] Note that the covariance is calculated under the domestic defaultable forward measure which is denoted by $\bar{\mathbb{T}}^d$.

Proof. First note that using (4.19) we have for $i = d, f$

$$
\begin{aligned}
\mathbb{E}_t^{\bar{\mathbf{T}}^i}[r_T^i] =& \frac{1}{\mathbf{1}_{\tau>t}\frac{\bar{P}^i(t,T)}{P^i(t,T)}}\mathbb{E}_t^{\mathbf{T}^i}[\mathbf{1}_{\tau>T}\frac{\bar{P}^i(T,T)}{P^i(T,T)}r_T^i] \\
=& \frac{1}{\mathbf{1}_{\tau>t}\frac{\bar{P}^i(t,T)}{P^i(t,T)}}\mathbb{E}_t^{\mathbf{T}^i}\left[\mathbf{1}_{\tau>T}\frac{\bar{P}^i(T,T)}{P^i(T,T)}\right]\mathbb{E}_t^{\mathbf{T}^i}[r_T^f] \\
&+ \frac{P^i(t,T)}{\mathbf{1}_{\tau>t}\bar{P}^i(t,T)}Cov_t^{\mathbf{T}^i}\left[\mathbf{1}_{\tau>T}\frac{\bar{P}^i(T,T)}{P^i(T,T)}, r_T^i\right] \\
=& f^i(t,T) + \frac{P^i(t,T)}{\mathbf{1}_{\tau>t}\bar{P}^i(t,T)}Cov_t^{\mathbf{T}^i}\left[\mathbf{1}_{\tau>T}, r_T^i\right].
\end{aligned}
\tag{4.34}
$$

From

$$
\mathbb{E}_t^{\mathbf{T}^i}[\mathbf{1}_{\tau>T}\, r_T^i] = \mathbf{1}_{\tau>t}\mathbb{E}_t^{\mathbf{T}^i}\left[\exp\left(-\int_t^T \lambda_u\, du\right) r_T^i\right] \tag{4.35}
$$

and

$$
\mathbb{E}_t^{\mathbf{T}^i}[\mathbf{1}_{\tau>T}] = \mathbf{1}_{\tau>t}\mathbb{E}_t^{\mathbf{T}^i}\left[\exp\left(-\int_t^T \lambda_u\, du\right)\right] \tag{4.36}
$$

we can get

$$
Cov_t^{\mathbf{T}^i}[\mathbf{1}_{\tau>T}, r_T^i] = \mathbf{1}_{\tau>t}Cov_t^{\mathbf{T}^i}\left[\exp\left(-\int_t^T \lambda_u\, du\right), r_T^i\right]. \tag{4.37}
$$

Using (4.37), we can write (4.34) as

$$
\mathbb{E}_t^{\bar{\mathbf{T}}^i}[r_T^i] = f^i(t,T) + \frac{P^i(t,T)}{\bar{P}^i(t,T)}Cov_t^{\mathbf{T}^i}\left[\exp\left(-\int_t^T \lambda_u\, du\right), r_T^i\right]. \tag{4.38}
$$

The expectation of λ_t under $\bar{\mathbb{T}}^f$ can be calculated using (4.21) as follows

$$\begin{aligned}
\mathbb{E}_t^{\bar{\mathbf{T}}^f}[\lambda_T] &= \frac{1}{\mathbf{1}_{\tau>t}\bar{X}(t,T)}\mathbb{E}_t^{\bar{\mathbf{T}}^d}[\mathbf{1}_{\tau>T}\bar{X}(T,T)\lambda_T] \\
&= \mathbb{E}_t^{\bar{\mathbf{T}}^d}[\lambda_T] + \frac{1}{\mathbf{1}_{\tau>t}\bar{X}(t,T)}Cov_t^{\bar{\mathbf{T}}^d}[\mathbf{1}_{\tau>T}\bar{X}(T,T),\lambda_T] \\
&= \mathbb{E}_t^{\bar{\mathbf{T}}^d}[r_T^d+\lambda_T] - \mathbb{E}_t^{\bar{\mathbf{T}}^d}[r_T^d] + \frac{1}{\mathbf{1}_{\tau>t}\bar{X}(t,T)}Cov_t^{\bar{\mathbf{T}}^d}[\mathbf{1}_{\tau>T}\bar{X}(T,T),\lambda_T] \\
&= f^d(t,T)+s^d(t,T) - \mathbb{E}_t^{\bar{\mathbf{T}}^d}[r_T^d] + \frac{1}{\mathbf{1}_{\tau>t}\bar{X}(t,T)}Cov_t^{\bar{\mathbf{T}}^d}[\mathbf{1}_{\tau>T}\bar{X}(T,T),\lambda_T] \\
&= f^d(t,T)+s^d(t,T) - \mathbb{E}_t^{\bar{\mathbf{T}}^d}[r_T^d] + \\
&\quad \frac{1}{\bar{X}(t,T)}Cov_t^{\bar{\mathbf{T}}^d}\Big[\exp\Big(-\int_t^T\lambda_u\,du\Big)\bar{X}(T,T),\lambda_T\Big]
\end{aligned} \tag{4.39}$$

where the equality

$$Cov_t^{\bar{\mathbf{T}}^d}[\mathbf{1}_{\tau>T}\bar{X}(T,T),\lambda_T] = \mathbf{1}_{\tau>t}Cov_t^{\bar{\mathbf{T}}^d}\Big[\exp\Big(-\int_t^T\lambda_u\,du\Big)\bar{X}(T,T),\lambda_T\Big]$$

can be obtained using the arguments used to show (4.37). Using (4.38) for $i=d$, (4.39) can be simplified to

$$\begin{aligned}
\mathbb{E}_t^{\bar{\mathbf{T}}^f}[\lambda_T] = s^d(t,T) - \frac{P^d(t,T)}{\bar{P}^d(t,T)}Cov_t^{\mathbf{T}^d}\Big[\exp\Big(-\int_t^T\lambda_u\,du\Big), r_T^d\Big]+ \\
+\frac{1}{\bar{X}(t,T)}Cov_t^{\bar{\mathbf{T}}^d}\Big[\exp\Big(-\int_t^T\lambda_u\,du\Big)\bar{X}(T,T),\lambda_T\Big].
\end{aligned} \tag{4.40}$$

Another way of calculating $\mathbb{E}_t^{\bar{\mathbf{T}}^f}[\lambda_T]$ is to use the following equation[5]:

$$f^f(t,T)+s^f(t,T) = \mathbb{E}_t^{\bar{\mathbf{T}}^f}[r_T^f+\lambda_T].$$

Therefore

$$\mathbb{E}_t^{\bar{\mathbf{T}}^f}[\lambda_T] = f^f(t,T)+s^f(t,T) - \mathbb{E}_t^{\bar{\mathbf{T}}^f}[r_T^f]. \tag{4.41}$$

[5] This equation can be shown to be true using similar arguments that were used to obtain (1.50) in Musiela and Rutkowski (2005).

Now using (4.38) for $i = f$, (4.41) can be written as

$$\mathbb{E}_t^{\bar{\mathbb{T}}^f}[\lambda_T] = s^f(t,T) - \frac{P^f(t,T)}{\bar{P}^f(t,T)} Cov_t^{\mathbb{T}^f}\left[\exp\left(-\int_t^T \lambda_u\, du\right), r_T^f\right]. \tag{4.42}$$

Using the equality $\bar{X}(T,T) = S_T$ and equating (4.42) to (4.40), we get

$$\begin{aligned} s^f(t,T) - \frac{P^f(t,T)}{\bar{P}^f(t,T)} Cov_t^{\mathbb{T}^f}\left[\exp\left(-\int_t^T \lambda_u\, du\right), r_T^f\right] \\ = s^d(t,T) - \frac{P^d(t,T)}{\bar{P}^d(t,T)} Cov_t^{\mathbb{T}^d}\left[\exp\left(-\int_t^T \lambda_u\, du\right), r_T^d\right] + \\ \frac{1}{\bar{X}(t,T)} Cov_t^{\bar{\mathbb{T}}^d}\left[\exp\left(-\int_t^T \lambda_u\, du\right) S_T, \lambda_T\right] \end{aligned} \tag{4.43}$$

From (4.2), (4.4) and (4.6), we get

$$\begin{aligned} s^f(t,T) &= \frac{d}{dT} \log_e \left(\frac{P^f(t,T)}{\bar{P}^f(t,T)}\right) \\ &= \bar{G}^f(t,T) \frac{d}{dT}\left(\frac{P^f(t,T)}{\bar{P}^f(t,T)}\right) \\ &= \bar{G}^f(t,T) \frac{d}{dT}\left(\frac{1}{\bar{G}^f(t,T)}\right) \end{aligned} \tag{4.44}$$

Substituting (4.44) in (4.43) and using (4.31), we now have

$$\begin{aligned} \bar{G}^f(t,T)\frac{d}{dT}\left(\frac{1}{\bar{G}^f(t,T)}\right) - \frac{1}{\bar{G}^f(t,T)} Cov_t^{\mathbb{T}^f}\left[\exp\left(-\int_t^T \lambda_u\, du\right), r_T^f\right] \\ = s^d(t,T) - \frac{1}{\bar{G}^d(t,T)} Cov_t^{\mathbb{T}^d}\left[\exp\left(-\int_t^T \lambda_u\, du\right), r_T^d\right] + \\ \frac{1}{\bar{X}(t,T)} Cov_t^{\bar{\mathbb{T}}^d}\left[\exp\left(-\int_t^T \lambda_u\, du\right) S_T, \lambda_T\right]. \end{aligned} \tag{4.45}$$

Dividing (4.45) by $\bar{G}^f(t,T)$, we get

$$\frac{d}{dT}\left(\frac{1}{\bar{G}^f(t,T)}\right) - Cov_t^{\mathbb{T}^f}\left[\exp\left(-\int_t^T \lambda_u\,du\right), r_T^f\right]\frac{1}{\bar{G}^f(t,T)^2}$$
$$= s^d(t,T)\left(\frac{1}{\bar{G}^f(t,T)}\right) - \frac{1}{\bar{G}^d(t,T)}Cov_t^{\mathbb{T}^d}\left[\exp\left(-\int_t^T \lambda_u\,du\right), r_T^d\right]\left(\frac{1}{\bar{G}^f(t,T)}\right)+$$
$$+\frac{1}{\bar{G}^f(t,T)}\frac{1}{\bar{X}(t,T)}Cov_t^{\bar{\mathbb{T}}^d}\left[\exp\left(-\int_t^T \lambda_u\,du\right)S_T, \lambda_T\right]. \quad (4.46)$$

Using (4.20) and (4.16), we get

$$\frac{1}{\bar{G}^f(t,T)}\frac{1}{\bar{X}(t,T)} = \frac{1}{\bar{G}^f(t,T)^2}\bar{G}^d(t,T)\frac{1}{X(t,T)}. \quad (4.47)$$

Substituting (4.47) in (4.46) and using the definition for Λ_t gives the result of the theorem. □

Theorem 4.6 provides us a method of getting the probability of survival $\bar{G}^f(t,T)$ from the extracted survival probability given by $\bar{G}^d(t,T)$ and the forward exchange rate provided we can get the covariances given by the theorem. Therefore we can interpret the covariances as the quanto adjustments.

We now give another method for calculating a quanto adjustment formula which can be used to calculate the survival probability of the obligor in the foreign market. Let us assume that we have a model where the default free short term rates, the foreign exchange rate and the intensity of default are $\mathcal{F}_t$ measurable i.e. continuous stochastic processes. Using the fact that the price of securities discounted by the savings account are martingales under $\mathbb{Q}^i$, it follows that the dynamics of $P^i(t,T)$ is given by

$$dP^i(t,T) = r_t^i\,dt + \eta^i(t,T)\,dW_t^i$$

where $\eta^i(t,T)$ denotes the instantaneous volatility of $P^i(t,T)$. Similarly the price of defaultable securities discounted by the defaultable savings account:

$$\exp\left(\int_0^t r_u^i + \lambda_u\,du\right)$$

is a martingale under $\mathbb{Q}^i$ so that we have

$$d\bar{P}^i(t,T) = (r_t^i + \lambda_t)\,dt + \bar{\eta}^i(t,T)\,dW_t^i$$

where $\bar{\eta}^i(t,T)$ denote the instantaneous volatility of $\bar{P}^i(t,T)$. Let $< F >$ denote the quadratic variation process of a stochastic process F then the Doléans-Dade exponential of F is denoted by $\mathcal{E}_t(F)$ and defined by

$$\mathcal{E}_t(F) = \exp\left(F_t - \frac{1}{2} < F >_t\right).$$

Theorem 4.7. *The following quanto adjustment of $\bar{G}^d(t,T)$:*

$$\mathbb{E}^{\bar{\mathbb{G}}^d}\left[\exp\left(\int_t^T \sigma_{\bar{g}}^d(u,T)\sigma_M(u,T)\,du\right)\Big|\mathcal{F}_t\right] \tag{4.48}$$

can be used to obtain $\bar{G}^f(t,T)$ i.e.

$$\bar{G}^f(t,T) = \mathbb{E}^{\bar{\mathbb{G}}^d}\left[\exp\left(\int_t^T \sigma_{\bar{g}}^d(u,T)\sigma_M(u,T)\,du\right)\Big|\mathcal{F}_t\right]\bar{G}^d(t,T)$$

where

$$\sigma_{\bar{g}}^d(u,T) = \bar{\eta}^i(u,T) - \eta^i(u,T)$$

and $\sigma_M(t,T)$ is the volatility of the forward exchange rate and $\bar{\mathbb{G}}^d$ is a measure with the following Radon-Nikodým density with respect to the foreign forward measure $\mathbb{T}^f$

$$\left.\frac{d\bar{\mathbb{G}}^d}{d\mathbb{T}^f}\right|_{\mathcal{G}_{T^*}} = \mathcal{E}_t\left(\int_0^{\cdot} \sigma_{\bar{g}}^d(u,T)\,dW_u^{\mathbb{T}^f}\right). \tag{4.49}$$

Proof. Since $\eta^i(t,T)$ is the volatility of $P^i(t,T)$ and $\bar{\eta}^i(t,T)$ is the volatility of $\bar{P}^i(t,T)$, we can use Ito's formula to show that the volatility of $\bar{G}^i(t,T)$ which is denoted by $\sigma_{\bar{g}}^i(t,T)$ satisfies the following equality:

$$\sigma_{\bar{g}}^i(t,T) = \bar{\eta}^i(t,T) - \eta^i(t,T). \tag{4.50}$$

We now consider

$$N_t^i := \mathbb{E}^{\mathbb{T}^i}\left[\exp\left(-\int_0^T \lambda_u\,du\right)\Big|\mathcal{F}_t\right] \tag{4.51}$$

which is shown to be a martingale under $\mathbb{T}^i$ by the following argument. Let $s \geqslant t$ then using the property of expectations, we have

$$\mathbb{E}^{\mathbb{T}^i}[N_s^i|\mathcal{F}_t] = \mathbb{E}^{\mathbb{T}^i}\left[\mathbb{E}^{\mathbb{T}^i}\left[\exp\left(-\int_0^T \lambda_u\, du\right)\Big|\mathcal{F}_s\right]\Big|\mathcal{F}_t\right] \tag{4.52}$$

$$= \mathbb{E}^{\mathbb{T}^i}\left[\exp\left(-\int_0^T \lambda_u\, du\right)\Big|\mathcal{F}_t\right] = N_t^i. \tag{4.53}$$

We can express the martingale N_t^i by

$$N_t^i = \exp\left(\int_0^t \lambda_u\, du\right)\bar{G}^i(t,T). \tag{4.54}$$

Now using Girsanov's theorem

$$W_t^{\mathbb{T}^i} := W_t^i - \int_0^t \eta^i(u,T)\, du \tag{4.55}$$

is a standard Brownian motion under the $\mathbb{T}^i$ measure. Hence using (4.50) the following can be shown:

$$dN_t^i = N_t^i\, \sigma_{\bar{g}}^i(t,T) dW_t^{\mathbb{T}^i} \tag{4.56}$$

The solution to (4.56) is the Doléans-Dade exponential which is given for

$$N_T^i = \exp\left(-\int_0^T \lambda_u\, du\right)$$

by

$$N_T^i = N_t^i \mathcal{E}_t\left(\int_t^{\cdot} \sigma_{\bar{g}}^i(u,T)\, dW_u^{\mathbb{T}^i}\right).$$

Therefore we have

$$\exp\left(-\int_0^T \lambda_u\, du\right) := \mathbb{E}_t^{\mathbb{T}^i}\left[\exp\left(-\int_t^T \lambda_u\, du\right)\Big|\mathcal{F}_t\right]\mathcal{E}_t\left(\int_t^{\cdot} \sigma_{\bar{g}}^i(u,T)\, dW_u^{\mathbb{T}^i}\right). \tag{4.57}$$

We can now show the relationship between

$$\mathbb{E}^{\mathbb{T}^d}\left[\exp\left(-\int_t^T \lambda_u\,du\right)\middle|\mathcal{F}_t\right] \text{ and } \mathbb{E}^{\mathbb{T}^f}\left[\exp\left(-\int_t^T \lambda_u\,du\right)\middle|\mathcal{F}_t\right]$$

by using the above and (4.17) which gives us that

$$dW_t^{\mathbb{T}^f} = dW_t^{\mathbb{T}^d} - \sigma_M(t,T)\,dt$$

is a standard Brownian motion under $\mathbb{T}^f$ where

$$\sigma_M(t,T) := \eta^f(t,T) - \eta^d(t,T) + \sigma^S(t)$$

is the volatility of the forward exchange rate process. Therefore we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{T}^f}\left[\exp\left(-\int_t^T \lambda_u\,du\right)\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{T}^f}\left[\mathbb{E}^{\mathbb{T}^d}\left[\exp\left(-\int_t^T \lambda_u\,du\right)\middle|\mathcal{F}_t\right]\mathcal{E}_t\left(\int_t^{\cdot} \sigma_{\bar{g}}^d(u,T)\,dW_u^{\mathbb{T}^d}\right)\middle|\mathcal{F}_t\right] \qquad (4.58)\\
&= \bar{G}^d(t,T)\mathbb{E}^{\mathbb{T}^f}\left[\mathcal{E}_t\left(\int_t^{\cdot} \sigma_{\bar{g}}^d(u,T)\,(dW_u^{\mathbb{T}^f} + \sigma_M(u,T)\,dt\right)\middle|\mathcal{F}_t\right]\\
&= \bar{G}^d(t,T)\mathbb{E}^{\mathbb{T}^f}\left[\mathcal{E}_t\left(\int_t^{\cdot} \sigma_{\bar{g}}^d(u,T)\,dW_u^{\mathbb{T}^f}\right)\exp\left(\int_t^T \sigma_{\bar{g}}^d(u,T)\,\sigma_M(u,T)\,du\right)\middle|\mathcal{F}_t\right]\\
&= \bar{G}^d(t,T)\mathbb{E}^{\bar{\mathbb{G}}^d}\left[\exp\left(\int_t^T \sigma_{\bar{g}}^d(u,T)\,\sigma_M(u,T)\,du\right)\middle|\mathcal{F}_t\right]. \qquad (4.59)
\end{aligned}$$

□

In a Gaussian setting where the volatilities $\sigma_{\bar{g}}^d(t,T)$ and are deterministic functions, it is clear that we have a deterministic quanto adjustment as

$$\mathbb{E}^{\bar{\mathbb{G}}^d}\left[\exp\left(\int_t^T \sigma_{\bar{g}}^d(u,T)\,\sigma_M(u,T)\,du\right)\middle|\mathcal{F}_t\right] = \exp\left(\int_t^T \sigma_{\bar{g}}^d(u,T)\,\sigma_M(u,T)\,du\right).$$

In a more general setting, the values $\sigma_{\bar{g}}^d(t,T)$ and $\sigma_M(u,T)$ are stochastic. Suppose we have a multi-factor model based on a n-dimensional continuous stochastic process $Z_t = (Z_{1t}, \ldots, Z_{nt})$ and the volatilities $\sigma_{\bar{g}}^d(t,T)$ and $\sigma_M(u,T)$ are continuous stochastic processes such that

$$d\sigma_{\bar{g}}^d(t,T) = \mu_1(Z_t,t)\,dt + \sigma_1(Z_t,t)\,dW_t^{\mathbf{T}^f} \tag{4.60}$$

$$d\sigma_M(t,T) = \mu_2(Z_t,t)\,dt + \sigma_2(Z_t,t)\,dW_t^{\mathbf{T}^f}. \tag{4.61}$$

Then the quanto adjustment

$$\mathbb{E}^{\bar{\mathbb{G}}^d}\left[\exp\left(\int_t^T \sigma_{\bar{g}}^d(u,T)\,\sigma_M(u,T)\,du\right)\middle|\mathcal{F}_t\right]$$

is similar to the formula for a default free bond price where

$$-\sigma_{\bar{g}}^d(u,T)\,\sigma_M(u,T)$$

can be seen as an equivalent interest rate. We can now use approximation schemes that have been suggested in the context of valuation of interest rate contingent claims such the asymptotic expansion approach of Kunitomo and Takahashi (2001) to approximate the quanto adjustment provided that μ_1, μ_2, σ_1 and σ_2 are measurable, bounded and smooth functions. Another approximation is to consider the process $\bar{g}_t$ with the dynamics

$$d\bar{g}_t = \bar{g}_t\,\sigma_{\bar{g}}^d(t,T)\,\sigma_M(t,T)dW_t^{\mathbf{T}^f} \tag{4.62}$$

and the orthogonal projection in the least squares sense onto the closed subspace of all square integrable martingales with deterministic covariation as described in Jamshidian (2004). We therefore have to make some assumptions with regard to the dynamics of default free and defaultable bonds as well as the foreign exchange rate process in order to obtain a quanto adjustment. If we are only interested in a quanto adjustment of default probabilities, it is convenient to assume a multi-factor model which leads to a tractable approximation of the quanto adjustment.

4.2 Valuation of Quanto Default Swaps

We now consider a two country defaultable market model based on state variables that follow a Gaussian Ornstein-Uhlenbeck process as in Chapter 3. Let Y_t be a multivariate Ornstein-Uhlenbeck process such that

$$dY_t = A\,Y_t + \Sigma\, dW_t^d \tag{4.63}$$

where

$$Y_t = (Y_{1t}, \ldots, Y_{nt})^\top, \quad Y_0 = (0, \ldots, 0)$$

and $\alpha(t)$ is a vector time dependent function used to calibrate to the default free and defaultable term structures.

We assume that r_t^d, r_t^f, λ_t are quadratic forms in Y_t

$$\begin{aligned} r_T^d &= (Y_t + \alpha(t))^\top I^d (Y_t + \alpha(t)) \\ r_T^f &= (Y_t + \alpha(t))^\top I^f (Y_t + \alpha(t)) \\ \lambda_t &= (Y_t + \alpha(t))^\top I^\lambda (Y_t + \alpha(t)). \end{aligned} \tag{4.64}$$

The matrices I^d, I^f and I^λ are taken to be diagonal matrices with 1 or 0 along the diagonal depending on which factors or coordinates of Y_t are used to model the process. The foreign exchange rate S_t is assumed to be a log-quadratic process as in Chapter 1. Thus S_t is the solution of the following SDE:

$$\frac{dS_t}{S_t^-} = (r_t^d - r_t^f)dt + (2C^S(t)\, Y_t + (B^S(t))^\top \Sigma\, dW_t^d$$

Under this assumption Y_t remains an Ornstein-Uhlenbeck process under the foreign risk neutral measure $\mathbb{Q}^f$. Therefore using the results of Cherif et al. (1994), the defaultable bonds $\bar{P}^i(t,T)$ for $i = d, f$ are log-quadratic Gaussian. The default free bonds $P^i(t,T)$ for $i = d, f$ are also log-quadratic Gaussian.

Definition 4.8. *A domestic quanto credit default swap (QCDS) is a security that guarantees the payment of a deterministic amount Z in foreign currency to the payer from the receiver at default time τ of a corporate if default occurs after $T_n \geqslant 0$ and before or at maturity $T = T_N > T_n$. This is called the default leg of the QCDS . In return for the default leg the payer pays a constant premium K in domestic currency at specified dates $\mathcal{T} = T_{n+1}, \ldots, T_N$ if default has not occurred by time T_i for $i = n+1, \ldots, N$. Assume $\zeta(\tau)$ is chosen from the index set $\{n+1, \ldots, N-1\}$ such that $T_{\zeta(\tau)}$ is the premium payment date immediately preceding $\tau \leqslant T$. Then if there is a default at time τ before the maturity T of the contract, then the contract is terminated after an accrued payment of $(\tau - T_{\zeta(\tau)})K$*

is made in domestic currency by the payer. We call this the premium leg of the $QCDS$.

In practice defaultable bonds do not have a value of zero upon default of the corporate. The value of the defaultable bond upon default will be assumed to be a fraction δ of the par value. The value δ is called the recovery rate of the defaultable bond and the deterministic amount Z in $QCDS$ is generally assumed to be equal to $1-\delta$ where δ is the recovery rate of the foreign defaultable bond. Let $\zeta(\tau) = \max[i : n+1 \leqslant i \leqslant N, T_i < \tau]$ and $\beta_i = T_i - T_{i-1}$. Then the value of a domestic quanto $QCDS$ at time $t \leqslant T_n$ to the payer is given by the following

$$\begin{aligned} QCDS(t,\mathcal{T},T,K,Z) &= S_t \mathbb{E}^{\mathbb{Q}^f}\Big[\Big(\exp\Big(-\int_t^\tau r_s^f\, ds\Big) Z\, \mathbf{1}_{T_n<\tau\leqslant T}\Big|\mathcal{F}_t\Big] \\ &- \mathbb{E}^{\mathbb{Q}^d}\Big[\exp\Big(-\int_t^\tau r_s^d\, ds\Big)\Big((\tau - T_{\zeta(\tau)})K\, \mathbf{1}_{T_n<\tau\leqslant T_N} \\ &\qquad - \sum_{i=n+1}^{N} \exp\Big(-\int_{T_n}^{T_i} r_s^d\, ds\Big)\beta_i\, K\, \mathbf{1}_{\tau>T_i}\Big)\Big|\mathcal{F}_t\Big]. \end{aligned}$$

As in the case of standard credit default swaps the value of K in $QCDS$ is chosen in such a way that $QCDS(0,\mathcal{T},T,K,Z)$ is equal to zero. This is because there are no payments exchanged at the initiation or start of the swap. Even though there are quanto credit default swaps that are traded at time $t=0$ in some markets, these trades are not liquid enough to be quoted on Bloomberg or Reuters. Under the assumptions of this section we can use the martingale hazard process of τ which is equal to the hazard process of τ i.e.

$$\Lambda_t = \Gamma_t = \exp\Big(-\int_0^t \lambda_u\, du\Big)$$

to give the value of $QCDS$:

$$QCDS(t,\mathcal{T},T,K,Z) = \mathbf{1}_{\tau>t} S_t Z \int_{T_n}^{T_N} \mathbb{E}^{\mathbb{Q}^f}\Big[\exp\Big(-\int_t^s r_k^f + \lambda_k\,dk\Big)\lambda_s\Big|\mathcal{F}_t\Big]\,ds$$

$$-\mathbf{1}_{\tau>t} K \int_{T_n}^{T_N} \mathbb{E}^{\mathbb{Q}^d}\Big[\exp\Big(-\int_t^s r_k^d + \lambda_k\,dk\Big)\lambda_s\Big|\mathcal{F}_t\Big](s - T_{\zeta(s)})\,ds$$

$$-\mathbf{1}_{\tau>t} K \sum_{i=n+1}^{N} \beta_i\, \mathbb{E}^{\mathbb{Q}^d}\Big[\exp\Big(-\int_t^{T_i} r_k^d + \lambda_k\,dk\Big)\Big|\mathcal{F}_t\Big]$$

Using the defaultable forward measure $\bar{\mathbb{T}}_s^i$ corresponding to using the defaultable bond of the ith economy

$$\bar{P}^i(s,T) = \mathbf{1}_{\tau>s}\mathbb{E}^{\mathbb{Q}^i}\Big[\exp\Big(-\int_s^T r_k^i + \lambda_k\,dk\Big)\Big|\mathcal{F}_s\Big]$$

as the numeraire, the price of a domestic quanto CDS is given by

$$QCDS(t,\mathcal{T},T,R,Z) = \mathbf{1}_{\tau>t} S_t Z \int_{T_n}^{T_N} \bar{P}^f(t,s)\, \mathbb{E}^{\bar{\mathbb{T}}_s^f}[\lambda_s|\mathcal{F}_t]\,ds$$

$$-\mathbf{1}_{\tau>t} K \int_{T_n}^{T_N} \bar{P}^d(t,s)\, \mathbb{E}^{\bar{\mathbb{T}}_s^d}[\lambda_s|\mathcal{F}_t]\,(s - T_{\zeta(s)})\,ds$$

$$-\mathbf{1}_{\tau>t} K \sum_{i=n+1}^{N} \beta_i\, \bar{P}^d(t,T_i). \quad (4.65)$$

Under the assumption made in this section where we model r_t^d, r_t^f, λ_t and S_t using quadratic Gaussian and log-quadratic Gaussian processes respectively, we can get $\bar{P}^j(t,T)$ and $\mathbb{E}^{\bar{\mathbb{T}}_s^i}[\lambda_s|\mathcal{F}_t]$ in closed form. Since λ_s is a quadratic form in Gaussian random variables, we can use the fact that we can calculate the mean and variance-covariance matrix of λ_s under $\bar{\mathbb{T}}_s^f$ using (1.118) and (1.119) from Chapter 1. The value of $\mathbb{E}^{\bar{\mathbb{T}}_s^d}[\lambda_s|\mathcal{F}_t]$ can be obtained using the closed form formulas given by (1.40) and (1.41).

4.3 A Two-Country Contagion-Type Reduced Form Model

We now consider a reduced form model of credit risk where corporations operating in the domestic and foreign economy might default in one economy but not necessarily in the other economy. We assume that a single corporation has issued defaultable bonds in the domestic economy and in the foreign economy and that the defaultable bonds issued by the corporation do not have a cross default provision. Hence we have two default times $\tau_i, i = d, f$ representing the default of the corporation in the domestic economy for $i = d$ and the default of the corporation in the foreign economy for $i = f$. Once again we postulate the existence of a background filtration $\mathbb{F}$ such that $\mathbb{F} = (\mathcal{F}_t)_{(0 \leqslant t \leqslant T^*)}$ is generated by n independent Brownian motions $W^d(t) = W_i(t), i = 1, \dots, n$ and satisfies the usual conditions. The initial filtration $\mathcal{F}_0$ is taken to be trivial and the time horizon is assumed to be finite so that $T^* > 0$ is some finite number. In the following statements involving subscripts or superscripts which are represented by i apply to both economies and we sometimes omit saying "for $i = d(f)$ corresponding to the domestic(foreign) economy" when it is clear that this is the intended implication. Let $H_t^i = \mathbf{1}_{\tau_i \leqslant t}, i = d, f$ and $\mathcal{H}_t^i = \sigma(H_u^i : u \leqslant t), i = d, f$. Let $\mathcal{G}_t$ denote the enlarged filtration such that

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^d \vee \mathcal{H}_t^f.$$

The filtration $\mathcal{G}_t$ consists of the default free market information and whether the corporate has defaulted in one or both of the economies. The intensities of H_t^i which we denote by $\lambda_i(t), i = d, f$ which are $\mathcal{G}_t$ adapted have the property that

$$M_t^i = H_t^i - \int_0^{t \wedge \tau_i} \lambda_u^i \, du \tag{4.66}$$

is a $\mathcal{G}_t$-martingale under $\mathbb{Q}^i$ for $i = d, f$. We assume that the intensities have the following form:

$$\lambda_t^i = \eta_t^i + \mathbf{1}_{\tau_i \leqslant t} \alpha^i, \quad \text{for } i = d, f \tag{4.67}$$

where η_t^i is an $\mathcal{F}_t$ adapted process. The quantity α^d is a constant representing the jump of the intensity η_t^d upon the default of the corporation in the foreign economy. Similarly the constant quantity α^f represent the jump of the intensity η_t^f upon the default of the corporation in the domestic economy. Hence this is similar to a contagion model[6] of default for two obligors. The construction of default times τ^i, for $i = d, f$ such that (4.66) is true can be achieved using the method specified in Yu (2004). We assume that α^i are positive numbers and hence default of the corporation in one economy will lead to an increased intensity of default in the other economy. The foreign exchange rate S_t is assumed to be a diffusion process that is $\mathcal{F}_t$ measurable as in the previous section (see equation 4.11). Also as in the previous section, the change of measure between the domestic risk neutral measure $\mathbb{Q}^d$ and the foreign risk neutral measure $\mathbb{Q}^f$ is once again given by

$$\left.\frac{d\mathbb{Q}^f}{d\mathbb{Q}^d}\right|_{\mathcal{G}_{T*}} = \exp\left(\int_0^{T*} \sigma^S(u).W_u^d - \frac{1}{2}\int_0^{T*} ||\sigma^S(u)||^2\, ds\right).$$

We use the same notation used in the previous section for the price of default free and defaultable bonds. Under the 'no jump condition at τ^i' of Duffie et al. (1996)(see also Jeanblanc & Rutkowski (2000)) we can use the intensity λ_t^i for valuation purposes. In fact we have under the risk neutral measure $\mathbb{Q}^i$

$$\mathbb{E}^{\mathbb{Q}^i}\left[\exp\left(-\int_t^T r_u\, du\right)\mathbf{1}_{\tau^i>T}Y|\mathcal{G}_t\right] = \mathbf{1}_{\tau^i>t}\mathbb{E}^{\mathbb{Q}^i}\left[\exp\left(-\int_t^T r_u + \lambda_u\, du\right)Y|\mathcal{G}_t\right] \tag{4.68}$$

where Y is a $\mathcal{G}$ integrable random variable and $i = d(f)$ corresponding to the domestic(foreign) economy. If the 'no jump condition at τ^i' does not hold there is a more general valuation formula given in Duffie et al. (1996) and Jeanblanc & Rutkowski (2000). Under the assumptions of this section the 'no jump condition at τ^i' is not satisfied. However we can still use the formula given in (4.68) provided we use a different probability measure than the risk neutral measure $\mathbb{Q}^i$ as shown in Collin-Dufresne et al. (2004). We denote once again by Y a $\mathcal{G}$ integrable random variable representing the payoff of a defaultable security at time T provided the obligor has not defaulted by time T. Let

$$\Lambda_t^i := \exp\left(\int_0^t \lambda_u^i\, du\right) \tag{4.69}$$

[6] The phrase contagion model is used here to describe the model of default where a default of an obligor will lead to an increase in the intensity of default of the other obligor (see for example Jarrow and Yu (2001)).

and $\mathbb{Q}^{i'}$ the absolutely continuous measure defined by the following Radon-Nikodým density

$$\left.\frac{d\mathbb{Q}^{i'}}{d\mathbb{Q}^{i}}\right|_{\mathcal{G}_t} = Z_t := \mathbf{1}_{\tau^i > t\wedge T}\Lambda_{t\wedge T} \tag{4.70}$$

for $i = d, f$. Furthermore let $\mathcal{G}^{i'} := (\mathcal{G}^{i'}_t)_{t\geqslant 0}$ be the augmentation of $\mathcal{G}$ by the null sets of $\mathbb{Q}^{i}$. Then the following result is established in Collin-Dufresne et al. (2004)

Proposition 4.9. *The exdividend price of a security that pays Y at time T provided there is no default by time T in the i^{th} economy which is denoted by V_t is given by*

$$V_t = \mathbf{1}_{t<T}\mathbf{1}_{\tau^i>t}\mathbb{E}^{\mathbb{Q}^{i'}}\left[\exp\left(-\int_t^T r^i_u + \lambda^i_u\,du\right)Y\middle|\mathcal{G}^{i'}_t\right]. \tag{4.71}$$

Moreover any local $\mathcal{G}$ martingale under $\mathbb{Q}^{i}$ that does not jump at τ^i remains a $\mathcal{G}^{i'}_t$ local martingale under $\mathbb{Q}^{i'}$. In particular W^i_t remains a Brownian motion under $\mathbb{Q}^{i'}$ according to the Lenglart-Girsanov absolutely continuous change of measure theorem [7]. Using the property that appropriately discounted values of tradeable securities are martingales under the risk neutral measure, the price of default free zero coupon bonds are now given by

$$P^i(t,T) = \mathbb{E}^{\mathbb{Q}^{i'}}\left[\exp\left(-\int_t^T r^i_u\,du\right)\middle|\mathcal{G}^{i'}{}_t\right] = \mathbb{E}^{\mathbb{Q}^{i'}}\left[\exp\left(-\int_t^T r^i_u\,du\right)\middle|\mathcal{G}^{i}{}_t\right] \tag{4.72}$$

and the price of defaultable zero coupon bonds are given by

$$\mathbf{1}_{\tau^i>t}\bar{P}^i(t,T) = \mathbf{1}_{\tau^i>t}\mathbb{E}^{\mathbb{Q}^{i'}}\left[\exp\left(-\int_t^T r^i_u + \lambda^i_u\,du\right)\middle|\mathcal{G}^{i'}{}_t\right]. \tag{4.73}$$

Instead of specifying the short term rate of interest rate and the intensity of default, we can also start from the specification of the default free forward rates and the forward credit spreads. However if we want to consider an HJM model of default we need to find the HJM condition that has to be satisfied by the forward credit spread process. In the following we derive the HJM condition for the forward credit process in a contagion model based on the work of Elouerkhaoui (2002). Let $G^i(t,T)$ be the probability of survival under $\mathbb{T}^i$ which is represented by the following formula

$$\mathbb{E}^{\mathbb{T}^i}[\mathbf{1}_{\tau^i>T}|\mathcal{G}_t].$$

[7] See Collin-Dufresne et al. (2004) for more detail where exact conditions are given.

In order to calculate $G^i(t,T)$ using the intensity λ_t^i, we define the absolutely continuous measure $\mathbb{T}^{i'}$ through the Radon-Nikodým density

$$\left.\frac{d\mathbb{T}^{i'}}{d\mathbb{T}^i}\right|_{\mathcal{G}_t} = Z_t := \mathbf{1}_{\tau^i > t\wedge T}\Lambda_{t\wedge T}. \tag{4.74}$$

Furthermore let $\mathcal{G}^{i'} := (\mathcal{G}_t^{'})_{t\geqslant 0}$ be the augmentation of $\mathcal{G}$ by the null sets of $\mathbb{T}^{i'}$. Then using the result of Collin-Dufresne et al. (2004), we have

$$G^i(t,T) = \mathbb{E}^{\mathbb{T}^i}[\mathbf{1}_{\tau^i>T}|\mathcal{G}_t] = \mathbf{1}_{\tau^i>t}\mathbb{E}^{\mathbb{T}^{i'}}\left[\exp\left(-\int_t^T \lambda_u^i\, du\right)\middle|\mathcal{G}_t^{'}\right]. \tag{4.75}$$

In Elouerkhaoui (2002) the quantity considered is the probability of survival under the risk neutral measure $\mathbb{Q}^i$ which is given by the formula

$$\mathbb{E}^{\mathbb{Q}^i}[\mathbf{1}_{\tau^i>T}|\mathcal{G}_t]. \tag{4.76}$$

Moreover Elouerkhaoui (2002) considers the forward intensity associated with the probability of survival under the risk neutral measure which is given by

$$\tilde{h}^i(t,T) = -\frac{d}{dT}\mathbb{E}^{\mathbb{Q}^i}[\mathbf{1}_{\tau>T}|\mathcal{G}_t]. \tag{4.77}$$

Here we consider the forward credit process $s^i(t,T)$ similar to the one in the previous section.

Definition 4.10. *If $\tau^i > t$, the forward credit spread process $s^i(t,T)$ is a bounded process which is defined by*

$$s^i(t,T) := -\frac{d}{dT}\log_e(G^i(t,T)).$$

If $\tau^i < t$ we set $s^i(t,T) = \infty$ so that $G^i(t,T) = 0$.

We have on the set $\tau^i > t$

$$s^i(t,T) = -\frac{d}{dT}\log_e(G^i(t,T)) = -\frac{d}{dT}\log_e\left(\mathbb{E}^{\mathbb{T}^{i'}}\left[\exp\left(-\int_t^T \lambda_u^i\, du\right)\middle|\mathcal{G}'_t\right]\right). \tag{4.78}$$

Since $G^d(t,T)$ jumps if $\tau^f < t$ the dynamics of $s^i(t,T)$ should have a jump component[8]. Hence the dynamics of $s^i(t,T)$ for $\tau^i > t$ is assumed to be

$$ds^i(t,T) = \alpha_s^i(t,T)\,dt + \sigma_s^i(t,T)\,dW_t^i + (1 - H_t^i)\eta_{ij}(t,T)dM_t^i. \tag{4.79}$$

If $\tau^i \leqslant t$ we do not need to consider the dynamics as we can assume that $s^i(t,T) = \infty$. Thus the forward credit spread $s^i(t,T)$ jumps by a quantity $\eta_{ij}(t,T)$ at time τ^j, $(j \neq i)$ which is the time of default of the corporation in the j^{th} economy. We assume that $\eta_{ij}(t,T)$ are positive deterministic functions. We now assume that $\alpha^i, \sigma_s^i, \eta_{ij}$ and λ^i satisfy conditions that allow measurability, integrability, and the interchangeability of the order of integration (see Elouerkhaoui (2002) for more detail). Using the dynamics of $s^i(t,T)$ given in (4.79) the following can be shown to be true by the derivation given in Elouerkhaoui (2002)

$$\begin{aligned} d\Big(- \int_t^T s^i(t,u)\,du\Big) &= s^i(t,t) - \int_t^T ds^i(t,u)\,du & (4.80)\\ &= \lambda_t^i\,dt - \alpha_s^{i*}(t,T)\,dt - \sigma_s^{i*}(t,T)\,dW_t^i & (4.81)\\ &\quad - \sum_{j\neq i}(1 - H^j(t))\eta_{ij}^*(t,T)dM_t^j & (4.82) \end{aligned}$$

where

$$\alpha_s^{i*}(t,T) = \int_t^T \alpha_s^i(t,u)\,du, \tag{4.83}$$

$$\sigma_s^{i*}(t,T) = \int_t^T \sigma_s^i(t,u)\,du. \tag{4.84}$$

and

$$\eta_{ij}^*(t,T) = \int_t^T \eta^{ij}(t,u)\,du. \tag{4.85}$$

As in the previous section let $f^i(t,T)$ denote the default free forward rate of the i^{th} economy for maturity T. We assume that $f^i(t,T)$ is a diffusion process and hence according to (Heath et al. 1992) the dynamics of $f^i(t,T)$ under the risk neutral measure $\mathbb{Q}^i$ is given by

$$df^i(t,T) = \sigma^i(t,T)^{\top}\sigma^{i*}(t,T)\,dt - \sigma^{i*}(t,T)\,dW_t^i \tag{4.86}$$

[8] Similarly $G^f(t,T)$ jumps if $\tau^d < t$.

where

$$\sigma^{i*}(t,T) = \int_t^T \sigma^i(t,u)\, du. \tag{4.87}$$

and we assume certain measurability, integrability conditions are satisfied by $\sigma^i(t,T)$ that also allow interchangeability of the order of integration(see Heath et al. (1992) for more detail). Moreover similar to (4.80) we have

$$d\Big(- \int_t^T f^i(t,u)\, du\Big) = r_t^i - \int_t^T \sigma^i(t,u)^{\mathrm{T}} \sigma^{i*}(t,u)\, du\, dt - \sigma^{i*}(t,u)\, dW_t^i. \tag{4.88}$$

First note that

$$\frac{d}{dT} G^i(t,T) = \mathbb{E}^{\mathbf{T}^{i'}}\left[\frac{d}{dT} \exp\Big(- \int_t^T \lambda_u^i\, du\Big)\Big|\mathcal{G}'_t\right] \tag{4.89}$$

$$= \mathbb{E}^{\mathbf{T}^{i'}}\left[- \lambda^i(T) \exp\Big(- \int_t^T \lambda_u^i\, du\Big)\Big|\mathcal{G}'_t\right] \tag{4.90}$$

and therefore

$$\frac{d}{dT} G^i(t,T)\Big|_{T=t} = \mathbb{E}^{\mathbf{T}^{i'}}\Big[- \lambda^i(t)|\mathcal{G}'_t\Big] = -\lambda^i(t).$$

On the other hand

$$\frac{d}{dT} G^i(t,T) = \frac{d}{dT} \exp\Big(- \int_t^T s^i(t,u)\, du\Big) = -s^i(t,T) \exp\Big(- \int_t^T s^i(t,u)\, du\Big) \tag{4.91}$$

is true so that

$$\frac{d}{dT} G^i(t,T)\Big|_{T=t} = -s^i(t,t) \tag{4.92}$$

is true. Therefore

$$\lambda^i(t) = s^i(t,t). \tag{4.93}$$

In the following we derive the HJM condition for the forward credit spread $s^i(t,T)$ which is similar to the HJM condition for the forward credit spread given in Schönbucher (2000) and Proposition 13.1.1 of Bielecki and Rutkowski (2002) except for an additional term depending on the default behavior of the corporation in the j^{th} economy where $j \neq i$. In Elouerkhaoui (2002) the HJM condition for the dynamics of $\tilde{h}^i(t,T)$ (see equation (4.77)) is derived. Here based on some of the results in Elouerkhaoui (2002), we derive the HJM condition for $s^i(t,T)$. In case of independence between the short term interest rate r_t and the intensity of default λ_t the quantities $h^i(t,T)$ and $s^i(t,T)$ are equal while in the case of non zero correlation between r_t and λ_t they are different from each other. Note that we consider the more general case of a time and maturity dependent jump size which is equal to $\eta_{ij}(t,T)$ instead of just a constant jump size one as in (4.67). Hence the forward credit spread in the i^{th} economy jumps by $\eta_{ij}(t,T)$ upon the default of the corporation in the j^{th} economy.

Theorem 4.11. *Under $\mathbb{Q}^i$, we have*

$$\begin{aligned}\alpha_s^{i*}(t,T) = \frac{1}{2}\|\sigma_s^{i*}(t,T)\|^2 - \int_t^T \sigma^i(t,u)^\top \sigma^{i*}(t,u)\,du + \frac{1}{2}\|\sigma^{i*}(t,T)\|^2 \\ + \sigma^{i*}(t,T)^\top \sigma_s^{i*}(t,T) + \sum_{j\neq i}(1-H_t^j)[\eta_{ij}^*(t,T) + \exp(-\eta_{ij}^*(t,T)) - 1]\lambda_t^j. \quad (4.94)\end{aligned}$$

We can differentiate the above with respect to T to get

$$\begin{aligned}\alpha_s^i(t,T) = \sigma^i(t,T)^\top \sigma_s^{i*}(t,T) + (\sigma_s^i(t,T) + \sigma^i(t,T))^\top \sigma_s^{i*}(t,T) \\ + \sum_{j\neq i}(1-H_t^j)[\eta_{ij}(t,T)(1-\exp(-\eta_{ij}^*(t,T))]\lambda_t^j. \quad (4.95)\end{aligned}$$

Proof. First note that the appropriately discounted price of the defaultable bond:

$$\exp\Big(-\int_0^t r_u^i + \lambda_u^i\,du\Big)\bar{P}^i(t,T) \quad (4.96)$$

is a $\mathcal{G}$ martingale under $\mathbb{Q}^i$ and therefore the drift term in the SDE of $\bar{P}^i(t,T)$ must be equal to

$$(r_t^i + \lambda_t^i)\,dt.$$

We have

$$\begin{aligned}\bar{P}^i(t,T) &= \mathbb{E}^{\mathbb{Q}^i}\Big[\exp\Big(-\int_0^t r_u^i\,du\Big)\mathbf{1}_{\tau^i>T}\Big|\mathcal{G}_t\Big] && (4.97)\\ &= P^i(t,T)\mathbb{E}^{\mathbf{T}^i}[\mathbf{1}_{\tau^i>T}|\mathcal{G}_t\Big] && (4.98)\\ &= P^i(t,T)G^i(t,T) && (4.99)\\ &= \exp\Big(-\int_t^T f^i(t,u) + s^i(t,u)\,du\Big). && (4.100)\end{aligned}$$

Using (4.80), (4.88) and applying the multivariate version of Ito's formula for jump diffusions, we can give the SDE that is satisfied by $\bar{P}^i(t,T)$.

$$d\bar{P}^i(t,T) = \bar{P}^i(t,T)\Bigg[\Bigg(r_t^i + \lambda_t^i - \alpha_s^{i*}(t,T) - \int_t^T \sigma(t,u)\sigma^{i*}(t,u)\,du + \frac{1}{2}||\sigma^{i*}(t,T)||^2$$
$$+ \frac{1}{2}||\sigma_s^{i*}(t,T)||^2 + \sigma^{i*}(t,T)^{\mathsf{T}}\sigma_s^{i*}(t,T)\Bigg)\,dt - \sigma^{i*}(t,T)\,dW_t^i - \sigma_s^{i*}(t,T)\,dW_t^i$$
$$+ \sum_{j\neq i}(1-H_t^j)(\exp(-\eta_{ij}^*(t,T)) - 1)(dH_t^j - \lambda_t^j\,dt)$$
$$+ \sum_{j\neq i}(1-H_t^j)(\exp(-\eta_{ij}^*(t,T)) - 1 + \eta_{ij}^*(t,T))\lambda_t^j\,dt. \quad (4.101)$$

Since we know the the drift of $\bar{P}^i(t,T)$ is equal to $r_t^i + \lambda_t^i$, it follows from (4.101) that

$$-\alpha_s^{i*}(t,T) - \int_t^T \sigma(t,u)^{\mathsf{T}}\sigma^{i*}(t,u)\,du + \frac{1}{2}||\sigma^{i*}(t,T)||^2 + \frac{1}{2}||\sigma_s^{i*}(t,T)||^2 + \sigma^{i*}(t,T)^{\mathsf{T}}\sigma_s^{i*}(t,T)$$
$$+ \sum_{j\neq i}(1-H_t^j)(\exp(-\eta_{ij}^*(t,T)) - 1 + \eta_{ij}^*(t,T))\lambda_t^j\,dt = 0. \quad (4.102)$$

Solving for $\alpha_s^{i*}(t,T)$ gives the result of the theorem. □

The drift condition for $\alpha_s^{i*}(t,T)$ given in (4.95) of Theorem (4.11) is similar to the one derived in Proposition 13.1.1 of Bielecki and Rutkowski (2002) and Schönbucher (2000), the only difference being the existence of an additional term that depends on on the default behavior of the corporation in the other economy where $j \neq i$ in H_t^j represents the default indicator function of the corporation in the other economy.

4.4 Pricing Quanto Default Swaps in a Contagion-Type Model

We now consider the price of a domestic quanto default swap(see definition 4.8) assuming no cross default holds. Without loss of generality we assume that the notional on which the payments are made is equal to one. Hence if the corporation defaults in the foreign economy an amount equal to Z units of foreign currency is paid to the payer from the receiver in return for premium payments in domestic currency at times $\mathcal{T} = \{T_{n+1}, \ldots, T_N\}$ provided the corporation has not defaulted in the foreign economy i.e. $\tau^f > T_i$. Under the assumption of zero recovery on defaultable bonds Z is equal to one unit of foreign currency. We denote by S_t the value of one unit of foreign currency in terms of domestic currency consistent with notation used in previous sections. Therefore $Z = S_{\tau^f}$ which is the value of S_t at the time of default of the corporation in the foreign economy. A quanto default swap is useful for an investor holding the foreign defaultable bond of the corporation whereby the investor's income is denominated in domestic economy. Therefore the default leg of the domestic quanto CDS has the following value

$$S_t\, Z\, \mathbb{E}^{\mathbb{Q}^f}\Big[\exp\Big(-\int_t^{\tau^f} r_u^f\, du\Big)\mathbf{1}_{t<\tau^f\leqslant T_N}|\mathcal{G}_t\Big] \tag{4.103}$$

in terms of domestic currency. The intensities of default of the corporation are assumed to be as in (4.67) which is given at the beginning of this section. Hence the intensity τ^f jumps by a constant amount α^f upon the default of the corporation in the domestic economy.

The value of the default leg can be decomposed into two parts depending on whether the corporation has not defaulted in the domestic economy by time T or the corporation has defaulted in the domestic economy by time T:

$$S_t\, Z\, \mathbb{E}^{\mathbb{Q}^f}\Big[\exp\Big(-\int_t^{\tau^f} r_u^f\, du\Big)\mathbf{1}_{t<\tau^f\leqslant T}\Big|\mathcal{G}_t\Big] = S_t\, Z\,(V_t + W_t) \tag{4.104}$$

where

$$V_t = \mathbb{E}^{\mathbb{Q}^f}\Big[\mathbf{1}_{\tau^d>T_N}\exp\Big(-\int_t^{\tau^f} r_u^f\, du\Big)\mathbf{1}_{t<\tau^f\leqslant T_N}\Big|\mathcal{G}_t\Big] \tag{4.105}$$

and

$$W_t = \mathbb{E}^{\mathbb{Q}^f}\Big[\mathbf{1}_{t<\tau^d\leqslant T_N}\exp\Big(-\int_t^{\tau^f} r_u^f\, du\Big)\mathbf{1}_{t<\tau^f\leqslant T_N}\Big|\mathcal{G}_t\Big]. \tag{4.106}$$

Recall that $\mathbb{Q}^{f'}$ denote the measure that is absolutely continuous with respect to $\mathbb{Q}^f$ and $\mathcal{G}^{f'} = (\mathcal{G}_t^{f'})_{t\geqslant 0}$ is the augmentation of $\mathcal{G}$ by the null sets of $\mathbb{Q}^{f'}$ (see (4.70)).

To calculate V_t in (4.105) we use Proposition 4.9 to get

$$V_t = \mathbf{1}_{\tau^f > t} \mathbb{E}^{\mathbb{Q}^{f'}} \left[\mathbf{1}_{\tau^d > T_N} \int_t^{T_N} \exp\left(- \int_t^u r_s^f + \eta_s^f \, ds \right) \eta_u^f \, du \Big| \mathcal{G}_t^{f'} \right] \tag{4.107}$$

$$= \mathbf{1}_{\tau^f \wedge \tau^d > t} \mathbb{E}^{\mathbb{Q}^{f'}} \left[\exp\left(- \int_t^{T_N} \eta_s^d \, ds \right) \int_t^{T_N} \exp\left(- \int_t^u r_s^f + \eta_s^f \, ds \right) \eta_u^f \, du \Big| \mathcal{G}_t^{f'} \right] \tag{4.108}$$

$$= \mathbf{1}_{\tau^f \wedge \tau^d > t} \mathbb{E}^{\mathbb{Q}^{f'}} \left[\int_t^{T_N} \exp\left(- \int_t^u r_s^f + \eta_s^f \, ds - \int_t^{T_N} \eta_s^d \, ds \right) \eta_u^f \, du \Big| \mathcal{G}_t^{f'} \right] \tag{4.109}$$

The equality (4.107) follows from the assumption $\mathbf{1}_{\tau^d > T_N}$ which implies that the intensity of τ^f does not jump up to time T_N while the equality (4.108) is obtained using the fact that under $\mathbb{Q}^{f'}$ we can consider the intensity of τ^d to be η_t^d since under this measure the set $\tau^f < T_N$ is a set of measure zero. To calculate (4.109) we use Fubini's theorem to exchange the integral and expectation signs since the integrand in (4.109) is positive. Let

$$\bar{P}^{f1}(t, u, T_N) := \mathbb{E}^{\mathbb{Q}^{f'}} \left[\exp\left(- \int_t^u r_s^f + \eta_s^f \, ds - \int_t^{T_N} \eta_s^d \, ds \right) \Big| \mathcal{G}_t^{f'} \right] \tag{4.110}$$

which can be viewed as the predefault value of a defaultable bond where the short term rate of interest r^f is equal to zero after time u and the intensity up to time u is $\eta_t^f + \eta_t^d$ but after time u it is equal to η_t^d. Let $\bar{\mathbb{T}}^{f1}$ denote the absolutely continuous measure with respect to $\mathbb{Q}^{f'}$ with the following Radon-Nikodým density

$$\frac{d\bar{\mathbb{T}}^{f1}}{d\mathbb{Q}^{f'}} \bigg|_{\mathcal{G}'_{T_N}} = \frac{\mathbf{1}_{\tau^d > T_N} \exp\left(- \int_0^u r_s^f + \eta_s^f \, ds \right)}{\mathbb{E}^{\mathbb{Q}^{f'}} \left[\mathbf{1}_{\tau^d > T_N} \exp\left(- \int_0^u r_s^f + \eta_s^f \, ds \right) \right]}. \tag{4.111}$$

Using the abstract Bayes theorem we can write for any $\mathcal{G}'_{T_N}$ measurable and $\mathbb{Q}^{f'}$ integrable random variable Y

$$\mathbb{E}^{\bar{\mathbb{T}}^{f1}}[Y | \mathcal{G}'_t] = \frac{\mathbb{E}^{\mathbb{Q}^{f'}} \left[\mathbf{1}_{\tau^d > T_N} \exp\left(- \int_0^u r_s^f + \eta_s^f \, ds \right) Y \Big| \mathcal{G}'_t \right]}{\mathbb{E}^{\mathbb{Q}'} \left[\mathbf{1}_{\tau^d > T_N} \exp\left(- \int_0^u r_s^f + \eta_s^f \, ds \right) \Big| \mathcal{G}'_t \right]} \tag{4.112}$$

Hence using (4.112) and Fubini's theorem we have

$$V_t = \mathbf{1}_{\tau^f \wedge \tau^d > t} \int_t^{T_N} \bar{P}^{f1}(t,u,T_N) \mathbb{E}^{\bar{\mathbb{T}}^{f1}} \left[\eta^f_u \middle| \mathcal{G}^{f'}_t \right] du. \tag{4.113}$$

We can calculate W_t in a similar way:

$$\begin{aligned}
W_t &= \mathbf{1}_{\tau^f < t} \mathbb{E}^{\mathbb{Q}^{f'}} \left[\mathbf{1}_{t<\tau^d \leqslant T_N} \int_t^{T_N} \exp\left(- \int_t^u r^f_s + \eta^f_s + \alpha^f \mathbf{1}_{\tau^d \leqslant s}\, ds \right) (\eta^f_u + \alpha^f \mathbf{1}_{\tau^d \leqslant u})\, du \middle| \mathcal{G}^{f'}_t \right] \\
&= \mathbf{1}_{\tau^f \wedge \tau^d > t} \mathbb{E}^{\mathbb{Q}^{f'}} \left[\int_t^{T_N} \int_t^{T_N} \exp\left(- \int_t^u r^f_s + \eta^f_s + \alpha^f \mathbf{1}_{w \leqslant s}\, ds \right) (\eta^f_u + \alpha^f \mathbf{1}_{w \leqslant u})\, du \right.
\end{aligned}$$

$$\left. \exp\left(- \int_t^w \eta^d_s\, ds \right) \eta^d_w\, dw \middle| \mathcal{G}^{f'}_t \right] = \mathbf{1}_{\tau^f \wedge \tau^d > t} \mathbb{E}^{\mathbb{Q}^{f'}} \left[\int_t^{T_N} \left(\int_t^w \exp\left(- \int_t^u r^f_s + \eta^f_s\, ds \right) \eta^f_u\, du + \right.\right.$$

$$\left.\left. \int_w^{T_N} \exp\left(- \int_t^u r^f_s + \eta^f_s\, ds - \alpha^f (u-w) \right) (\eta^f_u + \alpha^f)\, du \right) \exp\left(- \int_t^w \eta^d_s\, ds \right) \eta^d_w\, dw \middle| \mathcal{G}^{f'}_t \right]$$

(4.114)

The first equality in (4.114) follows from Proposition 4.9. Similarly the second equality in (4.114) follows from the fact that under $\mathbb{Q}^{f'}$, we can consider the intensity of τ^d to be η^d_t since under this measure the set $\tau^f < T_N$ is a set of measure zero and this implies there is no jump in the intensity of τ^d. We now use again Fubini's theorem to exchange the expectation and integral sign in (4.114) since the integrand is positive:

$$W_t = \mathbf{1}_{\tau^f \wedge \tau^d > t} \left(\int_t^{T_N} \int_t^w \mathbb{E}^{\mathbb{Q}^{f'}} \left[\exp\left(- \int_t^u r^f_s + \eta^f_s\, ds - \int_t^w \eta^d_s\, ds \right) \eta^f_u \eta^d_w \middle| \mathcal{G}^{f'}_t \right] du\, dw + \right.$$

$$\left. \int_t^{T_N} \int_w^{T_N} \mathbb{E}^{\mathbb{Q}^{f'}} \left[\exp\left(- \int_t^u r^f_s + \eta^f_s\, ds - \alpha^f (u-w) - \int_t^w \eta^d_s\, ds \right) (\eta^f_u + \alpha^f) \eta^d_w \middle| \mathcal{G}^{f'}_t \right] du\, dw \right).$$

(4.115)

Let

$$\bar{P}^{f2}(w,u) := \mathbb{E}^{\mathbb{Q}^{f'}}\Big[\exp\Big(-\int_w^u r_s^f + \eta_s^f\,ds\Big)\Big|\mathcal{G}_t^{f'}\Big]. \tag{4.116}$$

We now can write (4.115) as

$$\begin{aligned}
W_t =\ &\mathbf{1}_{\tau^f\wedge\tau^d>t}\Big(\int_t^{T_N}\int_t^w \mathbb{E}^{\mathbb{Q}^{f'}}\Big[\exp\Big(-\int_t^u r_s^f + \eta_s^f\,ds - \int_t^w \eta_s^d\,ds\Big)\eta_u^f\eta_w^d\Big|\mathcal{G}_t^{f'}\Big]\,du\,dw +\\
&\int_t^{T_N}\int_w^{T_N} \mathbb{E}^{\mathbb{Q}^{f'}}\Big[\exp\Big(-\int_t^w r_s^f + \eta_s^f\,ds - \alpha^f(u-w) - \int_t^w \eta_s^d\,ds\Big)\\
&\mathbb{E}^{\mathbb{Q}^{f'}}\Big[\exp\Big(-\int_w^u r_s^f + \eta_s^f\,ds\Big)\Big|\mathcal{G}_w^{f'}\Big](\eta_u^f + \alpha^f)\eta_w^d\Big|\mathcal{G}_t^{f'}\Big]\,du\,dw\Big)\\
=\ &\mathbf{1}_{\tau^f\wedge\tau^d>t}\Big(\int_t^{T_N}\int_t^w \mathbb{E}^{\mathbb{Q}^{f'}}\Big[\exp\Big(-\int_t^u r_s^f + \eta_s^f\,ds - \int_t^w \eta_s^d\,ds\Big)\eta_u^f\eta_w^d\Big|\mathcal{G}_t^{f'}\Big]\,du\,dw +\\
&\int_t^{T_N}\int_w^{T_N} \exp(-\alpha^f(u-w))\mathbb{E}^{\mathbb{Q}^{f'}}\Big[\exp\Big(-\int_t^w r_s^f + \eta_s^f\,ds - \int_t^w \eta_s^d\,ds\Big)\\
&\bar{P}^{f2}(w,u)(\eta_u^f + \alpha^f)\eta_w^d\Big|\mathcal{G}_t^{f'}\Big]\,du\,dw\Big).
\end{aligned} \tag{4.117}$$

We now use the following notations:

$$\bar{P}^{f1}(t,u,w) := \mathbb{E}^{\mathbb{Q}^{f'}}\Big[\exp\Big(-\int_t^u r_s^f + \eta_s^f\,ds - \int_t^w \eta_s^d\,ds\Big)\Big|\mathcal{G}_t^{f'}\Big] \text{ for } u < w \tag{4.118}$$

$$\bar{P}^{\widetilde{f1}}(t,w,w) := \mathbb{E}^{\mathbb{Q}^{f'}}\Big[\exp\Big(-\int_t^w r_s^f + \eta_s^f\,ds - \int_t^w \eta_s^d\,ds\Big)\Big|\mathcal{G}_t^{f'}\Big] \text{ for } w < u. \tag{4.119}$$

Note that (4.118) is the predefault value of the defaultable bond of maturity w which was introduced in (4.110). The value (4.119) is also the predefault value of the defaultable bond of shorter maturity which was introduced in (4.110) but here $w < u$ and hence the maturity is w and to avoid confusion we used a different notation. We define $\bar{\mathbb{T}}^{f1}_w$ and $\bar{\mathbb{T}}^{\widetilde{f1}}_w$ similar to (4.111) by:

$$\left.\frac{d\bar{\mathbb{T}}^{f1}_w}{d\mathbb{Q}^{f'}}\right|_{\mathcal{G}'_w} = \frac{\mathbf{1}_{\tau^d > w} \exp\left(-\int_0^u r^f_s + \eta^f_s \, ds\right)}{\mathbb{E}^{\mathbb{Q}^{f'}}\left[\mathbf{1}_{\tau^d > w} \exp\left(-\int_0^u r^f_s + \eta^f_s \, ds\right)\right]} \text{ for } u < w \tag{4.120}$$

$$\left.\frac{d\bar{\mathbb{T}}^{\widetilde{f1}}_w}{d\mathbb{Q}^{f'}}\right|_{\mathcal{G}'_u} = \frac{\mathbf{1}_{\tau^d > w} \exp\left(-\int_0^w r^f_s + \eta^f_s \, ds\right)}{\mathbb{E}^{\mathbb{Q}^{f'}}\left[\mathbf{1}_{\tau^d > w} \exp\left(-\int_0^w r^f_s + \eta^f_s \, ds\right)\right]} \text{ for } w < u. \tag{4.121}$$

Therefore we can claim

$$\begin{aligned} W_t &= \mathbf{1}_{\tau^f \wedge \tau^d > t}\Bigg(\int_t^{T_N}\int_t^{w} \bar{P}^{f1}(t,u,w)\mathbb{E}^{\bar{\mathbb{T}}^{f1}_w}\left[\eta^f_u \eta^d_w \middle| \mathcal{G}^{f'}_t\right] du\, dw \\ &+ \int_t^{T_N}\int_w^{T_N} \exp(-\alpha^f(u-w))\bar{P}^{f1}(t,w,w)\mathbb{E}^{\bar{\mathbb{T}}^{\widetilde{f1}}_w}\left[\bar{P}^{f2}(w,u)(\eta^f_u + \alpha^f)\eta^d_w \middle| \mathcal{G}^{f'}_t\right] du\, dw\Bigg) \end{aligned} \tag{4.122}$$

as by an application of the abstract Bayes theorem (4.122) is equivalent to (4.115).

We now consider the calculation of the premium leg of the quanto default swap. Ignoring the accrued premium, the premium leg consists of payments of $\beta_i K$ in domestic currency at times

$$T = T_{n+1}, \ldots, T_N$$

provided the corporation has not defaulted by time T_i for $i = n+1 \ldots, N$ where $\beta_i \doteq T_{i+1} - T_i$ and K is a constant premium amount set at time $T_n = T$. We value the premium leg in the foreign economy using the foreign exchange to convert the premium payments and finally converting the total value of the premium leg into domestic currency using the foreign exchange. The value of the premium leg can be decomposed into two parts depending on whether the corporation has not defaulted in the domestic economy by time T_N or the corporation has defaulted in the domestic economy by time T_N:

$$S_t \sum_{i=n+1}^{N} \mathbb{E}^{\mathbb{Q}^f}\left[\exp\left(-\int_t^{T_i} r^f_u \, du\right)\mathbf{1}_{\tau^f > T_i}\beta_i K \frac{1}{S_{T_i}}\middle|\mathcal{G}_t\right] = S_t(\bar{V}_t + \bar{W}_t) \tag{4.123}$$

where

$$\bar{V}_t = S_t K \sum_{i=n+1}^{N} \beta_i \mathbb{E}^{\mathbb{Q}^f}\Big[\mathbf{1}_{\tau^d > T_N} \exp\Big(-\int_t^{T_i} r_u^f\,du\Big)\mathbf{1}_{\tau^f > T_i}\frac{1}{S_{T_i}}|\mathcal{G}_t\Big] \tag{4.124}$$

and

$$\overline{W}_t = S_t K \sum_{i=n+1}^{N} \beta_i \mathbb{E}^{\mathbb{Q}^f}\Big[\mathbf{1}_{t<\tau^d\leqslant T_N} \exp\Big(-\int_t^{T_i} r_u^f\,du\Big)\mathbf{1}_{\tau^f > T_i}\frac{1}{S_{T_i}}|\mathcal{G}_t\Big]. \tag{4.125}$$

To calculate

$$\bar{V}_{it} := \mathbb{E}^{\mathbb{Q}^f}\Big[\mathbf{1}_{\tau^d > T_N} \exp\Big(-\int_t^{T_i} r_u^f\,du\Big)\mathbf{1}_{\tau^f > T_i}\frac{1}{S_{T_i}}|\mathcal{G}_t\Big] \tag{4.126}$$

we proceed in a similar fashion to the calculation of V_t in (4.107) by using once again Proposition 4.9 to obtain the following

$$\bar{V}_{it} = \mathbf{1}_{\tau^f > t}\mathbb{E}^{\mathbb{Q}^{f'}}\Big[\mathbf{1}_{\tau^d > T_N} \exp\Big(-\int_t^{T_i} r_s^f + \eta_s^f\,ds\Big)\frac{1}{S_{T_i}}\Big|\mathcal{G}_t^{f'}\Big] \tag{4.127}$$

$$= \mathbf{1}_{\tau^f\wedge\tau^d > t}\mathbb{E}^{\mathbb{Q}^{f'}}\Big[\exp\Big(-\int_t^{T_i} r_s^f + \eta_s^f - \int_t^{T_N}\eta_s^d\,ds\Big)\frac{1}{S_{T_i}}\Big|\mathcal{G}_t^{f'}\Big]. \tag{4.128}$$

Let

$$\bar{P}^{f3}(t,T_i) := \mathbb{E}^{\mathbb{Q}^{f'}}\Big[\exp\Big(-\int_t^{T_i} r_s^f + \eta_s^f + \eta_s^d\,ds\Big)\Big|\mathcal{G}_t^{f'}\Big] \tag{4.129}$$

and let the measure $\bar{\mathbb{T}}_i^{f3}$ which is absolutely continuous with respect to $\mathbb{Q}^{f'}$ be defined by the following Radon-Nikod'ym density

$$\frac{d\bar{\mathbb{T}}_i^{f3}}{d\mathbb{Q}^{f'}}\Big|_{\mathcal{G}'_{T_i}} = \frac{\mathbf{1}_{\tau^d > T_i}\exp\Big(-\int_0^{T_i} r_s^f + \eta_s^f\,ds\Big)}{\mathbb{Q}^{f'}\Big[\mathbf{1}_{\tau^d > T_i}\exp\Big(-\int_0^{T_i} r_s^f + \eta_s^f\,ds\Big)\Big]}. \tag{4.130}$$

Furthermore let

$$G^{f3}(T_i, T_N) := \mathbb{E}^{\mathbf{Q}^{f'}}\Big[\exp\Big(-\int_{T_i}^{T_N} \eta_s^d\, ds\Big)\Big|\mathcal{G}_{T_i}^{f'}\Big]. \tag{4.131}$$

We can now claim that

$$\begin{aligned}\bar{V}_{it} &= \mathbf{1}_{\tau^f \wedge \tau^d > t}\mathbb{E}^{\mathbf{Q}^{f'}}\Big[\exp\Big(-\int_t^{T_i} r_s^f + \eta_s^f + \eta_s^d\, ds\Big) \\ &\qquad \mathbb{E}^{\mathbf{Q}^{f'}}\Big[\exp\Big(-\int_{T_i}^{T_N} \eta_s^d\, ds\Big)\Big|\mathcal{G}_{T_i}^{f'}\Big]\frac{1}{S_{T_i}}\Big|\mathcal{G}_t^{f'}\Big] \\ &= \mathbf{1}_{\tau^f \wedge \tau^d > t}\bar{P}^{f3}(t, T_i)\,\mathbb{E}^{\bar{\mathbf{T}}_i^{f3}}\Big[G^{f3}(T_i, T_N)\frac{1}{S_{T_i}}\Big|\mathcal{G}_t^{f'}\Big].\end{aligned} \tag{4.132}$$

To calculate

$$\overline{W}_{it} = \mathbb{E}^{\mathbf{Q}^f}\Big[\mathbf{1}_{t<\tau^d\leqslant T_N}\exp\Big(-\int_t^{T_i} r_u^f\, du\Big)\mathbf{1}_{\tau^f > T_i}\frac{1}{S_{T_i}}|\mathcal{G}_t\Big] \tag{4.133}$$

we proceed similar to (4.114) using Proposition 4.9. Thus

$$\overline{W}_{it} = \mathbf{1}_{\tau^f > t}\mathbb{E}^{\mathbf{Q}^{f'}}\Big[\mathbf{1}_{t<\tau^d\leqslant T_N}\exp\Big(-\int_t^{T_i} r_s^f + \eta_s^f + \alpha^f \mathbf{1}_{\tau^d \leqslant s}\, ds\Big)\frac{1}{S_{T_i}}\Big|\mathcal{G}_t^{f'}\Big] \tag{4.134}$$

$$= \mathbf{1}_{\tau^f > t}\mathbb{E}^{\mathbf{Q}^{f'}}\Big[\mathbf{1}_{t<\tau^d\leqslant T_N}\exp\Big(-\int_t^{T_i} r_s^f + \eta_s^f\, ds - \alpha^f (T_N - \tau^d)\Big)\frac{1}{S_{T_i}}\Big|\mathcal{G}_t^{f'}\Big] \tag{4.135}$$

$$
\begin{aligned}
&= \mathbf{1}_{\tau^f \wedge \tau^d > t} \mathbb{E}^{\mathbb{Q}^{f'}} \Bigg[\exp\Big(- \int_t^{T_i} r_s^f + \eta_s^f \, ds \Big) \frac{1}{S_{T_i}} \\
&\qquad \int_t^{T_N} \exp(-\alpha^f (T_N - u)) \exp\Big(- \int_t^u \eta_s^d \, ds \Big) \eta_u^d \, du \Big| \mathcal{G}_t^{f'} \Bigg] \\
&= \mathbf{1}_{\tau^f \wedge \tau^d > t} \int_t^{T_N} \exp(-\alpha^f (T_N - u)) \\
&\qquad \mathbb{E}^{\mathbb{Q}^{f'}} \Bigg[\frac{1}{S_{T_i}} \exp\Big(- \int_t^{T_i} r_s^f + \eta_s^f \, ds - \int_t^u \eta_s^d \, ds \Big) \eta_u^d \Big| \mathcal{G}_t^{f'} \Bigg] du \\
&= \mathbf{1}_{\tau^f \wedge \tau^d > t} \Bigg(\int_t^{T_i} \exp(-\alpha^f (T_N - u)) \\
&\qquad \mathbb{E}^{\mathbb{Q}^{f'}} \Bigg[\frac{1}{S_{T_i}} \exp\Big(- \int_t^{T_i} r_s^f + \eta_s^f \, ds - \int_t^u \eta_s^d \, ds \Big) \eta_u^d \Big| \mathcal{G}_t^{f'} \Bigg] du + \\
&\quad + \int_{T_i}^{T_N} \exp(-\alpha^f (T_N - u)) \\
&\qquad \mathbb{E}^{\mathbb{Q}^{f'}} \Bigg[\frac{1}{S_{T_i}} \exp\Big(- \int_t^{T_i} r_s^f + \eta_s^f \, ds - \int_t^u \eta_s^d \, ds \Big) \eta_u^d \Big| \mathcal{G}_t^{f'} \Bigg] du \Bigg) \\
&= \mathbf{1}_{\tau^f \wedge \tau^d > t} \Bigg(\int_t^{T_i} \exp(-\alpha^f (T_N - u)) \\
&\mathbb{E}^{\mathbb{Q}^{f'}} \Bigg[\frac{1}{S_{T_i}} \exp\Big(- \int_t^u r_s^f + \eta_s^f + \eta_s^d \, ds \Big) \\
&\qquad \mathbb{E}^{\mathbb{Q}^{f'}} \Bigg[\exp\Big(- \int_u^{T_i} r_s^f + \eta_s^f \, ds \Big) \Big| \mathcal{G}_u^{f'} \Bigg] \eta_u^d \Big| \mathcal{G}_t^{f'} \Bigg] du \\
&\quad + \int_{T_i}^{T_N} \exp(-\alpha^f (T_N - u)) \mathbb{E}^{\mathbb{Q}^{f'}} \Bigg[\frac{1}{S_{T_i}} \exp\Big(- \int_t^{T_i} r_s^f + \eta_s^f + \eta_s^d \, ds \Big) \\
&\qquad \mathbb{E}^{\mathbb{Q}^{f'}} \Bigg[\exp\Big(- \int_{T_i}^u \eta_s^d \, ds \Big) \Big| \mathcal{G}_{T_i}^{f'} \Bigg] \eta_u^d \Big| \mathcal{G}_t^{f'} \Bigg] du \Bigg) \qquad (4.136)
\end{aligned}
$$

Let

$$\bar{P}^{f4}(t,u) := \mathbb{E}^{\mathbb{Q}^{f'}}\left[\exp\left(-\int_t^u r_s^f + \eta_s^f + \eta_s^d\,ds\right)\Big|\mathcal{G}_t^{f'}\right] \text{ where } u \leqslant T_i \tag{4.137}$$

$$\bar{P}^{f5}(u,T_i) := \mathbb{E}^{\mathbb{Q}^{f'}}\left[\exp\left(-\int_u^{T_i} r_s^f + \eta_s^f\,ds\right)\Big|\mathcal{G}_u^{f'}\right] \tag{4.138}$$

$$\tag{4.139}$$

$$G^{f6}(T_i,u) := \mathbb{E}^{\mathbb{Q}^{f'}}\left[\exp\left(-\int_{T_i}^u \eta_s^d\,ds\right)\Big|\mathcal{G}_{T_i}^{f'}\right]. \tag{4.140}$$

and let the measures $\bar{\mathbb{T}}_u^{f4}$ and $\bar{\mathbb{T}}_{T_i}^{f6}$ which are absolutely continuous with respect to $\mathbb{Q}^{f'}$ be defined by the following Radon-Nikod'ym density

$$\frac{d\bar{\mathbb{T}}_u^{f4}}{d\mathbb{Q}^{f'}}\bigg|_{\mathcal{G}_u'} = \frac{\mathbf{1}_{\tau^d>u}\exp\left(-\int_0^u r_s^f + \eta_s^f\,ds\right)}{\mathbb{Q}^{f'}\left[\mathbf{1}_{\tau^d>u}\exp\left(-\int_0^u r_s^f + \eta_s^f\,ds\right)\right]}. \tag{4.141}$$

We can now claim that

$$\begin{aligned}\overline{W}_{it} = \mathbf{1}_{\tau^f\wedge\tau^d>t}\Bigg(&\int_t^{T_i}\exp(-\alpha^f(T_N-u))\bar{P}^{f4}(t,u)\mathbb{E}^{\bar{\mathbb{T}}_u^{f4}}\left[\frac{1}{S_{T_i}}\bar{P}^{f5}(u,T_i)\eta_u^d\Big|\mathcal{G}_t^{f'}\right]du\\ &+\int_{T_i}^{T_N}\exp(-\alpha^f(T_N-u))\bar{P}^{f3}(t,T_i)\mathbb{E}^{\bar{\mathbb{T}}_{T_i}^{f3}}\left[\frac{1}{S_{T_i}}G^{f6}(T_i,u)\eta_u^d\Big|\mathcal{G}_t^{f'}\right]du\Bigg)\end{aligned} \tag{4.142}$$

as an application of the abstract Bayes theorem to (4.142) gives us (4.136). Note W_t in (4.115) requires the evaluation of a double integral with an integrand term containing $\mathbb{E}^{\mathbb{T}^{f1}}\left[\eta_u^f\eta_w^d|\mathcal{G}_t^{f'}\right]$ which is the expectation of a nonlinear function of η_t for varying t. In Kwok and Leung (2005) the valuation of a similar problem as the one considered here is shown to be possible with closed form formulas for the credit default swap premium under the assumption of constant r^f, λ^f and λ^d. However as can be seen here when we assume that r^f, λ^f and λ^d are stochastic, obtaining a closed form formula requires that we are to calculate the expectations involved in calculating V_t and especially W_t which is generally difficult to do. In the following we will show that the calculation of the expectations in $V_t, W_t, \bar{V}_{it}$ and $\overline{W}_{it}$ can be accomplished in analytically closed form where by closed form we mean up to numerical integration.

Let Y_t be a multivariate Ornstein-Uhlenbeck process such that

$$dY_t = A\, Y_t + \Sigma\, dW_t^f \tag{4.143}$$

where

$$Y_t = (Y_{1t}, \ldots, Y_{nt})', \; Y_0 = (0, \ldots, 0)$$

and $\alpha(t)$ is a vector time dependent function used to calibrate to the default free and defaultable term structures. Note unlike previous sections, we use constant parameters to specify the dynamics of the underlying Gaussian factor in the foreign economy but specifying constant parameters for the factor Y_t in the domestic economy can also be done with the same degree of tractability. We assume that $r_t^d, r_t^f, \lambda_t^d, \lambda_t^f$ are quadratic forms in Y_t

$$\begin{aligned} r_T^d &= (Y_t + \alpha(t))^\top I^d (Y_t + \alpha(t)) \\ r_T^f &= (Y_t + \alpha(t))^\top I^f (Y_t + \alpha(t)) \\ \lambda_t^d &= (Y_t + \alpha(t))^\top I^{\lambda^d} (Y_t + \alpha(t)) \\ \lambda_t^f &= (Y_t + \alpha(t))^\top I^{\lambda^f} (Y_t + \alpha(t)). \end{aligned} \tag{4.144}$$

The matrices I^d, I^f, I^{λ^d} and I^{λ^f} are taken to be diagonal matrices with 1 or 0 along the diagonal depending on which factors or coordinates of Y_t are used to model the process. The foreign exchange rate S_t is assumed to be a log-quadratic process as in Chapter 1. Thus S_t is the solution of the following SDE:

$$\frac{dS_t}{S_t^-} = (r_t^d - r_t^f)dt + (2C^S(t)\, Y_t + (B^S(t))^\top \Sigma\; dW_t^d$$

Under this assumption Y_t remains an Ornstein-Uhlenbeck process under the foreign risk neutral measure $\mathbb{Q}^f$. We will first discuss how to calculate

$$\bar{P}^{f1}(t, u, T_N) = \mathbb{E}^{\mathbb{Q}^{f'}}\left[\exp\left(-\int_t^u r_s^f + \eta_s^f + \eta_s^d\, ds - \int_u^{T_N} \eta_s^d\, ds\right)\middle|\mathcal{G}_t^{f'}\right]. \tag{4.145}$$

Note that $\bar{P}^{f1}(t, u, T_N)$ can be regarded as the predefault value of a zero coupon defaultable bond in the foreign economy where the short term interest rate is equal to zero after u and the intensity of default is equal to

$$\eta_s^f + \eta_s^d \text{ for } t \leqslant s \leqslant u$$

and is equal to

$$\eta_s^d \text{ for } u \leqslant s \leqslant T_N.$$

Therefore even though we started out by specifying constant parameters for the matrix defining the quadratic form for the short term rate of interest

$$r_t^f = (Y_t + \alpha(t))^{\mathsf{T}} I^f (Y_t + \alpha(t))$$

and the intensities of default

$$\lambda_t^i = (Y_t + \alpha(t))^{\mathsf{T}} I^{\lambda^i} (Y_t + \alpha(t)),$$

we now have to regard a piecewise constant matrix defined as follows

$$I_f^{f1}(s) := \begin{cases} I^f & t \leqslant s < u \\ 0 & u \leqslant s \leqslant T_N \end{cases} \tag{4.146}$$

$$I_\lambda^{f1}(s) := \begin{cases} I^{\lambda^f} + I^{\lambda^d} & t \leqslant s < u \\ I^{\lambda^d} & u \leqslant s \leqslant T_N \end{cases} . \tag{4.147}$$

Thus for calculating $\bar{P}^{f1}(t, u, T_N)$ we can assume the short term interest rate is given by

$$(Y_s + \alpha(s))^{\mathsf{T}} I_f^{f1}(s)(Y_s + \alpha(s)) \tag{4.148}$$

and the intensity of default is equal to

$$(Y_s + \alpha(s))^{\mathsf{T}} I_\lambda^{f1}(s)(Y_t + \alpha(s)) \tag{4.149}$$

in order to calculate $\bar{P}^{f1}(t, u, T_N)$. We can still calculate the value $\bar{P}^{f1}(t, u, T_N)$ in analytically closed form using the results of Theorem 1.20 in Chapter 1 but this would require more computation than the constant parmeter case where I_f^{f1} and I_λ^{f1} are not time dependent. In the following we will see that we can avoid computing $\bar{P}^{f1}(t, u, T_N)$ and therefore we start with a discussion of defaultable forward measures. So far we have considered the defaultable zero coupon bond of maturity T_N, in the following we need to introduce additional notations for defaultable zero coupon bonds of shorter maturity and the associated measures corresponding to using the bond values as the numeraire. Let

$$\left.\frac{d\bar{\mathbb{T}}_u^{f1}}{d\mathbb{Q}^{f'}}\right|_{\mathcal{G}'_u} = \frac{\mathbf{1}_{\tau^d>u}\exp\left(-\int_0^u r_s^f + \eta_s^f\,ds\right)}{\mathbb{E}^{\mathbb{Q}^{f'}}\left[\mathbf{1}_{\tau^d>u}\exp\left(-\int_0^u r_s^f + \eta_s^f\,ds\right)\right]} = \frac{\mathbf{1}_{\tau^d>u}\exp\left(-\int_0^u r_s^f + \eta_s^f\,ds\right)}{\bar{P}^{f1}(0,u,u)}. \tag{4.150}$$

From the definition of the Radon Nikodým density which is given by (4.111) we have

$$\left.\frac{d\bar{\mathbb{T}}^{f1}}{d\mathbb{Q}^{f'}}\right|_{\mathcal{G}'_{T_N}} = \frac{\mathbf{1}_{\tau^d>T_N}\exp\left(-\int_0^u r_s^f + \eta_s^f\,ds\right)}{\bar{P}^{f1}(0,u,T_N)}$$

and hence we can get the following

$$\frac{d\bar{\mathbb{T}}^{f1}}{d\bar{\mathbb{T}}_u^{f1}} = \frac{d\bar{\mathbb{T}}^{f1}}{d\mathbb{Q}^{f'}}\frac{d\mathbb{Q}^{f'}}{d\bar{\mathbb{T}}_u^{f1}} = \mathbf{1}_{\tau^d>T_N}\frac{\bar{P}^{f1}(0,u,u)}{\bar{P}^{f1}(0,u,T_N)}. \tag{4.151}$$

Moreover we have

$$\left.\frac{d\bar{\mathbb{T}}^{f1}}{d\bar{\mathbb{T}}_u^{f1}}\right|_{\mathcal{G}_t^{f'}} = \mathbb{E}^{\bar{\mathbb{T}}_u}\left[\mathbf{1}_{\tau^d>T_N}\frac{\bar{P}^{f1}(0,u,u)}{\bar{P}^{f1}(0,u,T_N)}\middle|\mathcal{G}_t^{f'}\right] = \frac{\mathbf{1}_{\tau^d>t}\frac{\bar{P}^{f1}(t,u,T_N)}{\bar{P}^{f1}(t,u,u)}}{\frac{\bar{P}^{f1}(0,u,T_N)}{\bar{P}^{f1}(0,u,u)}}. \tag{4.152}$$

Therefore for any $\mathcal{G}_u^{f'}$ measurable random variable Y we can use the abstract Bayes theorem to get the following equality

$$\begin{aligned}
\mathbb{E}^{\bar{\mathbb{T}}^{f1}}[Y|\mathcal{G}_t^{f'}] &= \mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}\left[\frac{\mathbf{1}_{\tau^d>T_N}}{\mathbf{1}_{\tau^d>t}\frac{\bar{P}^{f1}(t,u,T_N)}{\bar{P}^{f1}(t,u,u)}}Y\middle|\mathcal{G}_t^{f'}\right] && (4.153)\\
&= \mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}\left[\frac{1}{\mathbf{1}_{\tau^d>t}\frac{\bar{P}^{f1}(t,u,T_N)}{\bar{P}^{f1}(t,u,u)}}\mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}[\mathbf{1}_{\tau^d>T_N}|\mathcal{G}_u^{f'}]\,Y\middle|\mathcal{G}_t^{f'}\right] && (4.154)\\
&= \mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}\left[\frac{\mathbf{1}_{\tau^d>u}\frac{\bar{P}^{f1}(u,u,T_N)}{\bar{P}^{f1}(u,u,u)}}{\mathbf{1}_{\tau^d>t}\frac{\bar{P}^{f1}(t,u,T_N)}{\bar{P}^{f1}(t,u,u)}}Y\middle|\mathcal{G}_t^{f'}\right] && (4.155)
\end{aligned}$$

whereby we have used

$$\mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}[\mathbf{1}_{\tau^d>T_N}|\mathcal{G}_u^{f'}] = \mathbb{E}^{\mathbb{Q}^{f'}}\left[\frac{1}{\bar{P}^{f1}(u,u,u)}\mathbf{1}_{\tau^d>u}\exp\left(-\int_u^u r_s^f+\eta_s^f\,ds\right)\mathbf{1}_{\tau^d>T_N}\middle|\mathcal{G}_u^{f'}\right]$$

$$= \mathbb{E}^{\mathbb{Q}^{f'}}\left[\frac{1}{\bar{P}^{f1}(u,u,u)}\exp\left(-\int_u^{T_N}\eta_s^d\,ds\right)|\mathcal{G}_u^{f'}\right] = \mathbf{1}_{\tau>u}\frac{\bar{P}^{f1}(u,u,T_N)}{\bar{P}^{f1}(u,u,u)} \quad (4.156)$$

to get the last line (4.155). Note that we have

$$\mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}\left[\mathbf{1}_{\tau^d>u}\frac{\bar{P}^{f1}(u,u,T_N)}{\bar{P}^{f1}(u,u,u)}\middle|\mathcal{G}_t^{f'}\right] = \mathbf{1}_{\tau^d>t}\frac{\bar{P}^{f1}(t,u,T_N)}{\bar{P}^{f1}(t,u,u)} \quad (4.157)$$

so that the Radon-Nikodým density in (4.155) can be written as

$$\frac{\frac{\mathbf{1}_{\tau^d>u}\bar{P}^{f1}(u,u,T_N)}{\bar{P}^{f1}(u,u,u)}}{\mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}\left[\frac{\mathbf{1}_{\tau^d>u}\bar{P}^{f1}(u,u,T_N)}{\bar{P}^{f1}(u,u,u)}\middle|\mathcal{G}_t^{f'}\right]}. \quad (4.158)$$

Therefore we have

$$\mathbb{E}^{\bar{\mathbb{T}}^{f1}}\left[\eta_u^f\middle|\mathcal{G}_t^{f'}\right] = \mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}\left[\frac{\frac{\mathbf{1}_{\tau^d>u}\bar{P}^{f1}(u,u,T_N)}{\bar{P}^{f1}(u,u,u)}}{\mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}\left[\frac{\mathbf{1}_{\tau^d>u}\bar{P}^{f1}(u,u,T_N)}{\bar{P}^{f1}(u,u,u)}\middle|\mathcal{G}_t^{f'}\right]}\eta_u^f\middle|\mathcal{G}_t^{f'}\right] \quad (4.159)$$

which has the same structure as the change of measure in the default free case. We have just demonstrated how to change measures between the defaultable forward measures of different maturities. The first step is to calculate the mean and variance-covariance matrix of Y_u under $\mathbb{T}_u^{f1}$. We can calculate the mean and variance of Y_u under the the default free forward measure of maturity u i.e. the measure corresponding to using the default free bond of maturity u as the numeraire using lemma 1.3 from chapter 1 but this lemma is not immediately applicable to finding the mean and variance-covariance matrix of Y_u under the defaultable forward measure for maturity $\mathbb{T}_u^{f1}$. However we can still apply lemma 1.3 by changing to a different measure as follows. We define a measure that is equivalent to $\mathbb{Q}^{f'}$ by the following Radon-Nikodým density

$$\left.\frac{d\widetilde{\mathbb{T}}_u^{f1}}{d\mathbb{Q}^{f'}}\right|_{\mathcal{G}_u^{f'}} = \frac{\exp\left(-\int_0^u r_s^f + \eta_s^f + \eta_s^d\, ds\right)}{\mathbb{E}^{\mathbb{Q}^{f'}}\left[\exp\left(-\int_0^u r_s^f + \eta_s^f + \eta_s^d\, ds\right)\right]}. \tag{4.160}$$

Therefore for any $\mathcal{G}_u^{f'}$ measurable and $\mathbb{Q}^{f'}$ integrable random variable W, we can apply the abstract Bayes formula to get the following equality

$$\mathbb{E}^{\widetilde{\mathbb{T}}_u^{f1}}[W|\mathcal{G}_t^{f'}] = \mathbb{E}^{\mathbb{Q}^{f'}}\left[\frac{1}{\bar{P}^{f1}(t,u,u)}\exp\left(-\int_t^u r_s^f + \eta_s^f + \eta_s^d\, ds\right)W\middle|\mathcal{G}_t^{f'}\right]. \tag{4.161}$$

We now apply the definition of $\bar{\mathbb{T}}_u^{f1}$ to change measure back to $\mathbb{Q}^{f'}$ and then to $\widetilde{\mathbb{T}}_u^{f1}$ which gives us the following:

$$\begin{aligned}
\mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}[Y_u|\mathcal{G}_t^{f'}] &= \mathbb{E}^{\mathbb{Q}^{f'}}\left[\frac{1}{\bar{P}^{f1}(t,u,u)}\mathbf{1}_{\tau^d>u}\exp\left(-\int_t^u r_s^f + \eta_s^f\, ds\right)Y_u\middle|\mathcal{G}_t^{f'}\right] &(4.162)\\
&= \mathbf{1}_{\tau>t}\mathbb{E}^{\mathbb{Q}^{f'}}\left[\frac{1}{\bar{P}^{f1}(t,u,u)}\exp\left(-\int_t^u r_s^f + \eta_s^f + \eta_s^d\, ds\right)Y_u\middle|\mathcal{G}_t^{f'}\right] &(4.163)\\
&= \mathbf{1}_{\tau>t}\mathbb{E}^{\widetilde{\mathbb{T}}_u^{f1}}[Y_u|\mathcal{G}_t^{f'}]. &(4.164)
\end{aligned}$$

From Collin-Dufresne et al. (2004) we know that W_t^f in (4.63) remains a Brownian motion under $\mathbb{Q}^{f'}$ and therefore the dynamics of Y_u under $\widetilde{\mathbb{T}}_u^{f1}$ can be determined similar to the default free case by use of Girsanov's theorem. Therefore finding the mean of Y_u under $\bar{\mathbb{T}}_u^{f1}$ can be accomplished by finding the mean of Y_u under $\widetilde{\mathbb{T}}_u^{f1}$. To find the mean and the variance-covariance matrix of Y_u under $\widetilde{\mathbb{T}}_u^{f1}$ we can apply lemma 1.3 from chapter 1. Now we can calculate the mean and variance-covariance matrix of Y_u under $\bar{\mathbb{T}}^{f1}$ by substituting Y_u for η_u in (4.159) to get

$$\begin{aligned}
\mathbb{E}^{\bar{\mathbb{T}}^{f1}}\left[Y_u\middle|\mathcal{G}_t^{f'}\right] &= \mathbb{E}^{\widetilde{\mathbb{T}}_u^{f1}}\left[\frac{\frac{\mathbf{1}_{\tau^d>u}\bar{P}^{f1}(u,u,T_N)}{\bar{P}^{f1}(u,u,u)}}{\mathbb{E}^{\widetilde{\mathbb{T}}_u^{f1}}\left[\frac{\mathbf{1}_{\tau^d>u}\bar{P}^{f1}(u,u,T_N)}{\bar{P}^{f1}(u,u,u)}\middle|\mathcal{G}_t^{f'}\right]}Y_u\middle|\mathcal{G}_t^{f'}\right] &(4.165)\\
&= \mathbf{1}_{\tau^d>t}\frac{\bar{P}^{f1}(t,u,u)}{\bar{P}^{f1}(t,u,T_N)}\mathbb{E}^{\widetilde{\mathbb{T}}_u^{f1}}\left[\mathbf{1}_{\tau^d>u}\bar{P}^{f1}(u,u,T_N)Y_u\middle|\mathcal{G}_t^{f'}\right] &(4.166)
\end{aligned}$$

and changing measure to $\widetilde{\mathbb{T}}_u^{f1}$ as follows

$$
\begin{aligned}
&\mathbf{1}_{\tau^d>t}\frac{\bar{P}^{f1}(t,u,u)}{\bar{P}^{f1}(t,u,T_N)}\mathbb{E}^{\widetilde{\mathbb{T}}_u^{f1}}\left[\mathbf{1}_{\tau^d>u}\bar{P}^{f1}(u,u,T_N)Y_u\middle|\mathcal{G}_t^{f'}\right]\\
&=\mathbf{1}_{\tau^d>t}\frac{\bar{P}^{f1}(t,u,u)}{\bar{P}^{f1}(t,u,T_N)}\mathbb{E}^{\mathbb{Q}^{f'}}\left[\frac{\mathbf{1}_{\tau^d>u}\exp\left(-\int_t^u r_s^f+\eta_s^f\,ds\right)}{\bar{P}^{f1}(t,u,u)}\bar{P}^{f1}(u,u,T_N)Y_u\middle|\mathcal{G}_t^{f'}\right]\\
&=\mathbf{1}_{\tau^d>t}\frac{\bar{P}^{f1}(t,u,u)}{\bar{P}^{f1}(t,u,T_N)}\mathbb{E}^{\mathbb{Q}^{f'}}\left[\frac{\exp\left(-\int_t^u r_s^f+\eta_s^f+\eta_s^d\,ds\right)}{\bar{P}^{f1}(t,u,u)}\bar{P}^{f1}(u,u,T_N)Y_u\middle|\mathcal{G}_t^{f'}\right]\\
&=\mathbf{1}_{\tau^d>t}\frac{\bar{P}^{f1}(t,u,u)}{\bar{P}^{f1}(t,u,T_N)}\mathbb{E}^{\widetilde{\mathbb{T}}_u^{f'}}\left[\bar{P}^{f1}(u,u,T_N)Y_u\middle|\mathcal{G}_t^{f'}\right].
\end{aligned}
\tag{4.167}
$$

We now can show that we can calculate V_t in closed form avoiding the calculation of the more computationally intensive $\bar{P}^{f1}(t,u,T_N)$. Specifically

$$V_t=\mathbf{1}_{\tau^f\wedge\tau^d>t}\int_t^{T_N}\bar{P}^{f1}(t,u,T_N)\mathbb{E}^{\mathbb{T}^{f1}}\left[\eta_u^f\middle|\mathcal{G}_t^{f'}\right]du \tag{4.168}$$

$$=\mathbf{1}_{\tau^f\wedge\tau^d>t}\int_t^{T_N}\bar{P}^{f1}(t,u,T_N)\frac{\bar{P}^{f1}(t,u,u)}{\bar{P}^{f1}(t,u,T_N)}\mathbb{E}^{\widetilde{\mathbb{T}}_u^{f'}}\left[\bar{P}^{f1}(u,u,T_N)\eta_u^f\middle|\mathcal{G}_t^{f'}\right]du \tag{4.169}$$

$$=\mathbf{1}_{\tau^f\wedge\tau^d>t}\int_t^{T_N}\bar{P}^{f1}(t,u,u)\mathbb{E}^{\widetilde{\mathbb{T}}_u^{f'}}\left[\bar{P}^{f1}(u,u,T_N)\eta_u^f\middle|\mathcal{G}_t^{f'}\right]du \tag{4.170}$$

Therefore since we know the mean and variance-covariance matrix of Y_u under $\mathbb{T}_u^{f1}$ and $\bar{P}^{f1}(u,u,T_N)$ is log-quadratic Gaussian, we can apply lemma 1.14 from chapter 1 directly to the last line of (4.167). Note the calculation of $\bar{P}^{f1}(t,u,u)$ can be done easily since now the short term interest rate r_t^f is

$$(Y_t+\alpha(t))^{\top}I^f(Y_t+\alpha(t))$$

and the intensity of default $\eta_t^f+\eta_t^d$ is given by

$$(Y_t+\alpha(t))^{\top}(I^{\eta^f}+I^{\eta^d})(Y_t+\alpha(t))$$

and we can see that the matrices I^f, I^{η^f} and I^{η^d} are now constant matrices. The calculation of $\bar{P}^{f1}(u,u,T_N)$ also involves constant parameters as

$$\bar{P}^{f1}(u,u,T_N) = \mathbb{E}^{\mathbb{Q}^{f'}}\Bigg[\exp\Big(-\int_u^{T_N}\eta_s^d\,ds\Big)\Big|\mathcal{G}_t^{f'}\Bigg]$$

and we can see only the intensity η^d is used. We could have also established (4.170) from

$$\mathbf{1}_{\tau^d>t}\mathbb{E}^{\mathbb{Q}^{f'}}\Bigg[\exp\Big(-\int_t^{u} r_s^f+\eta_s^f\,ds-\int_t^{T_N}\eta_s^d\,ds\Big)Y_u\Big|\mathcal{G}_t^{f'}\Bigg] \tag{4.171}$$

through the following calculations

$$\begin{aligned}
&\mathbf{1}_{\tau^d>t}\mathbb{E}^{\mathbb{Q}^{f'}}\Bigg[\exp\Big(-\int_t^{u} r_s^f+\eta_s^f\,ds-\int_t^{T_N}\eta_s^d\,ds\Big)Y_u\Big|\mathcal{G}_t^{f'}\Bigg]\\
&=\mathbf{1}_{\tau^d>t}\mathbb{E}^{\mathbb{Q}^{f'}}\Bigg[\exp\Big(-\int_t^{u} r_s^f+\eta_s^f+\eta_s^d\,ds\Big)\mathbb{E}^{\mathbb{Q}^{f'}}\Bigg[\exp\Big(-\int_u^{T_N}\eta_s^d\,ds\Big)\Big|\mathcal{G}_u^{f'}\Bigg]\eta_u^f\Big|\mathcal{G}_t^{f'}\Bigg]\\
&=\mathbf{1}_{\tau^d>t}\mathbb{E}^{\mathbb{Q}^{f'}}\Bigg[\exp\Big(-\int_t^{u} r_s^f+\eta_s^f+\eta_s^d\,ds\Big)\bar{P}^{f1}(u,u,T_N)Y_u\Big|\mathcal{G}_t^{f'}\Bigg]\\
&=\mathbf{1}_{\tau^d>t}\bar{P}^{f1}(t,u,u)\mathbb{E}^{\tilde{\mathbb{T}}_u^{f'}}\Bigg[\bar{P}^{f1}(u,u,T_N)Y_u\Big|\mathcal{G}_t^{f'}\Bigg]
\end{aligned} \tag{4.172}$$

but this does not make it clear how to change measure between the defaultable forward measure of different maturities. In general it is useful to establish how to change measures between defaultable forward measures of different maturities as this can be useful to simplify valuation. By introducing the defaultable forward measures $\bar{\mathbb{T}}_u^{f1}$ and the change of measure from $\bar{\mathbb{T}}_u^{f1}$ to $\bar{\mathbb{T}}^{f1}$ we have shown that the formula is an exact analog of the change of measure for forward measures in the default free case (see Cherif et al. (1994) for the default free case). For the purposes of calculating expectations of random variables W which are $\mathcal{G}_u^{f'}$ measurable and $\mathbb{Q}^{f'}$ integrable we can then use the measure $\tilde{\mathbb{T}}_u^{f1}$ which is absolutely equivalent to $\mathbb{Q}^{f'}$ and hence we can apply the results established in Cherif et al. (1994) for calculating the mean and variance-covariance matrix in the defaultable case. It is now clear how to calculate

$$\mathbb{E}^{\bar{\mathbb{T}}^{f1}}[\eta_u^f|\mathcal{G}_t^{f'}]$$

as η_u^f is a quadratic Gaussian random variable and therefore can be determined from the variance-covariance matrix of Y_u. The calculation of W_t in (4.122) involves the expectation

$$\mathbb{E}^{\bar{\mathbb{T}}_w^{f1}}[\eta_u^f \eta_w^d | \mathcal{G}_t^{f'}]. \tag{4.173}$$

We now discuss how to calculate (4.173). First similar to (4.163) we can calculate

$$\mathbb{E}^{\bar{\mathbb{T}}_w^{f1}}[\eta_w^d | \mathcal{G}_u^{f'}] = \mathbb{E}^{\tilde{\mathbb{T}}_w^{f1}}[\eta_w^d | \mathcal{G}_u^{f'}] \tag{4.174}$$

in analytically closed form. Since η_w^d is a quadratic form in Gaussian random variable under $\widetilde{\mathbb{T}}_w^{f1}$, we can obtain (4.174) from the variance-covariance matrix of Y_u (see lemma 1.3 in chapter 1). Thus (4.174) consists of a part which is a quadratic function of Y_u and another part which is a deterministic function of u. Since (4.173) can be written as

$$\mathbb{E}^{\bar{\mathbb{T}}_w^{f1}}[\eta_u^f \, \mathbb{E}^{\tilde{\mathbb{T}}_u^{f1}}[\eta_w^d | \mathcal{G}_u^{f'}] | \mathcal{G}_t^{f'}], \tag{4.175}$$

it is clear that to calculate (4.173), we need to calculate not only first and second order moments of Y_u but also higher order moments of Y_u under $\bar{\mathbb{T}}_w^{f1}$. This can be achieved by first noting that we can calculate the mean and the variance-covariance matrix of $Y_u = (Y_{1u}, \ldots, Y_{nu})$ under $\bar{\mathbb{T}}_w^{f1}$ by changing measure to $\bar{\mathbb{T}}_u^{f1}$ and using $\widetilde{\mathbb{T}}_u^{f1}$ similar to what we did in (4.159). Hence it is sufficient to discuss how to calculate

$$\prod_{j=1}^{q} Y_{ju}^{r_j} \text{ for } j = 1, \ldots, n \text{ and } r_1 + \ldots + r_q = k$$

under $\bar{\mathbb{T}}_w^{f1}$. Thus we need to calculate

$$\mathbb{E}^{\bar{\mathbb{T}}_w^{f1}}\left[\prod_{j=1}^{q} Y_{ju}^{r_j} \middle| \mathcal{G}_t^{f'}\right] = \mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}\left[\frac{\frac{\mathbb{1}_{\tau^d > w} \bar{P}^{f1}(u,u,w)}{\bar{P}^{f1}(u,u,u)}}{\mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}\left[\frac{\mathbb{1}_{\tau^d > w} \bar{P}^{f1}(u,u,w)}{\bar{P}^{f1}(u,u,u)} \middle| \mathcal{G}_t^{f'}\right]} \prod_{j=1}^{q} Y_{ju}^{r_j} \middle| \mathcal{G}_t^{f'}\right] \tag{4.176}$$

$$= \frac{\bar{P}^{f1}(t,u,u)}{\bar{P}^{f1}(t,u,w)} \mathbb{E}^{\bar{\mathbb{T}}_u^{f1}}\left[\mathbb{1}_{\tau^d > w} \bar{P}^{f1}(u,u,w) \prod_{j=1}^{q} Y_{ju}^{r_j} \middle| \mathcal{G}_t^{f'}\right] \tag{4.177}$$

$$= \frac{\bar{P}^{f1}(t,u,u)}{\bar{P}^{f1}(t,u,w)} \mathbb{E}^{\tilde{\mathbb{T}}_u^{f1}}\left[\bar{P}^{f1}(u,u,w) \prod_{j=1}^{q} Y_{ju}^{r_j} \middle| \mathcal{G}_t^{f'}\right]. \tag{4.178}$$

First note that we can find the mean and the variance-covariance matrix of Y_u under the measure corresponding to using $\bar{P}^{f1}(u,u,w)$ as the numeraire by applying lemma 1.3. Calculating

$$\mathbb{E}^{\tilde{\mathbb{T}}_u^{f1}}\left[\bar{P}^{f1}(u,u,w)\prod_{j=1}^{q}Y_{ju}^{r_j}\middle|\mathcal{G}_t^{f'}\right] \tag{4.179}$$

involves finding higher moments of a multidimensional Gaussian random variable under the measure corresponding to using $\bar{P}^{f1}(u,u,w)$ as the numeraire. The higher order central moments of a multidimensional Gaussian random variable(or a multidimensional Gaussian random variable with zero mean vector) can be found in closed form in an efficient manner as shown in Triantafyllopoulos (2003). Hence we can find higher order moments of

$$Y_u - \mathbb{E}^{\tilde{\mathbb{T}}_u^{f1}}[\bar{P}^{f1}(u,u,T_N)Y_u|\mathcal{G}_t^{f'}\Big]$$

and therefore we can also find the noncentral higher order moments of Y_u which means we can calculate

$$\mathbb{E}^{\tilde{\mathbb{T}}_u^{f1}}\left[\bar{P}^{f1}(u,u,w)\eta_u^f\eta_w^d\middle|\mathcal{G}_t^{f'}\right]. \tag{4.180}$$

We now discuss how to calculate the expectation in (4.122) given by

$$\mathbb{E}^{\tilde{\mathbb{T}}_w^{f1}}\left[\bar{P}^{f2}(w,u)(\eta_u^f+\alpha^f)\eta_w^d\middle|\mathcal{G}_t^{f'}\right]. \tag{4.181}$$

Let

$$\left.\frac{d\tilde{\mathbb{T}}_w^{f1}}{d\mathbb{Q}^{f'}}\right|_{\mathcal{G}_w^{f'}} = \frac{\exp\left(-\int_0^w r_s^f+\eta_s^f+\eta_s^d\,ds\right)}{\mathbb{E}^{\mathbb{Q}^{f'}}\left[\exp\left(-\int_0^w r_s^f+\eta_s^f+\eta_s^d\,ds\right)\right]} \tag{4.182}$$

where $w<u$. We can now change measure back to $\mathbb{Q}^{f'}$ in (4.181) and apply the abstract Bayes formula (see for example (4.163)) to get the following equality,

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{T}}_w^{f1}}&\left[\bar{P}^{f2}(w,u)(\eta_u^f+\alpha^f)\eta_w^d\middle|\mathcal{G}_t^{f'}\right]\\ &= \mathbf{1}_{\tau^d>t}\mathbb{E}^{\tilde{\mathbb{T}}_w^{f1}}\left[\bar{P}^{f2}(w,u)(\eta_u^f+\alpha^f)\eta_w^d\middle|\mathcal{G}_t^{f'}\right]\end{aligned} \tag{4.183}$$

Moreover we can use the tower property of expectations (see, e.g., Karatzas and Shreve (1991)) to rewrite (4.183) in the following form

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{T}}_w^{f1}}\left[\bar{P}^{\widetilde{f2}}(w,u)(\eta_u^f+\alpha^f)\eta_w^d\middle|\mathcal{G}_t^{f'}\right] &= \mathbb{E}^{\tilde{\mathbb{T}}_w^{f1}}\left[\bar{P}^{\widetilde{f2}}(w,u)(\eta_u^f\eta_w^d+\alpha^f\eta_w^d)\middle|\mathcal{G}_t^{f'}\right] \\ &= \mathbb{E}^{\tilde{\mathbb{T}}_w^{f1}}\left[\bar{P}^{\widetilde{f2}}(w,u)(\eta_w^f\mathbb{E}^{\tilde{\mathbb{T}}_w^{f1}}[\eta_u^d|\mathcal{G}_w^{f'}]+\alpha^f\eta_w^d)\middle|\mathcal{G}_t^{f'}\right]. \quad (4.184)\end{aligned}$$

Once again

$$\mathbb{E}^{\tilde{\mathbb{T}}_w^{f1}}[\eta_u^d|\mathcal{G}_w^{f'}] \quad (4.185)$$

consists of a part that is a quadratic function of Y_w and a part that is a deterministic function of w. Therefore we need to calculate higher order moments of Y_w but we have already discussed how to perform this calculation in the context of how to calculate (4.173).

The calculation of $\bar{V}_{it}$ involves determining

$$\mathbb{E}^{\tilde{\mathbb{T}}_i^{f3}}\left[G^{f3}(T_i,T_N)\frac{1}{S_{T_i}}\middle|\mathcal{G}_t^{f'}\right]. \quad (4.186)$$

Let

$$\left.\frac{d\tilde{\mathbb{T}}_i^{f3}}{d\mathbb{Q}^{f'}}\right|_{\mathcal{G}_{T_i}^{f'}} := \frac{\mathbf{1}_{\tau^d>T_i}\exp\left(-\int_0^{T_i} r_s^f+\eta_s^f+\eta_s^d\,ds\right)}{\mathbb{E}^{\mathbb{Q}^{f'}}\left[\exp\left(-\int_0^{T_i} r_s^f+\eta_s^f+\eta_s^d\,ds\right)\right]}. \quad (4.187)$$

Then by an application of Bayes formula (4.185) is equal to

$$\mathbb{E}^{\tilde{\mathbb{T}}_i^{f3}}\left[G^{f3}(T_i,T_N)\frac{1}{S_{T_i}}\middle|\mathcal{G}_t^{f'}\right]. \quad (4.188)$$

Since we can use Lemma 1.3 in chapter 1 to calculate the mean and variance-covariance matrix of Y_u under $\tilde{\mathbb{T}}_i^{f3}$ and $G^{f3}(T_i,T_N)\frac{1}{S_{T_i}}$ is a product of log-quadratic Gaussian terms and therefore log-quadratic Gaussian, we can use Lemma 1.14 to calculate (4.188) in analytically closed form. The determination of $\overline{W}_{it}$ requires the calculation of the following two expectations expectations:

$$\mathbb{E}^{\tilde{\mathbb{T}}_u^{f4}}\left[\frac{1}{S_{T_i}}\bar{P}^{f5}(u,T_i)\eta_u^d\middle|\mathcal{G}_t^{f'}\right] \quad (4.189)$$

$$\mathbb{E}^{\tilde{\mathbb{T}}_i^{f3}}\left[\frac{1}{S_{T_i}}G^{f6}(T_i,u)\eta_u^d\middle|\mathcal{G}_t^{f'}\right]. \quad (4.190)$$

Let

$$\left.\frac{d\widetilde{\mathbb{T}}_u^{f4}}{d\mathbb{Q}^{f'}}\right|_{\mathcal{G}_u^{f'}} := \frac{\exp\left(-\int_0^u r_s^f + \eta_s^f + \eta_s^d \, ds\right)}{\mathbb{E}^{\mathbb{Q}^{f'}}\left[\exp\left(-\int_0^u r_s^f + \eta_s^f + \eta_s^d \, ds\right)\right]} \tag{4.191}$$

Then changing measure to $\mathbb{Q}^{f'}$ and applying the abstract Bayes formula, we can rewrite (4.189) and (4.190) into

$$\mathbb{E}^{\widetilde{\mathbb{T}}_u^{f4}}\left[\frac{1}{S_{T_i}}\bar{P}^{f5}(u,T_i)\eta_u^d\middle|\mathcal{G}_t^{f'}\right] \tag{4.192}$$

$$\mathbb{E}^{\widetilde{\mathbb{T}}_i^{f3}}\left[\frac{1}{S_{T_i}}G^{f6}(T_i,u)\eta_u^d\middle|\mathcal{G}_t^{f'}\right]. \tag{4.193}$$

Using lemma 1.3 from chapter 1 we can find the mean and variance-covariance matrix of Y_u under $\widetilde{\mathbb{T}}_u^{f4}$(and also $\widetilde{\mathbb{T}}_i^{f3}$ as discussed in the valuation of (4.186)). Since the foreign exchange rate $\frac{1}{S_{T_i}}$ is log-quadratic Gaussian as well as $\bar{P}^{f5}(u,T_i)$ and $G^{f6}(T_i,u)$, the products $\frac{1}{S_{T_i}}\bar{P}^{f5}(u,T_i)$ and $\frac{1}{S_{T_i}}G^{f6}(T_i,u)$ are also log-quadratic Gaussian. We can use (1.149) from chapter 1 to find the mean and variance-covariance matrix of Y_u under $\widetilde{\mathbb{T}}_u^{f4}$ as well as $\widetilde{\mathbb{T}}_i^{f3}$.

Therefore if we assume the quadratic Gaussian model given in (4.144), we can calculate all expectations involved in the calculation of the default leg and the premium leg of a quanto default swap in analytically closed form even if we assume a contagion model of default. The determination of the default leg involves a double integral as well as well as a single integral with the expectations as the integrands(see (4.122)). These integrals can be discretized and converted into sums. The determination of the premium leg involves only single integrals with the expectations as integrands and therefore can also be converted into sums. Though we have considered a single obligor, these calculations indicate what type of calculations are necessary in a contagion model of default where we have two different obligors. If we increase the number of obligors in a contagion model of default, then calculating the value of credit sensitive securities involves considering all permutations of the possible default sequences

$$\tau_{\sigma(1)}, \ldots, \tau_{\sigma(n)}$$

where σ is an element of the permutaion group of order n. Thus we have to consider $n!$ default sequences and for each such sequence, we will generally have to consider n-dimensional integrals whose integrands are higher order expectations of the factor Y_u under some defaultable forward measure. It seems that calculations for the different default sequences is a bigger challenge as we have to consider $n!$ such sequences.

APPENDIX

A. MATRIX RICCATI EQUATIONS

In this appendix we provide two theorems which are given in the survey article of Freiling (2002) (Theorem 3.1 and Theorem 3.5 of this reference). For more detail regarding matrix Riccati differential equations (RDE), we refer the reader to Freiling (2002).

Definition A.1. *Let $M_{11}(t), M_{12}(t), M_{21}(t)$ and $M_{22}(t)$ be piecewise continuous(or locally integrable), time-dependent real or complex square matrices of dimensions $n \times n, n \times m, m \times n$ and $m \times m$ respectively. Furthermore we make the assumption that the matrix initial value differential equation which is given by*

$$\dot{W} = M_{21}(t) + M_{22}(t)W - WM_{11}(t) - WM_{12}(t)W \tag{A.1}$$

is known as a matrix Riccati equations(RDE).

Let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

and

$$Y(t) = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} \tag{A.2}$$

where Q and P are real or complex matrices with dimensions $n \times n$ and $m \times n$ respectively. With (A.1), we associate the following linear ODE:

$$\dot{Y} = M(t)Y. \tag{A.3}$$

According to Freiling (2002), the solution of (A.1) was first given in Radon (1927) and Radon (1928).

Theorem A.2 (Radon's lemma—version 1). *(i) Let I be the $n \times n$ identity matrix and let W be on some interval $J \subset \mathbb{R}$ a solution of* (A.1) *with $W(t_0) =$*

W_0. If Q is for some $t \in J$ the unique solution of the initial value problem

$$\dot{Q} = (M_{11}(t) + M_{12}(t)W(t))Q, \quad Q(t_0) = I, \tag{A.4}$$

and $P(t) := W(t)Q(t)$, then $Y(t) = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix}$ defines for $t \in J$ the solution of (A.3) with $Y(t_0) = \begin{pmatrix} I \\ W_0 \end{pmatrix}$.

(ii) If $Y(t) = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix}$ is on some interval $J \subset \mathbb{R}$ the solution of (A.3) such that the determinant of $Q(t) \neq 0$ for $t \in J$, then

$$W : J \to \mathbb{C}^{m \times n}, \quad t \mapsto P(t)Q(t)^{-1} =: W(t)$$

is a solution of (A.1); in particular $W(t_0) = P(t_0)Q(t_0)^{-1}$.

Radon's lemma which is given above by Theorem A.2 is for the solution of initial value RDE. We can adapt Theorem A.2 to solve terminal value RDE. A terminal value RDE is a matrix Riccati differential equation where as in Definition A.1, we have

$$\dot{W} = M_{21}(t) + M_{22}(t)W - WM_{11}(t) - WM_{12}(t)W, \quad W(t_f) = W_f \tag{A.5}$$

for some value t_f representing the terminal end point of an interval $J \subset \mathbb{R}$. In this case we are looking for a solution of (A.5) over an infinite interval $(-\infty, t_f)$ or a finite interval $[t_0, t_f]$ for some finite $t_0 \in \mathbb{R}$. Provided a solution exists, we can also solve (A.5) by considering the associated terminal value linear ODE which is given by

$$\dot{Y} = M(t)Y, \quad Y(t_f) = \begin{pmatrix} I \\ W_f \end{pmatrix} \tag{A.6}$$

where $M(t)$ and $Y(t)$ are defined as in (A.2).

Theorem A.3. *If $Y(t) = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix}$ is on some interval $J = [t_0, t_f]$ or $J = (-\infty, t_f)$, the solution of (A.6) such that the determinant of $Q(t) \neq 0$ for $t \in J$, then*

$$W(t) := P(t)Q(t)^{-1}$$

is a solution of (A.5); *in particular* $W(t_f) = P(t_f)Q(t_f)^{-1}$.

Proof. Differentiating $W(t) = P(t)Q(t)^{-1}$ with respect to t shows that it is the solution of (A.5). □

Sufficient conditions on the initial value $W(t_0)$ guaranteeing the existence of a solution of the initial value RDE given by (A.1) are not known in the general case. For some particular cases, a summary of the results that have been obtained so far is given in Freiling (2002). Here we consider a special case of RDE known as hermitian (or symmetric) matrix Riccati differential equation where sufficient conditions for the existence of a solution can be given. Let A^* denote the conjugate transpose of the matrix A. Consider the terminal value RDE given by

$$\dot{P} = -A^*(t)P - PA(t) - Q(t) + PWS(t)P, \quad P(t_f) = P_f, \tag{A.7}$$

where Q, S and P_f are hermitian (or real symmetric) matrices. The terminal value RDE given by (A.7) is known as a hermitian (or symmetric) matrix Riccati differential equation (HRDE). The following theorem which is given in Freiling (2002) gives sufficient conditions for the existence of the solution of HRDE.

Theorem A.4. *Assume that $S(t), Q(t)$ and P_f are positive semi-definite, piecewise continuous and locally bounded for $t \leqslant t_f$. Let $A(t)$ be piecewise continuous and locally bounded for $t \leqslant t_f$. Then the (unique) solution P of* (A.7) *exists for $t \leqslant t_f$ with*

$$0 \leqslant P(t) \leqslant \widetilde{P}(t) \text{ for } t \leqslant t_f,$$

where $\widetilde{P}(t)$ is the solution of the linear equation

$$\dot{\widetilde{P}} = -A^*(t)\widetilde{P} - \widetilde{P}A(t) - Q(t), \quad \widetilde{P}(t_0) = P_0. \tag{A.8}$$

B. SOME RESULTS FOR MULTI-FACTOR QUADRATIC GAUSSIAN MODELS

In this appendix we first provide the characteristic function of a quadratic form in Gaussian random variables (see, e.g., Mathai and Provost (1992)). We then give some of the results that are given in Cherif et al. (1994) regarding multi-factor quadratic Gaussian factor models.

Lemma B.1. *Given a quadratic form in Gaussian random variables,*

$$\mathfrak{Q} = Y_T^{\top} \mathfrak{C}\, Y_T + \mathfrak{B}^{\top} Y_T + \mathfrak{A}$$

where $\mathfrak{C}$ is a square symmetric matrix, $\mathfrak{B}$ is a vector and $\mathfrak{A}$ is a constant, the moment generating function of $\mathfrak{Q}$ under the measure $\mathbb{T}$ and conditional on $\mathcal{F}_t$ is given by

$$\Phi(\mathfrak{Q}, z) :=\mid I - 2\,z\,\mathfrak{C}\,V(t,T) \mid^{-\frac{1}{2}} \exp\Bigg(-\frac{1}{2}(M(t,T)V(t,T)^{-1}M(t,T) - 2\,z\,\mathfrak{A}) + \frac{1}{2}(M(t,T) + zV(t,T)\mathfrak{B})^{\top}(I - 2\,z\,\mathfrak{C}\,V(t,T))^{-1}V(t,T)^{-1}(M(t,T) + z\,V(t,T)\,\mathfrak{B})\Bigg). \tag{B.1}$$

Proof. We assume that Σ in (1.3) is positive definite and therefore $V(t,T)$ is positive definite for $t \neq T$. Then we can use Theorem 3.2a.1 given in Mathai and Provost (1992) to find the moment generating function. □

Let r_t denote the instantaneous rate of interest for an economy. Let $D(t)$ denote the savings account which is the value of investing one unit of currency at time $t = 0$ and rolling over the account at the default free instantaneous rate of interest r_t:

$$D(t) = \exp\left(\int_0^t r_s\, ds \right).$$

Let $\mathbb{Q}$ denote the risk neutral measure which is the measure under which the value of a default free security discounted by the savings account is a martingale. Let W_t denote an n-dimensional standard Brownian motion under $\mathbb{Q}$. We now assume that we have an Ornstein-Uhlenbeck Gaussian process Y_t of dimension n which is a solution of the following SDE under the risk neutral measure:

$$dY_t = (\mu(t) + A(t)Y_t)\,dt + \Sigma(t)\,dW_t. \tag{B.2}$$

Hence in this appendix Y_t denotes a process with a non zero time dependent drift function given by $\mu(t)$. The matrices A and Σ are assumed to be time dependent. The first theorem gives the dynamics of a log quadratic Gaussian process.

Theorem B.2. *Let $\widetilde{E}_t$ be a stochastic process that is given by*

$$\widetilde{E}_t = \exp(-Y_t^\top C^{\widetilde{E}}(t)Y_t - B^{\widetilde{E}}(t)^\top Y_t - A^{\widetilde{E}}(t)),$$

where $C^{\widetilde{E}}(t)$ is a symmetric matrix, $B^{\widetilde{E}}(t)$ is a vector and $A^{\widetilde{E}}(t)$ is a scalar that are assumed to be differentiable with respect to t. Then $\widetilde{E}_t$ is the solution of the following SDE:

$$\frac{d\widetilde{E}_t}{\widetilde{E}_t} = (Y_t^\top \widehat{C}^{\widetilde{E}}(t)Y_t + \widehat{B}^{\widetilde{E}}(t)^\top Y_t + \widehat{A}^{\widetilde{E}}(t))\,dt - (2\,Y_t^\top C^{\widetilde{E}}(t) + B^{\widetilde{E}}(t)^\top)\Sigma\,dW_t \tag{B.3}$$

where

$$\widehat{C}^{\widetilde{E}}(t) = -\,A(t)^\top C^{\widetilde{E}}(t) - C^{\widetilde{E}}(t)A(t) - \partial_t C^{\widetilde{E}}(t) + 2\,C^{\widetilde{E}}(t)\Sigma(t)\Sigma(t)^\top C^{\widetilde{E}}(t) \tag{B.4}$$

$$\widehat{B}^{\widetilde{E}}(t) = -\,2\,C^{\widetilde{E}}(t)\mu(t) - A(t)^\top B^{\widetilde{E}}(t) - \partial_t B^{\widetilde{E}}(t) + 2\,C^{\widetilde{E}}\Sigma(t)\Sigma(t)^\top B^{\widetilde{E}}(t) \tag{B.5}$$

$$\widehat{A}^{\widetilde{E}}(t) = -\,Tr[\Sigma(t)\Sigma(t)^\top C^{\widetilde{E}}(t)] - B^{\widetilde{E}}(t)\mu(t) - \partial_t A^{\widetilde{E}}(t) + \frac{1}{2}|\Sigma(t)^\top B^{\widetilde{E}}(t)|^2. \tag{B.6}$$

Theorem B.2 is proved in Cherif et al. (1994) using Itô's formula (Itô (1946)). We now consider securities whose payoff at time T is log quadratic Gaussian. For such payoffs, we have the following lemma that is given in Cherif et al. (1994) (see also Cheng and Scaillet (2004).

Theorem B.3. *Let the payoff a security be equal to*

$$R_T = \exp(Y_T^\top C^{R}(T)Y_T + B^{R}(T)^\top Y_T + A^{R}(T))$$

at time T, then the discounted price[1] *of the domestic security which is denoted by R_t is log quadratic Gaussian and is given by*

$$R_t := \exp(Y_t^\top C^R(t) Y_t + B^R(t)^\top Y_t + A^R(t)).$$

where $C^R(t)$ solves the following terminal value symmetric matrix Riccati differential equation:

$$\frac{d}{dt} C^R(t) = -A^\top C^R(t) - C^R(t) A + 2C^R(t)\Sigma\Sigma^\top C^R(t) - I, \quad C^R(T) = C^{\bar{R}}(T) \tag{B.7}$$

and $B^R(t)$ and $A^R(t)$ solve the following ordinary differential equations

$$\begin{aligned} \frac{d}{dt} B^R(t) = -A^\top B^R(t) + 2C^R(t)\Sigma\Sigma^\top B^R(t) - \\ - 2C^R(t)\Sigma\Sigma^\top B^R(t) - 2\alpha(t), \quad B^R(T) = B^{\bar{R}}(T), \end{aligned} \tag{B.8}$$

$$\begin{aligned} \frac{d}{dt} A^R(t) = -Tr[\Sigma\Sigma^\top C^R(t)] - B^R(t)^\top \Sigma\Sigma^\top B^R(t) + \\ + \frac{1}{2} B^R(t)^\top \Sigma\Sigma^\top B^R(t) - \alpha(t)^\top \alpha(t), \quad A^R(T) = A^{\bar{R}}(T). \end{aligned} \tag{B.9}$$

[1] Here when we say discounted, it means by the savings account of the economy.

REFERENCES

Abramowitz, M. & Stegun, I. A. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover Publications, Inc., New York.

Ahn, D., Dittmar, R. & Gallant, A.R. (2002). Quadratic term structure models: theory and evidence. *The Review of Financial Studies* **15**-1, 243-288.

Amin, K. & Jarrow, R. (1992). Pricing options on risky assets in a stochastic interest rate economy. *Mathematical Finance* **2**, 217-237.

Amin, K. & Jarrow, R. (1991). Pricing foreign currency options under stochastic interest rates. *Journal of International Money and Finance* **10**, 310-329.

Anderson, L. (2003). Reduced-form models: curve construction and the pricing of credit swaps, options, and hybrids. In: *Credit Derivatives: The Definitive Guide*, ed. Gregory,

Andreasen, J. (1995). Pricing by arbitrage in an international economy. *Research in International Finance* **12**, 96-116.

Andreasen, J. (2006). Stochastic volatility for real. Working paper, Bank of America, London.

Assefa, S. (2007). Calibration and Pricing in a Multi-Factor Quadratic Gaussian Model. Quantitative Finance Research Centre Paper Series, University of Technology Sydney.

Beaglehole, D. & Tenney, M. (1991). General solutions of some interest rate-contingent claim pricing equations. *Journal of Fixed Income* **1**-2, 69-83.

Bensaid, B. & Bottazzi, J-M. (2001). Evaluation de quelques instruments quantos. *Finance* **22**(2), 25-50.

Bielecki, T.R. & Rutkowski, M. (2002). *Credit Risk: Modelling, Valuation and Hedging.* Springer-Verlag, Berlin, Heidelberg, New York.

Bielecki, T.R. & Rutkowski, M. (2001). Intensity-based valuation of credit derivatives. Working paper, Northeastern Illinois University and Technical University of Warsaw.

Black, F. (1976). The pricing of commodity contracts. *Journal of Financial Economics* **3**, 167-179.

Black, F. & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **81**, 637-653.

Boyle, P. & Tian, W. (1999). Quadratic interest rate models as approximations to effective rate models *Journal of Fixed Income*, December, 1999.

Brace, A., Gatarek, D. Musiela,M. (1997). The market model of interest rate dynamics. *Mathematical Finance* 4, 127-155.

Brémaud, P. (1981). *Point Processes and Queues, Martingale Dynamics* Springer-Verlag, Berlin.

Brigo, D. & Mercurio, F. (2006). *Interest Rate Models-Theory and Practice* Springer-Verlag, Berlin, Heidelberg, New York.

Brigo, D. & Alfonsi, A. (2004). Credit default swaps calibration and option pricing with the SSRD stochastic intensity and interest-rate model. Working paper, Banca IMI, San Paolo IMI Group.

Brigo, D. (2004). Constant maturity credit default swaps pricing with market models. Working paper, Banca IMI.

Carr, P. & Madan, D.B. (1998). Option valuation using the fast fourier transform. *Journal of Computational Finance* 2(2), 6173.

Chen, R., Filipović, D., & Poor, V. (2004). Quadratic term structure models for risk-free and defaultable rates. Working paper, Princeton University.

Chen, R., Cheng, X., Fabozzi, F. & Liu, B. (2006). An explicit, multi-factor credit default swap pricing model with correlated factors. Working paper, Rutgers University, Yale University and Fitch Ratings.

Cheng & Scaillet (2004). Linear-quadratic jump-diffusion modeling with application to stochastic volatility. Working paper, Crédit Suisse, FAME, HEC Genve and Universit de Genve.

Cherif, T., El Karoui, N. (1993). Arbitrage entre deux marchés: Application aux options quanto. Publication de la CAR No 4, June 1993.

Cherif, T., El Karoui, N., Myneni, R. & Viswanathan, R. (1994). Les option quanto "Un modèle quadratique Gaussien". Princeton University USA, March 1995, Journées Internationales de l'AFFI, Juin 1994.

Cherif, T. & Durand, Ph. (1995). Calcul d'option sur produits de taux "Un Modèle Quadratique Gaussien". Working paper Caisse des Dépots et Consignations(FMM) and Universiteé de Paris.

Collin-Dufresne, P. & Goldstein, R.S. (2002). Pricing swaptions within an affine framework. *Journal of Derivatives* **10**, 1-18.

Collin-Dufresne, P., Goldstein, R. & Hugonnier, J. (2004). A general formula for valuing defaultable securities. *Econometrica* **72**(5), 1307-1407.

Cools, R. & Haegemans, A. (2003). Algorithm 824: CUBPACK: A package for automatic cubature; Framework Description, *ACM Transactions on Mathematical Software* **29**-3, 287296.

Corrado, C.J. & Su,T. (1997). Implied volatility skews and stock return skewness and kurtosis implied by stock option prices. *The European Journal of Finance* **3**, 73-85.

Cox, J. C., Ingersoll, J. W. & Ross, S. A.(1985). A theory of the term structure of interest rates. *Econometrica* **54**, 385-407.

d'Aspremont, A. (2003). Interest rate model calibration using semidefinite Programme. *Applied Mathematical Finance* **10**, 183-213.

Duffie, D. (1998). First-to-default valuation. Working paper, Stanford University.

Duffie, D. & Singleton, K. J. (1999). Modelling term structures of defaultable bonds. *Review of Financial Studies* **12**, 687-720.

Duffie, D., Schroder, M. & Skiadas, C.(1996). Recursive valuation of defaultable securities and the timing of resolution of uncertainity. *Annals of Applied Probability* **6**(4), 1075-1090.

Ehlers, P. & Schönbucher, P. (2006). The influence of FX on credit spreads . Working paper, Department of Mathematics, ETH Zürich.

El Karoui, N., Viswanathan, R. & Myneni, R. (1991). Arbitrage pricing and hedging of interest rate claims with state variables. Working paper, Université de Paris IV and Stanford University.

El Karoui, N., Geman, H. & Rochet, J.C. (1995). Changes of numeraire, changes of probability measures and option pricing. *Journal of Applied Probability* **32**, 443-58.

Elouerkhaoui, Y. (2002) Correlated defaults: the conditional jump diffusion framework. Working paper, UBS Warburg and Université Paris Dauphine.

Filippov, A.F. (1988). *Differential equations with discontinuous righthand sides.* Kluwer Academic Publishers, Dordrecht, Boston.

Finkelstein, V. (2000). Pricing single name credit derivatives. Presentation slides, 7th annual CAP workshop, Columbia University, New York.

JP Morgan London (1999) The price of credit. *Risk* December, 1999, 68-71.

Frachot, A. (1995). Factor models of domestic and foreign interest rates with stochastic volatilities. *Mathematical Finance* **5**, 167-185.

Freiling, G. (2002). A survey of nonsymmetric ricatti equations. *Linear Algebra and its Applications* **351-352**, 243270.

Frey, R. & Sommer, D. (1996) A systematic approach to pricing and hedging international derivatives with interest rate risk: analysis of international derivatives under stochastic interest rates. *Applied Mathematical Finance* **3**, 295-317.

Grasselli, M. & Tebaldi,C. (2004). Solvable affine term structure models. Working paper, Universit degli Studi di Verona.

Harrison, J. M. & Pliska, S.R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications* **11**, 215-260.

Heath, D.C., Jarrow, R.A. & Morton,A. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claim valuation. *Econometrica* **60**, 77-105.

Hull, J. (2000). *Options, Futures, and other Derivative Securities,* 4th edition Prentice-Hall, Englewoods Cliffs, NJ.

Itô, K. (1946). On a stochastic integral equation. *Proc. Imperial Acad Tokyo* **22**, 32-35.

Jamshidian, F. (1993). Option and futures evaluation with deterministic volatilities. *Mathematical Finance* **3**(2), 149-159.

Jamshidian, F. (1996). Bond, futures and option evaluation in the quadratic Gaussian model. *Applied Mathematical Finance*3, 93-115.

Jamshidian, F. (1997). LIBOR and swap market models and measures. *Finance and Stochastics*1, 293-330.

Jamshidian, F. (2004). Valuation of credit default swaps and swaptions. *Finance and Stochastics* **8**, 343-371.

Jarrow, R. & Rudd, A. (1982). Approximate option valuation for arbitrary stochastic processes. *Journal of Financial Economics* **10**, 347369.

Jarrow, R., Lando, D. & Turnbull, S.A. (2001). A Markov model for the term Structure of credit spreads. *The Review of Financial Studies* **10**, 481-523.

Jarrow, R. & Yu, F. (2001). Counterparty risk and the pricing of defaultable securities. *Journal of Finance* **56**, 1765-1799.

Jeanblanc, M. & Rutkowski, M. (2000). Modelling of default risk: mathematical tools. Working paper, Université d'Evry Val d'Essonne and Warsaw University of Technology.

Karatzas, I. & Shreve, S (1991). *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York.

Kim, D. (2004). Time-varying risk and return in the quadratic-Gaussian model of the term Structure. Working Paper, Board of Governors of the Federal Reserve.

Kunitomo, N. & Takahashi, A. (2001). The asymptotic expansion approach to the valuation of interest rate contingent claims. *Mathematical Finance* **11**, 117-151.

Kusuoka, S. (1999). A remark on default risk models. *Advances in Mathematical Economics* **1**, 69-82.

Kwok, Y. & Leung, S. (2005). Credit default swap evaluation with counterparty risk. *The Kyoto Economic Review* **74**(1), 25-45.

Lando, D. (1998). On Cox processes and credit-risky securities. *Review of Derivatives Research* **2**, 99-120.

Lee, R. (2004). Option pricing by transform methods: extensions, unification, error control. *Journal of Computational Finance* **7**(3), 51-86.

Leippold, M. & Wu, L. (2002). Asset pricing under the quadratic class. *Journal of Financial and Quantitative Analysis* **37**, 271-295.

Leippold, M. & Wu, L. (2002). Multi-currency quadratic models:theory and evidence. Working Paper, University of Zurich and Fordham University.

Leippold, M. & Wu, L. (2003). Design and estimation of quadratic term structure models. *European Finance Review* **7**, 47-73.

Leonard, I.E. (1996). The matrix exponential. *SIAM Review* **38**(3), 507-512.

Levy, A. & Levin, R. (1999). Wrong way exposure-are firms underestimating their credit risk? *Risk, July 1999*, 52-55.

Martin, R., Thompson, K. & Browne, C. (2001). Price and probability. *Risk, January 2001*, 115-117.

Mathai, Provost (1992). *Quadratic Forms in Random Variables.* Marcel Dekker, New York.

Mellios, C. & Poncet, P. (2001). Valuation of options on bond spreads involving two currencies. *Finance* **22**(2), 75-111.

Mercurio,F. & Pallavivini,A. (2005). Mixing Gaussian models to price CMS derivatives. Working paper, Banca IMI.

Mercurio, F. & Pallavivini, A. (2006). Swaption skews and convexity adjustments. Working paper, Banca IMI.

Moler, C. & Van Loan, C. (1978). Nineteen dubious ways to compute the exponential of a matrix. *SIAM Review* **20**, 801-836.

Musiela, M. & Rutkowski, M. (2005). *Martingale Methods in Financial Modelling* Springer-Verlag, Berlin, Heidelberg, New York.

Norros, I. (1986). A compensator representation of multivariate life length distributions, with applications. *Scandinavian Journal of Statistics* **13**, 99-112.

Pelsser, A. (1997). A tractable interest rate model that guarantees positive interest rates. *Review of Derivatives Research* **1**, 269-284.

Pelsser, A. (2000). *Efficient Models for Valuing Interest Rate Derivatives* Springer-Verlag, Berlin, Heidelberg, New York.

Protter, P. (2004). *Stochastic Integration and Differential Equations.* Springer-Verlag, Berlin, Heidelberg, New York.

Provost, S. (2005). Moment-based density approximations. *The Mathematica Journal* **9**:4, 727-756.

Radon, J. (1927). Über oszillationstheorem der kunjugierten Punkte bei Probleme Lagrange, *Münchener Sitzberichte* **57**, 243-257.

Radon, J. (1928). Zum problem von Lagrange, *Hamburger Math. Einzelschr.* **6**.

Rebonato, R. (1998). *Interest Rate Option Models* , John Wiley and Sons Ltd., Chichester, UK.

Shaked, M. & Shanthikumar, G. (1987). The multivariate hazard construction. *Stochastic Processes and Their Applications* **24**, 241-258.

Schlögl, E. (2002). A multicurrency extension of the lognormal interest rate market rate models. *Finance Stochastics* **6**, 173-196.

Schlögl, E. & Schlögl, L. (2000). A square root interest rate model fitting discrete initial term structure data. *Applied Mathematical Finance* **7**(3), 183-209.

Schmidt, W.M. (1997). On a general class of one-factor models for the termstructure of interest rates. *Finance Stochastics* **1**, 3-24.

Schönbucher, P. (2000). The pricing of credit risk and credit risk derivatives, Working paper, University of Bonn.

Schönbucher, P. (2000). A LIBOR market model with default risk. Working paper, University of Bonn.

Schrager, D. & Pelsser, A. (2006). Pricing swaptions and coupon bond options in affine term structure models. *Mathematical Finance* **16**(4), 673-694.

Meza, A. & Satzschneider, W. (2002). ECIR model application on Mexican Treasury Certificates (CETES) through the steepest descent algorithm. Presentation at the International Congress of Actuaries, Cancún, México (ICA Cancun 2002).

Tanaka, K., Yamada, T. & Watanabe, T. (2005). A simple formulation of swaption prices by Gram-Charlier expansion. Working Paper, Graduate School of Economics, Kyoto University.

Triantafyllopoulos, K. (2003). On the central moments of the multidimensional Gaussian distribution. *Mathematical Scientist* **28**, 125-128.

Vaillant, N. (2001). A beginner's guide to credit derivatives. Working paper, Nomura International.

Yu, F. (2004) Correlated defaults and the valuation of defaultable securities Working paper, University of California, Irvine.

Zwillinger, D. (1998). *Handbook of Differential Equations.* Academic Press, San Diego.