

# SUMMATION OF RECIPROALS WHICH INVOLVE PRODUCTS OF TERMS FROM GENERALIZED FIBONACCI SEQUENCES—PART II

**R. S. Melham**

Department of Mathematical Sciences, University of Technology  
Sydney, PO Box 123, Broadway, NSW 2007 Australia

(Submitted May 1999)

## 1. INTRODUCTION

We consider the sequence  $\{W_n\}$  defined, for all integers  $n$ , by

$$W_n = pW_{n-1} + W_{n-2}, \quad W_0 = a, \quad W_1 = b. \quad (1.1)$$

Here  $a$ ,  $b$ , and  $p$  are real numbers with  $p \neq 0$ . Write  $\Delta = p^2 + 4$ . Then it is known [3] that

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \quad (1.2)$$

where  $\alpha = (p + \sqrt{\Delta})/2$ ,  $\beta = (p - \sqrt{\Delta})/2$ ,  $A = b - a\beta$ , and  $B = b - a\alpha$ . As in [3], we will put  $e_W = AB = b^2 - pab - a^2$ .

We define a companion sequence  $\{\bar{W}_n\}$  of  $\{W_n\}$  by

$$\bar{W}_n = A\alpha^n + B\beta^n. \quad (1.3)$$

Aspects of this sequence have been treated, for example, in [2] and [4].

For  $(W_0, W_1) = (0, 1)$ , we write  $\{W_n\} = \{U_n\}$  and, for  $(W_0, W_1) = (2, p)$ , we write  $\{W_n\} = \{V_n\}$ . The sequences  $\{U_n\}$  and  $\{V_n\}$  are generalizations of the Fibonacci and Lucas sequences, respectively. From (1.2) and (1.3) we see that  $\bar{U}_n = V_n$  and  $\bar{V}_n = \Delta U_n$ . Thus, it is clear that  $e_U = 1$  and  $e_V = -\Delta = -(\alpha - \beta)^2$ .

The purpose of this paper is to investigate the infinite sums

$$S_{k,m} = \sum_{n=1}^{\infty} \frac{\bar{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}}, \quad (1.4)$$

and

$$T_{k,m} = \sum_{n=1}^{\infty} \frac{1}{W_{kn}W_{k(n+m)}W_{k(n+2m)}W_{k(n+3m)}}, \quad (1.5)$$

where  $k$  and  $m$  are positive integers with  $k$  even. Indeed,  $S_{k,m}$  and the alternating sum derived from  $T_{k,m}$  have been studied in [5], where  $k$  and  $m$  were assumed to be odd positive integers. Both sums were expressed in terms of an infinite sum, and certain finite sums. Here, however, with the altered constraints on  $k$  and  $m$ , we express  $S_{k,m}$  and  $T_{k,m}$  in terms of finite sums only.

Now, if  $p > 0$ , then  $\alpha > 1$  and  $\alpha > |\beta|$ , so that

$$W_n \equiv \frac{A}{\alpha - \beta} \alpha^n \quad \text{and} \quad \bar{W}_n \equiv A\alpha^n. \quad (1.6)$$

On the other hand, if  $p < 0$ , then  $\beta < -1$  and  $|\beta| > |\alpha|$ , and so

$$W_n \equiv \frac{-B}{\alpha - \beta} \beta^n \quad \text{and} \quad \bar{W}_n \equiv B\beta^n. \tag{1.7}$$

Hence, assuming that  $a$  and  $b$  are chosen so that no denominator vanishes, we see from the ratio test that  $S_{k,m}$  and  $T_{k,m}$  are absolutely convergent.

**2. PRELIMINARY RESULTS**

We require the following, in which  $k$  and  $m$  are taken to be integers with  $k$  even.

$$\frac{\beta^{kn}}{W_{kn}} - \frac{\beta^{k(n+m)}}{W_{k(n+m)}} = \frac{AU_{km}}{W_{kn}W_{k(n+m)}}, \tag{2.1}$$

$$W_{k(n+m)}W_{k(n+2m)} - W_{kn}W_{k(n+3m)} = e_W U_{km} U_{2km}, \tag{2.2}$$

$$W_{n+k} - W_{n-k} = \bar{W}_n U_k, \tag{2.3}$$

$$B\beta^n = W_{n+1} - \alpha W_n. \tag{2.4}$$

Identities (2.1) and (2.2) are readily proved with the use of (1.2) and (1.3). Identity (2.3) is a special case of (75) in [2], while (2.4) can be obtained from (3.2) in [1].

We will also make use of the following lemma.

**Lemma 1:** Let  $k$  and  $m$  be positive integers with  $k$  even. Then

$$\sum_{n=1}^{\infty} \frac{1}{W_{kn}W_{k(n+m)}} = \frac{1}{e_W U_{km}} \left[ \sum_{n=1}^m \frac{W_{kn+1}}{W_{kn}} - m\alpha \right]. \tag{2.5}$$

**Proof:** If we sum both sides of (2.1), we obtain

$$\sum_{n=1}^{\infty} \frac{1}{W_{kn}W_{k(n+m)}} = \frac{1}{AU_{km}} \sum_{n=1}^m \frac{\beta^{kn}}{W_{kn}},$$

and (2.5) follows from (2.4) and the fact that  $e_W = AB$ . □

In fact, under the hypotheses of Lemma 1, Theorem 2' of [1] yields

$$\sum_{n=1}^{\infty} \frac{1}{W_{kn}W_{k(n+m)}} = \frac{1}{e_W U_k U_{km}} \left[ \sum_{n=1}^m \frac{W_{k(n+1)}}{W_{kn}} - m\alpha^k \right]. \tag{2.6}$$

To see that (2.6) reduces to (2.5), we use the identities  $\alpha^k = U_k \alpha + U_{k-1}$  and  $W_{k(n+1)} = U_k W_{kn+1} + U_{k-1} W_{kn}$ . From the first of these, which is easily proved by induction, we obtain the second if we first note that  $\alpha^{kn+k} = U_k \alpha^{kn+1} + U_{k-1} \alpha^{kn}$ , and write down the corresponding result involving  $\beta$ .

**3. THE MAIN RESULTS**

Our main results can now be given in two theorems.

**Theorem 1:** Let  $k$  and  $m$  be positive integers with  $k$  even. Then

$$S_{k,m} = \frac{1}{U_{km}} \sum_{n=1}^m \frac{1}{W_{kn}W_{k(n+m)}}. \tag{3.1}$$

**Proof:** Consider the expression

$$\frac{\beta^{kn}}{W_{kn}} - \frac{\beta^{k(n+m)}}{W_{k(n+m)}} + \frac{\beta^{k(n+2m)}}{W_{k(n+2m)}}. \tag{3.2}$$

Using (2.1), we can write this as

$$\frac{AU_{km}}{W_{kn}W_{k(n+m)}} + \frac{\beta^{k(n+2m)}}{W_{k(n+2m)}} \tag{3.3}$$

or as

$$\frac{\beta^{kn}}{W_{kn}} - \left[ \frac{\beta^{k(n+m)}}{W_{k(n+m)}} - \frac{\beta^{k(n+2m)}}{W_{k(n+2m)}} \right] = \frac{\beta^{kn}}{W_{kn}} - \frac{AU_{km}}{W_{k(n+m)}W_{k(n+2m)}}. \tag{3.4}$$

Now

$$\begin{aligned} \frac{AU_{km}}{W_{kn}W_{k(n+m)}} - \frac{AU_{km}}{W_{k(n+m)}W_{k(n+2m)}} &= \frac{AU_{km}}{W_{k(n+m)}} \left[ \frac{1}{W_{kn}} - \frac{1}{W_{k(n+2m)}} \right] \\ &= \frac{AU_{km}}{W_{k(n+m)}} \left[ \frac{W_{k(n+2m)} - W_{kn}}{W_{kn}W_{k(n+2m)}} \right] \\ &= \frac{AU_{km}^2 \overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}}, \text{ by (2.3).} \end{aligned} \tag{3.5}$$

But from (3.2)-(3.4), we then have

$$2 \left[ \frac{\beta^{kn}}{W_{kn}} - \frac{\beta^{k(n+m)}}{W_{k(n+m)}} + \frac{\beta^{k(n+2m)}}{W_{k(n+2m)}} \right] = \frac{\beta^{kn}}{W_{kn}} + \frac{\beta^{k(n+2m)}}{W_{k(n+2m)}} + \frac{AU_{km}^2 \overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}},$$

so that

$$\frac{AU_{km}^2 \overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}} = \left[ \frac{\beta^{kn}}{W_{kn}} - \frac{\beta^{k(n+m)}}{W_{k(n+m)}} \right] - \left[ \frac{\beta^{k(n+m)}}{W_{k(n+m)}} - \frac{\beta^{k(n+2m)}}{W_{k(n+2m)}} \right]. \tag{3.6}$$

Finally, summing both sides of (3.6), we obtain

$$AU_{km}^2 S_{k,m} = \sum_{n=1}^m \frac{\beta^{kn}}{W_{kn}} - \sum_{n=1}^m \frac{\beta^{k(n+m)}}{W_{k(n+m)}},$$

and (3.1) follows from (2.1).  $\square$

If we put  $W_n = F_n$  and  $W_n = L_n$ , and take  $k = 2$  and  $m = 1$ , (3.1) becomes, respectively,

$$\sum_{n=1}^{\infty} \frac{L_{2n+2}}{F_{2n}F_{2n+2}F_{2n+4}} = \frac{1}{3}, \tag{3.7}$$

and

$$\sum_{n=1}^{\infty} \frac{F_{2n+2}}{L_{2n}L_{2n+2}L_{2n+4}} = \frac{1}{105}. \tag{3.8}$$

**Theorem 2:** Let  $k$  and  $m$  be positive integers with  $k$  even. Then

$$e_W U_{km} U_{2km} T_{k,m} = \frac{1}{e_W} \left[ \frac{1}{U_{3km}} \sum_{n=1}^{3m} \frac{W_{kn+1}}{W_{kn}} - \frac{1}{U_{km}} \sum_{n=1}^m \frac{W_{kn+1}}{W_{kn}} \right] + \sum_{n=1}^m \frac{1}{W_{kn}W_{k(n+m)}} + \frac{m\alpha}{e_W} \left[ \frac{1}{U_{km}} - \frac{3}{U_{3km}} \right]. \tag{3.9}$$

**Proof:** From (2.2), we see that

$$\frac{e_W U_{km} U_{2km}}{W_{kn} W_{k(n+m)} W_{k(n+2m)} W_{k(n+3m)}} = \frac{1}{W_{kn} W_{k(n+3m)}} - \frac{1}{W_{k(n+m)} W_{k(n+2m)}}$$

Summing both sides we obtain, with the aid of (2.5),

$$e_W U_{km} U_{2km} T_{k,m} = \frac{1}{e_W U_{3km}} \left[ \sum_{n=1}^{3m} \frac{W_{kn+1}}{W_{kn}} - 3m\alpha \right] - \left[ \frac{1}{e_W U_{km}} \left[ \sum_{n=1}^m \frac{W_{kn+1}}{W_{kn}} - m\alpha \right] - \sum_{n=1}^m \frac{1}{W_{kn} W_{k(n+m)}} \right],$$

which is (3.9).  $\square$

If we put  $W_n = F_n$  and  $W_n = L_n$ , and take  $k = 2$  and  $m = 1$ , (3.9) becomes, respectively,

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n} F_{2n+2} F_{2n+4} F_{2n+6}} = \frac{60\sqrt{5} - 133}{576}, \tag{3.10}$$

and

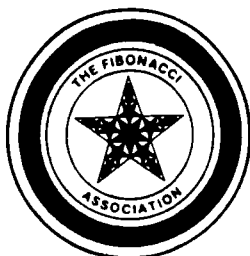
$$\sum_{n=1}^{\infty} \frac{1}{L_{2n} L_{2n+2} L_{2n+4} L_{2n+6}} = \frac{9\sqrt{5} - 20}{2160}. \tag{3.11}$$

**REFERENCES**

1. R. André-Jeannin. "Summation of Reciprocals in Certain Second-Order Recurring Sequences." *The Fibonacci Quarterly* **35.1** (1997):68-74.
2. G. E. Bergum & V. E. Hoggatt, Jr. "Sums and Products for Recurring Sequences." *The Fibonacci Quarterly* **13.2** (1975):115-20.
3. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.3** (1965):161-76.
4. C. T. Long. "Some Binomial Fibonacci Identities." In *Applications of Fibonacci Numbers 3*: 241-54. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1990.
5. R. S. Melham. "Summation of Reciprocals Which Involve Products of Terms from Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **38.4** (2000):294-98.

AMS Classification Numbers: 11B39, 11B37, 40A99





# The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

## TABLE OF CONTENTS

Some Basic Line-Sequential Properties of Polynomial Line-Sequences .....	<i>Jack Y. Lee</i>	194
On the Factorization of Lucas Numbers .....	<i>Wayne L. McDaniel</i>	206
On the Number of Maximal Independent Sets of Vertices In Star-Like Ladders .....	<i>Dragan Stevanović</i>	211
Reciprocal Sums of Second-Order Recurrent Sequences .....	<i>Hong Hu, Zhi-Wei Sun, and Jian-Xin Liu</i>	214
Remarks on the "Greedy Odd" Egyptian Fraction Algorithm .....	<i>Jukka Pihko</i>	221
Using Lucas Sequences to Factor Large Integers Near Group Orders .....	<i>Zhenxiang Zhang</i>	228
Rational Points in Cantor Sets .....	<i>Judit Nagy</i>	238
Diophantine Triplets and the Pell Sequence .....	<i>M.N. Deshpande and Ezra Brown</i>	242
An Algorithm for Determining $R(N)$ from the Subscripts of the Zeckendorf Representation of $N$ .....	<i>David A. Englund</i>	250
An Analysis of $n$ -Riven Numbers .....	<i>H.G. Grundman</i>	253
On the Representation of the Integers as a Difference of Nonconsecutive Triangular Numbers .....	<i>M.A. Nyblom</i>	256
Author and Title Index .....		263
Summation of Reciprocals Which Involve Products of Terms from Generalized Fibonacci Sequences—Part II .....	<i>R.S. Melham</i>	264
The Filbert Matrix .....	<i>Thomas M. Richardson</i>	268
Algorithmic Determination of the Enumerator for Sums of Three Triangular Numbers .....	<i>John A. Ewell</i>	276
Identities and Congruences Involving Higher-Order Euler-Bernoulli Numbers and Polynomials .....	<i>Guodong Liu</i>	279
New Problem Web Site .....		284
A New Recurrence Formula for Bernoulli Numbers .....	<i>Harunobu Momiyama</i>	285

VOLUME 39

JUNE-JULY 2001

NUMBER 3

## PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

## EDITORIAL POLICY

**THE FIBONACCI QUARTERLY** seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

## SUBMITTING AN ARTICLE

Articles should be submitted using the format of articles in any current issues of **THE FIBONACCI QUARTERLY**. They should be typewritten or reproduced typewritten copies, that are clearly readable, double spaced with wide margins and on only one side of the paper. The full name and address of the author must appear at the beginning of the paper directly under the title. Illustrations should be carefully drawn in India ink on separate sheets of bond paper or vellum, approximately twice the size they are to appear in print. Since the Fibonacci Association has adopted  $F_1 = F_2 = 1$ ,  $F_n + 1 = F_n + F_{n-1}$ ,  $n \geq 2$  and  $L_1 = 1$ ,  $L_2 = 3$ ,  $L_n + 1 = L_n + L_{n-1}$ ,  $n \geq 2$  as the standard definitions for The Fibonacci and Lucas sequences, these definitions *should not* be a part of future papers. However, the notations *must* be used. One to three *complete* A.M.S. classification numbers *must* be given directly after references or on the bottom of the last page. Papers not satisfying all of these criteria will be returned. See the new worldwide web page at:

<http://www.sdstate.edu/~wcsc/http/fibhome.html>

for additional instructions.

Three copies of the manuscript should be submitted to: **CURTIS COOPER, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, CENTRAL MISSOURI STATE UNIVERSITY, WARRENSBURG, MO 64093-5045.**

Authors are encouraged to keep a copy of their manuscripts for their own files as protection against loss. The editor will give immediate acknowledgment of all manuscripts received.

The journal will now accept articles via electronic services. However, electronic manuscripts must be submitted using the typesetting mathematical wordprocessor AMS-TeX. Submitting manuscripts using AMS-TeX will speed up the refereeing process. AMS-TeX can be downloaded from the internet via the homepage of the American Mathematical Society.

## SUBSCRIPTIONS, ADDRESS CHANGE, AND REPRINT INFORMATION

Address all subscription correspondence, including notification of address change, to: **PATTY SOLSAA, SUBSCRIPTIONS MANAGER, THE FIBONACCI ASSOCIATION, P.O. BOX 320, AURORA, SD 57002-0320. E-mail: [solsaap@itctel.com](mailto:solsaap@itctel.com).**

Requests for reprint permission should be directed to the editor. However, general permission is granted to members of The Fibonacci Association for noncommercial reproduction of a limited quantity of individual articles (in whole or in part) provided complete reference is made to the source.

Annual domestic Fibonacci Association membership dues, which include a subscription to **THE FIBONACCI QUARTERLY**, are \$40 for Regular Membership, \$50 for Library, \$50 for Sustaining Membership, and \$80 for Institutional Membership; foreign rates, which are based on international mailing rates, are somewhat higher than domestic rates; please write for details. **THE FIBONACCI QUARTERLY** is published each February, May, August and November.

All back issues of **THE FIBONACCI QUARTERLY** are available in microfilm or hard copy format from **BELL & HOWELL INFORMATION & LEARNING, 300 NORTH ZEEB ROAD, P.O. BOX 1346, ANN ARBOR, MI 48106-1346.** Reprints can also be purchased from **BELL & HOWELL** at the same address.

©2001 by

The Fibonacci Association

All rights reserved, including rights to this journal issue as a whole and, except where otherwise noted, rights to each individual contribution.

# The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)  
and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION  
DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

## EDITOR

PROFESSOR CURTIS COOPER, Department of Mathematics and Computer Science, Central Missouri State University, Warrensburg, MO 64093-5045 e-mail: cnc8851@cmsu2.cmsu.edu

## EDITORIAL BOARD

DAVID M. BRESSOUD, Macalester College, St. Paul, MN 55105-1899  
JOHN BURKE, Gonzaga University, Spokane, WA 99258-0001  
BART GODDARD, East Texas State University, Commerce, TX 75429-3011  
HENRY W. GOULD, West Virginia University, Morgantown, WV 26506-0001  
HEIKO HARBORTH, Tech. Univ. Carolo Wilhelmina, Braunschweig, Germany  
A.F. HORADAM, University of New England, Armidale, N.S.W. 2351, Australia  
STEVE LIGH, Southeastern Louisiana University, Hammond, LA 70402  
FLORIAN LUCA, Instituto de Matematicas de la UNAM, Morelia, Michoacan, Mexico  
RICHARD MOLLIN, University of Calgary, Calgary T2N 1N4, Alberta, Canada  
GARY L. MULLEN, The Pennsylvania State University, University Park, PA 16802-6401  
HARALD G. NIEDERREITER, National University of Singapore, Singapore 117543, Republic of Singapore  
SAMIH OBAID, San Jose State University, San Jose, CA 95192-0103  
ANDREAS PHILIPPOU, University of Patras, 26100 Patras, Greece  
NEVILLE ROBBINS, San Francisco State University, San Francisco, CA 94132-1722  
DONALD W. ROBINSON, Brigham Young University, Provo, UT 84602-6539  
LAWRENCE SOMER, Catholic University of America, Washington, D.C. 20064-0001  
M.N.S. SWAMY, Concordia University, Montreal H3G 1M8, Quebec, Canada  
ROBERT F. TICHY, Technical University, Graz, Austria  
ANNE LUDINGTON YOUNG, Loyola College in Maryland, Baltimore, MD 21210-2699

## BOARD OF DIRECTORS—THE FIBONACCI ASSOCIATION

G.L. ALEXANDERSON, *Emeritus*  
Santa Clara University, Santa Clara, CA 95053-0001  
CALVIN T. LONG, *Emeritus*  
Northern Arizona University, Flagstaff, AZ 86011  
FRED T. HOWARD, *President*  
Wake Forest University, Winston-Salem, NC 27109  
PETER G. ANDERSON, *Treasurer*  
Rochester Institute of Technology, Rochester, NY 14623-5608  
GERALD E. BERGUM  
South Dakota State University, Brookings, SD 57007-1596  
KARL DILCHER  
Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5  
ANDREW GRANVILLE  
University of Georgia, Athens, GA 30601-3024  
HELEN GRUNDMAN  
Bryn Mawr College, Bryn Mawr, PA 19101-2899  
MARJORIE JOHNSON, *Secretary*  
665 Fairlane Avenue, Santa Clara, CA 95051  
CLARK KIMBERLING  
University of Evansville, Evansville, IN 47722-0001  
JEFF LAGARIAS  
AT&T Labs-Research, Florham Park, NJ 07932-0971  
WILLIAM WEBB, *Vice-President*  
Washington State University, Pullman, WA 99164-3113