EVALUATION OF AMERICAN STRANGLES - EXTENDED ABSTRACT

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1. Introduction

It is common practice in financial markets to trade call and put option contracts not individually, but in portfolios. These portfolios typically involve familiar option trading strategies, such as straddles, strangles and butterflies. Pricing these European option portfolios using the Black-Scholes framework is a trivial matter. The properties of the Black-Scholes pricing formula are such that the price of a portfolio of European options is given by the sum of the Black-Scholes value for each component option within the portfolio.

Many option contracts traded in financial markets are of the American type, which allow the option to be exercised early. The purpose of this paper is to extend existing analytic American option pricing techniques to the case of American option portfolios. The example on which this paper focuses is the American strangle position. An American strangle is defined as an option position having the same payoff function as a European strangle, with the additional feature that the position may be exercised "optimally" at any time prior to expiry. More specifically, if the strangle is deep in-the-money on the call-side, then it may be exercised early, and in doing so the put-side payoff is foregone, and vice versa when the position is deep in-the-money on the put-side.

This is not financially equivalent to taking a long position in both an American call and an American put option on the same underlying asset. If the strangle were formed in this way, then exercising one option early would leave the holder with the other option still unexercised. This simple combination of an American call and put would fail to introduce the "knock-out" effect that a holder could find desirable in an American strangle position. It follows that the Black-Sholes price of an American strangle is not simply the sum of the Black-Scholes price for each component option. In light of this fact, the purpose of this research is to firstly derive the correct pricing formula for

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the American strangle position. It turns out that the free boundaries of the American strangle on the put and call side satisfy a pair of coupled integral equations. Having derived this coupled integral equation system, the next aim is to find and implement a numerical technique to solve for the free boundaries, so that the strangle's price can be determined.

2. Problem Formulation

Let $C(S,t)$ be the price of an American strangle position on the asset, S, at time t, with time to expiry $(T-t)$. The position's payoff is formed by combining a long put option with strike K_1, and a long call option with strike K_2, where $K_1 < K_2$. Let the early exercise boundary on the put side be denoted by $a_1(t)$, and the early exercise boundary on the call side be denoted by $a_2(t)$. Hence C satisfies the Black-Scholes PDE:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC = 0, \quad 0 \leq t \leq T \tag{1}$$

in the region $a_1(t) < S < a_2(t)$

where

$$\begin{aligned} \sigma &= \text{volatility of } S, \\ r &= \text{risk-free rate}, \\ q &= \text{dividend rate of } S \text{ (continuous)}, \end{aligned}$$

subject to the following final time and boundary conditions:-

$$C(S,T) = \max(K_1 - S, 0) + \max(S - K_2, 0), \quad 0 < S < \infty, \tag{2}$$

$$C(a_1(t),t) = (K_1 - a_1(t)), \quad t \geq 0 \tag{3}$$

$$C(a_2(t),t) = (a_2(t) - K_2), \quad t \geq 0 \tag{4}$$

$$\lim_{S\to a_1(t)} \frac{\partial C}{\partial S} = -1, \quad t \geq 0 \tag{5}$$

$$\lim_{S\to a_2(t)} \frac{\partial C}{\partial S} = 1, \quad t \geq 0. \tag{6}$$

To solve this PDE, the Fourier transform method of McKean (1965) shall be used. Details of this methodology for the American call option are provided in Kucera and Ziogas (2002). To facilitate the Fourier transform approach, the PDE (1) is transformed to a forward in time equation with constant coefficients.

Let

$$C(S,t) = V(x,\tau), \tag{7}$$

where

$$S = e^x, \quad t = T - \tau. \tag{8}$$

The transformed PDE is then

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + k\frac{\partial V}{\partial x} - rV, \quad 0 \le \tau \le T, \tag{9}$$

in the region $\ln c_1(\tau) < x < \ln c_2(\tau)$
where

$$\begin{aligned} c_1(\tau) &\equiv a_1(t), \\ c_2(\tau) &\equiv a_2(t), \quad \text{and} \\ k &= (r - q - \frac{1}{2}\sigma^2). \end{aligned}$$

The new initial and boundary conditions are

$$V(x,0) = \max(K_1 - e^x, 0) + \max(e^x - K_2, 0), \quad -\infty < x < \infty \tag{10}$$

$$V(\ln c_1(\tau), \tau) = K_1 - c_1(\tau), \quad \tau \ge 0, \tag{11}$$

$$V(\ln c_2(\tau), \tau) = c_2(\tau) - K_2, \quad \tau \ge 0 \tag{12}$$

$$\lim_{x \to \ln c_1(\tau)} \frac{\partial V}{\partial x} = -c_1(\tau), \tag{13}$$

$$\lim_{x \to \ln c_2(\tau)} \frac{\partial V}{\partial x} = c_2(\tau). \tag{14}$$

3. Applying the Fourier Transform

Expanding on the approach of McKean (1965), define the incomplete Fourier transform of $V(x,\tau)$ as follows:

$$\mathcal{F}^c\{V(x,\tau)\} = \int_{\ln c_1(\tau)}^{\ln c_2(\tau)} V(x,\tau) e^{i\eta x} dx.$$

In addition, denote

$$\hat{V}(\eta,\tau) \equiv \mathcal{F}^c\{V(x,\tau)\}.$$

Taking the incomplete Fourier transform of the PDE (9) with respect to x, the following ODE is obtained:

$$\frac{d\hat{V}}{d\tau} + (\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)\hat{V} = F(\eta,\tau), \tag{15}$$

where

$$\begin{aligned} F(\eta,\tau) &= e^{i\eta \ln c_2(\tau)}[\frac{\sigma^2 c_2(\tau)}{2} + (\frac{c_2'(\tau)}{c_2(\tau)} - \frac{\sigma^2 i\eta}{2} + k)(c_2(\tau) - K_2)] \\ &\quad + e^{i\eta \ln c_1(\tau)}[\frac{\sigma^2 c_1(\tau)}{2} + (\frac{c_1'(\tau)}{c_1(\tau)} - \frac{\sigma^2 i\eta}{2} + k)(c_1(\tau) - K_1)], \end{aligned} \tag{16}$$

and the initial condition is now

$$\mathcal{F}\{V(x,0)\} \equiv \hat{V}(\eta,0).$$

By solving this linear ODE, $\hat{V}(\eta,\tau)$ is given by

$$\hat{V}(\eta,\tau) = \hat{V}(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)\tau} + \int_0^\tau F(\eta,s)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)(\tau-s)}ds. \tag{17}$$

4. Inverting the Fourier Transform

Taking the inverse (complete) Fourier transform of (17) results in

$$V(x,\tau) = V_1(x,\tau) + V_2(x,\tau); \quad \ln c_1(\tau) < x < \ln c_2(\tau), \tag{18}$$

where $V_1(x,\tau)$ is given by

$$\begin{aligned} V_1(x,\tau) \;=\; & [K_1e^{-r\tau}N(d_2(K_1)) - e^xe^{-q\tau}N(d_1(K_1))] \\ & +[e^xe^{-q\tau}N(d_1(K_2)) - e^{-r\tau}N(d_2(K_2))] \\ & -[K_1e^{-r\tau}N(d_2(c_1(0))) - e^xe^{-q\tau}N(d_1(c_1(0)))] \\ & -[e^xe^{-q\tau}N(d_1(c_2(0))) - K_2e^{-r\tau}N(d_2(c_2(0)))], \end{aligned} \tag{19}$$

with

$$\begin{aligned} N(y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{\alpha^2}{2}}d\alpha, \\ d_1(K) &= \frac{x - \ln K + [k+\sigma^2]\tau}{\sigma\sqrt{\tau}}, \\ d_2(K) &= d_1(K) - \sigma\sqrt{\tau}, \end{aligned}$$

and $V_2(x,\tau)$ is given by

$$V_2(x,\tau) = \int_0^\tau \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}}[e^{-h_1(x,s)}Q_1(x,s) + e^{-h_2(x,s)}Q_2(x,s)]ds, \tag{20}$$

with

$$h_i(x,s) = \frac{(x - \ln c_i(s) + k(\tau-s))^2}{2\sigma^2(\tau-s)},$$

and

$$Q_i(x,s) = \frac{\sigma^2 c_i(s)}{2} + (\frac{c_i'(s)}{c_i(s)} + \frac{1}{2}[k - \frac{(x-\ln c_i(s))}{(\tau-s)}])(c_i(s) - K_i)$$

for $i = 1, 2$.

This expression turns out to be the linear sum of the valuation formulas for the component American call and put options which make up the American strangle position. The increased complexity arises when finding the early exercise boundaries $c_1(\tau)$ and $c_2(\tau)$, which are required before (18)-(20) can be applied.

To correctly find these free boundaries, the equation given by (18)-(20) must be evaluated at each free boundary. Recalling the boundary conditions (11) and (12), and noting that $V(x,\tau)$ is found by inverting the Fourier transform of a function that is discontinuous at both $x = c_1(\tau)$ and $x = c_2(\tau)$, the integral equations for the free boundaries are given by

$$\frac{K_1 - c_1(\tau)}{2} = V(c_1(\tau), \tau), \tag{21}$$

$$\frac{c_2(\tau) - K_2}{2} = V(c_2(\tau), \tau). \tag{22}$$

Since $V(x, \tau)$ is already a function of both $c_1(\tau)$ and $c_2(\tau)$, it will be necessary to solve the integral equation system (21)-(22) simultaneously.

From these analytical formulae it is clear that the free boundaries for the American strangle are not equivalent to the corresponding independent free boundaries associated with the position's component American call and put options. This is because exercising the position early will mean foregoing the out-of-the-money side of the strangle, which still has some intrinsic value.

5. Numerical Evaluation

Now that the integral equation system for the American strangle's free boundaries is known, the next step is to find a way to numerically solve the system for these free boundaries. The final version of the paper will give details on the algorithm, along with a collection of numerical results.

The integral equations on hand are non-linear Volterra equations of the second kind. One traditional approach to solving such integral equations is to divide the τ region into n sub-intervals of equal length, and then solve recursively for $c_1(\tau)$ and $c_2(\tau)$ at each discrete value of τ. The matter is complicated by the presence of the first derivatives of both $c_1(\tau)$ and $c_2(\tau)$ inside the integrand component, $V_2(x, \tau)$. It has been noted, however, that authors such as Kim (1990) and Carr, Jarrow and Myneni (1992) have found different integral representations for the American call and put price that do not involve these derivatives. It would be a simple matter to convert equation in (18)-(20) into one of these forms, and apply this proposed numerical technique.

Another approach that has been tentatively explored involves retaining the current form for the price in equation (18)-(20), and applying an iterative technique in which "reasonable" initial estimates are made for $c_1(\tau)$ and $c_2(\tau)$. Once again the τ region needs to be divided into n sub-intervals. The derivatives $c_1'(\tau)$ and $c_2'(\tau)$ can be estimated at each discrete τ value by fitting cubic splines to the approximations for $c_1(\tau)$ and $c_2(\tau)$ at each iteration. While some initial investigation has been conducted, the conditions for and rate of convergence is still unknown.

Once a numerical method has been successfully implemented, a comparison between the analytically correct American strangle price, and the "short cut" approach of simply adding the component American call and put values together shall be undertaken. In addition, an exploration will be conducted into how the parameter values (namely the strikes, volatility, risk-free rate and dividend rate) affect this pricing discrepancy.

References

[1] Carr, P., R. Jarrow, & R. Myneni (1992): "Alternative Characterizations of American Put Options", *Journal of Mathematical Finance*, 2, 87-106.

[2] Kim, I. J. (1990): "The Analytic Valuation of American Options", *Review of Financial Studies*, 3, 547-572.

[3] McKean, Jr., H. P. (1965): "Appendix: A Free Boundary Value Problem for the Heat Equation Arising from a Problem in Mathematical Economics", *Industrial Management Review*, 6, 32-39.

[4] Kucera, A & A. Ziogas (2002): "McKean's Problem Applied to an American Call Option", *working paper*, University of Technology, Sydney.