Linear Physical-layer Network Coding and Information Combining for the K-user Fading Multiple-access Relay Network

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Abstract

We propose a new linear physical-layer network coding (LPNC) and information combining scheme for the K-user fading multiple-access relay network, which consists of K users, one relay and one destination. The relay and the destination are connected via a rate-constraint wired or wireless backhaul. In the proposed scheme, the K users transmit signals simultaneously. The relay and the destination receive the superimposed signals distorted by fading and noise. The relay reconstructs L linear combinations of the K users' messages, referred to as L network coded (NC) messages, and forwards them to the destination. The destination then attempts to recover all K users' messages by combining its received signals and the NC messages obtained from the relay. We develop an explicit expression on the selection of the coefficients of the NC messages at the relay that minimizes the end-to-end error probability at a high signal to noise ratio. We develop a channel-coded LPNC scheme by using an irregular repeat-accumulate modulation-code over GF(q). An iterative belief-propagation algorithm is employed to compute the NC messages at the relay, while a new algorithm is proposed for the information combining decoding at the destination. We demonstrate that our proposed scheme outperforms benchmark schemes significantly in both un-channel-coded and channel-coded MARNs.

I. INTRODUCTION

Inter-user interference is becoming the dominating bottleneck of modern wireless systems, such as 4G/5G cellular networks and next generation WiFi networks [1], [2]. Physical-layer network coding (PNC) is a technique that can efficiently utilize inter-user interference to improve the throughput or reliability of some wireless networks [3] [4]. Most works of PNC focused on a two-way relay channel [5] - [13] and multi-input multi-output (MIMO) two-way relay channel

[14] - [16]. Recently, the impact of PNC on multiple-access networks draws a lot of attention. In [17], the authors proposed a two-layer bridging scheme which was termed network coded multiple-access (NCMA) to improve the throughput of the wireless network. In the scheme, PNC decoding and multi-user decoding (MUD) were jointly explored to achieve multi-reception at physical layer, and the multi-reception results were sent to the medium access control (MAC) layer to recover all users' messages. Implementation of NCMA scheme in [17] and [18] showed that the throughput of the wireless network can be boosted significantly. From the theoretical perspective, the authors in [19] generalized their compute-and-forward (CF) framework to the Gaussian multiple-access channel (GMAC). It was proved that the entire capacity region of two-user GMAC can be attained with a single-user decoder without time-sharing. The concept of PNC has also been used in random multiple-access network to resolve transmission contention. In [20] - [22], linear equations from the collided packets are explored to derive users' packets by solving the linear equations. But only one equation was formed in one time slot in [20] and [21]. Multiple linear equations were derived from the collided packets in one time slot in [22], however, the scheme used exhaustive decoding to find the recoverable linear equations, which leads to a high decoding complexity for a medium to high contention cardinality. Above works mainly focused on two-user multiple-access networks or random multiple-access networks. The impact of PNC on *K-user fading multiple-access relay networks* (MARN) remains unclear.

In this paper, we consider a MARN that consists of K single-antenna users, one relay node and one destination node. The relay node and destination node has N_R and N_D antennas, respectively. The relay node is connected to the destination via a wired or wireless backhaul (BH) link [23]. The destination is interested in recovering all K users' messages.

This MARN models a practical WiFi network operated with the wireless distribution system (WDS). In a typical setting, the destination node, referred to as the primary access point (AP), is connected to the Internet. The relay node, referred to as the auxiliary AP, is connected to the destination node via wired or wireless bridging. The two APs are usually placed reasonably far apart to provide coverage for the whole area. The K users attempt to access the Internet via the primary and auxiliary APs. Note that the MARN also applies to a cellular network with cooperative base stations, and small-cell systems or heterogeneous networks.

In the existing WiFi system with WDS, a simple AP switching method is used. That is, each user is allocated with only one of the APs based on the channel quality and other implementation constraints. This may result in several disadvantages such as: 1) *No information combining*, that is, the signal of a user received by the other AP (that is not allocated to this user) is completely ignored. This leads to reduced rate or reliability performance; 2) frequent AP switching is required when the channel condition changes, which leads to increased system overhead.

This paper contributes to the MARN in the following aspects:

1. We propose a new linear physical-layer network coding (LPNC) scheme for the un-channel-coded MARN, which consists of K single antenna users, one relay node and one destination node. In the proposed scheme, the relay node reconstructs L *linear combinations* of the K users' messages, which are referred to as L *network coded (NC) messages*, and forwards them to the destination. We investigate the optimal design of the proposed LPNC scheme and show that the end-to-end error performance of the MARN is determined by the choice of the coefficients w.r.t. the L NC messages.

2. We develop an explicit expression on the selection of the coding coefficients of the L NC messages at the relay that minimizes the end-to-end error probability at a high signal to noise ratio (SNR). The proposed NC coefficients selection approach does not require exhaustive decoding as used to reconstruct multiple NC messages in [22]. Thus, it has a much simpler relay operation. Furthermore, the design of L NC messages is very general and optimal, as it contains decoding of a single user message or an LPNC message as its special case. This is different from the NCMA scheme in [17] - [18], where MUD decoder, single-user decoder and PNC decoder are all equipped at the receiver to achieve multi-reception, and the optimality of multi-reception is not discussed.

3. We propose a new information combining scheme for the un-channel-coded MARN scheme at the destination node. In the proposed scheme, the destination node not only receives a distorted and noisy copy of the signals from the K users, but also receives *multiple* NC messages from the relay node. We propose an information combining decoding algorithm based on the maximum a posteriori probability (MAP) rule for the destination uses to recover all K users' messages, where the received NC messages are used as side-information. We show that, the side-information can effectively help the destination to decode the K users' signals, leading to a significant improvement in the system error performance.

4. We develop a practical channel-coded LPNC scheme. The underlying channel-code is an irregular repeat-accumulate (IRA) modulation code over GF(q). A symbol-wise NC message MAP detector and an iterative belief-propagation (BP) decoder are used to compute the NC message sequences at the relay. A symbol-wise information combining MAP detection algorithm is proposed to recover all K users' message sequences at the destination. We demonstrate that our proposed scheme outperforms complete decoding scheme by 3 - 7 dB at practical SNR values in an un-channel-coded wired backhaul MARN. We also show that our proposed scheme outperforms iterative multi-user detection and decoding (IDD) scheme by 2.95 - 4.5 dB at practical SNR values in a channel-coded wired or wireless backhaul MARN.

II. System Model (Un-channel-coded)

In this section, the MARN considered in this work will be introduced. We first focus on an un-channel-coded system over a real-valued channel model for two reasons. First, the signal constellations with LPNC can be clearly characterized in the un-channel-coded case. Only given that, the error probability of the proposed scheme can be expressed, leading to our asymptotically optimal LPNC design solution in Section IV[1]. Second, the complex-valued model can be represented by an equivalent real-valued model as shown in [16]. Given the equivalent real-valued model as in [16], we only need to consider the design for the real-valued model throughout this paper.

A. Real-valued Model

Consider a real-valued, un-channel-coded Rayleigh fading MARN. Let $u_k \in GF(q)$ be the *message symbol* of user k, $k \in \{1, 2, \cdots, K\}$. Each user employs zero mean uniform spacing q-PAM modulation, yielding the *modulated symbol*

$$x_k = \left(u_k - \frac{q-1}{2}\right)\frac{1}{\gamma},\ k \in \{1, 2, \cdots, K\}, \tag{1}$$

where γ is a normalization factor which ensures $E\left(x_k^2\right) = 1$. The K users transmit their modulated signals simultaneously.

[1]The performance analysis task may be intractable if channel coding and LPNC are jointly considered, making it very difficult to obtain the LPNC design insight.

The relay with N_R antennas receives

$$\mathbf{y}_R = \sum_{k=1}^{K} \mathbf{h}_{k,R} \sqrt{E_s} x_k + \mathbf{z}_R, \tag{2}$$

where E_s denotes the energy per-symbol, $\mathbf{h}_{k,R}$ is the size N_R-by-1 channel coefficient vector from user k to the relay and $\mathbf{z}_R$ is N_R-by-1 additive white Gaussian noise (AWGN) vector.

In the mean time, the destination node with N_D antennas receives

$$\mathbf{y}_D = \sum_{k=1}^{K} \mathbf{h}_{k,D} \sqrt{E_s} x_k + \mathbf{z}_D, \tag{3}$$

where $\mathbf{h}_{k,D}$ is the size N_D-by-1 channel coefficient vector from user k to the destination and $\mathbf{z}_D$ denotes the N_D-by-1 AWGN vector. Without loss of generality, we assume that the entries of $\mathbf{z}_R$ and $\mathbf{z}_D$ are i.i.d. and have zero mean and variance σ^2. The SNR is defined as $\rho \triangleq \frac{E_s}{\sigma^2}$.

For notation simplicity, let $\mathbf{u} = [u_1, u_2, \cdots, u_K]^T$, $\mathbf{x} = [x_1, x_2, \cdots, x_K]^T$, $\mathbf{H}_R = [\mathbf{h}_{1,R}, \cdots, \mathbf{h}_{K,R}]$ and $\mathbf{H}_D = [\mathbf{h}_{1,D}, \cdots, \mathbf{h}_{K,D}]$. Then, (2) and (3) are respectively rewritten as

$$\mathbf{y}_R = \mathbf{H}_R \sqrt{E_s} \mathbf{x} + \mathbf{z}_R \text{ and } \mathbf{y}_D = \mathbf{H}_D \sqrt{E_s} \mathbf{x} + \mathbf{z}_D. \tag{4}$$

Remark 1 (Degree of Channel State Information (CSI)): In this work, we consider an open-loop system, i.e., there is no feedback from a receiver to a transmitter. As such, each user does not have the transmitter-side CSI. We assume perfect local CSI at the receivers, i.e., the relay node has perfect knowledge of $\mathbf{H}_R$ and the destination node has perfect knowledge of $\mathbf{H}_D$.

The relay node is connected to the destination node via a wired or wireless BH link. Given the received signal vector $\mathbf{y}_R$, the relay node generates some function of it. The process can be represented as

$$\mathbf{y}_R \overset{\text{Relay}}{\longrightarrow} f\left(\mathbf{y}_R\right). \tag{5}$$

Here, the function $f\left(\mathbf{y}_R\right)$ needs to satisfy a BH rate constraint, as will be detailed momentarily. The relay node then sends $f\left(\mathbf{y}_R\right)$ to the destination. Denote by $\widehat{f}\left(\mathbf{y}_R\right)$ the function delivered and recovered at the destination.

The destination node attempts to recover all K users' messages based on its received signal $\mathbf{y}_D$ from the users and the function $\widehat{f}\left(\mathbf{y}_R\right)$ from the relay. The process at the destination can be represented as

$$\left(\mathbf{y}_D, \widehat{f}\left(\mathbf{y}_R\right)\right) \overset{\text{Destination}}{\longrightarrow} \widehat{\mathbf{u}}, \tag{6}$$

where $\widehat{\mathbf{u}} = [\widehat{u}_1, \widehat{u}_2, \cdots, \widehat{u}_K]^T$ denotes the K users' messages recovered by the destination.

B. Problem Statement

An end-to-end decoding error at the destination is declared if $\widehat{\mathbf{u}} \neq \mathbf{u}$. For the un-channel-coded model described above, we concern about the problem of designing an efficient communication strategy, subject to the rate constraint of the BH link, such that $\Pr\left(\widehat{\mathbf{u}} \neq \mathbf{u}\right)$ is small. A more ambitious object would be to find the scheme that minimizes $\Pr\left(\widehat{\mathbf{u}} \neq \mathbf{u}\right)$ subject to the rate constraint of the BH link. The key ingredients that are central to the design problem involves: 1) the processing at the relay, i.e., Eq. (5); 2) the processing at the destination, i.e., Eq. (6).

Remark 2 (Constraint of the BH link): In practice, the BH link between the relay and the destination can be either wired or wireless, depending on the specific application. For the case with a wired BH, we consider that the BH link is noiseless and subject to a rate constraint of R_{BH} bits/channel-use, as in the prior works [23] [27] [28]. To meet this rate constraint, the entropy of $f\left(\mathbf{y}_R\right)$ must be no greater than R_{BH}, i.e.,

$$H\left(f\left(\mathbf{y}_R\right)\right) \leq R_{BH}. \tag{7}$$

Given that the rate constraint is met, the function generated by the relay can be delivered to the destination free of error, i.e., $\widehat{f}\left(\mathbf{y}_R\right) = f\left(\mathbf{y}_R\right).$

Remark 3: For the case with a wireless BH, the destination will receive a noisy observation of the signal sent by the relay, as in the conventional single-user two-hop relay network. In this case, the rate constraint of the BH is dependent on the wireless channel between the relay and destination. Generally speaking, if a scheme performs better than the other scheme in the wired BH scenario, it will not perform worse than the other scheme in the wireless BH scenario. In this paper, we focus on the case with a wired BH in presenting our proposed LPNC scheme. Our method also applies to the MARN with a wireless BH. In fact, the performance of our scheme in an MARN with wireless BH is also evaluated in Section VI-B.

III. Proposed LPNC and Information Combining Scheme for an Un-channel-coded MARN

A. Preliminaries

Consider an MARN with a wired BH link of a rate constraint R_{BH}. In the proposed scheme, the relay constructs L linear combinations of the K users' messages over GF(q) based on its received signals from the K users. Recall (7), L is set to $\lfloor R_{BH}/\log_2^q \rfloor$ to meet the BH rate constraint, where $\lfloor x \rfloor$ obtains the integer part of x. For illustration convenience, we consider that q is a prime number so that the integer set $\{0, ..., q-1\}$ forms a finite field under modular addition and multiplication.

Let the L linear message-combinations over GF(q) be denoted by

$$w_l = \mathbf{g}_l^T \otimes \mathbf{u},\ l = 1, \ldots, L, \tag{8}$$

where "$\otimes$" denotes the modulo-q matrix multiplication operation. We refer to $w_1, \cdots, w_L$ as L linear *NC messages* and $\mathbf{g}_l$ as the *NC coefficient vector* associated with the lth NC message w_l.

For notation simplicity, let $\mathbf{w} = [w_1, \cdots, w_L]^T$, which is referred to as a *NC message vector*. In addition, let the coefficients w.r.t. the L NC messages be denoted by

$$\mathbf{G} = [\mathbf{g}_1, \cdots, \mathbf{g}_L]\,. \tag{9}$$

Then, (8) can be written as

$$\mathbf{w} = \mathbf{G}^T \otimes \mathbf{u}. \tag{10}$$

In this paper, we refer to $\mathbf{G}$ as an *NC generator matrix*. Throughout this paper, we consider that $\mathbf{g}_1, \cdots, \mathbf{g}_L$ are linearly independent in the finite field. This suggests that $L \leq K$ and the NC generator matrix $\mathbf{G}$ has a full column rank of L.

B. Relay Node's Processing (Linear Physical-layer Network Coding)

The relay node computes L NC messages following the notion of CF [26] or linear PNC [29]. The mechanism is described below:

Step 1. Based on the knowledge of $\mathbf{H}_R$, the relay selects an NC generator matrix $\mathbf{G}$.

Step 2. Given the received signal $\mathbf{y}_R$, the relay attempts to construct the NC message vector $\mathbf{w} = [w_1, \cdots, w_L]^T$ w.r.t. the selected $\mathbf{G}$. (The algorithm that the relay uses will be detailed momentarily.) Denote the NC message vector constructed by the relay as $\widehat{\mathbf{w}} = [\widehat{w}_1, \cdots, \widehat{w}_L]^T$.

Step 3. The relay sends the NC message vector $\widehat{\mathbf{w}}$ to the destination node via the BH link.

Remark 4: The selection of $\mathbf{G}$ is critical. Later we will show that the end-to-end error probability $\Pr(\widehat{\mathbf{u}} \neq \mathbf{u})$ at the destination is dependent on the choice of $\mathbf{G}$. We assume that $\mathbf{G}$ is delivered to the destination via a reliable link. This is done once per fading channel realization and the extra complexity and cost is not significant for a slow fading channel of a relatively large coherence time.

Here we turn back to Step 2 of the relay's processing and present two algorithms for the relay to construct the NC message vector $\mathbf{w}$ for a given $\mathbf{G}$. An algorithm based on a *joint maximum a posteriori probability* (J-MAP) rule is written as

$$\begin{aligned}\widehat{\mathbf{w}} &= \arg\max_{\mathbf{w}} p\left(\mathbf{w}|\mathbf{y}_R,\mathbf{G}\right) = \arg\max_{\mathbf{w}} \sum_{\mathbf{u}:\mathbf{G}^T\otimes\mathbf{u}=\mathbf{w}} p\left(\mathbf{u}|\mathbf{y}_R,\mathbf{G}\right) = \arg\max_{\mathbf{w}} \sum_{\mathbf{u}:\mathbf{G}^T\otimes\mathbf{u}=\mathbf{w}} p\left(\mathbf{y}_R|\mathbf{u}\right)\\ &= \arg\max_{\mathbf{w}} \sum_{\mathbf{u}:\mathbf{G}^T\otimes\mathbf{u}=\mathbf{w}} \exp\left(-\frac{\left\|\mathbf{y}_R-\mathbf{H}_R\left(\mathbf{u}-\frac{q-1}{2}\right)\frac{\sqrt{E_s}}{\gamma}\right\|^2}{2\sigma^2}\right). \end{aligned} \tag{11}$$

The algorithm specified in (11) first calculates the sum of the likelihood functions of all $\mathbf{u}$ vectors whose underlying NC message vectors are equal to a certain value of $\mathbf{w}$. Next, the sum of the likelihood functions w.r.t. different values of $\mathbf{w}$ are compared, and the one with the maximum value is decided as $\widehat{\mathbf{w}}$.

As an alternative, an algorithm based on a *componentwise MAP* (C-MAP) rule is given by

$$\begin{aligned}\widehat{w}_l &= \arg\max_{w_l} p\left(w_l|\mathbf{y}_R,\mathbf{g}_l\right) = \arg\max_{w_l} \sum_{\mathbf{u}:\mathbf{g}_l^T\otimes\mathbf{u}=w_l} p\left(\mathbf{u}|\mathbf{y}_R,\mathbf{g}_l\right) = \arg\max_{w_l} \sum_{\mathbf{u}:\mathbf{g}_l^T\otimes\mathbf{u}=w_l} p\left(\mathbf{y}_R|\mathbf{u}\right)\\ &= \arg\max_{w_l} \sum_{\mathbf{u}:\mathbf{g}_l^T\otimes\mathbf{u}=w_l} \exp\left(-\frac{\left\|\mathbf{y}_R-\mathbf{H}_R\left(\mathbf{u}-\frac{q-1}{2}\right)\frac{\sqrt{E_s}}{\gamma}\right\|^2}{2\sigma^2}\right), \text{ for } l=1,\cdots,L. \end{aligned} \tag{12}$$

The J-MAP algorithm finds the most likely $\mathbf{w}$ vector in a joint manner based on the received signal $\mathbf{y}_R$ and NC generator matrix $\mathbf{G}$. In contrast, the C-MAP algorithm finds the most likely NC messages $w_l, l=1,\ldots,L$ based on the received signal $\mathbf{y}_R$ and each NC coefficient vector $\mathbf{g}_l$, where the decisions on the L NC messages are made in a separate manner. For the case of $L=1$, the J-MAP algorithm is equivalent to the C-MAP algorithm. For a general L, the error probability based on the J-MAP algorithm is a lower bound of that based on the C-MAP algorithm. At a high SNR, the C-MAP algorithm has almost the same error probability of the J-MAP algorithm, while it can simplify the analysis and design of the proposed LPNC scheme. Therefore, the C-MAP algorithm will be used in our analysis and design.

Example 1: In this example, we show the received signal constellation at the relay for a specific channel realization and the relay's operation. We consider an MARN with $K=2$,

$N=1$, $L=1$, $q=3$, and the user-relay channel coefficients $[h_{1,R}, h_{2,R}] = [1.1, 1]$. There are $q^K = 9$ constellation points in total as shown in Fig. 1. The NC coefficient vector is selected to be $\mathbf{g} = [1,1]^T$. Those constellation points with identical underlying NC message are labelled with the same icon. Those points whose underlying NC messages are different are labelled with different icons. In the LPNC scheme, the relay will find the most likely NC message w w.r.t. the received signal $\mathbf{y}_R$ using (11) or (12). In this example we see that the minimum distance between the points with different icons are greater than the minimum distance w.r.t. all $q^K = 9$ points, which means that the error probability of decoding the NC message w at the relay using the NC coefficient vector $\mathbf{g} = [1,1]^T$ for the source-relay channel $\mathbf{H}_R = [1.1, 1]$ is small.

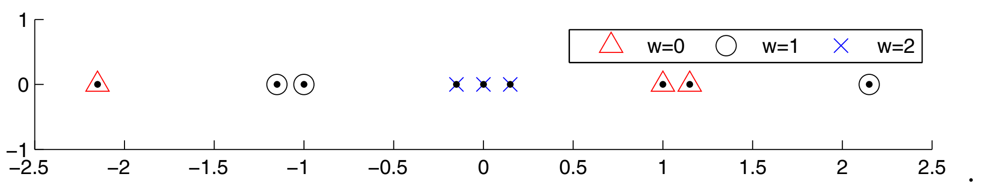

Fig. 1. Illustration of the constellations at the relay, where $\mathbf{H}_R = [1.1,\ 1]$ and $q = 3$.

C. Destination Node's Processing (Information Combining)

The destination node attempts to recover all K users' messages by combining its received signal $\mathbf{y}_D$ from the users and the NC message vector $\widehat{\mathbf{w}}$ obtained from the relay. The information combining (decoding) processing based on the MAP rule is given by

$$\begin{aligned}\widehat{\mathbf{u}} &= \arg\max_{\mathbf{u}} p\left(\mathbf{u}|\mathbf{y}_D, \widehat{\mathbf{w}}, \mathbf{G}\right) \underset{(a)}{=} \arg\max_{\mathbf{u}} p\left(\mathbf{y}_D|\mathbf{u}\right) p\left(\mathbf{u}|\widehat{\mathbf{w}}, \mathbf{G}\right) \\ &= \arg\max_{\mathbf{u}:\ \mathbf{G}^T \otimes \mathbf{u} = \widehat{\mathbf{w}}} \exp\left(-\frac{\left\|\mathbf{y}_D - \mathbf{H}_D\left(\mathbf{u} - \frac{q-1}{2}\right)\frac{\sqrt{E_s}}{\gamma}\right\|^2}{2\sigma^2}\right) \\ &= \arg\min_{\mathbf{u}:\ \mathbf{G}^T \otimes \mathbf{u} = \widehat{\mathbf{w}}} \left\|\mathbf{y}_D - \mathbf{H}_D\left(\mathbf{u} - \frac{q-1}{2}\right)\frac{\sqrt{E_s}}{\gamma}\right\|^2. \end{aligned} \tag{13}$$

Here, we assume that the users' messages $\mathbf{u}$ have equal probability to be transmitted in step (a), i.e., $p(\mathbf{u}) = \frac{1}{q^K}$. The fact that the selection of NC generator matrix $\mathbf{G}$ and the decoding of NC messages $\widehat{\mathbf{w}}$ at the relay are independent of the received signal $\mathbf{y}_D$ at the destination is also used in step (a).

In the algorithm specified in (13), among those message vectors $\mathbf{u}$ that satisfy $\mathbf{G}^T \otimes \mathbf{u} = \widehat{\mathbf{w}}$, the one that is closest to the received signal vector $\mathbf{y}_D$ is decided. Here, $\widehat{\mathbf{w}}$ provides side information which helps the destination to decode the transmitted signals. To be specific, consider the signal constellation of $\mathbf{H}_D \left(\mathbf{u} - \frac{q-1}{2}\right) \frac{\sqrt{E_s}}{\gamma}$ for all $\mathbf{u} \in \{0, \cdots, q-1\}^k$. In general, there are q^K points in the constellation to be distinguished if there is no side information provided. Given the side information $\widehat{\mathbf{w}}$ from the relay, the points whose underlying NC messages are not equal to $\widehat{\mathbf{w}}$ are expurgated as $p\left(\mathbf{u}|\widehat{\mathbf{w}}, \mathbf{G}\right) = 0$, $\forall \mathbf{u} : \mathbf{G}^T \otimes \mathbf{u} \neq \widehat{\mathbf{w}}$. Hence, there will be only q^{K-L} points in the constellation that are relevant to the decoding.

Example 2: In this example, we show the received signal constellation at the destination for a specific channel realization and the destination's operation. We consider an MARN with $K = 2$, $N_D = 1$, $L = 1$, $q = 3$, and the user-destination channel coefficients $[h_{1,D}, h_{2,D}] = [1.43, 0.75]$. There are $q^K = 9$ constellation points in total as shown in Fig. 2. Yet, there are only $q^{K-L} = 3$ points whose underlying NC messages are identical to $\widehat{w}$, the side information provided by the relay. We label these three points by "⊡". Upon receiving $\mathbf{y}_D$, the destination chooses the most likely point out of the three points labelled by "⊡". Those points without the icon "⊡" are not considered in the decoding process. In this example we see that the minimum distance w.r.t. the three points labelled by "⊡ " is much greater than the minimum distance w.r.t. all $q^K = 9$ points. This suggests that the end-to-end error probability will be reduced by exploiting the side information provided by the relay.

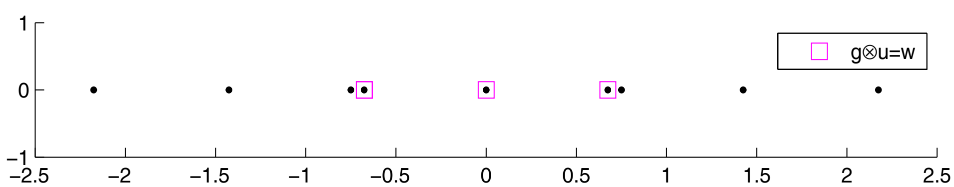

Fig. 2. Illustration of the constellations at the destination, where $\mathbf{H}_D = [1.43\,, 0.75]$ and $q = 3$

We note that in our proposed scheme, the relay tries to reconstruct *multiple* NC messages instead of only one message in [20]. Our work is also different from [24] and [25] where non-PNC scheme is used.

IV. On the Optimal Design of the Proposed LPNC Scheme

A. Design Problem Formulation

In this section, we investigate how to design the proposed LPNC scheme to minimize the end-to-end decoding error probability at the destination. Note that the freedom in designing the proposed scheme is the selection of the NC generator matrix $\mathbf{G}$ at the relay. Therefore, the design problem is formulated as finding

$$\mathbf{G}_{opt} = \arg\min_{\mathbf{G}} \Pr\left(\widehat{\mathbf{u}} \neq \mathbf{u} | \mathbf{G}, \mathbf{H}_R\right), \;\; s.t. \text{ Rank}(\mathbf{G}) = L \tag{14}$$

at the relay. We emphasize that the relay needs to find $\mathbf{G}_{opt}$ without the knowledge of $\mathbf{H}_D$, as local receiver-side CSI is considered in this work.

In general, finding the exact solution to (14) is very difficult. Here, we aim at finding a solution to (14) at a high SNR, i.e. $\rho \to \infty$. To this end, we first present a lemma.

Lemma 1: As $\rho \to \infty$, the solution to (14) satisfies

$$\mathbf{G}_{opt} = \arg\min_{\mathbf{G}} \Pr\left(\widehat{\mathbf{w}} \neq \mathbf{w} | \mathbf{G}, \mathbf{H}_R\right), \;\; s.t. \text{ Rank}(\mathbf{G}) = L. \tag{15}$$

Remark 5: Lemma 1 shows that at a high SNR, to minimize the end-to-end decoding error probability $\Pr\left(\widehat{\mathbf{u}} \neq \mathbf{u}\right)$ at the destination, one needs to find the matrix $\mathbf{G}$ that minimizes $\Pr\left(\widehat{\mathbf{w}} \neq \mathbf{w}\right)$ at the relay.

Proof: Since the relay has no knowledge of $\mathbf{H}_D$, the problem in (14) is written as

$$\mathbf{G}_{opt} = \arg\min_{\mathbf{G}} \int \Pr\left(\widehat{\mathbf{u}} \neq \mathbf{u} | \mathbf{G}, \mathbf{H}_R, \mathbf{H}_D\right) p\left(\mathbf{H}_D\right) d\mathbf{H}_D. \tag{16}$$

Using the law of total probability, we can further write (16) as

$$\mathbf{G}_{opt} = \arg\min_{\mathbf{G}} \left(P_1 + P_2\right), \tag{17}$$

where

$$\begin{aligned} P_1 &= \int \Pr\left(\widehat{\mathbf{u}} \neq \mathbf{u} | \widehat{\mathbf{w}} = \mathbf{w}, \mathbf{G}, \mathbf{H}_R, \mathbf{H}_D\right) \Pr\left(\widehat{\mathbf{w}} = \mathbf{w} | \mathbf{G}, \mathbf{H}_R, \mathbf{H}_D\right) p\left(\mathbf{H}_D\right) d\mathbf{H}_D, \\ P_2 &= \int \Pr\left(\widehat{\mathbf{u}} \neq \mathbf{u} | \widehat{\mathbf{w}} \neq \mathbf{w}, \mathbf{G}, \mathbf{H}_R, \mathbf{H}_D\right) \Pr\left(\widehat{\mathbf{w}} \neq \mathbf{w} | \mathbf{G}, \mathbf{H}_R, \mathbf{H}_D\right) p\left(\mathbf{H}_D\right) d\mathbf{H}_D. \end{aligned} \tag{18}$$

Here, P_1 and P_2 can be further rewritten as

$$\begin{aligned} P_1 &= \int \Pr\left(\widehat{\mathbf{u}} \neq \mathbf{u} | \widehat{\mathbf{w}} = \mathbf{G}^T \otimes \mathbf{u}, \mathbf{H}_D\right) p\left(\mathbf{H}_D\right) d\mathbf{H}_D \cdot \left(1 - \Pr\left(\widehat{\mathbf{w}} \neq \mathbf{w} | \mathbf{G}, \mathbf{H}_R\right)\right), \\ P_2 &= \int \Pr\left(\widehat{\mathbf{u}} \neq \mathbf{u} | \widehat{\mathbf{w}} \neq \mathbf{G}^T \otimes \mathbf{u}, \mathbf{H}_D\right) p\left(\mathbf{H}_D\right) d\mathbf{H}_D \cdot \Pr\left(\widehat{\mathbf{w}} \neq \mathbf{w} | \mathbf{G}, \mathbf{H}_R\right), \end{aligned} \tag{19}$$

where we have used the fact that $\Pr(\widehat{\mathbf{w}} \neq \mathbf{w}|\mathbf{G}, \mathbf{H}_R, \mathbf{H}_D)$ is independent of $\mathbf{H}_D$.

It is known that the deep fading probability of the user-destination channel approaches zero as $\rho \to \infty$, i.e., $\Pr(\widehat{\mathbf{u}} \neq \mathbf{u}|\mathbf{H}_D)\, p(\mathbf{H}_D)\, d\mathbf{H}_D \overset{\rho\to\infty}{\to} 0$. If the NC messages obtained from the relay are correct, i.e., $\widehat{\mathbf{w}} = \mathbf{G} \otimes \mathbf{u}$, they can improve error performance at the destination, and the error probability of recovering all K users' messages at the destination is expected to approach zero, i.e., the term $\Pr(\widehat{\mathbf{u}} \neq \mathbf{u}|\widehat{\mathbf{w}} = \mathbf{G} \otimes \mathbf{u}, \mathbf{H}_D)\, p(\mathbf{H}_D)\, d\mathbf{H}_D$ in P_1 vanishes to zero. On the other hand, if the NC messages obtained from the relay are erroneous, i.e., $\widehat{\mathbf{w}} \neq \mathbf{G} \otimes \mathbf{u}$, they will degrade the error performance at the destination, and the error probability of recovering all K users' messages at the destination can not approach zero, i.e., the term $\Pr(\widehat{\mathbf{u}} \neq \mathbf{u}|\widehat{\mathbf{w}} \neq \mathbf{G} \otimes \mathbf{u}, \mathbf{H}_D)\, p(\mathbf{H}_D)\, d\mathbf{H}_D$ in P_2 does not vanish to zero. Consider the convex combination of these two terms with combination factors given by $\Pr(\widehat{\mathbf{w}} \neq \mathbf{w}|\mathbf{G}, \mathbf{H}_R)$ and $1 - \Pr(\widehat{\mathbf{w}} \neq \mathbf{w}|\mathbf{G}, \mathbf{H}_R)$, we have

$$\arg\min_{\mathbf{G}} (P_1 + P_2) \overset{\rho\to\infty}{\to} \arg\min_{\mathbf{G}} \Pr(\widehat{\mathbf{w}} \neq \mathbf{w}|\mathbf{G}, \mathbf{H}_R). \tag{20}$$

This finishes the proof. ■

B. Set-Distance Spectrum and Error Probability

Previously, Lemma 1 shows that at a high SNR, the problem in (14) becomes finding the generator matrix $\mathbf{G}$ that minimizes $\Pr(\widehat{\mathbf{w}} \neq \mathbf{w}|\mathbf{G}, \mathbf{H}_R)$ at the relay, as specified in (15). A naive approach to find the solution would be to do the exhaustive search. However, this naive approach is computationally prohibitive even for a small K and L. The remainder of this section is devoted to finding an explicit solution to (15). To this end, we first characterize the cost function $\Pr(\widehat{\mathbf{w}} \neq \mathbf{w}|\mathbf{G}, \mathbf{H}_R)$ in the problem specified in (15) at a high SNR.

1) Signal Constellations at the Relay: Consider a deterministic user-relay channel realization of $\mathbf{H}_R$. Define

$$\mathbf{x}_s \triangleq \mathbf{H}_R \mathbf{x}, \tag{21}$$

which is referred to as a *superimposed* (SI) *symbol vector*. Let the set

$$\mathcal{X}_s = \{\mathbf{x}_s : \mathbf{x}_s = \mathbf{H}_R \mathbf{x}\} \tag{22}$$

collects all possible SI symbol vectors. The cardinality of the set is q^K.

For a given NC generator matrix $\mathbf{G}$, we define the following set

$$\mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}) \triangleq \left\{ \mathbf{x}_s : \mathbf{x}_s = \mathbf{H}_R \mathbf{x}, \mathbf{x} = \left(\mathbf{u} - \frac{q-1}{2} \right) \frac{1}{\gamma}, \mathbf{G}^T \otimes \mathbf{u} = \mathbf{w} \right\}. \tag{23}$$

Here, $\mathcal{X}_s^{(\mathbf{G})}(\mathbf{w})$ collects all the SI symbol vectors whose underlying NC codewords are identical and equal to $\mathbf{w}$. Clearly we have

$$\mathcal{X}_s = \left\{ \mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}), \text{ for all } \mathbf{w} \in \{0, 1, \cdots, q-1\}^L \right\}. \tag{24}$$

Here, (24) can be viewed as a partition of all possible SI symbol vectors into q^L sets, where each set corresponds to a specific NC codeword.

2) Conventional Distance Spectrum: Let $\mathbf{x}$ and $\widetilde{\mathbf{x}}$ be two different user signal vectors. Define

$$\boldsymbol{\delta} \triangleq (\mathbf{x} - \widetilde{\mathbf{x}}) \gamma \tag{25}$$

as the *difference vector* (DV) w.r.t. to the pair of ($\mathbf{x}$, $\widetilde{\mathbf{x}}$). Consider all possible transmitted signal vector pairs, the entries of $\boldsymbol{\delta}$ belong to $\{1-q, 2-q, ..., q-2, q-1\}$.

At the relay, the squared Euclidean distance w.r.t. $(\mathbf{x}, \widetilde{\mathbf{x}})$ is

$$E_s \left\| \mathbf{H}_R (\mathbf{x} - \widetilde{\mathbf{x}}) \right\|^2 = \frac{E_s}{\gamma^2} \left\| \mathbf{H}_R \boldsymbol{\delta} \right\|^2. \tag{26}$$

Let

$$d_1 = \min_{\boldsymbol{\delta}_1 \in \{1-q, \ldots, q-1\}^K, \ \|\boldsymbol{\delta}_1\| \neq 0} \frac{E_s}{\gamma^2} \left\| \mathbf{H}_R \boldsymbol{\delta}_1 \right\|^2 \tag{27}$$

represent the minimum squared Euclidean distance for all possible $(\mathbf{x}, \widetilde{\mathbf{x}})$ pairs, and

$$\boldsymbol{\Delta}_1 = \arg \min_{\boldsymbol{\delta}_1 \in \{1-q, \ldots, q-1\}^K, \ \|\boldsymbol{\delta}_1\| \neq 0} \frac{E_s}{\gamma^2} \left\| \mathbf{H}_R \boldsymbol{\delta}_1 \right\|^2 \tag{28}$$

denote the associated DV. It is noteworthy that d_1 is the minimum distance in a conventional scheme that completely decodes all individual messages $u_1, \cdots, u_K$.

Let d_2 and $\boldsymbol{\Delta}_2$ respectively be the second smallest squared Euclidean distance for all possible $(\mathbf{x}, \widetilde{\mathbf{x}})$ pairs and the associated DV, subject to

$$\text{Rank}(\text{mod}([\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2], q)) = 2. \tag{29}$$

Similarly, let d_l and $\boldsymbol{\Delta}_l$ respectively be the lth smallest squared Euclidean distance for all possible $(\mathbf{x}, \widetilde{\mathbf{x}})$ pairs, subject to

$$\text{Rank}(\text{mod}([\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2, \cdots, \boldsymbol{\Delta}_l], q)) = l, \text{ for } l = 1, 2, \cdots. \tag{30}$$

Let $A(d_l)$ be the total number of events associated with d_l, $l = 1, 2, \cdots$. We call $A(d_l)$ multiplicity for the distance d_l. Define

$$\mathcal{A} \triangleq \{A(d_1), A(d_2), \cdots\}. \tag{31}$$

We refer to $\mathcal{A}$ as the *conventional distance spectrum* of SI symbol vectors. Note that this is the distance spectrum if the relay attempts to distinguish all individual messages of K users.

3) NC Set-distance Spectrum: In light of the spirit of PNC, the relay does not need to distinguish all individual messages of K users. Instead, the relay only needs to distinguish the SI symbols whose underlying NC codewords are different. The effective distance and the distance spectrum for the NC codewords are different from the conventional ones, and are described as follows.

For a given NC generator matrix $\mathbf{G}$, let

$$d_1^{(\mathbf{G})} = \min_{\mathbf{x_s} \in \mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}),\ \mathbf{x}'_\mathbf{s} \in \mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}'),\ \mathbf{w} \neq \mathbf{w}'} E_s \left\| \mathbf{x_s} - \mathbf{x}'_\mathbf{s} \right\|^2 \tag{32}$$

be the minimum squared Euclidean distance for all possible SI symbol vector pairs $(\mathbf{x_s},\ \mathbf{x}'_\mathbf{s})$ with different underlying NC codewords. Let $d_i^{(\mathbf{G})}$ be the ith smallest squared Euclidean distance for all possible SI symbol vector pairs $(\mathbf{x_s},\ \mathbf{x}'_\mathbf{s})$ with different underlying NC codewords. It's obvious that $d_i^{(\mathbf{G})}$ is the distance between two sets $\mathcal{X}_s^{(\mathbf{G})}(\mathbf{w})$ and $\mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}')$ with different underlying NC codewords. We refer to $d_i^{(\mathbf{G})}$ as the *NC set-distance* for the given NC generator matrix $\mathbf{G}$.

For a given NC generator matrix $\mathbf{G}$, let

$$A^{(\mathbf{G})}(d_i) = \left| \left\{ \mathbf{x}, \widetilde{\mathbf{x}} : d\left(\mathbf{x}_s, \mathbf{x}'_s\right) = d_i^{(\mathbf{G})}, \mathbf{x}_s \in \mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}), \mathbf{x}'_s \in \mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}'), \mathbf{w} \neq \mathbf{w}' \right\} \right| \tag{33}$$

be the number of events that the squared Euclidean distance of two SI vectors, with different underlying NC codewords, is equal to $d_i^{(\mathbf{G})}$. We call $A^{(\mathbf{G})}(d_i)$ multiplicity for the set-distance $d_i^{(\mathbf{G})}$. Define

$$\mathcal{A}^{(\mathbf{G})} = \left\{ A^{(\mathbf{G})}(d_1), A^{(\mathbf{G})}(d_2), \cdots \right\}. \tag{34}$$

We referred to $\mathcal{A}^{(\mathbf{G})}$ as the *NC set-distance spectrum.*

4) Error Probability at a High SNR: For a given user-relay channel realization, we can compute the conventional distance spectrum of SI vectors. For a given NC generator matrix $\mathbf{G}$, some elements in the conventional distance spectrum of (31) will not appear in the NC set-distance spectrum of (34). This is due to that different SI vectors may generate the same NC codeword, which means their corresponding NC set-distance becomes zero. Different NC generator matrices $\mathbf{G}$ will have different SI vectors with zero NC set-distance in their set-distance spectrum, which leads to different NC error (NCE) probability. Intuitively, at a high SNR, the minimum NC set-distance determines the error probability, as we will see next.

The pair-wise error probability of confusing an NC message vector $\mathbf{w}$ with an erroneous one $\mathbf{w}'$ is given by

$$\begin{aligned} P_e\left(\mathbf{w}\rightarrow\mathbf{w}'\right) &\leq \sum_{\mathbf{x}_s\in\mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}),\mathbf{x}_s'\in\mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}'),\mathbf{w}\neq\mathbf{w}'} P_e\left(\mathbf{x}_s\rightarrow\mathbf{x}_s'\right) \\ &= \sum_{\mathbf{x}_s\in\mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}),\mathbf{x}_s'\in\mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}'),\mathbf{w}\neq\mathbf{w}'} Q\left(\sqrt{\frac{\rho}{2}d(\mathbf{x}_s,\mathbf{x}_s')}\right). \end{aligned} \tag{35}$$

Thus, the NCE probability of the proposed scheme, averaged over all NC message vector $\mathbf{w}$, can be upper bounded by

$$\begin{aligned} P_e &= \sum_{\mathbf{w}} p(\mathbf{w})P_e(\mathbf{w}) = \frac{1}{q^L}\sum_{\mathbf{w}}\sum_{\mathbf{w}'}P_e(\mathbf{w}\rightarrow\mathbf{w}') \\ &\leq \frac{1}{q^L}\sum_{\mathbf{w}}\sum_{\mathbf{w}'}\sum_{\mathbf{x}_s\in\mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}),\mathbf{x}_s'\in\mathcal{X}_s^{(\mathbf{G})}(\mathbf{w}'),\mathbf{w}\neq\mathbf{w}'} Q\left(\sqrt{\frac{\rho}{2}d(\mathbf{x}_s,\mathbf{x}_s')}\right). \end{aligned} \tag{36}$$

C. Asymptotic Solution to (14)

Here we first consider the solution to (14) for a special case with $L=1$. This applies to a practical scenario where the BH rate constraint is very stringent and only one NC message can be forwarded by the relay to the destination. Let

$$\mathbf{v}_l \triangleq \mathrm{mod}\left(\mathbf{\Delta}_l, q\right), l=1,2,\cdots K. \tag{37}$$

Theorem 1: As $\rho\rightarrow\infty$, a solution to (14) for $L=1$ satisfies

$$\mathbf{g}_{opt}^T \otimes \left[\mathbf{v}_1,\cdots,\mathbf{v}_{K-1}\right] = \mathbf{0}. \tag{38}$$

Corollary 1: When (38) is used to select the optimal NC coefficient vector $\mathbf{g}_{opt}$, the minimum set-distance $d_1^{(\mathbf{G})}$ for $\mathbf{g}_{opt}$ is equal to the Kth conventional minimum distance d_K.

Remark 6: The vectors $\mathbf{v}_1,\cdots,\mathbf{v}_{K-1}$ are derived from the DVs $\mathbf{\Delta}_1,\cdots,\mathbf{\Delta}_{K-1}$ as in (37), which correspond to the $K-1$ smallest values of distances in the conventional distance spectrum.

An NC coefficient vector $\mathbf{g}_{opt}$ satisfying (38) leads to

$$\mathbf{g}_{opt}^{T}\otimes\left(\bmod\left(\boldsymbol{\Delta}_{l},q\right)\right)=0,\ \text{for } l=1,2,\cdots,K-1. \tag{39}$$

Recall from (1), (25) and (28), the NC coefficient vector $\mathbf{g}_{opt}$ leads to

$$\mathbf{g}_{opt}^{T}\otimes\left(\bmod\left(\mathbf{u}_{l}-\mathbf{u}_{l}',q\right)\right)=0,\ \text{for } l=1,2,\cdots,K-1, \tag{40}$$

where $\mathbf{u}_l$ and $\mathbf{u}_l'$ represent the user message vectors lead to the lth minimum distance in the conventional distance spectrum. Recall the definition of NC message in (8), we can conclude from (40) that

$$\mathbf{w}_{l}-\mathbf{w}_{l}'=0,\ \text{for}\ \ l=1,2,\cdots,K-1. \tag{41}$$

It means that all the user message vectors lead to the conventional distance d_l, $l=1,2,\cdots,K-1$ have identical NC codeword. Therefore, the resultant minimum set-distance is d_K.

We next present the solution to (14) for a general L.

Theorem 2: As $\rho\to\infty$, a solution to (14) satisfies

$$\mathbf{G}_{opt}^{T}\otimes\mathbf{V}=\mathbf{T}, \tag{42}$$

where $\mathbf{T}$ is a lower-triangular matrix with non-zeros diagonal entries, i.e,

$$\mathbf{T}=\begin{bmatrix} a_{1,1} & 0 & \dots & 0 \\ a_{2,1} & a_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{L,1} & a_{L,2} & \dots & a_{L,K} \end{bmatrix} \tag{43}$$

where $a_{i,i}\neq 0$ for $i=1,\cdots,L$ and $\mathbf{V}=[\mathbf{v}_K,\mathbf{v}_{K-1},\cdots,\mathbf{v}_1]$.

Remark 7: The solution to (42) spans a L-dimension sub-space in the K-dimension finite field. The $\mathbf{G}_{opt}$ and the matrix that permutes the columns of $\mathbf{G}_{opt}$ are regarded as one solution.

Explanation: The first NC coefficient vector g_1 should be set in the way such that the underlying NC message is identical for the DVs $\boldsymbol{\delta}_1,\boldsymbol{\delta}_2,\cdots,\boldsymbol{\delta}_{K-1}$. The second NC coefficient vector g_2 should be set in the way such that the underlying NC message is identical for the DVs $\boldsymbol{\delta}_1,\boldsymbol{\delta}_2,\cdots,\boldsymbol{\delta}_{K-2}$ and Rank($[\mathrm{g}_1,\mathrm{g}_2]$)=2.

Proof of Theorem 2: Consider using the C-MAP algorithm. Without loss of generality, we order the NC coefficient vectors in the ascending order according to their error probabilities (i.e., the error probability w.r.t. g_1 is smallest and that w.r.t. $\mathbf{g}_L$ is highest).

The NC coefficient vector $\mathbf{g}_1$ satisfies

$$\mathbf{g}_1^T \otimes \mathbf{v}_l = 0, \forall l = 1, 2, \cdots, K-1. \tag{44}$$

The effective squared Euclidean distance w.r.t. the $\mathbf{g}_1$ is then d_K. Since there does not exist a K-dimension non-zero vector $\mathbf{g}_1$ which is perpendicular to all $\mathbf{v}_1, \cdots, \mathbf{v}_K$ in a K-dimension space, we have $a_{1,1} \neq 0$.

The NC coefficient vector $\mathbf{g}_2$ satisfies

$$\mathbf{g}_2^T \otimes \mathbf{v}_l = 0, \forall l = 1, 2, \cdots, K-2 \tag{45}$$

and $\mathrm{Rank}([\mathbf{g}_1, \mathbf{g}_2]) = 2$. The effective squared Euclidean distance w.r.t. $\mathbf{g}_2$ is then d_{K-1}. Since $\mathbf{g}_2$ and $\mathbf{g}_1$ span the two-dimension subspace that is orthogonal to the subspace spanned by $\mathbf{v}_1, \cdots,$ $\mathbf{v}_{K-2}$ and $\mathbf{g}_1^T \otimes \mathbf{v}_{K-1} = 0$, we have $\mathbf{g}_2 \otimes \mathbf{v}_{K-1} \neq 0$. This leads to $a_{2,2} \neq 0$.

Similarly, the NC coefficient vector $\mathbf{g}_l$ satisfies

$$\mathbf{g}_l^T \otimes \mathbf{v}_l = 0, \text{for } l = 1, 2, \cdots, K-l. \tag{46}$$

and $\mathrm{Rank}([\mathbf{g}_1, \cdots, \mathbf{g}_l]) = l$. Since $\mathbf{g}_l, \cdots, \mathbf{g}_1$ span the l-dimension subspace that is orthogonal to the subspace spanned by $\mathbf{v}_1, \cdots, \mathbf{v}_{K-l}$ and $\mathbf{g}_{l-1}^T \otimes \mathbf{v}_{K-l+1} = 0$, we have $\mathbf{g}_l \otimes \mathbf{v}_{K-l+1} \neq 0$. This leads to $a_{l,l} \neq 0$. Note that the squared Euclidean distance w.r.t. $\mathbf{g}_l$ is then d_{K-l+1}. Repeating this procedure, the effective distance w.r.t. $\mathbf{g}_L$ is d_{K-L+1}.

As $\rho \to \infty$, the error probability w.r.t. the lth NC coefficient vector $\mathbf{g}_l$ is upper-bounded by

$$P_e(w_l) \overset{\rho \to \infty}{\lesssim} A(d_{K-l+1}) Q\left(\frac{d_{K-l+1}}{2\sigma}\right), l = 1, \cdots, L, \tag{47}$$

where $A(d_{K-l+1})$ is the multiplicity w.r.t. the events with distance d_{K-l+1}, which is defined in (31). In addition, the error probability w.r.t. all NC coefficient vectors is given by

$$P_e \overset{\rho \to \infty}{\approx} \sum_{l=1,\cdots,L} A(d_{K-l+1}) Q\left(\frac{d_{K-l+1}}{2\sigma}\right). \tag{48}$$

It can be shown that other choice of $\mathbf{G}$ (except the matrices that permute the columns of $\mathbf{G}$) results in an error probability strictly no smaller than that in (48), as the effective minimum squared distance is strictly not greater than d_{K-L+1}. This completes the proof. ■

Remark 8: The solution of (42) may result in an NC coefficient vector $\mathbf{g}_l$ which contains one or multiple none-zero elements. When there is only one none-zero element in $\mathbf{g}_l$, relay decodes a *single user message* w.r.t $\mathbf{g}_l$. When there are multiple none-zero elements in $\mathbf{g}_l$, relay decodes an *NC message* w.r.t $\mathbf{g}_l$. Therefore, the optimal design of the proposed LPNC scheme is a generalization of single user decoding and PNC decoding.

V. Channel-coded LPNC and Information Combining scheme

In previous sections, we have presented an LPNC and information combining scheme for an un-channel-coded K-user MARN. In particular, we have shown an explicit optimal LPNC design solution that minimizes the end-to-end error probability at a high SNR. New design insights have been obtained from these developments.

In this section, we present a new channel-coded LPNC and information combining scheme for the MARN. Here, we focus on the real-valued model to present the scheme as the extension to the complex-valued model is straightforward. Block fading channel is considered in this section, where the channel fading coefficients remain the same within each channel-coded block while vary independently over different coded blocks. The proposed scheme is described below.

A. Encoding Using a Linear Code over GF(q) and q-PAM Modulation

In this work, we consider the symmetric rate case where all users have the same information rate. This scenario is particularly useful for the channel model under consideration where the transmitters do not have CSI.

Let $\mathbf{u}_k = [u_k[1], \cdots, u_k[m]] \in \{0, \cdots, q-1\}^m$ be a length-m *message* sequence of user k, $k \in \{1, \cdots, K\}$. In the channel coding, user k maps its message sequence to a length-n codeword sequence $\mathbf{c}_k = [c_k[1], \cdots, c_k[n]] \in \{0, \cdots, q-1\}^n$, $k \in \{1, \cdots, K\}$. The mapping can be represented as

$$\mathbf{c}_k = \mathbf{u}_k \otimes \mathbf{G}_c, k \in \{1, \cdots, K\}, \tag{49}$$

where the entries of the *channel coding generator matrix* $\mathbf{G}_c$ belongs to GF(q) and the size of $\mathbf{G}_c$ is m-by-n. We note that the same channel coding generator matrix $\mathbf{G}_c$ is used for all users following the spirit of CF [26]. The information rate for each user is given by $R = \frac{m}{n} \log_2 q$.

Remark 9: In this work, we propose to use the ensemble of random-coset IRA codes over GF(q) [13]. This ensemble of codes are practical as they have a simple encoding structure, where low-complexity iterative BP algorithm can be employed to directly compute the NC messages. Also, it has a flexible structure such that the code performance can be optimized using density evolution and a two-dimension extrinsic information transfer (EXIT) chart technique.

Similar to the un-channel-coded system, we consider q-PAM modulation in the channel-coded system. Then, the transmitted signal $\mathbf{x}_k[t]$ from user k and the received signal $\mathbf{y}_R[t]$ by the relay at time instant t is modelled similar as that in (1) and (2).

B. Iterative Decoding of NC Message Sequences at the Relay

Given $\mathbf{Y}_R = [\mathbf{y}_R[1], \mathbf{y}_R[2], \ldots, \mathbf{y}_R[n]]$, and the perfect knowledge of the channel matrix $\mathbf{H}_R$, the relay attempts to compute L linear message combinations. Here, let R_{BH} be the rate constraint of the BH per-block. To satisfy the BH rate constraint, it is obvious that L must satisfy

$$\frac{m}{n}\log_2^q \cdot L \leq R_{BH} \text{ or } L \leq \left\lfloor \frac{R_{BH}}{\frac{m}{n}\log_2^q} \right\rfloor. \tag{50}$$

For the ease of presentation, we will first illustrate the algorithms used at the relay for $L = 1$. After that, we will proceed to the general case with $L > 1$.

1) Processing at the Relay for $L = 1$: For a given NC coefficient vector $\mathbf{g} = [g_1, \cdots, g_K] \in \{0, \cdots, q-1\}^K$, the relay attempts to compute one linear message sequence combination

$$\mathbf{w} = \bigoplus_{k=1}^{K} g_k \otimes \mathbf{u}_k \tag{51}$$

associated with $\mathbf{g}$. We refer to $\mathbf{w}$ as an *NC message sequence*.

Let $\mathbf{U} = \left[\mathbf{u}_1^T, \cdots, \mathbf{u}_K^T\right]^T$. Then, the NC message sequence can be written as

$$\mathbf{w} = \mathbf{g}^T \otimes \mathbf{U}. \tag{52}$$

For $L = 1$, the MAP decoding rule for computing the NC message sequence is given by

$$\widehat{\mathbf{w}} = \arg \max_{\mathbf{w} \in \{0,\ldots,q-1\}^k} p\left(\mathbf{w} = \mathbf{g}^T \otimes \mathbf{U} | \mathbf{Y}_R\right). \tag{53}$$

Denote the decision made by the receiver as $\widehat{\mathbf{w}}$. An NC message sequence *error* is declared if $\widehat{\mathbf{w}} \neq \mathbf{w}$.

Now, let us turn our attention to the linear combination of the output of the channel encoder. Define the linear combination of the codeword sequence $\mathbf{c}_1, \ldots, \mathbf{c}_K$ as

$$\mathbf{c}_N = \bigoplus_{k=1}^{K} g_k \otimes \mathbf{c}_k. \tag{54}$$

We refer to $\mathbf{c}_N$ as an *NC codeword sequence*.

Let $\mathbf{C} = \left[\mathbf{c}_1^T, \cdots, \mathbf{c}_K^T\right]^T$. Then, the NC codeword sequence can be written as

$$\mathbf{c}_N = \mathbf{g}^T \otimes \mathbf{C}. \tag{55}$$

By using the same channel coding matrix $\mathbf{G}_c$ for all users, (53) is equivalent to

$$\widehat{\mathbf{c}}_N = \arg \max_{\mathbf{c}_N \in \{0,\ldots,q-1\}^n} p\left(\mathbf{c}_N = \mathbf{w} \otimes \mathbf{G}_c | \mathbf{Y}_R\right). \tag{56}$$

The complexity of the MAP decoding rule in (56) is exponential to the code length n. Here we suggest to use a practical decoding algorithm to approximate the MAP decoding solution with linear complexity to the code length n. This algorithm consists of a soft-input soft-output symbol-wise detector followed by an iterative BP decoder over GF(q), as detailed below.

The symbol-by-symbol detector calculates the symbol-wise a posteriori probability (APP) of $c_N[t]$, $t = 1, ..., n$, written as

$$\mathbf{p}\left(c_N[t] | \mathbf{y}_R[t]\right) = \mathbf{p}\left(\mathbf{g}^T \otimes \mathbf{c}[t] | \mathbf{y}_R[t]\right), t = 1, ...n, \tag{57}$$

where $\mathbf{c}[t]$ denotes the tth column of $\mathbf{C}$.

Note that for a linear code over GF(q), $\mathbf{p}\left(c_N[t] | \mathbf{y}_R[t]\right)$ is a q-dimension vector. Denote $p_i[t]$ as the detector's soft output when $c_N[t] = i, i = 0, ..., q-1$, then

$$\begin{aligned} p_i[t] &= p\left(\mathbf{g}^T \otimes \mathbf{c}[t] = i | \mathbf{y}_R[t]\right) = \frac{p\left(\mathbf{y}_R[t] | \mathbf{g}^T \otimes \mathbf{c}[t] = i\right) p\left(\mathbf{g}^T \otimes \mathbf{c}[t] = i\right)}{p\left(\mathbf{y}_R[t]\right)} \\ &= \sum_{\mathbf{c}[t] \in \{0,\ldots,q-1\}^K,\ \mathbf{g}^T \otimes \mathbf{c}[t] = i} \frac{p\left(\mathbf{y}_R[t] | \mathbf{g}^T \otimes \mathbf{c}[t]\right) p\left(\mathbf{g}^T \otimes \mathbf{c}[t]\right)}{p\left(\mathbf{y}_R[t]\right)}. \end{aligned} \tag{58}$$

In a fading channel with AWGN, the detector's soft output can be written as

$$p_i[t] = \frac{1}{\eta} \sum_{\mathbf{c}[t] \in \{0,\ldots,q-1\}^K,\ \mathbf{g}^T \otimes \mathbf{c}[t] = i} \exp\left(-\frac{\left\|\mathbf{y}_R[t] - \mathbf{H}_R\left(\mathbf{c}[t] - \frac{q-1}{2}\right)\frac{\sqrt{E_S}}{\gamma}\right\|^2}{2\sigma^2}\right), \tag{59}$$

where η is a normalization factor to ensure $\sum_{i=0}^{q-1} p_i[t] = 1$.

For simplicity, denote $\mathbf{p}\left(c_N[t] | \mathbf{y}_R[t]\right)$ as $\mathbf{p}[t] = \left[p_0[t], ..., p_{q-1}[t]\right], t = 1, ..., n$. With the detector's soft information $\mathbf{p}[t], t = 1, ..., n$, an iterative BP algorithm over GF(q) is employed to recover the NC message sequence $\mathbf{w}$ as in [13]. The details are omitted here.

2) Processing at the Relay for $L>1$: Given L linearly independent coefficient vectors $\mathbf{g}_1=[g_{1,1},\cdots,g_{K,1}],\cdots,\mathbf{g}_L=[g_{1,L},\cdots,g_{K,L}]$, the relay attempts to compute L NC message sequences

$$\mathbf{w}_l=\mathbf{g}_l^T\otimes\mathbf{U},\ l=1,2,\cdots L. \tag{60}$$

Denote L NC codeword sequences $\mathbf{c}_{N,l}$ corresponding to the L NC message sequences by

$$\mathbf{c}_{N,l}=\mathbf{g}_l^T\otimes\mathbf{C},\ l=1,2,\cdots L. \tag{61}$$

The relay first calculates the soft information $\mathbf{p}^{(l)}[t],t=1,...,n,l=1,...,L$ for L NC codeword sequences $\mathbf{c}_{N,l}$. For the tth symbol of the lth NC codeword sequences, the $(i+1)$th component of the soft information is calculated by

$$p_i^l[t]=\frac{1}{\eta}\sum_{\mathbf{c}[t]\in\{0,...,q-1\}^K,\ \mathbf{g}_l^T\otimes\mathbf{c}[t]=i}\exp\left(-\frac{\left\|\mathbf{y}_R[t]-\mathbf{H}_R\left(\mathbf{c}[t]-\frac{q-1}{2}\right)\frac{\sqrt{E_S}}{\gamma}\right\|^2}{2\sigma^2}\right),i=0,...,q-1, \tag{62}$$

where η is a normalization factor to ensure $\sum_{i=0}^{q-1}p_i^l[t]=1$, $\mathbf{g}_l$ is the lth NC coefficient vector calculated by (42). Then, soft information $\mathbf{p}^l[t],t=1,...,n,l=1,...,L$, are fed to L iterative BP decoders over GF(q) to recover the L NC messages sequences $\mathbf{w}_1,...,\mathbf{w}_L$.

3) Forwarding the NC Message Sequences via Backhaul: We assume that the relay is able to identify whether the decoding of a NC message sequence is successful (i.e. free of error) or not. In practice, this can be done by introducing a CRC check or using the cross entropy in the iterative decoding process for stoping criterion. Assuming $L',L'\leq L$ NC message sequences are successfully recovered, the relay re-encodes these NC message sequences by using the same channel code over GF(q) and forwards the NC codeword sequences to the destination via the BH. The BH is subjected to a rate constraint R_{BH}. It is obvious that the entropy of the L' NC message sequences should strictly meet the BH rate constraint. Thus, with a good channel code, the NC message sequences can be reliably delivered to the destination.

C. Information Combining and Iterative Decoding at the Destination

The signal received by the destination at time instant t is modelled similar as that in (3). Let $\mathbf{Y}_D=[\mathbf{y}_D[1],\mathbf{y}_D[2],\ldots,\mathbf{y}_D[n]]$. By combining the received signal $\mathbf{Y}_D$ with the side information of NC codeword sequences $\mathbf{c}_{N,1},...,\mathbf{c}_{N,L'}$ and the NC coefficient vectors $\mathbf{g}_1,...,\mathbf{g}_{L'}$ respectively, the destination aims to recover the messages of all K users.

The information combining process based on the MAP sequence decoding rule is given by

$$\widehat{\mathbf{U}} = \arg\max_{\mathbf{U}} p\left(\mathbf{U}|\mathbf{Y}_D, \mathbf{c}_{N,1}, ..., \mathbf{c}_{N,L'}, \mathbf{G}\right). \tag{63}$$

An user message sequence *error* is declared if $\widehat{\mathbf{U}} \neq \mathbf{U}$.

In our proposed decoding method, the destination first calculates the symbol-wise APP for K users. For the kth user's code symbol $t, t = 1, ..., n$, the symbol-wise APP is given by

$$\begin{aligned}
&\mathbf{p}^k\left(c_k[t]\,|\mathbf{y}_D[t], \mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t], \mathbf{G}\right)\\
&\underset{(a)}{=} \mathbf{p}^k\left(c_k[t]\,|\mathbf{y}_D[t]\right)\mathbf{p}^k\left(c_k[t]\,|\mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t], \mathbf{G}\right)\\
=&\frac{\mathbf{p}^k\left(\mathbf{y}_D[t]\,|c_k[t]\right)\mathbf{p}^k\left(c_k[t]\right)}{\mathbf{p}^k\left(\mathbf{y}_D[t]\right)}\frac{\mathbf{p}^k\left(\mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t], \mathbf{G}|c_k[t]\right)\mathbf{p}^k\left(c_k[t]\right)}{\mathbf{p}^k\left(\mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t], \mathbf{G}\right)}.
\end{aligned} \tag{64}$$

Here we used the fact that the received signal $\mathbf{y}_D[t]$ at the destination is independent from the side-information $\mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t]$ and $\mathbf{G}$ forwarded by the relay in step (a).

Denote the $(i+1)$th component of $\mathbf{p}^k\left(c_k[t]\,|\mathbf{y}_D[t], \mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t], \mathbf{G}\right)$ as $p_i^k[t]$, then

$$\begin{aligned}
p_i^k[t] &= \sum_{c_k[t]=i}\frac{p^k\left(\mathbf{y}_D[t]\,|c_k[t]\right)p^k\left(c_k[t]\right)}{p^k\left(\mathbf{y}_D[t]\right)}\frac{p^k\left(\mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t], \mathbf{G}|c_k[t]\right)p^k\left(c_k[t]\right)}{p^k\left(\mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t], \mathbf{G}\right)}\\
&\underset{(a)}{=} \sum_{\substack{\mathbf{c}[t]\in\{0,...,q-1\}^K,\ c_k[t]=i\\ \mathbf{g}_1^T\otimes\mathbf{c}[t]=c_{N,1}[t],\ ...,\ \mathbf{g}_{L'}^T\otimes\mathbf{c}[t]=c_{N,L'}[t]}}\frac{p^k\left(\mathbf{y}_D[t]\,|c_k[t]\right)p^k\left(c_k[t]\right)}{p^k\left(\mathbf{y}_D[t]\right)}.
\end{aligned} \tag{65}$$

In step (a) of (65), the side-information $\mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t]$ and $\mathbf{G}$ forwarded by the relay are combined with the received users' signals $\mathbf{y}_D[t]$ to improve the estimation of $p_i^k[t]$. In particularly, the information combining is done as that if the tth code symbol vector $\mathbf{c}[t]$ satisfy all the constraints $\mathbf{g}_1^T \otimes \mathbf{c}[t] = c_{N,1}[t], ..., \mathbf{g}_{L'}^T \otimes \mathbf{c}[t] = c_{N,L'}[t]$ introduced by the side-information, the conditional probability $p^k\left(\mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t], \mathbf{G}|c_k[t]\right)$ is set to one. Otherwise, the conditional probability $p^k\left(\mathbf{c}_{N,1}[t], \cdots, \mathbf{c}_{N,L'}[t], \mathbf{G}|c_k[t]\right)$ is set to zero. The underlying notion is the same as that in the un-channel-coded case described in Section III.

Given the user-destination channel realization $\mathbf{H}_D$, (65) can be written as

$$p_i^k[t] = \eta \sum_{\substack{\mathbf{c}[t]\in\{0,...,q-1\}^K,\ c_k[t]=i\\ \mathbf{g}_1\otimes\mathbf{c}[t]=c_{N,1}[t],\ ...,\ \mathbf{g}_{L'}\otimes\mathbf{c}[t]=c_{N,L'}[t]}} \exp\left(-\frac{\left\|\mathbf{y}_D[t] - \mathbf{H}_D\left(\mathbf{c}[t] - \frac{q-1}{2}\right)\frac{\sqrt{E_S}}{\gamma}\right\|^2}{2\sigma^2}\right), \tag{66}$$

where η is a normalization factor to ensure $\sum_{i=0}^{q-1} p_i^k[t] = 1$. Next, the symbol-wise APPs are forwarded to K iterative BP decoders to recover the K users' messages $\mathbf{u}_1, ..., \mathbf{u}_K$.

VI. Numerical Results

A. Un-channel-coded System

1) Real-valued Model: This section presents numerical results of our proposed LPNC and information combining scheme for the Rayleigh fading MARN where the real-valued model is considered. Fig. 3 presents the simulated average SER performance at the destination of the proposed scheme for a two-user MARN. Each of the relay node and destination node has single-antenna ($N = 1$). Here we consider that the relay forwards $L = 2$ NC messages to the destination. We also plot the SER of a benchmark scheme in which the relay attempts to completely decode and forward all K users' messages. It is demonstrated that for $q = 3$, our proposed scheme with NC generator matrix $\mathbf{G}$ given in Theorem 2 outperforms the benchmark scheme by about 3 dB at the SER of 10^{-3}. For $q = 7$, our proposed scheme outperforms the benchmark scheme by about 7 dB. We observe that as q increases, the performance improvement of the proposed scheme over the benchmark scheme becomes greater.

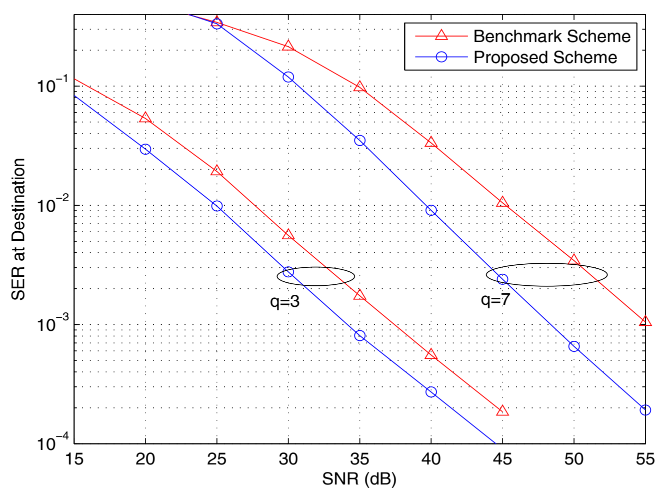

Fig. 3. SER at the destination for real-valued MARN with $K = 2$, $N = 1$, and $L = 2$.

We next consider a four-user ($K = 4$) real-valued MARN whose SER performance is shown in Fig. 4. Each of the relay node and destination node has two antennas ($N = 2$). The relay forwards $L = 4$ NC messages to the destination. Similar to Fig. 3, we include the SER of the complete decoding scheme. It is demonstrated that for $q = 3$, our proposed LPNC scheme outperforms the benchmark scheme by about 4 dB at the SER of 10^{-3}. For $q = 7$, our proposed scheme outperforms the benchmark scheme by about 7 dB. We again observe that the performance improvement of the proposed scheme over the benchmark scheme becomes greater as q increases.

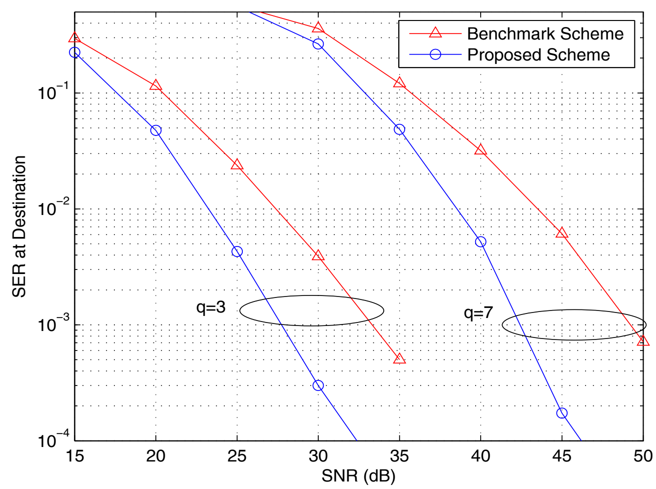

Fig. 4. SER at the destination for real-valued MARN with $K = 4$, $N = 2$, and $L = 4$.

2) Complex-valued Model: We next consider the complex-valued model. The equivalent real-valued model as presented in [16] is used in our simulations and LPNC design. In particular, we use Theorem 2 to obtain the optimized LPNC. Fig. 5 presents the simulated average SER performance at the destination of the two-user ($K = 2$) single-antenna ($N = 1$) complex-valued MARN, where the relay forwards $L = 4$ NC messages. We observe that our proposed scheme outperforms the benchmark scheme by more than 3 dB and 5 dB, for $q = 3$ and $q = 7$, respectively. We observe that the proposed scheme achieves a diversity order of $2N$ when $L = 2K$. Note that this is the maximal achievable diversity order of the scheme where the relay and the destination are co-located.

B. Channel-coded System

This section presents the numerical results of our proposed channel-coded LPNC and information combining scheme for MARN. We consider the end-to-end FER performance at the destination, i.e., $\Pr\left(\widehat{\mathbf{U}} \neq \mathbf{U}\right) = E_{\mathbf{H}_R,\mathbf{H}_D}\left\{\Pr\left(\widehat{\mathbf{U}} \neq \mathbf{U}|\mathbf{H}_{R,}\mathbf{H}_D\right)\right\}$. We evaluate the FER performance of our proposed LPNC scheme in MARN under different BH link constraints. More specifically, both wired and wireless BH links are considered in the simulations.

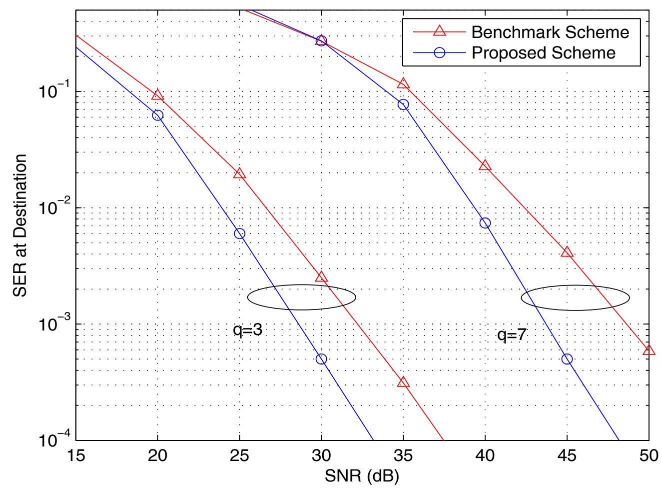

Fig. 5. SER at the destination for complex-valued MARN with $K = 2$, $N = 1$, and $L = 4$.

1) Wired Backhaul: In order to show the benefit of our proposed scheme, the FER performance of a benchmark scheme, which attempts to completely decode all users' messages at both relay and destination, is considered. Since the exact ML complete decoding has a forbidden complexity, an iterative multi-user detection and decoding (IDD) algorithm is used to approximate the ML complete decoding solution [30]. In particular, in the benchmark scheme, the relay does IDD with 8 receiver iterations to recover all K users' message sequences $\mathbf{u}_1, ..., \mathbf{u}_K$, and forwards all or part of the decoded users' message sequences to the destination, under the BH rate constraint. If the BH rate is limited such that only a part of the decoded users' message sequences can be forward to the destination, then the forwarded ones are selected randomly from all the decoded users' message sequences. The destination also does IDD to recover all K users' message sequences $\mathbf{u}_1, ..., \mathbf{u}_K$. If the message sequences successfully recovered by the destination and the message sequences forwarded by the relay contain all K users' message sequences, the end-to-end decoding is successful; otherwise, a decoding failure is declared.

Fig. 6 presents the simulated average FER performance at the destination of the two-user ($K = 2$) single-antenna ($N = 1$) complex-valued MARN with modulation level $q = 3$. We consider cases with $L = 1$, 2 and 4 NC messages or users' messages that can be forwarded to the destination through the BH link. It is demonstrated that our proposed and optimized LPNC scheme outperforms the benchmark scheme under all BH link constraints. For example, for $L = 4$, our proposed scheme outperforms the benchmark scheme by 2.95 dB at the FER of 10^{-3}

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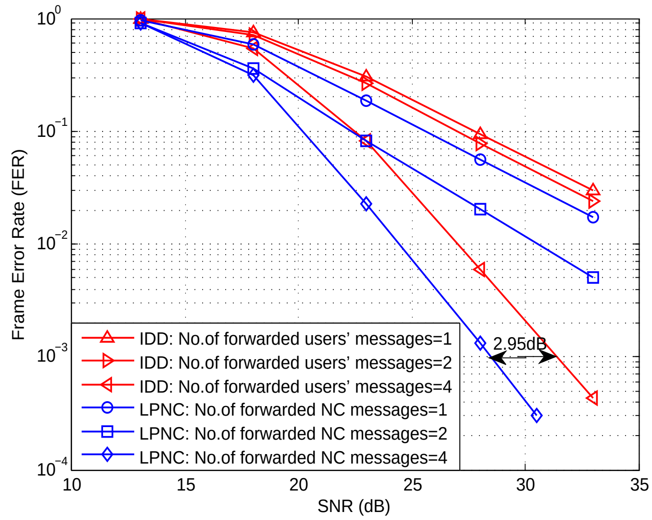

Fig. 6. FER at the destination for complex-valued MARN with wired backhaul, $K = 2$, $N = 1$, $L = 4$ and $q = 3$.

We next consider a two-user ($K = 2$) single-antenna ($N = 1$) complex-valued MARN with modulation level $q = 5$, whose FER performance is shown in Fig. 7. Here, $L = 1$, 2, and 4 NC message sequences or users' message sequences can be forward through the BH link. It is demonstrated that as the modulation level increases, the superiority of our proposed LPNC scheme becomes even more obvious compared to the benchmark scheme. In particular, for $L = 4$, our proposed scheme outperforms the benchmark scheme by 3.6 dB at the FER about 10^{-3}.

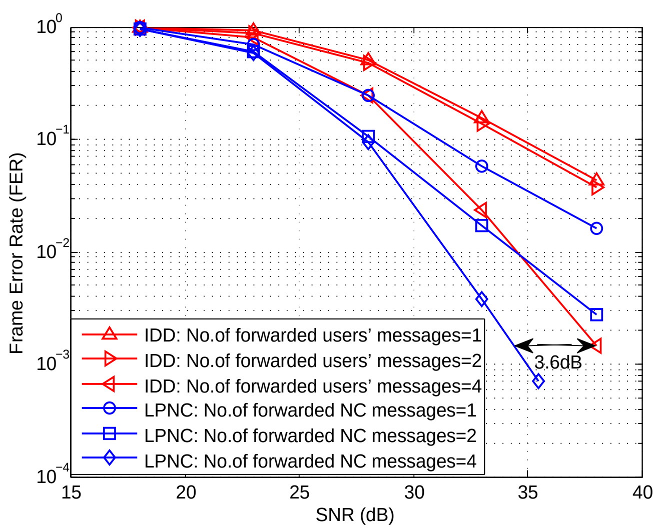

Fig. 7. FER at the destination for complex-valued MARN with wired backhaul, $K = 2$, $N = 1$, $L = 4$ and $q = 5$.

2) Wireless Backhaul: In this section, we show the FER performance of our proposed scheme in the MARN with wireless BH. We consider two cases where the CSI of the BH link is known or not known by the relay. When the relay has the CSI, the relay computes the instantaneous BH channel capacity and forwards an *appropriate* number of decoded NC message sequences/users' message sequences to the destination. The appropriate number of NC message sequences/users' message sequences have a sum rate less than the instantaneous BH capacity. When the relay does not have the CSI, the relay will send all the decoded NC message sequences/users' message sequences to the destination. If the sum rate of the decoded NC message sequences/users' message sequences is larger than the instantaneous BH channel capacity, we assume that all the message sequences forwarded from the relay are lost. If the sum rate of the decoded NC message sequences/users' message sequences is smaller than the instantaneous BH channel capacity, all the decoded NC messages/users' messages are assumed to be forwarded to the destination.

The two-user ($K = 2$) single-antenna ($N = 1$) complex-valued MARN with modulation level $q = 3$ and $q = 5$ are considered in the simulations. The wireless BH link SNR is set to be 5 dB higher than that of the user-relay (or user-destination) link in all the simulations. This is due to the fact that the BH link is usually much stronger than the user-relay (or user-destination) link.

Fig. 8 demonstrates the FER performance when the relay does not have the CSI of wireless BH link. It can be seen that, though the FER performance of both schemes are degraded compared to that of wired BH case, our proposed scheme still outperforms benchmark scheme significantly. In particulary, our proposed scheme outperforms benchmark scheme by more than 2.55 dB and 3 dB with modulation level $q = 3$ and $q = 5$, respectively, at the FER about 10^{-3}.

Fig. 9 shows the FER performance when the relay knows the CSI of the wireless BH link. It can be seen that, with CSI at the relay, the performance gain of our proposed scheme compared to the benchmark scheme is lager than that without CSI at the relay. In particulary, our proposed scheme outperforms benchmark scheme by more than 3.55 dB with modulation level $q = 3$ and 4.5 dB with modulation level $q = 5$ at the FER about 10^{-3}, respectively. This FER performance improvement suggests that with CSI of BH link at the relay, the relay can selects an appropriate number of decoded NC message sequences to be forwarded to the destination, and these forwarded NC message sequences will help the destination to decode the users' signals.

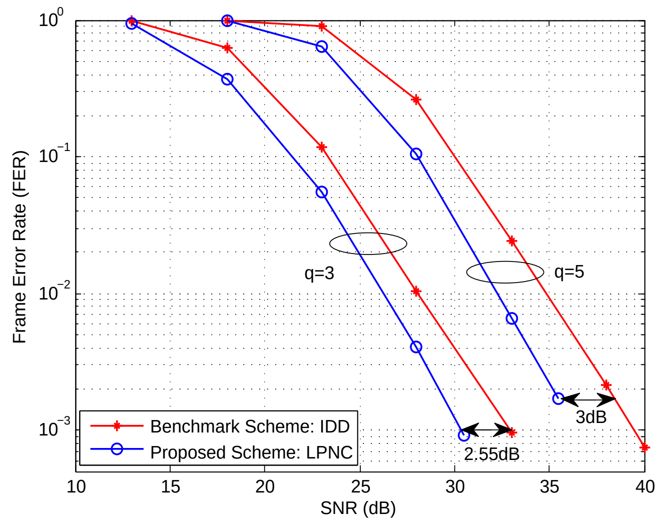

Fig. 8. FER at the destination for complex-valued MARN with wireless backhaul, $K = 2$, $N = 1$ and $L = 4$. The relay does not have the CSI of the backhaul link.

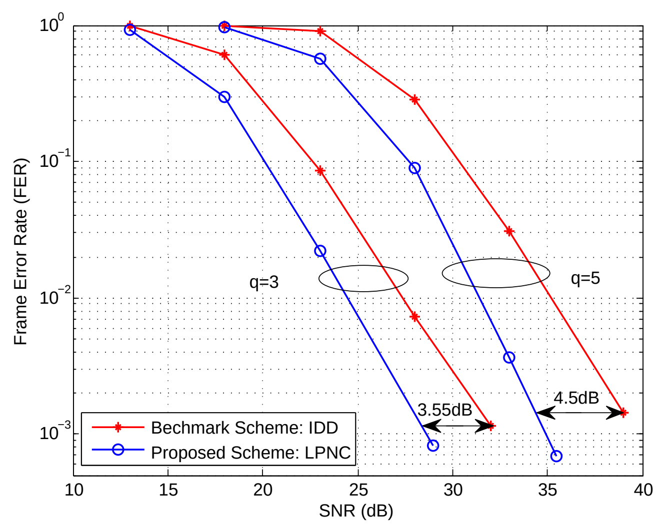

Fig. 9. FER at the destination for complex-valued MARN with wireless backhaul, $K = 2$, $N = 1$ and $L = 4$. The relay has the CSI of the backhaul link.

VII. Conclusions

We proposed a new LPNC and information combining scheme for the K-user MARN. In the proposed scheme, the relay reconstructs multiple NC messages and forwards them to the destination. Then, the destination attempts to recover all K users' messages by *combining* its received signal and the NC messages obtained from the relay. We developed an explicit expression on the selection of the NC coefficients that minimizes the end-to-end error probability.

We developed a channel-coded LPNC and information combining scheme with iterative BP algorithm for decoding at both nodes. Numerical results showed that our proposed scheme outperforms the complete decoding scheme by 3 - 7 dB in an un-channel-coded MARN and outperforms the IDD scheme by 2.95 - 4.5 dB in a channel-coded MARN, respectively, at practical SNR values.

The work in this paper can be enriched in various aspects. For example, how to extend the analysis and the design of the LPNC from a single relay node to multiple relay nodes and how to exploit the uplink-downland duality for the LPNC based scheme remain as challenging tasks.

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